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Developing insight into teachers' didactical practice in geometric proof problem solving

by
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To my mum and in loving memory of my dad.

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ABSTRACT

FACULTY OF LAW, ARTS & SOCIAL SCIENCES
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DEVELOPING INSIGHT INTO TEACHERS' DIDACTICAL PRACTICE IN GEOMETRIC PROOF PROBLEM SOLVING

The teaching of geometrical proof has received changing amounts of emphasis in recent curriculum reforms across many countries. Literature on classroom teaching and learning shows that geometry is quite demanding work for many school teachers, and students are not always successful, especially with formal proof in geometry. Research is needed towards developing fundamental understanding of the complexity of students' cognitive structure and the significance of teachers' instruction in the learning and teaching of geometry.

This study investigates how school teachers in China approach the teaching of proof in geometry. In particular, this study focuses on two aims: 1) To explore and elucidate the complexity of individual teacher's didactical practice towards the development of students' thinking for writing proofs in geometry; 2) To understand in what way the van Hiele model is a useful research tool to help analyse and interpret classroom teaching and learning of geometrical proof problem solving.

This study concentrates on three teachers' regular classes at Grade 8 in two common lower secondary schools in Shanghai (students' age, 13-14 years old). The van Hiele model is applied to analyzing data in this study. The data analysis consists of case study and statistical analysis of the three teachers' lessons which were on the topic of the quadrilateral family (parallelogram, rectangle, rhombus and square) for approximately 13 hours over a three week long school curriculum. Interview data regarding teachers' didactical views and small-scale survey data of Grade 8 students' learning results on formal proof writing was also collected.

Findings of the study indicate that students' geometric thinking in solving geometrical proof problems appears to be more complex than that ascribed by the van Hieles. On the one hand, across a set of proof problems, the visual, analytic and deductive thinking may concurrently grow up together; on the other hand, however, they may limit each other's development. Thus, this study proposes a dynamic view of the van Hiele levels of geometric thinking. Moreover, findings from the study suggest taking account of the role of the teacher in building the bridge between students and subject in effective learning. Last, the study proposes a pedagogical framework to elucidate four aspects of classroom instruction (visual approach, empirical/deductive approach, teacher's questioning and task variation) in the support of the dynamic development of students' geometric thinking for writing proofs.

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CHAPTER 1. INTRODUCTION

1.1 The context of this study

1.1.1 The teaching of geometry in the secondary school

The overall aim of this research is to develop fundamental understanding of the complexity of teachers' didactical practice towards the development of students' thinking for writing proofs in secondary school geometry classes. In particular, the key focus is to analyse and interpret, in depth, what kind of instructional approaches and strategies may support the development of students' geometric thinking to make the proofs.

The problem of teaching and learning of geometry in school has been the subject of considerable concern in mathematics education research over a number of decades (Morris, 1986; the International Commission on Mathematical Instruction, 1994; Mammana and Villani, 1998, Clement and Battista, 1992, Battista, 2007). In the first place, the teaching of geometrical proof has received varying amounts of emphasis in recent curriculum reforms across many countries (Educational Ministry of People's Republic of China, 2001; NCTM, 1989; 2000; Royal Society & Joint Mathematical Council, 2001; QCA, 2007). Secondly, literature on the perspective of classroom teaching shows that geometry is quite demanding work for many school teachers (Herbst, 2003; Lampert, 1993). Thirdly, a wide range of international comparative research, national surveys, and researchers' small-scale studies have uncovered that students are often unsuccessful, in particular with formal proof, in geometry (Hoffer, 1981; Senk, 1985; 1989; the study group of an experimental teaching process in school mathematics in Qingpu (TSGofQp), 1991; Tian, 1990; Usiskin, 1982; Wirszup, 1976; Xie and Tan, 1997). In addition, the analysis of classroom instruction reveals a far more devastating finding: students learn to separate the worlds of deductive and empirical geometry (Schoenfeld, 1988, 1989).

To understand better the complexity of students' cognitive structure in geometry, a considerable amount of research focuses on investigating students' visualisation and the use of visual approaches in geometry (Fuys, Geddes, and Tischler, 1988; Hershkowitz, 1989). "This question of the nature of visual abilities deserved more research" (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore, & Vinner, 1990, p.94). Second, in terms of school students' learning experience of generalising knowledge by reasoning, Mason (2002, p.16) questions

"Why do so many children find mathematics to be mysterious and without reason, a random collection of unjustified manipulations of symbols?"

Indeed, in respect of geometry teaching and learning, research is needed to develop understanding of the interplay of the empirical/deductive approaches (Jones, 1998; Schoenfeld, 1986, 1988, 1989). Third, the role of teachers' questions has been highlighted in the effective teaching and learning of mathematics. However, it is found that many teachers may use questions that involve low levels of mathematics thinking rather than higher levels of thinking (Huang and Leung, 2002; Johnston-Wilder and Mason, 2005; Mathematics Resource Project, 1978). It is essential to identify what types of questions may support thinking development in an advanced course such as geometric proof problem solving. Fourth, the importance of mathematical instructional tasks is highlighted, as are the difficulties associated with implementation ing high-level tasks and the ways of supporting the implementation ation of high-level tasks in effective teaching and learning of mathematics (Henningsen & Stein, 1997; Herbst, 2003; Mason and Johnston-Wilder, 2004). TSGofQp (1991) suggests that future research connects both cognitive research ideas, such as Vygotsky's ideas about the zone of proximal development, and scaffolding, and teachers' teaching tasks arrangement for interpreting the effectiveness of teaching and learning in geometry.

1.1.2 Effective classroom teaching and learning across cultures

Cross-cultural research in classroom teaching and learning also motivates this study. For instance,

- All countries are seeking to improve mathematics teaching through interpreting results of international surveys, such as TIMSS (Trends in International Mathematics and Science Study) and PISA (Programme for International Student Assessment) study;
- There is great international interest in how Chinese students are taught mathematics (Fan, Wong, Cai, and Li, 2004; Ma, 1999; Stevenson, and Stigler, 1992);
- There is considerable research on what it means to teach effectively (Muijs and Reynolds, 2005);
- There is considerable research on the global pattern of teaching (Alexander, 2000, 2001; Givvin *et al.*, 2005; Stigler and Hiebert, 1999).

In particular, this study is inspired by Muijs's and Reynolds's consideration of research in effective teaching. These researchers suggest that

> "... there is ... an urgent need to move beyond 'one size fits all' descriptions of effective teaching practices towards research which looks at whether there is 'context specificity' ... While in the field of effective teaching some factors apply across all social contexts (such as having high expectations of what children can achieve, or lesson structure), it may be that certain factors apply only in certain social contexts. ..." (Muijs and Reynolds, 2005)

1.2 The van Hiele model

Based on their pedagogical experience and their teaching experiments, the van Hieles proposed a psychological/pedagogical theory of thought levels in geometry (English version in Geddes, Fuys and Tischler, 1984). The first four levels of thought in geometry are characterised as *visual, descriptive/analytic, abstract/relational,* and *formal deductive*. Accompanying this model of thought levels, the van Hieles proposed a model of teaching that specifies five sequential phases of instruction: *information, guided orientation, explicitation, free orientation, and integration* (Clements and Battista, 1992). The van Hieles suggest that these five teaching phases are a universal means of enhancing students' thinking from one thought level to the next, not only in mathematics but also in other areas of the curriculum (van Hiele, 1986). Understanding students' cognitive structure in geometry, and the way to promote their geometrical thinking, is a critical issue. The van Hiele theory has been influential and extensively studied across many countries (for more details, see Chapters 2 and 3).

However, compared with the amount of research on the van Hiele levels of thinking, little research other than that conducted by the van Hieles themselves has directly examined the relationship of teachers' instruction (namely the five pedagogical phases), with the levels of thinking. While Crowley (1987), Presmeg (1991), Whitman (1995) and Pusey (2003) believe that the five teaching phases (called the van Hiele-based instruction), positively and effectively improve students' thinking levels and help students develop geometric reasoning on the teaching topic, a few empirical studies (Fuys *et al.*, 1988; Groth, 2005; Hoffer, 1994; Mistretta, 2000; Wai, 2005; and Whitman *et al.,* 1997) are mixed in terms of the validity and reliability of the phases and the sequential feature of the phases.

As a result, to date, there appears to be little research to link the van Hiele theory to teachers' actual practice in the classroom. The key issue for this study is to explore and elucidate the complexity of instructional approaches and strategies in the development of students' geometric thinking for writing proofs. This entails linking the van Hiele theory to teachers' classroom practice. In particular, this study goes beyond the general context, by linking the cognitive and pedagogical hypotheses of the theory to classroom teaching and learning in geometrical proof problem solving. First, the aim of this study is to develop a comprehensive understanding of the van Hiele theory. Second, the study looks at whether the teaching pattern and learning structure in the context of deductive geometry in the lower secondary school curriculum in Shanghai matches the theoretical pattern proposed by the van Hieles. Should the practical pattern mismatch the theoretical one, then the intention is to develop an explanatory framework.

1.3 Aims and research questions

The aims of this study are:

To explore and elucidate the complexity of individual teacher's didactical practice towards the development of students' thinking for writing proofs in geometry.

Questions to be addressed are:

- How and why are visual approaches used during the teaching process of geometric proof problem solving?
- How and why are inductive/deductive approaches used during the teaching process of geometric proof problem solving?
- How and why are questioning strategies used by teachers during the teaching process of solving geometrical proof problems?
- In what way, and for what purpose, are proof problems arranged by teachers during the teaching process of solving geometrical proof problems?

To understand in what way the van Hiele model is a useful research tool to help analyse and interpret classroom teaching and learning of geometrical proof problem solving.

Questions to be addressed are:

- In what way are the van Hiele levels useful in characterising students' learning responses and results on geometrical proof problem solving?
- In what way are the van Hiele teaching phases useful in characterising teachers' actual classroom instruction of geometrical proof problem solving?

Figure 1.1 presents an overview of this study. In the figure, the parts highlighted in bold are the main body of the study.

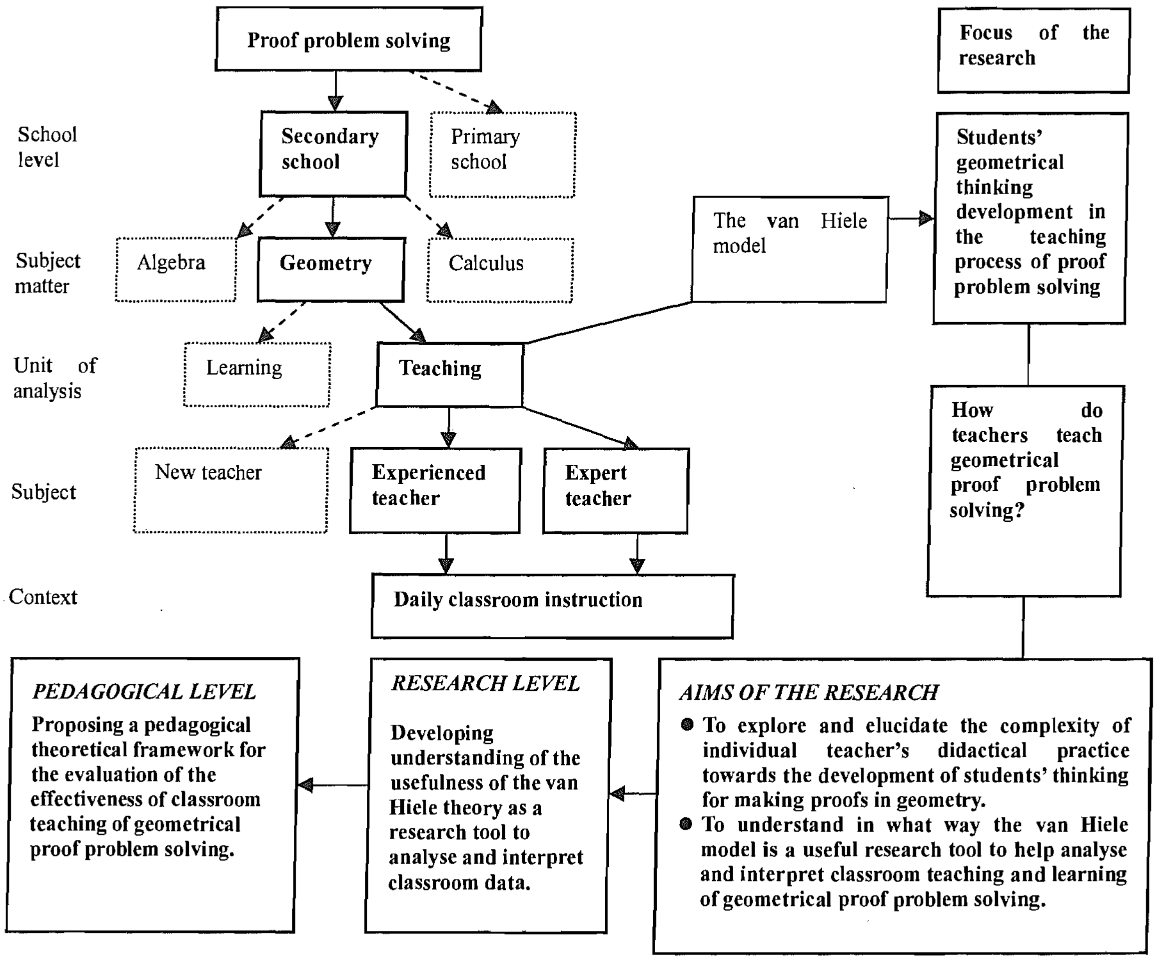

Figure 1.1 Overview of the research

1.4 Overview of the study

The present chapter, Chapter 1, provides the rationale for this research. First, a brief outline of the research context and motivation is given. Second, the reasons for using the van Hiele theory in this study are described. Third, the aims and key research questions of this study are stated. Finally, an overview of the whole thesis is outlined.

Chapter 2 is a review of the literature on research in geometry education. The key issues that motivate and inform this research are from studies of the variation and emphasis in school curricula, studies of teachers and their teaching practice and beliefs, and studies of students and their learning outcomes and understanding. Moreover, the main issues in the van Hiele theory that have been addressed by van Hiele-based research are summarised.

Chapter 3 outlines the theoretical framework and reviews the van Hiele theory and van Hiele-based research. A comprehensive understanding of the van Hiele theoretical framework for the analysis of data is developed.

Chapter 4 defines the methodology of the main study, which consisted of case studies of one expert teacher's and two experienced teachers' lessons at Grade 8 in Shanghai, and elaborates the reasons for the choice of particular research methods in relation to the aims of the study.

Chapter 5 presents a detailed cross-case analysis of examples selected from the three teachers' observed lessons, which link the classroom data to the van Hiele model. The analysis provides the ground for understanding the pattern match between theory and practice.

Chapter 6 discusses the results of the study of using the van Hiele theory to analyse and interpret the observation data from the three teachers' lessons. Questions are raised and a further explanation of the data is developed.

Chapter 7 provides an explanation of the issues and trends emerging from the data and proposes a pedagogical framework which brings together all issues which emerged from the cross-case analysis.

Chapter 8 draws the final conclusions from the study, provides reflections on the theory and methodology used in this study, and reveals the implications for future research.

CHAPTER 2. LITERATURE REVIEW

2.1. Overview

The overall aim of this research is to explore and elucidate the complexity of teachers' didactical practice towards the development of students' thinking for writing proofs in secondary school geometry classes. Thus, in this chapter, research work that addresses the key issues of classroom teaching and learning in secondary school geometry are presented. Moreover, as this study accepts the van Hiele's theoretical hypothesis that there is a strong relationship between teachers' instruction and students' geometric thinking development, relevant research issues using the van Hiele theory are also reviewed.

The chapter is divided into five sections. Section 2.2 focuses on international comparisons of the role of proof in secondary school geometry curricula development across many countries. In particular, the changing amount of emphasis on the teaching of proof in secondary school geometry in the U.S, England and China is highlighted. Section 2.3 summarises significant findings from classroom research on the teaching and learning of geometry. The next section (section 2.4) addresses four aspects of effective teaching and learning in geometry from both cognitive and pedagogical research in mathematics education. Section 2.5 provides a review of key research issues regarding the van Hiele theory relevant to the purposes of this study. In the final section (section 2.6), the main issues that are addressed in the previous sections of the chapter are summarised.

2.2 The role of proof in the secondary school geometry curriculum

At the secondary school level across the world, the study of geometry has been one of the most controversial issues debated by mathematicians and educators as well as researchers over the entire twentieth century (Fehr, 1973; Usiskin, 1987; Jones, 2000). For instance, in the U.S., Allendoerfer (1969, p.165) claimed that

> "the mathematics curriculum in our elementary and secondary schools faces a serious dilemma when it comes to geometry. It is easy to find fault with the traditional course in geometry, but sound advice on how to remedy these difficulties is hard to come by."

In fact, this statement reflects the dilemma in the geometry curriculum, not only in the U.S. at that time, but also in many countries across the world at present. According to a range of international studies, such as Morris (1986), the International Commission on

Mathematical Instruction (1994) and Mammana and Villani (1998), there is widespread agreement that geometry teaching should start at an early age, and continue in appropriate forms throughout the whole mathematics curriculum. However, what forms of geometry are appropriate, and what could be applied to different school levels, remain open questions.

Traditionally, the geometry class in its widest sense is the vehicle for introducing students to concepts such as axiom, conjecture, theorem, and deductive method. In this thesis, proof in the secondary school geometry curriculum is the main focus. The literature in this section concentrates on the issue currently being tackled of the role of proof in secondary school geometry.

The comparative study of the intended geometry curriculum for students aged 11 to 16 by Hoyles, Foxman, and Küchemann (2001) shows that the role of proof in geometry has received varying amounts of emphasis in different countries.

> "In some countries, given their orientation towards geometry (e.g. the Netherlands), proof is not mentioned at all; in others, students are encouraged to discover and use the results of proofs rather than to construct them for themselves (Ontario, Canada); another group of countries seem to encourage explanation as a basis for simple proofs (e.g. Poland). At the other extreme, students are expected to construct formal proofs (e.g. France and Japan, and selected students in Baden-Württemberg (Germany) and Lucerne (Switzerland). However many of these countries are changing their geometry curriculum (e.g. Japan) with an apparently changed emphasis on proof from construction to appreciation." (*ibid*, p.4)

In the first place, the traditional "two-column proof" (for details of the definition of two-column proof see Schoenfeld, 1988, pp.157-8) has received changing amounts of emphasis in recent curriculum reforms not only across many countries but especially in the U.S. For instance, in the U.S., the 1989 edition of the *Principles and Standards for School Mathematics* (NCTM, 1989) recommended that less emphasis be given to two-column proofs and to Euclidean geometry as a complete axiomatic system for grades 9-12 (*ibid*, p.127). Yet the revised 2000 edition (NCTM, 2000) returns to stressing the axiomatic nature of geometry, stating:

> "Geometry has long been regarded as a place in the school mathematics curriculum where students learn to reason and to see the axiomatic structure of mathematics. The Geometry Standard includes a strong focus on the development of careful reasoning and proof, using definitions and established

facts" (*ibid*, p.41)

In England, Mason (2002) points to concerns about the place of deductive reasoning in the school mathematics curriculum in England. In terms of the emphasis of coursework (for an explanation of coursework see Mason, 2002, p.12) in the school mathematics curriculum in England, Mason (2002, pp.12-13) argues that

> "... inductive generaliSation has, in my view, indeed predominated, while structural deduction is rarely found. Indeed, the topics used for coursework are migrating to data-handling, so that even induction is becoming rare, much less deduction."

Indeed, rather than the term geometry, the term "Shape, space, and measures" was used in the 1999 version of the National Curriculum in Mathematics for students aged 5-16 years (Department for Education and Employment Education, 1999, p.7). Moreover, while at Key Stage 3 (ages 11-14), there is a single curriculum for all pupils, the Key Stage 4 curriculum (ages 14-16) is divided into two parts, namely "mathematics foundation" and "mathematics higher". In terms of proof in that version of the national curriculum in England, Hoyles (1997, p.9) highlights the fact that,

> "The meaning of "to prove" has been replaced by social argumentation (which could mean simply giving some examples); justifying is largely confined to an archaic "investigations curriculum" separated from the body of mathematics content; and proof is labeled as inaccessible to the majority."

On recognising the declined status of geometry in the English mathematics curriculum, a certain number of issues, including the important position of theorems and proofs within mathematics, have been stressed in a report of the Royal Society & Joint Mathematical Council (2001), and the term "geometry and measures" is now being used in the newly-revised National Curriculum (QCA, 2007).

In China, the role of the traditional "two-column proof", together with the axiomatic Euclidean geometry, has received significant challenges in the recent geometry curriculum reform. According to Ma (2003), before 2001, geometry textbooks had been edited as closely as possible to the axiomatic system of Euclid. According to Zhang (2006), in the mathematics curriculum of 1963, it was stated that

> "geometry at the secondary school level is different from Euclidean geometry as a science. It should not and it is impossible to teach geometry in terms of the rigorous axiomatic system of Euclidean geometry. However, in order to help students more systematically understand geometrical knowledge,

> and to cultivate their ability to prove, the rigor of logic should be stressed as much as students are able to appreciate." (*ibid*, p.7, translated by Liping Ding.)

Indeed, the formal aspect of proof has been highly appreciated in school mathematics in China. Zhang, Li, and Tang (2004, p.198) state that

> "...it is useful in teaching rigorous deductive reasoning and formal proof. Compared to "verification", "proof" is more valued. For example, to prove Pythagoras Theorem, one needs to use rigorous algebraic or geometric methods, while demonstrating a cut-and-paste method is not acceptable".

However, in the mathematics curriculum of 2001 (Educational Ministry of People's Republic of China, 2001), the term geometry was replaced by the term "Space and Shape". The axiomatic system of geometry received little emphasis in the curriculum. Moreover, the geometry curriculum between Grades 7 and 9 is divided into four parts: "Figure recognition; Figure and transformation; Figure and coordinate; and Figure and proof" (*ibid*, p.11). The Research Working Group for the National Mathematics Curriculum Standards (1999, p.1) states the main educational considerations for such reforms as follows:

> "At the level of compulsory education, it is more essential to cultivate students' spatial conception, geometrical intuition and reasoning ability by geometry education. Students need to receive certain and necessary proof training through proving fundamental properties of basic figures. However, students should not focus on practicing the technique and speed of proof, but on understanding the necessity of a proof, the premise of a proof, and the ideas of proof. (translated by Liping Ding.)

Nevertheless, a considerable number of eminent Chinese mathematicians and scientists believe that the 2001 version of the National School Mathematics Curriculum is undermining mathematics education in China. For example:

> "The New Math Curriculum has been sharply criticized for betraying an excellent educational tradition, sacrificing mathematical thinking and reasoning for experiential learning, giving up disciplinary coherence in the name of inquiry learning, lowering expectations in the name of reducing students' burden, and causing confusion among teachers and students" (Zhao, 2005, p.219).

As such, geometry, in particular the axiomatic nature of Euclidean geometry, has become the centre of controversy in the current Chinese mathematics curriculum reform at the secondary school level. On the one hand, the curriculum developers are highly concerned about the following issues:

- The main aims of geometry education are not generally to maintain the axiomatic system of plane geometry, but to enrich students' recognition and experience of spatial figures;
- Based on the research survey, the majority of students are not able to understand the

"axiomatic" nature of geometrical thoughts; students should learn practical problems in geometry;

- Students should experience the exploration process by observation, manipulation, reasoning and visualisation, etc. They should appreciate and experience the application of transformation into practical life. Experimental activity such as "measuring", "drawing" should be involved in learning as much as possible;
- To focus on understanding proof itself, and not to pursue the quantity and technique of proof. (adapted from Zhang, 2005a, p.2, translated by Liping Ding)

On the other hand, mathematicians, mathematics educators and teachers have insisted on the following points:

- "Students should learn Euclidean geometry. …because students could experience the power of reasoning even in a simple situation. In terms of geometry, we will have significant problems without Euclidean geometry. Geometrical proof should not be deleted from the curriculum. The whole body of mathematics has been established by proofs. …Geometry should be more often considered when training students' reasoning, as it is a more visual subject…Ordinary students get used to calculation, but this is not the case for reasoning in geometry. However, Euclidean geometry should not be deleted just because students have difficulty in learning. On the contrary, it needs to be taught well in order to help students overcome such difficulty…" (Interview with the influential Chinese mathematician 陈省身 (Chen Xingshen）in 2002, Li, 2005, p.2, translated by Liping Ding)
- Plane geometry plays a significant role and function in training students' logical thinking (Chen, 2003; Wong and Deng, 2000; Tian and Li, 2005; translated by Liping Ding).
- "If geometrical concept and method are not greatly emphasised, and if it is merely to learn practical knowledge in geometry such as shape and measurement, how much difference is there between now and the flood period of the Nile River?" (Li, 2003, p.9, translated by Liping Ding)
- "Euclidean geometry should be taught. …The content could be reduced. …However, the system of Euclidean geometry should remain. Without this system, no logic could be found, let alone the logical thinking…(Zhang, Y.B., 2005, p.7, translated by Liping Ding)

- "Why reject the axiomatic system of geometry? Just because it is too difficult to learn? But what are students' difficulties? Is the axiomatic system itself a problem for students to understand? Might teachers concentrate too much on the technique of solving difficult problems in order to help students obtain a good mark in the standard examination? Certainly, students have great difficulty in solving those over-advanced problems. I do not think that this is an effective way to improve students' learning by merely giving up the axiomatic system of geometry (Gu, 2005, p.13, translated by Liping Ding)

Noticeably, there have been a few times that Euclidean geometry received less emphasis in the development of school mathematics curriculum in China, such as that in the 'Great Cultural Revolution", and the recent new curriculum. However, each time when it received less emphasis, serious criticism simultaneously followed and more emphasis was likely then to be given back. It might be understood from such changes of emphasis that the axiomatic nature of Euclidean geometry remains in its essential role in the Chinese school mathematics curriculum. Currently, according to personal conversations with Professors Zhang Dianzhou and Gu Lingyuan (in Shanghai, December, 2006), proof, with the axiomatic nature of plane geometry, will receive more emphasis in a newly modified version of the national curriculum.

2.3 The teaching and learning of geometrical proof in the classroom

Existing research related to mathematical proof is extensive (see for instance, Hazel and Sowder, 2007). For the purpose of this study, the literature review cites relevant studies of proof in the context of school geometry teaching and learning.

2.3.1 School teachers' classroom practice and their beliefs about proof in geometry

Literature on how students' acquire proof skills through classroom instruction is fairly sparse, yet the issue of classroom practice and its relation to students' understanding of geometrical proof has been increasingly emphasised in recent years.

In the first place, Lampert (1993) investigated teachers' points of view about using Geometric Supposer (geometric computer software) to substantially change the way they taught geometry. She describes what it is like to be "doing proofs" in a contemporary U.S. high school geometry classroom. Her account is likely to be considered as the prototypical geometry classroom.

> "... In a secondary school geometry lesson, some attention is given to the meaning of this mathematically significant translation from the conditions of universal logical to a particular empirical illustration, but whether the proof is about the particular elements of the diagram or the more general elements of the theorem becomes somewhat ambiguous. Whether students come to know that the proof is universal whereas the drawing is particular, probably depends on the authority asserted by the teacher rather than on their appreciation of the subtleties of logic.
> The list of statements and reasons make up the body of the proof. They are always written in parallel columns, and they must always begin with what is given and proceed through a series of steps so that the last is a statement of what one set out to prove. ...There is never any question that what needs to be proved will be proved; typically students write the first and last steps before going on to fill in the middle ones.
> What goes between the first and last statements is more of a problem, because that is where logical thinking comes in, or doesn't. ... Many students admit to memorizing the steps in proofs that are given in their books or demonstrated by their teachers. Teachers rarely ask students to write a proof that they have not seen before, except in honors or advanced placement sections." (*ibid*, pp.146-7)

According to this picture, students are likely to conceive of proof as a procedure in which they need to fit together the pieces of knowledge they have about the concepts involved to generate the desired sequence of steps. Some serious problems of learning raised from such instruction are illustrated in the following section.

Lampert (1993) found that the common methods of teaching geometry and the assumptions the teachers express about student learning mix a formalist philosophy of mathematics with a reliance on teachers' authority as the source of mathematical knowledge truth. The teachers in the study believed that

> "geometry is done without data collection and conjecturing on the part of learners; they saw doing geometry as a process of moving from teaching definitions and axioms and postulates to proving theorems, then using those theorems to prove more theorems." (*ibid*, p.160)

These statements underline the problems teachers actually had to deal with in the role of induction in geometry teaching and learning. On the one hand, these teachers recognised that this process did not work for many of their students. On the other hand, these teachers were still uncertain of how to relate inductive inquiry to traditional content, even though the teachers thought that students might be more likely to learn geometry from trying to find patterns in visual and numerical data, than from trying to deduce relationships among abstract figures.

Knuth (2002) examined sixteen in-service secondary school (Grades 9-12) mathematics teachers' conceptions of proof according to a certain number of theoretical suggestions of the role of proof in mathematics. These theoretical suggestions of the role of proof are: to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, or to systematise statements into an axiomatic system (e.g., Bell, 1976; de Villiers, 1998; Hanna, 1983, 1990; Hersh, 1993). The study reported that the teachers described a variety of roles that proof plays in mathematics. These roles suggest that teachers have a diverse and, pedagogically speaking, potentially powerful understanding of the function of proof in mathematics. However, the study indicates that there was no supporting evidence to suggest that the teachers viewed the promotion of understanding or insight as a role of proof in mathematics, in contrast to views espoused by many mathematicians (e.g., Hanna, 1990; Hersh, 1993). This result corresponds to what Chazan (1993) highlights that the focus of teachers' previous experiences with proof as students themselves, both at the secondary and collegiate levels, has tended to be primarily on the deductive mechanism or on the final product. As a result, "in most instructional contexts proof has no personal meaning or explanatory power for students" (Schoenfeld, 1994, p.75).

2.3.2 Students' beliefs, attitudes and performance in geometry

A considerable number of significant research findings about students' beliefs, attitudes and their understanding as well as performance in geometrical proof are cited in this section.

The analysis of classroom instruction by Schoenfeld (1988) reveals a potentially devastating finding: students learn to separate the worlds of deductive and empirical geometry.

> "... in these students' experience, proofs had always served as confirmation of information that someone (usually the teacher or mathematicians at large) already knew to be true; they provided the "justifications" for constructions. But ask these students to discover a construction, and they do not see that any proof arguments are relevant at all. For these students, a construction is right when it "works". They are in "discovery mode", and proofs have never helped them to discover. Confronted with a construction problem they make their best guess, and then test it by trying it out and seeing if their attempt meets their empirical standards." (*ibid*, pp.156-7)

Schoenfeld (1988) also uncovers one negative learning attitude of writing two-column proofs in that students came to believe that

> "it is the form of expression, as much as the substance of the mathematics, that is important" (*ibid*, p.158).

Moreover, Schoenfeld (1989) shows evidence of some students' misunderstandings of proof writing from the survey, as the following quote from a student shows:

> "... The key thing [in geometry proof] is to get the statement and reasons in the proper form. ...
> ... Memorizing is very important, and in geometry, especially for the final exam, because I am required to write proofs from memory." (*ibid*, p.344)

Studies such as Senk (1985, 1989) and Usiskin (1987) found that students have great difficulties in geometric reasoning, particularly in writing proofs. For instance, Senk (1985) tested the proof-writing ability of 1520 students (mean age sixteen years two months) in geometry classes that had studied the topic. A proof was considered correct if all the steps were followed logically, even if there were minor errors in notation, vocabulary, or names of theorems. On one item requiring an auxiliary line (see figure 2.1), 51% of the students were successful, with nearly all of these scoring the full 4 points, but 40% scored 0 on the item.

Write this proof in the space provided.
Given: Quadrilateral HIJK, HI=HK, IJ=JK.
Prove: $\angle I = \angle K$.

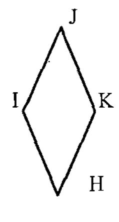

Figure 2.1. An item tested in Senk (1985, p.451)

The study shows that proofs of textbook theorems were even more difficult for many students.

> "Although 42% wrote a valid proof of the triangle sum theorem, another 40% failed to write a single valid deduction. Many scores of 0 were due to students' citing the statement of the theorem as the proof." (*ibid*, p.451)

Senk concluded that only about 30% of all students in full-year geometry courses that teach proof reach a 75% mastery level in proof writing. The study shows a rather low level of students' achievement in writing proofs. Interestingly, however, Senk (1985) also points out that a few of the geometry students did exceptionally well at writing proofs. For instance, fifty-four of the 1520 students received perfect scores. Nonetheless, she claims that the data from her study do not support the belief that some teachers hold, that proof is an "all or nothing" task.

In China, a number of national large scale surveys report students' learning attainment in school mathematics. For instance, data from Tian (1990) show that students' overall learning performance in algebra was lower than that in geometry in the general mathematics test (in algebra, the rate of students' correct answers was 65%, while in geometry, the rate was 70%). The study results from Xie and Tan (1997) show that the number of students (out of 2003 students) to attain the full marks in geometry was more than that in algebra (1424 students), though the rate of students' general correct answers in algebra was higher than that in geometry. Based on the analysis of students' such performance, Xie and Tan (*ibid,* p.238) pointed out that this might be due to the fact that students' attainment in the basic knowledge and skills of algebra was better than that of geometry. In view of such a surprise finding, Zhang (2005a) underlines two key facts about students' learning in geometry: 1) geometry is a more difficult subject than algebra for average students at the lower secondary school level; 2) there is a large learning gap between the best and the weakest students; the best students generally have good attainment in geometry.

There is very little existing research on students' proof writing in China. The only research informing this study is a small-scale quantitative study conducted by Professor Gu Lingyuan and his colleagues in Shanghai in 1979 (TSGofQp, 1991). In total, 335 students (14-15 years old) from seven schools in three counties of Shanghai attended the writing test, which was designed to provide information about students' learning attainment in plane geometry. Overall, the analysis of the attainment of good, average and weak students (not defined in the report) indicates that good students generally performed well in the test, while average and weak students' performance varied largely from one item to another. In one item (see figure 2.2), 35% of students gained full scores, yet 30% gained 0 scores (scoring method not described).

Given: Isosceles triangle ABC, AB=BC. Extend AB to D so that BD=AB.

Prove: Triangle ACD is a right triangle.

Figure 2.2 An item tested in TSGofQp (1991, p.29, translated by Liping Ding)

Among those successful students, five different solutions of proof (see *ibid*, p.36) which involve a certain range of definitions and theorems were demonstrated. Thus, the study indicates that good students had developed good logical reasoning and were capable of learning formal proof in plane geometry.

However, the study also indicated that weak students had great difficulty in proof writing. After the survey, for instance, some weak students received a further interview to identify their possible thought in proof writing. Extract 2.1 shows the interview of a boy student about a proof problem (see figure 2.3).

Given: In quadrilateral ABCD, AB=BC=CD=DA.

Prove: $\angle 1=\angle 2$.

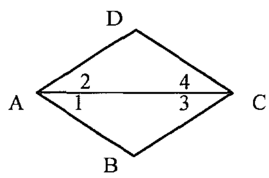

Figure 2.3 A proof problem used in an interview (TSGofQp, 1991, p.38, translated by Liping Ding)

Researcher (R): See this figure (figure 2.3), if AB=BC=CD=DA, could you prove $\angle 1=\angle 2$?
Student (S): Because triangle ABC is congruent to triangle ADC. So $\angle 1=\angle 2$.
R: Why are these two triangles congruent?
S: Because AB=AD, BC=DC, and $\angle 1=\angle 2$. You could use SAS to prove it.
R: $\angle 1=\angle 2$ is to be proved. It is not given.
S: Oh. You are right. OK, I could use $\angle 2=\angle 3$.
R: Why?
S: Because interior alternate angles are equal.

Extract 2.1. Data from an interview of TSGofQp (1991, p.38, translated by Liping Ding)

The analysis of this interview indicates that this boy student could not prove the congruent triangles by deductive reasoning. He misused the fact to be proved as the given and was not able to perceive the available properties of the figure (such as the common side AC in the figure). Moreover, he used some invalid facts such as the interior alternate angles. The study believes that the problems this boy had reflect the general difficulty a great number of students might have. The study further indicates the necessity to improve students' learning in plane geometry by developing classroom instruction and textbooks.

2.4. Research on the effective teaching and learning of geometry

In this section, four key aspects of research in the effective teaching and learning of geometry, which are related to the research questions of this study, are reviewed: 1) visualisation and visual approach; 2) the false dichotomy: empirical vs. deductive approach; 3) teachers' questioning; 4) the arrangement and implementation of tasks.

2.4.1 Visualisation and visual approach

There has been quite extensive research into visualisation in teaching and learning geometry (Bishop, 1983; Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore, & Vinner, 1990; Johnston-Wilder and Mason, 2005). Two examples of visualisation in teaching and learning geometry are cited in this section.

Hershkowitz (1989) investigates the role of visualisation (spatial ability) within the process of geometrical concept attainment of students (in Grades 5-8) from two schools, 142 preservice elementary teachers and 25 in-service senior elementary teachers. The concepts and tasks were sampled from the elementary school syllabus. Hershkowitz highlighted the complexity of visualisation and drew researchers' attention to the prototype phenomenon in learning geometrical concepts in that each concept has a set of critical attributes - relevant features - and a set of examples. The prototype is the example which is more popular than all others.

Findings from the examination of the prototypical examples further illustrate that in each prototypical example, attention is drawn to some specific attribute(s), in addition to the critical attributes of the concept (those attributes that each positive example of the concept must have). Thus, Hershkowitz suggests that visualisation is a necessary tool in geometrical concept formation; yet, on the contrary, it might put some limitations on the individual's ability to form all the concepts' examples.

Hershkowitz identifies three different types of behaviour to help understand the extension of students' concept images beyond the prototypical example(s), and link the development to the van Hiele levels as follows:

1. The prototypical example is used as the frame of reference and visual judgment is applied to other instances (van Hiele Level 1).
2. The prototypical example is used as the frame of reference but the subject bases his judgment on the prototype self attributes and tries to impose them on other concept examples (the

transition from van Hiele Level 1 to Level 2).

3. The critical attributes are used as a frame of reference in the formation of geometrical concepts (van Hiele Level 2 (and even Level 3)) (*ibid*, p.74).

Fuys, Geddes, and Tischler (1988) examine the ability of the van Hiele model to describe how students (sixth and ninth graders) learn geometry. These researchers developed and validated three instructional modules, which were largely based on the approaches and materials used by Dina van Hiele-Geldof in her doctoral research, as a research tool in clinical interviews. They found that some students have perceptual difficulties, including orientation and figure-ground problems. Moreover, their study substantiates some findings reported by other researchers. For instance,

- Turning or moving of figures to more customary positions by the students helped them identify such properties as right angles, parallel lines, congruent figures (Burger and Shaughnessy, 1986).
- Some students who know a correct verbal description of a concept but also have a special visual image associated tightly with the concept (e.g., a side of a right triangle must be horizontal), have difficulty applying the verbal description correctly (Vinner and Hershkowitz, 1980).

In terms of the use of a visual approach to support students in learning, Fuys *et al.* noted that all subjects in their study made extensive use of the concrete materials to explore relationships, discover patterns, or confirm hypotheses. Thus, they claim that

> "The use of manipulatives and other concrete materials allowed the students to try out their ideas, look at them, be reflective, and modify them. The visual approach seemed not only to maintain student interest but also to assist students in creating definitions and new conjectures, in gaining insight into new relationships and interrelationships." (*ibid*, p.138)

2.4.2 The false dichotomy: empirical vs. deductive approach

The controversial issues of the school geometry curriculum across many countries shown in section 2.2, and the approaches to the teaching and learning of geometry highlighted in section 2.3, reflect the education and research concerns of two extreme approaches of teaching and learning geometry. One approach focuses on the "empirical" or "intuitive" aspect of geometry, while the other approach focuses on the "deductive", "formal" or "axiomatic" aspect of geometry. There appear to be a number of ways of looking at the relationship between these two positions as explained by Jones (1998).

In the view of the traditional approach to teaching geometry, in which the process of inductive discovery was almost neglected, Freudenthal (1971, pp.417-418) points out that

> "The deductive structure of traditional geometry has never been a convincing didactical success. ... It failed because its deductivity could not be reinvented by the learner but only imposed."

Hershkowitz *et al.* (1990, p.89) highlight a common belief, held by many, of the necessity to have inductive, empirical discoveries in geometry as follows:

> "... because (a) they introduce a discovery aspect; (b) by regarding the generalization as a conjecture in itself, the learner feels the necessity to prove what he or she has conjectured to be true; and (c) inductive experiences are the intuitive base upon which the understanding and the generation of a deductive proof can be built."

Moreover, a considerable amount of cognitive research attempts have been tried to investigate the realisation of such a belief yet questions are raised (for more details see Hershkowitz *et al.*, 1990).

As discussed in section 2.3.2, Schoenfeld (1988, 1989) uncovered the other extreme of students' learning behaviours. That is, students generated hypotheses by purely empirical and intuitive means, even though the proof was available for them. A conclusion Schoenfeld (1986, p.241) reaches is that, as a false mathematical dichotomy,

> "The two ostensibly disjoint approaches to mathematics – a deductive approach to mathematical discovery, in which new pieces of information are logically deduced, and an empirical intuitive approach, in which "insight" plays the major role, are in fact mutually reinforcing."

Jones (1998) investigated how students approach geometrical problem solving by using Cabri (geometric computer software). The episode shown in his paper involves two pairs of recent mathematics graduates tackling a well-known geometrical problem. Findings from the analysis indicate that geometrical intuition has a role in the planning-implementation, and transition episodes of a problem-solving attempt. Evidence from Jones (1998) substantiates the view that students use a mixture of a deductive approach and an empirical intuitive approach (using Cabri in Jones' study) in solving geometrical problems.

2.4.3 Teachers' questioning

Three examples are cited in this section to demonstrate the important role of teachers' questions in the effective teaching and learning of mathematics, in particular the development of geometric thinking.

The Mathematics Resource Project (1978) points out that questioning is one of the most versatile and most used instructional tools. They suggest that teachers use questioning to promote learning as well as to evaluate whether students know specific information. In particular, they identify lower-level and higher-level types of questions as follows (*ibid*, p.43):

Lower level: Recall/recognition
Higher levels: Represent/interpret information
Explain
Analyze, interrelate, and apply information
Open search
-for solutions
-for problems

Their study indicates that many teachers' questions often call for recognition or recall types of thinking, and do not really involve students in higher levels of thinking.

To analyse how teachers teach Pythagoras' theorem in the Czech Republic, Hong Kong and Shanghai, Huang and Leung (2002, p.268) focus on one aspect of teaching: the patterns of classroom interaction. These researchers classify the teachers' questions into three categories: 1) "Yes/No" questions; 2) "Name" questions; 3) "Explanation" questions (details can be seen in *ibid*, p.269). The analysis of the teachers' questioning indicates that "Name" questions are mostly used. Moreover, the Czech teacher and the Hong Kong teacher adopted a similar pattern of questioning, with more than 70% of the questions requesting a simple yes or no response, and less than 15% of the questions in the other two types. In contrast, the Shanghai teacher asked less than 5% of the questions requesting a simple yes or no response; about half of the questions required student explanations.

Johnston-Wilder and Mason (2005) outline a question strategy to enhance learners' geometric thinking. For instance, they suggest that to ask questions like 'in how many ways can you …?' is more effective than to ask questions like 'can you find …?' or 'find a …'. Thus, they conclude that the questions which draw learners' attention to multiple methods help them engage more fully, more creatively and more readily than those questions which just ask to find an answer. Moreover, they suggest asking learners 'what is the same and what is different about' two or more objects or two or more figures, in order to help expand their perceptions of geometric figures.

2.4.4 The arrangement and implementation of tasks

> "The purpose of a task is to initiate mathematically fruitful activity that leads to a transformation in what learners are sensitised to notice and competent to carry out." (Mason and Johnston-Wilder, 2004, p.25)

Mason and Johnston-Wilder (2004) highlight the design and use of tasks in the development of teaching practices. Indeed, an important problem for research on instruction is that of examining teachers' arrangement and implementation of tasks that aim at high cognitive demands to be viable and sustainable in classrooms. Here, the term "arrangement" means that teachers design and set teaching/learning tasks. For the purposes of this study, this section draws some pedagogical ideas and empirical findings in mathematics, in particular in the geometry research domain (Fuys *et al.*, 1988; Henningsen & Stein, 1997; Herbst, 2003; Johnston-Wilder and Mason, 2005; TSGofQp, 1991).

Johnston-Wilder and Mason (2005) suggest that tasks be chosen to afford learners' experience of different aspects of geometrical thinking.

Herbst (2003) investigates the way a teacher manages students' mathematical work on novel tasks (as opposed to familiar tasks, for Herbst). Based on the study of a 6-week summer course in geometry for middle school graduates, Herbst argues that novel tasks intensify the complexity of a teacher's management of the development of knowledge. Three subject-specific tensions are identified to further explain that a teacher's actions might substantially shape the mathematical ideas that students have the opportunity to learn about.

Henningsen & Stein (1997) have drawn researchers' attention to the importance of mathematical instructional tasks, the difficulties associated with implementing high-level tasks and the ways of supporting implementation of high-level tasks. Findings from their study suggest that five factors appear to be prime influences associated with maintaining student engagement at the level of doing mathematics: task builds on students' prior knowledge; scaffolding; appropriate amount of time; modeling of high-level performance; and sustained pressure for explanation and meaning. In particular, these researchers highlight the role of the teacher not only in the selection of appropriate tasks but also in the implementation of those tasks for maintaining students' mathematical thinking at a high level.

The theory of variation proposed by TSGofQp (1991) links to Vygotsky's idea of the zone of proximal development. Vygotsky defined the zone of proximal development as follows:

> "It is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers." (Vygotsky, 1978, p.86)

In the study by TSGofQp (1991), 180 students (Grade 7-9, 12-15 years old) received two tests designed to measure the "distance" of the problem to be explored from the "anchoring part of knowledge" ("previous knowledge underpinning learning of the new knowledge and the exploration of the new problems", explained in Gu, Huang and Marton, 2004, p.325). Findings from their study indicate that the potential distance between students' previous knowledge and new knowledge in teaching and learning mathematics might be a significant fact to determine the difficulty of teaching and learning mathematics.

TSGofQp (1991) further linked the theory of variation to another of Vygotsky's theoretical ideas, namely scaffolding, ("*Pu Dian*", in Chinese). Bruner (1985, pp.24-25) notes the notion of scaffolding as follows:

> "If the child is enabled to advance by being under the tutelage of an adult or a more competent peer, then the tutor or the aiding peer serves the learner as a vicarious form of consciousness until such a time as the learner is able to master his own action through his own consciousness and control. When the child achieves that conscious control over a new function or conceptual system, it is then that he is able to use it as a tool. Up to that point, the tutor in effect performs the critical function of "scaffolding" the learning task to make it possible for the child, in Vygotsky's words, to internalize external knowledge and convert it into a tool for conscious control."

Thus, TSGofQp (1991) tried to use the Vygotsky's theoretical ideas, such as the zone of proximal development and scaffolding, to interpret the effects of teachers' arrangement and implementation of tasks in class. They suggest that teachers should be aware of adjusting the potential distance in the use of scaffolding or variation of problems, to ensure the nature of teaching and learning mathematics, which could be either an exploration and discovery process or a passive accepted process.

2.5. van Hiele theory and van Hiele-based research

As this study is largely inspired by the van Hieles' cognitive and pedagogical theory of students' geometric thinking development, in this section, the work of the van Hieles, together with a range of issues addressed in the van Hiele-based study, is presented.

2.5.1 Work of the van Hieles

As high school teachers in the Netherlands in the 1950s, Pierre and Dina van Hiele noticed the trouble their students had in geometry learning and were intrigued by the ineffectual communication between teachers and their students in the geometry teaching process. In 1957, the van Hieles completed companion doctoral dissertations which were based on the constructivism and geometric studies of Piaget. Pierre van Hiele formulated a system of levels of thinking in geometry, while Dina van Hiele-Geldof concentrated on an instructional experiment to develop students' thought levels.

For the purpose of the literature review, this section chooses a specification of the van Hiele model. The thought levels are described as follows (Hoffer, 1983, p.207):

Level 0 Students recognize figures by their global appearance. They can say *triangle, square, cube*, and so forth, but they do not explicitly identify properties of figures.

Level 1 Students analyze properties of figures: "rectangles have equal diagonals" and "a rhombus has all sides equal," but they do not explicitly interrelate figures or properties.

Level 2 Students relate figures and their properties: "every square is a rectangle," but they do not organize sequences of statements to justify observations.

Level 3 Students develop sequences of statements to deduce one statement from another, such as showing how the parallel postulate implies that the angle sum of a triangle is equal to 180°. However, they do not recognize the need for rigor nor do they understand relationships between other deductive systems.

Level 4 Students analyze various deductive systems with a high degree of rigor comparable to Hilbert's approach to the foundations of geometry. They understand such properties of a deductive system as consistency, independence, and completeness of the postulates.

The phases of learning are shown as follows (Hoffer, 1983, p.208):

Phase 1: inquiry The teacher engages the students in (two-way!) conversations about the objects of study. The teacher learns how the students interpret the words and gives the students some understanding of the topic to be studied. Questions are raised and observations made that use the vocabulary and objects of the topic and set the stage for further study.

Phase 2: directed orientation The teacher carefully sequences activities for student exploration by which students begin to realize what direction the study is taking, and they become familiar with the characteristic structures. Many of the activities in this phase are one-step tasks that elicit specific responses.

Phase 3: expliciting The students, building from previous experiences, with minimal prompting by the teacher, refine their use of the vocabulary and express their opinions about the inherent structures of the study. During this phase, the students begin to form the system of relations of the study.

Phase 4: free orientation The students now encounter multistep tasks, or tasks that can be completed in different ways. They gain experience in finding their own way or resolving the tasks. By orientation themselves in the field of investigation, many of the relations between the objects of the study become explicit to the students.

Phase 5: integration The students now review the methods at their disposal and form an overview. The objects and relations are unified and internalized into a new domain of thought. The teacher aids this process by providing global surveys of what the students already know, being careful not to present new or discordant ideas.

In summary, the key characteristics of the van Hiele "levels" are demonstrated by referring to P.M. van Hiele's work (1959/1984, p.246) that

a) At each level there appears in an extrinsic way which was intrinsic at the preceding level.
b) Each level has its own linguistic symbols and its own system of relations connecting these signs.
c) Two people who reason at two different levels cannot understand each other.
d) The maturation which leads to a higher level happens in a special way. Several stages can be revealed in it (this maturation must be considered above all as a process of apprenticeship and not as a ripening of a biological sort).

Broadly speaking, the van Hiele levels have been translated and interpreted by a considerable number of researchers. Both the number and numbering of the levels have varied. Some described the levels from level 0 to level 4 (for instance, Crowley, 1987; Fuys *et al.,* 1988; Hoffer, 1983; van Hiele-Geldof, 1957/1984, 1958/1984; van Hiele, P.M., 1959/1984), while others from level 1 to level 5 (for instance, Clements and Battista, 1992; Hoffer, 1981, 1994; Usiskin, 1982; Wirszup, 1976). The numbering of the levels from level 1 to level 5 is used in this study. Moreover, the original number of five levels of thinking is used in this study, with the focus on the first four levels of thinking. The reasons for this are, first, the existence of the fifth level has not yet been clearly established, and, second, the study mainly focuses on Grade 8 geometry lessons at the lower secondary school level.

The van Hiele theory has been influential and extensively studied across many countries (see Battista, 2007; Clements and Battista, 1992; Hoffer, 1983; Wirszup, 1976). The description of the van Hiele theory about both the levels of thinking and the pedagogical phases is carefully analysed in the next chapter (Chapter 3, theoretical framework). In the following sections, some significant research work based on the van Hiele theory, together with the issues implied, are cited.

2.5.2 Research work and implications of the van Hiele thinking levels

A significant number of studies have addressed a wide range of research issues on the thinking levels (for more details, see Battista, 2007; Clements and Battista, 1992; Hoffer, 1983; Wirszup, 1976.). For the purpose of this study, the literature review mainly concerns the following aspects of the model of levels: 1) The validity and reliability of the levels; 2) The discrete nature of the levels; 3) Type versus level of reasoning.

1) Validity and reliability of the levels

Empirical research such as Burger & Shaughnessy (1986), Fuys *et al.* (1988), Shaughnessy and Burger (1985), and Usiskin (1982) has confirmed that the van Hiele levels are useful in describing students' geometric concept development, from elementary school to college. In particular, the van Hiele levels 1, 2 and 3 are confirmed to be very useful in describing students' reasoning processes in geometry.

However, it does not mean that there is not a problem with the research on the reliability of the theorised levels. Usiskin (1982) substantiates P.M. van Hiele's disavowal of the existence of Level 5 in his more recent writings.

> "In the form given by the van Hieles, level 5 either does not exist or is not testable. All other levels are testable." (*ibid*, p.79)

Moreover, results from Fuys *et al.* (1988) not only support the original van Hiele levels but also the recent characterisation of the model in terms of three levels (van Hiele, 1986): visual (previously Level 1, according to these researchers), analytic (previously Level 2), and theoretical (previously Levels 3-5). Fuys *et al* declare that van Hiele agreed with this interpretation, but they are aware that it is sufficient to use the three-level model to characterise thinking, especially considering their findings that students progressed toward level 3 with no sign of axiomatic thinking. In terms of the number of the levels, Clements and Battista (1992, p.431) point out that

> "if levels can be changed and combined, their hypothesized discrete, hierarchical psychological nature should be questioned".

Gutiérrez, Jaime, and Fortuny (1991) suggest that students' thinking could be determined at multiple levels. These researchers use a vector approach to describe the degree of acquisition of van Hiele Levels 1 through 4 (for instance, one student might have a grade component for Level 1 of 96.67%; Level 2, 82.50%; Level 3, 50.00%; and Level 4, 3.75%;

the researchers could not measure Level 5 to their satisfaction). Nevertheless, in view of the validity of the levels, these researchers state that

> "In fact, we observed that not all students used a single level of reasoning, but some of them used several levels at the same time, probably depending on the difficulty of the problem. This does not imply a rejection of the hierarchical structure of the levels but rather suggests that we should better adapt the van Hiele theory to the complexity of the human reasoning process; people do not behave in a simple, linear manner, which the assignment of one single level would lead us to expect." (*ibid*, p.250)

Findings from the 3-year longitudinal investigation of students' reasoning about space and geometry by Lehrer, Jenkins and Osana (1998) indicate that, students' justifications about "how it looks" (the lowest level of the van Hiele hierarchy) often involved many distinctions, ranging from detection of features like fat or thin, to comparison to prototypical forms, to the action-based embodiment of pushing or pulling on one form to transform it into another. "These distinctions appear to defy description by a single, "visual" level of development" (*ibid*, p.142). In view of the validity of the levels, these researchers conclude that

> "... although the van Hiele levels may provide rough benchmarks about phenomenological transitions in children's reasoning about form, there is good reason to call into question the notion of a level or combination of levels as an adequate description of mental representation." (p.145)

Indeed, rather than invalidity of the levels, these researchers question the descriptive adequacy of the original van Hiele levels, and suggest elaborating and expanding the van Hiele levels towards significantly developing understanding of students' geometrical thinking development.

2) The discrete nature of the levels

> "The most distinctive property of the levels of thinking is their discontinuity, the lack of coherence between their networks of relations". (P.M. van Hiele, 1986, p.49)

Early research, such as Soviet research (see Wirszup, 1976), appears to confirm the discontinuity of the levels, yet results from other research are mixed on the whole. For instance, Shaughnessy and Burger (1985) found students in transition when they identified and defined a long shape rectangle.

> "... several students agreed that shape 12 (a long shape rectangle) had opposite sides that were parallel, just like their own definition, but that it still wasn't a parallelogram because it didn't look like one. If conflict occurred between the visual and the analytic levels of reasoning (level 1 and 2 in this study), the visual usually won." (*ibid*, p.423).

Burger and Shaughnessy (1986) reported the difficulties that reviewers had in deciding between levels. They considered such difficulties as evidence questioning the discrete nature of the van Hiele levels.

> "Thus, the levels appear to be dynamic rather than static and of a more continuous nature than their discrete descriptions would lead one to believe. Students may move back and forth between levels quite a few times while they are in transition from one level to the next." (*ibid*, p.45)

Fuys *et al.* (1988) report that on the one hand, performances of some students indicate that they are at a plateau for a level and cannot progress to the next level. On the other hand, results for other students suggest that movement between levels is likely to proceed in small steps.

Hershkowitz (1989) indicates that the different types of behaviour in her study can link to the van Hiele model (see section 2.4.1), yet some findings of her study contradict the discreteness of levels declared by van Hiele. Gutiérrez *et al.* (1991) has provided a careful and detailed description of the transition between van Hiele levels.

Lehrer *et al.* (1998) argue that "level mixture" was the most typical pattern of response for primary-age students. For these researchers, the metaphor "overlapping waves" ("where thinking is characterised by variation in forms of reasoning, with attendant selection of any particular form adapted to task, learning history, and related elements of context." *ibid*, p.163) accounts better for the simultaneous development of the different types of reasoning characteristic of their study.

In terms of the different views of the discrete features of the levels by Gutiérrez *et al.* (1991), Pegg and Davey (1998) suggest that

> "This conflict can in part be explained in terms of the closeness of the "microscope" to the analysis. When pulled back and taken in broad context, van Hiele's (1986) view of the discreteness of levels makes "logical" sense in that it represents an "ideal" theory. ... If the "microscope" is moved in more closely, a blurring occurs, and the starkness of the discrete nature is no longer apparent." (p.113)

While some researchers retain the "continuous growth" view of the levels (Battista, 2007), others like Fuys *et al.* (1988) and Hoffer (1983) consider that the observations of the "continuous growth" may not reflect continuity in learning but rather continuity in teaching. The question of the discrete nature of the levels still remains open.

3) Type versus level of reasoning

The question of the discrete nature of the levels seriously raised another question, namely type or level of reasoning of the levels.

Hoffer (1981) links the van Hiele levels to five "skills in geometry" (Visual, Verbal, Drawing, Logical and Applied skills) by describing the characteristics of each skill in each van Hiele level. Thus, all these skills are described as part of each level.

De Villiers (1987) identifies six "geometric thought categories" in specifying the van Hiele levels as follows (quoted from Gutiérrez and Jaime, 1998, p.29):

1) Recognition and represent of figure types (level 1)
2) Use and understanding of terminology (level 2)
3) Verbal description of properties of figure types (level 2)
4) Hierarchical classification (level 3)
5) One step deduction (level 3)
6) Longer deduction (level 4)

Gutiérrez and Jaime (1998, p.32) adopt an intermediate position between the work of Hoffer (1981) and De Villiers (1987), and identified different processes of reasoning as characteristics of each of the first four van Hiele levels: 1) Recognition; 2) Use of definitions; 3) Formulation of definitions; 4) Classification; 5) Proof. Thus, for Gutiérrez and Jaime, the van Hiele levels of reasoning include several abilities that students need to master. They suggest that

> "Students may have a higher or lower acquisition of the different abilities characterizing a given van Hiele level, so it is necessary to establish a scale to measure the quality of a student's reasoning." (ibid, p.45)

Moreover, to examine profiles of reasoning for individuals, Lehrer *et al.* (1998, pp.143-144) derive eight summary categories as follows: Resemblance, Size, Angle, Orientation, Morphing, Counting, Property, and Class. However, findings from their study do not support the discrete feature of the van Hiele model. These researchers state that

> "The children's thinking about shape can be characterized as appearance based, as suggested by van Hiele (1986). However, children distinguished among many different features of form, and the nature of the distinctions children made varied greatly with the contrast set involved in the similarity judgment." (*ibid*, p.145)

Clements and Battista (2001, p.126) suggest that students possess multiple types of geometric knowledge, for instance, visual/imagistic knowledge and nonvisual or verbal declarative knowledge ("knowing what") about shape. Sharing the view with Gutiérrez *et* *al.* (1991, 1998), Clements and Battista (2001) propose a synergistic approach for interpreting the van Hiele levels. Thus, in their view, neither imagistic knowledge nor verbal declarative knowledge should be merely associated with a single level, say Level 1 or Level 2.

In general, the distinction between type of reasoning and qualitatively different levels in the development of reasoning remains unclear. As stated by Battista (2007, p.853),

> "..., sometimes the visual-holistic level is used to refer to a type of reasoning that is strictly visual in nature, and sometimes it is used to refer to a period of development of geometric thinking when an individual's thinking is dominated and characterized by visual-holistic thinking."

Regarding the type or level of thinking, a number of questions are intermingled and remain not completely solved (for details see Battista, 2007, p.853-854).

2.5.3 Research work and implications of the van Hiele pedagogical phases

Compared with the amount of research on the levels of thinking, little research other than the van Hieles' has directly examined the relationship of teachers' instruction, namely the five pedagogical phases, with the levels of thinking. While Crowley (1987), Presmeg (1991), Whitman (1995) and Pusey (2003) believe that the five teaching phases, called the van Hiele-based instruction, positively and effectively improve students' thinking levels and help students develop geometric reasoning on the teaching topic, other empirical studies (Fuys *et al.*, 1988; Groth, 2005; Hoffer, 1994; Mistretta, 2000; Wai, 2005; and Whitman, Nohda, Lai, Hashimoto, Iijima, Isoda, and Hoffer, 1997) are mixed in the following two aspects of the features of the five phases: 1) the validity and reliability of the phases; 2) the sequential feature of the phases.

1) The validity and reliability of the phases

Groth (2005) applied the five phases to teaching the relationships among different types of quadrilaterals and the triangle inequality in a summer course for high school students who had failed geometry during the school year. In the report, two examples are shown of how the five-phase framework helped the teacher design instruction that moved students from having little understanding of relationships among quadrilaterals to seeing and reflecting on some of those relationships. The researcher claimed that results from the final exam show that most students successfully answered true/false statements regarding quadrilateral relationships after the course. However, the researcher noted that the van Hiele theory might not provide a method for serving the whole field of geometry to all students. It is not clear from the paper how long the summer course lasted. It seems that the length of each of the five phases was determined in the course by the different teaching and learning activities.

Mistretta (2000) describes a field assessment of a supplemental geometry unit intended to raise van Hiele thinking levels in a group of 23 eighth-grade students by having them become more adept at using higher order thinking skills (Level 3 thinking for the study). The pre-test and post-test of the study contained Level 1, 2 and 3 questions (Level 0, 1, 2 in the paper), involving geometric concepts, shapes and area. The geometry unit was developed by using teaching methods described by van Hiele (1984, same paper of P.M. van Hiele, 1959/1984) and Fuys *et al.* (1988). The unit was taught for one month and included three types of activities. The paper does not show the length of each of the five teaching phases in the unit. The study results indicated that the unit was successful in raising the van Hiele thinking levels of the students.

Wai (2005) investigated the effectiveness of van Hiele-based instruction. The study consists of two mathematics teachers and 132 students (14-15 years old) from four different classes (two control groups and two experimental groups) in a local male CMI (Chinese Media of Instruction) school in Hong Kong. The duration of the entire study comprises a series of four consecutive lessons (50 minutes for each) for both control and experimental groups. In the research, the first three van Hiele levels (Hoffer, 1981) and the five van Hiele teaching phases were adopted into different stages during the teaching in the experimental group. The stages were mainly formed by different activities. The result of this study presented a very similar outcome to other research studies, such as Bobango (1988), in raising high school students' van Hiele levels from level 1 to level 2. Interestingly, the study indicated that the van Hiele-based instruction produces different effects on different kinds of students. For instance, this study found that the van Hiele-based instruction is more effective in assisting students who are at van Hiele level one. That is, students who are starting at a lower van Hiele level seem more willing to gain the benefit from the van Hiele-based instruction. It is easier for them to be promoted in van Hiele levels. In contrast, for those students who are already at a high van Hiele level, there appears to be relatively less chance of promotion in van Hiele levels under the van Hiele-based instructions.

2) The sequential feature of the phases

Hoffer (1994) used the van Hiele thought levels and teaching phases to define a structure for observing, recording and evaluating both mathematics and science classes. Twenty mathematics lessons and twenty science lessons were observed. For the class observations, the instructional time was broken down into discernible activities lasting 3-20 minutes. These modules represented natural breaks in the teaching period, such as when a change of activity occurred. The study found that teachers moved through the phases in a manner that is non-hierarchical and student progress is temporary. Moreover, there were numerous occurrences of phase – level gap with students and teachers operating at different levels of understanding. In terms of a hierarchical model that dictates learning which may be inhibited when teachers and students operate on different levels of understanding, the study found that these gaps did not always have deleterious results. For example, the study found that one teacher lowered her instructional phases in order to raise the thinking level of her students. The study also found that there was a lack of contiguity in the instructional methods due to the different teaching style in mathematics and science. For instance, science teachers were more likely to use inquiry, visual cues and concrete materials in their teaching by employing Phase 1, Information, nearly twice as often as mathematics teachers. Mathematics teachers were three times as likely to employ Phase 3 – Explication, requiring formal terminology and exact linguistic symbols.

Whitman *et al.* (1997) investigated the relationship of geometry instruction in Japan and Hawaii with the appropriateness of the content and teaching strategies relative to the van Hiele level of the students, the NCTM Standards in the United States, and the Course of Study by Mombusho in Japan. These researchers used the coding schemes produced by Hoffer (1994) to analyse the geometry lessons observed in Japan and Hawaii. Questions were raised about the applicability of the van Hiele learning phases as follows:

> "In Japan, there was ambiguity in trying to identify the phase at which the teacher was teaching because it appeared that more than one interpretation was available. The teacher planned the lesson using the Problem-Solving Teaching Method, had to code using the Hoffer instrument. It is noted that the basic teaching procedure of teaching in Japan is different from that used in the Hoffer instrument. In the instrument, the order of procedure is Familiarization, Guided Orientation, Verbalization, Free Orientation, Integration. In the Japanese method of teaching, this order is changed as follows: Familiarization and Guided Orientation (review and posing), Free Orientation (time for solving, review

for some students in tutorial, sometimes the teacher provides the student with ideas), Verbalization (whole class discussion and explanation), Integration (summary and exercises)." (*ibid*, pp.229-230)

The statement demonstrates that these researchers might use the van Hiele learning phases to analyse the structure of a lesson and a set of lessons.

In general, Hoffer (1994) states that Pierre van Hiele views the instructional model as more a suggested process than a fixed formula. Thus, it is not clear whether it is necessary for the teacher to go through each and every phase in promoting students' learning development.

2.6 Summary

The discussion of the literature presented in this chapter has highlighted a certain number of key issues about proof teaching and learning in geometry, which form the basis for the rationale of this study.

Literature on school geometry curricula in section 2.2 has shown that there has been wide variation in approaches to geometry from the point view of school curricula across countries. In recent years, educational reformers and scholars have emphasised the notions of teaching proof for understanding and of providing students with authentic encounters with the ideas and practices of mathematical disciplines. In China, as in many other countries, significant curriculum reforms have been taking place in school mathematics. There has been "mathematics debate" on the issue of less emphasis on the axiomatic nature of geometry in the 2001 version of curriculum. Traditionally, geometry has been the subject to train students' mathematical thinking and logic, and therefore the axiomatic nature of geometry is strongly recommended to be retained in a dominant role at the lower secondary school level. Schoenfeld (1988) suggests that research in mathematics education needs to broaden the view of mathematics, of curricular goals, and of what students really learn in their instruction, in order to conceptualize and effect change. This motivates this thesis in inquiring, in great depth, into the extent that Chinese school teachers implement the educational goals set out in the curriculum to help young students gain insight into deductive geometry at the lower secondary school level.

The research on geometry classrooms highlighted in section 2.3.1 shows that students are largely drilled in learning on the deductive approach, particularly in the two-column form of geometry. This implies that instructional attempts at achieving the current goal stressed in the curriculum are failing. Geometry, in particular deductive geometry, is quite a demanding subject for many school teachers. Research on the teachers and their practical work has shown that teachers' instruction could also negatively influence students' learning attitudes, understanding and attainment.

Research on geometry classroom demonstrated in section 2.3.2 further shows that students' difficulty in learning geometry, in particular proof writing, with their relatively poor learning results, has received wide concern in research of mathematics education. Senk (1985) stresses that greater attention needs to be given to developing the meaning of proof by instruction and curriculum. This statement highlights the necessity for research to find more effective teaching strategies and approaches towards developing students' understanding and thinking in deductive geometry.

Research on the effective teaching and learning of geometry summarised in section 2.4 suggests that students' visualisation in geometry is extremely complex, and the visual approach is likely to play a significant role in supporting students' geometric concepts formation, geometric relationship exploration, patterns discovery and hypotheses confirmation. Second, the false dichotomy, namely an empirical/deductive approach, has been identified in geometry teaching and learning. There is a need for research to develop understanding of the interplay between these two approaches in geometry, in particular deductive geometry. Third, the role of teachers' questions has been highlighted in the effective teaching and learning of mathematics. However, it is found that many teachers may use questions that involve low-level mathematics thinking, rather than a higher level of thinking. It is essential to identify what types of questions may support thinking development in an advanced course such as geometric proof problem solving. Last, the importance of mathematical instructional tasks has been identified, including the difficulties associated with implementing high-level tasks and the ways of supporting the implementation of high-level tasks in effective teaching and learning of mathematics. In terms of the teaching tasks, this study particularly pays attention to the connection of the theory of variation with Vygotsky's theoretical ideas about the zone of proximal development, and scaffolding.

The van Hieles proposed an influential pedagogical and cognitive theory in geometry education in which a system of levels of thinking in geometry is formulated and an instructional experiment is described to address the relation of the five teaching phases with the development of students' thought levels. Section 2.5 briefly presented the work of the van Hieles and a number of research issues addressed by some significant van Hiele-based work, such as the validity and reliability of the levels and phases, the discrete nature of the levels and the sequential feature of the phases, and types versus level of reasoning.

The next chapter introduces the van Hiele theory in more detail, leading to a focus on developing a comprehensive understanding of the theoretical hypothesis on geometry teaching and learning.

CHAPTER 3. THEORETICAL FRAMEWORK – THE VAN HIELE MODEL

This chapter provides an in-depth account of the van Hiele model. In particular, the chapter focuses on developing understanding of the two main aspects of the model, the idea of levels of thinking in geometry and how these can be enhanced by teaching through a sequence of instructional phases.

In section 3.1, the elaborations of the original work of the van Hieles are reviewed. Next, a broad source of the van Hiele levels and phases is analysed. Finally, the version of the van Hiele model applied in this study is defined.

3.1. The elaborations of the original work of the van Hieles

3.1.1 The elaborations of the levels of thinking

Pierre M. van Hiele's original work on the levels of thinking was proposed in his unpublished doctoral thesis in Dutch (P.M. van Hiele, 1957). Wirszup (1976) first translated the work of van Hiele levels to American audiences. However, the translation by Wirszup (1976) was based on the Russian post-experimental description (Pyshikalo, 1968; Stolyar, 1965). For Wirszup (1976), the Russian description of the levels is more elaborate. This may be due to the fact that mathematics educators, methodologists and psychologists at the Soviet Academy of Pedagogical Sciences at that time organised intensive research and experimentation on the levels of development outlined by van Hiele, and between 1960 and 1964 they verified the validity of his assertions and principles.

Usiskin (1982, p.4) provided a summary of general descriptions, together with examples of the levels from the work of Hoffer (1979, 1981). In order to use the van Hiele theory to devise test instruments, a total of nine of van Hieles' writings, four originally written in English, five translated into English from Dutch, German, or French, were examined (for more details see Usiskin (1982, pp.9-12). Noticeably, Usiskin (1982, p.13) confessed that the available source of the van Hieles' work at that time did not include some significant work of the van Hieles, such as P.M.'s or Dina's complete dissertations (1957/1984) and the book "Begrip en inzicht" (P.M. van Hiele, 1973). In terms of the available source of the levels, Usiskin (1982, p.13) points out that:

> "There is a paucity of behaviours at level 5, and even those four behaviors listed are quite vague. ... A variety of behaviours is described for level 4, but the descriptions are often vague. ... At levels 1, 2, and 3 the behaviours are in sufficient quantity and detail to enable testing."

In view of the changes of the levels of thinking, Usiskin (1982, p.14) further argues that

> "Removing one level from the theory would not be disastrous to it. But removing two levels results in a theory that surely would not have been as attractive to the mathematics education community because it would not so clearly locate proof understanding and would, with three levels, be seen as too simplistic."

Furthermore, Fuys, Geddes, & Tischler (1984) translated some significant work of the van Hiele from Dutch and French into American English, which included Dina's dissertation, the article "*Didactics of geometry as learning process for adults*", the English summary which Pierre van Hiele prepared in 1957 for his thesis and the article "*La pensée de l'enfant et la géometrie*" ("*A child's thought and geometry*" (van Hiele, 1959/1984)). According to Fuys *et al.* (1984), it was this latter article (van Hiele, 1959/1984) which captured the attention of Soviet researchers who, in turn, developed ways of using the van Hiele principles to revise their school geometry curriculum. In the article, Pierre M. van Hiele described in detail the levels and phases within levels of his theoretical model for thought development in geometry (more details are shown in section 3.2.1).

The initial brief descriptions of the van Hiele levels in the project of Fuys *et al.* (1988) were based mainly on three articles (van Hiele & van Hiele-Geldof, 1958; van Hiele, 1959/1984; and Wirszup, 1976). In order to develop fuller characterisations of the levels and examples of how they are applied, Fuys *et al.* (1988) analysed several other van Hiele source documents, in particular, the doctoral dissertation of Dina van Hiele-Geldof (translated by Fuys *et al.*, 1984). Based on the analysis of the van Hiele sources (for more details see Fuys *et al.*, 1988, p.72), Fuys *et al.* (1988) claim that the validation of descriptors for levels 1, 2, 3 is particularly strong, as most descriptors are documented by several quotations of the van Hieles' original work. However, they also indicate that as the van Hieles were secondary school teachers and chiefly concerned about teaching and learning at these levels (Levels 1-3), there are relatively few references in their writings to levels 4 and 5. Moreover, the van Hieles tended to speak in general terms about the higher levels.

In view of the list of level descriptors from the work of the van Hieles, Fuys *et al.* (1988) point out that

> "... The Project's initial version of the van Hiele model was relatively simplistic ... While this version provided an adequate starting point, it lacked sufficient detail to be an operational model for the development of the Project's instructional/assessment modules and for the assessment of a student's level of thinking. Thus, the Project needed to flesh out this skeletal version." (*ibid*, p.56)

Thus, Fuys *et al.* (1988) formulated an operational version of the van Hiele model, together with examples of student responses cited for level descriptors. Noticeably, they pointed out that their project documentation of specific descriptors at level 4, and in particular, at level 5 is less precise. Their project regarded these descriptors as tentative (*ibid*, p.73). Nevertheless, they formulated and kept some descriptors in the operational model which were drawn upon project staff members' experiences learning geometry at the secondary, undergraduate, and graduate levels and their experiences teaching geometry at the secondary and college levels, as these descriptors reflected student performances that seemed to fill in a level (*ibid*, p.73).

According to Battista (2007, p.847), the levels of thinking have been increased to six levels, with an additional new level, Level 0. Nevertheless, the most original part of the levels, Level 1 to Level 4, which are mainly considered in this study, have remained with the same description (compared with the version of level descriptors in Clements and Battista, 1992, pp.427-428). Moreover, Battista (2007, pp.851-853) provided a new elaboration of the van Hiele levels to trace students' development of reasoning from informal intuitive conceptualizations of 2D geometric shapes to the formal property-based conceptual system used by mathematicians. This new elaboration considerably expands the van Hiele levels in two places – the development of property-based thinking, and the development of inference about properties (for more details, see section 3.2.1).

In addition, recent developments in van Hiele-based research have extended the level descriptors beyond 2D shapes. For instance, Gutiérrez and colleagues extended the van Hiele level descriptions to reasoning about 3D shapes (Gutiérrez, Jaime, & Fortuny, 1991). Johnson-Gentile, Clements, and Battista (1994), Lewellen (1992), and Jaime and Gutiérrez (1989) extended the descriptions to motions/transformations, etc. However, this study does not examine, in great depth, these research studies, as this thesis focuses on developing understanding of the level descriptors of 2D geometry.

Overall, there has been a considerable amount of research which translated and interpreted the nature of the van Hiele levels of thinking. In section 3.2.1, except for analysing the translation of some original work of the van Hieles, this study also further analyses the descriptions of Hoffer (1981, 1983; 1994), Burger and Shaughnessy (1986), Schoenfeld (1986), Fuys *et al.* (1988), Clements and Battista (1992), and Battista (2007), which have carefully elaborated and extended the van Hiele levels of thinking.

3.1.2 The elaborations of the instructional phases

Dina van Hiele-Geldof's original work on the teaching/learning phases was published in Dutch in 1958. When Wirszup (1976) introduced the work of the van Hieles, the phases were translated according to the original French version of P.M. van Hiele (1959/1984, translated into American English by Fuys *et al* in 1984). Fuys *et al* (1984) directly translated Dina's original article of the phases from Dutch into American English (see van Hiele-Geldof, 1958/1984).

In the original article (1958/1984), as translated by Fuys *et al.*, Dina van Hiele-Geldof stated that

> "The essence of didactics is the encounter of three elements: the pupil, the subject matter and the teacher. ... In this article I will not consider the pedagogical aspect – the relationship between pupil and teacher, nor will I discuss the sociological aspect – the relationship among the pupils. I wish to limit myself to the learning process – the relationship between pupil and subject matter – in order to focus attention on the particular structure of didactics." (p.217)

The five learning phases were described to develop understanding of the structure of geometry learning. Thus, Dina van Hiele-Geldof suggested the five phases as an effective means to help students to make the transition from Level 1 (original Level 0) to Level 2 (original Level 1) in geometry.

In P.M. van Hiele (1959/1984), the five phases, considered as a process of apprenticeship, are suggested as a means to lead to a higher level of thought in geometry. Moreover, P.M. van Hiele (1986) claims that

> "if we call the learning process leading from one level to the next a "period", then we find in one period the following phase: 1) information; 2) bound orientation; 3) explicitation; 4) free orientation;

5) integration." (*ibid*, p.176)

Such statement shows the development of the five phases as a means for generally structuring of students' learning experience not only for mathematics but also in other subjects.

Hoffer (1983) compared the van Hiele phases with the *learning cycle* by Dienes & Golding (1971). To highlight the role of teacher in supporting students to higher levels, Hoffer states that

> "... Quite clearly the ability to think at higher levels is not acquired from written materials alone and, at least for a while, not from computer materials alone. The phases suggest an interaction between student and teacher similar to one in which a vastly wise, knowledgeable, perceptive, and loving parent provides the child with the necessary and sufficient amount of help to enable the child to mature." (1983, p.225)

In terms of the function of the phases, Hoffer (1983) considers that

> "The van Hiele phases, currently explicated in outline form, provide a more complete teaching and learning plan than one finds in many existing programs. For example, so-called direct instruction models, such as the Montessori system or Distar, are based almost entirely on Phase 2: directed orientation. A perverse form of Phase 3, explicating, is called the lecture method. Phase 4, free orientation, is prominent in so called problem-solving curricula in which students are expected to find their own ways in diverse topics without adequately, according to the van Hiele model, setting the stage and building a system of relations from which students can operate effectively." (*ibid*, pp.225-6)

Fuys *et al.* (1988) developed and validated the instructional modules which embodied the spirit of Dina van Hiele-Geldof's work. These researchers used the instructional modules as a research tool to assess levels of thinking, and claimed that certain techniques and tasks particularly effective for developing and/or assessing student thinking. Moreover, they highlighted the role of the interviewer as follows:

> "The interviewer along with the instructional materials played a special role in helping students to progress within a level or to a higher level. The interviewer provided instruction designed to move students to a higher level. Also, the interviewer guided student responses through questioning and directives about the quality of responses, thus helping students to learn the rules of the game. For example, students needed to learn to observe relationships between parts of a figure and to make generalizations (level 1[1]) or to give deductive explanations (level 2[2]). ..."

Noticeably, the levels of thinking assessed by Fuys *et al.* (1988) through the instructional modules were mainly between Level 1 and Level 3.

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[1] Level 2 in this study.

[2] Level 3 in this study.

Clements and Battista (1992) suggest the five phases as an instructional process.

> "The van Hiele theory, though, does not support an "absorption theory" model of learning and teaching. The van Hieles claim that higher levels are achieved not via direct teacher telling, but through a suitable choice of exercises. Also, "children themselves will determine when the moment to go to the higher level has come" (P. van Hiele, personal communication, Sept. 27, 1988). Nevertheless, without the teacher, no progress would be made...." (*ibid*, pp.430-431)

Similarly, Hoffer (1994) views the five phases as an instructional process (see section 2.5.3).

Overall, it could be seen that the nature of the original phases proposed by Dina van Hiele-Geldof (1958/1984) has been significantly developed by P.M. van Hiele (1959/1984) from merely facilitating Level 1 to Level 2 thinking to any higher level of thinking, and by P.M. van Hiele (1986) from geometry to any other subjects. The elaboration of the nature of the learning process has been extended to the interpretation of the function of the teaching process by researchers (Hoffer, 1983, 1994; Fuys *et al.*, 1988; Clements and Battista, 1992). As a consequence, the instructional phases, as a learning/teaching process, have become quite complex and difficult to understand for researchers. This may be one reason that the phases have yet to be fully validated by the existing van Hiele-based research work. In section 3.2.2, the complexity of each phase is more fully analysed.

3.2 Developing a comprehensive understanding of the van Hiele model

3.2.1 Descriptions and understanding of the levels

> "Now we see why it is important to know which levels of thinking are necessary for which subject matter: Until a pupil has attained the needed level, the performing of his task is impossible. This line of thought has important consequences." (van Hiele, 1955, quoted in van Hiele, 1986, p.40)

In this section, a broad list of descriptions of the levels is first presented in date order of publication (see tables 3.1-4), followed by an analysis aimed towards developing understanding of each level. The aim of the broad analysis of the model of levels is to use its detailed characterisation of the levels to define an operational model for analysing audio-taped episodes of students' responses in the observed lessons and their learning results in their homework and test papers.

1. Level 1 thinking (Originally Level 0)

1) Descriptions of Level 1 thinking

Descriptions of Level 1 thinking	Source
Figures are judged by their appearance. A child recognizes a rectangle by its form and a rectangle seems different to him than a square. When one has shown a six-year-old child what a rhombus is, what a rectangle is, what a square is, what a parallelogram is, he is capable of reproducing these figures without error on a geoboard of Gattagno, even in difficult arrangements. At the base Level (Level 1), a child does not recognize a parallelogram in the shape of a rhombus. At this level, the rhombus is not a parallelogram, the rhombus seems to him a completely different thing. (p.245)	P.M. van Hiele (1959/1984, p.245) [translated from the original French version into American English]
The pupils do not see the parts of the figure, nor do they perceive the relationships among components of the figure and among the figures themselves. They cannot even compare figures with common properties with one another. The children who reason at this level distinguish figures by their shape as a whole.	Wirszup (1976, p.77)
The important thing on the basic level (Level 1) is that all the solutions that pupils are asked to find can be read from the structure. The problems the pupils have are purely visual; there are no rules. With the structure, the pupils are able to discover important principles of working.	van Hiele, P.M. (1980, p.2), quoted in Fuys *et al.* (1988, p.74)
Visual skill: Recognizes different figures from a picture. Recognizes information labelled on a figure. Verbal skill: Associates the correct name with a given figure. Interprets sentences that describe figures. Drawing skill: Makes sketches of figures accurately labelling given parts. Logical skill: Realizes there are differences and similarities among figures. Understands conservation of the shape of figures in various positions. Applied skill: Identifies geometric shapes in physical objects.	Hoffer (1981, table1)
"This figure is a rhombus." What is meant by this sentence depends on the speaker. If he is a naive beginner in mathematics, he probably does not mean any more than: "This figure has the shape I have learned to call 'rhomb'."	van Hiele, P.M. (1986, p.109)
The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.	Burger and Shaughnessy (1986, p.31)
Gestalt recognition of figures. Students recognize entities such as squares and triangles, but they recognize them as wholes; they do not identify the properties or determining characteristics of those figures.	Schoenfeld (1986, p.251)
The student identifies instances of a shape by its appearance as a whole: a) in a simple drawing, diagram or set of cut-outs; b) in different positions; c) in a shape or other more complex configurations. (p.58) The student constructs, draws, or copies a shape. (p.58) The student names or labels shapes and other geometric configurations and uses standard and/or non-standard names and labels appropriately. (p.58) The student compares and sorts shapes on the basis of their appearance as a whole. (p.59)	Fuys *et al.* (1988, pp.58-59)

The student verbally describes shapes by their appearance as a whole. (p.59) The student solves routine problems by operating on shapes rather than by using properties which apply in general. (p.59) The student identifies parts of a figure but a) does not analyze a figure in terms of its components; b) does not think of properties as characterizing a class of figures; c) does not make generalizations about shapes or use related language. (p.59)	
The end product of this reasoning (Level 1) is the creation of conceptualizations of figures that are based on the explicit recognition of their properties (that is, after this conceptual construction, the student is at Level 2).	Clements and Battista (1992, p.427)
Visual or verbal cues bring relevant responses; Students pursue personal characterizations vs. formal definitions.	Hoffer (1994, p.7)
Level 1: Visual-Holistic Reasoning Students may justify their responses using imagined visual transformations, saying for instance that a shape is a square because if it is turned it looks like a square. Orientation of figures may strongly affect Level 1 students' shape identifications. 1.1 Pre-recognition. Students are unable to identify many common shapes. 1.2 Recognition. Students correctly identify many common shapes.	Battista (2007, p.851)

Table 3.1 An overview of descriptions of Level 1 thinking.

2) Developing an understanding of Level 1 thinking

Hoffer (1981, table 1) used the word "*Recognition*" to characterise Level 1 thinking. Burger and Shaughnessy (1986, p.31) used the word "*Visualisation*" to describe this level. Clements and Battista (1992, p.427) and Battista (2007, p.849) used the word "*Visual*" to elaborate the nature of this level.

At this level, "figures are judged by their appearance" (P.M. van Hiele, 1959/1984, p.245). "The pupils do not see the parts of the figure, nor do they perceive the relationships among components of the figure and among the figures themselves." (Wirszup, 1976, p.77).

2. Level 2 thinking (Originally Level 1)

1) Descriptions of Level 2 thinking

Descriptions of Level 2 thinking (Originally Level 1)	Source
At this level a geometric shape is still interpreted as the totality of its geometric properties. The pupils are not yet capable of differentiating them into definitions and propositions. Logical relations are not yet a fit study-object for pupils who are at the first level of thinking (Level 2).	van Hiele, P.M. and van Hiele-Geldof, D. (1958, pp.77-78), quoted in Fuys *et al.* (1988, p.75)
The figures are bearers of their properties. That a figure is a rectangle means	P.M. van Hiele

that it has four right angles, diagonals are equal, and opposite sides are equal. Figures are recognized by their properties. If one tells us that the figure drawn on a blackboard has four right angles, it is a rectangle even if the figure is drawn badly. But at this level, properties are not yet ordered, so that a square is not necessarily identified as being a rectangle. (p.245)	(1959/1984, p.245) [translated from the original French version into American English]
The pupil who has reached the second level begins to discern the components of the figures; he also establishes relationships among these components and relationships between individual figures. At this level, he is therefore able to make an analysis of the figures perceived. This takes place in the process (and with the help) of observations, measurements, drawing, and model-making. The properties of the figures are established experimentally; they are described, but not yet formally defined. These properties which the pupil has established serve as a means of recognizing figures. ... However, these properties are still not connected with one another. For example, the pupil notices that in both the rectangle and the parallelogram of general type the opposite sides are equal to one another, but he does not yet conclude that a rectangle is a parallelogram.	Wirszup (1976, p.77-78)
Visual skill: Notices properties of a figure. Identifies a figure as part of a larger figure. Verbal skill: Describes accurately various properties of a figure. Drawing skill: Translates given verbal information into a picture. Uses given properties of figures to draw or construct the figures. Logical skill: Understands that figures can be classified into different types. Realizes that properties can be used to distinguish figures. Applied skill: Recognizes geometric properties of physical objects. Represents physical phenomena on paper or in a model.	Hoffer (1981, table 1)
A first level (Level 2) is attained when the pupil is able to apply operative properties known to him in a figure known to him. For instance, if a pupil knows that the diagonals of a rhombus are perpendicular, after having reached the first level (Level 2) he must be able to conclude that, if two equal circles have two points in common, the segment joining the centers of the circles are perpendicular to each other. It may be that he does not directly see the rhombus in the figure, or he should be able to finish after having his attention drawn to this rhombus. On the other hand, the pupil not having attained the level, does not see the importance of the knowledge of the figure containing the rhombus.	van Hiele, P.M. (1986, p.41), quoted in Fuys *et al.* (1988, p.75)
If someone has already studied mathematics for some time, he means by the statement "This figure is a rhombus" something different. The figure he refers to is a collection of properties, properties he has learned to call "rhombus."	van Hiele, P.M. (1986, p.109)
The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.	Burger and Shaughnessy (1986, p.31)
Analysis of individual figures.	Schoenfeld (1986, p.251)
The student identifies and tests relationships among components of figures (e.g., congruence of opposite sides of a parallelogram; congruence of angles in a tiling pattern). (p.60) The student recalls and uses appropriate vocabulary for components and relationships (e.g., opposite sides, corresponding angles are congruent, diagonals bisect each other). (p.60) The student a) compares two shapes according to relationships among their	Fuys *et al.* (1988, pp.60-63)

components; b) sorts shapes in different ways according to certain properties, including a sort of all instances of a class from non-instances. (p.60) The student a) interprets and uses a verbal description of a figure in terms of its properties and uses this description to draw/construct the figure; b) interprets verbal or symbolic statements of rules and applies them. (pp.59-60) The student discovers properties of specific figures empirically and generalizes properties for that class of figures. (p.61) The student a) describes a class of figures (e.g., parallelograms) in terms of its properties; b) tells what shape a figure is, given certain properties. (p.61) The student identifies which properties used to characterize one class of figures also apply to another class of figures and compares classes of figures according to their properties. (p.62) The student discovers properties of an unfamiliar class of figures. (p.62) The student solves geometric problems by using known properties of figures or by insightful approaches. (p.62) The student formulates and uses generalizations about properties of figures (guided by teacher/material or spontaneously on own) and uses related language (e.g., all, every, none) but a) does not explain how certain properties of a figure are interrelated; b) does not formulate and use formal definitions; c) does not explain subclass relationships beyond checking specific instances against given list of properties; d) does not see a need for proof or logical explanations of generalizations discovered empirically and does not use related language (e.g., if-then, because) correctly. (p.63)	
At this level, the objects about which students reason are classes of figures, thought about in terms of the sets of properties that the students associate with those figures. The product of this reasoning is the establishment of relationships between and the ordering of properties and classes of figures.	Clements and Battista (1992, p.427)
Properties may be organized in a formal way but certain inclusion relationships may be missing.	Hoffer (1994, p.7)
Level 2: Analytic-Componential Reasoning 2.1 Visual-informal componential reasoning. Students describe parts and properties of shapes informally and imprecisely; they do not possess the formal conceptualizations that enable precise property specifications. Descriptions and conceptualizations are visually based, focusing initially on parts of shapes then on spatial relationships between parts. ... In all cases, students describe parts and their relationships using strictly informal language, that is, language typically learned in everyday experience. (p.851) Students' informal language ranges greatly in precision and coherence, from using vague and incompletely formulated conceptualizations to informally describing a conceptualization that corresponds to a formal geometric concept. (p.851) 2.2 Informal and insufficient-formal componential reasoning. As students begin to acquire formal conceptualizations that can be used to "see" and describe spatial relationships between parts of shapes, they use a combination of informal and formal descriptions of shapes. The formal descriptions utilize standard geometric concepts and terms explicitly taught in mathematics curricula. However, the formal portions of students' shape descriptions are insufficient to completely specify shapes. ... Although students often recall properties that they have abstracted for classes of shapes, their reasoning is still visually based, and most of their descriptions	Battista (2007, p.851-852)

and conceptualizations still seem to occur extemporaneously as they are inspecting shapes. (p.852) 2.3 Sufficient formal property-based reasoning. Students explicitly and exclusively use formal geometric concepts and language to describe and conceptualize shapes in a way that attends to a sufficient set of properties to specify the shapes. Students have made a decided shift away from visually dominated reasoning because the major criterion for identifying a shape is whether it satisfies a precise set of verbally stated formal properties. (p.852) Students can use and formulate formal definitions for classes of shapes. However, their definitions are not minimal because forming minimal definitions requires relating one property to another using some type of inferential reasoning (which occurs at Level 3). Students do not interrelate properties or see that some subset of properties implies other properties. They simply think in terms of unconnected lists of formally described characteristics. The set of properties students give for a class of shapes is a list of all the visual characteristics the student has come to associate with that type of shape, described in terms of formal geometric concepts. (But students seem to recall these property-based specifications rather than discover them on the fly as they inspect shapes.) (p.852)	

Table 3.2 An overview of descriptions of Level 2 thinking.

2) Developing an understanding of Level 2 thinking

Hoffer (1981, table 1) and Burger and Shaughnessy (1986, p.31) used the word "*Analysis*" to characterise Level 2 thinking. Clements and Battista (1992, p.427) and Battista (2007, p.849) used the word "*Descriptive/Analytic*" to describe the nature of Level 2 thinking.

P.M. van Hiele (1959/1984, p.245) states that "the figures are bearers of their properties" at this level. Wirszup (1976, p.77-78) construes that "the properties of the figures are established experimentally; they are described, but not yet formally defined. These properties which the pupil has established serve as a means of recognising figures. ... However, these properties are still not connected with one another." Burger and Shaughnessy (1986, p.31) interpret that "the student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established."

3. Level 3 thinking (Originally Level 2)

1) Descriptions of Level 3 thinking

Descriptions of Level 3 thinking (Originally Level 2)	Source
The children discovered by reasoning that the angles of a triangle sum up to 180 degrees, the analogous facts for other polygons, and the interrelation between these facts. ... The logical relations were put into a logical pattern, using the implication arrow.	van Hiele, P.M. and van Hiele-Geldof, D. (1958, pp.71-72), quoted in Fuys *et al.*

	(1988, p.76)
Properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the intrinsic meaning of deduction is not understood by the students. The square is recognized as being a rectangle because at this level definitions of figure come into play. (pp.245-246)	P.M. van Hiele (1959/1984, pp.245-246) [translated from the original French version into American English]
At this second level (Level 3) of thinking a child knows how to reason in accordance with a deductive logical system: that is, its arguments now show an "intrinsic planning, fulfilling the laws of formal logic." This is not however identical with reasoning "on the strength" of formal logic.	van Hiele, P.M. (1959, p.8), quoted in Fuys *et al.* (1988, p.75)
At this level (Level 3) there occurs a logical ordering of the properties of a figure and of classes of figures. The pupil is now able to discern the possibility of one property following from another, and the role of definition is clarified. The logical connections among figures and properties of figures are established by definitions. However, at this level the student still does not grasp the meaning of deduction as a whole. The order of logical conclusion is established with the help of the textbook or the teacher. The child himself does not yet understand how it could be possible to modify this order, nor does he see the possibility of constructing the theory proceeding from different premises. He does not yet understand the role of axioms, and cannot yet see the logical connection of statements. At this level deductive methods appear in conjunction with experimentation, thus permitting other properties to be obtained by reasoning from some experimentally obtained properties. At the third level a square is already viewed as a rectangle and as a parallelogram.	Wirszup (1976, p.78)
Visual skill: Recognizes interrelationships between different types of figures. Recognizes common properties of different types of figures. Verbal skill: Defines words accurately and concisely. Formulates sentences showing interrelationships between figures. Drawing skill: Given certain figures, is able to construct other figures related to the given ones. Logical skill: Understands qualities of a good definition. Uses properties of figures to determine if one class of figures is contained in another class. Applied skill: Understands the concept of a mathematical model that represents relationships between objects.	Hoffer (1981, table 1)
A second level (Level 3) is attained when a pupil is able to apply operatively relations known to him between figures known to him. That means that a pupil having attained this level is able to apply congruence of geometrical figures to prove certain properties of a total geometrical figure of which the congruent figures are a part. It means also that the pupil can conclude from the parallelism of lines the equality of angles.	van Hiele, P.M. (1986, p.42), quoted in Fuys *et al.* (1988, p.75)
The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.	Burger and Shaughnessy (1986, p.31)
Analysis of relations.	Schoenfeld (1986, p.251)
The student a) identifies different sets of properties that characterize a class of figures and tests that these are sufficient; b) identifies minimum sets of properties that can characterize a figure; c) formulates and uses a definition for a class of figures. (p.64)	Fuys *et al.* (1988, pp.64-68)

The student gives informal argument (using diagrams, cutout shapes that are folded, or other materials): a) having drawn a conclusion from given information, justifies the conclusion using logical relationships; b) orders classes of shapes; c) orders two properties; d) discovers new properties by deduction; e) interrelates several properties in a family tree. (pp.64-66) The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument; b) gives a summary of variation of a deductive argument; c) gives deductive arguments on own. (pp.66-67) The student gives more than one explanation to prove something and justifies these explanations by using family trees. (p.67) The student informally recognizes difference between a statement and its converse. (p.67) The student identifies and uses strategies or insightful reasoning to solve problems. (p.67) The student recognizes the role of deductive argument and approaches problems in a deductive manner but a) does not grasp the meaning of deduction in an axiomatic sense (e.g., does not see the need for definitions and basic assumptions); b) does not formally distinguish between a statement and its converse (e.g., cannot separate the "Siamese twins" – the statement and its converse); c) does not yet establish interrelationships between networks of theorems. (p.68)	
As students discover properties of various shapes, they feel a need to organize the properties. One property can signal other properties, so definitions can be seen not merely as descriptions but as a method of logical organization. It becomes clear why, for example, a square is a rectangle. This logical organization of ideas is the first manifestation of true deduction. However, the students still do not understand that logical deduction is the method for establishing geometric truths. At this level, the objects about which students reason are properties of classes of figures. Thus, for instance, the "properties are ordered, and the person will know that the figure is a rhombus if it satisfies the definition of quadrangle with four equal sides" (van Hiele, 1986, p.109). The product of this reasoning is the reorganization of ideas achieved by interrelating properties of figures and classes of figures.	Clements and Battista (1992, p.427)
The student formulates and uses formal definitions; The student is able to perform one-step deductions;	Hoffer (1994, p.7)
Level 3: Relational-inferential property-based reasoning At Level 3, the spatial relationships described by formal property statements reach the second level of interiorization so that they can be symbolized by the statements, and so that students can reason meaningfully about the statements, in many cases, without having to visually re-present the actual spatial structurings that the statements describe. The verbally-stated properties themselves are interiorized so that they can be meaningfully decomposed, analyzed, and applied to various shapes. 3.1 Empirical relations. Students use empirical evidence to conclude that if a shape has one property, it has another. 3.2 Componential analysis. By analyzing how types of shapes can be built one-component-at-a-time, students conclude that when one property occurs, another property must occur. Students conduct this analysis by making	Battista (2007, pp.852-853)

drawings or imagining constructing shapes piece-by-piece. 3.3 Logical inference. Students make logical inferences about properties; they mentally operate on property statements, not images. For example, a student might reason that because a square has all sides equal, it has opposite sides equal. Such reasoning enables students to make the inferences needed for hierarchical classification. For instance, a student whose definition for a rectangle is "4 right angles and opposite sides equal" might infer that a square is a rectangle because "a square has 4 right angles, which a rectangle has to have; and because a square has 4 equal sides, it has opposite sides equal, which a rectangle has to have." But students do not use this inferencing ability to logically reorganize their conceptual networks about shapes, so they do not adopt a logical hierarchical shape classification system (i.e., they still resist the notion that a square is a rectangle even though they can follow the logic justifying such as a statement). Students' reasoning is "locally logical" in that they string together logical deductions based on "assumed-true" propositions, that is, propositions that they accept as true based on their experience, intuition, or authority. Thus, students at this level use logic, but they do not question the starting points for their logical analyses. 3.4 Hierarchical shape classification based on logical inference. Students use logical inference to *reorganize* their classification of shapes into a logical hierarchy. They fundamentally restructure their shape classification networks (as opposed to merely making additional connections here and there). It becomes not only clear why a square is a rectangle, but a necessary part of reasoning. Students give logical arguments to justify their hierarchical classifications. Finally, students' use of logic to draw conclusions provides them with a new way to accumulate knowledge. That is, new knowledge can now be generated not merely through empirical or intuitive means, but through logical deduction.	

Table 3.3 An overview of descriptions of Level 3 thinking.

2) Developing an understanding of Level 3 thinking

Hoffer (1981, table 1) used the word "*Ordering*" to characterise the nature of Level 3 thinking. Burger and Shaughnessy (1986, p.31) used the word "*Abstraction*" to describe Level 3 thinking. Clements and Battista (1992, p.427) and Battista (2007, p.849) used the word "*Abstract/Relational*" to interpret the characterisation of thinking at this level.

P.M. van Hiele (1959/1984, pp.245-246) points out that "At this level the intrinsic meaning of deduction is not understood by the students." Wirszup (1976, p.78) elaborates that at this level, "The order of logical conclusion is established with the help of the textbook or the teacher. The child himself does not yet understand how it could be possible to modify this order, nor does he see the possibility of constructing the theory proceeding from different premises. He does not yet understand the role of axioms, and cannot yet see the logical connection of statements. At this level deductive methods appear in conjunction with experimentation, thus permitting other properties to be obtained by reasoning from some experimentally obtained properties."

4. Level 4 thinking (Originally Level 3)

1) Descriptions of Level 4 thinking

Descriptions of Level 4 thinking (Originally Level 3)	**Source**
At the third level (Level 4) it would be possible to develop an axiomatic system of geometry, but the axiomatics themselves belong to the fourth level (Level 5).	van Hiele, P.M. and van Hiele-Geldof, D. (1958, p.75), quoted in Fuys *et al.* (1988, p.76)
Thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions. (p.246)	P.M. van Hiele (1959/1984, p.246) [translated from the original French version into American English]
The third level (Level 4), that of discernment in geometry, or the essence of mathematics. The aim of instruction is now to understand what is meant by logical ordering (what do we mean by: one property "precedes" another property?). The material is made up of geometric theorems themselves. In the ordering of these theorems certain ideas will become apparent, namely: the link between a theorem and its converse, why axioms and definitions are indispensable, when a condition is necessary and when sufficient. Students can now try to order new domains logically, as for example when they first study the cylinder.	van Hiele, P.M. (1959, p.250), quoted in Fuys *et al.* (1988, p.76)
At the fourth level, the students grasp the significance of deduction as a means of constructing and developing all geometric theory. The transition to this level is assisted by the pupils' understanding of the role and the essence of axioms, definitions, and theorems; of the logical structure of a proof; and of the analysis of the logical relationships between concepts and statements. The students can now see the various possibilities for developing a theory proceeding from various premises. For example, the pupil can now examine the whole system of properties and features of the parallelogram by using the textbook definition of a parallelogram: A parallelogram is a quadrilateral in which the opposite sides are parallel. But he can also construct another system based, say, on the following definition: A parallelogram is a quadrilateral, two opposite sides of which are equal and parallel.	Wirszup (1976, p.78)
Visual skill: Uses information about a figure to deduce more information. Verbal skill: Understands the distinctions among definitions, postulates, and theorems. Recognizes what is given in a problem and what is required to find or do. Drawing skill: Recognizes when and how to use auxiliary elements in a figure. Deduces from given information how to draw or construct a specific figure. Logical skill: Uses rules of logic to develop proofs. Is able to deduce consequences from given information. Applied skill: Is able to deduce properties of objects from given or obtained information. Is able to solve problems that relate objects.	Hoffer (1981, table 1)

A [fourth] level must be connected with the possibility of comparing, transposing, and operating with relations.	van Hiele, P.M. (1986, p.44)
The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.	Burger and Shaughnessy (1986, p.31)
Deductive competence, the goal state of 10th grade geometry. If asked to prove (for example) that the inscribed angle subtending a given arc of a circle has a measure half that of the central angle subtending the same arc, the student can do so by producing a series of statements that logically justify the conclusion as a consequence of the "givens."	Schoenfeld (1986, p.251)
The student recognizes the need for undefined terms, definitions, and basic assumptions (e.g., postulates). (p.69) The student recognizes characteristics of a formal definition (e.g., necessary and sufficient conditions) and equivalence of definitions. (p.69) The student proves in an axiomatic setting relationships that were explained informally on level 2 (Level 3). (p.69) The student proves relationships between a theorem and related statements (e.g., converse, inverse, contrapositive). (p.69) The student establishes interrelationships among networks of theorems. (p.69) The student compares and contrasts different proofs of theorems. (p.70) The student examines effects of changing an initial definition or postulate in a logical sequence. (p.70) The student establishes a general principle that unifies several different theorems. (p.70) The student creates proofs from simple sets of axioms frequently using a model to support arguments. (p.70) The student gives formal deductive arguments but does not investigate the axiomatics themselves or compare axiomatic systems. (p.70)	Fuys *et al.* (1988, pp.69-70)
At this level (Level 4), students can reason formally by logically interpreting geometric statements such as axioms, definitions, and theorems. The objects o their reasoning are relationships between properties of classes of figures. The product of their reasoning is the establishment of second-order relationships – relationships between relationships – expressed in terms of logical chains within a geometric system.	Clements and Battista (1992, p.428)
The student is able to work within an axiomatic system and prove propositions formally.	Hoffer (1994, p.7)
Students can understand and construct formal geometric proofs. That is, within an axiomatic system, they can produce a sequence of statements that logically justifies a conclusion as a consequence of the "givens." They recognize differences among undefined terms, definitions, axioms, and theorems.	Battista (2007, p.853)

Table 3.4 An overview of descriptions of Level 4 thinking.

2) Developing an understanding of Level 4 thinking

Hoffer (1981, table 1) and Burger and Shaughnessy (1986, p.31) used the word "*Deduction*" to describe the nature of Level 4 thinking. Clements and Battista (1992, p.427) and Battista (2007, p.849) used the word *"Formal Deduction"* to help readers to more exactly understand the nature of this level.

van Hiele, P.M. and van Hiele-Geldof, D. (1958, p.75) state that "At the third level (Level 4) it would be possible to develop an axiomatic system of geometry, but the axiomatics themselves belong to the fourth level (Level 5)." P.M. van Hiele (1959/1984, p.246) points out that at this level, "Thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions."

3.2.2 The descriptions and understanding of the instructional phases

In this section, first a broad range of work on the van Hiele phases is reviewed (see table 3.5-9). In each table, the descriptions are listed in date order. An analysis of the different descriptions of each phase follows. The two main purposes of the analysis of the source of the model of phases are: 1) to develop a comprehensive understanding of the nature and function of the phases; 2) to use its detailed characterisation of the phases in analysing audio-taped episodes of teachers' actual instruction in the observed lessons.

1. The first phase

1) Descriptions of the first phase

Descriptions of the first phase	Source
(Information/Inquiry): My first geometry lesson at the secondary school is information for me. It is a fact that man is able to perceive structure in almost any material however unordered it may be, and that this structure can be perceived in the same way by different people. This allows man to discover the intrinsic ordering in the material that is presented to him. For example, the knowledge of shapes is developed through manipulation of material objects. (pp.217-218) ... I wish to know which shapes have already been differentiated in a geometric sense by the twelve-year-old pupils. ... I ask them, for instance, to tell me what regularities they perceive in a cube. I show them several cubes of different sizes. ... I ask the pupils why they are so convinced that this equality exists. It appears that the pupils are proposing as a method, the action of "fitting." ... The pupils themselves make the relation of equality explicit. For congruence is the means by which we order geometrically. Goal and means are clearly brought forward. (p.218) I know from the inquiry (first geometry lesson) that the pupils are capable of purposeful action. We can call these first class conversations the informative phase in the learning process of the pupils. They discover which aspect, out of the multitude of experiences they have already had, we are dealing with. The action of fitting allows the observation of equalities of parts in figures. (pp.218-219)	van Hiele-Geldof (1958/1984, p.217-218, 223) [translated from the original Dutch version into American English]

Information (Inquiry) by means of representative material gathered from the existing substratum of empirical experiences in order to bring the pupils to purposeful action and perception. (p.223)	
Inquiry: the student learns to know the field under investigation by means of the material which is presented to him. This material leads him to discover a certain structure.	P.M. van Hiele (1959/1984, p.247) [translated from the original French version into American English.]
Information: the student learns to recognize the field of investigation by means of the material which is presented to him. This material causes him to discover a certain structure.	Wirszup (1976, p.83)
Inquiry: The teacher engages the students in (two-way!) conversations about the objects of study. The teacher learns how the students interpret the words and gives the students some understanding of the topic to be studied. Questions are raised and observations made that use the vocabulary and objects of the topic and set the stage for further study.	Hoffer (1983, p.208)
In the first phase (Information), by placing at the children's disposal (putting into discussion) material clarifying the context.	van Hiele, P.M. (1986, p.177)
Inquiry, in which the dialogue between teacher and student introduces the objects of discourse and establishes the students' current understandings of those objects.	Schoenfeld (1986, p.251)
Information: The student gets acquainted with the working domain (e.g., examines examples and non-examples). (p.7) In an activity on area of parallelograms in Module 3, the activity opens with informal work with area of parallelograms to acquaint students with this topic (Information). (p.13)	Fuys *et al.* (1988, p.7, 13)
Information: The teacher discusses materials clarifying this content, placing them at the child's disposal. Through this discussion, the teacher learns how students interpret the language and provides information to bring students to purposeful action and perception.	Clements and Battista (1992, p.431)
Familiarization: The teacher acquaints the pupils with the working domain including the vocabulary and motivates the subject. (p.2) The student becomes acquainted with the working domain. (p.6) ● The teacher introduces problems which help in the discovery process (p.6) ● The teacher "sets the stage" for upcoming topics by introducing questions that incite curiosity (p.6) ● The teacher has students use visual cues and manipulatives (p.6)	Hoffer (1994, p.2, 6)

Table 3.5 An overview of descriptions of the first phase.

2) Developing an understanding of the first phase

Table 3.5 shows that Wirszup (1976, p.83) translated the first phase as an "information" phase, while Hoffer (1983, p.208) translated this phase as an "inquiry" phase. Noticeably, Fuys *et al.* (see van Hiele-Geldof, 1958/1984) used "Inquiry" to characterise the first phase, yet they used "information" from this phase in their work (1988). Such a change of the use of the word may be due to the use of "information" by P.M. van Hiele (1986, p.177) to describe why a pupil may gain nothing by the information given in a textbook of geometry.

The use of both "inquiry" and "information" may reflect the researchers' different concerns about the role of teacher and students in the teaching/learning process. When the word "information" is used, the researchers seem to indicate the essential role of the teacher, which may entail helping students receive relevant information. For instance, Wirszup (1976, p.83) indicates that "the student learns to recognise the field of investigation ...."; P.M. van Hiele (1986, p.177) states "by placing at the students' disposal material..."; Fuys *et al.* (1988, p.7) suggest that "the student gets acquainted with the working domain..."; and Clements and Battista (1992, p.431) clearly highlight that "the teacher discusses materials clarifying this content...".

When the word "inquiry" is used by the researchers, the role of students is emphasised. As has been pointed by Hoffer (1983),

> "... to help students raise their thought levels, the van Hieles specified a sequence of phases that moves from very direct instruction to the students' independence of the teacher." (*ibid*, p.207)

Indeed, P.M. van Hiele (1959/1984, p.247) interpreted that "this material leads him (the student) to discover a certain structure". Hoffer (1983, p.208) stressed that "the teacher engages the students in (two-way!) conversations about the objects of study..." during the first phase. Schoenfeld (1986, p.251) also suggests paying attention to "...the dialogue between teacher and student introduces the objects of discourse ..." during this phase.

In terms of the nature of the phases, Wirszup (1976, p.83) and P.M. van Hiele (1959/1984, p.247) interpreted the phases as the process of apprenticeship. Hoffer (1994, p.2) used the word "familiarization" to characterise the first phase. His descriptions of this phase (see Hoffer, 1994, p.6) indicate that the teacher provides necessary instruction to help students become acquainted with the working domain.

According to van Hiele-Geldof (1958/1984), empirical work is the main focus of this phase (see key words "manipulation of material objects, measuring, folding, fitting", p.218). van Hiele (1986, p.177) suggests using information to clarify the context at this phase. Wirszup (1976, p.83) and P.M. van Hiele (1959/1984, p.247) interpreted this phase as a discovery process of a certain structure to be studied. Hoffer (1983, p.208; 1994, p.2) considered this phase as helping students become acquainted with the vocabulary and objects of the topic and set the stage for further study. Schoenfeld (1986, p.251) considered using this phase to establish "the students' current understanding of those objects", and Clements and Battista (1992, p.431) highlighted the key words "purposeful action and perception" of this phase, which was translated in van Hiele-Geldof (1958/1984, p.223).

Overall, this phase may be understood as a phase involving activities of either information or inquiry mainly to develop students' understanding of the topic to be studied and to lead them to discover a certain structure which is to be explored at the next phase of teaching.

2. The second phase

1) Descriptions of the second phase

Descriptions of the second phase	Source
Directed Orientation: They now look at figures in a certain way. Figures are not only being folded, but in the process of doing so, one purposefully looks at what equalities of parts of the figure are being revealed. (p.219) ... The pupils are actively engaged in cutting out figures and in subsequently checking in what way those figures fit in the openings; networks of known spatial figures are being made and these are checked by actually making the figure; it is also investigated in how many different ways plane figures can be folded in two through actual folding. (p.219) This period during which the manipulation is prominent and is being required of the pupils, we call directed orientation. The empirical experiences are broadened through manipulations. There is continual investigation of how one part of a figure can take the place of another part. (p.219) Directed orientation which is possible when the child demonstrates a disposition towards exploration and is willing to carry out the assigned operations. (p.223)	van Hiele-Geldof (1958/1984, p.219, 223) [translated from the original Dutch version into American English]
Directed orientation: the student explores the field of investigation. He already knows in what direction the study is directed; the material is chosen in such a way that the characteristic structures appear to him gradually.	P.M. van Hiele (1959/1984, p.247) [translated from the original French version into American English.]
Directed orientation: the student explores the field of investigation by means of the material. He knows then in what direction the study is geared; the matter is chosen in such a way that the characteristic structures progressively appear to him.	Wirszup (1976, p.83)
Directed orientation: Many of the activities in this phase are one-step tasks that elicit specific responses.	Hoffer (1983. p.208)

In the second phase (Bound orientation), by supplying the material by which the pupils learn the principal connections in the field of thinking.	van Hiele, P.M. (1986, p.177)
Directed orientation, in which the student undertakes a carefully designed sequence of exercises chosen to unfold the structures of the objects of inquiry.	Schoenfeld (1986, p.251-252)
Guided orientation: The student does tasks involving different relations of the network that is to be formed (e.g., folding, measuring, looking for symmetry).	Fuys *et al.* (1988, p.7)
Guided orientation: students become acquainted with the objects from which geometric ideas are abstracted. The goal of instruction during this phase is for students to be actively engaged in exploring objects (for example, folding, measuring) so as to encounter the principal connections of the network of relations that is to be formed. The teachers' role is to direct students' activity by guiding them in appropriate explorations – carefully structured, sequenced tasks (often one-step, eliciting specific responses) in which students manipulate objects so as to encounter specific concepts and procedures of geometry. teachers should choose materials and tasks in which the targeted concepts and procedures are salient.	Clements and Battista (1992, p.431)
Guided orientation – The teacher guides the pupils with one-step tasks to explore and become familiar with the network of relations that is to be formed in the area of the subject. (p.2) The student uncovers the links that form relationships. (p.6) • The teacher steers the students' responses to the specific subject matter or discipline they are studying (p.6) • The teacher leads students in discussing the material in a narrow framework of topics (p.6)	Hoffer (1994, p.6)

Table 3.6 An overview of descriptions of the second phase.

2) Developing an understanding of the second phase

Table 3.6 illustrates that three terms are used by different researchers to characterise the second phase: directed orientation, bound orientation and guided orientation. These three terms appear to emphasise different aspects of this phase. For instance, Wirszup (1976, p.83), Hoffer (1983, p.208), van Hiele-Geldof (1958/1984, p.223), P.M. van Hiele (1959/1984. p.247) and Schoenfeld (1986, p.251) used the term "directed orientation" for this phase.

> "During directed orientation, which involves expanding empirical experiences, the teacher introduces new material. The function of the material then is such that it should be able to contribute to the discoveries of the pupils. ... During this phase of the learning process, the pupil is dependent on the ability of the teacher to find the appropriate tasks." (van Hiele-Geldof, 1958/1984, pp.220-221)

Here, the word "directed" may mean that students are directed to learn new knowledge by the well-designed materials during the second phase. Moreover, Dina van Hiele-Geldof (1958/1984) emphasises that

> "...The exploration of the teacher should concern the didactic approach – by generating tasks, by

> creating favourable learning situations – but not the exploring of the material." (van Hiele-Geldof, 1958/1984, pp.220-221)

This statement may help link to her didactical view that the student "purposefully searches for results" and is "willing to carry out the assigned operation" (*ibid,* p.223).

Interestingly, however, Pierre van Hiele (1986) uses the term "bound orientation" to describe this phase in the learning process. He provides an example from a language lesson of the meaning of the term "bound orientation" as follows:

> "We find an example of bound orientation in language lessons when the pupils are asked to transpose into the past tense a paper that has been written in the present tense. Here the intention of the pupils is directed at accomplishing the task: putting the paper into the past tense correctly. The teaching aim differs: it consists in confronting the pupils with the differences of the verb forms. Here the intention cannot be directed, for the task is only aimed at calling attention to the verb forms. Before the task has been done, these verb forms do not appear to the pupils in this sense. Just as in this case, it is practically always the situation in bound orientation that the aim is to render visible the principal relations between the concepts, but the students' intentions are not aimed at this target, because at the beginning of this phase the target is invisible." (*ibid*, pp.179-180)

Here, the word "bound" appears to link to the teachers' intention towards the subject to be learned.

Fuys *et al.* (1988), Clements and Battista (1992, p.431) and Hoffer (1994) use the term "guided orientation" to characterise this phase. As such, these researchers emphasise the significant role the teacher plays in facilitating student's progress to higher levels of thinking.

In terms of the function of the second phase, van Hiele-Geldof (1958/1984) clearly shows that material manipulation is prominent during the second phase. Key words such as "folded, cutting out, making the figure, etc." were used to describe the learning activities of this phase (*ibid*, p.219). Noticeably, Hoffer (1983) and Clements and Battista (1992) suggest that the activities of this phase consist mostly of one-step tasks.

Overall, the nature of the second phase may be understood as a phase in which students are directed by learning materials, or guided by the teacher, to discover and explore the principal connections of new learning materials.

3. The third phase

1) Descriptions of the third phase

Descriptions of the third phase	Source
Explicitation: The results of the manipulation of material objects are now expressed in words. Equalities that have been observed are enumerated. Each child need not find out everything for himself. Subjective experiences are exchanged. In this way the figures acquire geometric properties. The theorems are expressed by the pupils. The role of the teacher here consists of introducing the necessary technical terms. (p.219) During the information – conversations, the pupils speak their own language. What is crucial in these conversations is to get to know what knowledge the pupils bring with them. (p.219) The goal of explicitation is to establish properties of figures. As a result, the shape as a whole becomes less important and the figure becomes a conglomerate of properties. (p.219) Explicitation through which subjective experiences are objectified and geometric symbols are formed. (p.223)	van Hiele-Geldof (1958/1984, p.219, 223) [translated from the original Dutch version into American English]
Explicitation: at this phase, explicitation takes place. Acquired experience is linked to exact linguistic symbols and the students learn to express their opinions about the structures observed during discussions in class. The teacher takes care that these discussions use the habitual terms. It is during this third phase that the system of relations is partially formed.	P.M. van Hiele (1959/1984, p.247) [translated from the original French version into American English.]
Explanation takes place in the course of the third phase. The acquired experiences are linked to exact linguistic symbols, and the students learn to express themselves in the course of discussions about these structures which take place in class. The teacher sees to it that the customary terms are employed in the discussions. It is during the course of this third phase that the network of relations is partially formed.	Wirszup (1976, p.83)
Expliciting: The students, building from previous experiences, with minimal prompting by the teacher, refine their use of the vocabulary and express their opinions about the inherent structures of the study. During this phase, the students begin to form the system of relations of the study.	Hoffer (1983. p.208)
In the third phase (Explicitation), by leading class discussions that will end in a correct use of language.	van Hiele, P.M. (1986, p.177)
Explicitation, in which the newly discovered properties are discussed and codified.	Schoenfeld (1986, p.252)
Explicitation: The student becomes conscious of the relations, tries to express them in words, and learns technical language which accompanies the subject matter (e.g., expresses ideas about properties of figures).	Fuys *et al.* (1988, p.7, 13)
Explicitation: Students become conscious of the relations and begin to elaborate on their intuitive knowledge. Thus, in this phase, children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. The teacher's role is to bring the objects of study (geometric objects and ideas, relationships, patterns, and so on) to an explicit level of awareness by leading students' discussion of them in their own language. Once students have demonstrated their awareness of an object of study and have discussed it in their own words, the teacher introduces	Clements and Battista (1992, p.431)

the relevant mathematical terminology.	
Verbalization – The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (p.2) The student becomes aware of relations that they try to express in words with increasing accuracy. Students learn the technical language of the topic. (p.6) • The teacher helps students express and clarify formal definitions (p.6) • The teacher helps students link experience with exact linguistic symbols (p.6) • In discussions, the teacher promotes the use of terminology specific to the subject matter (p.6)	Hoffer (1994, p.2, 6)

Table 3.7 An overview of descriptions of the third phase.

2) Developing an understanding of the third phase

Three main terms, explanation, explicitation and verbalisation, were found in the translations and interpretations of the third phase. When comparing the translations given by Wirszup (1976, p.83) with that by Fuys *et al.* (see P.M. van Hiele (1959/1984, p.247), it can be seen that the words "explanation" and "explicitation" described the same function of the third phase. However, Hoffer (1983, p.208) points out that

> "Phase 3 has been incorrectly translated as explanation by other writers. It is essential here that students make the observations explicitly rather than receive lectures (explanations) from the teacher."

Moreover, key words used to describe the third phase are "exchanged", "expressed", "introducing the necessary technical terms" in van Hiele-Geldof (1958/1984, p.219), "linked to exact linguistic symbols" in Pierre van Hiele (1959/1984, p.247), "vocabulary" in Hoffer (1983, p.208), "technical language" in Fuys *et al.* (1988, p.7), "traditional mathematical language" in Clements and Battista (1992, p.431) and "terminology" in Hoffer (1994, p.6). These key words indicate the function of the third phase, which is likely to encourage students to express their opinions and to correct their use of formal language for the study. Indeed, Hoffer (1983) suggests that at the third phase, students build from previous experiences and refine their use of the vocabulary and express their opinions about the inherent structures of the study with "minimal prompting by the teacher" (p.208). Moreover, Hoffer (1994) used the term "verbalization" to highlight the correct use of language during this phase.

Pierre van Hiele (1986) explained the meaning of explicitation of this phase in terms of physics teaching as follows:

> "We find a good example of explicitation in physics when at the beginning of the chapter "Heat" the pupils are asked to give their opinions about the ways in which transfer of heat can take place. Here we are dealing with an explicitation of knowledge the pupils have acquired by means of a development, for they have acquired this knowledge before the guided learning process has begun. By imagining how heat is distributed in living rooms, the pupils can easily find the three types of distribution: radiation, convection, and conduction. Various simple relations can be deduced from remembered experiences. The intention of the pupils is to give opinions about the relations they have seen; the object of teaching is still the recording of such relations in the proper language." (*ibid*, p.180)

Here, explicitation seems to mean "an explicitation of knowledge the pupils have acquired".

In terms of the function of the third phase, van Hiele-Geldof (1958/1984, p.219) emphasises that this phase is to "establish properties of figures". Pierre van Hiele (1959/1984, p.247) puts the subject in a universal context by stating that "It is during this third phase that the system of relations is partially formed". Clements and Battista (1992, p.431) explain that students begin to "elaborate on their intuitive knowledge" at the third phase.

4. The fourth phase

1) Descriptions of the fourth phase

Descriptions of the fourth phase	**Source**
The Fourth phase (Free Orientation): By comparing symbols with one another, by searching for similarities and differences, the pupils orient themselves in the domain of symbols. ... During this phase there is not yet a real problem setting. It is rather an ordering of the manipulations that have to be carried out: I first have to do this, than that, in order to obtain the intended result. A new type of manipulation now develops: the drawing of figures of which elements are given or chosen. During free orientation the teacher appeals to the inventive ability of his pupils. (p.221) Empirical experiences can be expanded still more by joining figures that are already known, for example, by mirroring a triangle along one of its sides, or by rotating a triangle half a turn around the midpoint of one side. (p.221) Free orientation which is the willful activity to choose one's own actions as the object of study in order to explore the domain of abstract symbols. (p.223)	van Hiele-Geldof (1958/1984, p.221, 223) [translated from the original Dutch version into American English]
Free orientation: The field of investigation is for the most part known, but the student must still be able to find his way there rapidly. This is brought about by giving tasks which can be completed in different ways. All sorts of signposts are placed in the field of investigation: they show the path towards symbols.	P.M. van Hiele (1959/1984, p.247) [translated from the original French version into American English.]
Free orientation: (description is similar to that of P.M. van Hiele (1959/1984,	Wirszup (1976,

p.247)	p.83)
Free orientation: The students now encounter multistep tasks, or tasks that can be completed in different ways. They gain experience in finding their own way or resolving the tasks. By orientation themselves in the field of investigation, many of the relations between the objects of the study become explicit to the students.	Hoffer (1983. p.208)
In the fourth phase (Free orientation), by supplying materials with various possibilities of use and giving instructions to permit various performances.	van Hiele, P.M. (1986, p.177)
Free orientation, in which students refine knowledge and develop facility in the domain by engaging in problem solving.	Schoenfeld (1986, p.252)
Free orientation: The student learns, by doing more complex tasks, to find his/her own way in the network of relations (e.g., knowing properties of one kind of shape, investigates these properties for a new shape, such as kites). A variety of problems embodying the concept just learned are presented to the student for exploration (Free Orientation).	Fuys *et al.* (1988, p.7, 13)
Free orientation: Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. The teacher's role is to select appropriate materials and geometric problems (with multiple solution paths), to give instructions to permit various performances and to encourage students to reflect and elaborate on these problems and their solutions, and to introduce terms, concepts, and relevant problem-solving processes as needed.	Clements and Battista (1992, p.431)
Free orientation – The pupils learn by multi-step tasks and problems to find their own way in the network of relations of the subject. (p.2) The students learn their own way in a network of relations. (p.6) ● The teacher presents multi-step or open-ended problems that help students find their way in the system of relationships (p.6) ● The teacher utilizes problems that may have multiple solutions (p.6) ● The teacher urges students to think for themselves (p.6) ● The teacher encourages students to generate their own problems (p.6)	Hoffer (1994, p.2, 6)

Table 3.8 An overview of descriptions of the fourth phase.

2) Developing an understanding of the fourth phase

The fourth phase is consistently named as "free orientation" in the translations and interpretations of the phases. P.M. van Hiele (1986, p.180) explains the meaning of free orientation as follows:

> "We have a good example of free orientation when pupils are asked to write an essay about some aspect of a subject they have studied. Here they have to bring well-known relations into an often somewhat new connection. The intention of the pupil is to express the materials of the essay in the correct sequence; the object of teaching is to constitute a coherent network of relations."

Here, free orientation may link to the different ways that students "bring well-known relations into an often somewhat new connection".

According to van Hiele-Geldof (1958/1984, p.221), the function of this phase is to appeal to students' "inventive ability" through the empirical work. P.M. van Hiele (1959/1984) and other researchers also consider that at this phase, students need to "find their own way", "different ways" or to "orient themselves" to complete tasks (Hoffer, 1983; Fuys *et al.*, 1988, Clements and Battista, 1992).

Moreover, both van Hiele-Geldof (1958/1984) and P.M. van Hiele (1959/1984) claim that during this phase, students should have most of the knowledge to complete tasks. In terms of problem solving, Schoenfeld (1986, p.252) highlights that the fourth phase is for students to "refine knowledge". Yet Clements and Battista (1992, p.431) point to the necessity to "introduce terms, concepts, and relevant problem-solving processes" during this phase.

In terms of the nature of tasks at this phase, Hoffer (1983, p.208; 1994, p.6) suggest "multistep tasks", "open-ended problems", "problems that may have multiple solutions" or students "generate their (students) own problems". P.M. van Hiele (1986, p.177) discusses "materials with various possibilities of use and giving instructions to permit various performances". Fuys *et al.* (1988, p.7) consider "complex tasks". Clements and Battista (1992, p.431) mean those "problems whose solution requires the synthesis and utilization of those concepts and relations previously".

Noticeably, P.M. van Hiele (1986, p.180) points out that

> "In many cases, free orientation will have a less distinct character and will coincide with the bound orientation of a new subject."

If this statement indicates the bound nature of free orientation, it remains a question of how such bound nature of free orientation links to those open-ended tasks suggested by other researchers. A further question is what those open-ended tasks mean at this phase.

4. The fifth phase

1) Descriptions of the fifth phase

Descriptions of the fifth phase	**Source**
Integration: The manipulations are understood, there is insight into the operation. The concept congruent has acquired a geometric context. Congruent	van Hiele-Geldof (1958/1984, p.221,

triangles can be recognized and can be constructed. The concepts which one has formed of the figures play an important role. (p.222) Integration which can be recognized as being oriented in the domain, as being able to operate with the figures as a totality of properties.	223) [translated from the original Dutch version into American English]
Integration: the student has oriented himself, but he must still acquire an overview of all the methods which are at his disposal. Thus he tries to condense into one whole the domain that his thought has explored. At this point, the teacher can aid this work by furnishing global surveys. It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows.	P.M. van Hiele (1959/1984, p.247) [translated from the original French version into American English.]
Integration: the student has been oriented, but he must still acquire an overview of the methods which are at his disposal.	Wirszup (1976, p.83)
Integration: The students now review the methods at their disposal and form an overview. The objects and relations are unified and internalized into a new domain of thought.	Hoffer (1983. p.208)
In the fifth phase (Integration), by inviting the pupils to reflect on their actions, by having rules composed and memorized, and so on. In integration the intention is for a summary of the things that have been learned. The way in which the past tense of a verb may be formed is fixed; the multiplication tables are memorized; a rule giving the values of x for which a given quadratic function of x is positive is memorized. This is the phase in which intention coincides with the object of teaching.	van Hiele, P.M. (1986, p.177, 180)
Integration, in which the teacher helps to consolidate the students' knowledge by means of coherent summary presentations.	Schoenfeld (1986, p.252)
Integration: The student summarizes all that he/she has learned about the subject, then reflects on his/her actions and obtains an overview of the newly formed network of relations now available (e.g., properties of a figure are summarized). The student summarizes this in a family tree (Integration).	Fuys *et al.* (1988, p.7, 13)
Integration: Students build a summary of all they have learned about the objects of study, integrating their knowledge into a coherent network that can easily be described and applied. The language and conceptualizations of mathematics are used to describe this network. The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, increasing emphasis on the use of mathematical structures as a framework for consolidation. Finally, the consolidated ideas are summarized by embedding them in the structural organization of formal mathematics.	Clements and Battista (1992, p.431)
Integration – The pupils build an overview of all they have learned of the subject, of the newly formed network of relations and can move freely among the objects. (p.2) The students build an overview of the subject. (p.6) • The teacher asks questions, assessing student understanding of the topic (p.6) • The teacher helps students evaluate their knowledge and illuminate gaps they may have in fully understanding the subject matter (p.6) • The teacher designs question that apply and extend the accumulated knowledge of the subject (p.6)	Hoffer (1994, p.2, 6)

Table 3.9 An overview of descriptions of the fifth phase.

2) Developing an understanding of the fifth phase

The term "integration" is used to characterise the final phase. The function of this phase is reflected by the key words in the translations and interpretations. For instance, "overview", "global surveys", "no new knowledge", "reflect", "summary", "consolidate", "evaluate and assess". These key words are consistent with the meaning of this phase given by P.M. van Hiele (1986) as follows:

> "In integration the intention is for a summary of the things that have been learned. The way in which the past tense of a verb may be formed is fixed; the multiplication tables are memorized; a rule giving the values of x for which a given quadratic function of x is positive is memorized. This is the phase in which intention coincides with the object of teaching." (p.180)

Noticeably, however, while the translation by Wirszup (1976, p.83) clearly shows that "the student has been oriented", the translation by Fuys *et al.* (see P.M. van Hiele, 1959/1984, p.247) demonstrates that "the student has oriented himself" during this phase. Clements and Battista (1992, p.431) suggest that teacher plays an important role to "encourage students to reflect on and consolidate their geometric knowledge ...". Hoffer (1994, p.6) highlights that a teacher assesses students' understanding of the topic and helps them to evaluate their knowledge during this phase.

3.3 Developing an operational model

3.3.1 The levels of thinking

Based on the analysis of the broad descriptions listed in section 3.2.1, the levels of thinking accepted in this study are defined as follows:

> *Level 1*: Figures are judged by their appearance. The pupils do not see the parts of the figure, nor do they perceive the relationships among components of the figure and among the figures themselves.
>
> *Level 2*: The figures are bearers of their properties. The properties of the figures are established experimentally; they are described, but not yet formally defined. These properties which the pupil has established serve as a means of recognizing figures. However, these properties are still not connected with one another.
>
> *Level 3*: Properties are ordered. At this level the intrinsic meaning of deduction is not understood by the students.
>
> *Level 4*: Thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions.

Moreover, the operational model developed by Fuys *et al.* (1988), together with the cited examples, is to be used to analyse audio-taped episodes of students' responses in the observed lessons and their learning results in their homework and test papers.

3.3.2 The teaching phases

Based on the analysis of the broad descriptions listed in section 3.2.2, this study considers the phases as a teaching process which is defined as follows:

Information: The teacher introduces the objects to be studied and provides information to bring students to purposeful action and perception.

Guided orientation: The teacher guides students to discover and explore the principal connections of new materials.

Explicitation: The teacher ensures students correctly use mathematical language to elaborate the relations of knowledge.

Free orientation: The teacher ensures students explore the network of relations in their own way.

Integration: The teacher helps students to summarise the knowledge they have learned.

Moreover, the descriptions by Clements and Battista (1992) and Hoffer (1994) are to be used to analyse audio-taped episodes of teachers' actual instruction in the observed lessons.

3.4 Summary

In section 3.1, the elaborations of the original work of the van Hieles are carefully reviewed. In section 3.2, a broad source of the descriptions of the two key aspects of the work of the van Hieles, namely, levels and phases, are analysed. In section 3.3, an operational version of the van Hiele levels and phases is defined.

The main aim of this study is to develop a fundamental understanding of the didactical practice of effective teachers towards the development of students' thinking for writing proofs in geometry class. Chapter 5 presents a way of using the operational version of the van Hiele levels and phases to analyse audio-taped episodes of students' responses in the observed lessons and their learning results in their homework and test papers, and to analyse audio-taped episodes of teachers' actual instruction in the observed lessons.

CHAPTER 4. METHODOLOGY

4.1 Overview

The two main aims of this study are as follows:

- To explore and elucidate the complexity of individual teacher's didactical practice towards the development of students' thinking for writing proofs in geometry.
- To understand in what way the van Hiele model is a useful research tool to help analyse and interpret classroom teaching and learning of geometrical proof problem solving.

Given these aims, this chapter presents the methodological considerations of the study. The chapter is divided into three main parts. The first part (section 4.2) focuses on the methodological issues raised from van Hiele-based research. In particular, the section concerns with the key methods used for data collection, the general findings, the limitations and suggestions of the methods reflected in van Hiele-based studies. The second part (section 4.3) presents the pilot study. The last part (section 4.4) presents the design of the main field study in which the research strategy for the study is identified (section 4.4.1), followed by the units of analysis (section 4.4.2), data collection (section 4.4.3), data analysis (section 4.4.4), reliability, validity and generalisability of the study (section 4.4.5) and ethical issues (section 4.4.6).

4.2 Methodological considerations on van Hiele-based research

4.2.1 Key methods and findings of the assessment of students' thinking levels

Three methods have been mainly used in the assessment of students' thinking levels in the van Hiele-based research in the most recent 20 years:

1) Individual student interviews (e.g., Burger and Shaughnessy, 1986; Clements and Battista, 2001; Fuys *et al.*, 1988; Lehrer *et al.*, 1998).
2) Paper-and-pencil tests (e.g., Clements and Battista, 2001; Gutiérrez and Jaime, 1998; Senk, 1989; Whitman *et al.*, 1997).
3) Classroom observation (e.g. Clements and Battista, 2001; Hoffer, 1994; Whitman *et al.*, 1997).

In the first place, an overview of the sample and subject areas of these studies are summarised in table 4.1. Pertinent information from the studies listed in table 4.1 is then provided.

Methods	van Hiele-based research	Sample	Subject areas
Individual student interview	Burger and Shaughnessy (1986)	45 American students from early primary (Grades K-1), primary (Grades 2-3), middle (Grades 4-8), Algebra 1 (pregeometry), geometry, Algebra 2 (post-geometry), and college mathematics majors.	Geometric shapes (triangles, quadrilaterals): drawing shapes, identifying and defining shapes, sorting shapes, and engaging in both informal and formal reasoning about geometric shapes.
	Fuys *et al.* (1988)	16 sixth grade and 16 ninth grade American students.	Properties of Quadrilaterals, Angle Relationships for Polygons, and Area of Quadrilaterals.
	Lehrer *et al.* (1998)	37 American students from an elementary school (from first to third grade) at the beginning of the 3-year longitudinal study.	a) two- and three-dimensional Euclidean forms, including angle; b) the measurement of length and area; and c) related skills such as mental manipulation of imagines, drawing, and graphing.
Individual student interview and Paper-and-pencil tests	Clements and Battista (2001)	The 1987-88 field test involved 324 American students in Grades K-7 in Ohio; The 1988-89 field test involved 1,300 American students in Grades K-6 in Ohio and New York.	Logo geometry curriculum: Paths, Shapes, and Motions.
Paper-and-pencil tests	Gutiérrez and Jaime (1998)	309 Spanish students from grades 6th to 8th of Primary (aged 11 to 14) and first to fourth grades of Secondary (aged 14 to 18).	Knowledge and reasoning of polygons and other related concepts.
	Senk (1989)	241 American students: over 60% of these students were 10th graders, with most of the rest divided about equally between the 9th and 11th grades.	Proof-writing in geometry.
Paper-and-pencil tests and Classroom observation	Whitman *et al.* (1997)	Test data involved 649 American students from Grade 3, 6, 8 and 9-12, and 444 Japanese students from Grades 3, 7, 9 and 11. Classroom data included five class periods from one teacher with 40 Grade 8 Japanese students in one classroom in Japan, and 11 class periods from one teacher with 13 Grade 10 American students in one classroom in Hawaii.	Test content was the same content generally found in the curricula of both Japan and Hawaii. Classroom observation content in both Japanese and American classroom was congruence of triangles.

Classroom observation and Individual teacher interview	Hoffer (1994)	6 mathematics teachers and 6 science teachers at Grade 7 in three local American schools were involved. In total, twenty mathematics lessons and twenty science lessons were observed.	Curriculum guides in the school districts.

Table 4.1. An overview of sample and subject areas of some significant van Hiele-based research.

Burger and Shaughnessy (1986) used an audiotaped clinical interview approach to investigate the van Hiele levels in school geometry. Some data on the drawing tasks of their study suggest that students have different visual qualities such as the orientation of the figure on the page or the "skinniness" of the figure (Level 1 for these researchers (Level 0 in their study)), a focus on the components of the shapes (Level 2), or interrelating of shapes (Level 3). These researchers noted a number of imprecise visual qualities that some students used in describing and reasoning the shapes on the identifying and defining tasks and sorting tasks. Thus, they conclude that the students' behaviour on the tasks was consistent with the van Hieles' original general description of the levels, yet the discreteness of levels, particularly of analysis and abstraction, was not confirmed.

Fuys *et al.* (1988) investigated sixth and ninth graders' potential for progress in geometry in terms of the van Hiele levels. They found from the videotaped interviews that in both groups, some students were strictly Level 1 thinkers (Level 0 in their study, 'learning basic concepts and terms', (*ibid*, p.133)). Some were in the progress to Level 1 and into Level 2 ('using these concepts to describe shapes and to formulate properties for some classes of shapes', (*ibid*, p.133)). Some showed their thinking at Levels 1 and 2 at the start of the study, and then made progress toward Level 3 ('following and then summarising arguments', (*ibid*, p.134)). Moreover, some grade nine students were assessed at Level 2 as entry level, and then showed characteristics of level 3 thinking ('not only followed arguments but provided simple deductive explanations', (*ibid*, p.135)). In addition, the study identified a certain number of factors that affected students' performance on modules, such as language, visual perception, misconceptions and prior learning, etc.

To develop a contemporary and widespread portrait of students' emerging skills in reasoning about space, Lehrer *et al.* (1998) tested the adequacy of the van Hiele model as a description of the progression of students' thinking. These researchers developed nine triads of two-dimensional shapes to assess students' conceptions of planar figures. They found from the videotaped interviews that students' justifications involved many distinctions about form that seemed to involve several different types of mental operations, ranging from detection of features like fat or thin, to comparison to prototypical forms, to the action-based embodiment of pushing or pulling on one form to transform it into another. Thus, these distinctions appeared to defy description by a single "visual" level of development. Moreover, they found that many students also referred to properties of figures, such as the number of sides or the number of vertices or "corners", as well as to classes of figures, (e.g., squares and rectangles). Of the 37 students in the first wave (from the first grade to the second grade) of the study, only 4 (3 in the third grade) never referred to either conventional properties or classifications of shape in their justifications of similarity. Thus, these researchers claim that level mixture was the most typical pattern of response, and students' justifications often "jumped" across nonadjacent levels of the van Hiele hierarchy.

Senk (1989) investigated the cognitive factors for explaining why doing proofs is so difficult for many students. Specifically, the study addressed the relations between van Hiele levels, geometry proof-writing achievement, and achievement on objective tests of standard geometry content not involving proof. The study consisted of three main paper-and-pencil tests: the CDASSG Proof Test (the test consisted of two short-answer items and four full proofs for the student to write.), the van Hiele Geometry Test (the test contained five subtests, each with five multiple-choice items based on one van Hiele level), and Tests for Knowledge of Standard Content (the test contained 19 multiple- choice items covering geometry facts and concepts). The data of this study show that the higher the student's van Hiele level on entering, the greater the probability that the student mastered proof writing later in the year and the lesser the likelihood that he or she failed to learn to write proofs. Noticeably, this study only partially supported the van Hieles' prediction that students below Level 3 should not be able to do proofs at all other than by memorisation, and that students at Level 3 might be able to do short proofs based on empirically derived premises, but only students at Level 4 or 5 should be expected to write formal proofs consistently.

Gutiérrez and Jaime (1998) were aware of the strengths and weaknesses of both individual interview and paper-and-pencil tests in the assessment of students' van Hiele thinking levels. Thus, they sought to develop written items to be as close as possible to an item for a semi-structured clinical interview. They found that most students in the sample, except 6^{th} graders, had a high or complete acquisition of the first level and they were progressing in the acquisition of level 2. The chart of data from their study shows that the higher the course, the better the acquisition of the levels. Interestingly, data from their study demonstrated that many students in the sixth and second grades had not completed the acquisition of level 1, but they were progressing toward acquisition of level 2. Thus, they suggested that as the van Hiele levels of reasoning are integrated by several abilities, a student may progress in the mastering of some abilities but not of others, so the student cannot demonstrate complete acquisition of a level of reasoning, but he/she can show progress in some ability on the higher level.

To investigate how elementary school students learn geometric concepts and how Logo programming in turtle graphics might affect this learning, Clements and Battista (2001) used multiple assessment techniques, including pre-tests and post-tests that sampled overall curriculum objectives, unit tests, structured interview scripts, and classroom observations. Some data from the Pre-Post tests shown that *LG* students (using Logo Geometry curriculum), especially younger students, gained more than control students, not in their ability to identify figures that are triangles, but in their ability to correctly identify as nontriangles those figures that share spatial characteristics with triangles (e.g., the chevron, or deltoid). Further interview data indicated that after working with *LG*, the student interviewed made progress from attending to the visual aspects of the shapes to the properties of each shape. Some classroom observation data further support the finding that *LG* activities, including the computer environment and classroom dialogue, support students' development of higher levels of geometric thinking.

Whitman *et al.* (1997) examined how students in Hawaii compared to Japan at the start of Grades 4, 7, 9, and 11 are distributed with respect to the levels in the van Hiele theory. The percentage of correct responses on the geometry test of their study showed that the Japanese students were ahead of the Hawaiian students by about 2 years. They considered that part of the difference can be accounted for by the geometry curriculum and instruction in verbal communication in both places. Thus, they claimed that language and context could influence how a student responds to a test item.

4.2.2 Key methods and findings of the relationship of teaching phases with levels of thinking

Little research, other than that carried out by the van Hieles, has examined the phases directly. Nevertheless, two main methods found in the studies are as follows:

1) Classroom observation (e.g. Hoffer, 1994; Whitman *et al.*, 1997))
2) Individual teacher interview (e.g. Hoffer, 1994)

Hoffer (1994) used student thought levels and teaching phases to define a structure for observing, recording and evaluating mathematics classes. The study also attempted to apply the model to science classes in order to understand more about the barriers of transfer between mathematics and science. The analysis of the study showed that sixty-one percent of the modules were coded as Phase 2, Guided Orientation, in which the teachers' propensity for asking questions required one-word answers. The study revealed that a large amount of Level 1 behaviour is directly linked to a preponderance of phase 2 teaching. Moreover, Hoffer found that teachers often interrupt student progress toward higher levels in order to return to phase 2 instruction. There was a lack of contiguity in the instructional sequence. In addition, interview data revealed that most of teachers demonstrated a lack of clarity as to the meaning of deductive reasoning on the part of their students.

Whitman *et al.* (1997) reported that classroom observation data from the American class showed multiple phases and levels within a given module. Though the style of instruction was whole class instruction, the teacher used Phase 2 instruction. Students' thinking was identified at Level 3 due to doing proofs in geometry, something which was written in the textbook and expected in the curriculum content of high school geometry. However, their study encountered difficulties in coding the instruction in Japan in terms of the van Hiele phases and the levels of thinking (see section 2.5.3).

4.2.3 Strengths, limitations and some suggestions about the methods

The strengths and limitations of individual interviews in the assessment of students' thinking are summarised in table 4.2.

Strengths	Limitations
● Examine in depth students' thinking on geometrical concepts (Burger and Shaughnessy, 1986); ● Enable assessment of student's "potential" level in more dynamic form (Fuys *et al.*, 1988);	● A relatively small sample of students representing a very broad range of ages (Burger and Shaughnessy, 1986);

• Enable tracking of continuity and change in students' reasoning (Lehrer *et al.*, 1998); • Provide a more valid assessment of van Hiele levels (Battista, 2007).	• More time-consuming to administer (Battista, 2007).

Table 4.2. The strengths and limitations of individual student interview.

The strengths and limitations of paper-and-pencil tests in the assessment of students' thinking are summarised in table 4.3.

Strengths	Limitations
• Less time-consuming and more practical for many research projects and classroom teachers; • Enable dealing with large numbers of students (Battista, 2007; Gutiérrez and Jaime, 1998)	• Rarely give as complete a picture of student reasoning (Battista, 2007; Senk, 1989); • Multiple choice items have not yet been successful in assessing students' thinking (Battista, 2007; Crowley, 1989; Gutiérrez and Jaime, 1998; Fuys *et al.*, 1988)

Table 4.3. The strengths and limitations of paper-and-pencil tests.

This study is highly motivated by the spirit of van Hiele-Geldof (1957/1984) that

> "... more attention should be paid to receptive-structuring moments of thinking and that the phenomenon of learning should not be reduced to a cognitive process – even though differentiating may have already taken place." (p.65)

Thus, for the educational purpose, it is more essential to investigate the relationship between teachers' instruction with students' thinking development. In order to answer better the research questions in this study, the strengths of classroom observation and teacher interview addressed by Clements and Battista (2001) and Hoffer (1994) are largely considered:

- Classroom observation provides data for interpreting students' dynamic transition of geometric thinking (Clements and Battista, 2001).
- Classroom observation enables the description of teaching strategies and students' patterns of learning in the context of the educational setting (Hoffer, 1994).
- Teacher interviews can form a bridge between the curriculum and the teaching observed in the classroom. Interviewing can also help to determine the teaching style of the instructor, and link observed student behaviour to the type of educational environment (Hoffer, 1994).

4.3 Implications of the pilot study

A pilot study was conducted in December, 2005, after the research inquiry for the study had been generally formulated. The main aims of the pilot study were to refine research questions, to determine the data sampling and the time duration as well as the methods of data collection and analysis for the main study. The main results from the pilot study are presented in a paper (Ding and Jones, 2006). This section mainly discusses the implications for the main study, particularly from the methodological and theoretical point of view.

In the pilot study, the researcher visited a considerable number of schools in two districts of Shanghai and different types of lessons over a variety of topics of mathematics from Grade 6 (students age, 11-12 years old) to Grade 8 (students age, 13-14 years old) (see table 4.4), as she needed to determine which grade and which teachers' class she would focus on in the main study.

Grades	Schools	Teachers	Classrooms	Subjects	Textbooks	Type of lessons
Sixth	3	3	3	Number/ algebra/ geometry	II (1)	Open (3)
Seventh	1	3	3	Geometry	I (2)	Open
Eighth	1	3	3	Geometry	I	Open
	3	7	7	Geometry	I	Regular (4)

(1) During the period of the pilot study, the second term of curriculum and textbooks reforms had been taking place in the city. The researcher therefore had a chance to observe the use of the second term version of textbook (II). (2) The first version of textbook (I) was used in the classrooms observed. (3) An open lesson is a lesson open to other teachers for demonstrating teaching or for teaching competitions in the city. (4) A regular lesson is a daily lesson at school.

Table 4.4 The outline of schools and classrooms observed in the pilot study

The similarities and differences between open lessons and regular lessons are summarised in table 4.5.

	Open lesson	Regular lesson
Differences	Teacher might not teach her/his own classes in her/his school; Teacher might not know students in the class; Teacher's lesson plan has been discussed with colleagues; Teacher strictly follows the lesson plan; Computer and software are used; New instructional practises are shown.	Teacher teaches her/his own classes in the school; Teacher knows her/his students well in the class; Teacher devises her/his own lesson plan; Teacher adjusts the lesson plan according to students' responds; Computer and software are rarely used; Traditional classroom instruction is used;
Similarities	Same age students; Same textbook; Same time duration of lesson.	

Table 4.5. The similarities and differences of open and regular lessons

Such a wide lesson observation certainly helped the researcher to make a decision about the sampling for the main study. First, the pilot study implies that lesson observations focused on teachers' regular lessons could help the researcher not only to develop understanding of teachers' daily classroom instruction, but also to further analyse students' thinking responses to teachers' instruction in their daily learning environment. Secondly, lesson observations focused on teachers' use of the first version of textbooks could help the researcher not only to investigate classroom instruction on plane geometry (in the second term version of textbook of the city, geometry curriculum has been significantly reformed. The term "Plane geometry" was replaced by "Shape and Space"), but also to motivate the researcher to gain insight into the significance of the current curriculum and textbook reforms. Moreover, the aim for establishing research collaboration with school teachers for the main study required the researcher to concentrate on developing links with a small number of teachers and their classrooms. Indeed, the pilot study indicated that lesson observation, by focusing on a small number of teachers with the same classes over topics, could provide rich data of the dynamic nature of the classroom teaching and learning.

In the pilot study, one expert teacher (Lily, a pseudonym) was introduced to me by the school district mathematics researcher (he is a school mathematics expert, and was responsible for school mathematics teaching and learning at the lower secondary school level in the school district during the time the researcher visited). The observation of Lily's lesson indicated that she played a significant role in the classroom, and therefore it would be possible for the researcher to investigate, in great depth, the characteristics of her instruction. Moreover, the wider classroom observation broadened the researcher's horizon in classroom instruction, as even in one local school, different teachers might have totally different teaching styles and applied alternatively instructional approaches in their class. Therefore, the researcher decided to choose a senior experienced teacher who was Lily's colleague at the time and also had a very good reputation for her teaching in that school. The third teacher selected from the pilot study was a young experienced teacher in another school. In her lessons, students played a very active role in the classroom; therefore a study on her lessons provided essential information which might not be able to be observed in the expert teacher's and senior experienced teacher's lessons, particularly from the students' learning perspective. The details of the three teachers are given in section 4.4.2.

In terms of the time duration for the main study, the pilot study implied that three weeks field study would be sufficient to observe the teaching of a whole unit of lessons (new lessons and exercises lessons) of quadrilateral family (parallelogram, rectangle, rhombus and square) according to the school curriculum at Grade 8.

The pilot study indicated that it was difficult to keep track of every student and manage all the data and relationships presented in the whole classroom (each classroom size is of 40 students). It was therefore very necessary to focus on a small number of students in order to examine in detail these students' thinking behaviours over a unit of lessons. In the main study, the researcher required the three teachers to arrange for her to sit in a special position in the classroom in which three types of students' (good, average and weak students) interactions could be easily recorded by an audio recorder. More details of students involved in the study are provided in section 4.4.2. In order to capture the interactions between teachers and students in classrooms, the pilot study suggested that the researcher brought an audio recorder to record fully these selected students' interactions with their teacher in the main study. Moreover, a camera was suggested to record teachers' work on the blackboard, which was not able to be fully copied using field notes alone.

In terms of teachers' interviews, interview questions about teachers' beliefs and ideas in the pilot study were fixed. The teachers' responses were helpful in understanding what teachers were really concerned about in their classroom teaching. However, the pilot study suggested that teachers' interviews be unstructured to provide data to extend and modify observation data in the main study.

In respect of the theoretical point of view, the implications are as follows.

In the pilot study, the van Hiele model of thinking levels and teaching phases was used to direct the data collection and analysis (for details of the pilot study result, see Ding and Jones, 2006). However, it was found that there was a certain pattern of mathematics lessons (Introduction/review – New content – Exercises – Summary – Homework) which was not compatible to the van Hiele five phases. Such a finding suggested that the main study focused on the teaching process of new definitions/theorems and individual proof problems rather than the lesson structure. Moreover, the analysis of observed data suggested that the study carefully analysed both the original work of the van Hieles, and the significant work of other van Hiele researchers, in order to develop a comprehensive understanding of the use of the model as a research tool for this study. In particular, the analysis of data on students' learning results in homework and test paper indicated that there was a need to formulate an operational version of the van Hiele model before the analysis of the main data. In the pilot study, teaching tasks such as proof problems and the use of a visual approach were focused on. In order to analyse and interpret the complexity of teachers' instruction, the pilot study suggested that the researcher be open to other instructional elements possibly emerging from the main data.

4.4 The main study

4.4.1 Identifying the research strategy for the study

Given the overall aim of this study to develop fundamental understanding of the complexity of teachers' didactical practice towards the development of students' thinking for writing proofs, this study applies mixed methods, namely, qualitative case study approaches combined with quantitative analysis methods.

According to Johnson and Onwuegbuzie (2004, p.17), mixed methods research is formally defined as "the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study". These researchers point out that the goal of mixed methods research is not to replace either qualitative approach or quantitative approach, but rather to draw from the strengths and minimise the weaknesses of both in single research study.

In the first place, Yin (2003, pp.13-14) highlights the case study strategy as follows:

1. A case study is an empirical inquiry that
- investigates a contemporary phenomenon within its real-life context, especially when
- the boundaries between phenomenon and context are not clearly evident.

2. The case study inquiry
- copes with the technically distinctive situation in which there will be many more variables of interest than data points, and as one result
- relies on multiple sources of evidence, with data needing to converge in a triangulating fashion, and as another result
- benefits from the prior development of theoretical propositions to guide data collection and analysis.

These features of case study well address the needs of this study. First of all, the case studies help the researcher focus on exploring and elucidating the pedagogical phenomenon of geometric proof problem solving through teachers' actual didactical practice in their daily classroom. Secondly, the researcher considered the case study as "a comprehensive research strategy" ("not either a data collection tactic or merely a design feature alone" (Yin, 2003, p.14)). In this study, the cases to be studied are three individual teachers. Information about each individual teacher has been collected through classroom observation, teacher interview, students' responses during the teaching process, and students' learning outcomes on their homework and test paper. Thus, to study in depth the teachers' classroom instruction, each single case included not only what the teacher actually taught in the classroom, but also the teachers' didactical view of their lessons and students' actual learning responses and outcomes during and after the observed lessons. Finally, this study developed the van Hiele theoretical framework for guiding the design of the study, and for collecting and analysing the relevant data.

Moreover, quantitative methods were considered for a diversity of types of data to explore in this study. As highlighted by Mason (1996, p.25),

> "because you want to use different methods or sources to corroborate each other so that you are using some form of methodological triangulation".

As discussed above, to obtain full answers for this study, a larger number of students' learning outcomes in their homework and test papers were collected in this study. Silverman (2005, p.128) suggests that "quantitative measures may sometimes be used to infer from one case to a larger population ...". Hammersley (1992) argues that the comparisons with a larger sample may allow the researchers to establish some sense of the representativeness of their single case. Indeed, the researcher considered the use of quantitative methods to complement the weaknesses of the qualitative case study approach in data collection and analysis. For instance, data collection using a small-scale survey in students' learning outcomes in these three teachers' whole class and in other same grade classes was relatively easy and quick. The analysis of students' responses in class could be compared with that of a larger number of students in homework and test papers. Numerical data, such as time duration and frequency, is more effective to show the significance of the instructional strategies and approaches used by teachers in the observed lessons. Data collected by quantitative methods (for instance, students' learning results in the test paper) is fairly independent of the researcher.

In general, considering the complexity of the research questions of teachers' didactical practices, this study combined qualitative case study of classroom observation with teachers' interview and quantitative analysis of students' learning outcomes in homework and test papers. The researcher believes that triangulation improves the reliability of a single method, but with caution that "we cannot simply aggregate data in order to arrive at an overall 'truth'" (Silverman, 2005, p.122).

4.4.2 The units of analysis

1) The selection of individual teachers for the study

The main focus of the study is a deep and fine analysis of the individual teacher's instruction of geometrical proof problem solving, together with an analysis of students' responses on the instructional strategies and approaches. This focus determined the unit of analysis for the study. Three teachers and their classes within two schools were selected for the data collection. The selection of teachers was based on a number of design strategies for case study discussed by Bogdan and Biklen (1998, pp.60-2) and practical criteria in the field, which are shown as follows. In general, the researcher wanted the lessons (geometrical proof problem solving in the quadrilateral family) teachers presented were in their daily classroom environment and followed the school curriculum, so that she would be able to observe teachers' instruction and students' responses in authentic classroom settings. More importantly, the researcher would be able to observe how teachers implemented certain amounts of content embedded in the school curriculum through a unit of lessons. Some of the main considerations of the selection of teachers were discussed in the pilot study (see section 4.3). Here, some other main criteria are given:

- The teachers needed to have experience in deductive geometry teaching at the lower secondary school level (not the first time to use the first term version of textbook to teach deductive geometry);
- The reputation the teachers needed to have in effective teaching in school mathematics in their school or the school district. In particular, the expert teacher should have an extremely good reputation for her/his teaching in the school district;

- The teachers needed to teach proof problems at the beginning of a deductive geometry course in the curriculum;
- Their willingness to be included in the study.

The criteria listed above meant that teachers taught at Grade 8 needed to be chosen (see table 4.6), as students start to learn deductive geometry at this grade according to the mathematics curriculum in Shanghai.

Schools (pseudonyms (1))	**Teachers (pseudonyms (2))**	**Types of teachers**	**Teaching experience**
Century school (South-west of the city)	Lily	Expert teacher	20 years
	Spring	Senior experienced teacher	20 years
Garden school (North-east of the city)	Nana	New experienced teacher	7 years

(1) Name was used according to the history or location of the schools. (2) Name was used according to the translation of one single Chinese word of the teachers' first name (usually one or two words of the first name in Chinese).

Table 4.6. The teachers involved in the research

The researcher had worked in the Garden school for nine years both as a mathematics teacher and as a head of the mathematics department of the school, before she came to England to undertake her Masters study and PhD study. This meant that at the Garden school, the researcher was able to gain access to conduct the study. While the teacher studied (Nana) was a respected and trusted colleague of the researcher, nevertheless, the researcher clearly demonstrated to the teacher the nature of the study, and what the researcher wished to learn and why.

Teachers in Century school (Lily and Spring) were introduced to the researcher by one of the researcher's previous colleagues. As a consequence, the researcher introduced herself to the teachers during the first day when she visited the school for the pilot study and kept in touch with them by telephone prior to the main study. In China, 'Guanxi' is essential for developing a network relationship. The researcher was very satisfied with this relationship for her study. Similarly, the researcher clearly demonstrated to the teachers what the nature of the study was, and what the researcher wished to learn and why.

In the Chinese culture, it is important to establish relationships in which people trust and support each other. The selected teachers understood that the researcher wished very much to understand in depth the form of daily classroom teaching in deductive geometry in their classrooms. The researcher emphasised to the teachers the natural setting for her study. They, therefore, did not ask the researcher what she particularly wished to see. They supported the researcher very much. For instance, when the researcher needed to talk with them, they were always available. When the researcher wanted to have some information about students' learning outcomes, they permitted her to further see students' homework and test papers.

The main study concentrated on observing the teachers' lessons in the quadrilateral family (see Appendix D) which usually takes place at the second term of Grade 8 in Shanghai (there are two terms each school year in Shanghai.). According to the school curriculum, the lessons are expected to be implemented for no more than three weeks long. The researcher was therefore suggested by these three teachers to observe these lessons between 8 and 26 May, 2006. As all schools followed the City standard curriculum, the researcher was able to observe these teachers' lessons on the same topics by using the same textbooks during that period. However, it became difficult for the researcher to travel across the city every day to simultaneously observe these teachers' lessons, given the different teachers' lesson timetables (the teachers and their classes focused on in the main study) in the two schools (see Table 4.7 and 4.8) (in the lower secondary school in Shanghai, a mathematics teacher is usually responsible for the daily teaching of two classes at a certain grade).

Century school		Monday	Tuesday	Wednesday	Thursday	Friday
Morning (am)	8:40 – 9:20	★		●		●
	9:30 – 10:10		●			
	10:25 – 11:05		●			★
	11:15 – 11:55			★	★	
Afternoon (pm)	1:15 – 1:55	●				
	2:10 – 2:50		★		●	
	3:00 – 3:40	★				

Table 4.7 Century school curriculum and lesson timetable of Lily in Class 3 (●) and Spring in Class 5 (★)

Garden school		Monday	Tuesday	Wednesday	Thursday	Friday
Morning (am)	8:05 – 8:50	◆				◆
	9:00 – 9:45		◆	◆		
	10:00 – 10:45					
	11:00 – 11:45					
Afternoon (pm)	1:00 – 1:45					
	2:00 – 2:45				◆	
	2:55 – 3:40					

Table 4.8 Garden school curriculum and Nana's lesson timetable in Class 1 (◆)

The researcher decided to focus on Century school mainly due to three considerations: 1) the study aimed to understand and interpret an expert teachers' classroom instruction, and Lily was recommended as an expert teacher who has an extremely good reputation for her mathematics teaching in her school district; 2) the researcher would be easily able to concentrate on both Lily's and Spring's lessons due to their different lesson timetables in the same school; 3) Century school has a very good reputation in the city for students' successful learning attainment in annual city standard examinations. Nevertheless, when Lily and Spring arranged exercises lessons and they had one day off for a school trip, the researcher went to Garden school to observe Nana's lessons. Therefore, Nana's lessons were not the main part of the study, yet provided very essential data for the comparison with the main study in Century school and eventually for the generalisation of the study.

As for a representation of the population, the opportunity from these three cases appears small, but the researcher focused more closely on cases that seemed to offer the opportunity to learn in great depth (Stake, 2006).

2) The selection of examples from the observed lessons

Overall, the content of the lessons observed in the three teachers' classes during the main field study are outlined in Appendix A-C. In general, Lily's second and third lessons were selected for the analysis of students' levels of thinking and the characteristics of teaching phases. Two episodes of teaching proof problem solving from Spring's second lesson, and one episode of teaching proof problem solving from Nana's first lesson were also selected for the analysis of students' levels of thinking and the characteristics of teaching phases. These examples were selected from each single case to demonstrate the complexity of individual teacher's didactical practices.

Stake (2006) stated that

> "For qualitative fieldwork, we will usually draw a purposive sample of cases, a sample tailored to our study; this will build in variety and create opportunities for intensive study." (p.24)
> "For multicase studies, selection by sampling of attributes should not be the highest priority. Balance and variety are important; relevance to the quintain [the word "quintain" used by Stake to describe an object or phenomenon or condition for the cases to be studied] and opportunity to learn are usually of greater importance." (pp.25-26).

Before the selection of these examples for analysing the main data (an overview of the selected examples from the observed lessons see tables 1-3 in Appendix E), the researcher had tried very hard to link the van Hiele model to teachers' practice by initially analysing the expert teacher's (Lily) two lessons of teaching each individual proof problem solving (for details see Ding and Jones, 2007). Next, to develop a comprehensive understanding of the van Hiele model, the original work of the van Hieles was carefully analysed, together with other significant van Hiele-based work. As a result, the researcher developed the van Hiele theoretical framework (see section 3.3) for thoroughly analysing all of the three teachers' observed lessons. However, findings from the analysis of either the expert teacher's lessons or the other two teachers' lessons indicated that it was likely that more than one interpretation was available to analyse the teachers' instruction. Consequently, one main analytic strategy was to select examples from the individual cases to demonstrate the different interpretations of the teachers' actual instruction on similar proof problems (involving the same geometric knowledge) by the van Hiele model. Moreover, to further explain the theoretical ideas of teachers' such didactical practice on geometrical proof problem solving, examples were selected to present the instructional strategy and approach which dominated each teacher's instruction.

Consequently, three main criteria were considered for selecting these examples:

- All examples are relevant to the main research questions;
- The examples represent individual teacher's different teaching strategies and approaches which dominated in their own class, and therefore provide diversity across the three teachers' lessons;
- The examples provide good opportunities to learn about the complexity of teaching and learning of geometrical proof problem solving.

3) The individual students involved in the study

This section describes the details of these three teachers' classrooms and students involved in this study. In general, students' learning attainments in the three classes were above average. This information was gained mainly based on these students' learning outcomes in the local standard exams during the school year. In general, students involved in the study are firstly summarised in table 4.9.

Schools	Teachers	Number of students in class	Students involved in this study (pseudonyms (1))	Students' age (Grade 8)
Century school	Lily	39	Boys Liuliu and Linlin, girls Beibei and Youyou, plus other students	13-14 years old
	Spring	41	Boy Jiajia, girl Meimei, boy Minmin, plus other students	
Garden school	Nana	38	Boy Fanfan, girl Qinqin, boy Jianjian, plus other students	

(1) Students' name was labelled by the common students' name in Chinese.

Table 4.9. The classrooms and students involved in the research

As informed by the pilot study, the researcher was arranged in a special position in the classroom in order to observe the behaviours of three different types of students during the main study. In Lily's classroom (see figure 4.1), the researcher was arranged to sit in a place to focus on three students (all names are pseudonyms) - boy Wenwen (good student), boy Liuliu (above average), girl Sisi (weak student). However, Wenwen and Sisi were not active (generally very quiet) in all the lessons the researcher observed. Fortunately, Liuliu was very active and talked very often to his neighbour (girl Beibei, above average) during the time of the observed lessons. Moreover, during the three week long lesson observation in the main study, boy Linlin (above average) and girl Youyou (above average) were very active in the classroom, and their voices were clearly recorded on the audio recorder. Therefore, the researcher decided to take a look in depth at these four students' responses to Lily's instruction over the unit of lessons for the main data analysis. Other students' responses were also taken into account in the analysis, particularly when Lily asked them to present their thoughts in the class or when the four focused students did not show any response to the teachers' instruction.

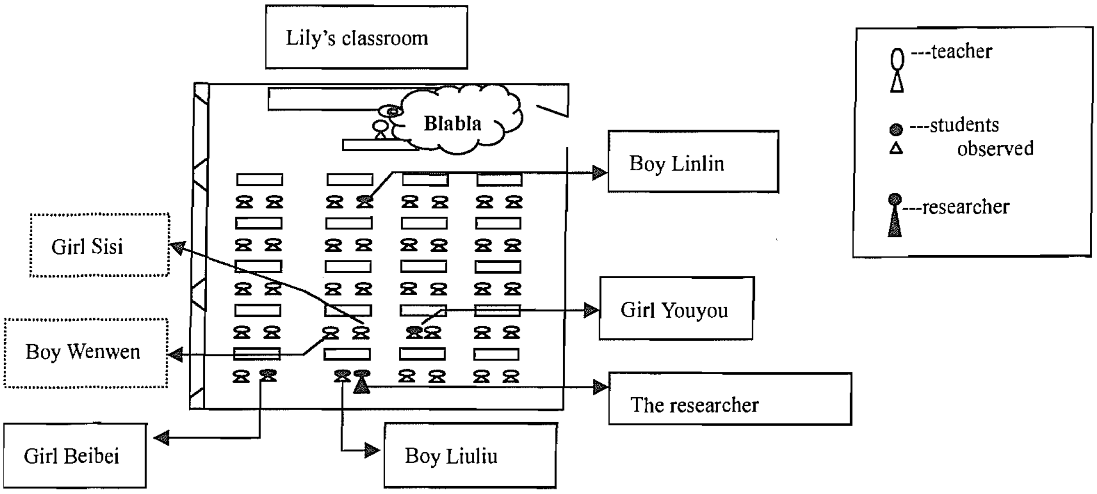


Figure 4.1 Students involved in Lily's classroom

In Spring's classroom (see figure 4.2), the researcher was arranged to sit in a place to focus on three students (all names are pseudonyms) - boy Jiajia (good student), girl Meimei (above average), boy Minmin (weak student). However, in Spring's observed lessons, students answered the teachers' questions according to their school ID number. Some times during the class observation, the teacher asked student's ID number, but most times, students just automatically stood up and answered the questions. The researcher guessed that students generally knew who was the next to answer questions, so the teacher did not need to frequently mention the ID number at all in the class. It was therefore difficult to identify individual students' response with the large size. Moreover, as Jiajia, Meimei, Minmin only responded loudly to the teachers' instruction when it was on their turn, the audio recorder recorded very limited voices from these selected students. The researcher decided to use the general interaction between the teacher and students in the class when analysing students' responses of Spring's lessons.

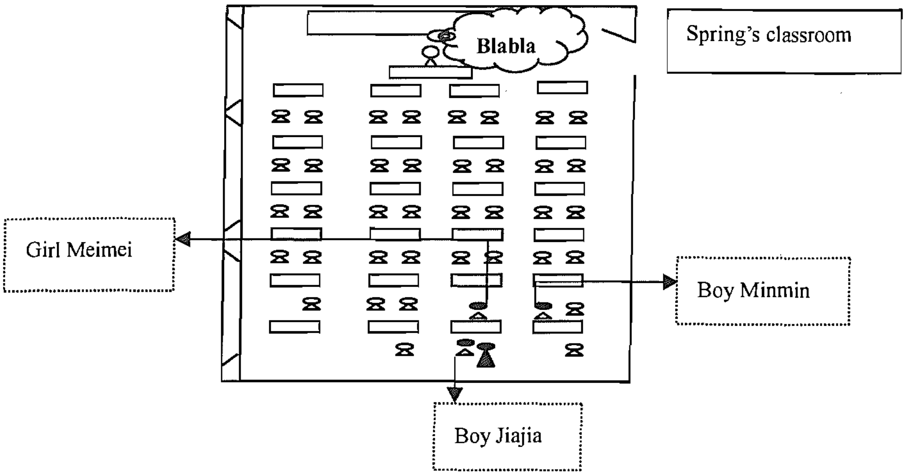


Figure 4.2. Students involved in Spring's classroom

In Nana's classroom (see figure 4.3), the researcher was arranged to sit in a place to focus on three students (all names are pseudonyms) - boy Fanfan (good student), girl Qinqin (above average), boy Jianjian (weak student). In general, the three students were well disciplined students. This meant that they did not talk but listen, while the teacher encouraged others to stand up to present their thoughts in the class. Consequently, a number of students' discussion, rather than that of the focused students, was clearly recorded in the audio-recorder. The researcher decided to generally use the students' voices who presented their thoughts in the class to analyse students' responses of Nana's lessons.

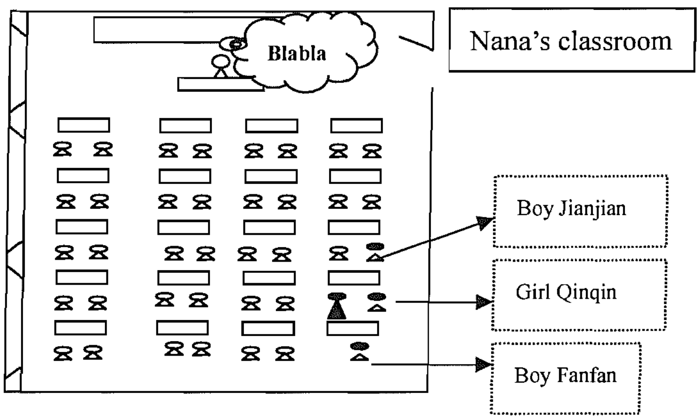


Figure 4.3. Students involved in Nana's classroom

4.4.3 Data collection

1. Classroom Observation

As discussed in section 4.2.2, classroom observation is the main method of data collection in this study. The main research aims required the analysis of the teachers' teaching process, together with students' levels of thinking, therefore the researcher needed to have access to both the teaching process and students' learning progress. The teaching/learning process is constituted by both external practices and internal practices. The external practices are what teachers and their students did and talked (e.g. the teacher may draw a figure on the blackboard, present a problem, and ask students questions, etc. Students may observe a figure presented by the teacher on the blackboard, answer the teachers' questions, discuss with others, do exercises, etc.), and can be directly observed in the classroom. The internal practices are the nature of teachers' teaching processes and students' thinking progress over a unit of lessons, and can only be inferred on the basis of the observations. In terms of methods, choices needed to be made as far as the means of observations of these practices.

1) Audio-recording

The audio-recorder was considered to be the most suitable tool to record the classroom conversations by naturally observing lessons in teachers' daily class. The three teachers generally did not mind recording their voices. They agreed that an audio-recorder would not trouble their natural instruction and would not disturb students' concentration in learning. The researcher could use an audio recorder to gain details of the teachers' interactions with the whole class and responses from a small number of students on which the researcher focused.

The researcher came to the class and put the audio recorder on the student's study desk where she arranged to take a seat. No student noticed the mini-sized audio recorder which provided very high quality of voice record. The researcher was pleased with the clear classroom conversation she wanted to gain, and felt positive about using an audio recorder to record each lesson she observed.

2) Field notes

While the researcher used the audio-recorder to mainly gain classroom conversation data, she largely concentrated on writing down what she observed of the lessons which could not be recorded by the audio recorder, in particular the instructional content (proof problems). In the field notes, one schedule was designed and was consistently used for each observed lesson. The schedule is demonstrated in table 4.10.

School title:________ Date of lesson observation:________ Grade:____ Class size:____			
Teacher name:________ Textbook:______			
Title of the lesson:________ Duration of lesson:______			
Time duration	Teaching content	Students' responses	Note

Table 4.10. The schedule for classroom observation

To make sure that the researcher would be able to recognise the students' voices, the researcher wrote down the time and which student she focused on was talking. Moreover, the researcher wrote down on the field notes of the teachers' main instructional action such as drawing a figure, and details of the figures presented over the time such as which colour was used by the teacher for highlighting which part of a figure at a moment, etc. In addition, the researcher was able to pay attention to the whole class and got to notice some other active students in the class who eventually replaced the students the researcher planned to observe yet failed to gain any information she needed to learn from them (see details in section 4.4.2. *3)*).

3) Camera recording

The three teachers agreed that the researcher could take photos of the observed lessons. A digital camera was therefore used to record information the researcher needed yet was not able to be recorded either by audio-recorder or field notes, such as the teachers' and students' work and practices on the blackboard.

2. Teachers' Interviews

Given the aim of this study to interpret in great depth teachers' actual classroom instruction, there was a need not only to learn what teachers did and said, but also why they did and said so. The researcher therefore considered interview techniques as a useful method for providing this kind of data (see the consideration of the strengths of teacher interview in section 4.2.2). Apart from teachers' interview in the pilot study, the researcher conducted another two times of teachers' interview with these three teachers. Teachers' interviews were firstly conducted during the main field study time (May, 2006). The researcher interviewed Lily once, Spring and Nana each twice after their lessons. The questions were unstructured and mainly of some observed yet not focused students' information, of the teachers' themselves view of the planned and implemented lessons, and of the consideration of the instructional approaches used in the lessons, etc. Each interview took

10 to 30 minutes long. The researcher was also able to attend once the Grade 8 teachers' meeting in Century school to gain information from Lily, Spring and their other three colleagues about the feedback from the implemented lessons (the same topics the researcher focused on in this study) and their discussion of the following week's teaching plan (the second week of the field study).

Teachers' interviews were secondly conducted in December, 2006, after the researcher had an initial analysis of the main data of the three teachers' lessons. The initial study results and the researcher's thoughts about these results were further discussed with these teachers by interview, to have their agreement on the way the researcher interpreted their lessons. The interview was conducted once with each of these three teachers for one hour length.

3. Small-scale Survey of students' learning results

Data about students' learning outcomes in their homework and test papers provided information that the researcher was not able to catch in the class observation. Students' learning results were collected in two different periods. The first period was in the period of the main field study. The researcher gained permission from the three teachers to take a look at their students' homework after the lessons and the students' learning outcomes in the unit test paper. As students' homework and test papers could be read only after the teachers marked them and before they were returned to students during a school day, the researcher had to ask the teachers to provide some information for guiding her to examine these students' learning outcomes. For instance, in the proof problems the teachers found that the mistakes were made by a certain number of students. The researcher also considered focusing on basic proof problems (particularly simple steps proof problems) to mainly see students' understanding of new theorems and definitions, and to understand their basic capability to write formal proofs. The researcher then wrote down the main points she wanted to gain and kept these recorded in a file, such as the methods different students used in solving a same proof problem, the errors highlighted by their teachers, etc. Similarly, the teachers provided useful guidance to the researcher to examine a number of test items in which students' different thoughts were shown. The researcher mainly took a look at students' learning outcomes in test papers in Spring's class, as students' test learning results in Spring's class were available to the researcher during the main field study, and as students' general learning attainments in Spring's class and in Lily's class were constantly similar in a range of unit tests and school standard examinations according to the statement from both Lily and Spring. Nana's class was also available to the researcher to see students' test attainment, though the test items used in the two schools were different. Nevertheless, for the researcher's study purpose, students' learning results in a number of individual test items were more interesting to learn than their general attainment in the test. The researcher was happy with the learning results data she collected, and gained general understanding of a larger number of students' actual learning in the class she observed.

The second time to collect students' learning results in proof writing was conducted in December, 2006. After a personal discussion with Professor Zhang Dianzhou about students' proof writing ability at lower secondary school level, it was suggested to the researcher to use a small-scale survey to gain data of students' proof writing ability of the well-known theorem of the media of triangles. In total, 328 Grade 9 students from ten classes in Garden school and 41 Grade 9 students in Spring's class (the class the researcher focused on in the main study) were involved in this survey. The researcher clearly described to Nana (head teacher of Grade 9 mathematics teachers working group in Garden school at that time) and Spring the proof problem and the time (5-10 minutes) to finish it during an exercise lesson. However, students in seven classes in Garden school were actually allowed to finish the proof problem as homework (due to the limited time to arrange the survey in school time). Lily left Century school due to a work promotion in another school at that time, therefore, her class the researcher focused on was not able to attend to this survey. Nevertheless, the researcher gained a general sense of a larger population of students in writing formal proof and had the copy of all these students' worksheets.

4. Data collected

The raw data collected through the previously described research methods, appropriately transformed, constituted the main corpus of data to be analysed then. It consisted of field notes, transcripts from the audio recording of observed lessons and teachers' interview, photos, students' learning results record files and worksheets, as shown in table 4.11.

Methods of data collection	Data collected
Audio-recording	Transcripts of audio-tapes of lessons
Taking field notes	Field notes
Digital camera	Photos of lessons
Teachers' interview	Transcripts of audio-tapes of teachers' interview
Collections of students' homework and test papers	Files of students' learning results
Small-scare survey	Students' worksheets of writing a formal proof.

Table 4.11. Collecting data

In particular, the main data collected from the three teachers in the two schools are shown in table 4.12 and 4.13.

Teacher	**Lily**	**Spring**
Number of observed lessons (in total)	12	8
Transcripts of lessons (in Chinese)	12 lessons	8 lessons
Number of photos (in total)	40	38
Transcripts of teachers' interview	10 minutes long (9 May, 2006) 30 minutes long (27 December, 2006)	15 minutes long (10 May, 2006) 15 minutes long (15 May, 2006) 20 minutes long (27 December, 2006)
Files of students' homework	37 students' homework on one proof problem (10 May, 2006) 35 students' homework on one proof problem (17 May, 2006) 38 students' homework on two proof problems (24 May, 2006)	40 students' homework on one proof problem (10 May, 2006) 41 students' homework on one proof problem (19 May, 2006)
Files of students' test papers	-	37 students' learning results on six test items (26 May, 2006)
Students' worksheets	-	40 students' worksheets of one proof problem (27 December, 2006)
Field notes and transcripts of teachers' interview at the teacher meeting	5 teachers' regular meeting, 45 minutes long, field notes (9 May, 2006) 5 teachers' regular meeting, 15 minutes long audio recording (15 May, 2006)	

Table 4.12. Data collected in Lily's and Spring's classroom in Century school

Teacher	**Lily**
Number of observed lessons (in total)	3
Transcripts of lessons (in Chinese)	3 lessons
Number of photos (in total)	44
Transcripts of teachers' interview	10 minutes long (12 May, 2006) 10 minutes long (18 May, 2006)
Files of students' homework	29 students' homework on one proof problem (18 May, 2006)
Files of students' test papers	37 students' learning results on 11 test items (25 May, 2006)
Students' worksheets	41 students' worksheets of one proof problem (29 December, 2006)
290 students' worksheets from nine other classes of one proof problem (29 December, 2006)	

Table 4.13. Data collected in Nana's classroom in Garden school

5. The role of the researcher

The spectrum of possible roles for observers to play has been highlighed by Bogdan and Biklen (1998, p.81): at one extreme is the complete observer, and at the other end is complete involvement at the site. Given the aim of the classroom observation is to mainly collect data of teachers' instruction and a small group of students' responses, the researcher's role in the classes observed was between the two extremes, yet closer to the complete observer.

The researcher observed these three teachers' class in the pilot study once. In the schools in Shanghai, it is very usual for teachers to observe lesson of one another for improving teaching. The researcher was therefore regarded by students in the three teachers' classes as their teacher's colleague in the pilot study. When the researcher came into the class at the first day of the main study (after six months of the pilot study), students in the three classes were friendly and recognised her. They said to her, "You come again." The researcher told some students around her that she needed to learn teaching from their teacher. When students saw the researcher almost every day in their mathematics lesson during the three weeks long main study, they regarded her as a new teacher who came to learn teaching from their teacher. Generally, the longer the researcher stayed with them in their class, the more relaxed students were.

The researcher did not clearly tell students of her research in the observed class, as she thought (agreed by the three teachers) that students would not behave in a natural manner if they knew that they were actually being focused on by the researcher. Particularly, if students understood that there was an audio recorder to record their voice, they might choose totally not to talk at all in the class. Moreover, these students might pay more attention to the researcher's responses to their behaviours instead of the lessons they should be concentrated on.

The class size the researcher studied was around 40 students, and the researcher was sat in the back of each classroom (see figure 4.1-3). The researcher was therefore not likely to change students' behaviour in the whole class by her presence, but was likely to be unobtrusive. Generally, the researcher tried very hard to establish classmate relationships with students around her. She sometimes repeated teachers' questions if students around her did not hear clearly. Sometimes, students showed the researcher the pages of textbook when they did mathematical exercises. However, the researcher tried not to answer any questions students asked about solving the problems, but suggested to them either to concentrate on their teacher's instruction or discuss with their neighbours. When students discussed solving the proof problem with their neighbours, the researcher joined the focused group. Basically, the researcher listened to students' discussions, but sometimes when she could not understand students' thoughts she asked them why. In addition, the researcher received the permission from the teachers to look at what students practiced during classroom exercises. Generally, the researcher had to stay in the back of the classroom, in order to observe and record the whole teaching process and students' responses. The longer the time the researcher stayed with students in their classroom, the more likely they came to regard the researcher as one of their teachers.

4.4.4 Data Analysis

To elucidate the teachers' different instructional strategies and approaches, together with students' responses, this study applies a cross-case analysis strategy. In total, three teachers were selected for the study (details for the selection see section 4.2.2.1). Each teacher was the subject of an individual case study, but the study as a whole covered different teachers' instructional strategies and approaches on the similar proof problem solving, and students' various responses and learning outcomes which were related to their teacher's instructions.

Next, this study used two specific techniques, the "pattern matching and explanation building", which are suggested by Yin (2003, p.109) for analysing case studies. The analysis process of this study started from developing the van Hiele theoretical framework. To understand the usefulness of the van Hiele theoretical framework to analyse and interpret teachers' instruction towards the development of students' geometrical thinking, the theoretical framework was used to analyse three individual teacher's teaching processes on a set of proof problem solving. The analytic technique "pattern matching" (*ibid*, p.116) was used to compare the empirically based pattern (the results from the analysis of the data) with the predicted one (the van Hiele theoretical framework). Moreover, to interpret in depth the pedagogical phenomenon of teachers' actual instruction on geometrical proof problem solving, it is not enough to only identify whether teachers' practical patterns matched the theoretical patterns or not. The study concentrated on building up an explanation of how and why the outcome occurred across the individual cases.

The cases selected to be part of the corpus of data for the analysis are outlined in table 4.14.

Case study	Data collected	Teaching/learning topics	Students	Setting
Case 1. Lily	Proof problems in two lessons (L2&3)	Parallelogram	Liuliu, Beibei, Youyou, Linlin and other students	Lily's class
	Teacher interview	Proof in geometry		
	Students' learning results during homework	Rectangle	All students of Lily's observed class	
Case 2. Spring	Proof problems S-Ex4 and S-Ex6 in one lesson (S2)	Parallelogram	Some students	Spring's class
	Teacher interview	Proof in geometry		
	Students' learning results during homework	Rectangle	All students of Spring's observed class	
Case 3. Nana	Proof problems N-Ex1 in one lesson (N1)	Parallelogram	Some students	Nana's class
	Teacher interview	Proof in geometry		
	Students' learning results in test paper	One test item of rectangle	All students of Nana's observed class	

Table 4.14 The corpus of data for the analysis

1) Data reduction

> "Data reduction refers to the process of selecting, focusing, simplifying, abstracting, and transforming the data that appear in field-notes or transcriptions" (Miles & Huberman, 1994, p.10).

The first step of data analysis was actually started immediately during the main field study period. The researcher transcribed from listening to the audio recorder, wrote down on paper mainly the classroom interactions[3] between the teacher and students, or between individual students, the details of how the figure was drawn or presented by the teacher, and details of students' responses to the teachers' instruction in Chinese. There were mainly two reasons for immediately transcribing the observed lessons. First, the researcher needed to immediately get familiar with the voice of the teacher and the students she

[3] The researcher did not follow any particular transcription rules, because the kind of analysis she wanted to carry out did not require very much precision in the transcripts; what was important for the researcher was to have all the talk and actions recorded. The rules she adopted for the transcriptions were: when the teacher's questions and students' responses simultaneously happened, the researcher ordered their sentences by numbering 1 and 1.1, for instance. Both the teacher's and students' actions are in brackets, and their talk is in Normal.

focused on, for instance, whose voice is who. More importantly, she needed to understand which students' voices were dominant in the audio recorder and then were likely to replace those students she focused on, yet she found that she could not actually recollect their voice. Secondly, the researcher needed to immediately understand and write down exactly what the teachers talked about on the audio recorder, together with those relevant data collected by the field notes or the camera. For instance, when teacher Spring talked about "red colour" of a figure, the researcher needed to clearly understand which line segment the teacher talked about at that moment with data written down in the field notes or relevant photos.

After completing the first draft of all lesson transcripts (12 of Lily's lessons (each 40 minutes long), 8 of Spring's lessons (each 40 minutes long), and 3 of Nana's lessons (each 45 minutes long)) and had become familiar with both teachers' and focused students' voices, the second step was to determine the suitable form for presenting the transcripts so that it would be more useful for the data analysis. Firstly, after trying out many different forms, the researcher chose the form (see table 4.15) for the analysis and the presentation of data. In this form, a table is constructed in which the first column from the left is used for the time duration of teaching each individual proof problem solving. The second column is used for the teachers' instruction, including the teachers' talk and actions as well as the figure drawn by the teacher. The third column is generally for the responses from students who were not initially focused by the researcher. In Lily's class, as boy Linlin's and girl Youyou's voice were dominant in the audio recorder, their name was largely used to represent other students' similar responses. The fourth and fifth columns are used for each of the focused students. This layout allows seeing at a glance how significant a role the teacher played during the classroom interactions, and the activeness of not only the focused individual students but also other students during each teaching process of the individual proof problem solving. All these things were helpful for the research aims. Secondly, each transcript was divided into episodes, in which the focus was the teaching process of each individual proof problem. Each transcript has two versions, Chinese and English (episodes were translated into English when the episodes were selected to be presented in the thesis). Finally, the lesson transcripts were all written down (see examples in the next chapter) as a first "write-up" for each case, which allowed the researcher "to become familiar with each case as a stand-alone entity" (Eisenhardt, 2002).

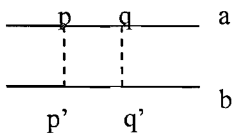
	Teacher (Lily)	Other students (Youyou and Linlin)	Liuliu	Beibei
Proof 1				
2 '50	8. Now lets say that a and b are a pair of parallel lines. There are two points on line a. They could be anywhere on it. Or we could say that these two points are moving on line a. How do they move then? They move regularly. From p constantly draw a vertical line to line b at p', from q draw a vertical line to line b at q'. Now, let us think, if p and q are moving, do p' and q' follow or stay? 			
		9. Students: Follow.		
	10. So they are also moving. They follow p and q. (Lily gradually drew the figure on the blackboard)			
	11. So we get two line segments, pp' and qq'. Now, I want you to think, what location relation could they (pp', qq') have?			
		12. Youyou: Parallel.	12.1 Parallel.	

Table 4.15. Example of the layout of a transcript

2) Data categories

In general, the data categories for lessons observed were based on the categories in the van Hiele theoretical framework: students' thinking levels and teaching phases (see section 3.3). Further explanation of the lessons was based on the categories developed from the literature review and emerging from the data analysis: the visual approach, the deductive/inductive approach, types of teachers' questioning, and the arrangement of proof problems.

3) Data coding

"L" was used to code the lessons observed in Lily's class. "L1" represented the first observed lesson, "L2" represented the second observed lesson, and so on. Similarly, "S" was used to code the lessons observed in Spring's class, and "N" was used to code the lessons observed in Nana's class. "S1" represented the first observed lesson in Spring's class, and "N1" represented the first observed lesson in Nana's class.

The observed lessons were all of geometrical proof problem solving. "PR" was used to code new geometrical theorems in the observed lessons. "PR1" meant the first new theorem taught in the lesson and "PR2" meant the second new theorem taught in the lesson, and so on. As the three teachers used the same textbook and taught the same topics, "PR1" represented the same theorem taught in the three teachers' lessons.

As the three teachers arranged different geometric proof problems, or same proof problem provided in a different order during the observed lessons, "L1-Ex1" meant the first proof problem observed as a whole and took place in the first observed lesson in Lily's class, "S1-Ex1" meant the first proof problem observed as a whole and took place in the first observed lesson in Spring's class, and "N1-Ex1" meant the first proof problem observed as a whole and took place in the first observed lesson in Nana's class. "L1-Ex1", "S1-Ex1" and "N1-Ex1" did not necessarily mean the same proof problem. Moreover, in each teacher's lesson, different proof problems were numbered as "L1-Ex1", "L1-Ex2" and so on.

In addition, the van Hiele's levels of thinking and teaching phases were used to code students' responses and teachers' instruction during the lessons (see table 4.16).

van Hiele Levels	van Hiele teaching phases
Level 1 – Figures are judged by their appearance.	Information/Familiarization – Phase 1
Level 2 – Figures are bearers of their properties.	Guided Orientation – Phase 2
Level 3 – Properties are ordered.	Explicitation/Verbalization – Phase 3
Level 4 – Formal deduction.	Free Orientation – Phase 4
	Integration – Phase 5

Table 4.16. Coding levels and phases of the observed lessons.

4) The cross-case analysis

Yin (2003, p.47) suggests that the multiple cases follow a "replication" logic rather than "sampling" logic. The replication approach used in the cross-case analysis of this study is illustrated in figure 4.4.

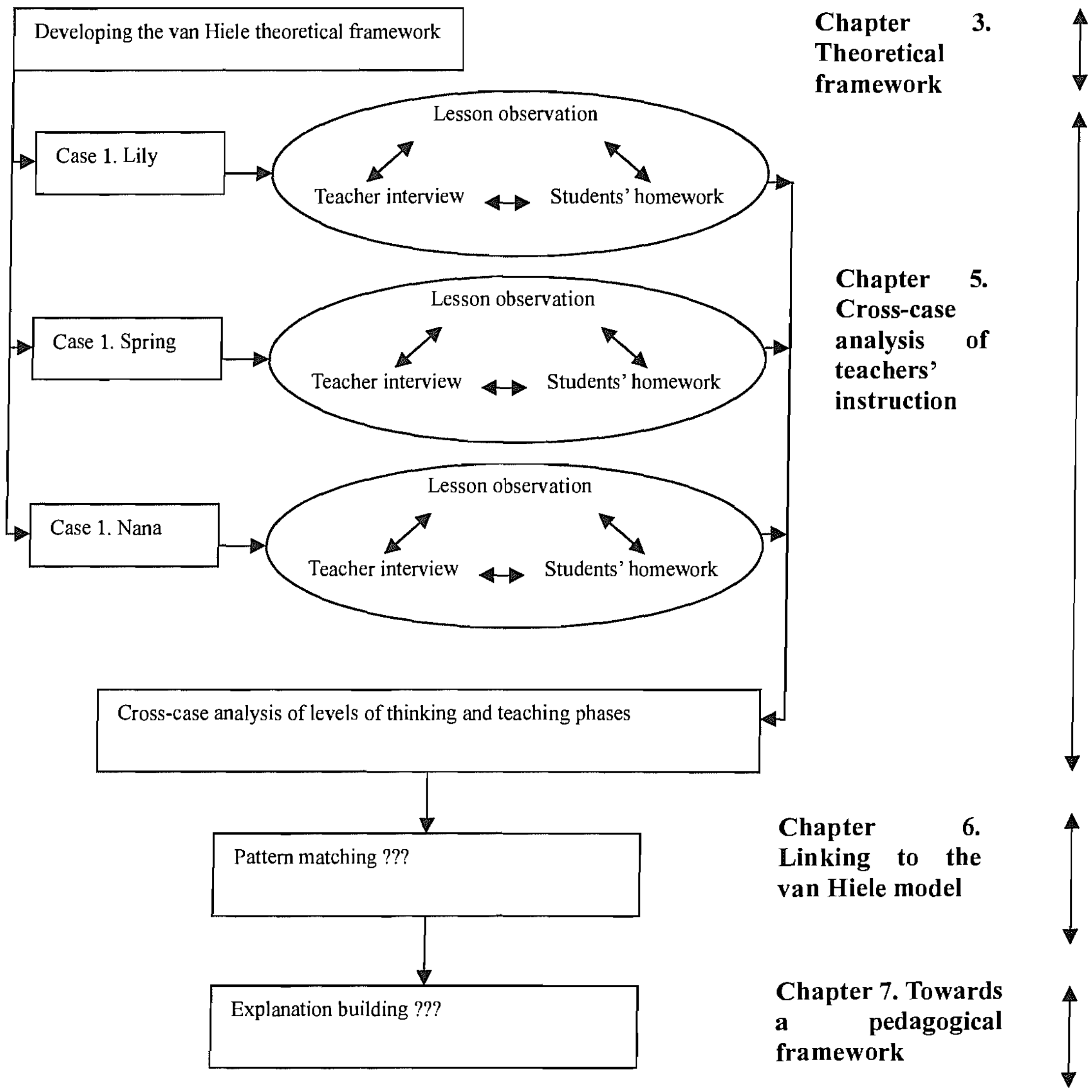


Figure 4.4. The cross-case analytical process

As shown in figure 4.4, the first step of the analysis was to develop a rich theoretical framework for guiding the data analysis. Given the main purpose of this study is to develop fundamental understanding of the pedagogical phenomenon of geometrical proof problem solving, the van Hiele model was considered. As has been analysed in chapter 3, Dina van Hiele-Geldof proposed the five phases mainly to facilitate students' thinking from Level 1 to Level 2. Pierre van Hiele suggested that the five phases is a universal means to develop students' thinking to any a higher level. To date, little research appeared to link the proposed theory to the teachers' actual practice in classroom (see section 2.5.3). One key aim of this study was to contribute to understanding the usefulness of the van Hiele model to analyse and interpret teachers' didactical practice on geometrical proof problem solving. Moreover, Pierre van Hiele proposed the hierarchical and discrete levels of students' geometrical thinking development. While the levels have provided the significant cognitive structure of students' geometrical thinking, more research work is needed to elaborate and extend the van Hiele's model, particularly to elaborate the levels structure with geometrical proof. By understanding teachers' didactical practice in the classroom, this study aimed to develop understanding of the dynamic development of students' thinking in the geometrical proof problem solving teaching process.

The second step of the data analysis was to select the three case studies for analysing and interpreting the levels of thinking and the teaching phases according to the van Hiele model. Case 1 presented the analysis of observation data of Lily's instruction of different geometrical proof problem solving (both teaching new theorems and using the new theorems to solve other proof problems) and students' thinking responses to Lily's instruction, of interview data with Lily, and of small-scale survey data of students' learning outcomes in Lily's class. Cases 2 and 3 respectively presented the analysis of Spring's and Nana's instruction on the proof problems as similar to those posed by Lily and students' thinking responses to the teachers' instruction, of interview data with Spring and Nana, and of small-scale survey data of students' learning outcomes in Spring's and Nana's class.

The third step of the analysis was to compare the outcome of the analysis across the three cases with the van Hiele model. As a result, an elaboration of the van Hiele model was presented. Finally, based on the findings from the analysis, an explanation was established to interpret the teachers' actual instruction across the three cases, which was not adequately elucidated by the van Hiele model.

4.4.5 Reliability, validity and generalisability

This study is a case study of three teachers' actual instruction of geometrical proof problem solving in their daily classroom. Kidder & Judd (1986, pp.26-29) suggest four tests to establish the quality of case studies: construct validity, internal validity, external validity, and reliability.

In the first place, Yin (2003, p.36) suggests three tactics to increase construct validity when conducting case studies: using multiple sources of evidence, establishing chain of evidence, and having key informants review draft case study report. This study focused on studying three types of teachers' instruction: expert teacher, senior experienced teacher and junior experienced teacher. Next, the researcher collected data by classroom observation, teachers' interview and students' learning outcome during their homework and test paper. Classroom observation helped the researcher to learn how the teacher taught and students learned in the class. Teachers' interviews helped to interpret why the teacher used some strategies and approaches. Students' learning results during their homework and test papers provided information which was not able to be gained by classroom observation and teachers' interviews. Last, the analysis results had been reviewed by the three studied teachers, and the researcher's academic colleagues through the analysis process.

Yin (2003, p.36) further suggests four analytic tactics to increase internal validity when conducting case studies: pattern matching, explanation building, addressing rival explanations, and using logic models. The following chapters (chapter 5, 6 and 7) will provide an account of how two analytic techniques, pattern matching and explanation building, were used to analyse data, so that the reader will be able to see how internal validity was constructed.

In terms of the external validity, Yin (2003, p.34) suggests two tactics to increase external validity: using theory in single-case studies and using replication logic in multiple-case studies. This study started from developing a comprehensive understanding of the van Hiele theory and comparing the data in three cases with the theory. As a result, a process of explanation building took place. A pedagogical framework was proposed in chapter 7, which identified a range of instructional strategies and approaches to the support of the development of students' geometrical thinking for writing proofs. This theoretical framework will be further tested by replicating the findings in the future research.

In terms of establishing data reliability, Croll (1986, p.148) points out that the main difficulty is "the conceptual problem of deciding what is meant by reliability in this context". The goal of reliability in the sense of case study is to minimise the errors and biases in a study (Yin, 2003). In this study, reliability was obtained by the variety of settings observed and by providing the details of the data collection procedure in section 4.4.3.

In terms of generalisability on the basis of case studies, Flyvbjerg (2004, p.425) considers that "formal generalisation is overvalued as a source of scientific development, whereas 'the force of examples' is underestimated." Moreover, Gobo (2004, p.435) points out two kinds of generalisation in research: a generalisation about a specific group or population (which aims at estimating the distribution in a population) and a generalisation about the nature of a process. The second kind of generalisation was applied in this study. In this sense, the goal of this study was not to represent a "sample", but to expand and generalise theories. Yin (2003, p.10) called this kind of generalisation as "analytic generalization". In addition, in respect of the van Hiele-based study, Battista (2007, p.847) particularly stresses that "generalization typically requires reconceptualization and refinement." The pedagogical framework proposed in the study is based on the explanation of theoretical ideas, supported by the data of the teachers' actual classroom instruction. The theoretical explanations for the observed teachers' didactical practice provide generality but the researcher was aware of the specificity of the setting and process observed. A balance between the details of the data and the generality of the theory being constructed was attempted throughout the research process.

4.4.6 Ethical issues

Ethical consideration may arise from classroom observation and interviewing teachers, and especially from the intention to audio-record and photograph the classroom observation and interview sessions. These issues are explicitly addressed by formally following the ethical procedures of the University of Southampton, School of Education.

First, the researcher complied with the Chinese school culture, in which the school head teacher and classroom teachers are mainly responsible for students in school time. Therefore, the purposes and methods of the study were clearly explained to people, such as school head teachers, classroom teachers, and mathematics teachers involved in the study, of the purposes, methods and intended and possible uses of the research. The researcher gained permission from the teachers for the use of an audio recorder and camera for the study. The researcher developed field relations for reducing the possible pressure for teachers on the use of an audio recorder and camera, and the possible influence on students of the researcher's presence in class in the study.

The confidentiality of the lessons observed, together with the interviews, the audio-recordings, photographic materials, and students' practices on the worksheet and homework, can be assured. Teachers selected are all willing to be studied in their daily classroom. In the study, the anonymity of schools, teachers, as well as students is protected by using pseudonyms. I also obtained the agreement from all teachers for the use of data to be presented in books, journals and conferences.

CHAPTER 5. CROSS-CASE ANALYSIS OF THREE TEACHERS' INSTRUCTION IN DEDUCTIVE GEOMETRY

5.1 Overview

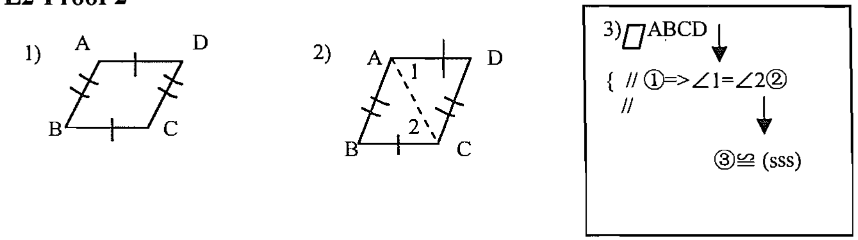
This chapter presents a detailed cross-case analysis of examples selected from the three teachers' observed lessons that link the classroom data to the van Hiele model. The analysis provides the ground for understanding the form of pattern match between the van Hiele theory and classroom practice. The chapter is divided into four main sections. The first three sections (sections 5.2-5.4) demonstrate the analysis of the three individual case studies, and the final section (section 5.5) shows the cross-case analysis of the three cases.

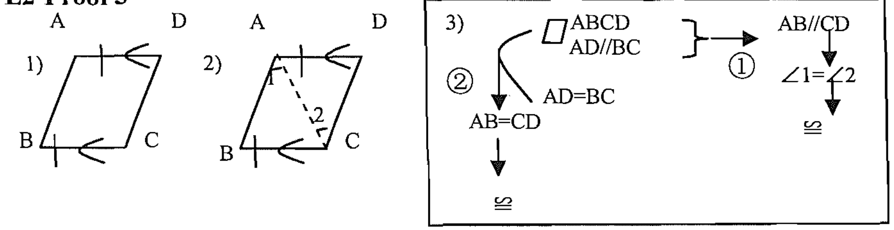
5.2 The case of teacher Lily

5.2.1 An analysis of Lily's lessons

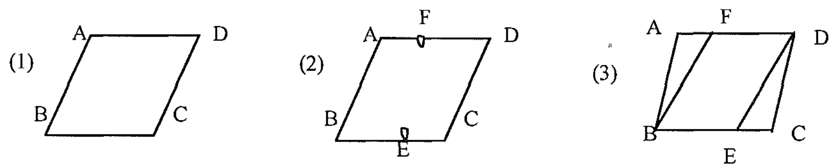
This section presents the analysis of teaching episodes of two new theorems of parallelogram (Proof 2 & 3) and five geometric proof problems solving (Ex4-Ex8, about proving a parallelogram) which took place in the second and the third observed lessons (L2 & L3) in Lily's class. The explaination for selecting examples is shown in section 4.4.2(2). The details of the two lesson structures and content can be seen in figure L2 and L3 in Appendix A.

Table 5.2 contains the analysis in terms of the van Hiele levels of thinking and the van Hiele teaching phases. This analysis, together with the analysis of lessons by teachers Spring and Nana, is subject to cross-case analysis in section 5.5.

Transcripts (L2)	Levels of thinking	Teaching phases
L2-Proof 2  **Figure 5.1. L2-Proof2.** This problem was one of the students' homeworks (It was also discussed in lesson 1). Therefore, the teacher briefly presented the statement, and asked one student to present its converse statement (#23-32). She then demonstrated the requirement for a formal procedure of proving a statement (#33). She drew a parallelogram on the blackboard, and presented and put the marks of the given and the problem to be proved on the figure (#34, see figure 5.1(1) and (3), parallelogram ABCD). 35 Lily: So far, we have only known one way to prove a shape is a parallelogram. …? (waiting students' response) 36 Some students: Use the definition. (To prove two pairs of parallel opposite sides.) (The teacher wrote down the signals of parallel lines on the blackboard, see figure 5.1(3)①, (#37).) 39 Lily: To prove two lines are parallel, what methods did we learn early at grade 7? 40 Some students: The alternate interior angles are equal. (The teacher encouraged students to establish the 'three lines and eight angles' in the parallelogram (three lines means two parallel lines are intersected by the third line. Eight angles are the acute angles formed by the three lines). Some students said to link A and C, some said B and D. Then the teacher chose to link A and C. see figure 5.1(2). (#41-44).) 45 Lily: Now, I linked A and C. If I want to prove AD//BC, what actually must I turn to prove here? (Some students shortly answered equal angles, (The teacher put 1 in ∠DAC, and 2 in ∠ACB on the blackboard, see figure 5.1(2). (#46).) 49 Lily: If to prove ∠1=∠2, what should I turn to prove first? (Some students answered congruent triangles (#50). Then the teacher briefly guided	**#36, 40, 46 (Others)** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66)	**#35-49. *Guided orientation*** The teacher steers the students' responses to the specific subject matter or discipline they are studying; The teacher leads students in discussing the material in a narrow framework of topics. (Hoffer, 1994, p.6) **#35-49. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#35-49. *Explicitation*** … children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#35-49. *Integration*** The teacher asks questions, assessing student understanding of the topic; The teacher designs question that apply and extend the accumulated knowledge of the subject. (Hoffer, 1994, p.6)

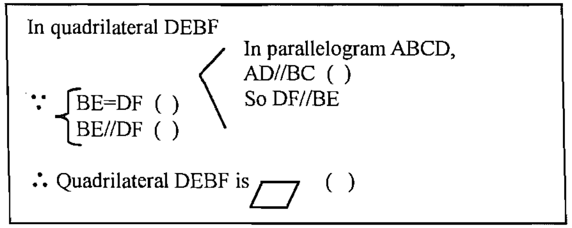
students to prove two congruent triangles, see figure 5.1(2-3), (#51-52).) (While the teacher asked students these questions, she gradually wrote down an analytic path for the proof on the blackboard (see figure 5.1(3). See photo 1 in Appendix F.). **Extract 5.1. L2-Proof2, figure 5.1, #35-49**		
L2-Proof 2 68 Lily: Is this new theorem about the property of a parallelogram? Or about verifying a parallelogram? 68.1 Some students: Verify a parallelogram. 69 Lily: You see, the conclusion is a parallelogram. 70 Lily: So it is used to verify whether a quadrilateral is a parallelogram or not. (The teacher then taught students to use this fact as a theorem in proof. The teacher wrote down the lesson title on the blackboard. (#71-73)) 74 Lily: Well, who could use words to state this new theorem again? (The teacher repeated several times, (#75)). 76 Lily: And who could use mathematical language to state the theorem? (The teacher asked a student to stand up and present the theorem. After then, the teacher showed students how to use mathematical language to present the theorem on the blackboard. (#77-86)) **Extract 5.2. L2-Proof2, figure 5.1(1), #68-76**		**#68-76. *Verbalization*** The teacher helps students express and clarify formal definitions. (Hoffer, 1994, p.6) **#68-76. *Explicitation*** … children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#68-76. *Integration*** The teacher asks questions, assessing student understanding of the topic. (Hoffer, 1994, p.6)
L2-Proof 3  **Figure 5.2. L2-Proof3.** This problem was also one of the students' homeworks. The teacher therefore	**#93, 97, 99, 104, 113, 133, 136, 138, 140 (Youyou); #102, 103, 127, 128 (Others); #116 (Linlin); #140.1 (Liuliu); #141 (Beibei).** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66)	**#89-99. *Integration*** The teacher asks questions, assessing student understanding of the topic. (Hoffer, 1994, p.6) **#89-99. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. Through this discussion, the teacher learns how students interpret the language and provides information to bring

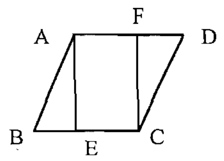
briefly presented the given and proof of the statement (Proof 3), and put marks of the given on the figure on the blackboard (see figure 5.2(1), #87-88). 89 Lily: Who would like to prove this parallelogram (ABCD)? (Some students said "SAS". Liuliu thought that the method was the as same as that used in Proof2. (#90).) 91 Lily: Well. So far, how many methods did we learn to verify a parallelogram? 92 Liuliu: Many ways. 93 Youyou: Two methods. (The teacher asked again to make sure that most students knew the two methods. (#94-95).) 96 Lily: Which two? 97 Youyou: The definition. 98 Lily: more? 99 Youyou: The first theorem[4]. (The teacher repeated this. (#100)). 101 Lily: OK. Now I consider, if I need to prove that this is a parallelogram, what is given? 102 Some students: A pair of opposite sides is parallel. 103 Some students: A pair of opposite sides is equal. 104 Youyou: AD=BC. (The teacher repeated students' answers. (#105).) 106 Lily: OK. Now, according to the given, which method will you consider, the definition or the first theorem of verifying a parallelogram? 107 Lily: How do you make a decision? (Some students considered the definition, while others considered the first theorem of verifying a parallelogram. (#108-109).) 110 Lily: OK. If I consider this condition (AD//BC), ...This reminds me and some students, ..., which method could be considered? 113 Youyou: Use the definition. (Some students responded with the same thought. (#114).) 115 Lily: If I use the definition to prove, what should I prove first? 116 Linlin: Parallel. (The teacher asked the question again to make sure that students understood how to prove AB//CD. (#117-119).) 120 Lily: How to prove the parallel lines? (AB//CD). (Students decided to link AC. The teacher used the triangle ruler to link AC. See figure 5.2(2). (#121-125).) 126 Lily: To prove AB//CD, what should I turn to prove first? 127 Some students: To prove two angles are equal.		students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#89-99. *Familiarization*** The teacher introduces problems which help in the discovery process. (Hoffer, 1994, p.6) **#89-141. *Guided orientation*** The teacher steers the students' responses to the specific subject matter or discipline they are studying; ... The teacher leads students in discussing the material in a narrow framework of topics. (Hoffer, 1994, p.6) **#89-141. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#89-141. *Explicitation*** ... children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#89-141. *Free orientation*** The teacher utilizes problems that may have multiple solutions. (Hoffer, 1994, p.6) Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves

128 Some students: Alternate interior angles are equal. 129 Lily: Which pair of angles? (According to students' answers, the teacher used number 1 and 2 to highlight angles BAC and ACD. See figure 5.2(2)) 132 Lily: To prove ∠1=∠2, what should we turn to prove first…? 133 Youyou: To prove congruent triangles. (The teacher repeated. (#134).) 135 Lily: Which pair of triangles should we prove then? 136 Youyou: ABC. 137 Lily: ABC…and…? 138 Youyou: CDA. 139 Lily: Congruent triangles. What is given? 140 Youyou: SAS. 140.1 Liuliu: SAS. 141. Beibei: Two sides, a pair of angles. (While the teacher asked students these questions, she gradually wrote down an analytic path for the proof on the blackboard (see figure 5.2(3). See photo 1 in Appendix F.) **Extract 5.3, L2-Proof3, figure 5.2(1-3), #89-141**		within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431)
L2-Ex4  **Figure 5.3. L2-Ex4.** 197 Lily: Now, in the figure, the given. Look at the figure carefully. (The teacher drew a parallelogram ABCD on the blackboard, see figure 5.3(1)). 198 Lily: This is a parallelogram. Now the given is that on it's (ABCD) four sides, there are two points. For instance, E is on BC. F is on AD. (The teacher put F and E on AD and BC, see figure 5.3(2).). (Liuliu guessed the possible given (AF=CE or FD=BE), #199). 200 Lily: E and F are two dynamic points. They move regularly. This means that BE is always equal to DF. Well, now, if I link like this… (The teacher linked BF and DE, action omitted. See figure 5.3(3). She briefly repeated #198 and 200, students simultaneously exchanged their thoughts of the proof to one another. See Extract 5.6)		**#197-211. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. …and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#197-211. *Familiarization*** The teacher introduces problems which help in the discovery process. … The teacher has students use visual cues … (Hoffer, 1994, p.6) **#197-211. *Guided orientation*** The teachers' role is to direct students' activity by guiding them in appropriate explorations … (Clements and Battista, 1992, p.431)

206 Lily: Is quadrilateral BEDF changeable? 210 Lily: What does quadrilateral BEDF look like? 211 Students: A parallelogram. **Extract 5.4. L2-Ex4, figure 5.3(1-3), #197-211.**		The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6)
L2-Ex4 212 Lily: OK. I will ask one student to tell us how he might prove this problem. How to decide the way to prove this problem? (Students discussed in the class. See extract 5.6.) 214 Lily: How many methods could be used to prove that this is a parallelogram? 216 Youyou: Three methods. 217 Lily: Which three? 218 Youyou: The definition. 219 Lily: The definition. Anymore? 220 Youyou: The first and the second theorems. (Proof 2 and 3) 222 Lily: There are three ways to prove. Which way will you consider? 223 Many students: The second. 224 Lily: Good. Why do you consider the second theorem? 225 Students: BE=DF... 226 Lily: It is already given that a pair of opposite sides is equal. A pair of opposite sides is equal, isn't it? **Extract 5.5. L2-Ex4, figure 5.3(3), #212-226.**	**#216, 218, 220 (Youyou); #223, 225 (Others)** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66) The student recognizes the role of deductive argument and approaches problems in a deductive manner…(Fuys *et al.*, 1988, p.68)	**#212-226. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. Through this discussion, the teacher learns how students interpret the language and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#212-226. *Familiarization*** The teacher introduces problems which help in the discovery process. (Hoffer, 1994, p.6) **#212-226. *Guided orientation*** The teacher steers the students' responses to the specific subject matter or discipline they are studying; The teacher leads students in discussing the material in a narrow framework of topics. (Hoffer, 1994, p.6) **#212-226. *Integration*** The teacher asks questions, assessing student understanding of the topic; The teacher designs question that apply and extend the accumulated knowledge of the subject. (Hoffer, 1994, p.6)

L2-Ex4 (Lily repeated #198 and 200, see Extract 5.4.) 203 Liuliu: (spoke to himself) ABF is equal to ECD. It must be. (see figure 5.3(3)). 206.1 Beibei: If a pair of opposite sides is equal and parallel, then…. 207 Some students: Parallel and also equal. (The researcher guessed that this statement might mean that FD//=BE. see figure 5.3(3)). 208 Linlin: Oh, I see. 209 Liuliu: (responded to Beibei #206.1) Yeh, parallel and equal…??? (The teacher repeated her questions (see Extract5.4, #210), and encouraged students to present their thoughts of the proof, dialogue omitted.) 215 Liuliu: Opposite sides are equal. I could use this to prove this problem. (The researcher guessed that this statement might mean FD=BE, BF=DE). (The teacher encouraged students to outline different methods to prove a parallelogram, see extract 5.5, #214-217.) 221.1 Beibei: Why? (Response to Liuliu #215) 221.2 Liuliu: You could see here. Firstly, to calculate that ABF and ECD are congruent. Next, BF and DE are congruent. Oh, equal. BE and FD are already known. 221.3 Liuliu: This is to prove quadrilateral BEDF is a parallelogram. 221.4 Beibei: It is already given that a pair of opposite sides is equal. 221.5 Liuliu: You need to calculate that its opposite sides are equal. One pair of sides is given, yet you need to know another pair of sides. 221.6 Beibei: It is already given that BE=FD. 221.7 Liuliu: BE=DF. But you need to prove that BF=DE. 221.8 Beibei: If a pair of opposite sides of a quadrilateral is not only equal, but also parallel, then it is a parallelogram. (Liuliu considered Beibei's idea and did not reply immediately.) 221.10 Liuliu: OK. How do you prove it then? (Beibei did not reply immediately. Both of them listened to a girl Yang DH's presentation of proof see Extract 5.7. After Yang DH's presentation, Beibei talked to Liuliu again as follow.) 253.1 Beibei: You did not notice this, did you? (FD//BE). 253.2 Liuliu: No. I did not notice it. **Extract 5.6. L2-Ex4, figure 5.3(3), #203-253.2.**	**#203-253.2 (Liuliu and Beibei)** ***Level 3*** The student recognizes the role of deductive argument and approaches problems in a deductive manner…(Fuys *et al.*, 1988, p.68)	**#203-253.2. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#203-253.2. *Explicitation*** … children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#203-253.2. *Free orientation*** The teacher utilizes problems that may have multiple solutions. (Hoffer, 1994, p.6) Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431) **#203-253.2. *Integration*** The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, increasing emphasis on the use of mathematical structures as a framework for consolidation. (Clements and Battista, 1992, p.431)

L2-Ex4  **Figure 5.4. L2-Ex4.** (The teacher asked a girl student, Yang DH, to present her ideas about the proof to the whole class). 230 Yang DH: (stood up). In quadrilateral DEBF… 232 Yang DH: Because BE=DF. 234 Yang DH: the given. 236 Yang DH: and because BE//DF. 237 Lily: And because BE//DF? …But this is not directly given. 239.2 A boy student: because quadrilateral ABCD. 239.3 Yang DH: because of parallelogram ABCD. AD//BC. 242 Lily: AD//BC. What should you write here? 243 Yang DH: The definition of parallelogram. 244 Lily: The definition of parallelogram? Well. What should you write more precisely? 245 Yang DH and some students: In a parallelogram, its opposite sides are parallel. (The teacher helped the student to correctly prove FD//BE. Dialogue omitted.) 254 Yang DH and some students: so BEDF is a parallelogram. 255 Lily: What reason could you write here? (The teacher and students together presented the theorem of verifying a parallelogram. Dialogue omitted.) (During the student's presentation, the teacher repeated and wrote down the key words the student said on the blackboard. See figure 5.4.) **Extract 5.7. L2-Ex4, figure 5.3(3) and 5.4, #230-255.**	**#230-254 (Others)** ***Level 3*** The student recognizes the role of deductive argument and approaches problems in a deductive manner…(Fuys *et al.*, 1988, p.68)	**#230-255. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#230-255. *Explicitation*** … children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#230-255. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431) **#230-255. *Integration*** The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, increasing emphasis on the use of mathematical structures as a framework for consolidation. (Clements and Battista, 1992, p.431)
L2-Ex4 259 Lily: OK. Now, I change the problem. If F and E are dynamic points, what location and quantity relation do BF and DE have? 260 Students: Parallel and equal. 261 Lily: Parallel and equal. Please prove your conjecture.	#260, 262 (others) ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the	**#259-273. *Guided orientation*** The teacher steers the students' responses to the specific subject matter or discipline they are studying; The teacher leads students in discussing the material in a narrow

262 Students: You just write after that parallelogram...(discussed in class) 263 Lily: You mean to firstly prove it is a parallelogram. And then, you could explain that the opposite sides of a parallelogram are parallel and equal. 266 Lily: All right. Now, what method could we also use, if I want to prove two line segments are equal or parallel? 267 Students: To prove a parallelogram. 268 Lily: Previously, to prove parallel lines, we have to use the three lines and eight angles. But now, what method may be used? 269 Students: To prove parallelogram. 272 Lily: If to prove two equal line segments, what method could be considered? 273 Youyou: To prove parallelogram. **Extract 5.8. L2-Ex4, figure 5.3(3), #259-267**	argument. (Fuys *et al.*, 1988, p.66)	framework of topics. (Hoffer, 1994, p.6) **#259-273. *Integration*** The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, increasing emphasis on the use of mathematical structures as a framework for consolidation. (Clements and Battista, 1992, p.431)
L2-Ex5  **Figure 5.5. L2-Ex5.** 275 Lily: I draw two lines like this. (The teacher rubbed out BF and DE in figure 5.3(3), and drew AE and CF on the blackboard. see figure 5.5.) 276 Lily: I do not change other condition (BE=DF). This is still a parallelogram. If I change the condition (means AE and CF). What location and quantity relation do AE and CF have? 277 Students: Parallel and equal. 280 Lily: What should we turn to prove first? 281 Linlin: The distance of two parallel lines is always equal. Next, to prove parallel lines. (The teacher repeated Linlin's answer to the whole class. Dialogue omitted.) 283 Some students: (to Linlin.) How do you know that they are vertical? (AE⊥BC, FC⊥BC are not given.) (The teacher encouraged students to present their thoughts about the proof (#288).) 290.2 Beibei: BE=DF. You see, the larger one is a parallelogram. So, AD=BC. You subtract them. The same amount is subtracted by the same amount. The difference is equal.	**#281 (Linlin); #290.2 (Beibei); #283, 304, 307-309.1, 313 (Others)** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66) The student recognizes the role of deductive argument and approaches problems in a deductive manner...(Fuys *et al.*, 1988, p.68)	**#275-277. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. ...and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#275-277. *Familiarization*** The teacher introduces problems which help in the discovery process. ... The teacher has students use visual cues ... (Hoffer, 1994, p.6) **#281-313. *Guided orientation*** The teacher steers the students' responses to the specific subject matter or discipline they are studying. The teacher leads students in discussing the material in a narrow framework of topics. (Hoffer, 1994, p.6) **#281-313. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided

303 Lily: To prove that this (AECF) is a parallelogram, what method do you consider? 304 Wang XW: (girl) Use the second theorem. (The teacher highlighted the key elements of the second theorem (#306-309).) 306.1 Lily: Next, what should we prove? 307- 309.1 Wang XW: AF//CE. …and AF=CE. 310 Lily: How to prove they are equal? (AF=CE) 312 Lily: (asked Wang XW) What method do you use? 313 Wang XW: The property of equation. (AD=BC, BE=FD, so AD-FD=BC-BE.) **Extract 5.9. L2-Ex5, figure 5.5, #275-313.**		orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#281-313. *Explicitation*** … children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, …. (Clements and Battista, 1992, p.431) **#281-313. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431) **#281-313. *Integration*** The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, increasing emphasis on the use of mathematical structures as a framework for consolidation. (Clements and Battista, 1992, p.431)

L2-Ex6

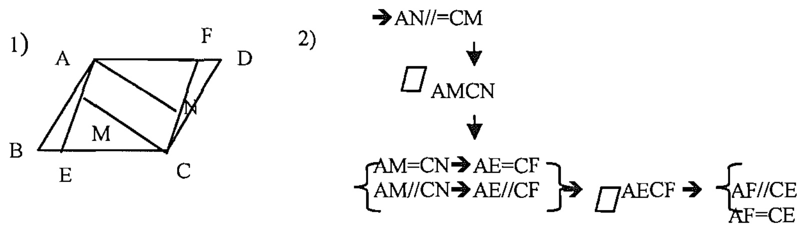


Figure 5.6. L2-Ex6.

(While the teacher presented the given and the proof problem (#330-339), students discussed their ideas of the proof to each other in the class. They made the conjecture that AN//=CM (#340). The teacher then encouraged them to present their thoughts about the proof (#342), many students suggested using the second theorem (proof 3) to prove parallelogram AMCN (#343).)

349 Liuliu: A pair of opposite sides is parallel and equal.

353 Lily: Now I ask a student to present the idea of the proof.

354 Lily: To prove AN//=CM, what should you turn to prove first?

355 Youyou: parallelogram.

356 Some students: AECF is a parallelogram.

357 Some students: AMCN is a parallelogram.

358 Lily: To prove AMCN is a parallelogram. (The teacher appeared not to clearly hear the students' different answers. She wrote down parallelogram AMCN on the blackboard:)

359 Lily: To prove AMCN is a parallelogram, what should you turn to prove first?

360 Beibei: AM//CN, AM=CN.

361 Students: AM//CN, AM=CN. (Probably some said AN//=CM.)

362 Liuliu: To prove AN//CM,...

363 Lily: To prove AM=CN, and AM//CN, right? (The teacher wrote AM=CN, and AM//CN on the blackboard:)

364 Liuliu: Ah?

365 Linlin: Ah? You wrote AM (Linlin thought AN and CM).

366.1 Students: It is about AM=CN. (laughed at Linlin.)

366.2 Liuliu: Oh, Yes, it is AM=CN (also laughed.)

(The analytic path for the proof was written by the teacher on the blackboard, while she used a sequence of questions to guide students to discuss their thoughts

#349, 362, 366.2 (Liuliu); #365 (Linlin); #360 (Beibei); #361, 366 (Others)

Level 3

The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66)

The student recognizes the role of deductive argument and approaches problems in a deductive manner but does not yet establish interrelationships between networks of theorems. (Fuys *et al.*, 1988, p.68)

#349-366.2. *Guided orientation*

The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6)

#349-366.2. *Verbalization*

The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2)

#349-366.2. *Explicitation*

... children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431)

#349-392. *Free orientation*

The teacher presents multi-step or open-ended problems that help students find their way in the system of relationships. (Hoffer, 1994, p.6)

#349-392. *Integration*

The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, increasing emphasis on the use of mathematical structures as a framework for consolidation. (Clements and Battista, 1992, p.431)

about the proof. see figure 5.6(2), #370-394) **Extract 5.10. L2-Ex6, figure 5.6, #349-392.**		
L2-Ex6 396 Lily: To prove parallel and equal lines, now we have one more method. If the line segments are in a quadrilateral, then we turn to prove it is a parallelogram. To prove a parallelogram, there are three methods. Here, we use the second theorem. 396.1 Lily: So, let's see. The definition of parallelogram is that two pairs of opposite sides are respectively parallel, isn't it? 396.2 Lily: The first theorem of verifying a parallelogram is that two pairs of opposite sides are respectively equal, isn't it? 396.3 Lily: The second theorem of verifying a parallelogram is shared part of the conditions of both the definition and the first theorem. It is that a pair of opposite sides is not only parallel but also equal, isn't it? 396.4 Lily: Therefore, the second theorem is used very often. 398 Lily: Well, What do you learn from solving this problem? 400 Lily: The use of the theorems of verifying a parallelogram, right? 402 Lily: If I could prove a parallelogram, I could use its property, such as its opposite sides are parallel and equal. 403 Lily: And, in this problem, how many times did we use the theorems of verifying a parallelogram? 404 Youyou and Liuliu: Twice. 406 Lily: Which theorem did we use then? 407 Youyou: The second one. 408 Lily: The second theorem. It is not easy to use the theorem twice to prove in a problem. It is quite complicated, isn't it? It's like that we could use congruent triangles twice to prove a problem, isn't it? 409 Lily: What did you learn from the use of congruent triangles? 410 Lily: We could use congruent triangle to prove two line segments are equal, two corresponding angles are equal, right? 411 Lily: The use of the theorems is quite similar here. 412 Lily: Now, lets consider. Here, through proving a parallelogram, I could solve a problem. Now we know one more way that might be used to prove parallel lines, equal line segments. **Extract 5.11. L2-Ex6, figure 5.6, #396-412.**		**#396-412. *Integration*** The teacher designs questions that apply and extend the accumulated knowledge of the subject. (Hoffer, 1994, p.6)

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L2-Ex7

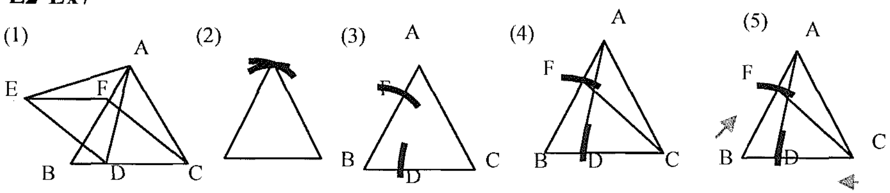

Figure 5.7. L2-Ex7.

424.1 Lily: Look. This figure on the blackboard is as same as that in the book (figure 5.7(1)). It is a whole figure. But you did not see the process by which I drew the figure. (The teacher used compasses and a triangle ruler to draw an equilateral triangle on the blackboard. dialogue omitted. see figure 5.7(2))

425.1 Lily: Next, CD=BF. What does this mean? (Students appeared not to understand the question, dialogue omitted).

427 Lily: It means that D and F are dynamic points, aren't they? (The teacher repeated the question a couple of times, dialogue omitted).

430 Lily: OK. CD=BF. This means that D and F are dynamic points. D could be here, could be here, could be here, right? (The teacher used compasses to draw D and F and made CD=BF. See figure 5.7(3). She then used a ruler to link CF and AD. see figure 5.7(4). dialogue omitted).

434 Lily: D and F are dynamic points. Now they move such that CD=BF. So if D goes this way. F goes that way. ··· The different dynamic points go in different directions at the same speed, right? ··· So the length they (D and F) moved should be the same, shouldn't they? (The teacher put red arrows in the figure on the blackboard, see figure 5.7(5))

435 Lily: If you are told like this statement, you might understand that this means CD=BF. We could describe a problem in different way, yet the meaning could be same. In this problem, it means that CD=BF.

436 Lily: Well. Now, are you familiar with this figure? (The teacher pointed out figure 5.7(5) on the blackboard.)

(The teacher encouraged students to compare figure 5.7(5) with (1) on the blackboard (see photo 2 in Appendix D), and to recognise which lines could be rubbed away from figure 5.7(1). Some students discussed to rub EF, ED and AE. (#439).).

440 Lily: You could think about this figure (see figure 5.7(5)) during the lesson break. You learnt about the equilateral triangle at Grade 7. In the process

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#439 (Others)
Level 1
The student compares and sorts shapes on the basis of their appearance as a whole. (Fuys *et al.*, 1988, p.59)

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#424.1-440. *Information*
The teacher discusses materials clarifying this content, placing them at the child's disposal. ...and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431)

#424.1-440. *Familiarization*
The teacher introduces problems which help in the discovery process. The teacher "sets the stage" for upcoming topics by introducing questions that incite curiosity. The teacher has students use visual cues ... (Hoffer, 1994, p.6)

#424.1-440. *Guided orientation*
The teachers' role is to direct students' activity by guiding them in appropriate explorations ... (Clements and Battista, 1992, p.431)
The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6)

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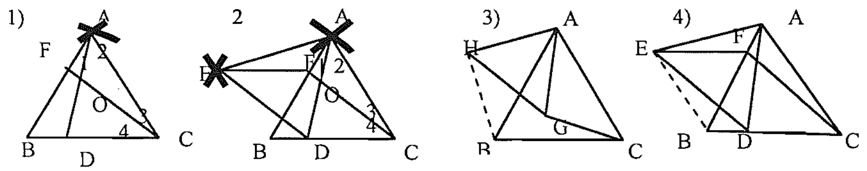
of the movement of D and F, D and F move regularly. Could you find what is never changed in the movement? (The teacher's instructional intention was to support students in discovering that AD=FC, the angle formed by AD and CF is 60°.) (The figure the teacher drew on the blackboard which can be seen on the right part of photo 2 in Appendix F) **Extract 5.12. L2-Ex7, figure 5.7, #424.1-440.**		

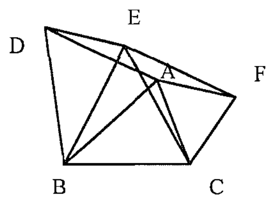
Table 5.1. An analysis of levels of thinking and teaching phases of L2.

Transcripts (L3)	**Levels of thinking**	**Teaching phases**
L3-Ex7  **Figure 5.8. L3-Ex7.** Most students in the class had great difficulty to discover the hidden property of the figure asked by the teacher in Z2 (see extract 5.12, #440.) (#1). The teacher therefore firstly guided students to consider the possible methods to prove the problem by outlining three known laws of verifying a parallelogram, namely the definition and two theorems (Proof 2 and 3) (#2-7). Next, the teacher guided students to evaluate the proper use of the three known laws by highlighting the key elements of the premise of the three laws, in order to relate to the given and to decide what they had to prove next (mainly to prove either CF//=DE or CD//=EF) (#8-9). The teacher then guided students to prove CF//=DE (#22) by guiding students to recognise the hidden property of the figure constructed in Z2 (figure 5.7(5) and figure 5.8(1)). 37. Lily: In this figure, could you find what is not changed, when D and F are moving? (see figure 5.8(1)) (Students discussed in the classroom (#38).) 40. Students: DC=BF. 41. Lily: DC=BF? This is already given. Except this, what else is not changed?	**#40 (Others)** ***Level 1*** The student identifies parts of a figure but a) does not analyze a figure in terms of its components. (Fuys *et al.*, 1988, p.59) **#42 (Others)** ***Level 2*** The student b) sorts shapes in different ways according to certain properties. (Fuys *et al.*, 1988, p.60) **#44, 50 (Others); #45 (Beibei)** ***Level 3*** The student discovers new properties by deduction. (Fuys *et al.*, 1988, p.65)	**#37-50. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. ...and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#37-50. *Familiarization*** The teacher introduces problems which help in the discovery process; The teacher "sets the stage" for upcoming topics by introducing questions that incite curiosity. (Hoffer, 1994, p.6) **#37-50. *Guided orientation*** The teachers' role is to direct students' activity by guiding them in appropriate explorations ... (Clements and Battista, 1992, p.431) The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6) **#37-50. *Explicitation*** ... children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and

42. Students: Oh, AF=BD. Because AB=BC. 43. Lily: AB=BC? This is given, as it is an equilateral triangle (ABC). 44. Students: CF=AC. 45. Beibei: CF=AC. (Students kept observing and guessing the possible fact (#48). The teacher asked a boy student, Wang WY. (#49)). 50. Wang WY: (stood up). Two triangles are congruent. AD and CF are always equal. **Extract 5.13. L3-Ex7, figure 5.8(1), #37-50.**		learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#37-50. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2)
L3-Ex7 56. Lily: Now in the class, the time is too limited. But when you go home, you might draw the figure to have a look. I drew it here. D and F are at different location, and you see this figure is just a moment of a dynamic figure. I suggest you to measure it. It must be that CF=AD. How about their location relationship? 57. One boy: 60°. 58. Lily: Obviously, they (AD, CF) are not parallel. They are intersected, aren't they? How is the angle they formed? Will it change? You could use a protractor to measure the figure on your book. You could measure the angle before and after the movement. 59.1. Some students: It will be the same. 59.2. Liuliu: 60°, 60°. Only need to prove two parallel lines. (probably CF//ED) 60. Lily: How do you explain that they are equal. No any change? How much is the angle then? 61. Some students: 60°. 62. Liuliu: 60°. 63. Beibei: Why is it 60°? 64. Lily: If this angle (AOF, ∠1) is 60°. How to prove? 66.One boy asked his neighbor student: Why is it 60°? 68. Beibei: (asked Liuliu) Why is it 60°? Parallel? 69. Liuliu: If both of them are 60°, then they are always parallel. (Probably if ∠1=∠ADE=60°, then FC//ED.) 70. Linlin: Oh, in the middle, there is a pair of straight opposite angles! (Probably∠1=∠COD)	**#59.2, 69 (Liuliu); #63, 68 (Beibei); #66, 72, 76 (Others); #70 (Linlin).** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66) The student recognizes the role of deductive argument and approaches problems in a deductive manner but does not yet establish interrelationships between networks of theorems. (Fuys *et al.*, 1988, p.68)	**#56-76.1. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. …and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#56-76.1. *Familiarization*** The teacher introduces problems which help in the discovery process; The teacher "sets the stage" for upcoming topics by introducing questions that incite curiosity. (Hoffer, 1994, p.6) **#56-76.1. *Guided orientation*** The teachers' role is to direct students' activity by guiding them in appropriate explorations … (Clements and Battista, 1992, p.431) The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6) **#56-76.1. *Explicitation*** … children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical

71. Lily: Zheng YQ. (The teacher asked a boy student to present his ideas.) 72 Zheng YQ (boy): Because ∠1=∠DAC+∠ACF. (The teacher used number 2 to present ∠DAC.) 75.1 Students, Linlin and Liuliu: Ah? It is ∠ACF??? 76. Zheng YQ: Because of the congruent triangles (ADC and FBC), ∠2=∠FCB. 76.1 Some students: Oh, the bottom angle! (Probably ∠ACD) **Extract 5.14. L3-Ex7, figure 5.8(1), #56-76.1.**		language for the subject matter. (Clements and Battista, 1992, p.431) **#56-76.1. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2)
L3-Ex7 99. Lily: Now, lets see how the problem is formed? Look at here. I am drawing on the blackboard. (The teacher used a pair of compasses to draw equilateral triangle ADE. See figure 5.8(2). See photo 3 in Appendix F.). Equilateral triangle has a special property, as its three sides are equal. So they could be replaced to each other, right? We know AD=CF. Now where could AD be replaced to? 100. Liuliu: DE. 101. Lily: Isn't AD DE? An equilateral triangle has three equal sides. See, DE=CF. The replacement of equal elements. The equal element of CF and DE is AD. ED=AD, because of the equilateral triangle. CF=AD, because of congruent triangles. So, ED=CF. 103. Lily: We have proved one condition of the parallelogram. The opposite sides are equal. We need to go further to prove they are parallel. 104. Youyou: If the corresponding angles are equal, then two lines are parallel. 104.1 Some students: If alternate interior angles are equal, then two lines are parallel. 105. Beibei: 60°. If the corresponding angles are equal, then two lines are parallel. 106. Lily: How much are these two angles? (∠1, ∠ADE) 107. Students: 60° (The teacher led students to prove ∠1=∠ADE, and then CF//ED (#108-110) **Extract 5.15. L3-Ex7, figure 5.8(2), #99-107.**	**#100 (Liuliu); #104 (Youyou); #104.1 (Others); #105 (Beibei)** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66)	**#99-110. *Guided orientation*** The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6) **#99-110. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#99-110. *Explicitation*** ... children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#99-110. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431)

		#99-110. *Integration* The teacher designs question that apply and extend the accumulated knowledge of the subject. (Hoffer, 1994, p.6)
L3-Ex7 113. Lily: Well. See here. (The teacher guided students back to see the original figure, see figure 5.7(1)). Now, consider, is there any more method? 114. Lily: To prove this is a parallelogram. Consider, in this figure, is there any more basic figure? I drew another figure here. This is an equilateral triangle ABC. This is a small equilateral triangle AHG. (The researcher added letters to the figure for the purpose of clear description. See figure 5.8(3). See photo 3 in Appendix F.) 115. Youyou: Oh, two small angles are equal. 116. Lily: This small equilateral triangle AHG is rotated around A. It turns a certain angle here, right? (More students discussed two equal angles.) 117 Youyou: Oh, ∠EAB=∠DAC. 118. Lily: In this rotation, this triangle (AGC) is always…? 119. Students: congruent. 121 Some students: So, link EB. (see figure 5.8(4).) (The teacher guided students to recognise congruent triangles AHB, AGC. (#122)) 123. Lily: So, which triangle is congruent to triangle ADC? (see figure 5.8(4)) 124. Students: AEB. 125. Lily: This figure is not a full figure. We modify it here (The teacher linked EB). If we link EB, these two triangles (AEB, ADC) are congruent. SAS, right? (see figure 5.8(4).) 127. Lily: If congruent, how much is ∠EBA? 128. Students: 60° 130. Lily: And, CD=BF. This is given. Congruent triangles, EB=DC. So here accurs a small equilateral triangle EBF. 131. Liuliu: Discovered all of them! 132. Youyou: so EF=CD. 133. Lily: So EF=CD. A pair of opposite sides are equal (EF=CD). But we still need to prove they are parallel. How to prove? 134. Students: No need. Two pairs of opposite sides are equal. 135. Lily: I want to prove EF//DC. (Students repeated their views. See #134.)	**#115, 117, 132 (Youyou);** **#119, 121, 124, 128, 134 (Others);** **#138, 143 (Beibei).** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66)	**#113-155. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. …and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#113-155. *Familiarization*** The teacher introduces problems which help in the discovery process; The teacher "sets the stage" for upcoming topics by introducing questions that incite curiosity. (Hoffer, 1994, p.6) **#113-155. *Guided orientation*** The teachers' role is to direct students' activity by guiding them in appropriate explorations … (Clements and Battista, 1992, p.431) The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6) **#113-155. *Explicitation*** … children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#113-155. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided

136. Lily: I still use the second theorem. I do not use these congruent triangles (BFC, ADC). I proved these congruent triangles (AEB, ADC). 138. Beibei: Congruent triangles, twice. 139 Lily: I do not prove twice congruent triangles. It's too much. I have already proved these congruent triangles (AEB, ADC). EF=CD. To prove EF//CD, then I could go the second way to prove (The teacher pointed out EF//=CD on the blackboard). 140. Lily: I want to prove parallel lines. So we have to see the location. Could you have a look? Is there "three lines eight angles"? (Students observed the figure on the blackboard.) 142. Lily: None of you find it? What triangle is BEF? 143. Beibei: Equilateral triangle. One is 60°, another is also 60°. Alternate interior angles are equal. (Boy Zheng YQ also recognised the equilateral triangle EBF, therefore. The teacher asked him to stand up and present his findings, which were $\angle EFB=60^{\circ}$, and $\angle FBC=60^{\circ}$. (#144-152).) (The teacher finally reviewed the ideas of the two key methods with the figures of proof and emphasised the importance of accumulating basic figures through problem solving, dialogue #154-155 omitted.) **Extract 5.16. L3-Ex7, figure 5.8(3-4), #113-155.**		orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#113-155. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431) **#113-155. *Integration*** The teacher designs question that apply and extend the accumulated knowledge of the subject. (Hoffer, 1994, p.6)
L3-Ex8 (The teacher provided students with time to read the problem and to analyse the figure and the problem on their own (#156). See figure 5.9).  **Figure 5.9. L3-Ex8.** 160. Liuliu: All are 60°. (probably the angles of an equilateral triangle.) (The teacher reminded students to pay attention to the rotatable relationship of the equilateral triangles (#161).) 162. Liuliu: I see. It is equilateral triangle. One of its sides could be replaced by another. (The teacher encouraged students to draw the figure on their exercise book and	**#160, 162, 170, 173, 175, 185 (Liuliu)** **#171, 174, 186.2 (Beibei)** ***Level 3*** The student identifies and uses strategies or insightful reasoning to solve problems. (Fuys *et al.*, 1988, p.67) The student recognizes the role of deductive argument and approaches problems in a deductive manner...(Fuys *et al.*, 1988, p.68)	**#160-186.2. *Guided orientation*** The teachers' role is to direct students' activity by guiding them in appropriate explorations ... (Clements and Battista, 1992, p.431) The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6) **#160-186.2. *Explicitation*** ... children become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn some of the traditional mathematical language for the subject matter. (Clements and Battista, 1992, p.431) **#160-186.2. *Verbalization*** The pupils attempt to explicitly verbalize the

explore the problem on their own (#164-167)). 170. Liuliu: Which two congruent triangles could we prove? 171. Beibei: (while thinking the problem, she very slowly answered Liuliu) ... BAC and... EFC. 172. Liuliu: (Liuliu thought for a second) Oh, yeh. 173. Liuliu: AC=CF, BC=EC. What's more? 174. Beibei: ... and one angle. 175. Liuliu: AD=EF, but we need another condition. (The teacher asked students whether they had proved one pair of equal opposite sides such as DE=AF, some students (Linlin) answered positively (#176-180).) 181. Liuliu: (answered the teacher) Yes. I have proved it. Oh, no. But AD=EF is OK. (The teacher asked students to raise their hand if they had proved one pair of equal opposite sides. Many students raised their hands. The teacher encouraged them to prove another pair of equal opposite sides (#182-184).) 185. Liuliu: (thought shortly) Both are to use 60° to subtract EBA ($\angle$DBA-$\angle$EBA=$\angle$EBC-$\angle$EBA). 186.2. Beibei: Oh, yeh. It's same. **Extract 5.17. L3-Ex8, figure 5.9, #160-186.2.**		relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#160-186.2. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431) **#160-186.2. *Integration*** The teacher designs question that apply and extend the accumulated knowledge of the subject. (Hoffer, 1994, p.6)
L3-Ex8 (After providing students with about 8 minutes to explore the problem on their own, the teacher dynamically explained the figure.) 195. Lily: Look at the figure on the blackboard. Through drawing the figure, we generally understand the formation process of this figure. 196. Lily: Firstly, the given triangle is not a special triangle. This means that its three sides are not equal. You should not draw this triangle ABC as an isosceles triangle or an equilateral triangle. Otherwise, additional given, which is not of this proof, will be created by you. 197. Lily: Well. Based on the three sides of this triangle (ABC), three equilateral triangles will be drawn. (The teacher put marks to highlight the three sides of triangle ABC, omitted). Based on AC, triangle ACF is drawn. This is point F. Based on AB, triangle ABD is drawn. This is point D. Based on BC, triangle EBC is drawn. This is point E. Three points are D, E, F. I haven't yet linked them. (The teacher repeated #197.) 200. Lily: We should have drawn 39 triangles ABC. (There were 39 students in		**#195-206. *Information*** The teacher discusses materials clarifying this content, placing them at the child's disposal. ...and provides information to bring students to purposeful action and perception. (Clements and Battista, 1992, p.431) **#195-206. *Familiarization*** The teacher introduces problems which help in the discovery process. The teacher "sets the stage" for upcoming topics by introducing questions that incite curiosity. ... The teacher has students use visual cues ... (Hoffer, 1994, p.6) **#195-206. *Guided orientation*** The teachers' role is to direct students' activity

the class.) 201. Liuliu: 39??? (It appeared that Liuliu did not understand why there were 39 triangles ABC.) 202. Lily: Unless coincidence, triangle ABC drew by two of you may be congruent. But this chance is really small. 204. Lily: So points D, E, F are three dynamic points, right? Different ABC would produce different points D, E, F. 205 Liuliu: Much easier than last problem. 206. Lily: Well. Actually, this problem is about quadrilateral DAFE produced by the moment of a dynamic movement of points D, A, F and E. Now, to prove it's a parallelogram. The teacher highlighted the use of rotatable relation of equilateral triangles to prove congruent triangles in Ex 7 (see figure 5.13(3)) (#210, 216) and in this problem (see figure 5.14) (#223, 231). Some students were only able to recognise part of the rotatable relation of equilateral triangles and then found one pair of congruent triangles (either they could prove DA=EF, or DE=AF) (#221, 222, 232), they could not immediately fully recognise prove both DA=EF and DE=AF (#229, 235). **Extract 5.18. L3-Ex8, figure 5.9, #195-206.**		by guiding them in appropriate explorations ... (Clements and Battista, 1992, p.431)

Table 5.2. An analysis of levels of thinking and teaching phases of L3.

5.2.2 Students' learning results in their homework

One proof problem, A1, in the exercises textbook A (Mathematics A, 1996, p.28), is shown in figure 5.10.

2. 如图，BF、BE 分别是 $\angle ABC$ 与它的邻补角的平分线，$AE \perp BE$，点 E 是垂足，$AF \perp BF$，点 F 是垂足，求证：四边形 $AEBF$ 是矩形.

Given: (see figure 5.10), BF, BE are respectively bisector of angle ABC and its supplementary angle. AE ⊥BE at E, AF⊥BF at F.
Prove: Quadrilateral AEBF is a rectangle. (translated by Liping Ding)

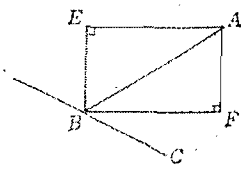


Figure 5.10. Proof problem A1.

21 out of 38 students correctly made the proof during their homework (see one example in figure 5.11). Another 17 out of 38 students could make the general structure of the proof. Interestingly, among the 17 students, 6 students including Liuliu only proved one right angle and then claimed the rectangle (see figure 5.12 and 5.13). For these students, angles E and F might be already seen as right angles in the given figure (see figure 5.10), therefore they did not prove them.

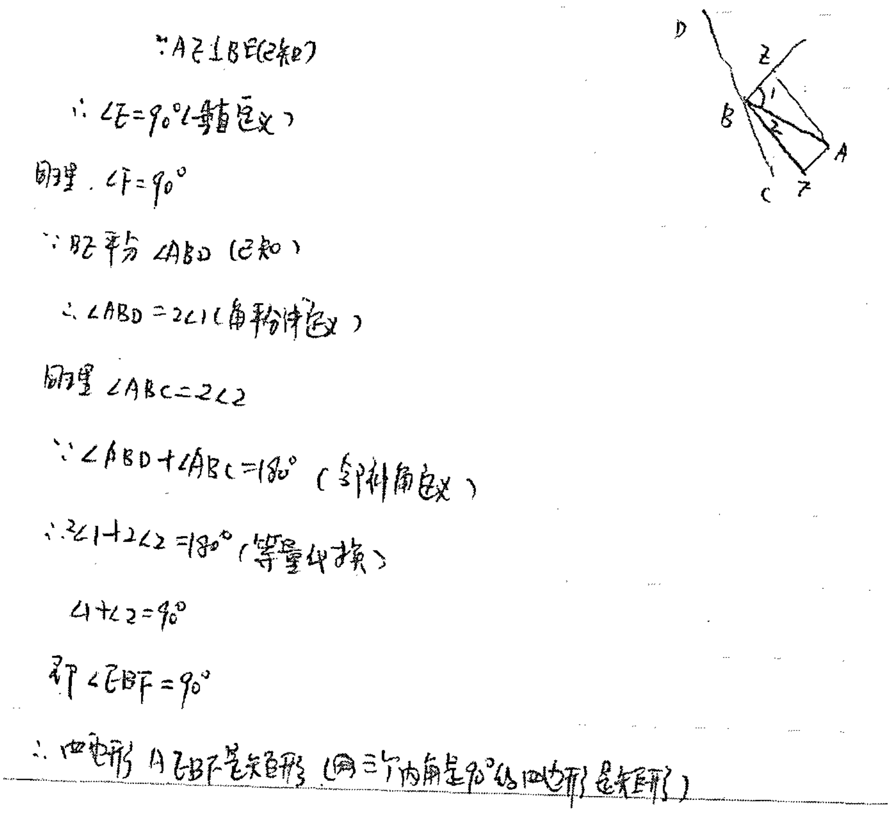


Figure 5.11. An example of the correct result of students' homework on proof problem A1.

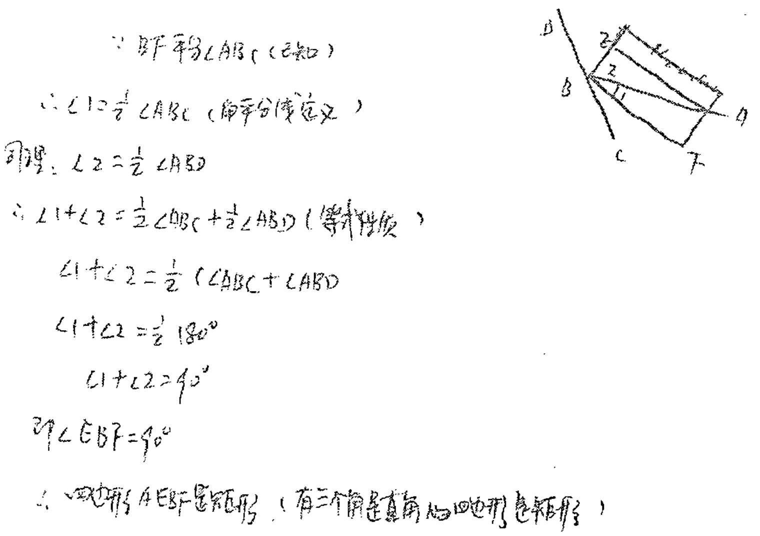


Figure 5.12. An example of proving only one right angle on proof problem A1.

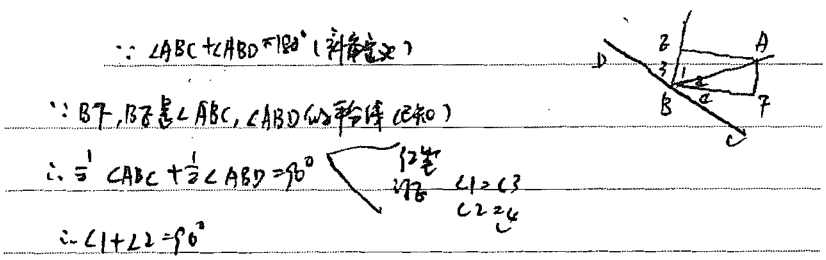


Figure 5.13. Liuliu's homework on proof problem A1.

Students' thinking shown in figure 5.11, 5.12 and 5.13 was identified at Level 3, "The student recognises the role of deductive argument and approaches problems in a deductive manner" (Fuys *et al.*, 1988, p.68). In fact, these students' thinking might be a sign of their thinking to reach Level 4. "The student proves in an axiomatic setting relationships that were explained informally on level 2 (Level 3). (Fuys *et al.*, 1988, p.69).

5.2.2 The interviews with Lily

According to the interview with Lily in the pilot study, Lily considered the following mathematical thoughts which should be emphasised in geometry teaching at Grade 8.

> "Mathematical thoughts such as transformation, classification, motion, and combination of figure and number are very important. Actually, such thoughts should be highlighted from grade 6. These thoughts will be useful for students in their future life, though mathematics might not be used in everyday. Teacher must know very much the mathematical thoughts hidden in the problems." (Lily, interviewed on 24th December, 2005, translated by Liping Ding.)

Extract 5.19. Interview with Lily

In terms of difficulty of geometry teaching and learning at Grade 8, Lily considered that

> "It is particularly difficult to teach how to add an auxiliary line in proving. Students also have great difficulties in understanding so. It is difficult to analyse a complicated figure to students as well. The way to solve a problem is sometimes quite difficult to explain. In deductive geometry, writing proof is very abstract, so it is difficult for students to learn. I think that it is more difficult to train students to observe and analyse the figure than to prove. The skill to make deductive reasoning could be trained gradually." (Lily, interviewed on 24th December, 2005, translated by Liping Ding).

Extract 5.20. Interview with Lily

Lily viewed the van Hiele five phases as follows:

> "Teaching new knowledge should have these five phases. When teaching problem solving, the lesson must focus on how to use knowledge. Then the problems are varied according to a certain problem. Problem variation is aimed to explore a certain problem. When analysing the teaching process of individual problems, we could understand the teachers' instruction in class. For instance, why does the teacher set this problem? Which problem is this problem varied from? What aim is expected to achieve? Except from the points of knowledge, what does the teacher tend to sort out of the skills in solving problems? What mathematical thinking does the teacher tend to develop? In mathematics lesson, there is another type of lessons, which focus on how to introduce new knowledge. In those lessons, the teaching process is a discovery process. But you just observed lessons of teaching proof problem solving in class. (Lily, interviewed on 27th December, 2006, translated by Liping Ding.)

Extract 5.21. Interview with Lily

In terms of linking Lily's classroom instruction to the van Hiele phases such as "Free orientation" and "Integration", Lily explained that

> "Geometry is different from other subjects. It is very difficult for students to learn. It is easy to provide students time to discuss the problem in other subject, particularly when you ask a question which does not need to think in depth, or the context of the problem, I mean the information is known and is easy to discuss. But I think in mathematics, students should first learn how to think. For example, I could provide them a question to discuss, but they wouldn't know what and how to discuss. So, when a problem is provided, students need to think on their own for a while. I could then guide them to discover the analytic path for the problem. Finally, they could in turn prove the problem by the analytic path. According to the given, what should turn to prove first. So, here, students must not discuss immediately to each other, but learn to think on their own. In the class, there are some students who are able to present the solving strategies of the problem, while others could not. In teaching geometrical proof problem solving, at best, those good students could play a role to lead other students to learn. So when I ask students questions, I actually have this purpose. So, you see, when a new theorem is taught, I could encourage them to discuss to each other how they deduce the new theorem. But, now, it is grade 8, we are mainly dealing with proof problems which have very strong logic. What do you want them to discuss on their own? If I distribute different problems to different students for presenting their thoughts in the class, I could enhance the efficiency of classroom instruction. When teaching a new lesson, you could provide students more free time to discuss, to explore the problem. For instance, when teaching the "GouGu" (Pythagoras) theorem, I could encourage them to do experiments such as to fit figures together for the discovery. Students were very interested in the exploratory process. But now, I already started to teach how to use theorems to solve problems. The logic is very strong. So it is the problem to organize the discussion, I meant to consider who might be able to present the logical thought of the problem in the class." (Lily, interviewed on 27th December, 2006, translated by Liping Ding.)

Extract 5.22. Interview with Lily

Lily viewed the statement of "the teacher needs to lead students in learning proof" as follows:

> "You mean the teacher guides students in learning proof? Yes, I think that students need to be carefully guided in learning geometrical proof. Students need to learn to write formal proof and learn to present their thoughts of the proof. If s/he does not fully present the proof, the teacher could help correct her/is presentation. When students write proof, they need to use mathematical language to illustrate the truth of a problem. So I emphasise the analytic path for proof, because students wouldn't know where to start when writing a proof. At the beginning of teaching geometrical proof, it is very essential to guide students. Direct instruction is very useful. If you do not give direct instruction, but let them to freely write the proof, they might lose the direction to go. Teacher need to teach students the way s/he think about a mathematical problem and the way s/he solve the problem. For example, in mathematics, we must teach students how to analyse a problem. Actually, the methods s/he learned in mathematics would be very useful for solving any other problems in their daily life." (Lily, interviewed on 27th December, 2006, translated by Liping Ding.)

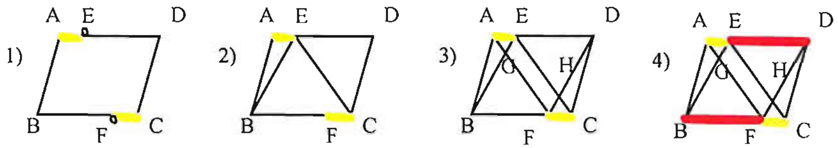
Extract 5.23. Interview with Lily

5.3 The case of teacher Spring

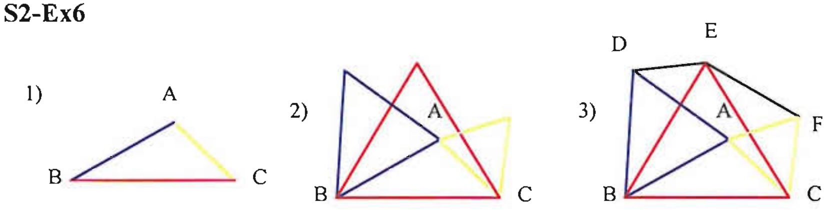
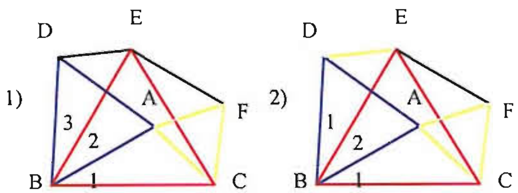
5.3.1 An analysis of Spring's lessons

This section presents the analysis of teaching episodes of two geometric proof problems solving (Ex4, Ex6, about proving a parallelogram) which took place in the second observed lesson (S2) in Spring's class. The explanation for selecting examples is shown in section 4.4.2(2). The details of the lesson structure and content can be seen in figure S2 in Appendix B.

The analysis presented in table 5.3, together with the analyses of the teaching of teachers Lily and Nana, is subject to cross-case analysis in section 5.5.

Transcripts (S2)	Levels of thinking	Teaching phases
S2-Ex4  **Figure 5.14. S2-Ex4.** (The teacher first drew parallelogram ABCD and used yellow colour to highlight AE and CF (#23). See figure 5.14(1) and photo 1 in Appendix G. Students were encouraged to draw their own figure on the exercises book according to the guidance of the teacher (#24).) 25. Spring: Next, who? What do you see? (see figure 5.14(1)) 26. Girl 12: (observed the figure on the blackboard.) Parallelogram ABCD. 27. Spring: Any more? 28. Girl 12: AE=CF. (The teacher then guided students to draw AE=CF, and link BE, EC (see figure 5.14(2)) and DF, AF (see figure 5.14(3)). She encouraged students to make a conjecture about quadrilateral EGFH (#29). Many students guessed 'a parallelogram' (#30).) 31. Spring: (heard students' discussion in the class.) Oh, you think this (GFHE) is a parallelogram? 31.1 One student: It does not look like a parallelogram. 31.2 some students: It is a parallelogram. (The teacher encouraged students to prove parallelogram EGFH (#31.3). Students were in discussion in the class. (#32)) 33. Spring: (asked students in the class.) What do you think? 33. Some students and Junjun: Use the definition. 34. Spring: To prove parallelogram, which method do you use? (Students were in discussion about the method to prove the parallelogram. (#35)) 37. Spring: (heard students' discussion in the class.) I see, some of you may consider proving congruent triangles. These two triangles (BGF, EHD) look like congruent triangles, right? If they are congruent, corresponding sides (GF, EH) are equal, right? 37.1 Some students: Prove congruent triangles. 38. Some students: I could prove they are congruent triangles.	**#26, 28, 31.1, 47.1. (students)** ***Level 1*** The student identifies instances of a shape by its appearance as a whole. (Fuys *et al.*, 1988, p.58) **#47, 51, 53, 57 (Students)** ***Level 2*** The student a) interprets and uses verbal description of a figure in terms of its properties…(Fuys *et al.*, 1988, p.59) **#33, 37.1, 38, 61 (Jiajia, Junjun, and other students)** ***Level 3*** The student gives informal deductive arguments a) follows a deductive argument and can supply parts of the argument. (Fuys *et al.*, 1988, p.66)	**#23-31.2.** ***Familiarization*** The teacher has students use visual cues … (Hoffer, 1994, p.6) **#23-69.** ***Guided orientation*** The teachers' role is to direct students' activity by guiding them in appropriate explorations … (Clements and Battista, 1992, p.431) The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6)

39. Spring: (heard students' discussion in the class.) Oh, these two small triangles (AGE, FCH) also look like congruent triangles. If they are congruent, corresponding sides (GE, FH) are equal. 40. Spring: But, could you really prove they are congruent? You see, to prove congruent triangles, the givens are not enough here. Maybe it is not the most appropriate method here to use congruent triangles to prove parallelogram. See, is it easy to prove these two congruent triangles (AGE, FCH)? 41. Some students: It is not easy to prove (congruent triangles AGE, FCH). (More students said to use the definition of parallelogram to prove (#43-45).) 46. Spring: Right. Some students considered the definition of parallelogram. You see, here, what relation is the pair of yellow (AE, CF)? (see figure 5.14(3).) 47. Some students: Parallel. 47.1 Some students: Equal. 50. Spring: This means AFCE is a parallelogram. . So, what relation is another pair of sides (AF, EC)? 51. Students and Jiajia: Parallel. 52. Spring: All right. AD-AE, I use red colour here (The teacher used red colour to highlight ED). Which one should also be red? (see figure 5.14(4)) 53. Students and Junjun: BF. 56. Spring: So, what is this quadrilateral (BFDE) if BF is its (ED) opposite side? 57. Students and Jiajia: Parallelogram. 58. Spring: If it (BFDE) is a parallelogram, is this pair of sides (BE, DF) parallel? 59. Students: Oh, I see. 60. Spring: This pair of sides (BE, DF) is parallel, is GE//FH? So, what method do we use here? 61. Students and Jiajia: The definition. (The teacher guided students to establish the structure of the proof by presenting to them the formal proof writing (#63-69).) **Extract 5.24. S2-Ex4, figure 5.14, #23-69.**		

S2-Ex6  **Figure 5.15. S2-Ex6.** 100. Spring: Listen to me before you draw the figure, please. I firstly drew an obtuse triangle ABC. Then I used different colours (red, yellow and blue) to highlight three equilateral triangles. (see figure 5.15(1)) 101. Spring: Based on side AC, I draw an equilateral triangle. What does this mean? 102. Students: Three sides are equal. (The teacher then gradually used yellow, red and blue colours to draw equilateral triangles AFC, EBC, and EBA (#103-105). See figure 5.15(2). Students observed and discussed in the class when the teacher drew the figure (#104).) 106. Spring: Right. Now, if you draw the figure correctly, what will it be when linking D, E, F, and A? (see figure 5.15(3)) 107. Some students: A quadrilateral. (The teacher encouraged students to more carefully observe the figure (#108). (see photo 2 in Appendix G.) Some students said 'a parallelogram'. (#109) When the teacher encouraged students to prove parallelogram DAFE, many students like Jiajia and Junjun were very surprised and might feel very difficult by saying 'Wa…' (#110).) **Extract 5.25. S2-Ex6, figure 5.15 #100-110.**	**#102. (students)** ***Level 2*** The student interprets verbal or symbolic statements of rules and applies them. (Fuys *et al.*, 1988, p.61) **#107. (students)** ***Level 1*** The student identifies instances of a shape by its appearance as a whole. (Fuys *et al.*, 1988, p.58)	**#100-110. *Familiarization*** The teacher introduces problems which help in the discovery process. … The teacher has students use visual cues … (Hoffer, 1994, p.6) **#100-110. *Guided orientation*** The teachers' role is to direct students' activity by guiding them in appropriate explorations … (Clements and Battista, 1992, p.431) The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6)
S2-Ex6  **Figure 5.16. S2-Ex6.**	**#120, 122. (students)** ***Level 2*** The student interprets verbal or symbolic statements of rules and applies them. (Fuys *et al.*, 1988, p.61) **#124. (students)** ***Level 3*** The student gives informal deductive arguments a) follows a deductive	**#119-143. *Guided orientation*** The teacher steers the students' responses to the specific subject matter or discipline they are studying. (Hoffer, 1994, p.6)

(The teacher provided students with time to draw and think on their own on the problem on their exercise book (#111-118).) 119. Spring: There is so much information of an equilateral triangle. Lets see the largest one (EBC). Three sides are equal. OK. How much is 1+2? (The teacher put number 1 and 2 in the figure to present angles ABC, EBA. See figure 5.16(1)) 120. Students: 60°. 121. Spring: Similarly, in the blue triangle (DBA), how much is 2+3? (The teacher put number 3 in the figure to present angle DBE. See figure 5.16(1)) 122. Students: 60°. 123. Spring: 2+3=60°, 2+1=60°. 3 is ...? 124. Students: 1. (The teacher changed the number 3 to 1 to present angle DBE in the figure. See figure 5.16(2).) 125. Spring: Now, you see. In triangle ABC, there are blue, 1 and red. 126. Spring: Next, you see. The above triangle DBE. There is also blue, 1, and red. Are they congruent? ($\triangle$ABC, $\triangle$DBE) 127. Students: Oh, ... 128. Spring: Congruent triangles. In $\triangle$DBE, there is small side (DE), right? Because yellow (AC), then yellow (DE). They should be the same colour. Now it has been proved. 129. Some students and Junjun: All right. 130. Spring: So in this quadrilateral (DEFA), there is already a pair of equal sides (DE, AF). (Similarly, the teacher guided students to see the relation of DA and EF (#132-143). Finally, the teacher guided students to establish the structure of the proof by presenting to them the formal proof writing (#145-146). **Extract 5.26. S2-Ex6, figure 5.16, #119-143.**	argument and can supply parts of the argument. (Fuys *et al*., 1988, p.66)	

Table 5.3. An analysis of levels of thinking and teaching phases of S2.

5.3.2 Students' learning results in their homework and test paper

In total, 41 students in Spring's class submitted their homework on 19th May, 2006 (after Spring's fifth lesson, S5). Findings from their homework on proof problem A1 (see figure 5.10) show that 26 out of 41 students could correctly make the proof (see one example of these students' proof of this problem in figure 5.17).

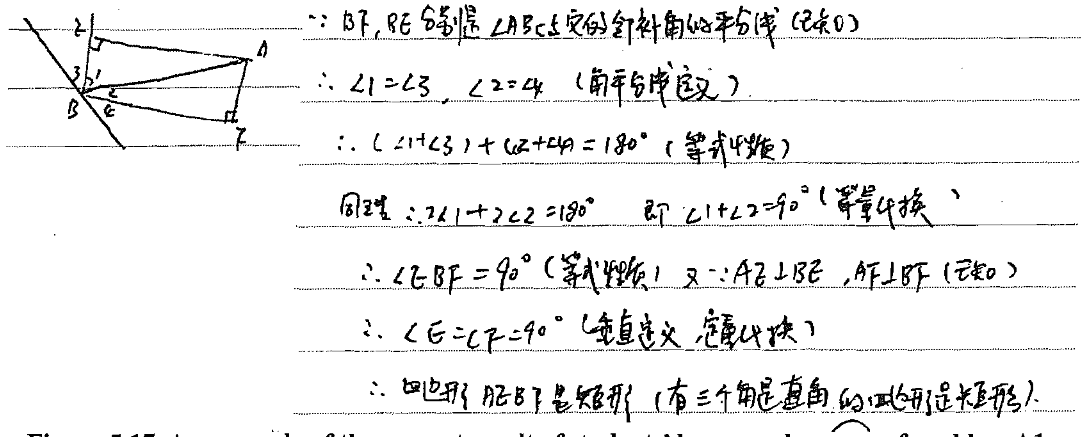


Figure 5.17. An example of the correct result of students' homework on proof problem A1.

Interestingly, according to the teacher's comment, some good students might choose quite complicated way to prove the problem (see one example of a good student's proof of this problem in figure 5.18).

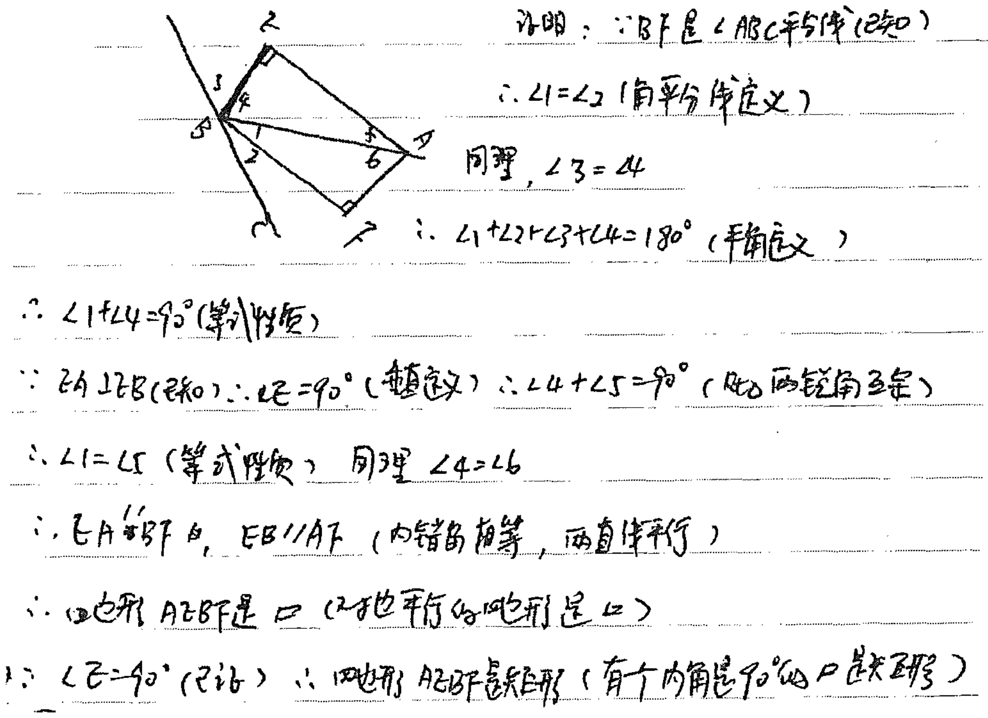


Figure 5.18. An example of a good student' proof of the proof problem A1.

Similar to that of Lily's class (see 5.12 and 5.13), 14 out of 41 students could make the general structure of the proof, but they only prove one right angle and then claimed the rectangle.

Students' thinking shown in figures 5.17 and 5.18 was identified at Level 3, "The student recognises the role of deductive argument and approaches problems in a deductive manner" (Fuys *et al.*, 1988, p.68). In fact, these students' thinking might also be a sign of their thinking to reach Level 4. "The student proves in an axiomatic setting relationships that were explained informally on level 2 (Level 3). (Fuys *et al.*, 1988, p.69).

5.3.3 The interviews with Spring

According to the interview with Spring in the pilot study, Spring identified that the main educational aim for teaching geometry at Grade 8 is

> "To establish mathematical knowledge foundation for preparing students to go to high school. The logical thinking needs to be greatly emphasised. Geometry is an ideal subject to cultivate students' spatial thinking and develop their creative ability." (Spring, interviewed on 26th December, 2005, translated by Liping Ding.).

Extract 5.27. Interview with Spring.

In terms of difficulty of geometry teaching and learning at Grade 8, Spring considered that

> "Students have great difficulty in writing the formal proof."(Spring, interviewed on 26th December, 2005, translated by Liping Ding.).

Extract 5.28. Interview with Spring.

Spring viewed the van Hiele five phases as follows:

> "There is a process from visual to abstract thinking, as described by the van Hieles. It is very essential to guide students at the beginning of this process." (Spring, interviewed on 27th December, 2006, translated by Liping Ding.).

Extract 5.29. Interview with Spring.

In terms of linking Spring's classroom instruction to the van Hiele phases such as "Free orientation" and "Integration", Spring explained that

> "The van Hiele first phases were completed in primary school. It is not because we do not have these phases. Students even in kindergarten started to draw figures and to fold pieces of papers. Geometry teaching and learning at Grade 8 are not mainly about experimental work. That's not our key instructional purpose. We started to teach geometrical proof at Grade 8. You see, there is a long process to prepare students to learn proof before Grade 8. When you are on the first floor, you might think it's really high to reach the 100th floor. But if you are already on the 99th floor, it is considerably easy to go to the 100th floor, right?" (Spring, interviewed on 27th December, 2006, translated by Liping Ding.).

Extract 5.30. Interview with Spring.

Spring viewed the statement of "teacher needs to lead students in learning proof" as follows:

> "For example, when teaches a new theorem, if you teach wrongly, students wouldn't follow you. Theorems are commonly agreed. It's not necessary and might be impossible to encourage students to spend the whole lesson to explore how the theorem was discovered. We need to give students direct instruction and guide them to accept knowledge. So we say that we need to lead students to learn proofs. But once students learned well of the theorems, they do not need the teachers' guidance to do homework. They might then be able to use the known theorems to explore and solve problems on their own." (Spring, interviewed on 27th December, 2006, translated by Liping Ding.).

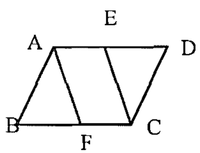
Extract 5.31. Interview with Spring.

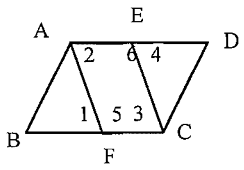
5.4 The case of teacher Nana

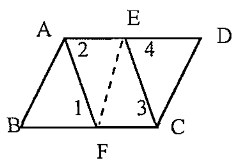
5.4.1 An analysis of Nana's lessons

This section presents the analysis of teaching episodes of one geometric proof problem solving (Ex1, about proving a parallelogram) which took place in the first observed lesson (N1) in Nana's class. The explanation for selecting examples is shown in section 4.4.2(2). The details of the lesson structure and content can be seen in figure N1 in Appendix C.

The analysis presented in table 5.4, together with the analysis of the teaching of teachers Lily and Spring, is subjected to cross-case analysis in section 5.5.

Transcripts (N1)	Levels of thinking	Teaching phases
N1-Ex1  **Figure 5.19. N1-Ex1.** (Nana firstly guided students to review each theorem they learned about proving a parallelogram in last lesson, and drew a figure of each theorem on the blackboard (see photo 1 in Appendix H). She then presented to students the "givens" and "proof" of the problem.) 11. Nana: The given, in parallelogram ABCD, E and F are respectively the middle point of AD and BC. Is quadrilateral AFCE a parallelogram? Could you prove it? (see figure 5.19) 12. Students: Yes. 13. Nana: How to prove it? (The teacher asked a boy student, Qiaoqiao, to present his thought of the proof. (#15)) 16-18. Qiaoqiao: Because quadrilateral ABCD is a parallelogram. AD=BC. (The teacher heard AD//BC (#19), students responded that they needed to prove AD//BC as well (#20.1-.2).) 23. Qiaoqiao: Because E, F are middle points of AD, BC. 25. Qiaoqiao: so AE=CF. (The teacher corrected students to prove AE=CF by demonstrating AE=1/2AD, CF=1/2BC (#26-32).) 35. Qiaoqiao: The replacement of equal elements. 36. Qiaoqiao: So quadrilateral AFCE is a parallelogram. (The teacher encouraged others to present their thoughts of the proof (#41).) 46. Haohao: I use the first theorem. 48. Haohao: Because E, F is parallelogram..., oh, no, they are the middle points. 50. Haohao: So AE=1/2AD, CF=1/2BC. 52. Haohao: So AE=FC. 54. Haohao: Because in triangle ..., no, because ABCD is a parallelogram, so AB=CD. 56. Haohao: $\angle B = \angle D$. 58. Haohao: BF=DE. 60. Haohao: Next, could prove congruent triangles. $\triangle ABF \cong \triangle CDE$. So,	**#23, 25, 35, 36 (Qiaoqiao)** ***Level 3*** The student recognizes the role of deductive argument and approaches problems in a deductive manner. (Fuys *et al.*, 1988, p.68) **#46, 48, 50, 52, 54, 56, 58, 60, 62. (Haohao)** ***Level 3*** The student recognizes the role of deductive argument and approaches problems in a deductive manner. (Fuys *et al.*, 1988, p.68)	**#11-62. *Guided orientation*** The teacher leads students in discussing the material in a narrow framework of topics. (Hoffer, 1994, p.6) **#11-62. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#11-62. *Explicitation*** The teacher's role is to bring the objects of study (geometric objects and ideas, relationships, patterns, and so on) to an explicit level of awareness by leading students' discussion of them in their own language. (Clements and Battista, 1992, p.431) **#11-62. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431) **#11-62. *Integration*** The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, ... (Clements and Battista, 1992, p.431)

AF=CE. 62. Haohao: Because AF=CE. AE=CF. So, AFCE is a parallelogram. **Extract 5.32. N1-Ex1, figure 5.19, #11-62.**		
N1-Ex1  **Figure 5.20. N1-Ex1.** (The teacher continuously encouraged other students to present their thoughts of the proof (#65).) 69. Taotao: I use the fourth theorem. 71. Taotao: Because AD//CB, so, ∠AFB=∠FA... (The teacher encouraged Taotao to go to the blackboard to put numbers on the figure, see figure N1.2. (#72).) 74. Taotao: (He first put number 1(angle AFB), 2 (angle FAE), 3 (angle ECF), 4 (angle DEC).) Because parallel (AD//BC), so ∠1=∠2, ∠3=∠4. Next, ... (He put number 5 (angle AFC), 6 (angle AEC). (#75) (see figure 5.20)) 76. Taotao: Because ∠5 is supplementary angle of ∠1. ∠6 is supplementary angle of ∠4. 80. Nana: Because ∠6+∠4=180°, ∠1+∠5=180°. But how do you get ∠5=∠6? 81. Taotao: Because congruent. (△ABF, △ECD) 82. Nana: Oh, you still use the congruent triangle proved by Haohao. (∠6+∠4=180°, ∠1+∠5=180°) So ∠5=∠6. But how to prove ∠2=∠3? 83. Taotao: Alternate interior angles are equal. (see #74)) The replacement of equal elements. (∠2=∠3) 84. Yinyin: Corresponding angles. (The teacher and some students are confused by the argument between Taotao and Yinyin (#85, 86). Taotao thought to prove congruent triangles (see #81) (#87, 89), but Yinyin thought to directly prove corresponding angles.) 92. Yinyin: Teacher, do not need to prove congruent triangles. Alternate interior angles, corresponding angles. Again, alternate interior angles, corresponding angles.	**#69, 71, 74, 76, 81, 83, 106 (Taotao)** **#84, 92, 95, 97 (Yinyin)** **#93, 99 (Bingbing)** **#96 (others)** ***Level 3*** The student recognizes the role of deductive argument and approaches problems in a deductive manner. (Fuys *et al.*, 1988, p.68)	**#69-106. *Guided orientation*** The teacher leads students in discussing the material in a narrow framework of topics. (Hoffer, 1994, p.6) **#69-106. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#69-106. *Explicitation*** The teacher's role is to bring the objects of study (geometric objects and ideas, relationships, patterns, and so on) to an explicit level of awareness by leading students' discussion of them in their own language. (Clements and Battista, 1992, p.431) **#69-106. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431) **#69-106. *Integration*** The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, ... (Clements and Battista, 1992,

93. Bingbing: How? Corresponding angles? (The teacher asked Yinyin which corresponding angles and alternate interior angles. (#94).) 95. Yinyin: Because ∠1=∠2, ∠3=∠4. And because ∠1=∠3, ∠2=∠4. 96. Some students: But why ∠1=∠3? You do not know parallel (AF,CE). 97. Yinyin: I just think so. Because ∠1=∠2, ∠2=∠4, ∠4=∠3. 99. Bingbing: Why? You do not know parallel (AF,CE). If you know, it is directly a parallelogram. (Students all laughed at Yinyin's thought in the class (#100).) (Because Taotao spoke very fast in #83, and the proof involved many angles, the teacher then asked Taotao to explain slowly his thoughts again of ∠2=∠3 to the class (#103).) 106. Taotao: (a bit impatient.) ∠1=∠2, right? ∠3=∠4, right? Congruent, right? ∠1=∠4, right? The replacement of equal elements, right? ∠2=∠3, right? (The whole class then understood Taotao's thoughts, and they laughed (#108).) **Extract 5.33. N1-Ex1, figure 5.20, #69-106.**		p.431)
N1-Ex1  **Figure 5.21. N1-Ex1.** (The teacher continuously encouraged other students to present their thoughts of the proof (#109).) 112. Tingting: Link EF. (see figure 5.21) (Bingbing argued not necessary to add the auxiliary line EF (#114).) 116. Tingting: Already proved ∠2=∠3. 118. Tingting: ∠AEF=∠EFC, ….next, the sum of angles of a triangle is 180°. … next, ∠AF… 119. Nana: ∠2=∠3, ∠AEF=∠EFC. Next, …? 120. Tingting: Next, prove ∠AFE=∠FEC. So, parallel. (AF, EC). 121. One boy: could prove two congruent triangles (probably means △AFE, △EFC)	**#112, 116, 118, 120 (Tingting)** **#114, 123, 126 (Bingbing)** ***Level 3*** The student recognizes the role of deductive argument and approaches problems in a deductive manner. (Fuys *et al.*, 1988, p.68)	**#112-159. *Guided orientation*** The teacher leads students in discussing the material in a narrow framework of topics. (Hoffer, 1994, p.6) **#112-159. *Verbalization*** The pupils attempt to explicitly verbalize the relations that they observe in the guided orientation phase as they learn to use correctly the technical language of the subject. (Hoffer, 1994, p.2) **#112-159. *Explicitation*** The teacher's role is to bring the objects of study (geometric objects and ideas, relationships, patterns, and so on) to an explicit level of awareness by leading students' discussion of them in their own language. (Clements and Battista, 1992,

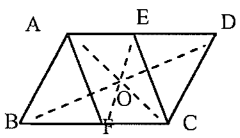
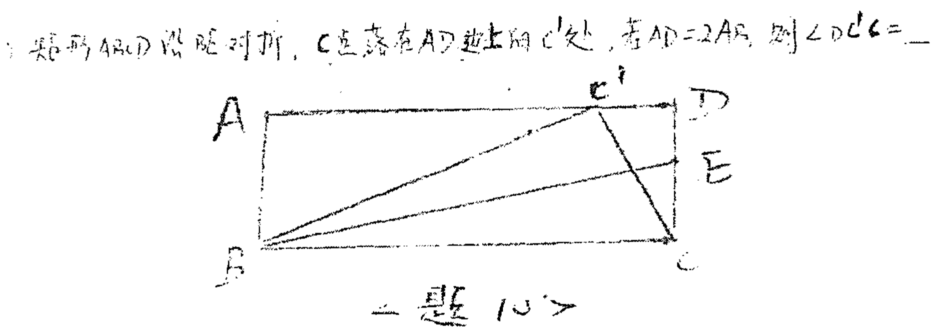
(Other students thought Tingting's method was not brief (#121.1). The teacher encouraged Bingbing to present his thoughts #122.) 123. Bingbing: Just change a bit of the last method (see #81 in extract 5.33). After congruent triangles ($\triangle$ABF, $\triangle$ECD), … (The teacher encouraged Bingbing to go to the blackboard and to show his thoughts to the class (#125).) 126. Bingbing: After proved congruent triangles, $\angle 2=\angle 4$ ($\angle 1=\angle 2$, see#74 in extract 5.33). So AF//CE. **Extract 5.34. N1-Ex1, figure 5.21, #112-126.**		p.431) **#112-159. *Free orientation*** Children solve problems whose solution requires the synthesis and utilization of those concepts and relations previously elaborated. They learn to orient themselves within the "network of relations" and to apply the relationships to solving problems. (Clements and Battista, 1992, p.431)
N1-Ex1  **Figure 5.22. N1-Ex1.** (Haohao raised his hand and wanted to try the third theorem (#130). The teacher then encouraged him to present his thoughts to the class (#133).) 134. Nana: So we must link AC, EF. The third theorem is about diagonals… 135. Haohao: Ah? Teacher, link BD. 136. Nana: Which one do you want to link? 136.1 Some students: Three. They all need to link. (see figure 5.22. Other students felt too complicated of the proof method (#138).) (The teacher then guided students to discuss why E, O, F were on the same line (#145). Students then suggest not first link AC (#151). Haohao then presented his thoughts of congruent triangles OBF, OED.) 153. Haohao: BF=DE, right? 155. Haohao: $\angle$BOF=$\angle$EOD. 157. Haohao: AD//BC, so $\angle$DEF=$\angle$BFE. 159. Haohao: I prove two congruent triangles ($\triangle$OBF,$\triangle$OED). EO=FO. **Extract 5.35. N1-Ex1, figure 5.22, #134-159.**	**#135, 153, 155, 157, 159 (Haohao)** **#136.1 (others)** ***Level 3*** The student recognizes the role of deductive argument and approaches problems in a deductive manner. (Fuys *et al.*, 1988, p.68)	**#112-159. *Integration*** The teacher's role is to encourage students to reflect on and consolidate their geometric knowledge, …. (Clements and Battista, 1992, p.431)

Table 5.4. An analysis of levels of thinking and teaching phases of N1.

5.4.2 Students' learning results in test paper

In total, 37 out of 38 students' learning results in the unit test paper were available for the analysis of this study. In item I10 (see figure 5.23), 11 students correctly made the answer, angle DC'C 75°. Interestingly, however, 18 students gave the same wrong answer, angle DC'C 60°. According to the teacher Nana's comment, there might be three facts which caused students to make the mistake: 1) the figure drawn on the test paper was not precise, AD looked over twice longer than AB; 2) students might have thought AD=2AB, AD=BC=BC'. They might deduce angle AC'B 30°. Because angle BC'C looked like 90° due to the imprecise figure, these students therefore might deduce angle DC'C 60°; 3) angle DC'C looked like 60° according to the imprecise figure. Moreover, some of these 18 students showed their correct deductive reasoning by writing proofs on other test items.



Rectangle ABCD is folded along BE, and C is then on AD at C'. If AD=2AB, then angle DC'C=__.
(translated by Liping Ding)

Figure 5.23. Item I10 of the unit test used in Nana's class.

5.4.3 The interviews with Nana

According to the interview with Nana in the pilot study, Nana concerned that the main educational aim for teaching geometry at Grade 8 is

> "Mainly to develop students' logical thinking ability" (Nana, interviewed on 15 December, 2005, translated by Liping Ding)

Extract 5.36. Interview with Nana.

In terms of difficulty of geometry teaching and learning at Grade 8, Nana identified that

> "It's very difficult for students to learn how to add an auxiliary line to prove. It's also too difficult for students to understand the dynamic nature of a static figure in problem solving." (Nana, interviewed on 15 December, 2005, translated by Liping Ding)

Extract 5.37. Interview with Nana.

When asked about the van Hiele five phases, Nana said that she did not know the van Hiele theory so that she did not have any thought about the theory.

In terms of instructional strategies used in Nana's lessons, Nana explained that

> "The class you observed is a good class and students are very clever. I feel that I should not use the approach of direct instruction to confine students' thinking. So most time of my lesson, I encourage them to first think on their own. Instead of my explanation, I encourage those students who are able to present their thoughts for solving the problems in class. We need to cultivate students' free thinking and creative ability."(Nana, interviewed on 29 December, 2006, translated by Liping Ding)

Extract 5.38. Interview with Nana.

Nana viewed the statement of "teacher needs to lead students in learning proof" as follows:

> "That's right. This is experience, the Chinese tradition. The teacher leads students in learning. For example, the analytic and synthetic methods in proof, students wouldn't be able to think about these mathematical methods on their own. Particularly, in geometrical proof problem solving, we need to teach students how to make deduction step by step to link the problem and the givens.
> The lessons you observed were quite simple. I was actually in the control of the class, though I provided opportunities for students to discuss their thoughts of the proof. I knew what methods they were likely to discuss. But if the problem needs the teacher's guidance, usually it is difficult problem. I actually wouldn't let them to discuss a difficult problem in the class. I will lead them to explore the difficult problem. See, I actually do not have so much confidence to let them freely discuss a difficult problem. I mean if students discuss some thought I do not expect, how could I control the class? Next, if I do not prepare for their thought, I must work together with them in the class and try to understand their thoughts then. It is too time consuming. Or some students may discuss impossible questions. If, say, I really spend a lot of time with a few students in the free discussion of the problem in the class, what will the majority of students do then? It would be too wasteful for most students. After all, only very few student has creative ability.
> However, it is said that "famous teacher produce excellent students". I believe so. Clever teacher does lead clever students in learning."(Nana, interviewed on 29 December, 2006, translated by Liping Ding.)

Extract 5.39. Interview with Nana.

5.5 Cross-case analysis

5.5.1 The levels of students' thinking

From the pool of 39 students in Lily's class across 12 lessons, responses of four students (Liuliu, Beibei, Linlin and Youyou) in two lessons (L2&3) were presented for interpreting the levels of students' thinking (table 5.2). Other students' responses were also taken into account in the analysis, particularly when Lily asked them to present their thoughts in the class or when the four focused students did not show any response to the teacher's instruction. From the pool of 41 students in Spring's class across eight lessons, some students' responses in Spring's one lesson (S2) were presented for interpreting the levels of students' thinking (table 5.3). Similarly, from the pool of 38 students in Nana's class across three lessons, some students' responses in Nana's one lesson (N1) were shown for interpreting the levels of students' thinking (table 5.4). Moreover, some learning results of the majority of students in the three teachers' classes during homework or test papers were also presented (sections 5.2.2, 5.3.2, 5.4.2). Overall, the samples were chosen for the variety of responses the students exhibited during the lessons observed or their homework and test papers, a variety that is representative of each individual teacher's class.

According to the types of codes of levels of thinking (see section 4.4.4.3), the students' responses on the teachers' instruction are summarised in table 5.5.

van Hiele levels	Individual proof problems in the three teachers' lessons												
	L2-PR2	L2-PR3	L2-Ex4	L2-Ex5	L2-Ex6	L2-Ex7	L3-Ex7	L3-Ex8		S2-Ex4	S2-Ex6		N1-Ex1
Level 1						✯	✯			●	●		
Level 2							✯			●	●		
Level 3	✯	♠ ◇ ♡ ♣ ✯	♠ ◇ ♣✯	♡◇ ✯	♠ ◇ ✯ ♡		♠ ◇♡♣ ✯ ✯	♠◇		●	●		○
Level 4													

Liuliu - ♠; Beibei - ◇; Youyou - ♣; Linlin - ♡; Some other student (not necessary the same student in Lily's class) - ✯; Some student (not necessary the same student in Spring's class) - ●; Some student (not necessary the same student in Nana's class) - ○.

Table 5.5. An overview of students' responses from Lily's, Spring's and Nana's lessons.

As shown in table 5.5, these students' responses to the teachers' instructions about solving different proof problems could link to van Hiele levels from Level 1 to Level 3 as follows: 1) linking to Level 1 thinking; 2) linking to Level 2 thinking; 3) linking to Level 3 thinking.

1. Linking to Level 1 thinking

In Lily's two lessons (L2&3), students' responses which are linked to Level 1 thinking were observed in L2-Ex7 and L3-Ex7. Some students showed that they observed the figure by its appearance. For instance, in L2-Ex7 (see extract 5.12 in table 5.1), the teacher understood the difficulty students had in the analysis of the figure, so she dynamically constructed the figure step-by-step and guided students to observe the figure (#424.1-436). The teacher guided students to compare two figures (#439), and students were able to observe that EF, AE, and ED were rubbed away in the newly drawn figure (see figure 5.7(5)). Under the guidance of the teacher, students were only able to generally perceive the part of the complex figure.

Some students perceived the figure by the given of the problem. In L3-Ex7 (see extract 5.13 in table 5.2), for instance, the teacher guided students to analyse the figure by the question 'In this figure, could you find what is not changed, when D and F are moving?" (#37). Some students responded "DC=BF" (#40). This response was based on the given of the figure.

In Spring's lessons, the teacher constantly encouraged students to first perceive the given of the figure and to make conjectures about the relation of the figure, before she provided the givens and problem to them. For instance, in S2-Ex4 (see extract 5.24 in table 5.3), the teacher asked "What do you see?" (#25). Students perceived the figure by the colour the teacher used (see #26, 28). While some students made conjectures about parallelogram EGFH (#31), one student perceived figure EGFH by saying "It does not look like a parallelogram." (#31.1). Moreover, in S2-Ex6 (see extract5.25 in table 5.3), some students made conjectures about figure DAFE by it appearance, "A quadrilateral." (#107).

2. Linking to Level 2 thinking

When students described some properties of figures by the definitions and theorems they knew, their responses are linked to Level 2 thinking. For instance, in L3-Ex7 (see extract 5.13 in table 5.2), when the teacher asked the relation of figure (#37), some students responded "AF=BD, because AB=BC" (#42). According to these students' answers, students started to perceive the property of equilateral triangle ABC.

In S2-Ex4 (see extract 5.24 in table 5.3), the teacher guided students to observe the relation of the figure. She asked "what relation is the pair of yellow (AE, CF)?" (#46). Some students responded "Parallel" (#47). For these students, they perceived the property of parallelogram ABCD. Moreover, Spring's guidance (#50, 52, 56) was likely to check students' understanding (#51, 53, 57) of some properties of the figure.

3. Linking to Level 3 thinking

Students' responses which are linked to Level 3 thinking are summarised as follows:

1) Some students could correctly write formal proof or establish the general proof structure during their homework (see figure 5.11, 5.12, 5.13, 5.17 and 5.18).
2) Some students could formally present their thoughts about the proof in class (see extract 5.7 in table 5.1; #46, 48, 50, 52, 54, 56, 58, 60, 62 in extract 5.32 in table 5.4).
3) Some students could order the relation of theorems and discuss their thoughts about the proof in a deductive manner (see students Liuliu's and Beibei's responses in extract 5.6 in table 5.1).
4) Some students could follow the teachers' guidance of an analytic path for the proof.
   a. some students knew a set of theorems to prove a parallelogram (see #93, 97, 99 in

extract 5.3 in table 5.1) ;

b. some students were able to choose an appropriate theorem to prove a simple figure (see #113, 133, 136, 138 in extract 5.3; #223, 225 in extract 5.5; #304, 307-309.1, 313 in extract 5.9 in table 5.1; #33 in 5.24 in table 5.3);

c. some students discovered hidden properties of figures by deduction (see #44, 45 in extract 5.13 in table 5.1; #72, 76 in extract 5.14 in table 5.2);

d. some students recognised corresponding figures in a complex figure and knew how to add an auxiliary line for providing a proof (see 115, 117, 121, 124, 132 in extract 5.16; #171, 174, 186.2 in extract 5.17 in table 5.2);

e. some students could support part of the analytic path for the proof, but they could not fully see the whole relation of a complex figure (see #100, 104, 105, 107 in extract 5.15; #132, 134, 138, 143 in extract 5.16 in table 5.2).

5) Some students made deductions by visual image of theorems (see #281 in extract 5.9 in table 5.1).

6) Some students saw the relation of figures but could not distinguish the difference of a theorem and its converse theorem (see #349, 362, 365 in extract 5.10 in table 5.1).

7) Some students could order the relations of figure, but the order was not based on the givens of the problem, but on the visual image of the figure (see #59.2, 69, 70 in extract 5.14 in table 5.2).

5.5.2 The phases of teachers' instruction

According to the types of codes of teaching phases (see section 4.4.4.*3)*), the teaching phases occurred in the three individual teacher's instruction are summarised in table 5.6.

van Hiele teaching phases	**Individual proof problems in the three teachers' lessons**												
	L2-PR2	**L2-PR3**	**L2-Ex4**	**L2-Ex5**	**L2-Ex6**	**L2-Ex7**	**L3-Ex7**	**L3-Ex8**		**S2-Ex4**	**S2-Ex6**		**N1-Ex1**
Information		*	**	*		*	***	*					
Familiarization		*	**	*		*	***	*		*	*		
Guided orientation	*	*	***	*	*	*	** **	**		*	**		***
Explicitation	**	*	**	*	*		** **	*					***
Verbalization	**	*	**	*	*		** **	*					***
Free orientation		*	**	*	*		**	*					***
Integration	**	*	** **	*	**		**	*					***

*** - occurred in the teaching process.**

Table 5.6. An overview of teaching phases in Lily's, Spring's and Nana's lessons.

Table 5.6 shows that some features of these three teachers' classroom instruction during the teaching process of each individual proof problem could link to the characteristics of the van Hiele's five phases.

1) Linking to Information/ Familiarization phase

When the teacher led students to review previous knowledge for preparing students for the new topic, it was considered at the *Information* phase. In Lily's lessons, for instance, before discussing the thoughts of a new proof problem, the teacher led students to review known theorems by asking a set of questions as follows: *"How many methods did we learn to ...?"* (#91 in extract 5.3), *"Which two?"* (#96 in extract 5.3), *"Why do you consider the second theorem?"* (#224 in extract 5.5).

During the *Information* phase, the teacher introduced teaching tasks and involved students' perception and interest into the proof problem solving process. In Lily's lessons, for instance, the teacher dynamically presented the drawing process of the figure (see extract 5.4 and 5.12) or interpreted to students the dynamic nature of a static figure (see extract 5.18). Before teaching the proof problem N1-Ex1, Nana guided students to review the theorems and drew students' attention to the critical attribution of the theorems by putting some marks on each single visual figure example (see extract 5.32).

The *Familiarization* phase was considered when the teacher mainly had students use visual cues while introducing a new proof problem. In Spring's lessons, for instance, the teacher encouraged students to perceive the given and then to make a conjecture about the figure at the beginning of the proof problem teaching process (see #23-31.1 in extract 5.24). When introducing a complex figure to students at this phase, Spring used colours to help students understand the given and problem of the task (see extract 5.25).

2) Linking to Guided Orientation phase

When the teacher guided students to explore the principal connections of new tasks, it was considered at the *Guided Orientation* phase. For instance, during this phase, Lily guided students to establish an analytic path for the proof by a set of questions: "*To prove ..., what methods did we learn ...?*", "*If to prove ..., what actually must I turn to prove here?*", "*If to prove ..., what is given?*", etc. (see extract 5.1 and 5.3).

During the *Guided Orientation* phase, the teacher led students to purposefully observe the variables and invariables of the figure and to learn the use of same theorems with different figures. For instance, from L2-Ex4 to L2-Ex6 (see extract 5.4, 5.9 and 5.10), Lily gradually changed the figure and increased the complexity of the figure, in order to deepen students' understanding of the use of known theorems of verifying a parallelogram.

When posing a complex proof problem, the teacher guided students to discover the hidden properties of the figure and to establish links between the given and the problem. For instance, in L2-Ex7 (see extract 5.12), Lily dynamically drew again the figure for guiding students to discuss the property of a basic figure hidden in the complex figure.

During the *Guided Orientation* phase, the teacher guided students to present their thoughts on the formal proof writing. To help students develop proof writing ability, Lily frequently asked a set of questions like *"what should you write here?", "what reason could you write here?", "what method do you consider?", "what should we prove?", "what method do you use?"*, etc. (see extract 5.7 and 5.9). In Spring's observed lessons, the teacher constantly guided students to first discuss their thoughts on the proof, and then students learned to write down the formal proof writing by the teacher's guidance (see extract 5.24). Nana also guided students to present their thoughts on the proof in class by using the questioning strategy, such as *"Could you prove it?", "How to prove it?", and "Any more method to prove it?"*, etc. (see extract 5.32 to 5.35).

3) Linking to Explicitation/Verbalization phase

During the *Explicitation* phase, students became aware of the relations of the problem and learned to use mathematical language to present their thoughts. For instance, while Lily guided students to establish an analytic path for the proof, students actively followed the teacher's guidance in a deductive manner (see extract 5.1 and 5.3). In L3-Ex7 (see extract 5.13-16), students received insight of the hidden relations of the complex figure through the teacher' guidance and encouragement.

During this phase, the teacher largely emphasised the correct use of mathematical language to present a formal theorem, particularly after taught a new theorem (see extract 5.2). Students were also encouraged to openly present their thoughts on the proof in class, though their thoughts might be corrected by other students or the teacher (see extract 5.6, 5.7, 5.9 and extract 5.32-35)

4) Linking to Free Orientation phase

Free Orientation phase was suggested when students freely discussed their different ways for solving the proof problems. Some of these open discussions were guided by the teacher (see extract 5.3, 5.9, 5.10 and 5.16), while others happened during the class discussion (see extract 5.6 and 5.32-35).

5) Linking to Integration phase

When the teacher asked students questions to assess their understanding of previous knowledge, it was linked to the *Integration* phase (see #35-49 in extract 5.1). Moreover, at this phase, the teacher guided students to link new knowledge just learned to previous knowledge. For instance, the teacher Lily asked students *"Is the new theorem about ..., or about ...?"* (see extract 5.2), *"How many methods did we learn ...?"* (see extract 5.3), "*What method could we also use, if I want to prove ...?*" (see extract 5.8), etc.

During this phase, the teacher led students to establish an overview of what they had learned of the theorems of a certain figure, or different strategies to solve a proof problem (see extract 5.11 and 5.16).

In addition, when the teacher designed questions to help students to apply and to extend the accumulated knowledge for solving proof problems, it was also considered at *Integration* phase (see extract 5.8, 5.17 and extract 5.32-35).

CHAPTER 6. LINKING FINDINGS TO THE VAN HIELE MODEL – A DISCUSSION

This chapter presents the "pattern match" results based on the analysis outcomes across the three cases according to the van Hiele model, as presented in chapter 5. The chapter is divided into three parts. The first part (section 6.1) presents the findings of assigning observation data and small-scale survey data to the van Hiele thinking levels. An elaboration of the van Hiele thinking theory is given. The second part (section 6.2) presents the findings of assigning observation data to the van Hiele phases. The problems and difficulty raised from the analysis are further discussed. The final part (section 6.3) summarises the key findings highlighted in the previous two sections, and demonstrates the need for developing an explanation of the data emerging from the analysis of this study.

6.1 The van Hiele levels

6.1.1 Assignment of levels

Previous research studies have drawn some conclusions relevant to this study. Fuys *et al.* (1988) and Mayberry (1983) found that the van Hiele levels appear to be hierarchical in nature. Usiskin (1982) found that individual students can be assigned a van Hiele level but that students in transition from one level to the next are difficult to classify reliably. Burger and Shaughnessy (1986) noted a number of imprecise visual qualities that some students used in describing and reasoning the shapes. Similarly, studies by Clements and Battista (2001), Lehrer et al. (1998), Gutiérrez and Jaime (1998) questioned the discrete feature of the van Hiele levels.

In this study, students' responses to each individual proof problem in the observed lessons and their learning results during homework and test paper were first assigned to van Hiele levels of thinking by the researcher. The initial analysis of results were then discussed with members of a research team at research meetings. As this study focuses on analysing students' responses during the teaching process of geometrical proof problems, students' responses assigned to Level 1 were observed mainly when the teacher Lily provided complex figures (see L2-Ex7, #439 in extract 5.12 and L2-Ex7, #40 in extract 5.13), and when the teacher Spring asked questions which encouraged students to perceive the figure by colours or its appearance (see S2-Ex4, #26, 28, 31.1 in extract 5.24).

Students' responses at Level 2 thinking were mainly observed in Sping's lessons, as the teachers' questions focused on checking students' understanding of individual theorems (#51, 53, 57 in extract 5.24).

Students' responses at Level 3 thinking were observed in all the three teachers' lessons, most likely due to the lesson topics of proof problem solving. On the proof problem N1-Ex1 in Nana's lesson, the researcher noted that students used a set of theorems to prove formally parallelogram AFCE (see extract 5.32, 5.33, 5.34 and 5.35). On the problem L2-Ex4 in Lily's lesson, one student formally presented her thought of the proof (see extract 5.7). On the problem L2-Ex5, students followed Lily's questions by presenting the main theorems and definition to make the proof (see #304, 313 in extract 5.9). Moreover, students' learning results during their homework showed that some students could correctly make the proof writing or generally establish the proof structure (see figure 5.11, 5.12, 5.13, 5.17 and 5.18). These students' thinking was eventually assigned to Level 3 rather than Level 4 after a considerable amount of discussions at the research meetings. This decision was made mainly due to the lack of evidence from the study to clearly show that these students grasped the significance of deduction as a means of constructing and developing all geometric theory, or that they understood the role and the essence of axioms, definitions, and theorems.

When students could follow the teachers' instruction for establishing an analytic path for the proof or the general structure of the proof, their responses were considered at Level 3. Similarly, when students discussed their thoughts of the proof in a deductive manner, like Liuliu and Beibei did in extract 5.6, their thoughts were assigned to Level 3.

Thus, the hierarchical nature of the levels, noted by Fuys *et al.* and Mayberry, were confirmed in this study. However, as observed by Usiskin, the researcher also encountered the difficulty of assigning some students' responses to a certain level, as these students were likely to be in the transition between levels. That is, some disagreement occurred in the research team, particularly assigning some students' responses between Levels 2 and 3.

While analysing students' responses on different proof problems, the researcher detected that some students reasoned deductively based on the visual image of the theorem or of the figure. Linlin in L2-Ex5 (see #281 in extract 5.9), for example, used the theorem to prove

AE=FC. For Linlin, AE and CF looked like vertical lines to AD and BC, even though it was not the given. The researcher decided to assign Linlin's thinking to Level 3 thinking as his visual thinking was based on the visual image of the theorem, not merely on the appearance of the figure. Students' similar reasoning was also found in Nana's class (see Yinyin's responses in extract 5.33).

On the problem L3-Ex7 (see extract 5.14), some students could discover angle AOF 60° by deduction. While some students, like Liuliu and Linlin, tried to connect logically the finding to other properties of the figure, Beibei could not understand the logical relation of the figure suggested by Liuliu. These students' responses were assigned to Level 3 as their intention was the exploration of the logical relation of figure.

Overall, it was noted that when solving a simple proof problem, students' responses were generally assigned to Level 3. However, when solving a complex proof problem, students needed the teachers' guidance to establish an analytic path for the proof which usually started from perceiving and analysing individual properties of the figure. Noticeably, when the figure became complex, some students observed the figure as a whole by its appearance (see extract 5.12).

According to the interview with Lily (see extract 5.19 and 5.20), the teacher considered that it was essential to develop students' mathematical thinking by teaching proof in geometry at Grade 8, such as transformation, classification, and motion, etc. She thought that it was very difficult to help students to understand how to add an auxiliary line in proof. In particular, Lily considered that "it is more difficult to train students to observe and analyse the figure than to make a proof. The skill to do deductive reasoning could be trained gradually." In Lily's lessons, it was noted that the teacher largely concentrated on guiding students to establish an analytical path for the proof problem (see extract 5.1, 5.3 and 5.10). It is noted that Lily posed some considerably difficult problems for developing students' visual and analytic thinking of the construction of the proof problem, together with the complex figure (see extract 5.12, 5.13-16, and 5.18).

According to the interview with Spring (see extract 5.27 and 5.28), the teacher considered that it was important to develop students' deductive reasoning and spatial thinking as well as creative thinking by teaching proof in geometry at Grade 8. She thought that it was very difficult to teach students to write formal proofs. In Spring's lessons, the teacher constantly encouraged students to perceive the figure she drew on the blackboard and used questions to check students' understanding of the use of theorems. Noticeably, Spring spent a considerable amount of time in guiding students to write formal proofs (see extract 5.24 and 5.26).

According to the interview with Nana (see extract 5.36 and 5.37), the teacher thought that teaching proof in geometry at Grade 8 was mainly to develop students' logical thinking. Nana considered that it was very difficult to help students to understand how to add an auxiliary line in proof and to solve proof problems which involved the motion of the figure. In Nana's lessons, the teacher strictly followed the school curriculum and textbook, and provided students with time to freely make deductive reasoning about considerably simple proof problems (compared with those problems posed by Lily and Spring in their lessons.).

In summary, the analysis of observation data and teacher interview data indicates that students' thinking development appears to depend on the teachers' didactical thoughts of the subject and their actual instruction in the class.

6.1.2 Interpretation of levels

During the course of the study, several features of the levels emerged that the researcher was not aware of initially. First, although the van Hieles have theorised that the levels are a discrete structure, this study did not detect that feature. The analysis of students' learning results on test items (see figure 5.23) confirms findings by Shaughnessy and Burger (1985, p.423) that "If conflict occurred between the visual and the analytic levels of reasoning (level 1 and 2 in this study), the visual usually won." Moreover, the difficulties that the researcher and her colleagues had in deciding between levels while making level assignments can be considered as evidence questioning the discrete nature of the van Hiele levels.

Second, evidence from observed lessons, students' homework and test papers indicate that a considerable number of students at Grade 8 appeared to be able to make formal deductive reasoning on relatively simple proof steps (see figures 5.11-13, 5.17-18). However, findings from Lily's lessons (see extract 5.12-16) indicate that students were not able to do proofs on complex tasks by themselves, but required teachers' guidance on the visual and analytic analysis of the figure. Thus, the levels appear to be dynamic rather than static and of a more continuous nature than their discrete descriptions would lead one to believe. Data from Burger and Shaughnessy (1986) particularly support the fact that students may move back and forth between levels quite a few times while they are in transition from Levels 2 and 3 (Levels 1 and 2 in their study). Data from this study particularly support this phenomenon of the transition.

Last, the analysis of students' responses during the teaching process in this study suggests that students' geometric thinking in solving proof problems is likely to involve multiple mental operations. That is, students' geometric thinking at a higher level appears to be an extremely complex process involving the concurrent development of visual, analytic and deductive thinking. As discussed by Gutiérrez *et al.* (1991, 1998), Lehrer *et al.* (1998) and Clements and Battista (2001), students' geometric thinking development is not likely to follow a simple, linear model as ascribed by the van Hieles. In terms of visual thinking development, Clements and Battista (2001, p.131) state that

> "Imagery has a number of psychological layers, from more primitive to more sophisticated (each of which connect to a different level of geometric thinking) that play different (but always crucial) roles in thinking, depending on which layer is activated. Thus, even at the highest levels, geometric relationships are intertwined with images, though these may be abstract images."

During the analysis of data in this study, the researcher detected that students' visual thinking development appeared to be much more complex than that ascribed by the levels. During the teaching process of proof problem solving, for instance, some students seemed to possess certain imagistic prototypes of theorems, without distinguishing the critical attributes of the theorems. The "prototype phenomenon" has been highly analysed by Hershkowitz (1989). As discussed by Clements and Battista (2001, p.127), "these prototypes are not absolutely rigid, but they have constraints." The analysis of the data in this study shows that some students used deductive reasoning based on such imagistic prototypes of the theorems (see Linlin's response (#281) in extract 5.9, and Yinyin's responses (#84, 92, 95, 97) in extract 5.33).

Moreover, extracts 5.13, 5.16, 5.17 and 5.18 show that the teacher guided students to discover the hidden basic figures in the complex figures. These basic figures could be considered as abstract images of the network of a certain number of theorems. To recognise these abstract images hidden in the complex figure, students' analytic reasoning and deductive reasoning needed to be firstly developed.

Under the guidance by the teacher, some students demonstrated the growth of their visual thinking from observing the figure by its appearance to recognising certain properties of the figure by making deductive reasoning. For instance, in extract 6.1, Liuliu showed such visual thinking development (see #177, 178, 181.1, 198).

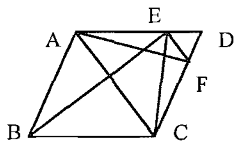


Figure 6.1. L1-Ex3.

176. Lily: If this is a parallelogram ABCD. AC is a diagonal. Here, a parallel line. (EF//AC) (see figure 6.1)
177. Liuliu: Teacher, if you link BF, there is a pentagon. (ABCFE)
177.1 Lily: E, F are dynamic points. EF is a dynamic line segment. But it is always parallel to AC.
178. Liuliu: Slope. (probably meant EF//AC)
179. Lily: Now, please find as many triangles as possible whose area is equal to that of △ABE.
180. Linlin: AEC.
181. Youyou: CAE.
181.1 Liuliu: ECA.
(The teacher encouraged students to explore further of the figure (#182-190).)
187.2 A boy: AEC. And AFC.
197. Linlin: Yes, CAF.
198. Liuliu: CAF? ...Oh, yes.
202. Youyou and Beibei: Why CAF?

Extract 6.1. L1-Ex3, L1, figure 6.1, #176-178.

Moreover, extract 6.1 shows that while Liuliu gradually recognised triangles AEC and AFC which were equal to the area of triangle ABE, Beibei and Youyou could not understand why the area of triangle ABE was equal to that of AFC. This illustrates a complex process of the growth of visual thinking which involves the development of deductive reasoning. Thus, students' visual thinking on geometric proof tasks is not likely to merely rely on empirical work as suggested by the van Hieles, but also rely on their understanding of the abstract interrelation of formal theorems and definitions (other examples, see #50 in extract 5.13, and #72 in extract 5.14).

Thus, the development of visual, analytic and deductive thinking in solving geometric proof problems is extremely complex. On the one hand, the visual, analytic and deductive thinking may concurrently grow up together; on the other hand, however, they may limit each other's development. For instance, some students, like Beibei in Lily's lessons, could perceive the visual geometric structure of the proof task and then use formal deductive reasoning on some tasks (see extract 5.3 and 5.6). However, Beibei who had shown her ability for deductive reasoning on some tasks could not do some other proofs, as she was not able to perceive certain visual geometric figures hidden in those complex figures (see #63 in extract 5.14 and #202 in extract 6.1). In contrast, some students, like Linlin in Lily's lesson, might wrongly use deductive reasoning of the visual image of a theorem on some proof tasks (see #281 in extract 5.9). However, Linlin could also visualise the hidden property in a complex figure and provided insightful strategies to solve some very difficult proof tasks (see extract 7.14 in the next chapter). Such evidence particularly substantiates the observations by Burger and Shaughnessy (1986), Fuys *et al.* (1988), Hershkowitz (1989), Mayberry (1983) that students' thinking behaviours changed from one proof problem to the other.

As pointed out by Pegg and Davey (1998, p.115),

> "... the van Hiele model represents "a psychology of learning" (van Hiele, 1986), the underlying purpose of which is to see the role of instruction as the development of insight in students. The model represents a broad unidimensional approach to learning and does not take into account how individuals may proceed, other than by their rate of progress. Van Hiele did not endeavor, as have later investigators, to describe the sophisticated and varying intellectual competencies that students exhibit in their development."

In summary, as a result of the analysis of students' responses in the classes and their learning outcomes in the homework and test papers, this study proposes a dynamic view of the van Hiele levels of geometric thinking, as proposed in figure 6.2. As figure 6.2 indicates, students' geometric thinking development in geometric proof problem solving might globally link to the van Hiele Level 3 and the transition toward Level 4. However, when the problem becomes difficult, students' thinking may go back to the analysis of the geometric figure and properties at Levels 1 and 2. Thus, this study claims that students' geometric thinking development is extremely complex in solving geometric proof problems, as such development appears to involve the simultaneous development of a variety of kinds of thinking, such as visual, analytic and deductive thinking across levels. It is fundamental for this study to further examine how the teachers used instructional strategies and approaches to involve and facilitate these kinds of thinking through solving proof problems.

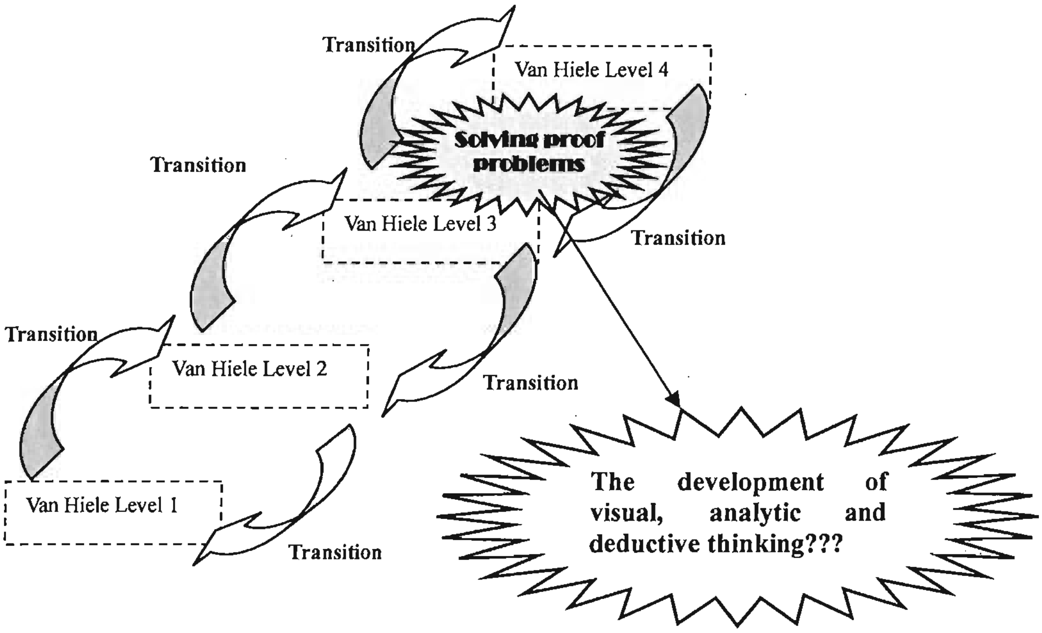


Figure 6.2. Hypothesised dynamic view of the levels of geometric thinking.

6.2 The van Hiele teaching phases

6.2.1 Assignment of phases

Whitman *et al.* (1997) have reported some findings relevant to this study about the van Hiele teaching phases. These researchers stated that different results might emerge if data were coded by different researchers. They also found that the basic teaching procedure in Japan was different from the van Hiele teaching phases as exemplified by Hoffer (1994).

In this study, individual teacher's instruction on each proof problem solving task was first assigned to van Hiele phases by the researcher. The different order of the phases, as noted by Whitman *et al.* (1997) in the Japanese classroom, was confirmed by this study. Indeed, findings from the analysis of the study indicated that teachers might not necessarily start teaching proof problem solving from the Information/Familiarization phase. Moreover, when the researcher discussed how data were coded and analysed with colleagues in the research meetings, different interpretations of the teachers' instruction occurred.

When teaching a geometrical proof problem, Lily dynamically drew and interpreted the figure to help students to gain insight into the dynamic nature of the static figure. The researcher assigned Lily's such instruction to the *Information/Familiarization* phase, as the teacher helped students to understand the proof problem and its figure, and involved them in the problem solving process. The researcher noticed that when teaching some proof problems, this information phase occurred at the beginning of proof problem solving (see extract 5.4, 5.9 and 5.12). However, during the instruction of some other proof problems, Lily constantly guided students to discover hidden information through exploring the relation of the figure during the teaching process (see extract 5.13, 5.14 and 5.16). Sometimes, Lily deepened students' understanding of the problem after students had explored and solved the problem on their own (see extract 5.18). During this phase, Lily asked questions such as "*Is ... changeable?*", "*what does ... look like?*" (see #206, 210 in extract 5.4), to learn what conjectures students might make; or questions such as "*in this figure, could you find what is not changed, when ...?*" (see #37 in extract 5.13) to encourage students to discover the hidden property of the figure. However, some of the researcher's colleagues suggested that Lily's use of the visual approach (to be discussed in section 7.2) is at the *Guided Orientation* phase. This is mainly due to the significant role the teacher played in guiding students to observe and analyse the geometric figures.

Moreover, when the visual approach was used by Lily, different students perceived different structures of the figure and exchanged their different thoughts about the proof (see extract 5.6). Consequently, it is possible to argue that this might be regarded as the phase of either *Explicitation* or *Free Orientation*. Indeed, the researcher had great difficulty to clearly identify these two phases. The *Explicitation* phase appeared to happen after Lily dynamically interpreted the figure, as students became aware of the figure they observed, and they started to discuss the use of known theorems to prove the problem (see extract 5.6). Thus, when students became aware of the geometric structure of the figure, and tried to establish the logical relation of the figure, they were likely to be at the *Explicitation* phase. However, when students presented different ways for solving a problem, they were also likely to be at the *Free Orientation* phase (see extract 5.32 to 5.35). The analysis of Lily's and Nana's lessons indicated that these two phases were likely to occur simultaneously in some of the proof problem solving processes.

When the teacher encouraged students to present their thoughts about the proof, the researcher noticed that Lily used questioning strategies to guide students to present their thoughts. For instance, when students stood up and presented their thoughts about the proof, Lily constantly asked "*What method do you consider?*", "*what should we prove?*", "*How to prove ...?*", "*What method do you use?*", etc. (see extract 5.9). These questions were likely to guide students to refine the use of known theorems. Moreover, Lily guided students to establish an analytic path for the proof by questions such as "*To prove ..., what methods did we learn ...?*", "*If to prove ..., what actually must I turn to prove here?*", "*If to prove ..., what is given?*", etc. (see extract 5.1 and 5.3). The researcher considered Lily's such instruction as Explicitation/Verbalization phase, as students learned to refine their knowledge and to use analytic techniques to present their ideas of proof problem solving in a formal way. However, some colleagues thought Lily's such instruction as a *Guided Orientation* phase due to the significant role the teacher played.

The researcher noted that sometimes when Lily used the visual approach, her instruction could also be assigned to the *Integration* phase. While Lily dynamically drew the figure, students were actually expected to recognise some basic figures they had just learned (see extracts 5.4, 5.9 and 5.10) or they learned before (see extract 5.12).

Thus, this study substantiates the findings by Whitman *et al.* (1997) that there is ambiguity in trying to identify exactly the phase at which the teacher was teaching.

Moreover, according to the interview with Lily (see extract 5.21), the teacher considered that the van Hiele five phases do not describe the process of teaching problem solving, but of teaching new knowledge. When teaching problem solving, Lily highlighted the instructional strategy of problem variation. Moreover, in terms of the use of the phase "*Free Orientation*", Lily particularly pointed out the distinction of teaching geometric proof problem solving from teaching an initial course of geometry or other subjects (see extract 5.22). For Lily, it was more important to guide students to analyse the problem on their own rather than to encourage students to discuss immediately the problem with each other, as geometrical proof is a strong logical subject. Even in a classroom discussion, Lily further emphasised the significance of teachers' questioning. In terms of the leading role of teacher in teaching proof, Lily agreed that teacher should play a significant role (see extract 5.23). For Lily, it was fundamental for teachers to guide students to learn how to analyse and solve a problem, with its geometric figures, and how to use mathematical language to write a formal proof.

According to the interview with Spring (see extracts 5.29 and 5.30), the teacher considered that the van Hiele five phases describe long-term learning in geometry. For Spring, the experimental work such as drawing, or using paper to make a model played a less important role during teaching geometrical proof problem solving at Grade 8. Furthermore, in terms of the leading role of the teacher (see extract 5.31), Spring was confused by how to teach a new theorem without direct instruction from the teacher. For Spring, the teacher played a significant role in effectively guiding students to establish the systematic knowledge foundation within the limited lesson time.

According to the interview with Nana, the teacher did not know the van Hiele theory. For Nana (see extract 5.38), when teaching able students like her class selected for this study, she would like to encourage, on the one hand, students to freely present their thoughts in the class, as she is concerned with the development of students' creative thinking. However, on the other hand, Nana also appreciated the leading role of the teacher in teaching proof (see extract 5.39). For Nana, when teaching simple proof problems, she would be able to ensure that students explored the problem in different ways which she understood. When teaching considerably more complex problems, Nana considered directly guiding students to analyse the problem mainly due to the limited lesson time and the efficiency of using direct instruction to guide the majority of students' learning.

Overall, the researcher encountered great problems in assigning the phases. First, there was more than one interpretation of an instructional strategy or approach applied by a single teacher according to the van Hiele phases. Secondly, the van Hiele theory was not known to every teacher, and even it was known to some teachers, their understanding of the phases could be different due to the use of the phases on different subjects, different topics or types of mathematics lessons.

6.2.2 Interpretation of phases

As has already been mentioned, during the course of the study, the researcher had difficulty in elaborating the nature of the phases by the available data. First, as analysed in section 3.1.2, the phases appeared to be complex structures mixed with both teaching and learning processes. Dina van Hiele-Geldof (1958/1984) proposed the five phases to describe the learning structure of students with learning materials in an initial geometry course. This study did not detect this child-centered learning feature. Findings from this study indicate that when teaching geometrical proof problem solving, the teachers played a significant role. That is, the teacher deepened students' understanding of the geometric figures and the proof problems; guided students to establish an analytic path for the proof; developed students' verbal skill in the use of mathematical language in proof writing; or led students to overview the different methods for a proof, etc. Thus, this study suggests that the description of students' learning in geometrical proof problem solving not only focus on the relation of students and subject, but also the role of the teacher in building up the bridge between students and subject in effective learning.

Second, the original phases by Dina van Hiele-Geldof (1958/1984) highly emphasised the sequential nature of the phases to help students to make progress from Level 1 to Level 2. Yet P.M. van Hiele (1959/1984) suggested the five phases not as a fixed model but as an effective means to help students to make progress to any higher levels. This study did not detect the sequential feature of the phases. Moreover, findings from the study indicate that there was more than one interpretation of teachers' actual practice by the phases.

Third, as has been shown in the literature review (section 2.5.3), it is not clear how long a teaching phase may last. According to Dina van Hiele-Geldof (1958/1984, p.217), one phase could mean one lesson. In analysing the geometry lessons observed in Shanghai (see sections 5.2.1, 5.3.1, and 5.4.1), it proved possible to identify the van Hiele phases in the teaching process of each geometrical theorem and of each problem solving episode. Such an analysis of the instructional structure of the teaching of individual proof problems enabled the study to interpret, in depth, the Chinese teachers' instructional strategies and approaches in geometrical proof problem solving.

Last, during the teaching process of geometrical proof problem solving, it was noted that teachers appeared to apply different instructional strategies and approaches across a set of proof problems. Thus, this study indicates that, though the five phases may present a way to facilitate students' geometrical thinking to a higher level, there are other ways which may support the development of students' thinking to write proofs.

6.3 Summary

Section 6.1.1 presented the assignment of students' responses and learning outcomes to the first three van Hiele levels. It was found that the majority of students' responses in the teaching process of individual geometric proof problem solving could link to Level 3 thinking. Thus, the study detected the hierarchical nature of the van Hiele levels. However, this study could not detect the discrete feature of the levels. Findings shown in section 6.1.2 indicate that students' geometric thinking at a higher level (geometrical proof problem solving) appeared to be more complex that that ascribed by the van Hieles. On the one hand, across a set of proof problems, the visual, analytic and deductive thinking may concurrently grow up together; on the other hand, however, they may limit each other's development. As a result of the analysis, a dynamic view of the van Hiele levels of geometric thinking was proposed in section 6.1.2.

Section 6.2.1 demonstrated the ambiguity of trying to identify exactly the phase at which the teacher was teaching in this study. That is, there was more than one interpretation of an instructional strategy or approach applied by a single teacher according to the van Hiele phases. Moreover, this study did not detect the sequential feature of the phases to help students to make transitions to a higher level. Discussion in section 6.2.2 further suggests focusing on the role of the teacher in building the bridge between students and subject in effective learning.

In the next chapter, an explanation of the three cases is built up, in order to elucidate how and why the different strategies and approaches were actually used by the teachers to develop students' geometric thinking for solving proof problems.

CHAPTER 7. TOWARDS A PEDAGOGICAL FRAMEWORK

7.1 Overview

This chapter seeks to develop an explanation of the relationship between teachers' instructional approaches and strategies with students' thinking development in teaching geometrical proof problem solving that emerges from the observation data as these do not match the description of the van Hiele pedagogical theory (see the discussion in section 6.2). This chapter is divided into five main parts. Section 7.2 presents the analysis of visual approach. Section 7.3 gives an explanation of teachers' unique use of an empirical/deductive approach. Section 7.4 identifies the key types of teachers' questioning. Section 7.5 shows the teachers' arrangement of problems. Section 7.6 summarises the key findings in each previous section and then proposes a pedagogical framework of teaching geometrical proof problem solving.

7.2 The visual approach

This section addresses the analysis of the relationship between the teachers' use of a visual approach with students' geometric thinking development in solving proof problems. In this study, the *visual approach* encompasses individual teacher's instructional strategies for using visual geometric figures as a means to involve and develop students' various kinds of thinking for writing proofs in geometry. For instance, the teacher may show a visual example of a theorem, use colours to highlight part of a figure, or present a figure in a dynamic way during the teaching process for geometrical proof problem solving.

van Hiele-Geldof (1957/1984) hypothesised that students need to have certain visual geometric structures in their mind for ordering the geometrical properties. van Hiele-Geldof (*ibid*, p.152) defined the visual geometric structure as follows:

> "... the visual geometric structure, is established through the analysis of objects in a geometrical context on the basis of empirical truths."

As discussed in section 6.1.2, however, when ordering the interrelation of theorems for writing geometrical proofs, the visual geometric structures students need to have may be much more sophisticated than that ascribed by the van Hiele levels. Clements and Battista (1992, p.437) point out that

> "investigations need to consider how visual thinking is manifested when higher levels are achieved. ... it is doubtful that it is untransformed and merely "pushed into the background" by more sophisticated ways of thinking.

In this study, evidence of Linlin's response (#281) in extract 5.9, and Yinyin's responses (#84, 92, 95, 97) in extract 5.33 substantiate findings of Hershkowitz (1989) that when students learn a new geometric theorem, there might be "the prototype phenomenon". That is, there is the prototypical example of a theorem, in which attention is drawn to some specific attribute(s), in addition to the critical attributes of the theorem. Concerning students' learning experiences of basic geometrical concepts (e.g., angles, quadrilaterals, triangles), Hershkowitz *et al.* (1990, p.85) summarise the following principal characteristics of teaching strategies in students' learning situations:

a) lack of completeness, in which only some of the examples and attributes are presented;
b) lack of awareness as well as absence of knowledge of the existence of further elements on the part of the teacher or even the textbook;
c) lack of awareness of student difficulties and misconceptions in constructing concepts; and
d) generalization of concept attributes (definitions) given (if at all) by the teacher or the textbook, with the learner seen as a passive receptor.

These principal characteristics of teaching strategies highlight the significance of helping students to appreciate the visual aspect of geometric concepts in learning. The analysis of the three teachers' instruction through chapters 5 and 6 demonstrates the teachers' instructional intentions to help students gain insight into geometric figures. That is, the three teachers in this study largely used the visual approach. However, it is found that they use the visual approach differently.

In the first place, when teaching N1-Ex1, Nana guides students to perceive the basic figures hidden in the geometric problem (see extract 5.32). The basic figures are the visual figure example of a set of theorems. Before teaching the proof problem N1-Ex1, Nana guides students to review the theorems and draws students' attention to the critical attribution of the theorems by putting some marks on each single visual figure example (see the left part of figure 7.1). Those visual examples drawn on the blackboard are largely based on the figures provided by the textbook. Students are then encouraged to order the relation of the figure for writing a proof by perceiving the corresponding visual figure example of the theorems in the figure of the problem (see the right part of figure 7.1). The function of the visual approach used by Nana is shown in figure 7.1.

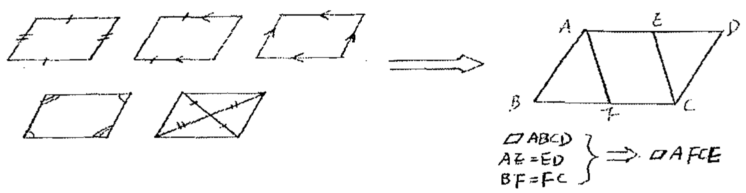


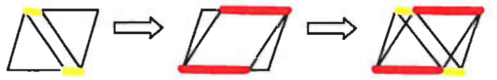
Figure 7.1. The function of visual approach used by Nana in N1-Ex1.

In terms of students' geometric thinking development, van Hiele-Geldof (1957/1984, p.191) suggests that,

> "... By letting pupils analyze at their level, an ordering of certain relations evolves. Known relations can be a consequence of other known relations and new relations can be discovered from known relations. ... Through this analysis it becomes possible for pupils to expand their visual geometric structures into structures that belong to the second level of thinking [Level 3 in this study]."

Thus, extracts 5.32-35 show how students extend their visual geometric structure of a single geometric figure by developing their deductive reasoning of the relation between a set of known theorems (Level 3, "properties are ordered." P.M. van Hiele, 1959/1984, p.245).

Next, when Spring teaches the similar proof problem S2-Ex4 (both N1-Ex1 and S2-Ex4 involved using similar theorems to prove a parallelogram and shared the basic geometric figure, see N1-Ex1in figure N1 in Appendix C and S2-Ex4 in figure S2 in Appendix B), she uses colours and draws students' attention to the construction process of the geometric figure. It is noted that Spring first uses yellow colours to highlight the equal sides AE and CF. However, rather than linking AF and EC, Spring first links BE and EC and then AF, FD (see extract 5.24). While some students seem to perceive parallelograms AFCE and EBFD by the yellow colours, other students might ignore the yellow colours by perceiving different geometric structures of the figure, such as congruent triangles. Spring eventually uses yellow and red colours to guide students to recognise parallelogram AFCE and BFDE (see extract 5.24). Thus, colours seem to be helpful to support students to visualise what they might not be able to visualise about the hidden geometric objects in the figure. Moreover, two visual examples of a parallelogram hidden in the figure are highlighted by different colours (see the first two figures in figure 7.2). In general, two functions of the visual approach used by Spring are summarised in figure 7.2.



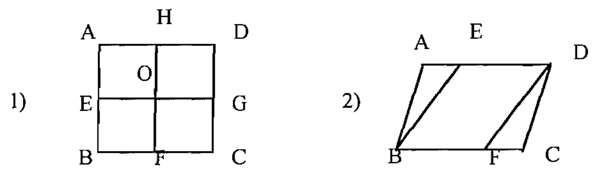
- *Visualizing the hidden geometric objects.*
- *Perceiving the critical attributes of a theorem with different visual figure examples.*

Figure 7.2. The functions of visual approach used by Spring in S2-Ex4.

In terms of students' geometric thinking development, it is noted that a certain number of students in this case are not able to visualise parallelograms AFCE and EBFD by the red or yellow colours until Spring highlights these two objects (see #59 in extract 5.24). This is to show students' visual thinking does not seem to merely remain at Level 1 ("Figures are judged by their appearance." P.M. van Hiele, 1959/1984, p.245), but needs to be developed with the growth of analytic thinking (Level 2, "the figures are bearers of their properties." *ibid*, p.245), and deductive reasoning (Level 3, "properties are ordered." *ibid*, p.245). Moreover, the analysis of the case of Spring in this study further substantiates findings by TSGofQp (1991) that although colours support students to visualise the hidden geometric objects, colours themselves seem to have limitations to explain how analytic and deductive thinking might limit students' visual thinking. Thus, this case particularly indicates that the development of visual thinking, analytic thinking and deductive thinking is extremely complex in the geometric proof problem solving process. On the one hand, the different kinds of thinking may simultaneously grow up together; on the other hand, however, they may limit each other's development.

To draw students' attention to the critical attributes of a theorem, Lily applies a different strategy. She dynamically presents to students a variety of visual examples of the same theorem across a set of proof problems. Indeed, according to the interview with Lily (see extract 5.21), she accepts and applies the theoretical idea of teaching with variation suggested by Gu *et al.* (2004) into her teaching practice. For Gu *et al.* (2004), there are two forms of variations: conceptual variation and procedural variation (see Gu *et al.*, 2004, p.315). In terms of conceptual variation, they address the significance of varying visual and concrete instances to help students to more fully understand abstract concepts. Indeed, similar to students' responses in Nana's case (see extracts 5.32-35), the discussion of Liuliu and Beibei during the teaching process of L2-Ex4 (see extract 5.6) indicates that students may perceive different visual geometric examples of different theorems in a geometric figure due to their attention on the different attributes of the theorems. The analysis across the teaching episodes of L2-Ex5 and L2-Ex6 shows that Lily varies the proof problems to draw students' attention to the invariants of theorems. Moreover, the variation of visual figures across a set of proof problems reveals the complexity of students' geometric thinking in the geometric proof problem solving process. For instance, when Lily slightly changes the figure in L2-Ex5, Linlin wrongly perceives the visual geometric example of the theorem of two parallel lines (see #281 in extract 5.9). This example again indicates the complexity of students' visualisation of the network of theorems in geometric proof problem solving. During the teaching process of L2-Ex6, students show their difficulty in distinguishing the interrelationship of a set of theorems (see #362, 364, 365 in extract 5.10). This case substantiates the findings of Gutiérrez *et al.* (1991) that students' thinking is not developed in a single linear way. Rather, students' visual, analytic and deductive thinking may concurrently develop when they solve a considerably easy problem (see Liuliu's responses in extract 5.6). However, when they encounter a more difficult problem, their visual, analytic and deductive thinking may limit each other's development and therefore they may not be able to order the relation of the figure for writing a proof.

In particular, the critical attributes of the theorem, with its more abstract visual example, are emphasised by Lily when she teaches L12-Ex45 (see figure 7.3).



Given: (see figure 7.3(1)) in square ABCD, E, F,G, and H are respectively the middle point of AB, BC, CD and DA. EG and HF are crossed at O. Prove: quadrilaterals AEOH, EBFO, FCGO, OGDH are square.

Figure 7.3. L12-Ex45.

85. A girl student: I use two pairs of parallel sides. (Probably AH//EO, AE//HO.) (see figure 7.3(1))
86. Lily: To prove this parallelogram (AEOH), she chose the definition.
(The student's voice was very low in the audio recorder (#87). The researcher used the teacher's repeat as follows).
88. Lily: Very good. She recognised that AD//EG. To prove AD//EG, turn to prove parallelogram AEGD. What method do you use then? (see figure 7.3(1))
89. The girl student: Because of the square, a pair of opposite sides is equal and parallel.
91. Lily: Good. In fact, this figure, its outlook is a square. (The teacher drew a parallelogram. See figure 7.3(2).) AE=CF. Though E, F might not be the middle points, EBFD must be a parallelogram.

Extract 7.1. L12-Ex45, figure 7.3, #70-91.

Thus, with the growth of students' knowledge in geometry, Lily also tends to develop students' visual thinking to an abstract level. The function of the visual approach particularly used by Lily is summarised in figure 7.4.

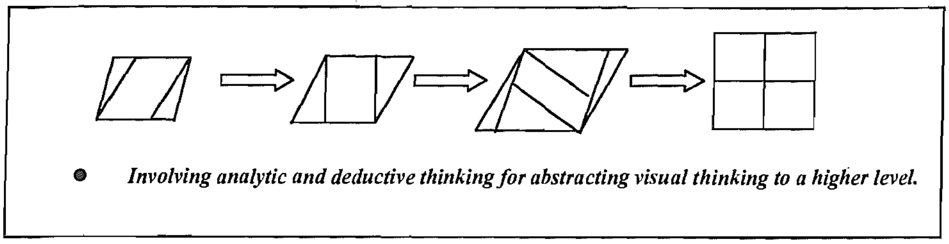


Figure 7.4. The function of visual approach used by Lily in L2-Ex4, L2-Ex5, L2-Ex6, and L12-Ex45.

In addition, it is noted that when Lily guides students to perceive the hidden geometric objects in a complex figure, she does not use colours as much as Spring does, but dynamically shows students some basic figures which construct the complex figure. For instance, when Lily teaches L2-Ex7 (see extract 5.12), she dynamically shows students the motion of F and D on two sides AB and BC of equilateral triangle ABC, and guides students to perceive the hidden geometric objects of the figure. By dynamically presenting the static figure, students are encouraged to perceive hidden geometric properties such as congruent triangles ADC and FBC, CF=AD, and $\angle AOF=60^\circ$ (see extracts 5.13-14). Lily's such instructional strategy is briefly highlighted in figure 7.5(1). Moreover, Lily continuously guides students to perceive other geometric objects hidden in the complex figure by dynamically drawing two rotatable relationships of equilateral triangles (see figure 5.8(3)), in which two congruent triangles need be perceived by deductive reasoning (see #121 in extract 5.16). Lily's such instructional strategy is briefly highlighted in figure 7.5(2). The function of visual approach particularly used by Lily is summarised in figure 7.5.

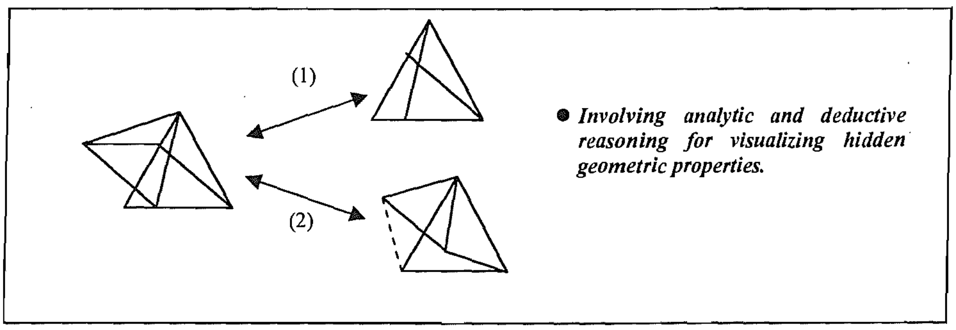


Figure 7.5. Function of the visual approach used by Lily in L2-Ex7, L3-Ex7.

Laborde (2005, p.26) points out that,

> "Learning geometry seems to involve not only how to use theoretical statements in deductive reasoning but also learning to recognize visually relevant spatio-graphical invariants attached to geometrical invariants."

This statement shows the significance for students to learn the implicit rules for using diagrams [figures] through good teaching in geometry. In this section, five functions of the use of the visual approach are identified for the development of students' geometric thinking for making a proof:

1) Perceiving the visual figure example of theorems for making a proof.
2) Visualising the hidden geometric objects.
3) Perceiving the critical attributes of a theorem with different visual figure examples.
4) Involving analytic and deductive reasoning for abstract visual reasoning to a higher level.
5) Involving analytic and deductive reasoning for visualising hidden geometric properties.

Overall, this section indicates that teachers may vividly use the visual approach in very different ways in supporting students' geometric thinking for writing proofs. While Nana focuses on guiding students to appreciate the visual example of a single theorem in the network of theorems in a single figure, Spring largely uses colourful chalks to highlight the hidden geometric objects in the figure, and Lily presents a static geometric figure in a dynamic way and varies the visual examples of a theorem across a set of proof problems.

As a result of such different ways of using the visual approach, this section further uncovers the complexity of students' geometric thinking development in solving geometric proof problems, in particular of their visualisation development. In the first place, while Nana guides students to perceive a set of visual figure examples of theorems, students' visual, analytic, and deductive thinking might be simultaneously developed for writing the proofs at van Hiele Level 3 (see extracts 5.32-35). However, the teacher might be not aware of the complexity of visualisation in geometric proof problem solving, such as the prototypical example(s) of a theorem (see Yinyin's responses (#92, 95, 97) in extract 5.33).

Indeed, Spring' use of colours and Lily's dynamic presentation of a set of geometric figures highlight the complexity of visualisation in geometric proof problem solving. Spring largely used colours to guide students to perceive the hidden geometric shapes and properties (at van Hiele Levels 1 and 2) (see extracts 5.24-26). However, colours themselves seem to have limitations to explain how analytic and deductive thinking might limit students' visual thinking.

Though students' responses in Lily's case (see extracts 5.1-18) are generally assigned to van Hiele Level 3, the quality of their thinking seems to be different, from being able to make deductive reasoning of the relationship of the figure (see extracts 5.6, 5.7, 5.17), to not being able to distinguish the difference of a theorem and its converse theorem (see extract 5.10), as observed by Battista (2007).

In addition to the different uses of a visual approach with different students' learning responses, the analysis of the case of Lily indicates that it was the teacher, who constantly guides students to pay attention to the invariants of the theorem through the variable visual geometric figures, and guides students to abstract their visual thinking at a higher level across a set of geometric proof problems (see extract 7.1).

Finally, Fujita and Jones (2002) highlight the role of *geometrical eye* (a term defined by Godfrey (1910, p.197) as "the power of seeing geometrical properties detach themselves from a figure") as a potent tool for building effectively on geometrical intuition. The five functions of the use of visual approach identified in this section could be considered as the ways to train students' such geometrical eye through geometric proof problem solving.

7.3 The empirical (inductive)/deductive approaches

This section addresses the issue of the use of empirical (inductive) /deductive approaches in teaching geometric proof problem solving. The van Hieles (1957/1984, 1959/1984) suggest that a solid empirical grounding is necessary for apprehending and then manipulating abstract geometric objects. They consider that the objects themselves will not be meaningful to the students unless they have the appropriate experiential foundation. In view of an initial geometry course, Van Hiele-Geoldof (1957/1984) points out that

> "It is only after this period of active formation of visual geometric structures of geometric objects (figures) that it becomes meaningful to insert a period during which associations are formed. ... If one introduces this period too early, an association formation will most probably be established, but the pupil is not on the level of thinking – he therefore misses the opportunity to make operational use of his knowledge." (p.186)

Thus, for the Hieles, it is more important to develop the diversity of students' thinking, before developing their deductive reasoning in an axiomatic system of geometry.

In view of the role of empirical and deductive approaches in geometry, in particular in an advanced geometry course like geometric proof problem solving, Jones (1998) suggests that

> "… a deductive and an intuitive approach can prove to be mutually reinforcing when solving geometrical problems." (1998, p.83)

The analysis of the three teachers' actual instruction of geometric proof problem solving in chapter 5 substantiates Jones' (1998) view of the use of an empirical (inductive) /deductive approach. While a deductive approach is largely used by these three teachers to support students to write proofs, empirical approaches seem to play an essential role to help develop students' insights into the interrelationship of the network of theorems. Evidences from the teachers' use of the visual approach discussed in the previous section illustrate that to support students to write proofs, the teachers largely use instructional tools such as a triangle ruler, a compass, colourful chalks, to draw students' attention to the geometric construction and motion of a figure. Thus, the study indicates that it is not likely to be a question about whether inductive activity is needed or not when teaching geometrical proof problem solving, but a question of how those inductive activities should be implemented in the class for effectively supporting students' thinking to write proofs.

This section shows two examples of Lily's unique instructional strategy of dealing with the inductive/deductive approach for the development of students' geometric thinking for solving geometric proof problems. The first example is given as follows. To develop students' insight into the hidden relation of the figure, the angle formed by AD and CF, the teacher, Lily, suggests students do some measurement after the lesson as follows:

56. Lily: Now in the class, the time is too limited. But when you go home, you might draw the figure to have a look. I drew it here. D and F are at different locations, and you see this figure is just a moment of a dynamic figure. I suggest you to measure it. It must be that CF=AD. How about their location relationship?

58. Lily: Obviously, they (AD, CF) are not parallel. They are intersected, aren't they? How is the angle they formed? Will it change? You could use a protractor to measure the figure on your book. You could measure the angle before and after the movement.

Extract 7.2. L3-Ex7 (quoted from extract 5.14)

Responses from students (see #59.2, 63, 69, 70, 75.1 in extract 5.14) to the teachers' instruction indicate that a certain number of students do not actually gain insight into the figure from what the teacher asked. In fact, students' responses such as "Why is it 60° ?" (see #63, 66 in extract 5.14) show that they have difficulty in understanding the relationship of congruent triangles ADC and FBC with the equilateral triangle ABC by observing the abstract figure on the blackboard and by just thinking about the teacher's suggestion (see figure 5.8(1)). On the other hand, however, there are some students like boy Zheng YQ (#72, 76 in extract 5.14) who are able to discover the hidden relation of the figure by deduction.

Wu (1996, pp.223-224) demonstrates the view of the role of hands-on experiments and proof in mathematics as follow:

> "… Mathematics is concerned with statements that are true, forever and without exceptions, and there is no other way of arriving as such statements except through the construction of proofs."

The teacher's intention shown in extract 5.14 appears to draw students' attention to such a powerful role of proof in generalising the truth of mathematical knowledge, which is not dependent on empirical activity available by drawing and measuring a figure, but rely on deductive reasoning by analysing and interpreting a figure. Thus, in the class, while a deductive approach is highly emphasised in the teaching process of solving proof problems, an inductive approach is suggested as an after lesson learning activity for students to explore the relation of the figure.

Here is the second example. Lily presents a static figure in a dynamic manner:

200. Lily: We should have drawn 39 triangles ABC. (There were 39 students in the class.)
201. Liuliu: 39???
202. Lily: Unless coincidence, triangle ABC drew by two of you may be congruent. But this chance is really small.
204. Lily: So points D, E, F are three dynamic points, right? Different ABC would produce different points D, E, F.
205 Liuliu: Much easier than last problem (L3-Ex7).

Extract 7.3. L3-Ex8 (quoted from extract 5.18)

When Lily dynamically presents the static figure to students, Liuliu seems not immediately to understand Lily's explanation (see #201 in extract 7.3). However, Lily's further interpretation of the figure might develop Liuliu's insight into the figure (see the connection of Liuliu's responses in extract 5.17 and #205 in extract 7.3). This example again shows that an inductive approach might not necessarily mean to let students experiment. Rather, the teacher might dynamically interpret a static figure as a way of using an inductive approach to support the development of students' insight into solving geometric proof problems.

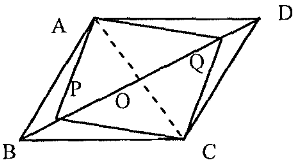
When teaching geometry, Chinese teachers appreciate four functions of using a visual geometric figure (according to the interview with Lily's and Spring's colleagues in the teachers' regular meeting): drawing and measuring a figure, observing a figure, analysing and interpreting the invariants of a figure, and thinking of a figure in the mind. Thus, students' geometric intuition might be developed not only from what geometric objects they manipulate, but also from what they observe, understand, think of the meaningful geometric figures in their minds.

In summary, this section indicates that the teachers in this study use the deductive approach a lot to emphasise the role of proof in generalising the truth of mathematical knowledge. When the inductive approach is used in geometric proof problem solving, the instructional intention appears not to emphasise very much drawing figures or measuring figures, but guiding students to interpret and to think of the invariants of figures.

7.4 Teachers' questioning

This section presents the analysis of the three teachers' questioning in teaching geometric proof problem solving. This study substantiates the idea that questioning is one of the most versatile and most used instructional tools, and teachers might use different questions to involve students thinking at different levels (Mathematics Resource Project, 1978). Interview data with Lily (see extract 5.22) particularly highlights that teachers' questioning plays a significant role in the involvement of individual students in learning and in the development of their mathematical thinking.

During the teaching process of proof problem solving, Lily frequently uses a set of questions to organise an analytic path for the proof. For instance, in L4-Ex9 (see figure 7.6), after Lily dynamically presents the figure to students, she encourages students to present their thought on the proof as follows:



Given: In parallel ABCD, BP=QD.
Prove: Quadrilateral APCQ is a parallelogram.

Figure 7.6. L4-Ex9.

96. Lily: I need to prove this is a parallelogram. ***So far, how many methods do we have?*** (to prove a parallelogram)

(Students discussed five methods (one definition and four theorems) in the whole class (#96.1).)

98. Lily: ***How to choose? Which one is more suitable here?*** We need to consider about the given.

(Many students then focused on the diagonals of the quadrilateral and suggested linking AC (#99).)
100. Lily: Don't tell me which two points should be linked. ***Please tell me which method you choose.***
(Students replied the third theorem, the theorem of diagonals of parallelogram (#101).)
102. Lily: The third theorem. ***Why do you consider this theorem?***
(Students already noticed the diagonal BD (#103) and suggested adding an auxiliary line (AC) (#105). Lily then encouraged students to prove AO=OC, OP=OQ (#113).)
115. Zhao L: (Boy) Because ABCD is a parallelogram, AO=OC. The diagonals of a parallelogram are bisected each other.
117. Lily: ***How to prove this?*** (OP=OQ) Xu YN? (The teacher asked a girl student to stand up and present her thought.)
118. Xu YN: (Girl) Because ABCD is a parallelogram, ...
119. Lily: Now, to prove OP=OQ, ***tell me what method do you use?***
120. Xu YN: (girl) The property of equotation.
121. Lily: Here, ***what should we turn to prove first?***
(The girl student then presented to prove OB=OD first, then to prove BP=QD (#122, 124))

Extract 7.4. L4-Ex9, figure 7.6, #96-124.

This is a particular example to demonstrate that rather than encouraging students to present the proof structure by themselves, the intention of Lily's questioning is likely to involve students to refine their analytic thinking of the relation of theorems (see #100, 119 in extract 7.4). Indeed, during teaching many proof problems (for instance, see #101, 106, 107, 110, 115, 126, 139 in extract 5.3; #222 in extract 5.5; #280, 303, 306.1, 312 in extract 5.9, etc.) across Lily's lessons, the teacher often first asks a general question such as "*How to prove it?*", and then immediately follows with a set of questions, such as "*What method could we use? How many methods do you know to prove...? Why do you choose this method? What should we first turn to prove...?*".

Schoenfeld (1986, p.259) states that

> "Much of the intrinsic power of mathematics comes from the perception of structure – from seeing connections and exploiting them. One of the primary reasons for learning mathematics is to develop such skills"

The set of questions demonstrated above appear to emphasise the development of such skills of perceiving the mathematical structure. In the case of proof problem solving, the mathematical structure is the analytic path for a proof. In particular, these types of questions are likely to support important mathematical thinking – transformation, as illustrated in figure 7. 7. This is further discussed in the next section.

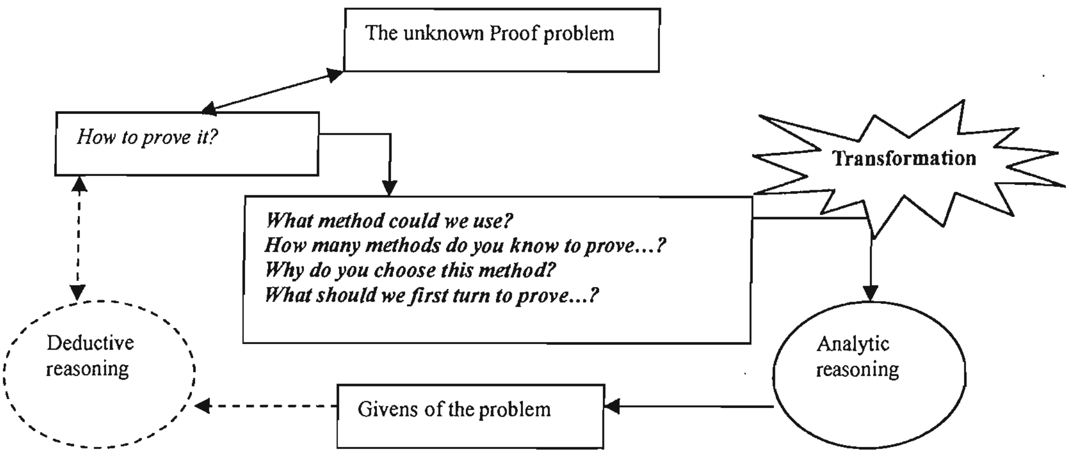


Figure 7.7 The function of teachers' questions in proof problem solving

Next, during the teaching process of proof problem solving, the teachers use a set of questions to guide students to make meaningful connections to previous knowledge. For instance,

39 Lily: To prove two lines are parallel, ***what methods did we learn early at grade 7?*** (see extract 5.1)
91 Lily: Well. So far, ***how many methods did we learn to verify a parallelogram?*** (see extract 5.3)
96 Lily: ***Which two?*** (see extract 5.3)
46. Spring: Right. Some students considered the definition of parallelogram. You see, here, ***what relation is the pair of yellow (AE, CF)?*** (see extract 5.24)

Extract 7.5 Examples of the teachers' questions

Moreover, after being taught a new theorem or definition, the teachers use a set of questions to help students to appreciate the critical attributes of a new theorem or the function of the theorem. For instance,

68 Lily: ***Is this new theorem about the property of a parallelogram? Or about verifying a parallelogram?*** (see extract 5.2)
74 Lily: Well, ***who could use words to state this new theorem again?*** (see extract 5.2)
266 Lily: All right. Now, ***what method could we also use,*** if I want to prove two line segments are equal or parallel? (see extract 5.8)

Extract 7.6 Examples of Lily's questions

The teachers also use a set of questions to encourage students to make conjectures of a problem during teaching the proof problem solving. For instance,

206 Lily: ***Is quadrilateral BEDF changeable?*** (see extract 5.4)
210 Lily: ***What does quadrilateral BEDF look like?*** (see extract 5.4)
259 Lily: OK. Now, I change the problem. If F and E are dynamic points, ***what location and quantity relation do BF and DE have?*** (see extract 5.8)
276 Lily: I do not change other condition (BE=DF). This is still a parallelogram. If I change the condition (means AE and CF). ***What location and quantity relation do AE and CF have?*** (see extract 5.9)
56. Lily: ... ***How about their location relationship?*** (see extract 5.14)

Extract 7.7 Examples of Lily's questions

Noticeably, the three teachers largely encourage students to make meaningful explanations by presenting their thoughts about a proof during the proof problem solving instructional process. For instance,

120 Lily: ***How to prove the parallel lines?*** (see extract 5.3)
224 Lily: Good. ***Why do you consider the second theorem?*** (see extract 5.5)
64. Lily: If this angle (AOF, ∠1) is 60°. ***How to prove?*** (see extract 5.14)
133. Lily: So EF=CD. A pair of opposite sides are equal (EF=CD). But we still need to prove they are parallel. ***How to prove?*** (see extract 5.16)
33. Spring: (asked students in the class.) ***What do you think?*** (see extract 5.24)
13. Nana: ***How to prove it?*** (see extract 5.32)

Extract 7.8 Examples of the teachers' questions

In addition, as discussed in section 7.2 about the different uses of visual approaches to demonstrate the motion and construction of the figure, Spring and Lily use a set of different types of questions to involve various kinds of thinking for the support of students' visual thinking for writing a proof. For instance,

436 Lily: Well. Now, ***are you familiar with this figure?*** (see extract 5.12)
440 Lily: You could think about this figure during the lesson break. ... In the process of the movement of D and F, D and F move regularly. ***Could you find what is never changed in the movement?*** (see extract 5.12)
37. Lily: In this figure, ***could you find what is not changed, when D and F are moving?*** (see extract 5.13)
58. Lily: Obviously, they (AD, CF) are not parallel. They are intersected, aren't they? ***How is the angle they formed? Will it change?*** (see extract 5.14)
60. Lily: ***How do you explain that they are equal. No any change? How much is the angle then?*** (see extract 5.14)
99. Lily: Now, lets see how the problem is formed? ... We know AD=CF. Now ***where could AD be replaced to?*** (see extract 5.15)
106. Lily: ***How much are these two angles?*** (see extract 5.15)
118. Lily: ***In this rotation, this triangle (AGC) is always...?*** (see extract 5.16)
123. Lily: So, ***which triangle is congruent to triangle ADC?*** (see extract 5.16)
140. Lily: I want to prove parallel lines. So we have to see the location. ***Could you have a look? Is there "three lines eight angles"?*** (see extract 5.16)
142. Lily: None of you find it? ***What triangle is BEF?*** (see extract 5.16)

Extract 7.9 Examples of Lily's questions

25. Spring: Next, who? ***What do you see?*** (see extract 5.24)
106. Spring: Right. Now, if you draw the figure correctly, ***what will it be when linking D, E, F, and A?*** (see extract 5.25)
126. Spring: Next, you see. The above triangle DBE. There is also blue, 1, and red. ***Are they congruent?*** (see extract 5.26)

Extract 7.10 Examples of Spring's questions

Comparing Lily's questions in extract 7.9 with Spring's questions in extract 7.10, it is noted that though the intention of the two teachers' questions is likely to develop students' visual thinking, their questions seem to involve the visual thinking at different levels. The teacher's interview with Lily (see extract 5.20) indicates that she is largely concerned with how to develop students' ability to observe a geometric figure by solving proof problems. Thus, to develop students' insight into the geometric figure, analytic and deductive reasoning appears to be involved (Level 2, "the figures are bearers of their properties." Level 3, "properties are ordered." P.M. van Hiele, 1959/1984, p.245). On the contrary, Spring is concerned with how to develop students' spatial thinking and geometric intuition by solving proof problems (see the interview with Spring in extract 7.11).

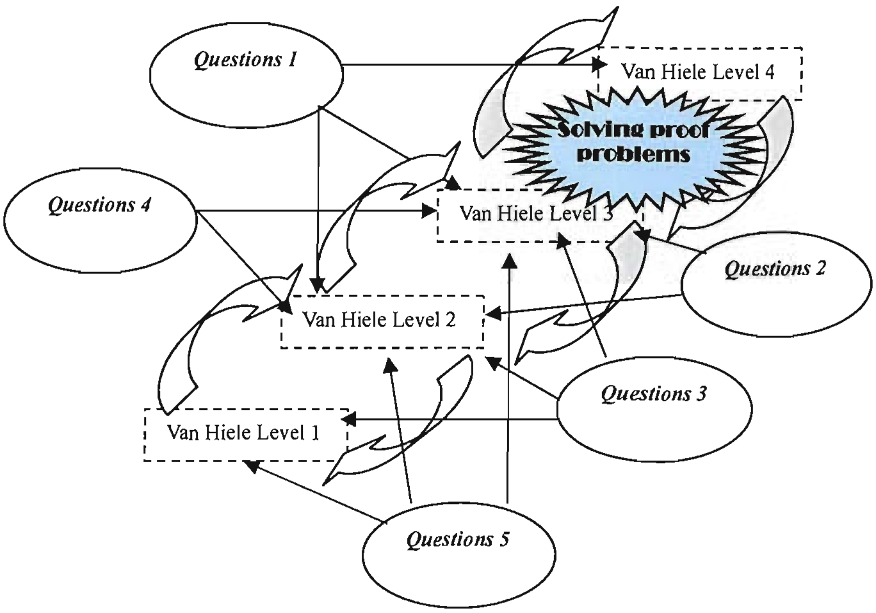
The researcher: I noticed that when teaching geometric proof problem solving, you often gradually draw a figure on the blackboard, and encourage students to perceive what you have drawn. Why?

Spring: Observing the drawing process might help students perceive the figures. In daily life, they (students) would not necessarily see any regular geometric figure, wouldn't they? When teaching geometric proof problem solving, we need to develop students' geometric intuition and spatial thinking as well. (Interviewed on 10 May, 2006. translated by Liping Ding)

Extract 7.11 An interview with Spring.

Thus, students' attention is largely drawn to perceive the appearance and properties of a geometric figure (Level 1, "Figures are judged by their appearance." Level 2, "the figures are bearers of their properties." (P.M. van Hiele, 1959/1984, p.245)).

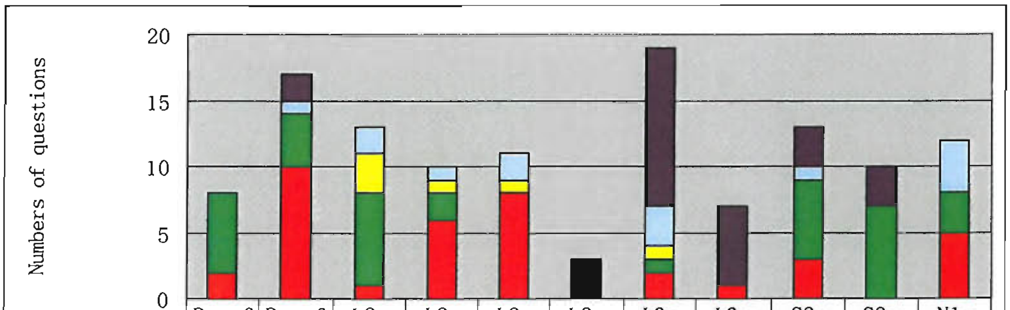
Overall, the different types of teachers' questions analysed in this section reflect the complexity of the development of students' geometric thinking for writing proofs. The van Hieles claim that to make a proof, students' thinking need to reach Level 4. The analysis of teachers' questions in this section indicates that students are likely to be supported to solve proof problems at Level 3. Sometimes when solving a difficult problem, students' thinking might be guided from Level 1 towards Level 3. Thus, during the teaching process of solving a proof problem, the teachers' questions not only maintain the development of deductive reasoning at Level 3, but also involve the development of visual and analytic reasoning from Levels 1 and 2. A model of the relation of teachers' questions with the dynamic development of students' geometric thinking for writing proofs is proposed in figure 7.8.



Question 1 – to develop skills for analysing and solving a mathematical problem.
Question 2 – to make meaningful connections for previous knowledge or deepen understanding of new knowledge.
Question 3 – to make conjectures about a problem.
Question 4 – to make meaningful explanations of solving a problem.
Question 5 – to facilitate visual thinking.

Figure 7.8. A model of the relationship of teachers' questions with the dynamic development of geometric thinking for making a proof.

In addition, it is noted that the questioning strategy is varied from one proof problem to another proof problem by an individual teacher, and from one individual teacher to another teacher, both in the number of questions and in the application of the types of questions (see figure 7.9).



	Proof 2	Proof 3	L2-Ex4	L2-Ex5	L2-Ex6	L2-Ex7	L3-Ex7	L3-Ex8	S2-Ex4	S2-Ex6	N1-Ex1
Question 5	0	2	0	0	0	3	12	6	3	3	0
Question 4	0	1	2	1	2	0	3	0	1	0	4
Question 3	0	0	3	1	1	0	1	0	0	0	0
Question 2	6	4	7	2	0	0	1	0	6	7	3
Question 1	2	10	1	6	8	0	2	1	3	0	5

Proof problems

■ Question 1 ■ Question 2 □ Question 3 □ Question 4 ■ Question 5

(The interpretation of the five types of questions see figure 7.8)

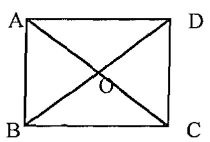
Figure 7.9. Examples of the questions used by the three teachers.

Figure 7.9 indicates that both the quality and the quantity of teachers' questions are likely to support the development of students' geometric thinking for writing proofs.

7.5 The arrangement of proof problems

This section illustrates the main findings of an analysis of the arrangement of proof problems during the teaching process of proof problem solving. This study confirms the significant role of the teacher not only in the selection of appropriate tasks but also in the implementation of those tasks for maintaining students' mathematical thinking at a high level, as observed by Henningsen & Stein (1997) and Herbst (2003). In each observed lesson in this study, the teachers arrange and implementation a set of proof problems. The term "arrangement" used in this section links to two theoretical considerations: teaching with variation (变式教学) (see the definition in Gu *et al.* (2004, p.315) and scaffolding (see Bruner, 1985, pp.24-25). The theoretical idea of teaching with variation has been linked to the analysis of Lily's use of visual approaches in section 7.2 (see figure 7.4, for instance). This section concentrates on analysing and interpreting the use of scaffolding to support students' thinking development for writing proofs.

In lesson L5, for instance, Lily first leads students to learn two new theorems about the basic properties of a rectangle, Proofs 6 & 8 (see figure L5 in Appendix A). She then arranges four problems (L5-Ex12, L5-Ex13, L5-Ex14, L5-Ex15, see figure L5 in Appendix A) to guide students to understand the use of the two new theorems. Firstly, when Lily teaches L5-Ex12 (see figure 7.10), she largely encourages students to present their thoughts about the proof structure, which are at Level 3 ("properties are ordered." P.M. van Hiele, 1959/1984, p.245) (see extract 7.12).



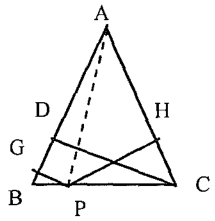
Given: in rectangle ABCD, AC and BD are diagonals.
Prove: AO=BO=CO=DO.

Figure 7.10. L5-Ex12.

(The teacher encouraged students to prove OA=OB (#66). See figure 7.10)
67. Liuliu: First, to prove two pairs equal (probably AC, BD). Next, to say diagonals are bisected each other. And then, is of half. These two are equal. (probably OA=OB)
69.1. Liuliu: OA=OC=1/2AC. Next, OB=OD=1/2BD.
(The teacher asked students what method they used to prove the problem (#70).)
74.1. Liuliu: The replacement of equal elements.[5]

Extract 7.12. L5-Ex12, figure 7.10, #67-74.1.

After L5-Ex12, the teacher guides students to review a previous problem, L5-Ex13 (see figure 7.11).



Given: in isosceles triangle ABC, AB=AC. CD⊥AB, GP⊥AB, and PH⊥ AC. D and G are on AB. H is on AC. Prove: DC=GP+PH.

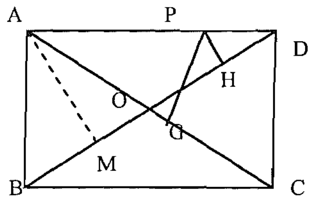
Figure 7.11. L5-Ex13.

Surprisingly, many students simultaneously respond to the teacher's question about the quantitative relationship of GP and PH, which involves both Level 2 thinking ("The student solves geometric problems by using known properties of figures or by insightful approaches." (Fuys *et al*, 1988, p.62)), and Level 3 thinking ("The student identifies and uses strategies or insightful reasoning to solve problems." (Fuys *et al*, 1988, p.67)) (see extract 7.13).

108. Lily: What relation is among the three lines (PG, PH, CD)?
109. Youyou and Liuliu: DC=GP+PH.
110. Lily: This means the sum of the distances from a point on the bottom side to the other two legs is the attitude of one leg of the isosceles triangle. Could you prove it?
113. Youyou: Use area.
114. Liuliu: Use area. Link AP. (see figure 7.11)
122.1. Liuliu: Area. Two small triangles are equal to a large one. (probably meant that S△ABP+S△APC=S△ABC.)

Extract 7.13, L5-Ex13, figure 7.11, #108-122.1.

After L5-Ex12 & 13, the teacher guides students to consider L5-Ex14 (see figure 7.12).



Given: in rectangle ABCD, P is a dynamic point on AD. When P moves along AD, whether is the sum of the distances from P to AC and BD (PG and PH) changeable? If it is not changeable, how much is the sum of PG and PH?

Figure 7.12. L5-Ex14.

The teacher first encourages students to consider the quantitative relation of PG and PH, when the dynamic point P is moving on AD (#143, L5). Many students respond 'no change' to the sum of PG and PH (#144, 146, L5), but they do not know what the sum of PG and PH is (#148, L5). The teacher suggests students draw the whole figure in their exercise books and think about the problem on their own (#149, L5). After thinking shortly, some students recognise the isosceles triangle AOD in figure 7.10, and relate the figure to the isosceles triangle ABC discussed in L5-Ex13 (see figure 7.11).

167 Wang YN (girl student): (stood up and presented her thought in the class) △AOD is an isosceles.
169 Wang YN: Next, I mean, it is as same as last problem.

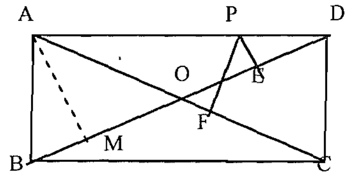
Extract 7.14, L5-Ex14, figure 7.12, #167-169.

However, students in the class respond differently to the girl student's finding (see extract 7.15).

170.1 Youyou: (Surprised) Ah?
170.2 Liuliu: (talked to himself.) One (PG) is the altitude of AO. One (PH) is the altitude of DO.
171. Lily: Is this an isosceles triangle? A point (P) on the bottom of the isosceles triangle (AOD) (The teacher talked very slowly.)
171.1 Youyou: Oh! Right!
171.2 Beibei: Why?

Extract 7.15. L5-Ex14, figure 7.12, #170.1-171.2.

After guiding students to recognise the relation of the isosceles triangle ADO (PG+PH=AM) in L5-Ex14, the teacher encourages students to consider L5-Ex15 (see figure 7.13).



Given: in rectangle ABCD, P is a dynamic point on AD. PF⊥AC, PE⊥BD. E is on BD, and F is on AC. AD=12. AB=5. The problem: GP+PH=? Prove your result.

Figure 7.13. L5-Ex15.

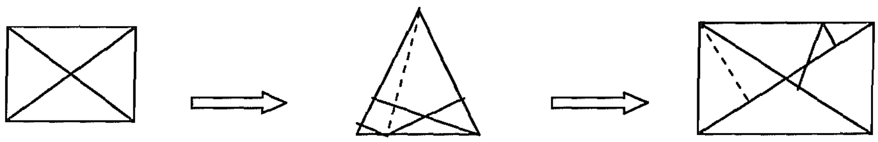
Students now understand how to calculate the sum of PF and PE by turning to firstly calculate AM, the altitude of OD in isosceles triangle AOD (#188, 189, L5). To calculate AM, the teacher guides them to turn to firstly calculate BD in right triangle ABD (#204, 207, L5). Beibei suggests using the area to calculate AM (S △ ABD=1/2AD*AB=1/2BD*AM, #207, L5). The teacher then guides students to analyse what they need to prove before proving PE+PF=AM (#212, L5). Interestingly, while many students respond to prove AO=OD to get isosceles triangle AOD (#215, L5), Linlin demonstrates his different thoughts as follows:

230. Linlin: ***We need to prove OB=OD as well.***
231. Lily: (surprised.) Why?
232. Linlin: ***Because, this one, .. . the sum of area of △AOP and △POD is △AOD. Next, prove OB=OD. Next, ABO=AOD. The replacement of equal elements.***

Extract 7.16. L5-Ex15, figure 7.13, #230-232.

Extract 7.16 shows that Linlin recognises the equal area of triangles ABO and AOD, and tries to use triangle ABO to replace triangle AOD.

Examples of students' learning responses across the different proof problems (L5-Ex12 to L5-Ex15) illustrate the complexity of students' geometrical thinking development in solving proof problems. First, students' thinking behaviours are different in different proof problems, as observed by Battista (2007), Burger & Shaughnessy (1986), Fuys *et al.*, (1988), Gutiérrez and Jaime (1988), and Senk (1989), etc. For some students, like Liuliu, Youyou and Beibei, while they are able to present deductive thoughts and insightful strategies for proving the problem in L5-Ex12 and L5-Ex13 (see extracts 7.12 and 7.13), they are not able to perceive the hidden geometric object, isosceles triangle ADO in L5-Ex14 (see extract 7.15). Thus, the visual thinking is not likely to be automatically developed with the growth of analytic and deductive thinking. It therefore seems very essential for the teacher to arrange appropriate problems to extend students' visual thinking to a higher level. For instance, to visualise the isosceles triangle ADO in L5-Ex14, students need to combine the visual imagine of L5-Ex12 and L5-Ex13 (see figure 7.14).



L5-Ex12 (see figure 7.10) | L5-Ex13 (see figure 7.11) | L5-Ex14 (see figure 7.12)

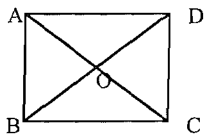
Figure 7.14. The combination of the visual images of two proof problems.

Next, as addressed by Fujita and Jones (2002), the analysis of a set of tasks (from L5-Ex12 to L5-Ex15) in this section substantiates the significance of designing certain tasks (exercises) to help develop students' geometric eyes ("the power of seeing geometrical properties detach themselves from a figure", Godfrey (1910, p.197)). That is, Lily arranges these tasks to help students to be aware of certain geometric objects, with the invariants of these objects. For instance, in L5-Ex13 (see figure 7.14), the visual image of the isosceles triangle is an acute triangle. In L5-Ex14 (see figure 7.14), the visual image of the isosceles triangle is an up-side-down obtuse triangle. Thus, students need to first appreciate the invariants of geometric objects of L5-Ex13 and L5-Ex14, namely, the isosceles triangle.

Thirdly, this study found that the scaffolding provided by Lily seems to support the development of students' insights into problem solving. van Hiele (1973, quoted in Hoffer, 1983, p.205) defines *insight* as follows:

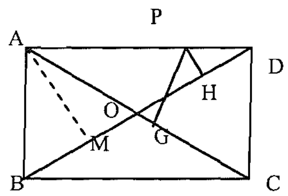
> "A person shows insight if the person (a) is able to perform in a possibly unfamiliar situation; (b) performs competently (correctly and adequately) the acts required by the situation; and (c) performs intentionally (deliberately and consciously) a method that resolves the situation. To have insight, students understand what they are doing, why they are doing it, and when to do it. They can apply their knowledge in order to solve problems."

Linlin's responses to L5-Ex15 (see extract 7.14) indicate that his insight into problem solving of L5-Ex15 is developed by the support from the teacher to explicate the visual, analytic and deductive relation of the figures in L5-Ex12, L5-Ex13 and L5-Ex14 (see figure 7.15).



Rectangle ABCD ➔ AO=BO=CO=DO.
(L5-Ex12, see figure 7.10)

Linlin perceives that the area of four isosceles triangles is equal: S△AOD=S△AOB=S△BOC=S△COD.



Isosceles triangle AOD, PG⊥AC, PH⊥BD, AM⊥BD ➔ PG+PH=AM.
(L5-Ex14, see figure 7.12)

Linlin perceives that
S△AOD=S△APG+ S△POD, and
S△AOD=S△AOB, so S△AOB=S△APG+ S△POD.
For Linlin, either 1/2AO*PG+1/2OD*PH=1/2AM*OD ➔ PG+PH=AM; or,
1/2AO*PG+1/2OD*PH=1/2AM*OB ➔ PG+PH=AM

Figure 7.15. An explanation of Linlin's insight shown in extract 7.14 (according to the interview with Linlin)

In addition, as discussed in the use of questioning to develop students' mathematical thinking in section 7.4, the arrangement of proof problems across L5-Ex12 to L5-Ex15 shown in this section actually demonstrates how the teachers facilitate students' mathematical thinking in problem solving, namely, the transformation of an unknown problem to a known problem, a complex problem to a simple problem (see figure 7.16).

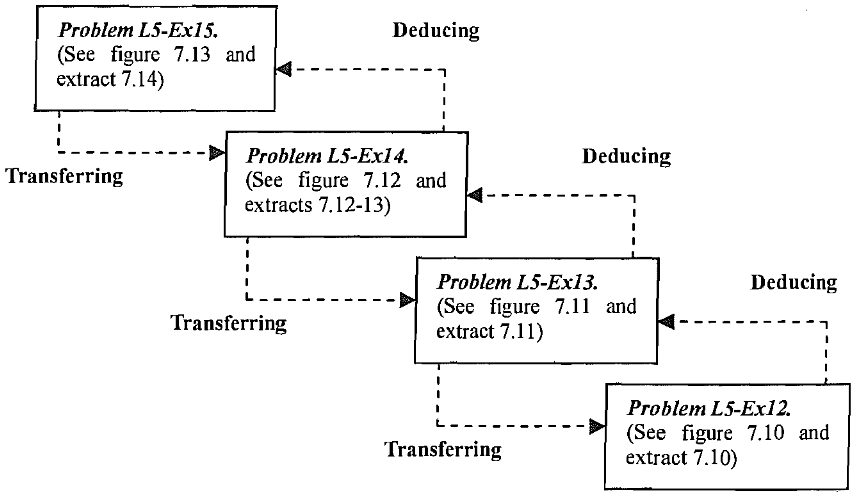


Figure 7.16. Scaffolding for solving problems.

As has been suggested by TSGofQp (1991) and Clements and Battista (1992), the analysis of the arrangement of problems in this section proves the necessity to further link theoretical ideas of the van Hiele levels and teaching with variation to Vygotsky's notions of "zone of proximal development" and "scaffolding", in order to elucidate the complexity of students' geometrical thinking development.

7.6 Towards developing a pedagogical framework

Section 7.2 provided an explanation of the relationship of the teachers' use of visual approaches with students' geometric thinking development in solving proof problems. It is found that teachers apply the theoretical idea of teaching with variation to address the significance of varying visual and concrete instances to help students to more fully understand abstract concepts. The five functions of the use of a visual approach identified in this section highlight the significance of developing students' geometric intuition through solving geometric proof problems, and suggest that it is an essential stage for teachers to help students to learn to recognise and analyse the visual geometric figures in proof problem solving activity.

Section 7.3 presented the unique ways of teachers' strategies in dealing with the use of empirical (inductive) /deductive approaches in teaching geometric proof problem solving. While the deductive approach is highly emphasised to draw students' attention to the powerful role of proof in generalising the truth of mathematical knowledge, the inductive approach is used to help develop students' insight into the interrelationship between the network of theorems. Moreover, the inductive approach might not necessarily mean letting students do experiments. In an advanced geometry course such as proof problem solving, the teacher might dynamically interpret a static figure as a way of using the inductive approach to support the development of students' insight into solving geometric proof problems.

Section 7.4 identified five types of teachers' questioning in the involvement of individual students in learning and in the development of their mathematical thinking. It was found that during the teaching process of solving a proof problem, the teachers' questions not only maintain the development of deductive reasoning at Level 3, but also involve the development of visual and analytic reasoning from Levels 1 and 2. Moreover, both the quality and the quantity of teachers' questions are likely to support the development of students' geometric thinking for writing proofs.

Section 7.5 concentrated on analysing and interpreting the use of scaffolding to support students' thinking development for writing proofs. The study demonstrates an appropriate design of tasks (exercises) may help develop students' geometric intuition in proof problem solving. Moreover, the study indicates that the scaffolding provided by teachers may support the development of students' insight into problem solving.

Based on the relevant literature and evidence from this study, a pedagogical framework is proposed to elucidate four key aspects of teachers' didactical practice in classrooms (visual approach, empirical/deductive approach, teachers' questioning and the arrangement of problems) towards the dynamic development of geometrical thinking in geometric proof problem solving (see figure 7.17).

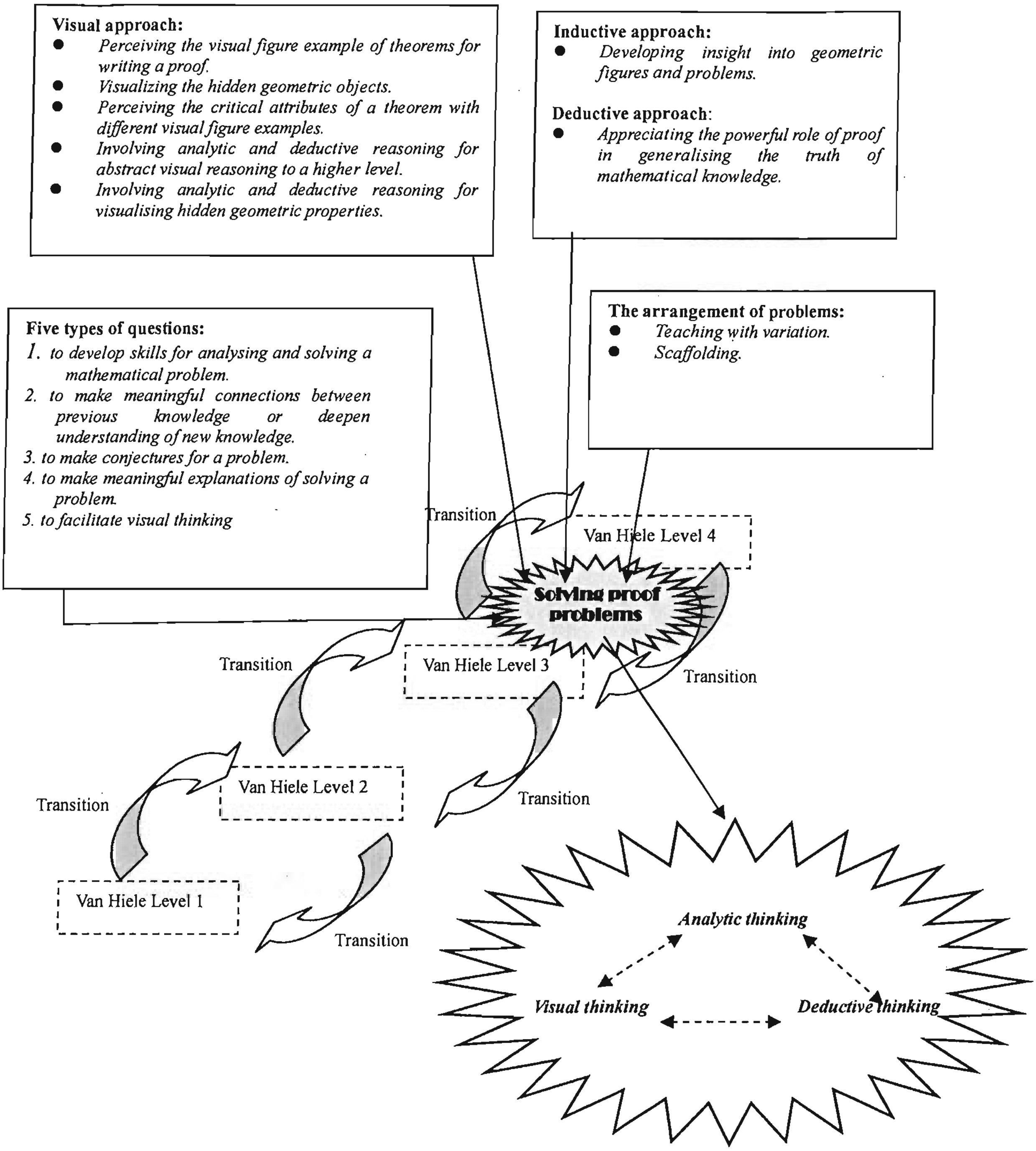


Figure 7.17. Proposing a pedagogical framework for teaching geometrical proof problem solving

CHAPTER 8. CONCLUSION

8.1. Overview

The two main aims of this thesis are:

- To explore and elucidate the complexity of individual teacher's didactical practice towards the development of students' thinking for writing proofs in geometry.
- To understand in what way the van Hiele model is a useful research tool to help analyse and interpret classroom teaching and learning of geometrical proof problem solving.

To explore the complex relationship of teachers' instruction with students' geometric thinking development, this study first developed the van Hiele theoretical framework for guiding the design of the study, and for collecting and analysing the relevant data. The study applied mixed methods, namely, qualitative case study approaches combined with quantitative analysis methods. Across three cases, analysis of this study presented the "match up" of the observation and small-scale survey data with the van Hiele theory. As a result of this analysis, an elaboration of the van Hiele model was presented. Finally, based on the findings from the analysis, an explanation was established to interpret the teachers' actual instruction across the three cases, which was not adequately elucidated by the van Hiele model.

In this chapter, the significant findings of the thesis are first summarised, which presents the usefulness of the van Hiele model and the complexity of individual teacher's didactical practice. The theoretical framework and the methodology of this study are then reflected upon. Finally, future research work which is growing directly from results of this study is given.

8.2. Summary and significance of findings

8.2.1 The usefulness of the van Hiele model

One key aim of this study is to understand whether the van Hiele model is a useful research tool to help analyse and interpret classroom teaching and learning of geometrical proof problem solving. Two questions are asked as follows:

- *Are the van Hiele levels useful in characterising students' learning responses and results of geometrical proof problem solving?*

- *Are the van Hiele phases useful in characterising teachers' actual classroom instruction of geometrical proof problem solving?*

In terms of the van Hiele levels, this study substantiates the usefulness of the levels in characterising the hierarchical nature of students' geometric thinking in solving geometric proof problems. The analysis of students' responses in the observed lessons and their learning results in homework and test papers in chapter 5 show that, the different types of students' learning responses and results could be assigned to the van Hiele Levels 1 to 3. Some students' learning responses and results might even link to the van Hiele Level 4.

However, further analysis of the different individual students' learning responses to the teachers' different instruction strategies and approaches across a range of proof problems illustrates a much more complex geometric thinking development than that ascribed by the van Hieles. In general, two aspects of such a complexity are addressed in this study as follows:

1) This study does not detect the discrete feature of the levels, which was noted by Usiskin (1982), Burger and Shaughnessy (1986), and Fuys *et al.* (1988), etc. Based on the analysis of observation data and small-scale survey data, this study proposes a dynamic view of the van Hiele levels of geometric thinking. That is, students' geometric thinking development in geometric proof problem solving might globally link to the van Hiele Level 3 and the transition toward Level 4. However, when the problem becomes difficult, students' thinking may go back to the analysis of the geometric figure and properties at Levels 1 and 2.
2) As observed by Gutiérrez *et al.* (1991, 1998), Lehrer *et al.* (1998), Pegg and Davey (1998) and Clements and Battista (2001), students' geometric thinking development is not likely to follow a simple, linear model as ascribed by the van Hieles. This study claims that students' geometric thinking development is extremely complex in solving geometric proof problems, as such development appears to involve the simultaneous development of a variety of kinds of thinking, such as visual, analytic and deductive thinking across levels. Moreover, findings of the study indicate that on the one hand, the visual, analytic and deductive thinking may concurrently grow up together; on the other hand, however, they may limit each other's development. In addition, this study also proves findings by Burger and Shaughnessy (1986), Fuys *et al.* (1988), Hershkowitz (1989), Mayberry (1983) that students' thinking

behaviours changed from one proof problem to another.

In terms of the van Hiele phases, this study shows that there is ambiguity in trying to identify exactly the phase at which the individual teachers were teaching, as it appeared that more than one interpretation was available (see the analysis in chapter 5 and the discussion in section 6.2). Moreover, this study does not detect the sequential feature of the phases to help students to make transitions to a higher level. The different order of the phases, as noted by Whitman *et al.* (1997) and Hoffer (1994), was confirmed by this study. Moreover, the analysis of observation and interview data from this study suggests that the description of students' learning in geometrical proof problem solving not only focuses on the relation of students and subject, but also the role of the teacher in building the bridge between students and subject in effective learning. Finally, the mismatch of the van Hiele phases with the teachers' actual classroom instruction of geometrical proof problem solving indicates that there may be other instructional strategies and approaches which support the development of students' thinking to write proofs.

8.2.2 The complexity of individual teacher's didactical practice

Another key aim of this study is to explore and elucidate the complexity of individual teacher's didactical practice towards the development of students' thinking for writing proofs in geometry. Four questions are addressed as follows:

- *How the visual approach is used during the teaching process of geometric proof problem solving? Why?*
- *How the inductive/deductive approaches are used during the teaching process of geometric proof problem solving? Why?*
- *How questioning strategies are used by teachers during the teaching process of solving geometrical proof problems? Why?*
- *How proof problems are arranged by teachers during the teaching process of solving geometrical proof problems? Why?*

This study provides an explanation of the relationship between the teachers' use of visual approach and students' geometric thinking development in solving proof problems. It was found that teachers apply the theoretical idea of teaching with variation to address the significance of varying visual and concrete instances to help students to more fully understand abstract concept. The five functions of the use of a visual approach identified in this study highlight the significance of developing students' geometric intuition through solving geometric proof problems, and suggest that it is an essential stage for teachers to help students to learn to recognise and analyse the visual geometric figures in proof problem solving activity.

This study presents the unique ways of teachers' strategies for dealing with the use of empirical (inductive) /deductive approaches in teaching geometric proof problem solving. While deductive approaches are highly emphasised to draw students' attention to the powerful role of proof in generalising the truth of mathematical knowledge, an inductive approach is used to help develop students' insight into the interrelationship of the network of theorems. Moreover, this study suggests that an inductive approach might not necessarily mean letting students do experiments. In an advanced geometry course such as proof problem solving, the teacher might dynamically interpret a static figure as a way of using an inductive approach to support the development of students' insight into solving geometric proof problems.

This study identifies five types of teachers' questioning in the involvement of individual students in learning and in the development of their mathematical thinking. It is found that during the teaching process of solving a proof problem, the teachers' questions not only maintain the development of deductive reasoning at Level 3, but also involve the development of visual and analytic reasoning from Levels 1 and 2. Moreover, both the quality and the quantity of teachers' questions are likely to support the development of students' geometric thinking for writing proofs.

This study also concentrates on analysing and interpreting the use of scaffolding to support students' thinking development for writing proofs. The study demonstrates that an appropriate design of tasks (exercises) may help develop students' geometric intuition in proof problem solving. Moreover, the study indicates that the scaffolding provided by teachers may support the development of students' insight into problem solving.

8.3 Reflections on the theoretical framework and the methodology of this study

This section reflects on the theory and methodology used in this study and discusses how well the research questions were answered.

In the first place, the researcher found that

- The development of a comprehensive understanding of the van Hiele model helps the study to focus on the observation and the analysis in depth of the dynamic thinking development through the teaching process of individual geometrical proof problem solving.
- The combination of case study and quantitative data in this study helps the study address better the research questions. To study in depth the teachers' classroom instruction, each single case included not only what the teacher actually taught in the classroom, but also the teachers' didactical view of their lessons and students' actual learning responses and outcomes during and after the observed lessons. To gain some sense of the representativeness of a larger population, the analysis of focused individual students' responses in the three cases was then compared with a small-scale survey data of students' learning outcomes in these three teachers' whole classes. Numerical data, such as time duration and frequency, helped to more effectively show the significance of the instructional strategies and approaches used by teachers in the observed lessons.
- The study confirms the usefulness of the analytic strategies of case study, namely, the "replication" logic, and the case studies analytic techniques, namely, "pattern matching" and "explanation building" (Yin, 2003), to establish the connection between the van Hiele theory and classroom practice. Thus, by using the case study analytic strategies and techniques, the study not only contributes to developing the view of the van Hiele theory, but also proposes a pedagogical framework for further study of the effectiveness of classroom instruction in learning geometrical proof problem solving.

The strengths of the methods to answer the research questions in the study are summarised as follows:

- Constant classroom observation over three weeks in selected individual teacher's classes provides rich data for the study to explore and elucidate in detail the significant function of a set of instructional strategies and approaches which may support the development of geometric thinking for writing a proof in natural classroom settings.

Thus, the results of the study contribute to fundamentally understanding the teachers' specific didactical efforts to improve students' cognitive processes in geometrical proof problem solving.

- Classroom observation enables the study to demonstrate the dynamic development of students' geometric thinking.
- Individual teacher's interviews help the study not only to examine what teachers did and said in classrooms, but also why they did and said so. It enables the study to highlight the role of the teacher in building the bridge between students and subject in effective learning..
- Small-scale surveys of students' learning results allow the study to infer from one case to a considerably larger population. Thus, the comparisons with a larger sample allow the study to establish some sense of the representativeness of the single case.

The limitations of the methods to answer the research questions in the study are summarised as follows:

- Observation data and small-scale survey data from homework and unit test papers are descriptive, but not sufficient to reveal fully the complexity of students' cognitive processes about solving different geometric proof problems.
- The study results contribute to elucidating the complexity of teachers' didactical practice towards the development of students' geometric thinking for writing proofs. However, the observation and small-survey data do not enable the study to interpret in great depth the pedagogical effects on students' cognitive processes.

8.4 Implications for future research

This section offers suggestions for future research on the van Hiele model and classroom practice in geometry, which arise directly from results of this study. In general, future research may relate to the following three aspects: 1) van Hiele-based research on students' geometric thinking development; 2) cognitive research in geometry; 3) international comparative study on curriculum and instruction in geometry.

1. van Hiele-based research on students' geometric thinking development

One future research aim is to extend the descriptions with examples of the van Hiele thinking levels in the context of geometric proof problem solving. This particular concern arises directly from the analysis of results from the classroom observations in this study. As demonstrated in section 5.5.1 and discussed in section 6.2, the majority of students' responses in the teaching process of individual geometric proof problem solving could be assigned to Level 3 thinking. However, the quality of their thinking seems to be different, from being able to make deductive reasoning of the relationship of the figure to not being able to distinguish the difference between a theorem and its converse theorem. Moreover, students' geometric thinking development in geometric proof problem solving might globally link to the van Hiele Level 3 and the transition toward Level 4. However, when the problem becomes difficult, these students' thinking may go back to the analysis of the geometric figure and properties at Levels 1 and 2. Thus, clinical interviews with a considerable number of students (same age but different learning attainments) appears to be necessary to make explicit the complexity of students' geometrical thinking in solving proof problems.

2. Cognitive research in geometry

Future cognitive research is needed to develop a fundamental understanding of students' visualisation (geometric intuition) development in geometry. This study substantiates "the prototype phenomenon" recognised by Hershkowitz (1989) and demonstrates the complexity of students' visualisation in geometric proof problem solving processes. The analysis of teachers' instructional approaches and strategies, such as the use of visual approaches, empirical/deductive approaches, teachers' questioning and the arrangement of problems (see in sections 7.2-7.5) further emphasises that it is an essential stage to develop students' visualisation (geometric intuition) in the teaching process of proof problem solving. Future work will focus on linking Godfrey's notion of geometrical eye which is emphasised by Fujita and Jones (2002) and Fischbein's (1993) theoretical idea of "figural concepts" to teachers' practical data demonstrated in this study, such as the five functions of teachers' use of visual approaches and the unique ways teachers dealing with the empirical/deductive approach.

Research that builds on the strengths of the theory of teaching with variation and Vygotsky's notions of the "zone of proximal development" and "scaffolding" in students' development might have potential. This study demonstrates that Chinese teachers apply theoretical ideas of the teaching with variation into teaching proof problem solving. According to Gu *et al.* (2004), Vygotsky's theory of scaffolding is linked to one type of teaching with variation, namely, procedural variation, which is commonly used in Chinese classroom teaching (in Chinese "scaffolding" is called "Pu Dian" (铺垫)). Thus, the theory of teaching with variation and Vygotsky's notions of the "zone of proximal development" and "scaffolding" in students' development possesses the commonalities in their view of knowledge construction. A synthesis of these would possibly yield a richer, more veridical pedagogical model. Ideally, such a model would have the explication of teaching with variation perspective and the developmental aspects of Vygotsky's notions of the "zone of proximal development" and "scaffolding".

3. International comparative study on curriculum and instruction in geometry

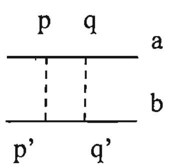
This study proposes a pedagogical framework which is based on teachers' didactical practice in geometric proof problem solving in naturalistic Chinese secondary school classrooms in Shanghai. Though teachers use the same textbook and follow the same city mathematics curriculum, a diversity of teaching approaches and strategies are identified in this study. Thus, the study indicates that students' thinking development appears to depend on individual teacher's didactical thoughts about the subject and their actual instruction in the class. It would be very interesting to use the pedagogical framework proposed in this study as a research tool to analyse and interpret teachers' didactical practice in geometric proof problem solving across countries. Such an international comparative study will help understanding about what instructional approaches and strategies are more universal across cultures, and what might be more local phenomena. In addition, such a study will develop understanding of the significance of curricula in relation to teachers' didactical views and practice and students' learning attitudes and outcomes in geometry.

APPENDIX A. The structure of Lily's 12 lessons (L1-12)
L1

Introduction 0'00-2'50 (2'50)

Teaching Proof 2'50-12'20 (9'30)

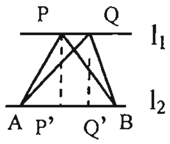
1) PR1



Given: lines a//b, p and q are on a, both pp' and qq' are vertical to line b. p' and q' are on b.
Prove: pp' //= qq'

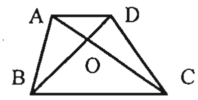
Exercises 12'20 -33'05 (20'45)

1) Ex1
(5'00)



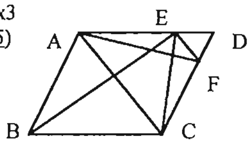
Given: lines l1// l2. P and Q are on l_1, A, P',Q' and B are on l_2.
Prove: S△PAB=S△QAB

2) Ex2
(9'00)



Given: In a quadrilateral ABCD, AD//BC. S△AOD=4, and S△ABO=5.
Prove: The area of the quadrilateral ABCD=?

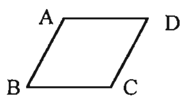
3) Ex3
(6'45)



Given: In a parallelogram ABCD, EF//AC.
Prove: Which triangle is equal to the area of △ABE?

Teaching Proof 33'05 – 40'30 (7'25)

2) PR2



Given: In quadrilateral ABCD, AB=CD, AD=BC.

Prove: Quadrilateral ABCD is a parallelogram.

Figure L1.

L2

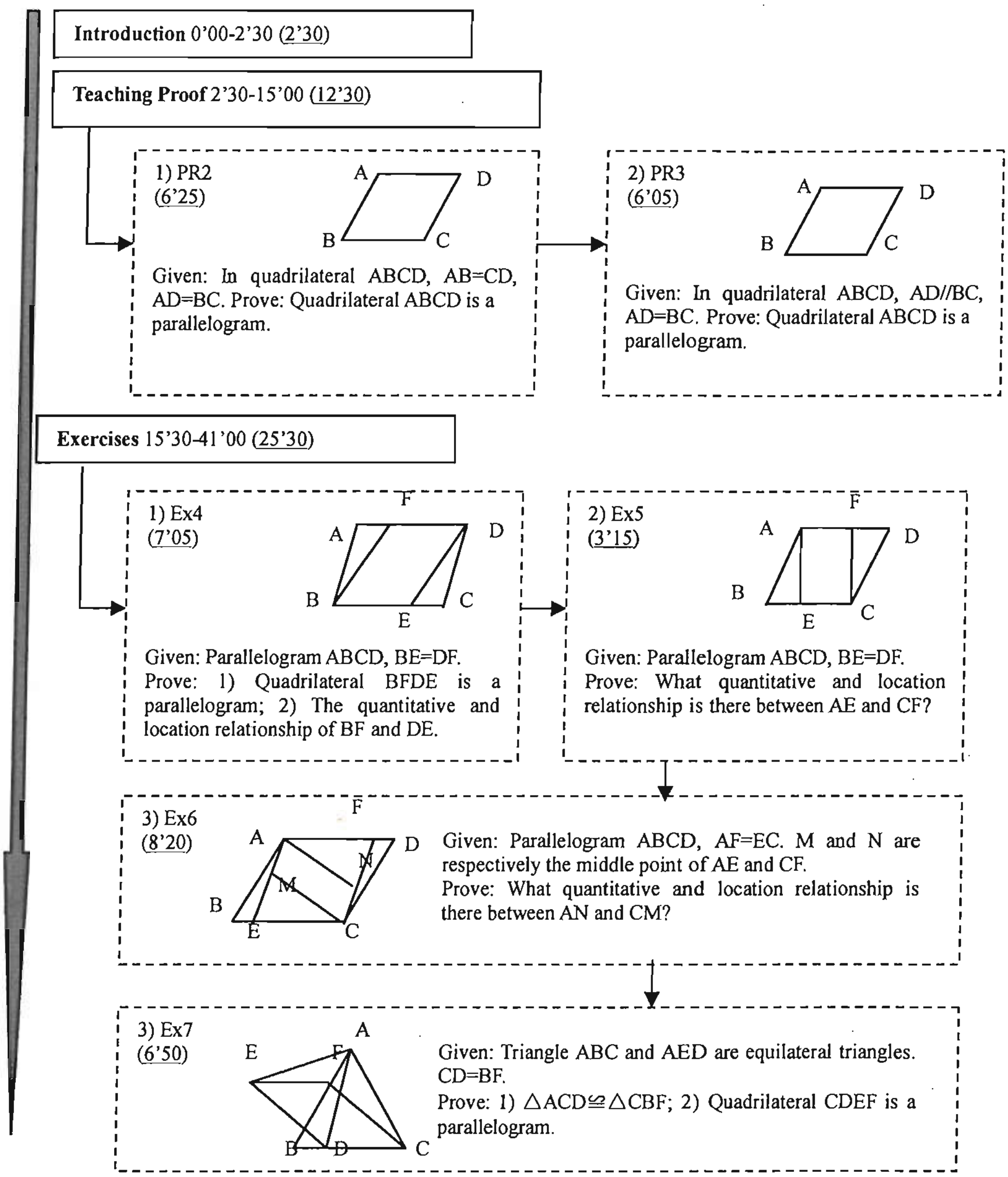


Figure L2.

L3

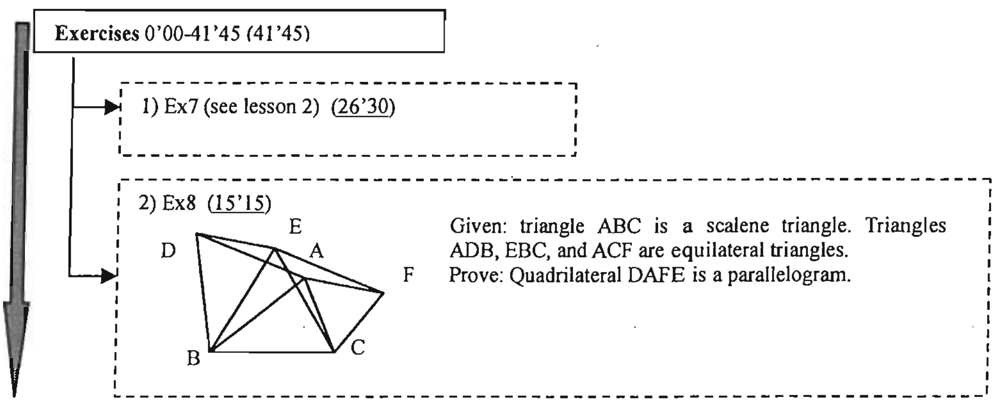


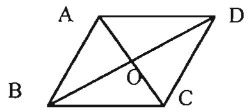
Figure L3.

L4

Review 0'00-6'50 (6'50)

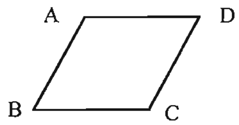
Teaching Proof 6'50- 23'20" (16'30)

1) PR5
(11'00)



Given: In quadrilateral ABCD, AO=OC, BO=OC.
Prove: Quadrilateral ABCD is a parallelogram.

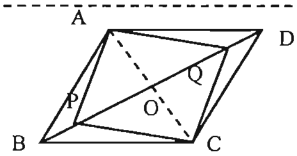
2) PR4
(5'30)



Given: In quadrilateral ABCD, ∠A=∠C, ∠B=∠D.
Prove: Quadrilateral ABCD is a parallelogram.

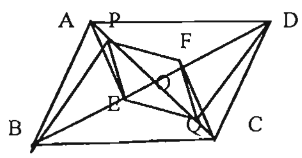
Exercises 23'20- 42'10 (18'50)

1) Ex9
(6'30)



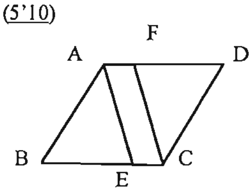
Given: In parallel ABCD, BP=QD.
Prove: Quadrilateral APCQ is a parallelogram.

2) Ex10
(7'10)



Given: In parallel ABCD, BP⊥AC, DQ⊥AC, AE ⊥ BD, CF ⊥ BD. Prove: Quadrilateral PEQF is a parallelogram.

3) Ex11 (5'10)



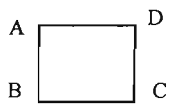
Given: In parallel ABCD, AE is the bisector of angle BAD, CF is the bisector of angle BCD. Prove: Quadrilateral AECF is a parallelogram.

Figure L4.

L5

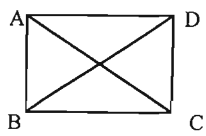
Teaching Proof 0'00- 12'20 (12'20)

1) PR6 and the definition of rectangle (6'20)



Given: In rectangle ABCD, $\angle ABC=90°$. Prove: Each interior angle of rectangle ABCD is 90°.

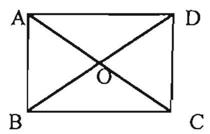
2) PR8 (6'00)



Given: In rectangle ABCD, AC and BD are its diagonals. Prove: AC=BD.

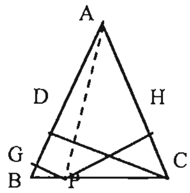
Exercises 12'20- 40'20 (28'00)

1) Ex12 (5'30)



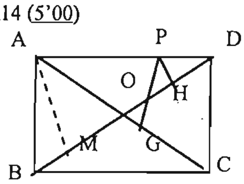
Given: in rectangle ABCD, AC and BD are diagonals. Prove: AO=BO=CO=DO.

2) Ex13 (7'10)



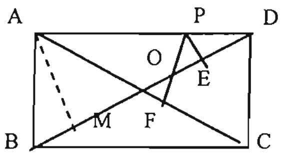
Given: in isosceles triangle ABC, AB=AC. $CD\perp AB$, $GP\perp AB$, and $PH\perp AC$. D and G are on AB. H is on AC. Prove: DC=GP+PH.

3) Ex14 (5'00)



Given: in rectangle ABCD, P is a dynamic point on AD. When P moves along AD, whether is the sum of the distances from P to AC and BD (namely, PG and PH) changeable? If it is not changeable, how much is the sum of PG and PH?

4) Ex15 (10'20)



Given: in rectangle ABCD, P is a dynamic point on AD. $PF\perp AC$, $PE\perp BD$. E is on BD, and F is on AC. AD=12. AB=5. The problem: GP+PH=? Prove your result.

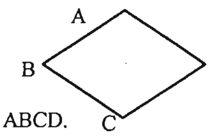
Figure L5.

L6

Review 0'00-4'15 (4'35)

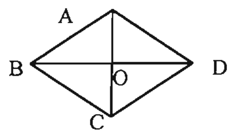
Teaching the definition of rhombus, PR7 and PR9 4'35-8'10, 8'10-14'05, 22'15-23'55 (3'35+5'55+1'40)

1) PR7 (3'35)



Given: Rhombus ABCD.
Prove: AB=BC=CD=AD.

2) PR9 (5'55+1'40)



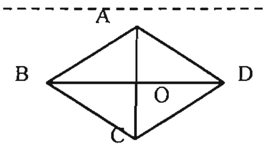
Given: Rhombus ABCD.
Prove: 1) AC ⊥ BD; 2) AC and BD are bisectors of angles BAD, ACD, ABC, and ADC.

Review (14'05 – 15'40 (1'35)) and Proof 9.1 15'40 – 22'15 (6'35)

PR9.1 (See figure in PR9)
Given: Rhombus ABCD.
Prove: The area of rhombus ABCD is 1/2AC*BD.

Exercises 23'55- 29'30 (5'35)

1) Ex16
(4'35)



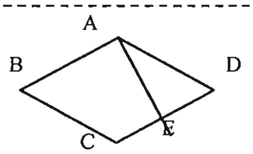
Given: ∠ DBC=25°, ∠ ABD=X°, ∠ CAD=Y°. The problem: X=? Y=?

2) Ex17 (1'00)
Which is the correct one of the property of rhombus?
a) Two diagonals are equal; b) Two diagonals are vertical to each other; c) Four angles are equal; d) Its' symmetry axis is a straight line which crosses the middle points of any pair of its opposite sides.

Teaching new knowledge 29'30-32'40 (3'10)

Exercises 32'40- 40'20 (7'40)

3) Ex18
(7'40)



In rhombus ABCD, AE is the height on CD. E is on CD. AE also bisects CD. Which is the right one of the largest angle of the rhombus?
a) 105°; b) 120°; c) 135°; d) 150°.

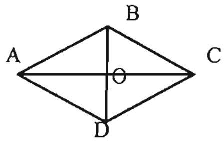
Figure L6.

L7

Review 0'00- 2'30 (2'30)

Exercises 2'30- 40'35 (37'35)

1) Ex19 (5'45)

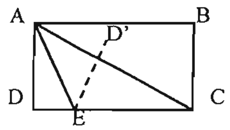


Given: In rhombus ABCD, AB=13cm, AC=24cm. The problem: calculate the area of rhombus ABCD, and prove your result.

2) Ex20 (4'50)
Given: In rhombus ABCD, one of its sides is 6cm. One of its angles is 60°. The problem: calculate the length of the diagonals of rhombus ABCD, and prove your result.

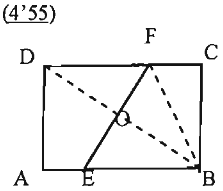
3) Ex21 (1'10)
Given: In rhombus ABCD, its perimeter is 32cm. Its shorter diagonal BD is 8cm. The problem: calculate the area of rhombus ABCD, and prove your result.

4) Ex22 (5'10)



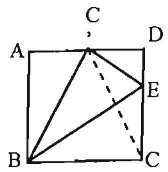
Given: In rectangle ABCD, AB=2, BC=1. E is on DC. ∠DAE=∠EAC.
The problem: Calculate the length of EC, and prove your result.

5) Ex23 (4'55)



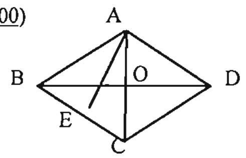
Given: in rectangle ABCD, AB=8cm, BC=6cm. If rectangle ABCD is folded, and B meets D, EF is crease.
The problem: calculate the length of EF, and prove your result.

6) Ex24 (9'15)



Given: in rectangle ABCD, AB=12cm, ∠C'BE=30°. To fold rectangle ABCD, and to make C meet C'. C' is on AD. BE is 折痕. The problem: calculate the length of BE, and prove your result.

7) Ex25 (7'00)



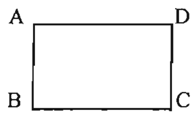
Given: in rhombus ABCD, its diagonals AC and BD are met at O. AE⊥BC. E is on AD. AE=OB. The problem: calculate the degree of ∠CAE, and prove your result.

Figure L7.

L8

Teaching proof 0'00- 18'30 (18'30)

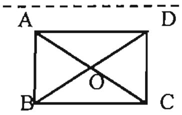
1) Proof 11 (9'30)



Given: in quadrilateral ABCD, ∠A=∠B=∠C=90°. To prove: quadrilateral ABCD is a rectangle.

→

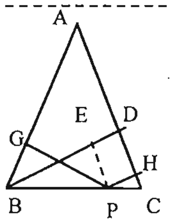
2) Proof 11 (9'00)



Given: in parallelogram ABCD, AC=BD. To prove: Parallelogram ABCD is a rectangle.

Exercises 18'30 – 41'00 (22'30)

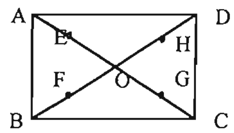
1) Ex26 (Ex13) (12'30)



Given: in isosceles triangle ABC, AB=AC. CD⊥AB, GP⊥AB, and PH⊥AC. D and G are on AB. H is on AC. Prove: DC=GP+PH.

→

2) Ex27 (10'00)



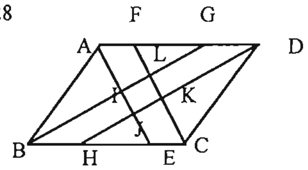
Given: in rectangle ABCD, AC and BD are its diagonals. E and G are on AC, F and H are on BD. They are dynamic points. AE=BF=CG=DH.
To prove: quadrilateral EFGH is a rectangle.

Figure L8.

L9

Exercises 0'00-7'30 (7'30)

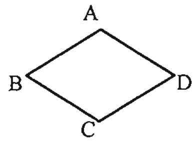
Ex28



Given: in parallelogram ABCD, AE, CF, BG, and DH are respectively bisectors of ∠BAD, ∠BCD, ∠ABC, and ∠ADC. They are respectively crossed at I, J, K and L. F and G are on AD. H and E are on BC.
Prove: quadrilateral IJKL is a rectangle.

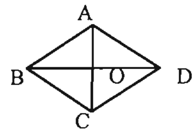
Teaching proofs 7'30-15'50 (8'20)

1) PR10 (4'20)



Given: In quadrilateral ABCD, AB=BC=CD=AD.
Prove: quadrilateral ABCD is a rhombus.

2) PR13 (4'00)



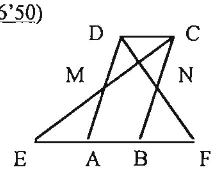
Given: In parallelogram ABCD, its diagonals AC and BD are vertical to each other.
Prove: parallelogram ABCD is a rhombus.

Exercises 15'50-40'00 (24'10)

Ex29 (4'10)

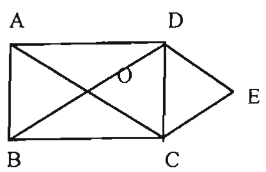
In parallelogram ABCD, its diagonals AC and BD are crossed at O.
If AB=AD, then parallelogram ABCD is ____.
If AC=BD, then parallelogram ABCD is ____.
If ∠ABC is a right angle, then parallelogram ABCD is ____.
If ∠BAO=∠DAO, then parallelogram ABCD is ____.

Ex30 (16'50)



Given: in parallelogram ABCD, AD=2AB. AE=AB=BF.
Prove: quadrilateral DMNC is a rhombus.

Ex31 (3'10)



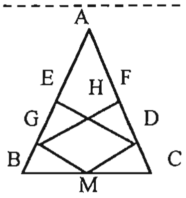
Given: in rectangle ABCD, its diagonals are crossed at O. DE//AC, CE//DB.
Prove: quadrilateral OCED is a rhombus.

Figure L9.

L10

Exercises 0'00-5'00 (5'00)

Ex32

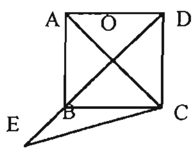


Given: in triangle ABC, AB=AC. M is the middle point of BC. MG ⊥AB, MD⊥AC. G and D are respectively on AB and AC. GF⊥AC, DE⊥AB. E and F are respectively on AB and AC. DE and GF are crossed at H.
Prove: quadrilateral HGMD is a rhombus.

Teaching the definition and property of square 5'00-14'50 (9'50)

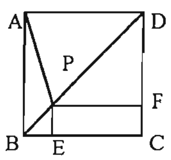
Exercises 14'50-40'00 (25'10)

Ex33 (8'30)



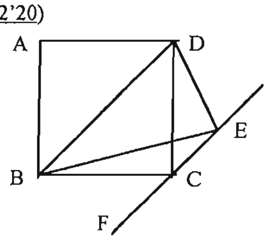
Given: in square ABCD, its diagonals AC and BD are crossed at O. E is on the extended line of OB. ∠ECB=15 .
Prove: CE=BD.

Ex34 (4'20)



Given: in square ABCD, BD is its diagonal. P is a dynamic point on BD. To connect A and P. PF⊥DC, PE ⊥BC.
Prove: PA=EF.

Ex35 (12'20)



Given: in square ABCD, BD is it diagonal. BD//EF. BD=BE.

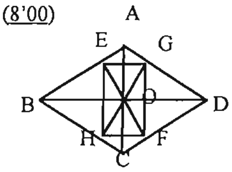
Prove: DG=DE.

Figure L10.

L11

Homework feedback 0'00-9'20 (1'20)

Ex36 (8'00)

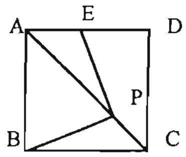


Given: in rhombus ABCD, AC and BD are its diagonals. They are crossed at O. EF is vertical to both AB and CD. E and F are respectively on AB and CD. GH is vertical to both AD and BC. G and H are respectively on AD and BC. EF and GH are crossed at O.
Prove: EHFG is a rectangle.

Review the property of square 9'20-12'00 (2'40)

Exercises 12'00-20'20 (8'20)

Ex37



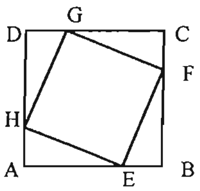
Given: in square ABCD, AC is its diagonal. P is a dynamic point on AC. BP$\perp$EP.
Prove: BP=EP.

Proof 14 & 15 20'20-24'20 (4'00)

Proof 14: If one angle of a rhombus is a right angle, then it is a square.
Proof 15: If a pair of adjacent sides of a rectangle is equal, then it is a square.

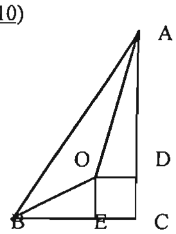
Exercises 24'20-40'00 (15'40)

Ex38 (6'00)



Given: in square ABCD, AE=BF=CG=DH.
Prove: quadrilateral EFGH is a square.

Ex39 (2'10)



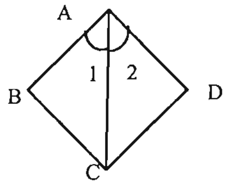
Given: in right triangle ABC, $\angle C=90$. The bisectors of $\angle A$ and $\angle B$ are met at D. DE$\perp$AC, DF$\perp$BC. E and F are respectively on AC and BC.

Prove: quadrilateral DFCE is a square.

Ex40 (2'30)

Is a quadrilateral a square, if its diagonals are equal and vertical to as well as bisect each other? Why?

Ex41 (0'45)



Given: in parallelogram ABCD, $\angle 1=\angle 2=45$.
Prove: quadrilateral ABCD is a square.

Ex42 (4'15)

In parallelogram ABCD, its diagonals AC and BD are crossed at O.

1) If AB=BC, and AC=BD, then parallelogram ABCD is ___.
2) If OA=OB, and OA $\perp$ OB, then parallelogram ABCD is ___.
3) If $\angle$ AOB=90 , $\angle$ BAD=90 , then parallelogram ABCD is ___.
4) If $\angle$BAO=$\angle$DAO, $\angle$BAO=$\angle$ABO, then parallelogram ABCD is ___.

Figure L11.

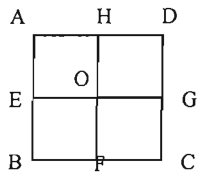
L12

Exercises 0'00- 40'00 (40'00)

Ex43 (4'45)
To prove: if the diagonals of a rectangle are vertical to each other, then this rectangle is a square.

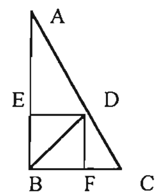
Ex44 (1'15)
To prove: if four sides of a quadrilateral and its diagonals are equal, then it is a square.

Ex45 (10'40)



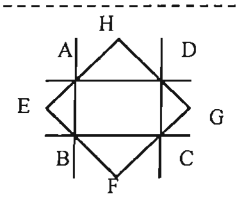
Given: in square ABCD, E, F,G, and H are respectively the middle point of AB, BC, CD and DA. EG and HF are crossed at O.
Prove: quadrilaterals AEOH, EBFO, FCGO, OGDH are square.

Ex46 (1'20)



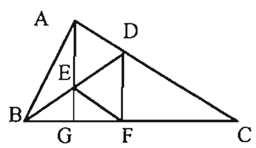
Given: in triangle ABC, $\angle$ABC=90 . BD is the bisector of $\angle$ABC. BD and AC are crossed at D. DE$\perp$AB. DF$\perp$BC. E is on AB. F is on BC.
Prove: quadrilateral BFDE is a square.

Ex47 (4'15)



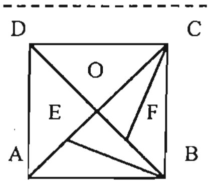
Given: in rectangle ABCD, the bisectors of its external angles are crossed at E, F, G and H.
Prove: quadrilateral EFGH is a square.

Ex48 (9'15)



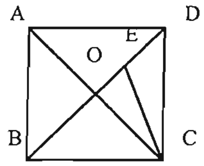
Given: in right triangle ABC, $\angle$BAC=90 . AG$\perp$BC. G is on BC. BD is the bisector of $\angle$ABC. DF$\perp$BC. F is on BC.
Prove: quadrilateral AEFD is a rhombus.

Ex49 (1'30)



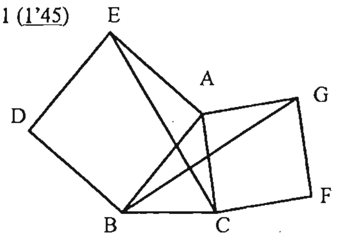
Given: in square ABCD, its diagonals AC and BD are crossed at O. OE=OF.
Prove: $\angle$ACF=$\angle$DBE.

Ex50 (<u>1'30</u>)



Given: in square ABCD, AC and BD are its diagonals. E is on BD. BE=BC.
The problem: ∠BEC=? ∠ACE=? Prove your results.

Ex51 (<u>1'45</u>)



Given: in triangle ABC, based on AB and AC, squares ABDE and ACFG are drawn.
Prove: 1) CE=BG; 2) CE⊥BG

Figure L12.

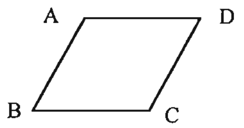
APPENDIX B. The structure of Spring's 8 lessons (S1-8)

S1

Review 0'00"- 6'20 (6'20)

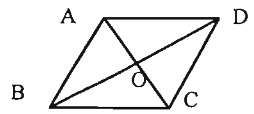
Teaching Proof 6'20- 14'40 (8'20)

1) PR5
(4'50)



Given: In quadrilateral ABCD, ∠A=∠C, ∠B=∠D.
Prove: Quadrilateral ABCD is a parallelogram.

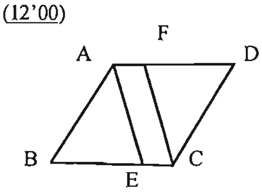
2) PR4
(3'30)



Given: In quadrilateral ABCD, AO=OC, BO=OC.
Prove: Quadrilateral ABCD is a parallelogram.

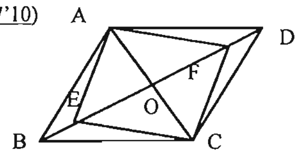
Exercises 14'4- 40'00 (25'20)

1) Ex1 (12'00)



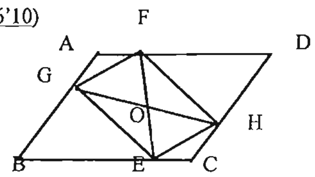
Given: In parallel ABCD, AE is the bisector of angle BAD, CF is the bisector of angle BCD. Prove: Quadrilateral AECF is a parallelogram.

2) Ex2 (7'10)



Given: In parallel ABCD, BE=EF=FD.
Prove: Quadrilateral AECF is a parallelogram.

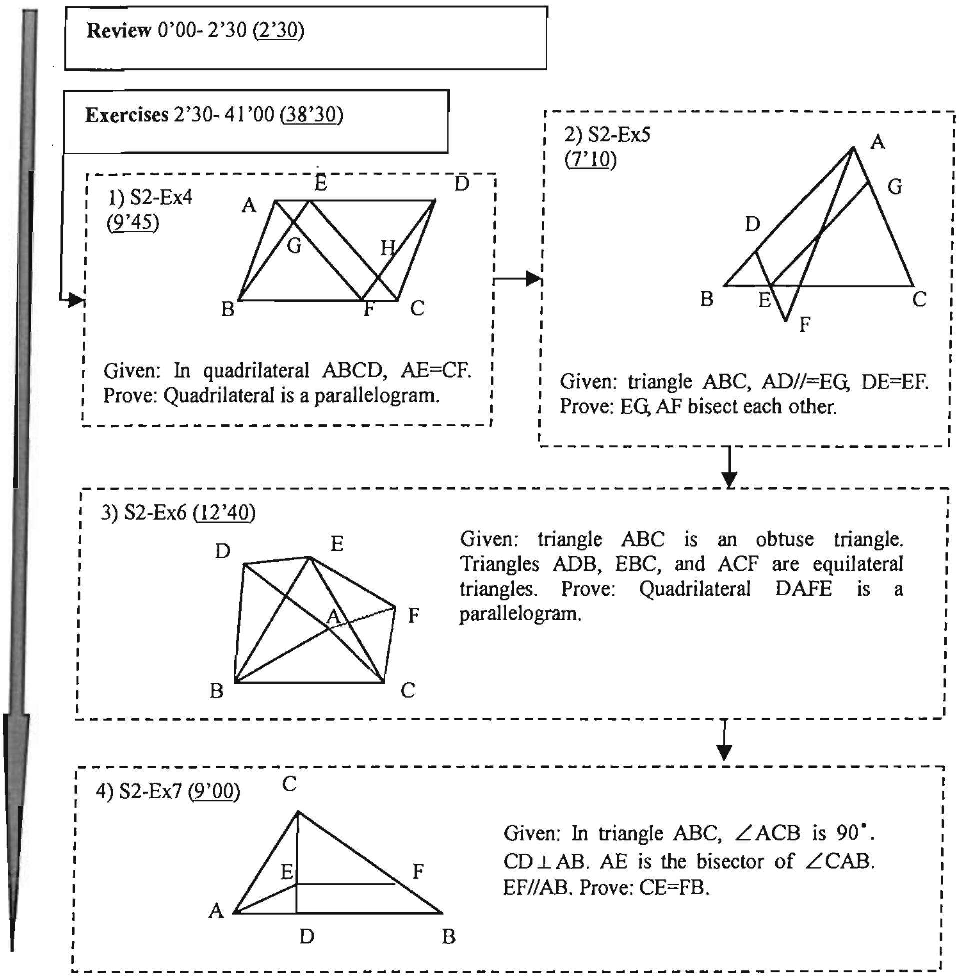
3) Ex3 (6'10)



Given: In parallel ABCD, AE is the bisector of angle BAD, CF is the bisector of angle BCD. Prove: Quadrilateral AECF is a parallelogram.

Figure S1.

S2



Review 0'00- 2'30 (2'30)
Exercises 2'30- 41'00 (38'30)
1) S2-Ex4 (9'45)
A
E
D
G
H
B
F
C
Given: In quadrilateral ABCD, AE=CF.
Prove: Quadrilateral is a parallelogram.
2) S2-Ex5 (7'10)
A
G
D
B
E
C
F
Given: triangle ABC, AD//=EG, DE=EF.
Prove: EG, AF bisect each other.
3) S2-Ex6 (12'40)
D
E
A
F
B
C
Given: triangle ABC is an obtuse triangle. Triangles ADB, EBC, and ACF are equilateral triangles. Prove: Quadrilateral DAFE is a parallelogram.
4) S2-Ex7 (9'00)
C
E
F
A
D
B
Given: In triangle ABC, ∠ACB is 90°. CD⊥AB. AE is the bisector of ∠CAB. EF//AB. Prove: CE=FB.

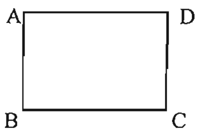
Figure S2.

S3

Teaching definition of rectangle and rhombus 0'00- 10'00 (10'00)

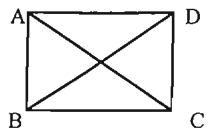
Teaching Proof 6 and 8 10'00- 23'20 (13'20)

1) PR6



Given: In rectangle ABCD, ∠ABC=90°.
Prove: Each interior angle of rectangle ABCD is 90°.

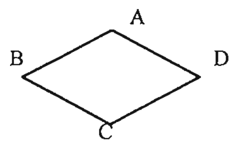
2) PR8



Given: In rectangle ABCD, AC and BD are its diagonals.
Prove: AC=BD.

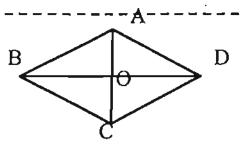
Teaching proof 7 and 9 23'20– 31'20 (8'00)

1) PR7



Given: Rhombus ABCD. AB=AD.
Prove: AB=BC=CD=AD.

2) PR9

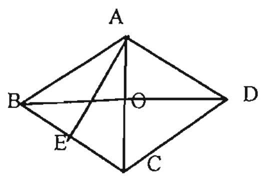


Given: Rhombus ABCD. AC and BD are its diagonals. They are intersected at O.
Prove: AC and BD are vertical to one another, and they bisect opposite angles of rhombus ABCD.

Teaching new knowledge 31'20 – 34'10 (2'50)

Exercises 34'10 – 42'20 (8'10)

1) Ex8



Given: in rhombus ABCD, AC and BD are its diagonals. They are intersected at O. AE is the height of BC. E is on BC. AE=BO. The problem: calculate the degree of ∠BAE, and prove the result you obtain.

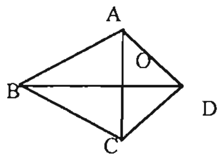
Figure S3.

S4

Review 0'00-1'20 (1'20)

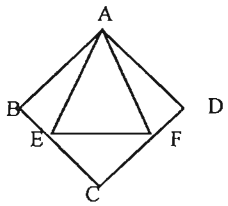
Teaching new knowledge 1'20-10'20 (9'00)

Exercises 10'20-41'00 (30'40)



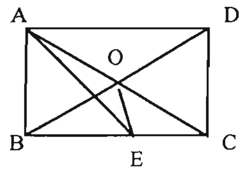
1) Ex 9 (1'00)
a) if AC=7, BD=9, then the area of ABCD is ____.
b) if AC=5, BD=9, then the area of ABCD is ____.

2) Ex10 (15'00)



Given: rhombus ABCD, equilateral triangle AEF. E is on BC. F is on CD. AB=AE.
The problem: how much is each angle of rhombus ABCD? Prove your result.

3) Ex11 (14'40)



Given: In rectangle ABCD, its diagonals AC and BD are crossed at O. AE is the bisector of angle BAD. Connect OE. Angle ADB is 30 .
The problem: how much is angle BOE? Prove your result.

Figure S4.

S5

Teaching new knowledge 0'00-22'00 (22'00)
1) The definition of rectangle; 2) Proof 11 & 12; 3) The definition of rhombus; 4) Proof 10 & 13.

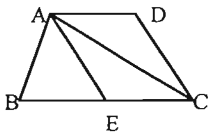
Exercises 22'00-40'00 (18'00)

1) Ex12 (6'00)
a. In __, if one of its angle is a right angle, then it is a rectangle; in __, if its adjacent sides are equal, then it is a rhombus.
b. If the diagonals of a parallelogram are __, then it is a rhombus; if the diagonals of a parallelogram are __, then it is a rectangle.
c. If four sides of a quadrilateral are equal, then it is a __; if three angles of a quadrilateral are right angles, then it is a __.
d. If the diagonals of a quadrilateral are not only vertical but also bisect to each other, then it is a __; if the diagonals of a quadrilateral are not only equal but also bisected to each other, then it is a __.
e. If one angle of a quadrilateral is a right angle, and its diagonals are __, then it is a rectangle; if a pair of adjacent sides of a quadrilateral is equal, and its diagonals are __, then it is a rhombus.

2) Ex13 (3'00)

A carpenter wants to make a frame of a door. He has two pieces of equal long woods, and two pieces of equal short woods. Could you help him to make sure that the door frame is a rectangle? How?

2) Ex14 (9'00)



Given: in quadrilateral ABCD, AD=CD. AD//BC. Angle BAC=90°. AE is a bisector of BC.
To prove: quadrilateral AECD is a rhombus.

Figure S5.

S6

Teaching new knowledge (30'00)
1) The definition of a square; 2) The property of a square; 3) Proof 14 & 15

Exercises
1) Ex15 (10'00)
In parallelogram ABCD, its diagonals AC and BD are crossed at O.

1) If AB=BC, and AC=BD, then parallelogram ABCD is ___.
2) If OA=OB, and OA⊥OB, then parallelogram ABCD is ___.
3) If ∠AOB=90 , ∠BAD=90 , then parallelogram ABCD is ___.
4) If ∠BAO=∠DAO, ∠BAO=∠ABO, then parallelogram ABCD is ___.

Figure S6.

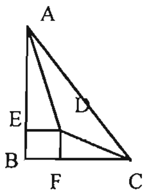
S7

Review (4'20)

Exercises (38'20)

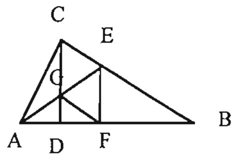
1) Ex15 (4'30) (see figure S6 above)

2) Ex16 (8'00)



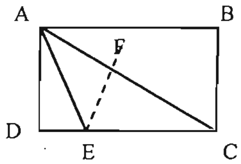
Given: in right triangle ABC, ∠C=90 . The bisectors of ∠A and ∠B are met at D. DE⊥AC, DF⊥BC. E and F are respectively on AC and BC.
To prove: quadrilateral DFCE is a square.

3) Ex17 (10'50)



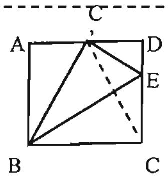
Given: in right triangle ABC, ∠ACB=90 . CD ⊥AB. AE is the bisector of ∠CAB.
EF⊥AB.
To prove: quadrilateral CGFE is a rhombus.

4) Ex18 (9'50)



Given: in rectangle ABCD, AB=2, BC=1. E is on DC. ∠DAE=∠EAC.
The problem: Calculate the length of EC, and prove your result.

5) Ex19 (5'00)



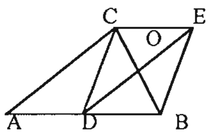
Given: in rectangle ABCD, AB=12cm, ∠C'BE=30. To fold rectangle ABCD, and to make C meet C'. C' is on AD. BE is 折痕.
The problem: calculate the length of BE, and prove your result.

Figure S7.

S8

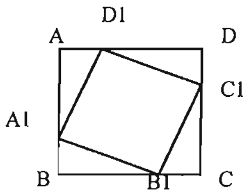
Review homework (20'00)

1) Ex20 (5'10)



Given: in right triangle ABC, ∠ACB=90°. CD is the bisector of AB. DE is the bisector of ∠CDB. DE=AC.
To prove: quadrilateral BECD is a rhombus.

2) Ex21 (3'20)
See Ex17 in figure S7.

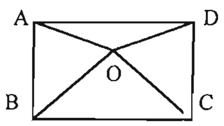


3) Ex22 (10'30)

Given: in square ABCD, each size is 1. How to cut off its four angles to get square A1B1C1D1? Prove your result.

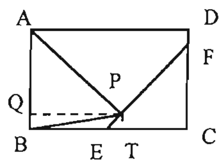
Exercises (20'00)

4) Ex23 (7'40)



Given: in rectangle ABCD, ∠ OAD= ∠ ODA=1/4∠BOC. To prove: OB=OC=AB.

5) Ex24 (12'20)



Given: in rectangle ABCD, AB=4, BC=6, CE=CF=3. P is on EF. EP=a. The problem: how large is the area of triangle APB?

Figure S8.

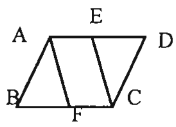
APPENDIX C. Nana's lesson structure (N1-3)

N1

Introduction 0'00- 3'40 (3'40)

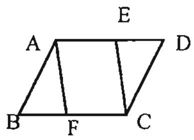
Exercises 3'40- 46'00 (42'20)

1) Ex1
(12'00)



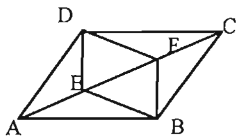
Given: In quadrilateral ABCD, E and F are respectively the middle point of AD and BC.
Prove: Quadrilateral AFCE is a parallelogram.

2) Ex2
(5'40)



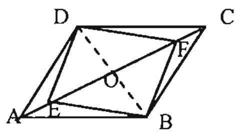
Given: In quadrilateral ABCD, E and F are respectively on AD and BC.
Prove: What are needed, in order to prove that quadrilateral AFCE is a parallelogram.

3) Ex3
(10'00)



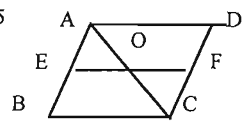
Given: In parallelogram ABCD, AE=EF=FC.
Prove: Quadrilateral DEBF is a parallelogram.

4) Ex4
(4'40)



Given: In parallelogram ABCD, AE=CF.
Prove: Quadrilateral DEBF is a parallelogram.

5) Ex5
(7'00)



Given: In parallelogram ABCD, E and F are respectively the middle point of AB and CD.
Prove: EF//BC.

Figure N1.

N2

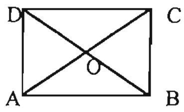
Review 0'00- 2'00 (2'00)

Teaching new knowledge 2'00- 32'00 (30'00)
1) The definition of rectangle; 2) The definition of rhombus; 3) Proof 6, 7, 8, 9.

Review 32'00- 34'00 (2'00)

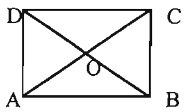
Exercises 34'00- 46'00 (12'00)

1) Ex1
(3'00)



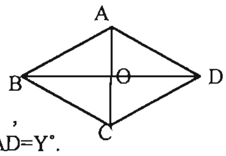
Given: ∠DAC=65 , ∠DBA=Y , ∠ACB=X .
The problem: X=? Y=?

2) Ex2
(3'00)



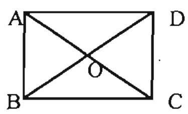
Given: OA=2x, OC=2y+4, OB=x-y.
The problem: x=? y=?

3) Ex3
(2'00)



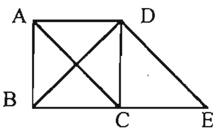
Given: ∠DBC=25 , ∠ABD=X , ∠CAD=Y°.
The problem: X=? Y=?

4) Ex4
(2'00)



Given: in rectangle ABCD, ∠AOB=60°.
Which choice is correct?
a) AC+BD=AB+BC+DC+DA;
b) BD=2AB;
c) AC+BD=AB+BC;
d) None is correct.

5) Ex5
(1'00)



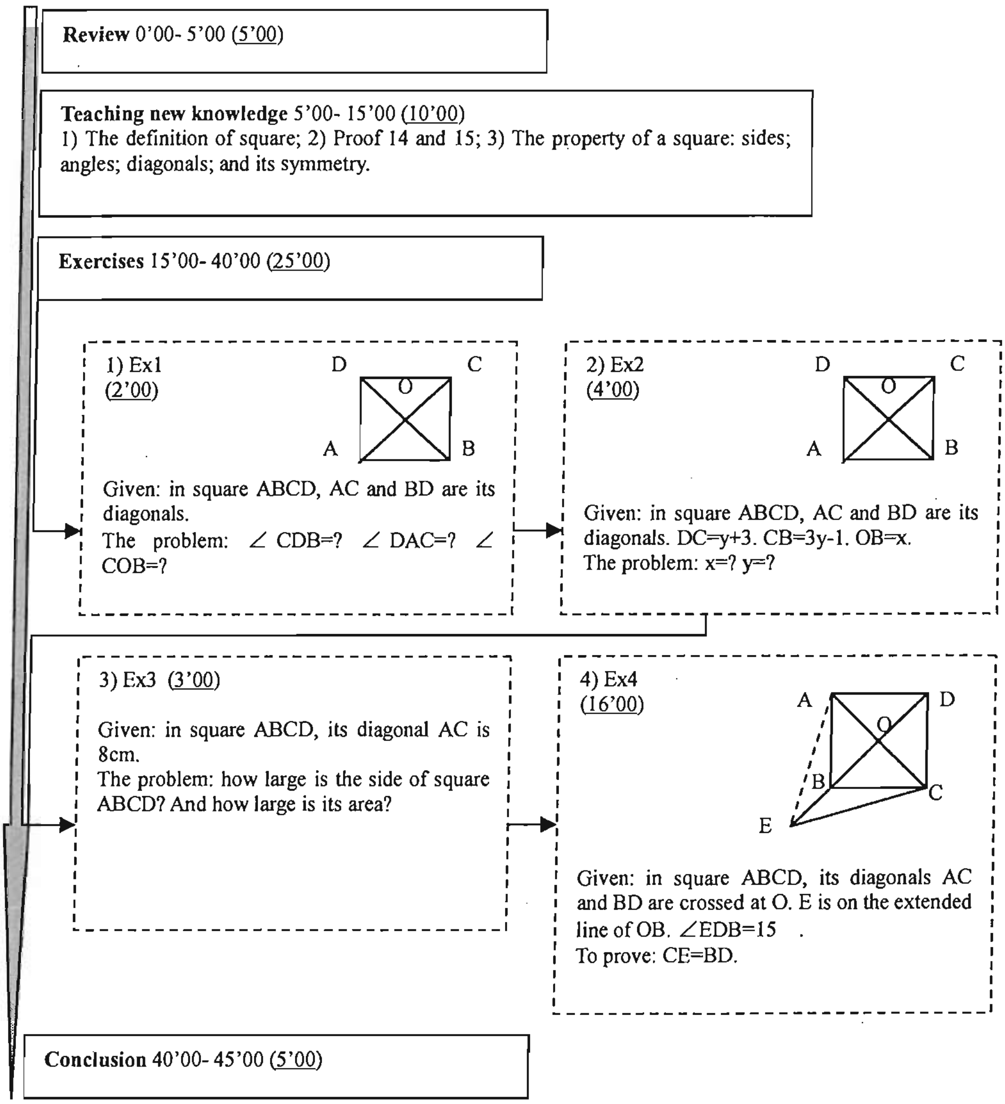
Given: rectangle ABCD. AC//DE.
To prove: triangle DBE is an isosceles triangle.

6) Ex6 (1'00)

A rectangle is divided into four small triangles by its diagonals. If the sum of the perimeter of the four triangles is 86 cm, and the diagonal of the rectangle is 13 cm, then the perimeter of the rectangle is ___cm.

Figure N2.

N3



Review 0'00- 5'00 (5'00)
Teaching new knowledge 5'00- 15'00 (10'00)
1) The definition of square; 2) Proof 14 and 15; 3) The property of a square: sides; angles; diagonals; and its symmetry.
Exercises 15'00- 40'00 (25'00)
1) Ex1 (2'00)
D C O A B
Given: in square ABCD, AC and BD are its diagonals.
The problem: ∠ CDB=? ∠ DAC=? ∠ COB=?
2) Ex2 (4'00)
D C O A B
Given: in square ABCD, AC and BD are its diagonals. DC=y+3. CB=3y-1. OB=x.
The problem: x=? y=?
3) Ex3 (3'00)
Given: in square ABCD, its diagonal AC is 8cm.
The problem: how large is the side of square ABCD? And how large is its area?
4) Ex4 (16'00)
A D O B C E
Given: in square ABCD, its diagonals AC and BD are crossed at O. E is on the extended line of OB. ∠EDB=15 .
To prove: CE=BD.
Conclusion 40'00- 45'00 (5'00)

Figure N3.

APPENDIX D. Content of the quadrilateral family in the textbook
Chapter 26. Quadrilateral

I. Parallelogram
26. 1 Proving a parallelogram
The definition of distance between two parallel lines.
Theorem 1 (PR1): The distance between two parallel lines is always equal.
Theorem 2 (PR2): If two pairs of opposite sides are equal, then it is a parallelogram.
Theorem 3 (PR3): If one pair of opposite sides is equal and parallel, then it is a parallelogram.
Theorem 4 (PR4): If the diagonals are bisected each other, then it is a parallelogram.
Theorem 5 (PR5): If two pairs of opposite angles are equal, then it is a parallelogram.

26. 2 The property of rectangle and rhombus
The definition of rectangle and rhombus.
Theorem 6 (PR6): A rectangle has four right angles.
Theorem 7 (PR7): A rhombus has four equal sides.
Theorem 8 (PR8): The diagonals of a rectangle are equal.
Theorem 9 (PR9): The diagonals of a rhombus are not only vertical to each other, but also bisect the opposite angles.

The area of a rhombus is half of the product of multiplication of its diagonals.

26. 3 Proving a rectangle and a rhombus
Theorem 10 (PR10): If four sides of a quadrilateral are equal, then it is a rhombus.
Theorem 11 (PR11): If a quadrilateral has three right angles, then it is a rectangle.
Theorem 12 (PR12): If the diagonals of a parallelogram are equal, then it is a rectangle.
Theorem 13 (PR13): If the diagonals of a parallelogram are vertical, then it is a rhombus.

26. 4 Square
The definition of square.
Theorem 14 (PR14): If a rhombus has one right angle, then it is a square.
Theorem 15 (PR15): If a pair of adjacent sides of a rectangle are equal, then it is a square.

APPENDIX E. Details of the selected examples from the observation data.

Lesson topics		Coding lessons	Coding Proof problems	Levels of thinking	Instructional phases
Parallelogram	Verifying parallelogram — the property of parallel lines	L1	Proof 1		
			L-Ex1		
			L-Ex2		
			L-Ex3	*	
			Proof 2		
	The first and second theorems of verifying parallelogram	L2	Proof 2	*	*
			Proof 3	*	*
			L-Ex4	*	*
			L-Ex5	*	*
			L-Ex6	*	*
			L-Ex7	*	*
	Exercises of verifying parallelogram	L3	L-Ex7	*	*
			L-Ex8	*	*
	The third and fourth theorems of verifying parallelogram	L4	Proof 4		
			Proof 5		
			L-Ex9		
			L-Ex10		
			L-Ex11		
Rectangle and Rhombus	The property of rectangle	L5	Proof 6		
			Proof 8		
			L-Ex12		
			L-Ex13		
			L-Ex14		
			L-Ex15		
	The property of rhombus	L6	Proof 7		
			Proof 9		
			Proof 9.1		
			L-Ex16		
			L-Ex17		
			L-Ex18		
	Exercises of rectangle and rhombus	L7	L-Ex19		
			L-Ex20		
			L-Ex21		
			L-Ex22		
			L-Ex23		
			L-Ex24		
			L-Ex25		
	Theorems of verifying rectangle	L8	Proof 11		
			Proof 12		
			L-Ex26		
			L-Ex27		
	Theorems of verifying rhombus	L9	L-Ex28		
			Proof 10		
			Proof 13		
			L-Ex29		
			L-Ex30		
			L-Ex31		
Square	The property of square	L10	L-Ex32		
			L-Ex33		
			L-Ex34		
			L-Ex35		
	Theorems of verifying square	L11	L-Ex36-37		
			Proof 14&15		
			L-Ex38-42		
Conclusion and exercises	Exercises of rectangle, rhombus and square	L12	L-Ex43-44		
			L-Ex45		
			L-Ex46-50		

Table 1. Twelve lessons observed in Lily's class. *- selected examples for the data analysis.

Lesson topics		Coding lessons	Coding Proof problems	Levels of thinking	Instructional phases
Parallelogram	The third and fourth theorems of verifying parallelogram	S1	Proof 5		
			Proof 4		
			S-Ex1		
			S-Ex2		
			S-Ex3		
	Exercises of verifying parallelogram	S2	S-Ex4	*	*
			S-Ex5		
			S-Ex6	*	*
			S-Ex7		
Rectangle and Rhombus	The property of rectangle and rhombus	S3	Proof 6		
			Proof 8		
			Proof 7		
			Proof 9		
			S-Ex8		
	Exercises of rectangle and rhombus	S4	Proof 9.1		
			S-Ex9		
			S-Ex10		
			S-Ex11		
	Theorems of verifying rectangle and rhombus	S5	Proof 12		
			Proof 11		
			Proof 13		
			Proof 10		
			S-Ex12		
			S-Ex13		
			S-Ex14		
Square	The property of square and theorems of verifying square	S6	Proof 14&15		
			S-Ex15		
Exercises	Exercises of parallelogram, rectangle, rhombus and square	S7	S-Ex15		
			S-Ex16		
			S-Ex17		
			S-Ex18		
			S-Ex19		
	Exercises of parallelogram, rectangle, rhombus and square	S8	S-Ex20		
			S-Ex21		
			S-Ex22		
			S-Ex23		
			S-Ex24		

Table 2. Eight lessons observed in Spring's class. *- selected examples for the data analysis.

Lesson topics		Coding lessons	Coding Proof problems	Levels of thinking	Instructional phases
Parallelogram	Exercises of verifying parallelogram	N1	N-Ex1	*	*
			N-Ex2		
			N-Ex3		
			N-Ex4		
			N-Ex5		
Rectangle and Rhombus	The property of rectangle and rhombus	N2	Proof 6		
			Proof 8		
			Proof 9		
			N-Ex6		
			N-Ex7		
			N-Ex8		
			N-Ex9		
			N-Ex10		
Square	The property of square	N3	Proof 14		
			Proof 15		
			N-Ex11		
			N-Ex12		
			N-Ex13		
			N-Ex14		
			N-Ex15		

Table 3. Three lessons observed in Nana's class. *- selected examples for the data analysis.

APPENDIX F. Photos of Lily's lessons

Photo 1. L2-Proof 2 and L2-Proof 3.

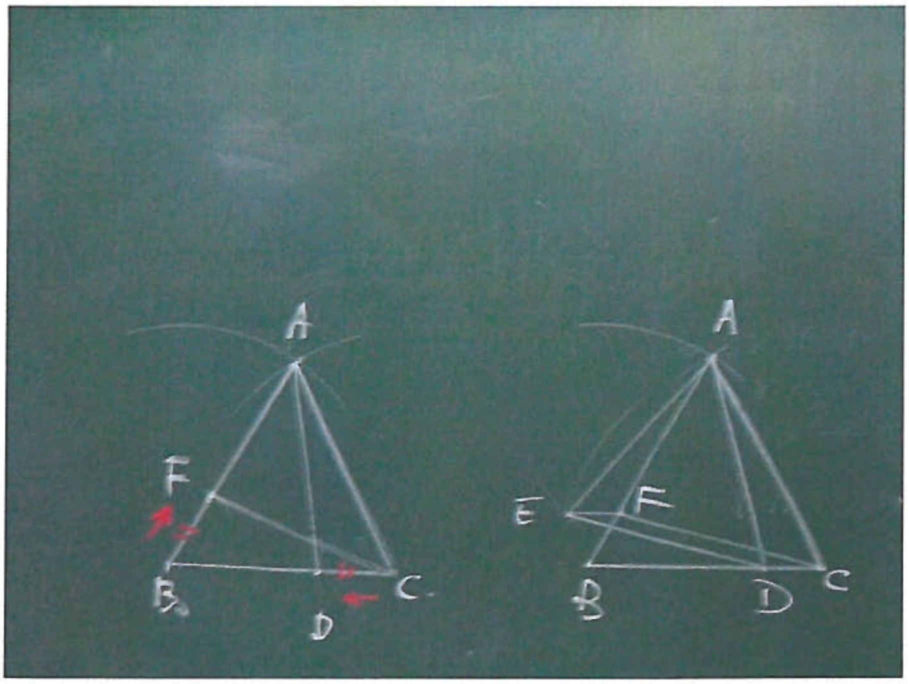


Photo 2. L2-Ex7.

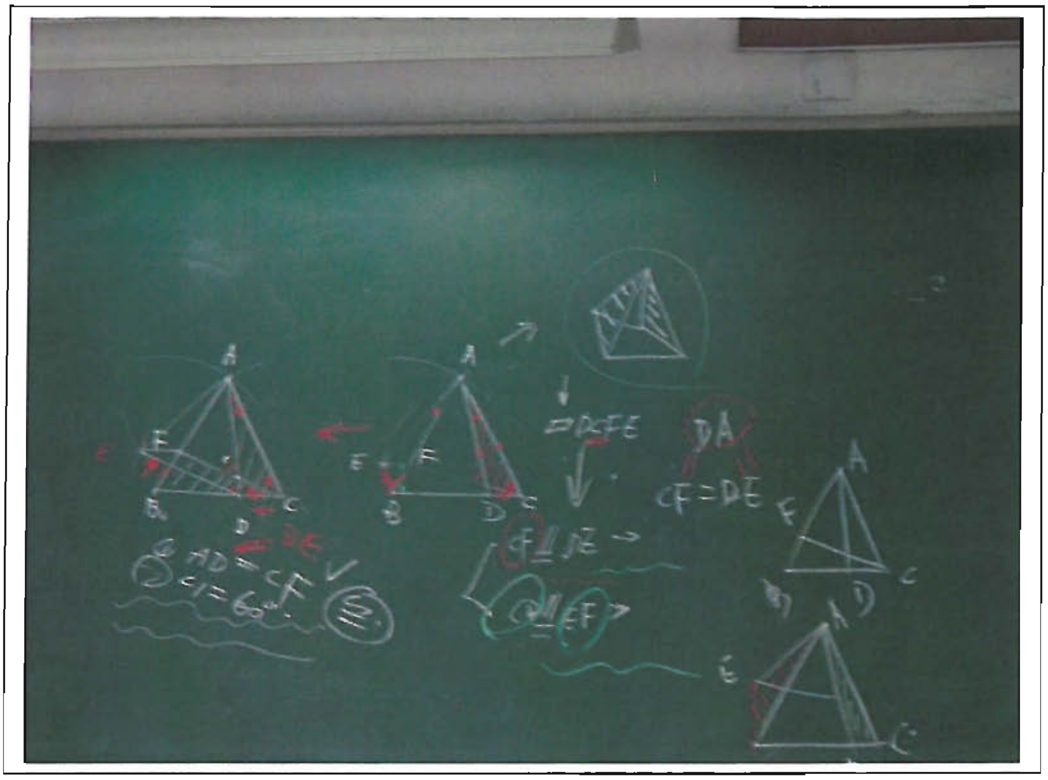


Photo 3. L3-Ex7.

APPENDIX G. Photos of Spring's lessons

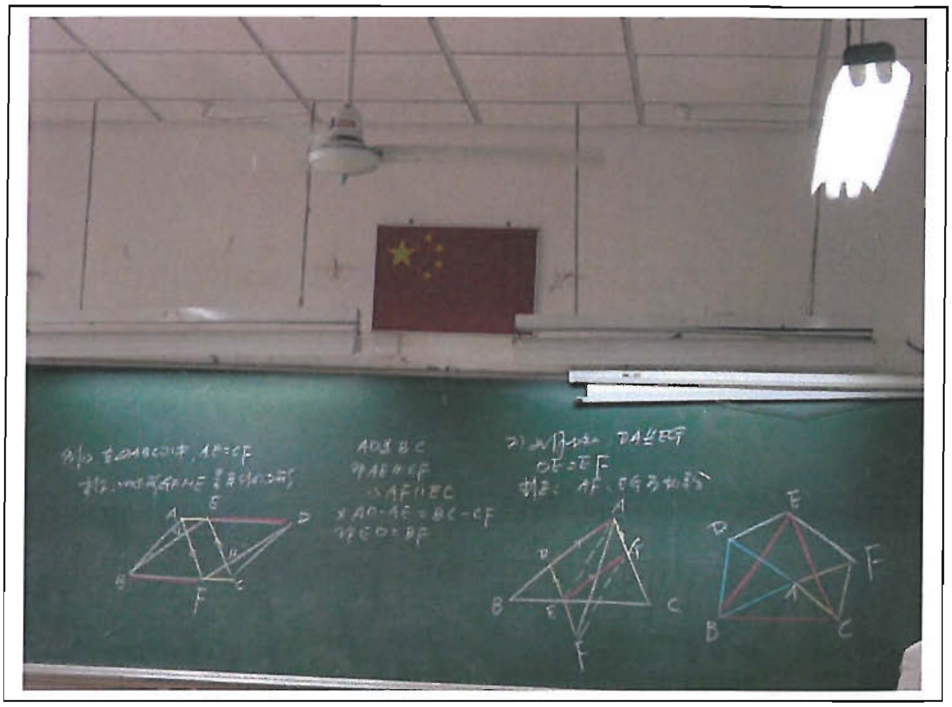


Photo 1. S2-Ex4

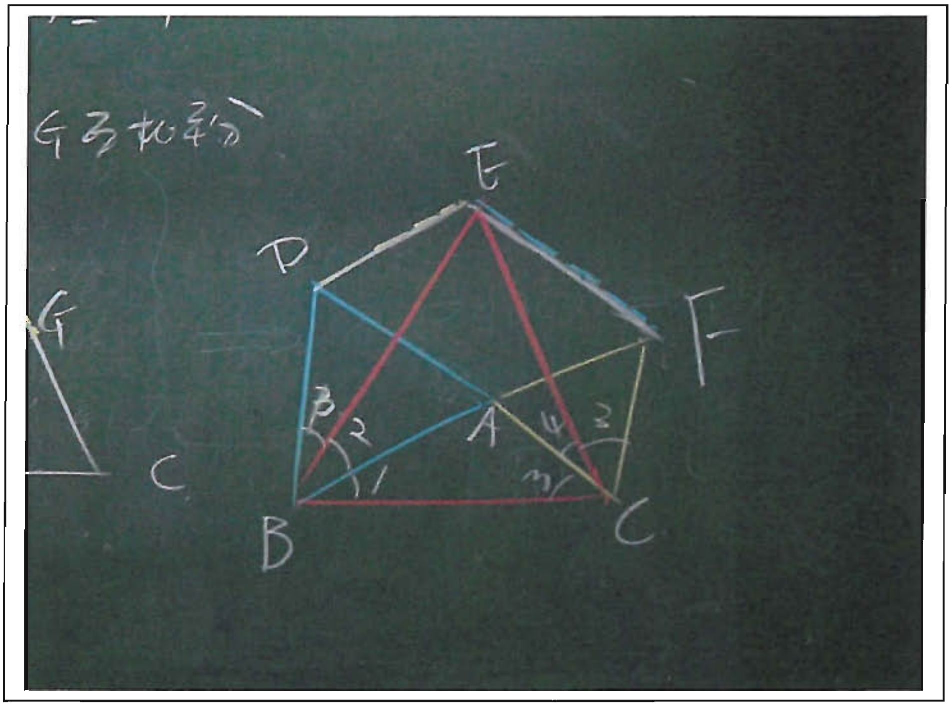


Photo 2. S2-Ex6

APPENDIX H. Photos of Nana's lessons

Photo 1. N1-Ex1

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