UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

School of Mathematics

Dold-Puppe Complexes and the Derived Functors of the Third Symmetric Power Functor

by

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ABSTRACT

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For a chain complex $C.$ a Dold-Puppe complex is a complex of the form $NF\Gamma(C.)$, i.e. the image of $C.$ under the composition of the functors Γ, F and N; here Γ and N are the functors given by the Dold-Kan correspondence and F is a not-necessarily linear functor between two abelian categories. When $C.$ is a projective resolution of a module the i^{th} homology of this Dold-Puppe complex is the i^{th} derived functor of the functor F.

The definition of Γ is quite abstract and combinatorial. The first half of the first chapter of this thesis gives an algorithm that streamlines the calculation of $\Gamma(C.)$. The second half of the first chapter gives algorithms that allow the explicit calculation of the Dold-Puppe complex in terms

The second chapter produces a partial proof of Köck's predictions of the derived functors of the third symmetric power functor Sym^3. This is achieved by comparing certain cross-effect modules of the predictions and of the derived functors.

Contents

Annotations to the declaration of authorship

The work presented in this thesis was done wholly during my candidature for the degree of PhD at the University of Southampton, 2004-2008.

A shortened version of Chapter 1 has been submitted for publication, and has been made available to the public via the preprint archive at http://eprints.soton.ac.uk/45903/

All published sources that I have used are listed in the bibliography and, where applicable, quoted within the text.

Acknowledgements

'This view may have been originally suggested by notation, and if so, that is much in its favor, for a good notation has a subtlety and suggestiveness which at times make it seem almost like a live teacher.' Bertrand Russell, *Introduction to the Tractatus Logico-Philosophicus.*

'A problem shared is a problem halved.' Proverb

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Introduction

Let R and S be rings. The construction of the left derived functors L_kF : R-**mod** $\to$ S-**mod** of any covariant right-exact functor F : R-**mod** $\to$ S-**mod** is achieved by applying three functors. The first functor constructs a projective resolution $P.$ of the R-module M that we wish to calculate the derived functor of. Then the functor F is applied to the resolution $P.$ giving the chain complex $F(P.)$. Lastly the k^{th} derived functor L_kF is defined to be $H_k(F(P.))$, the k^{th} homology of the chain complex $F(P.)$. However for a given module M the projective resolution of M is unique only up to chain-homotopy equivalence, so this contruction crucially depends on the fact that F preserves chain-homotopies. In general this fact does not hold when F is a non linear functor such as the l^{th} symmetric power functor, Sym^l, or the l^{th} exterior power functor, Λ^l. In the paper [DP] Dold and Puppe overcome this problem and define the derived functors of non-linear functors by passing to the category of simplical complexes using the Dold-Kan correspondence.

The Dold-Kan correspondence gives a pair of functors Γ and N that provide an equivalence between the category of bounded chain complexes and the category of simplicial complexes; under this correspondence chain homotopies correspond to simplicial homotopies. Furthermore in the simplicial world all functors preserve simplicial homotopy (not just linear functors). Because of this the above definition of the derived functors of F becomes well defined for any functor when $F(P.)$ is replaced by the complex $NF\Gamma(P.)$. We call chain complexes of the form $NF\Gamma(C.)$ Dold-Puppe complexes, for any bounded chain complex $C.$.

Let R be a Noetherian commutative ring and let I be an ideal in R that is locally generated by a non-zero divisor. If $P.$ is the length 1 R-projective resolution of a projective R/I-module V then the homology of the Dold-Puppe complex $N\,\mathrm{Sym}^k\,\Gamma(P.)$, $k \geq 1$, has been explicitly computed in [Kö]. These computations yield a very natural and new proof of the classical Adams-Riemann-Roch theorem for regular closed immersions and hence a new approach to the seminal Grothendieck-Riemann-Roch theorem avoiding the comparitively involved deformation to the normal cone, see [Kö].

The purpose of this thesis is two-fold. In Chapter 1 we shed some light on the combinatorial structure of Dold-Puppe complexes in general. In Chapter 2 we study the homology of the Dold-Puppe complex $N\,\mathrm{Sym}^3\,\Gamma(P.)$ when I is an ideal which is locally generated by a regular sequence of length 2 and when, as above, $P.$ is a projective resolution of a projective R/I-module V.

We now describe Chapter 1 in more detail.

If $C.$ is a chain complex of length ≥ 2 then the calculation of the Dold-Puppe complex $NF\Gamma(C.)$ is normally too complicated to be performed on a couple of pieces of paper, and the nature of the calculation means that errors easily creep in. The purpose of Chapter 1 is to develop an algorithm that computes such Dold-Puppe complexes in a manner that is both efficient and easy to check. We hope that the explicit description of the Dold-Puppe complex that the algorithm provides will help later work in calculating its homology.

In section 1.1 we introduce an ordering in the set $\mathrm{Mor}([n],[k])$ of order preserving maps between $[n] := \{0 < 1 < \ldots < n\}$ and $[k] := \{0 < 1 < \ldots < k\}$. We show that composition with the face maps $\delta_i : [n-1] \to [n]$ and degeneracy maps $\sigma_i : [n] \to [n-1]$ is "well-behaved" with respect to this ordering.

The simplicial complex $\Gamma(C.)$ is defined by

$$\Gamma(C.)_n = \bigoplus_{k=0}^{n} \bigoplus_{\mu \in \mathrm{Sur}([n],[k])} C_k,$$

so we have a copy of the direct summand C_k for each surjective order preserving map $\mu : [n] \to [k]$. The face and degeneracy operators in the simplicial complex $\Gamma(C.)$ are defined in terms of composition of μ with the maps δ_i and σ_i. In section 1.2 we show how the results in section 1.1 can be used to streamline the calculation of the the face and degeneracy operators in the simplicial complex $\Gamma(C.)$.

In section 1.3 we summarise the results on cross-effect functors that are needed for the sections that follow.

The Dold-Puppe complex $NF\Gamma(C.)$ is constructed by modding out the images of the degeneracy operators in $F\Gamma(C.)$. To calculate this we apply the theory of cross-effect functors to decompose both the numerator and denominator into the direct sum of cross-effect modules; the non-degenerate modules correspond to the terms that appear in the numerator but not in the denominator. However the decomposition produces many, many terms and seeing which are non-degenerate is far from obvious. In section 1.4 we introduce a criterion to distinguish between the non-degenerate and the degenerate terms; this criterion is defined in terms of the ordering we introduced in section 1.1. Later we introduce an algortithm that constructs all terms that satisfy this criterion; thus avoiding the need to check each of the many terms one by one.

In section 1.5 we calculate the Dold-Puppe complex $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)$, where $C \to B \to A$ is a complex of length 2 concentrated in degrees 0, 1 and 2. The purpose of this calculation is to elucidate how the results of this paper can be applied to calculate virtually any other Dold-Puppe complex.

We now describe Chapter 2 in greater detail.

In [Kö] Köck made predictions for the derived functors of Sym^3. In section 2.7 we give a partial proof of these predictions. To produce this partial proof we use a method Köck discovered which tells us that two functors are isomorphic if they agree on certain data given by their cross-effect functors (see section 2.6) i.e. we show that some of the preconditions of the relevant theorem holds. The earlier sections of Chapter 2 introduce the various tools we use in this calculation.

Section 2.1 introduces a spectral sequence for the functor hypertor, which allows us to calculate the homology of the tensor product of two chain complexes that we know the homology of. Section 2.2 introduces Koszul complexes which we use both as projective resolutions and also to define Schur functors of hook type. Schur functors are introduced in section 2.3, they are important because Köck's predictions are given in terms of Schur functors and also because of their role in the Cauchy decomposition of $\mathrm{Sym}^3(F \otimes G)$.

In [ABW] a filtration is put on $\mathrm{Sym}^n(F \otimes G)$, the successive quotients of which are isomorphic to modules of the form $L_\lambda(F) \otimes L_\lambda(G)$ where λ is a partition of weight n, and L_λ is the Schur functor of shape λ. This is known as the the Cauchy decomposition of $\mathrm{Sym}^n(F \otimes G)$. This decomposition gives us a number of short exact sequences that allow us to calculate the information that we need about the first cross-effect functor of the derived functors of Sym^3.

In section 2.4 we introduce the Eilenberg-Zilber Theorem and extend it to suit our needs. The Eilenberg-Zilber theorem gives a suprising homotopy equivalence between the diagonal of a bisimplicial complex and the total complex of the associated double complex. We will use it extensively in our calculations in section 2.7.

In section 2.7 we perform the calculations that give us our partial proof. Let R be a Noetherian commutative ring and let I be an ideal in R which is locally generated by a regular sequence of length 2. Let G_k be the k^{th} derived functor of Sym^3.

In [Kö] Köck made the following predictions about the functor G_k, if V is a finitely generated projective R/I-module then:

$$G_k(V) \cong \begin{cases} \mathrm{Sym}^3(V) & k = 0 \\ L_{(2,1)}(V) \otimes I/I^2 & k = 1 \\ L_{(2,1)}(V) \otimes I/I^2 \otimes \Lambda^2(I/I^2) & k = 3 \\ D^3(V) \otimes \Lambda^2(I/I^2)^{\otimes 2} & k = 4 \\ 0 & k \geq 5 \end{cases}$$

and for the case when $k = 2$ he suggests that there exists an exact sequence:

$$0 \to D^2(V) \otimes V \otimes \Lambda^2(I/I^2) \to H_2 N \operatorname{Sym}^3 \Gamma(P.(V)) \\ \to \Lambda^3(V) \otimes \operatorname{Sym}^2(I/I^2) \to 0,$$

where D^i is the i^{th} divided power functor. For any non-negative integer k that is not equal to 2 we let F_k be the functor that Köck predicted for G_k to be. We let F_2 be any functor that fits in a short exact sequence

$$0 \to D^2(V) \otimes V \otimes \Lambda^2(I/I^2) \to F_2(V) \to \Lambda^3(V) \otimes \operatorname{Sym}^2(I/I^2) \to 0.$$

Provided that I is globally generated by a regular sequence we prove that $G_k(R/I) \cong F_k(R/I)$ i.e. that these predictions hold if $V = R/I$. Moreover, regardless of whether I is globally generated or not, we prove a similar statement for the higher cross-effects of G_k and F_k namely that for all k and $l > 1$ we have

$$\operatorname{cr}_l(G_k)(R/I, \ldots, R/I) \cong \operatorname{cr}_l(F_k)(R/I, \ldots, R/I).$$

These results are a major step toward allowing us to apply Theorem 2.6.2 to the functors F_k and G_k, which will show that the predictions are true in general.

Notations

Let Δ be the category whose objects are the finite totally ordered sets $[n] := \{0 < 1 < ... < n\}$ where $n \in \mathbb{N}$ and the set of morphisms, $\operatorname{Mor}([n],[k])$, between $[n]$ and $[k]$ consists of all the order preserving maps between them. Recall for each $i \in \{0, \ldots, n\}$ the face map $\delta_i : [n-1] \to [n]$ is the unique injective order preserving map with $\delta_i^{-1}(i) = \emptyset$ and for each $i \in \{0, \ldots, n-1\}$ the degeneracy map $\sigma_i : [n] \to [n-1]$ is the unique order preserving surjective map with $\sigma_i^{-1}(i) = \{i, i+1\}$. For a category

$\mathcal{A}$, a simplicial object A in $\mathcal{A}$ is a contravariant functor $A : \Delta \to \mathcal{A}$. We write A_n for $A([n])$, d_i for the *face operator* $A(\delta_i) : A_n \to A_{n-1}$, s_i for the *degeneracy operator* $A(\sigma_i) : A_{n-1} \to A_n$ and $\mathrm{Sur}([n],[k])$ for the set of surjective morphisms between $[n]$ and $[k]$.

Chapter 1

An Algorithmic approach to Dold-Puppe complexes

1.1 Partitions and composition with face maps and degeneracy maps in Δ

For the whole of this section let us fix the natural numbers n and k. In this section we introduce an ordering on $\mathrm{Mor}([n],[k])$, investigate the maps $x \mapsto x\delta_i$ and $y \mapsto y\sigma_i$ between $\mathrm{Mor}([n],[k])$ and $\mathrm{Mor}([n-1],[k])$ and show that these maps behave in a nice way with respect to the ordering on $\mathrm{Mor}([n],[k])$.

This ordering will be used again throughout this paper. In chapter 1.2 it will allow us to describe algorithms that streamline the calculation of the face and degeneracy operators in the simplicial complex $\Gamma(C.)$. In chapter 4 the ordering will be used to define the notion of honourability, and thereby to help us give a description of the Dold-Puppe complex $NF\Gamma(C.)$.

Definition 1.1.1. Let an n-tuple $x := (x_1, \ldots, x_n) \in \mathbb{N}^n$ be called *a partition of m of length n* if $\sum_{l=1}^{n} x_l = m$. If each $x_i \neq 0$ we call it a *proper partition*, otherwise we call it an *improper partition.* We write x_i for the i^{th} entry of x.

A function $f : [n] \to [k]$ is determined by the following sets $f^{-1}(0)$, $f^{-1}(1), \ldots, f^{-1}(k)$. If f is a monotonically increasing function then the sets $f^{-1}(0)$, $f^{-1}(1)$, $\ldots$, $f^{-1}(k)$ consist of consecutive elements of $[n]$. Because of this it is sufficient to know the sizes of these sets to determine an element of $\mathrm{Mor}([n],[k])$. Hence we can think of a morphism $f : [n] \to [k]$ as a partition of $n+1$ of length $k+1$. A surjective morphism would correspond to a proper partition and a non-surjective morphism would correspond to an improper partition.

Notation 1.1.2. For a morphism f in $\mathrm{Mor}([n],[k])$ we write f^* to denote the following partition obtained from f, $(|f^{-1}(0)|, |f^{-1}(1)|, \ldots, |f^{-1}(n)|)$. Note that $f_i^* = |f^{-1}(i-1)|$.

Lemma 1.1.3. $|\,\mathrm{Sur}([n],[k])| = \binom{n}{k}$

Proof. If $f : [n] \to [k]$ is a surjective morphism then the sets $f^{-1}(0)$, $f^{-1}(1)$, $\ldots$, $f^{-1}(k)$ are non-empty, disjoint, their union is $[n]$ and each set consists of consecutive elements of $[n]$. So if we know the smallest elements of $f^{-1}(1)$, $f^{-1}(2)$, $\ldots$, $f^{-1}(k)$ then we have determined f. Since we know $0 \in f^{-1}(0)$ the smallest elements are in the set $\{1, ..., n\}$. So there are as many elements of $\mathrm{Sur}([n],[k])$ as there are ways of choosing k elements from a set of size n. □

Notation 1.1.4. For $i \in \{0, \ldots, n\}$ define $\overline{\delta}_i : \mathrm{Mor}([n],[k]) \to \mathrm{Mor}([n-1],[k])$ by $x \mapsto x\delta_i$ and for $i \in \{0, \ldots, n-1\}$ define $\overline{\sigma}_i : \mathrm{Mor}([n-1],[k]) \to \mathrm{Mor}([n],[k])$ by $y \mapsto y\sigma_i$. (See diagram below). By abuse of notation we write $\mathrm{Im}(\overline{\sigma}_i)$ for $\overline{\sigma}_i(\mathrm{Sur}([n],[k]))$.

Note we will ocassionally extend these functions to partitions in the obvious way.

$$\begin{tikzcd}
{[n-1]} \arrow[d, "\delta_i"'] \arrow[dr, "x\delta_i"] & \\
{[n]} \arrow[r, "x"'] & {[k]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[n-1]} \arrow[dr, "y"] & \\
{[n]} \arrow[u, "\sigma_i"] \arrow[r, "y\sigma_i"] & {[k]}
\end{tikzcd}$$

Lemma 1.1.5. *For all $i \in \{0, \ldots, n-1\}$ we have $\overline{\delta}_i \overline{\sigma}_i = \mathrm{id}$, and hence $\overline{\sigma}_i$ is injective and $\overline{\delta}_i$ is surjective; also $\overline{\delta}_n$ is surjective.*

Proof. Result follows directly from $\sigma_i \delta_i = \mathrm{id}$ for $i \in \{0, \ldots, n-1\}$ and from $\sigma_{n-1}\delta_n = \mathrm{id}$ □

Since knowing the effects of $\overline{\delta}_i$ and $\overline{\sigma}_i$ are essential in calculating d_i and s_i it is useful to have a quick way of working out $(x\delta_i)^*$ and $(x\sigma_i)^*$ from x^*.

Lemma 1.1.6. *(a) For each $i \in \{0, \ldots, n-1\}$ we have $(x\sigma_i)^*_l = x^*_l$ unless $\sum_{m=1}^{l-1} x^*_m < i+1 \leq \sum_{m=1}^{l} x^*_m$ in which case $(x\sigma_i)^*_l = x^*_l + 1$.*

(b) Also for each $i \in \{0, \ldots, n\}$ we have $(x\delta_i)^*_l = x^*_l$ unless $\sum_{m=1}^{l-1} x^*_m < i+1 \leq \sum_{m=1}^{l} x^*_m$ in which case $(x\delta_i)^*_l = x^*_l - 1$.

Proof. Recall for every f we have $f^*_l = |f^{-1}(l-1)|$ and $(x\sigma_i)^{-1}(l-1) = \sigma_i^{-1}x^{-1}(l-1)$. Recalling σ_i is the unique surjective map $[n-1] \to [n]$ with $\sigma_i^{-1}(i) = \{i, i+1\}$ we see $|(x\sigma_i)^{-1}(l-1)| = |x^{-1}(l-1)|$ if and only if $i \notin x^{-1}(l-1)$. If $i \in x^{-1}(l-1)$ then $(x\sigma_i)^*_l = |(x\sigma_i)^{-1}(l-1)| = |x^{-1}(l-1)| + 1 = x^*_l + 1$. Remembering that i is the $(i+1)^{\text{th}}$ element of $[n]$ we get our result concerning $\overline{\sigma}_i$.

We similarly get our results for δ_i. □

Corollary 1.1.7. *For all i and for all $x \in \mathrm{Sur}([n], [k])$ the map $x\delta_i$ is not surjective if and only if there is some $l \in [k]$ with $x^{-1}(l) = \{i\}$.*

Proof. Obvious. □

Lemma 1.1.8. *Let $x \in \mathrm{Sur}([n], [k])$ and let $i \in \{0, \ldots, n\}$. Suppose that $x\delta_i$ is not surjective. Then write $x\delta_i = \delta_j \hat{x}$ for some $\hat{x} \in \mathrm{Sur}([n-1], [k-1])$ and some $j \in \{0, \ldots, k\}$. We have $i = 0$ if and only if $j = 0$*

Proof. Suppose $x\delta_0 = \delta_j \hat{x}$ for some $\hat{x} \in \mathrm{Sur}([n-1], [k-1])$ and some $j \neq 0$, then the above corollary tells us that $x^{-1}(0) = \{0\}$, so we see that $x\delta_0(0) = x(1) = 1$. But because $\hat{x}$ is surjective $\delta_j \hat{x}(0) = \delta_j(0) = 0$, So we have contradiction.

Suppose $x\delta_i = \delta_0\hat{x}$ with $i \neq 0$ then, recalling x is surjective $x\delta_i(0) = x(0) = 0$. But $\delta_0\hat{x}(0) = \delta_0(0) = 1$, so we have a contradiction. □

Definition 1.1.9. Let a be a partition of length k. If x is a partition of length $l \leq k$ with $x_i = a_i$ for $1 \leq i \leq l$ then we call x an *initial partition of a*. We write $a = (x, y)$ where y is the partition of length $k - l$ defined by $y_i = a_{i+l}$ for $1 \leq i \leq k - l$. (Note we may allow either x or y to be the empty partition, so a is an initial partition of itself).

If a and b are both partitions of the same number over the same number of places and x is an initial partition of both then we call x a *common initial partition of a and b*. Because a and b are of finite length there must be some longest common initial partition (even if it is of length 0, or it is equal to a).

Definition 1.1.10. If x is the longest common initial partition of $a = (x, y)$ and $b = (x, z)$ then we say $a < b$ if and only if $y_1 < z_1$.

This gives the lexicographic ordering on the partitions and we use it to define an order on $\mathrm{Mor}([n], [k])$ also.

Proposition 1.1.11. *For each* $i \in \{0, \ldots, n-1\}$ *the map* $\overline{\sigma}_i : \mathrm{Mor}([n-1], [k]) \to \mathrm{Mor}([n], [k])$ *is strictly order preserving.*

Proof. Suppose $x, y \in \mathrm{Mor}([n-1], [k])$ and $x < y$. Let a be the longest common partition of x^* and y^* and set $x^* = (a, b)$ and $y^* = (a, c)$. Lemma 1.1.6 tells us that $(x\sigma_i)^*_l = x^*_l$ for all l except one and for that l we have $(x\sigma_i)^*_l = x^*_l + 1$. Let L stand for the l for which we have $(x\sigma_i)^*_l = x^*_l + 1$. We will show that whatever the value of L we have $(x\sigma_i)^* < (y\sigma_i)^*$. Let p be the length of a.

If $L \leq p$ then we have $(x\sigma_i)^* = (a', b)$ and $(y\sigma_i)^* = (a', c)$ for some a'. Since we know $b_1 < c_1$ we have $(x\sigma_i)^* < (y\sigma_i)^*$.

If $L > p$ then we have $(x\sigma_i)^* = (a, b')$ and $(y\sigma_i)^* = (a, c')$ for appropriate b' and c'.

If $L > p + 1$ then $b'_1 = b_1 < c_1 \leq c'_1$ so $(x\sigma_i)^* < (y\sigma_i)^*$.

If $L = p+1$ then we have $b_1' = b_1 + 1$ Let m be the number a is a partition of. We have $m < i < m + b_1 < m + c_1$, thus the value of l we have $(y\sigma_i)_l^* = y_l^* + 1$ for is also $p+1$. Therefore $b_1' = b_1 + 1 < c_1 + 1 = c_1'$ and so we get $(x\sigma_i)^* < (y\sigma_i)^*$. □

Notation 1.1.12. For $i \in \{0 \ldots, n\}$ let $S_i^n := \{x \in \mathrm{Sur}([n],[k]) | x^*$ has an initial partition of $i+1\}$ and for a a partition of $i+1$ let $S_{i,a}^n := \{x \in \mathrm{Sur}([n],[k]) : x^*$ begins with $a\}$.

Obviously $S_{i,a}^n \subset S_i^n$ and $\cup_a S_{i,a}^n = S_i^n$ where a ranges over all partitions of $i+1$ of whatever length.

Lemma 1.1.13. *For each $i \in \{0, \ldots, n-1\}$ we have $|S_i^n| = \binom{n-1}{k-1}$. Also for each $i \in \{1, \ldots, n-1\}$ we have $|\bigcup_z S_{i,(z,1)}^n| = \binom{n-2}{k-2}$ where z ranges over all partitions of i and finally $|\bigcup_z S_{n,(z,1)}^n| = \binom{n-1}{k-1}$ where z ranges over all partitions of n.*

Proof. For $i \in \{0, \ldots, n-1\}$ if $x \in S_i^n$ then, for some l, we have i is the maximal element of $x^{-1}(l)$. Furthermore we know that n is the maximal element of $x^{-1}(k)$, therefore choosing an element x of S_i^n amounts to the same as choosing the maximal elements for all but one (since we already know one meximal element must be i) of the sets $x^{-1}(0), \ldots, x^{-1}(k-1)$ from the $n-1$ remaining elements of $[n]$. Therefore $|S_i^n| = \binom{n-1}{k-1}$.

For $i \in \{1, \ldots, n-1\}$ if $x \in \bigcup_z S_{i,(z,1)}^n$ then for some l we have $i-1$ is the maximal element of $x^{-1}(l)$ and also i is the maximal element of $x^{-1}(l+1)$, i.e. choosing an element x of $\bigcup_z S_{i,(z,1)}^n$ amounts to the same as choosing the maximal elements for all but two of the sets $x^{-1}(0), \ldots, x^{-1}(k-1)$ from the $n-2$ remaining elements of $[n]$. Therefore $|\bigcup_z S_{i,(z,1)}^n| = \binom{n-2}{k-2}$.

For the remaining statement we merely observe that $\bigcup_z S_{n,(z,1)}^n = S_{n-1}^n$ and use the first result. □

Theorem 1.1.14. *For each $i \in \{0, \ldots, n-1\}$ the set $\mathrm{Sur}([n],[k])$ is the disjoint union of S_i^n and $\mathrm{Im}\,\overline{\sigma}_i$:*

$$\mathrm{Sur}([n],[k]) = S_i^n \amalg \mathrm{Im}\,\overline{\sigma}_i$$

Note $S_n^n = \operatorname{Sur}([n],[k])$ and there is no map $\overline{\sigma}_n$.

Proof. First we prove S_i^n and $\operatorname{Im}\overline{\sigma}_i$ are disjoint. Let $x \in \operatorname{Sur}([n],[k])$. The partition x^* has an initial partition of $i+1$ if and only if there is some l such that i is the maximal element of $x^{-1}(l)$ (remember i is the $(i+1)^{\text{th}}$ element of $[n]$). If i is the maximal element of $x^{-1}(l)$ then $i+1$ is the minimal element of $x^{-1}(l+1)$ which means $x(i) \neq x(i+1)$. But $x \in \operatorname{Im}\overline{\sigma}_i$ means that for some $y \in \operatorname{Sur}([n-1],[k])$ we have $x = y\sigma_i$. So $x(i) = y\sigma_i(i) = y(i) = y\sigma_i(i+1) = x(i+1)$ therefore x cannot be both in S_i^n and $\overline{\sigma}_i \operatorname{Sur}([n-1],[k])$.

Now we prove that the union of S_i^n and $\operatorname{Im}\overline{\sigma}_i$ form the whole of $\operatorname{Sur}([n],[k])$ by using a counting argument. We know that $S_i^n \cap \operatorname{Im}\overline{\sigma}_i = \emptyset$ so $|S_i^n \cup \operatorname{Im}\overline{\sigma}_i| = |S_i^n| + |\operatorname{Im}\overline{\sigma}_i|$. Lemma 1.1.5 tells us that $\overline{\sigma}_i$ is injective, from this we see that $|S_i^n| + |\operatorname{Im}\overline{\sigma}_i| = |S_i^n| + |\operatorname{Sur}([n-1],[k])|$ and by Lemmas 1.1.3 and 1.1.13 we see that $|S_i^n| + |\operatorname{Sur}([n-1],[k])| = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} = |\operatorname{Sur}([n],[k])|$. □

Proposition 1.1.15. *For all $i \in \{0,\dots,n-1\}$ we have $\overline{\delta}_i$ is strictly order preserving on both $\operatorname{Im}\overline{\sigma}_i$ and S_i^n, also $\overline{\delta}_n$ is strictly order preserving on $\operatorname{Sur}([n],[k]) = S_n^n$.*

Note that while $\overline{\delta}_i$ is order preserving on these two complementary sets of $\operatorname{Sur}([n],[k])$ it is *not* order preserving on the whole of $\operatorname{Sur}([n],[k])$, for an illustration of this look at the calculation in section 1.2.

Proof. That $\overline{\delta}_i$ is order preserving on $\operatorname{Im}\sigma_i$ follows from directly Lemma 1.1.5 and Proposition 1.1.11.

Let $x, y \in S_i^n$ with $x < y$, a be the longest common partition of x^* and y^*, set $x^* = (a,b)$, $y^* = (a,c)$. Lemma 1.1.6 tells us that $(y\delta_i)_l^* = y_l^*$ for all l except one and for that l we have $(y\delta_i)_l^* = y_l^* - 1$. Let L stand for the l for which we have $(y\delta_i)_l^* = y_l^* - 1$. We will show that whatever the value of L we have $(x\delta_i)^* < (y\delta_i)^*$. Let p be the length of a.

If $L \leq p$ then for appropriate a' we have $x^* = (a', b)$, $y^* = (a', c)$, so since we know $b_1 < c_1$ we have $(x\delta_i)^* < (y\delta_i)^*$.

If $L > p$ then for appropriate b' and c' we have $x^* = (a, b')$, $y^* = (a, c')$.

Let m be the number a is a partition of. If $L > p+1$ then $i+1 > m + c_1 > m + b_1$ so the l for which we have $(x\delta_i)^*_l = x^*_l - 1$ is also greater than $p+1$. So since $c'_1 = c_1 > b_1 = b'_1$ we have $(x\delta_i)^* < (y\delta_i)^*$.

If $L = p+1$ then by Lemma 1.1.6 we see that $m < i+1$. Recalling y^* begins with a partition of $i+1$ and using Lemma 1.1.6 again we see $i+1 = m + c_1$. We know $b_1 < c_1$ so $i+1 = m + c_1 > m + b_1$, so the l for which we have $(x\delta_i)^*_l = x^*_l - 1$ is greater than $p+1$. Therefore $b'_1 = b_1 < c_1 = c'_1 + 1$, so $b'_1 < c'_1 + 1$, i.e. $b'_1 \leq c'_1$.

If $b'_1 < c'_1$ we have $(x\delta_i)^* < (y\delta_i)^*$.

If $b'_1 = c'_1$ then $b_1 = b'_1 = c'_1 = c_1 - 1 = (i+1-m) - 1 = i - m$, recalling x^* begins with a partition of $i+1$ we see $b_2 = 1$. Therefore by Lemma 1.1.6 $b'_2 = 0 < c_2 = c'_2$ so we see $(x\delta_i)^* < (y\delta_i)^*$. □

1.2 The face and degeneracy operators in the simplical object $\Gamma(C.)$

For an abelian category $\mathcal{A}$ the Dold-Kan correspondence gives two mutually inverse functors Γ and N between the category of bounded chain complexes, $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, and the category of simplicial objects in $\mathcal{A}$, $S\mathcal{A}$. For a chain complex $C. \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ the functor $\Gamma(C.)$ is usually defined by $\Gamma(C.)_n = \bigoplus_{k=0}^n \bigoplus_{\sigma \in \mathrm{Sur}([n],[k])} C_k$. So $\Gamma(C.)$ contains $|\mathrm{Sur}([n],[k])|$ copies of C_k and these copies are indexed by elements of $\mathrm{Sur}([n],[k])$.

The effect of the degeneracy operator $s_i : \Gamma(C.)_{n-1} \to \Gamma(C.)_n$ on the copy of C_k indexed by $\mu \in \mathrm{Sur}([n-1],[k])$ is to identify it with the copy of $C_k \in \Gamma(C.)_n$ indexed by $\overline{\sigma}_i(\mu)$ (c.f. Notation 1.1.4).

The effect of the face operator $d_i : \Gamma(C.)_n \to \Gamma(C.)_{n-1}$ on the copy of C_k indexed by $\mu \in \mathrm{Sur}([n],[k])$ depends on the nature of $\overline{\delta}_i(\mu)$ (c.f. Notation

1.1.4):

- If $\overline{\delta}_i(\mu)$ is surjective then C_k is identified with the copy of C_k indexed by $\overline{\delta}_i(\mu)$;
- if $\overline{\delta}_i(\mu)$ is not surjective and $\overline{\delta}_i(\mu) = \delta_j\hat{\mu}$ for some $\hat{\mu} \in \mathrm{Sur}([n-1],[k-1])$ and for some $j \neq 0$ then C_k is mapped to 0;
- if $\overline{\delta}_i(\mu)$ is not surjective and $\overline{\delta}_i(\mu) = \delta_0\hat{\mu}$ for some $\hat{\mu} \in \mathrm{Sur}([n-1],[k-1])$ then d_i maps the copy of C_k indexed by μ to the copy of C_{k-1} indexed by $\hat{\mu}$ with the same action as the differential of $C.$.

This can be expressed more concisely in symbols rather than in words. For $\mu \in \mathrm{Sur}([n],[k])$ we write $C_{k,\mu}$ to denote the copy of C_k in $\bigoplus_{\sigma\in\mathrm{Sur}([n],[k])} C_k$ that is contributed by μ and also, for $m \in C_k$, we write (m,μ) to denote $m \in C_{k,\mu}$. The face and degeneracy maps in $\Gamma(C.)$ are defined as follows:

$$s_i(m,\mu) := (m, \overline{\sigma}_i(\mu))$$

$$d_i(m,\mu) := \begin{cases} (m, \overline{\delta}_i(\mu)) & \text{if } \overline{\delta}_i(\mu) \text{ is surjective} \\ (\partial(m), \hat{\mu}) & \text{if } \overline{\delta}_i(\mu) = \delta_0\hat{\mu} \text{ with } \hat{\mu} \in \mathrm{Sur}([n-1],[k-1]) \\ 0 & \text{if } \overline{\delta}_i(\mu) = \delta_j\hat{\mu} \text{ with } \hat{\mu} \in \mathrm{Sur}([n-1],[k-1]), j \neq 0 \end{cases}$$

The object of this section is to rewrite these expressions using results from the previous section and thereby make the calculation of these operators simpler.

Lemma 1.1.3 tells us that for natural numbers n and k

$$\Gamma(C.)_n = C_0^{\binom{n}{0}} \bigoplus C_1^{\binom{n}{1}} \bigoplus C_2^{\binom{n}{2}} \bigoplus \cdots;$$

(note this is always a finite sum, since if $k > n$ then $\binom{n}{k} = 0$) again each copy of C_k indexed by the element of $\mathrm{Sur}([n],[k])$ that contributes it. But now we can use the ordering defined in Section 2 on $\mathrm{Sur}([n],[k])$ to order the copies of C_k. Because of this we will tend to use the ordinal associated to $\mu \in \mathrm{Sur}([n],[k])$ instead of μ to index a copy of C_k, i.e. if μ is the m^{th} element of $\mathrm{Sur}([n],[k])$ we will usually write $C_{k,m}$ instead of $C_{k,\mu}$.

Combining various results from the previous section we get the following proposition which allows us to simplify the cases. We write A^C for the complement of A in the set $\mathrm{Sur}([n],[k])$.

Proposition 1.2.1. *For $n, k \geq 0$ the following statements hold:*

(a) (i) for each $i \in \{0,\dots,n-1\}$ the sets $\mathrm{Sur}([n-1],[k])$ and $(S_i^n)^C$ have the same cardinality;

(ii) for each $i \in \{1,\dots,n\}$ the sets S_{i-1}^{n-1} and $S_i^n \setminus \bigcup_z S_{i,(z,1)}^n$ have the same cardinality (where z ranges over all partitions of i of length k or less);

(iii) the sets S_0^n and $\mathrm{Sur}([n-1],[k-1])$ have the same cardinality.

(b) For each $i \in \{0,\dots,n-1\}$ the map $\overline{\sigma}_i : \mathrm{Sur}([n-1],[k]) \to \mathrm{Sur}([n],[k])$ sends the l^{th} element of $\mathrm{Sur}([n-1],[k])$ to the l^{th} element of $(S_i^n)^C$.

(c) (i) For each $i \in \{0,\dots,n\}$ and $x \in \mathrm{Sur}([n],[k])$ the morphism $\overline{\delta}_i(x)$ is a non-surjection if and only if $x \in \bigcup_z S_{i,(z,1)}^n$ where z ranges over all partitions of i of length k or less.

(ii) If $x \in S_0^n$ then for some $\hat{x} \in \mathrm{Sur}([n-1],[k-1])$ we have $\overline{\delta}_0(x) = \delta_0\hat{x}$. Moreover the map $x \mapsto \hat{x}$ acts on S_0^n by sending the l^{th} element of S_0^n to the l^{th} element of $\mathrm{Sur}([n-1],[k-1])$.

(iii) For each $i \in \{0,\dots,n-1\}$ the map $\overline{\delta}_i : \mathrm{Sur}([n],[k]) \to \mathrm{Mor}([n-1],[k])$ acts on the set $(S_i^n)^C$ by sending the l^{th} element of $(S_i^n)^C$ to the l^{th} element of $\mathrm{Sur}([n-1],[k])$.

(iv) For each $i \in \{1,\dots,n\}$ the map $\overline{\delta}_i : \mathrm{Sur}([n],[k]) \to \mathrm{Mor}([n-1],[k])$ acts on the set $S_i^n \setminus \bigcup_z S_{i,(z,1)}^n$ by sending the l^{th} element of $S_i^n \setminus \bigcup_z S_{i,(z,1)}^n$ to the l^{th} element of S_{i-1}^{n-1}.

Proof. Part (a) of this proposition ensures that the later statements are well defined. Part (a) (i) follows from Theorem 1.1.14 and the injectivity of $\overline{\sigma}_i$ (Lemma 1.1.5). Lemma 1.1.13 tells us that for $i \in \{1, \dots, n-1\}$ we have $|S_i^n| = \binom{n-1}{k-1}$ and that $|\bigcup_z S^n_{i,(z,1)}| = \binom{n-2}{k-2}$, and therefore $|S_i^n \setminus \bigcup_z S^n_{i,(z,1)}| = \binom{n-1}{k-1} - \binom{n-2}{k-2} = \binom{n-2}{k-1} = |S_{i-1}^{n-1}|$ (the final step is given by Lemma 1.1.3). Furthermore $S_n^n = \mathrm{Sur}([n],[k])$ and by Lemma 1.1.13 we see that $|\bigcup_z S^n_{n,(z,1)}| = \binom{n-1}{k-1}$ so $|S_n^n \setminus \bigcup_z S^n_{i,(z,1)}| = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k} = |S_{n-1}^{n-1}|$ (the final step is given by Lemma 1.1.3). So we have shown part (a) (ii) of the second half of this theorem for all $i \in \{0, \dots, n\}$. The remaining part of part (a) follows from Lemmas 1.1.3 and 1.1.13.

Part (b) is seen by applying Proposition 1.1.11 to part (a) (i).

Let $i \in \{0, \dots, n\}$ and a be a partition of $i+1$ of length p. Define a' by $a'_l = a_l$ for $l < p$ and $a'_l = a_l - 1$ for $l = p$. Lemma 1.1.6 tells us that if $x = (a,b)$ then $\overline{\delta}_i(x) = (a', b)$. So if $a_p > 1$ then $\overline{\delta}_i(x) \in S_{i-1}^{n-1}$ (note that this case never occurs for $i = 0$ since if $i = 0$ then $a = (1)$). On the other hand we have $\overline{\delta}_i(x) = (a', b)$ is a non-surjection if and only $a_p = 1$, and hence we have part (c) (i) of this proposition.

If we take $x \in S_0^n$ we have (by Lemma 1.1.8) $\overline{\delta}_0(x) = \delta_0 \hat{x}$ for some $\hat{x} \in \mathrm{Sur}([n-1],[k-1])$. For any morphism y we see that $\delta_0(y(l)) = y(l) + 1$ so $(\delta_0 y)^* = ((0), y^*)$. So the map $x \mapsto \hat{x}$ takes a morphism whose associated partition is of the form $(1, \hat{x})$ to a morphism whose associated partition is $\hat{x}$ and thus this map is strictly order preserving. So using (a) (iii) we get (c) (ii).

By applying $\overline{\delta}_i \overline{\sigma}_i = \mathrm{id}$ (Lemma 1.1.5) to part (b) we get part (c) (iii).

Now the remaining statement follows by applying the fact that for all $i \in \{1, \dots, n\}$ $\overline{\delta}_i$ is strictly order preserving on S_i^n (Proposition 1.1.15) to part (a) (ii) of this statement. □

Theorem 1.2.2. *Let $n > 0$.*

(a) Let $i \in \{0, \dots, n-1\}$ and let $\underline{c} = (c_{k,l})_{k=0,\dots,n-1; l=1,\dots,\binom{n-1}{k}} \in \Gamma(C.)_{n-1}$. Write $s_i(\underline{c}) = (b_{k,l})_{k=0,\dots,n; l=1,\dots,\binom{n}{k}} \in \Gamma(C.)_n$ then we get the following relations:

(i) *If the* l^{th} *element of* $\mathrm{Sur}([n],[k])$ *is the* m^{th} *of* $(S_i^n)^C$ *then* $b_{k,l} = c_{k,m}$ *;*

(ii) *If the* l^{th} *of* $\mathrm{Sur}([n],[k])$ *is an element of* S_i^n *then then* $b_{k,l} = 0$ *.*

(b) *Let* $\underline{c} = (c_{k,l})_{k=0,\dots,n;l=1,\dots,\binom{n}{k}} \in \Gamma(C.)_n$ *.*

Write $d_0(\underline{c}) = (b_{k,l})_{k=0,\dots,n-1;l=1,\dots,\binom{n-1}{k}} \in \Gamma(C.)_{n-1}$ *then we get the following relation:* $b_{k,l} = \partial(c_{k+1,l}) + c_{k,\binom{n-1}{k-1}+l}$ *.*

(c) *Let* $i \in \{1,\dots,n-1\}$ *and let* $\underline{c} = (c_{k,l})_{k=0,\dots,n;l=1,\dots,\binom{n}{k}} \in \Gamma(C.)_n$ *.*

Write $d_i(\underline{c}) = (b_{k,l})_{k=0,\dots,n-1;l=1,\dots,\binom{n-1}{k}} \in \Gamma(C.)_{n-1}$ *then we get the following relations:*

(i) *If the* l^{th} *element of* $\mathrm{Sur}([n-1],[k])$ *is the* m^{th} *element of* S_{i-1}^{n-1} *then* $b_{k,l} = c_{k,\alpha(l)} + c_{k,\beta(m)}$ *where* $\alpha(l)$ *is the ordinal associated to the* l^{th} *element of* $(S_i^n)^C$ *and* $\beta(m)$ *is the ordinal associated with the* m^{th} *element of* $S_i^n \setminus \bigcup_z S_{i,(z,1)}^n$ *;*

(ii) *If the* l^{th} *element of* $\mathrm{Sur}([n-1],[k])$ *is an element of* $(S_{i-1}^{n-1})^C$ *then* $b_{k,l} = c_{k,\alpha(l)}$ *, where* $\alpha(l)$ *is the ordinal associated with the* l^{th} *element of* $(S_i^n)^C$ *.*

(d) *Let* $\underline{c} = (c_{k,l})_{k=0,\dots,n;l=1,\dots,\binom{n}{k}} \in \Gamma(C.)_n$ *.*

Write $d_n(\underline{c}) = (b_{k,l})_{k=0,\dots,n-1;l=1,\dots,\binom{n-1}{k}} \in \Gamma(C.)_{n-1}$ *then we get the following relation:*

If the l^{th} *element of* $\mathrm{Sur}([n],[k])$ *is the* m^{th} *element of* $S_n^n \setminus \bigcup_z S_{n,(z,1)}^n$ *then* $b_{k,m} = c_{k,l}$ *.*

Proof. This is a direct corollary of Proposition 1.2.1. □

When using Theorem 1.2.2 we can instantly describe the action of d_0 (part b) but to describe the action of the other face operators and the degeneracy operators we need to do some calculation. For each n that we are concerned with (the position in the simplicial complex) and each

$k \in \{1, \ldots, \min(n, l)\}$ (where l stands for the length of the chain complex) we need to know the sets S_i^n and $S_i^n \setminus \bigcup_z S_{i,(z,1)}^i$ for each $i \in \{0, \ldots, n\}$. So for each n and k we draw a table to help us determine these sets (see Example 1.2.3 below). The columns of the table we label by the possible values of i (0 through to n). The rows of the table we label with both the partition and the ordinal associated with the elements of $\mathrm{Sur}([n],[k])$.

If a cell in the table has its column labelled by i and its row is labelled by a partition x^* that has an initial partition of $i+1$ then we mark the cell with a $\times$ mark, if that initial partition ends with a 1 then we also mark the cell with a *. So if a cell is marked with a $\times$ mark then the surjection x is an element of the set S_i^n, if the cell is also marked with a * then x is an element of the set $\bigcup_z S_{i,(z,1)}^n$.

Having made the tables for n we can use Theorem 1.2.2 (a) to calculate the degeneracy operators $s_0, \ldots, s_{n-1} : \Gamma(C.)_{n-1} \to \Gamma(C.)_n$. If we have already made tables for $n-1$ then we can use Theorem 1.2.2 (c) to calculate the face operators $d_1, \ldots, d_{n-1} : \Gamma(C.)_n \to \Gamma(C.)_{n-1}$ and Theorem 1.2.2 (d) to calculate $d_n : \Gamma(C.)_n \to \Gamma(C.)_{n-1}$.

For $i = 1, \ldots, n$ we (obviously) have that $i \neq n+1$, but $\{(n+1)\} = \mathrm{Sur}([n],[0])^*$. So Theorem 1.2.2 (c) tells us that for each $i \in \{1, \ldots, n\}$ the face operator d_i acts on the single copy of C_0 in $\Gamma(C.)_n$ by sending it identically to the single copy of C_0 in $\Gamma(C.)_{n-1}$. Similarly Theorem 1.2.2 (a) tells us that each of the degeneracy operators act on the single copy of C_0 in $\Gamma(C.)_{n-1}$ by sending it identically to the single copy of C_0 in $\Gamma(C.)_n$. So often we won't bother to repeat describing the action of d_i on the copy of C_0 when $i \neq 0$, or the action of any degeneracy operator on C_0.

Example 1.2.3. To help elucidate these results we now look at the chain complex of length 2 $C \to B \to A$ placed in degrees 0, 1 and 2 with differential ∂. For each $n \in \{0,1,2,3,4\}$ we calculate all the degeneracy maps $s_i : \Gamma(C \to B \to A)_n \to \Gamma(C \to B \to A)_{n+1}$ and all the face maps $d_i : \Gamma(C \to B \to A)_{n+1} \to \Gamma(C \to B \to A)_n$.

Observe that Theorem 1.2.2 (c) tells us that for $i \in \{1, \ldots, n\}$ (i.e. when

$i \neq 0$) the face operator $d_i : \Gamma(C \to B \to A)_n \to \Gamma(C \to B \to A)_{n-1}$ acts by sending copies of C to copies of C, copies of B to copies of B and copies of A to copies of A, i.e. the differential ∂ plays no role. So when $i \neq 0$ we'll describe the action of d_i on the copies of C, B and A seperately. Theorem 1.2.2 (a) tells us that all the degeneracy operators act similarly. So we'll describe the action of s_i on the copies of C, B and A separately.

For $n = 1$ we get the following table:

	0	1
1 (1, 1)	$\times$	$\times^*$

So the face operator d_1 between $\Gamma(C \to B \to A)_1 = B \oplus A$ and $NF\Gamma(C \to B \to A)_0 = A$ acts by:

$$d_1 : (b_1) \mapsto 0.$$

The face operator d_0 between $\Gamma(C \to B \to A)_1 = B \oplus A$ and $NF\Gamma(C \to B \to A)_0 = A$ acts by:

$$d_0 : (b_1, a) \mapsto (a + \partial(b_1))$$

For $n = 2$ we get the following tables:

	0	1	2
1 (1, 2)	$\times$		$\times$
2 (2, 1)		$\times$	$\times^*$

	0	1	2
1 (1, 1, 1)	$\times$	$\times^*$	$\times^*$

So the face operators d_1, d_2 between and $\Gamma(C \to B \to A)_2 = C \oplus B^2 \oplus A$

and $\Gamma(C \to B \to A)_1 = B \oplus A$ act by:

$$d_1 : (b_1, b_2) \mapsto (b_1 + b_2)$$
$$d_2 : (b_1, b_2) \mapsto (b_1)$$
$$d_1 : (c) \mapsto 0$$
$$d_2 : (c) \mapsto 0$$

The face operator d_0 between $\Gamma(C \to B \to A)_2 = C \oplus B^2 \oplus A$ and $\Gamma(C \to B \to A)_1 = B \oplus A$ acts by:

$$d_0 : (c, b_1, b_2, a) \mapsto (\partial(c) + b_2, \partial(b_1) + a)$$

The degeneracy operators s_0, s_1 between $\Gamma(C \to B \to A)_1 = B \oplus A$ and $\Gamma(C \to B \to A)_2 = C \oplus B^2 \oplus A$ act by:

$$s_0 : (b_1) \mapsto (0, b_1)$$
$$s_1 : (b_1) \mapsto (b_1, 0)$$

For $n = 3$ we get the following tables:

	0	1	2	3
1 $(1,3)$	$\times$			$\times$
2 $(2,2)$		$\times$		$\times$
3 $(3,1)$			$\times$	$\times^*$

	0	1	2	3
1 $(1,1,2)$	$\times$	$\times^*$		$\times$
2 $(1,2,1)$	$\times$		$\times$	$\times^*$
3 $(2,1,1)$		$\times$	$\times^*$	$\times^*$

So the face operators d_1, d_2, d_3 between $\Gamma(C \to B \to A)_3 = C^3 \oplus B^3 \oplus A$

and $\Gamma(C \to B \to A)_2 = C \oplus B^2 \oplus A$ act by:

$$d_1 : (b_1, b_2, b_3) \mapsto (b_1 + b_2, b_3)$$
$$d_2 : (b_1, b_2, b_3) \mapsto (b_1, b_2 + b_3)$$
$$d_3 : (b_1, b_2, b_3) \mapsto (b_1, b_2)$$
$$d_1 : (c_1, c_2, c_3) \mapsto (c_2 + c_3)$$
$$d_2 : (c_1, c_2, c_3) \mapsto (c_1 + c_2)$$
$$d_3 : (c_1, c_2, c_3) \mapsto (c_1)$$

The face operator d_0 between $\Gamma(C \to B \to A)_3 = C^3 \oplus B^3 \oplus A$ and $\Gamma(C \to B \to A)_2 = C \oplus B^2 \oplus A$ acts by:

$$d_0 : (c_1, c_2, c_3, b_1, b_2, b_3, a) \mapsto (c_3, \partial(c_1) + b_2, \partial(c_2) + b_3, \partial(b_1) + a)$$

The degeneracy operators s_0, s_1, s_2 between $\Gamma(C \to B \to A)_2 = C \oplus B^2 \oplus A$ and $\Gamma(C \to B \to A)_3 = C^3 \oplus B^3 \oplus A$ act by:

$$s_0 : (b_1, b_2) \mapsto (0, b_1, b_2)$$
$$s_1 : (b_1, b_2) \mapsto (b_1, 0, b_2)$$
$$s_3 : (b_1, b_2) \mapsto (b_1, b_2, 0)$$
$$s_0 : (c_1) \mapsto (0, 0, c_1)$$
$$s_1 : (c_1) \mapsto (0, c_1, 0)$$
$$s_2 : (c_1) \mapsto (c_1, 0, 0)$$

For $n = 4$ we get the following tables:

	0	1	2	3	4
1 (1, 4)	$\times$				$\times$
2 (2, 3)		$\times$			$\times$
3 (3, 2)			$\times$		$\times$
4 (4, 1)				$\times$	$\times^*$

	0	1	2	3	4
1 (1, 1, 3)	$\times$	$\times^*$			$\times$
2 (1, 2, 2)	$\times$		$\times$		$\times$
3 (1, 3, 1)	$\times$			$\times$	$\times^*$
4 (2, 1, 2)		$\times$	$\times^*$		$\times$
5 (2, 2, 1)		$\times$		$\times$	$\times^*$
6 (3, 1, 1)			$\times$	$\times^*$	$\times^*$

So the face operators d_1, d_2, d_3, d_4 between $\Gamma(C \to B \to A)_4 = C^6 \oplus B^4 \oplus A$ and $\Gamma(C \to B \to A)_3 = C^3 \oplus B^3 \oplus A$ act by:

$$d_1 : (b_1, b_2, b_3, b_4) \mapsto (b_1 + b_2, b_3; b_4)$$
$$d_2 : (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2 + b_3, b_4)$$
$$d_3 : (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, b_3 + b_4)$$
$$d_4 : (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, b_3)$$
$$d_1 : (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (c_2 + c_4, c_3 + c_5, c_6)$$
$$d_2 : (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (c_1 + c_2, c_3, c_5 + c_6)$$
$$d_3 : (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (c_1, c_2 + c_3, c_4 + c_5)$$
$$d_4 : (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (c_1, c_2, c_4)$$

The face operator d_0 between $\Gamma(C \to B \to A)_4 = C^6 \oplus B^4 \oplus A$ and $\Gamma(C \to B \to A)_3 = C^3 \oplus B^3 \oplus A$ acts by:

$$d_0 : (c_1, c_2, c_3, c_4, c_5, c_6, b_1, b_2, b_3, b_4, a) \mapsto$$
$$(c_4, c_5, c_6, \partial(c_1) + b_2, \partial(c_2) + b_3, \partial(c_3) + b_4, \partial(b_1) + a).$$

The degeneracy operators s_0, s_1, s_2, s_3 between $\Gamma(C \to B \to A)_3 = C^3 \oplus B^3 \oplus A$ and $\Gamma(C \to B \to A)_4 = C^6 \oplus B^4 \oplus A$ act by:

$$s_0 : (b_1, b_2, b_3) \mapsto (0, b_1, b_2, b_3)$$
$$s_1 : (b_1, b_2, b_3) \mapsto (b_1, 0, b_2, b_3)$$
$$s_2 : (b_1, b_2, b_3) \mapsto (b_1, b_2, 0, b_3)$$
$$s_3 : (b_1, b_2, b_3) \mapsto (b_1, b_2, b_3, 0)$$
$$s_0 : (c_1, c_2, c_3) \mapsto (0, 0, 0, c_1, c_2, c_3)$$
$$s_1 : (c_1, c_2, c_3) \mapsto (0, c_1, c_2, 0, 0, c_3)$$
$$s_2 : (c_1, c_2, c_3) \mapsto (c_1, 0, c_2, 0, c_3, 0)$$
$$s_3 : (c_1, c_2, c_3) \mapsto (c_1, c_2, 0, c_3, 0, 0)$$

For $n = 5$ we get the following tables:

	0	1	2	3	4	5
1 $(1,5)$	$\times$					$\times$
2 $(2,4)$		$\times$				$\times$
3 $(3,3)$			$\times$			$\times$
4 $(4,2)$				$\times$		$\times$
5 $(5,1)$					$\times$	$\times^*$

and

	0	1	2	3	4	5
1 (1,1,4)	$\times$	$\times^*$				$\times$
2 (1,2,3)	$\times$		$\times$			$\times$
3 (1,3,2)	$\times$			$\times$		$\times$
4 (1,4,1)	$\times$				$\times$	$\times^*$
5 (2,1,3)		$\times$	$\times^*$			$\times$
6 (2,2,2)		$\times$		$\times$		$\times$
7 (2,3,1)		$\times$			$\times$	$\times^*$
8 (3,1,2)			$\times$	$\times^*$		$\times$
9 (3,2,1)			$\times$		$\times$	$\times^*$
10 (4,1,1)				$\times$	$\times^*$	$\times^*$

So the face operators d_1, d_2, d_3, d_4, d_5 between $\Gamma(C \to B \to A)_5 = C^{10} \oplus B^5 \oplus A$ and $\Gamma(C \to B \to A)_4 = C^6 \oplus B^4 \oplus A$ act by:

$$d_1 : (b_1, b_2, b_3, b_4, b_5) \mapsto (b_1 + b_2, b_3, b_4, b_5)$$
$$d_2 : (b_1, b_2, b_3, b_4, b_5) \mapsto (b_1, b_2 + b_3, b_4, b_5)$$
$$d_3 : (b_1, b_2, b_3, b_4, b_5) \mapsto (b_1, b_2, b_3 + b_4, b_5)$$
$$d_4 : (b_1, b_2, b_3, b_4, b_5) \mapsto (b_1, b_2, b_3, b_4 + b_5)$$
$$d_5 : (b_1, b_2, b_3, b_4, b_5) \mapsto (b_1, b_2, b_3, b_4)$$
$$d_1 : (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}) \mapsto (c_2 + c_5, c_3 + c_6, c_4 + c_7, c_8, c_9, c_{10})$$
$$d_2 : (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}) \mapsto (c_1 + c_2, c_3, c_4, c_6 + c_8, c_7 + c_9, c_{10})$$
$$d_3 : (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}) \mapsto (c_1, c_2 + c_3, c_4, c_5 + c_6, c_7, c_9 + c_{10})$$
$$d_4 : (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}) \mapsto (c_1, c_2, c_3 + c_4, c_5, c_6 + c_7, c_8 + c_9)$$
$$d_5 : (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}) \mapsto (c_1, c_2, c_3, c_5, c_6, c_8)$$

We know that $\Gamma(C \to B \to A)_5 = (C^{10} \oplus B^5 \oplus A)$ and also that $\Gamma(C \to B \to A)_4 = (C^6 \oplus B^4 \oplus A)$. The face operator d_0 between $\Gamma(C \to B \to A)_5$ and $\Gamma(C \to B \to A)_4$ acts by: sending the element $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, b_1, b_2, b_3, b_4, b_5, a)$ to the element $(c_5, c_6, c_7, c_8, c_9, c_{10}, \partial(c_1) + b_2, \partial(c_2) + b_3, \partial(c_3) + b_4, \partial(c_4) + b_5, \partial(b_1) + a)$

The degeneracy operators s_0, s_1, s_2, s_3, s_4 between $\Gamma(C \to B \to A)_4 = C^6 \oplus B^4 \oplus A$ and $\Gamma(C \to B \to A)_5 = C^{10} \oplus B^5 \oplus A$ act by:

$$
\begin{aligned}
s_0 &: (b_1, b_2, b_3, b_4) \mapsto (0, b_1, b_2, b_3, b_4)\\
s_1 &: (b_1, b_2, b_3, b_4) \mapsto (b_1, 0, b_2, b_3, b_4)\\
s_2 &: (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, 0, b_3, b_4)\\
s_3 &: (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, b_3, 0, b_4)\\
s_4 &: (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, b_3, b_4, 0)\\
s_0 &: (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (0, 0, 0, 0, c_1, c_2, c_3, c_4, c_5, c_6)\\
s_1 &: (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (0, c_1, c_2, c_3, 0, 0, 0, c_4, c_5, c_6)\\
s_2 &: (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (c_1, 0, c_2, c_3, 0, c_4, c_5, 0, 0, c_6)\\
s_3 &: (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (c_1, c_2, 0, c_3, c_4, 0, c_5, 0, c_6, 0)\\
s_4 &: (c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (c_1, c_2, c_3, 0, c_4, c_5, 0, c_6, 0, 0)
\end{aligned}
$$

1.3 Cross-effect functors

Recall a functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called linear if for all pairs of objects in $A, B \in \mathcal{A}$ we have the relation $F(A \oplus B) = F(A) \oplus F(B)$. For a nonlinear functor $G : \mathcal{A} \to \mathcal{B}$, with the property that $G(0_{\mathcal{A}}) = 0_{\mathcal{B}}$, the theory of cross-effect functors allows us to decompose $G(A \oplus B)$ into the direct sum of objects in $\mathcal{B}$, and also gives us analogues of other nice properties of linear functors. In this section I summarise some results from the paper 'On the groups $H(\Pi, n)$, II' [EM] that are relevant to my work.

For the rest of this section let G and H be commutative groups and

$f : G \to H$ a function.

Definition 1.3.1. We define the composition $\top$ on $\mathbb{Z}[G]$, the group ring of G, by saying it acts on elements of G as follows

$$\top : \mathbb{Z}[G] \times \mathbb{Z}[G] \to \mathbb{Z}[G] \tag{1.1}$$

$$(x_1, x_2) \mapsto (x_1 +_G x_2) -_{\mathbb{Z}[G]} (x_1) -_{\mathbb{Z}[G]} (x_2) \tag{1.2}$$

where $+_G$ stands for addition in G and $-_{\mathbb{Z}[G]}$ stands for subtraction in $\mathbb{Z}[G]$. By requiring $\top$ to be distributive we define it on the whole of $\mathbb{Z}[G]$

It can easily be seen that this composition $\top$ on $\mathbb{Z}[G]$ is associative. The commutativity of $+_G$ means that $\top$ is also commutative.

Definition 1.3.2. (a) Extend f to $\mathbb{Z}[G]$ linearly i.e. as follows

$$\mathbb{Z}[G] \to H$$

$$m_1(x_1) + \ldots + m_n(x_n) \mapsto \sum_{i=1}^{n} m_i f(x_i)$$

where $m_i \in \mathbb{Z}, x_i \in G$ and call this extension f.

(b) The composition $f(- \top -)$ is called *the first deviation of* f and acts on elements of G^2 as follows

$$\mathbb{Z}[G] \times \mathbb{Z}[G] \to H$$

$$(x, y) \mapsto f(x +_G y) -_H f(x) -_H f(y),$$

the linearity of f on $\mathbb{Z}[G]$ and the distributativity of $\top$ means the first deviation is linear in each variable and is distributive.

The first deviation of f is the zero function exactly when $f : G \to H$ is a homomorphism, so in some sense the first deviation of f measures how close f is to being a homomorphism. The commutativity and associativity of $\top$ mean that the first deviation of f is also commutative and associative.

The associativity of the first deviation of f gives rise to a unique second, third and further deviations, which will all be associative and symmetric and linear in each variable. By induction we see that the $(n-1)^{th}$ deviation $f(- \top \ldots \top -) : \mathbb{Z}[G]^n \to H$ acts on elements of G^n as follows

$$f(x_1 \top \ldots \top x_n) = \sum_{k=1}^{n} \sum_{j_1 < \ldots < j_k} (-1)^{n-k} f(x_{j_1} +_G \ldots +_G x_{j_k}) \tag{1.3}$$

its action on the rest of $\mathbb{Z}[G]^n$ is given by its multilinearity.

It is easy to see that the n^{th} deviation of a sum of functions is the sum of their n^{th} deviations.

Rearranging $f(x \top y) = f(x +_G y) -_H f(x) -_H f(y)$ we get the following equation $f(x +_G y) = f(x \top y) +_H f(x) +_H f(y)$ and by induction we get the following:

$$f(x_1 + \ldots + x_n) = \sum_{k=1}^{n} \sum_{j_1 < \ldots < j_k} f(x_{j_1} \top \ldots \top x_{j_k}). \tag{1.4}$$

Lemma 1.3.3. *If* $f(0) = 0$ *then* $f(x_1 \top \ldots \top x_n) = 0$ *whenever any of* $x_1, \ldots, x_n$ *are* 0.

Proof. Without loss of generality we may assume that $x_n = 0$, then using equation 1.3 we see:

$$\begin{aligned} f(x_1 \top \ldots \top x_n) &= \sum_{k=1}^{n} \sum_{j_1 < \ldots < j_k} (-1)^{n-k} f(x_{j_1} +_G \ldots +_G x_{j_k}) \\ &= (-1)^{n-1} f(x_n) + \\ &\left(\sum_{k=1}^{n-1} \sum_{j_1 < \ldots < j_k < n} (-1)^{n-k} f(x_{j_1} +_G \ldots +_G x_{j_k}) + \right. \\ &\left. \sum_{k=1}^{n-1} \sum_{j_1 < \ldots < j_k < n} (-1)^{n-k-1} f(x_{j_1} +_G \ldots +_G x_{j_k} +_G x_n). \right) \end{aligned}$$

The first term is zero by assumption, the two double sums are the same except for a change of sign, so the sum of these two is zero. Hence $f(x_1 \top \dots \top x_n) = 0$. □

In an abelian category each Hom set is an abelian group. For the rest of this section we will be applying what we have learned about deviations to construct cross-effect functors. We let $F : \mathcal{A} \to \mathcal{B}$ be a functor between an additive category $\mathcal{A}$ and an abelian category $\mathcal{B}$ with $F(0_{\mathcal{A}}) = 0_{\mathcal{B}}$. The condition $F(0_{\mathcal{A}}) = 0_{\mathcal{B}}$ is equivalent to the condition that the image of any zero homomorphism in $\mathcal{A}$ under F is a zero homomorphism in $\mathcal{B}$.

If we have $f_1, \dots, f_n \in \mathrm{Hom}(A, B)$, then applying the definitions above we see that $F(f_1 \top \dots \top f_n) = \sum_{k=1}^{n} \sum_{j_1 < \dots < j_k} (-1)^{n-k} F(f_{j_1} + \dots + f_{j_k})$.

The functoriality of F and the distributativity of composition in an abelian category tells us that for $g \in \mathrm{Hom}(B, B')$ and $h \in \mathrm{Hom}(A', A)$ we get the following relations:

$$F(g)F(f_1 \top \dots \top f_n) = F(gf_1 \top \dots \top gf_n) \tag{1.5}$$

$$F(f_1 \top \dots \top f_n)F(h) = F(f_1 h \top \dots \top f_n h).$$

Notation 1.3.4. Let $A = A_1 \oplus \dots \oplus A_n$. For each non-empty set $\alpha = \{j_1 < \dots < j_k\}$ that is a subset of $\{1, \dots, n\}$, and each $j \in \alpha$ we write: $A^\alpha = \bigoplus_{k \in \alpha} A_k$; i^α for the canonical injection $A^\alpha \to A$; p^α for the canonical projection $A \to A^\alpha$; ψ_j^α for the map $A^\alpha \to A^\alpha$, $(a_{i_1}, \dots, a_{i_k}) \mapsto (0, \dots, 0, a_j, 0 \dots, 0)$.

When β is a subset of α we write ψ_β^α for the sum $\sum_{j \in \beta} \psi_j^\alpha$, in particular we write $\psi_{\{k\}^C}$ for the sum $\psi_1 + \dots + \hat{\psi}_k + \dots + \psi_n$.

For convenience when $\alpha = \{1, \dots, n\}$ we suppress the superscript α on A^α, and ψ_j^α for each $j \in \alpha$.

Because each A^α is a direct sum we get the following relations for each $\alpha \subset \{1, \dots, n\}$ and $i, j \in \alpha$:

$$\psi_i^\alpha \psi_j^\alpha = \begin{cases} \psi_j^\alpha & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.6}$$

$$\sum_{k \in \alpha} \psi_k^\alpha = \mathrm{id}_{A^\alpha} \tag{1.7}$$

$$\psi_j = i^\alpha \psi_j^\alpha p^\alpha \tag{1.8}$$

Definition 1.3.5. The n^{th} cross-effect of F is a functor $\mathcal{A}^n \to \mathcal{B}$. It acts on objects by

$$\mathrm{cr}_n(F)(A_1, \ldots, A_n) = F(\psi_1 \top \ldots \top \psi_n)F(A).$$

For morphisms $f_1 : A_1 \to B_1, \ldots, f_n : A_n \to B_n$ the morphism $\mathrm{cr}_n(F)(f_1, \ldots, f_n)$ is defined to be the following restriction

$$F(f_1 \oplus \ldots \oplus f_n) : \mathrm{cr}_n(F)(A_1, \ldots, A_n) \to \mathrm{cr}_n(F)(B_1, \ldots, B_n).$$

The action on morphisms is well defined because for each $i \in \{1, \ldots, n\}$ the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "f_1\oplus\ldots\oplus f_n"] \arrow[d, "\psi_i"] & B \arrow[d, "\mu_i"] \\ A \arrow[r, "f_1\oplus\ldots\oplus f_n"'] & B \end{tikzcd}$$

where $B = \oplus_{i=1}^n B_i$ and μ_i is the map $(b_1, \ldots, b_n) \to (0, \ldots, 0, b_i, 0, \ldots, 0)$, and hence $F(f_1 \oplus \ldots \oplus f_n)F(\psi_1 \top \ldots \top \psi_n) = F(\mu_1 \top \ldots \top \mu_n)F(f_1 \oplus \ldots \oplus f_n)$.

The fundamental property of cross-effect functors is the decomposition they give of F evaluated on a direct sum, as is shown in the following theorem.

Theorem 1.3.6. *For each subset $\alpha = \{j_1 < \ldots < j_{|\alpha|}\}$ of $\{1, \ldots, n\}$ write $(A_j, j \in \alpha)$ for the tuple $(A_{j_1}, \ldots, A_{j_{|\alpha|}})$.*

(a) *The restriction*

$$F(i^\alpha) : \mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha) \to F(A_1 \oplus \ldots \oplus A_n)$$

has kernel zero and image $F(\psi_{j_1} \top \cdots \top \psi_{j_{|\alpha|}})F(A)$*; the inverse of this map is the map induced by* $F(p^\alpha)$.

(b) *The image of the direct sum* A *under the functor* F *can be decomposed into a direct sum by the following isomorphism*

$$F(A_1 \oplus \ldots \oplus A_n) \cong \bigoplus_{\alpha \subset \{1,\ldots,n\}} \mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha)$$

more precisely $F(A)$ *is the following direct sum*

$$F(A_1 \oplus \ldots \oplus A_n) = \bigoplus_{\alpha \subset \{1,\ldots,n\}} F(i^\alpha)\,\mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha).$$

Proof. Since $p^\alpha i^\alpha = \mathrm{id}_{A_\alpha}$ we see that $F(p^\alpha)F(i^\alpha) = F(\mathrm{id}_{A_\alpha})$. The following shows that the restriction of the projection $F(p^\alpha)$ to $F(\psi_{j_1} \top \cdots \top \psi_{j_k})F(A)$ has image $\mathrm{cr}_k(F)(A_j, j \in \alpha)$:

$$\begin{aligned} F(p^\alpha)F(\psi_{j_1} \top \cdots \top \psi_{j_k})F(A) &= F(p^\alpha)F(i^\alpha\psi^\alpha_{j_1}p^\alpha \top \cdots \top i^\alpha\psi^\alpha_{j_k}p^\alpha)F(A) \\ &= F(p^\alpha)F(i^\alpha)F(\psi^\alpha_{j_1} \top \cdots \top \psi^\alpha_{j_k})F(p^\alpha)F(A) \\ &= F(\psi^\alpha_{j_1} \top \cdots \top \psi^\alpha_{j_k})F(p^\alpha)F(A) \\ &= F(\psi^\alpha_{j_1} \top \cdots \top \psi^\alpha_{j_k})F(A^\alpha) = \mathrm{cr}_k(F)(A_j, j \in \alpha) \end{aligned}$$

To see the inverse of $F(p^\alpha)$ on $\mathrm{cr}_k(F)(A_j, j \in \alpha)$ is $F(i^\alpha)$ we apply it to both sides of the equation:

$$\begin{aligned} F(i^\alpha)\,\mathrm{cr}_k(F)(A_j, j \in \alpha) &= F(i^\alpha)F(p^\alpha)F(\psi_{j_1} \top \cdots \top \psi_{j_k})F(A) \\ &= F(i^\alpha p^\alpha \psi_{j_1} \top \cdots \top i^\alpha p^\alpha \psi_{j_k})F(A) \\ &= F(\psi_{j_1} \top \cdots \top \psi_{j_k})F(A) \end{aligned}$$

So we have shown part (a).

We now show part (b) by showing that for every pair of subsets $\alpha = \{i_1, \ldots, i_k\}$, $\beta = \{j_1 \ldots, j_l\}$ of $\{1, \ldots, n\}$ the composition $F(\psi_{i_1} \top \cdots \top \psi_{i_k})F(\psi_{j_1} \top \cdots \top \psi_{j_l})$ is zero if $\alpha \neq \beta$, and is $F(\psi_{i_1} \top \cdots \top \psi_{i_k})$ if $\alpha = \beta$ (i.e. we show that $F(\psi_{i_1} \top \cdots \top \psi_{i_k})$ and $F(\psi_{j_1} \top \cdots \top \psi_{j_l})$ are orthogonal).

When $\alpha \neq \beta$ we can, without loss of generality, say there is some element $i \in \beta \setminus \alpha$. Applying equation 1.6 we see that

$$\begin{aligned}
&F(\psi_{j_1} \top \cdots \top \psi_{j_k})F(\psi_{j'_1} \top \cdots \top \psi_{j'_l}) \\
&\qquad = F(\psi_{j_1}\psi_{\{i\}^C} \top \cdots \top \psi_{j_k}\psi_{\{i\}^C})F(\psi_{j'_1} \top \cdots \top \psi_{j'_l}) \\
&\qquad = F(\psi_{j_1} \top \cdots \top \psi_{j_k})F(\psi_{\{i\}^C})F(\psi_{j'_1} \top \cdots \top \psi_{j'_l}) \\
&\qquad = F(\psi_{j_1} \top \cdots \top \psi_{j_k})F(\psi_{\{i\}^C}\psi_{j'_1} \top \cdots \top \psi_{\{i\}^C}\psi_{j'_l}).
\end{aligned}$$

By definition $i \notin \alpha$, so by equation 1.6 one of $\psi_{\{i\}^C}\psi_{j'_1}, \ldots, \psi_{\{i\}^C}\psi_{j'_l}$ is zero. Hence Lemma 1.3.3 tells us that this composition is zero.

Now from equation (1.4) we see that

$$F(\mathrm{id}_A) = F(\psi_1 + \ldots + \psi_n) = \sum_{k=1}^{n} \sum_{j_1 < \ldots < j_k} F(\psi_{j_1} \top \cdots \top \psi_{j_k}).$$

and using part (a), which we have already shown, we see that for any $\{j'_1 < \ldots < j'_l\}$ that is a subset of $\{1, \ldots, n\}$ we get

$$\begin{aligned}
F(\psi_{j'_1} \top \cdots \top \psi_{j'_l}) &= F(\psi_{j'_1} \top \cdots \top \psi_{j'_l})F(\mathrm{id}_A) \\
&= F(\psi_{j'_1} \top \cdots \top \psi_{j'_l}) \sum_{k=1}^{n} \sum_{j_1 < \ldots < j_k} F(\psi_{j_1} \top \cdots \top \psi_{j_k}) \\
&= \sum_{k=1}^{n} \sum_{j_1 < \ldots < j_k} F(\psi_{j'_1} \top \cdots \top \psi_{j'_l})F(\psi_{j_1} \top \cdots \top \psi_{j_k}) \\
&= F(\psi_{j'_1} \top \cdots \top \psi_{j'_l})F(\psi_{j'_1} \top \cdots \top \psi_{j'_l})
\end{aligned}$$

i.e. the maps $F(\psi_{j'_1} \top \dots \top \psi_{j'_l})$ form a complete set of pairwise orthogonal projectors. □

Proposition 1.3.7. *(a) Whenever any of the objects A_j for $j \in \{1,\dots,n\}$ are the zero object then $\mathrm{cr}_n(F)(A_1,\dots,A_n)$ is also the zero object.*

(b) For each permutation π in S_n, the group of permuations of n, we get the natural isomorphism:

$$\mathrm{cr}_n(F)(A_1,\dots,A_n) \cong \mathrm{cr}_n(F)(A_{\pi(1)},\dots,A_{\pi(n)}).$$

Proof. For the first part we note that if A_j is the zero object then ψ_j is the zero map. Hence by Lemma 1.3.3 the map $F(\psi_1 \top \dots \top \psi_n)$ is the zero map, so its image, $\mathrm{cr}_n(F)(A_1,\dots,A_n)$, must be the zero object.

Now we prove the second part. For each $\pi \in S_n$ we also write π for the ismorphism $A_1 \oplus \dots \oplus A_n \to A_{\pi(1)} \oplus \dots \oplus A_{\pi(n)}$ which acts by $(a_1,\dots,a_n) \mapsto (a_{\pi(1)},\dots,a_{\pi(n)})$, and we write $\mu_{\pi(i)}$ for the map $A_{\pi(1)} \oplus \dots \oplus A_{\pi(n)} \to A_{\pi(1)} \oplus \dots \oplus A_{\pi(n)}$ that acts by $(a_1,\dots,a_n) \mapsto (0,\dots,0,a_{\pi(i)},0,\dots,0)$. Then for each $\pi \in S_n$ the following diagram is commutative:

$$\begin{tikzcd}
A_1 \oplus \dots \oplus A_n \arrow[r, "\pi"] \arrow[d, "\psi_i"] & A_{\pi(1)} \oplus \dots \oplus A_{\pi(n)} \arrow[d, "\mu_{\pi(i)}"] \\
A_1 \oplus \dots \oplus A_n \arrow[r, "\pi"'] & A_{\pi(1)} \oplus \dots \oplus A_{\pi(n)}
\end{tikzcd}$$

And so the following diagram is commutative:

$$\begin{tikzcd}
F(A_1 \oplus \dots \oplus A_n) \arrow[r, "F(\pi)"] \arrow[d, "F(\psi_1 \top \cdots \top \psi_n)"] & F(A_{\pi(1)} \oplus \dots \oplus A_{\pi(n)}) \arrow[d, "F(\mu_{\pi(1)} \top \cdots \top \mu_{\pi(n)})"] \\
\mathrm{cr}_n(F)(A_1,\dots,A_n) \arrow[r, "F(\pi)"'] & \mathrm{cr}_n(F)(A_{\pi(1)},\dots,A_{\pi(n)})
\end{tikzcd}$$

So $F(\pi)$ induces a homomorphism between $\mathrm{cr}_n(F)(A_1, \ldots, A_n)$ and $\mathrm{cr}_n(F)(A_{\pi(1)}, \ldots, A_{\pi(n)})$. Because $\pi\pi^{-1} = \mathrm{id}_A$ we see that $F(\pi)$ is actually an isomorphism. $\square$

Lemma 1.3.8. *Suppose* $\mathrm{cr}_n(F)(A, \ldots, A) = 0$. *Let* B *and* C *be modules. Then for any* n *homomorphisms* $f_i : A \to B$ *we have*

$$F(f_1 \top \cdots \top f_n) = 0$$

and similarly for any n *homomorphisms* $g_i : C \to A$

$$F(g_1 \top \cdots \top g_n) = 0$$

Proof. Let d be the diagonal map $A \to A^n$. The morphisms $f_1, \ldots, f_n$ define a morphism $f : A^n \to B$ by $f(a_1, \ldots, a_n) = f_1(a_1) + \ldots + f_n(a_n)$. For each $i \in \{1, \ldots, n\}$ let $\lambda_i : A^n \to A^n$ be the homomorphism which acts by $(a_1, \ldots, a_n) \mapsto (0, \ldots, 0, a_i, 0, \ldots, 0)$. Now $f_i = f\lambda_i d$. So we get

$$F(f_1 \top \cdots \top f_n) = F(f\lambda_1 d \top \cdots \top f\lambda_n d) = F(f)F(\lambda_1 \top \cdots \top \lambda_n)F(d)$$

but $0 = \mathrm{cr}_n(F)(A, \ldots, A) = F(\lambda_1 \top \cdots \top \lambda_n)F(A^n) = 0$ so $F(\lambda_1 \top \cdots \top \lambda_n) = 0$ hence we see the first part of the result.

Define the map $+_n : A^n \to A, (a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i$. The other half follows similarly by noting that $g_1, \ldots, g_n$ define a map $g : C \to A^n$ by $g = (g_1 \oplus \ldots \oplus g_n)d$, and that $g_i = +_n\lambda_i g$ and following through the argument given above. $\square$

Theorem 1.3.9. *Let* n *be a non-negative integer. The following are equivalent conditions on the functor* F:

(i) the functor $\mathrm{cr}_n(F)$ *is the zero functor;*

(ii) for every $M, M' \in \mathcal{A}$ *and any morphisms* $f_1, \ldots, f_n : M \to M'$ *the morphism* $F(f_1 \top \cdots \top f_n)$ *is the zero morphism;*

(iii) for every $M \in \mathcal{A}$ the object $\mathrm{cr}_n(F)(M, \ldots, M)$ is the zero object.

Proof. The first clearly implies the third part. Given the third part we can see the second part by using the above Lemma, taking $A = M, B = M'$ and $f_1, \ldots, f_n$ the maps between them. Given the second part we can see the first if we take $M = M' = A$ and $f_1, \ldots, f_n$ to be $\psi_1, \ldots, \psi_n$. $\square$

Definition 1.3.10. If F satisfies any of the conditions in Theorem 1.3.9 then we say that F *is a functor of degree less than* $n+1$. If a functor is of degree less than n then, because of the iterative way that deviations are defined, it is also a functor of degree less than $n+1$. So every covariant functor between $\mathcal{A}$ and $\mathcal{B}$ that sends the zero object in $\mathcal{A}$ to the zero object in $\mathcal{B}$ has a well-defined degree. The degree of a functor is either a non-negative integer or infinity.

Proposition 1.3.11. *Let $\alpha = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$. Let E be a functor $\mathcal{A}^k \to \mathcal{B}$ which is zero whenever any of its arguments is zero. Let $g_\alpha : E(A_{i_1}, \ldots, A_{i_k}) \to F(A_1 \oplus \ldots \oplus A_n)$ be a homomorphism which is natural when both sides are regarded as covariant functors from $\mathcal{A}^n$ to $\mathcal{B}$. Then we have the following inclusion:*

$$g_\alpha E(A_{i_1}, \ldots, A_{i_k}) \subset F(\psi_{i_1} \top \cdots \top \psi_{i_k}) F(A)$$

Proof. For each non-empty subset β of α and each $j \in \alpha$ define morphisms $\lambda_j^\beta : A_j \to A_j$ by $\lambda_j^\beta = \mathrm{id}_{A_j}$ if $j \in \beta$ and $\lambda_j^\beta = 0$ if $j \notin \beta$. Then because g_α is natural the following diagram is commutative for each β.

$$\begin{tikzcd}
E(A_{i_1}, \ldots, A_{i_k}) \arrow[r, "g_\alpha"] \arrow[d, "E(\lambda_{i_1}^\beta, \ldots, \lambda_{i_k}^\beta)"'] & F(A_1 \oplus \ldots \oplus A_n) \arrow[d, "F(\sum_{j \in \beta} \psi_j)"] \\
E(A_{i_1}, \ldots, A_{i_k}) \arrow[r, "g_\alpha"'] & F(A_1 \oplus \ldots \oplus A_n)
\end{tikzcd}$$

Because of this commutivity by adding in signs we get the following sum:

$$\begin{aligned} g_\alpha \sum_{\beta\subset\alpha}(-1)^{|\alpha|-|\beta|}E(\lambda^\beta_{i_1},\dots,\lambda^\beta_{i_k}) &= \sum_{\beta\subset\alpha}(-1)^{|\alpha|-|\beta|}F(\sum_{j\in\beta}\psi_j)g_\alpha \\ &= F(\psi_{i_1}\top\cdots\top\psi_{i_k})g_\alpha \end{aligned}$$

Now if β is a proper subset of α then $E(\lambda^\beta_{i_1},\dots,\lambda^\beta_{i_k})$ is zero, as one argument is zero. If $\beta=\alpha$ then $E(\lambda^\beta_{i_1},\dots,\lambda^\beta_{i_k})$ is the identity as each argument is the identity. Hence we get $g_\alpha = F(\psi_{i_1}\top\cdots\top\psi_{i_k})g_\alpha$, as required. □

Corollary 1.3.12. *Let E be a covariant functor between $\mathcal{A}$ and $\mathcal{B}$. Let $h: E\to F$ be a natural transformation. Then the natural homorphism*

$$h(A_1\oplus\dots\oplus A_n): E(A_1\oplus\dots\oplus A_n)\to F(A_1\oplus\dots\oplus A_n)$$

induces a natural homomorphism

$$\mathrm{cr}_n(E)(A_1,\dots,A_n)\to\mathrm{cr}_n(F)(A_1,\dots,A_n).$$

Proof. By definition $\mathrm{cr}_n(E)(A_1,\dots,A_n)$ is a submodule of $E(A_1\oplus\dots\oplus A_n)$. Hence the injection of $\mathrm{cr}_n(E)(A_1,\dots,A_n)$ into $E(A_1\oplus\dots\oplus A_n)$ is a natural transformation, composing this with the natural homomorphism $h(A_1\oplus\dots\oplus A_n): E(A_1\oplus\dots\oplus A_n)\to F(A_1\oplus\dots\oplus A_n)$ we get a natural homomorphism between $\mathrm{cr}_n(E)(A_1,\dots,A_n)$ and $F(A_1\oplus\dots\oplus A_n)$ that satisfies the conditions of the above Theorem. So by invoking the above Theorem we get a natural homorphism $\mathrm{cr}_n(E)(A_1,\dots,A_n)\to\mathrm{cr}_n(F)(A_1,\dots,A_n)$. □

The following theorem gives us a characterisation of the cross-effect functors of F by their appearance in a direct sum decomposition as in Theorem 1.3.6.

Corollary 1.3.13. *For each subset $\alpha=\{j_1<\dots<j_{|\alpha|}\}$ of $\{1,\dots,n\}$ write$(A_j, j\in\alpha)$ for the tuple $(A_{j_1},\dots,A_{j_{|\alpha|}})$, and for each α let E_α be a covariant functor between $\mathcal{A}^{|\alpha|}$ and $\mathcal{B}$, which is zero when any of its argu-*

ments is zero. If we have a natural isomorphism:

$$h: \bigoplus_{\alpha \subset \{1,\ldots,n\}} E_\alpha(A_j, j \in \alpha) \cong F(A_1 \oplus \ldots \oplus A_n)$$

then the natural transformation h *maps each of the* $E_\alpha(A_j, j \in \alpha)$ *isomophically to* $F(\psi_{j_1}, \ldots, \psi_{j_{|\alpha|}})F(A_1, \ldots, A_n)$. *Hence we get the natural isomorphism:*

$$F(p^\alpha)h : E_\alpha(A_j, j \in \alpha) \to \mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha)$$

Proof. Theorem 1.3.6 and our assumptions give us the pair of direct sum decompositions:

$$\bigoplus_{\alpha \subset \{1,\ldots,n\}} E_\alpha(A_j, j \in \alpha) \overset{h}{\cong} F(A_1 \oplus \ldots \oplus A_n)$$
$$\overset{1.3.6}{\cong} \bigoplus_{\alpha \subset \{1,\ldots,n\}} F(\psi_{j_1}, \ldots, \psi_{j_{|\alpha|}})F(A_1, \ldots, A_n)$$

but Theorem 1.3.11 tells us that for each α

$$hE_\alpha(A_j, j \in \alpha) \subset F(\psi_{j_1}, \ldots, \psi_{j_{|\alpha|}})F(A_1, \ldots, A_n)$$

so we see that for each α we have the isomorphism

$$h : E_\alpha(A_j, j \in \alpha) \to F(\psi_{j_1}, \ldots, \psi_{j_{|\alpha|}})F(A_1, \ldots, A_n).$$

Now applying 1.3.6 (a) get the isomorphism

$$F(p^\alpha)h : E_\alpha(A_j, j \in \alpha) \to \mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha)$$

□

1.4 Expressing Dold-Puppe complexes in terms of cross-effect modules

Let $\mathcal{A}$ be an abelian category. Previously we have worked with the functor $\Gamma : \mathrm{Ch}_{\geq 0}\mathcal{A} \to \mathcal{SA}$, now we will discuss its inverse $N : \mathcal{SA} \to \mathrm{Ch}_{\geq 0}\mathcal{A}$. The normalized chain complex $N(X.)$ of a simplicial object $X.$ in an abelian category $\mathcal{A}$ is given by

$$N(X.)_n := X_n \Big/ \sum_{i=0}^{n-1} \mathrm{Im}\, s_i;$$

with its differential defined by the alternating sum of the face maps of X.

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : X_n \to X_{n-1}$$

for all $n \geq 0$. An important application of the Dold-Kan correspondence is the construction of Dold-Puppe complexes i.e. complexes of the form $NF\Gamma(C.)$ where $C.$ is a bounded below chain complex and $F : \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories (that has been extended to the category $\mathcal{SA}$ in the obvious way).

In his paper 'Computing the Homology of Koszul complexes.' [Kö] Köck used cross-effect functors to give a description of the Dold-Puppe complex of a chain complex $C. = (P \to Q)$ of length one (i.e. $C_n = 0$ when $n > 1$) in the category $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. Lemma 2.2 of [Kö] proved that

$$NF\Gamma(P \to Q)_n \cong \mathrm{cr}_n(F)(P, \ldots, P) \oplus \mathrm{cr}_{n+1}(F)(Q, P, \ldots, P)$$

and gave an explicit description of the differential. The aim of this section is to generalise this result and give a similar description of Dold-Puppe complexes in terms of cross-effect functors when the original complex is longer.

It will be useful to introduce another way of denoting elements of $\mathrm{Sur}([n], [k])$, which will be more useful when dealing with the problems in this section.

For the rest of this section we fix the value of n to be some positive integer.

Definition 1.4.1. Let $k \in \{0, \ldots, n\}$. For a surjection $f \in \mathrm{Sur}([n],[k])$ we write $f^{\triangle}$ for its image under the following bijective map:

$$\begin{aligned} \mathrm{Sur}([n],[k]) &\to \{x \cup \{n\} | x \subset \{0,1,\ldots,n-1\} \text{ and } |x| = k\} \\ f &\mapsto \{\max f^{-1}(0), \max f^{-1}(1), \ldots, \max f^{-1}(k)\} \end{aligned}$$

where max is the function that gives the maximum element of a set.

The reason why n is always in $f^{\triangle}$ is because the surjectivity of f means $\max f^{-1}(k) = n$. Let $f \in \mathrm{Sur}([n],[k])$ then f^* begins with a partition of $i+1$ if and only if $i \in f^{\triangle}$. We will be using this observation extensively to apply the results in Section 1.2, reading '$i \in f^{\triangle}$' whenever it said 'f^* begins with a partition of $i+1$'.

We consider the set of sets $\{x \cup \{n\} | x \subset \{0,1,\ldots,n-1\} \text{ and } |x| = k\}$ to be ordered in the following manner. If $x, y \in \{x \cup \{n\} | x \subset \{0,1,\ldots,n-1\} \text{ and } |x| = k\}$ and we write $x \cup \{n\} = \{x_1 < \ldots < x_k < n\}$ and $y \cup \{n\} = \{y_1 < \ldots < y_k < n\}$, then we say that $x < y$ *in the* $^{\triangle}$ *ordering* if $(x_1, \ldots, x_k) < (y_1, \ldots, y_k)$ in the lexicographic ordering described in Notation 1.1.2. It is not difficult to see that for surjections $f, g \in \mathrm{Sur}([n],[k])$ we have $f^{\triangle} < g^{\triangle}$ if and only if $f^* < g^*$.

Definition 1.4.2. Let α be a subset of the disjoint union $\amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$. We say α is an *honourable* index set if and only if $\cup_{f \in \alpha} f^{\triangle} = \{0, 1, \ldots, n\}$.

Notation 1.4.3. Let $\alpha \subset \amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$. For each $k \in \{0, \ldots, n\}$ let $\alpha_k = \alpha \cap \mathrm{Sur}([n],[k])$, and write $\alpha_k = \{\alpha_{k,1} < \ldots < \alpha_{k,|\alpha_k|}\}$. For $C_0, \ldots, C_n \in \mathcal{A}$ we write $(C_{0,\alpha_0}, \ldots, C_{n,\alpha_n})$ for the following $|\alpha|$-tuple: $(C_{0,\alpha_{0,1}}, \ldots, C_{0,\alpha_{0,|\alpha_0|}}, \ldots, C_{n,\alpha_{n,1}}, \ldots, C_{n,\alpha_{n,|\alpha_n|}})$

Proposition 1.4.4. *Let $\alpha \subset \amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$ and for each $k \in \{0,\dots,n\}$ let $\alpha_k = \alpha \cap \mathrm{Sur}([n],[k])$ then the module $\mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\dots,C_{n,\alpha_n})$ is a direct summand of $NF\Gamma(C.)_n$ if and only if α is honourable. In other words*

$$NF\Gamma(C.)_n \cong \bigoplus_{\alpha\subset\amalg_{k=0}^{n}\mathrm{Sur}([n],[k]),\,\alpha \text{ is honourable}} \mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\dots,C_{n,\alpha_n})$$

Proof. Using the definition of N and Γ we see that

$$NF\Gamma(C.)_n = F\Big(\bigoplus_{k=0}^{n}\bigoplus_{\mu\in\mathrm{Sur}([n],[k])} C_k\Big)\Big/\sum_{i=0}^{n-1}\mathrm{Im}\,F(s_i).$$

Theorem 1.2.2 (a) tells us $\mathrm{Im}\,s_i = \bigoplus_{k=0}^{n}\bigoplus_{\mu\in(S_i^n)^C} C_k$ which is a subsum of the sum $\bigoplus_{k=0}^{n}\bigoplus_{\mu\in\mathrm{Sur}([n],[k])} C_k$, so Theorem 1.3.6 tells us that $F(\mathrm{Im}\,s_i) \cong \mathrm{Im}\,F(s_i)$. So we get

$$NF\Gamma(C.)_n \cong F\Big(\bigoplus_{k=0}^{n}\bigoplus_{\mu\in\mathrm{Sur}([n],[k])} C_k\Big)\Big/\sum_{i=0}^{n-1}F(\mathrm{Im}\,s_i).$$

Expanding the numerator in terms of cross-effects we get the following formula

$$F\Big(\bigoplus_{k=0}^{n}\bigoplus_{\mu\in\mathrm{Sur}([n],[k])} C_k\Big) = \bigoplus_{\alpha\subseteq\amalg_{k=0}^{n}\mathrm{Sur}([n],[k])} \mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\dots,C_{n,\alpha_n}),$$

where $\alpha_k = \alpha\cap\mathrm{Sur}([n],[k])$. Now using Theorem 1.2.2 to give us an expression for $\mathrm{Im}(s_i)$ we expand the denominator in terms of cross-effects and we see that:

$$F(\mathrm{Im}\,s_i) = F\Big(\bigoplus_{k=0}^{n}\bigoplus_{\mu\in\mathrm{Sur}([n],[k])\setminus S_i^n} C_{k,\mu}\Big) = \bigoplus_{\alpha}\mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\dots,C_{n,\alpha_n}),$$

where the last sum ranges over all subsets $\alpha \subset \amalg_{k=0}^{n}\mathrm{Sur}([n],[k])$ where $i \notin \cup_{f\in\alpha} f^{\triangle}$. From this we see that $\mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\dots,C_{n,\alpha_n})$ is not a direct summand of $\mathrm{Im}\,F(s_i)$ if and only if $i \in \cup_{f\in\alpha} f^{\triangle}$. A module is a direct summand of $NF\Gamma(C.)_n$ if and only if it is not a direct summand of $\sum_{i=0}^{n-1} \operatorname{Im} F(s_i)$ and hence we see the desired result. $\square$

Definition 1.4.5. Let $\alpha \subset \amalg_{k=0}^{n} \operatorname{Sur}([n],[k])$ and let α be honourable. We say that $f \in \alpha$ is *superfluous in* α (or when context is obvious just *superfluous*) if $\alpha \setminus \{f\}$ is also honourable. We say that α is a *minimal honourable set* (or often just *minimal*) if it contains no superfluous surjections.

If $\alpha \subset \beta \subset \amalg_{k=0}^{n} \operatorname{Sur}([n],[k])$ then $\cup_{f\in\alpha} f^{\triangle} \subset \cup_{f\in\beta} f^{\triangle}$ and hence if α is honourable then β is also honourable. Because of this if we know all the *minimal* honourable sets that are a subset of $\amalg_{k=0}^{n} \operatorname{Sur}([n],[k])$ then it is easy to find all the honourable sets that are subsets of $\amalg_{k=0}^{n} \operatorname{Sur}([n],[k])$.

Proposition 1.4.6. *(a) Let $\alpha \subset \amalg_{k=0}^{n} \operatorname{Sur}([n],[k])$ and let α be honourable. For each $k \in \{0,\dots,n\}$ define $\alpha_k = \alpha \cap \operatorname{Sur}([n],[k])$. Then we have the inequality $\sum_{k=0}^{n} k|\alpha_k| \geq n$.*

(b) Conversely let $(a_0,\dots,a_n) \in {\mathbb{N}_0}^{n+1}$ with $a_k \leq \binom{n}{k}$ for each $k \in \{0,\dots,n\}$. Then if $\sum_{k=0}^{n} ka_k \geq n$ then there is some honourable index set $\alpha \subset \amalg_{k=0}^{n} \operatorname{Sur}([n],[k])$ with $a_k = |\alpha \cap \operatorname{Sur}([n],[k])|$.

Note in part (b) the condition requiring $a_k \leq \binom{n}{k}$ is there because $|\operatorname{Sur}([n],[k])| = \binom{n}{k}$.

Proof. Firstly we prove part (a). We know α is honourable, so by definition

$$\cup_{k=0}^{n} \cup_{f\in\alpha_k} f^{\triangle} = \{0,1,\dots,n\}.$$

Hence we also have $\cup_{k=0}^{n} \cup_{f\in\alpha_k} (f^{\triangle} \setminus \{n\}) = \{0,1,\dots,n-1\}$, and so

$$\sum_{k=0}^{n} \sum_{f\in\alpha_k} |f^{\triangle} \setminus \{n\}| \geq |\{0,1,\dots,n-1\}| = n$$

Now if $f \in \operatorname{Sur}([n],[k])$ then $f^{\triangle} \setminus \{n\}$ is a subset of $\{0,\dots,n-1\}$ of cardinality k and hence we see our result.

Now we need to prove part (b). Because $|\{0,\ldots,n-1\}| = n \leq \sum_{l=0}^{n} la_l$ we can cover the set $\{0,\ldots,n-1\}$ using: a_1 sets of cadinality 1, a_2 sets of cardinality 2, …, a_{n-1} sets of cardinality $n-1$ and a_n sets of cardinality n. Take such a covering β and define $\alpha := \{f \in \amalg_{k=0}^{n} \mathrm{Sur}([n],[k]) : f^{\triangle} = g \cup \{n\}$ for some $g \in \beta\}$, then α is an honourable set. ◻

Corollary 1.4.7. *Let $C.$ be a chain complex of length l and F a functor of degree d. The length of the Dold-Puppe complex $NF\Gamma(C.)$ is less than or equal to ld, equality is achieved if $\mathrm{cr}_d(F)(C_l,\ldots,C_l)$ is not the zero module.*

Proof. Lemma 1.4.4 tells us that

$$NF\Gamma(C.)_n \cong \bigoplus_{\alpha\subset\amalg_{k=0}^{n}\mathrm{Sur}([n],[k]),\,\alpha\text{ is honourable}} \mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\ldots,C_{n,\alpha_n}),$$

if $|\alpha| > d$ then $\mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\ldots,C_{n,\alpha_n})$ vanishes. Also the properties of cross-effects tell us if any of the modules are zero then cross-effect modules involving them will also vanish, in particular any which involve any copies of $C_{l'}$ where $l' > l$ vanish. So the only non-zero cross-effect modules in $NF\Gamma(C.)_n$ are those which correspond to subsets of $\amalg_{k=0}^{\min\{n,l\}} \mathrm{Sur}([n],[k])$ that are honourable and of cardinality d or less.

Proposition 1.4.6 (a) tells us that if $\alpha \subset \amalg_{k=0}^{n}\mathrm{Sur}([n],[k])$, α is honourable, and if for each $k \in \{0,\ldots,n\}$ we write α_k for $\alpha \cap \mathrm{Sur}([n],[k])$, then we have the inequality $\sum_{k=0}^{n} |\alpha_k|k \geq n$. On the other hand we have

$$\sum_{k=0}^{\min\{n,l\}} |\alpha_k|k \leq \sum_{k=0}^{\min\{k,l\}} |\alpha_k| \min\{n,l\} = \min\{n,l\} \sum_{k=0}^{\min\{k,l\}} |\alpha_k| = \min\{n,l\}|\alpha|.$$

So if $|\alpha| \leq d$ then $\sum_{k=0}^{\min\{n,l\}} |\alpha_k|k \leq d\min\{n,l\}$. So if $n > ld$ then there can't be any honourable sets in $\amalg_{k=0}^{\min\{n,l\}} \mathrm{Sur}([n],[k])$. So for $n > ld$ we get $NF\Gamma(C.)_n = 0$

To prove equality is achieved if $\mathrm{cr}_d(F)(C_l,\ldots,C_l)$ is not the zero module, we set $n = dl$, $a_l = d$ and $a_k = 0$ if $k \neq l$. Proposition 1.4.6 (b) tells us that there is some honourable set $\alpha \subset \amalg_{k=0}^{n}\mathrm{Sur}([n],[k])$ with $a_k = |\alpha \cup$

$\mathrm{Sur}([n],[k])|$ for each $k \in \{0,\ldots,n\}$, this condition on $|\alpha \cup \mathrm{Sur}([n],[k])|$ tells us that $\alpha \subset \mathrm{Sur}([n],[l])$. So $\mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\ldots,C_{n,\alpha_n}) = \mathrm{cr}_{|d|}(F)(C_{l,\alpha_l})$, this is non-zero by assumption and non-degerate in $NF\Gamma(C.)_n$ because of our choice of α. $\square$

Honourable index sets are fairly abstract objects, we will now introduce a way of representing them pictorially in the hope of making the combinatorial conditions more digestible. We associate $\alpha \subset \amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$ to a table with n columns. The headings of the columns are $0,1,\ldots,n-1$. Each row will represent one of the surjections in α; if f is in α the row in the table representing f will contain a mark in the column headed by i whenever $i \in f^{\triangle}$.

Let $\alpha^{\triangle} = \{\{1,3\},\{2,3\},\{0,1,2,3\}\}$. We could choose to represent α by

0	1	2
	$\backslash$	
		$\backslash$
$\backslash$	$\backslash$	$\backslash$

where the first row represents the set $\{1,3\}$, the second row represents the set $\{2,3\}$ and the final row represents the set $\{0,1,2,3\}$. But we could also represent α by any of the following tables:

0	1	2
	$\backslash$	
		$\backslash$
$\backslash$	$\backslash$	$\backslash$

0	1	2
		$\backslash$
	$\backslash$	
$\backslash$	$\backslash$	$\backslash$

0	1	2
$\backslash$	$\backslash$	$\backslash$
		$\backslash$
	$\backslash$	

0	1	2
	$\backslash$	
$\backslash$	$\backslash$	$\backslash$
		$\backslash$

However it would be nice if there were a unique table to represent any given index set. To give this unique representation we require the rows to have a particular order.

Definition 1.4.8. (a) For each $i \in \{1,\ldots,m\}$ let T_i be a subset of

$\{0, 1, \ldots, n-1\}$ with $T_i \neq T_{i'}$ whenever $i \neq i'$, then we call the ordered set $T = (T_1, \ldots, T_m)$ *a (formal) table with* n columns. For $i \in \{1, \ldots, m\}$ we say that T_i is the i^{th} row of T. Associated with this formal table we draw a table with n columns (with headings $0, 1, \ldots, n-1$) which has a mark in the $(i, j)^{th}$ cell of the diagram if $j - 1 \in T_i$ (it is $j-1$ because the first column is headed by 0).

(b) If $\alpha \subset \amalg_{k=0}^{n} \mathrm{Sur}([n], [k])$ and $\{T_1 \cup \{n\}, \ldots, T_m \cup \{n\}\} = \alpha^{\triangle}$ then we say that *the table* $(T_1, \ldots, T_m)$ *represents* α.

(c) Let $T = (T_1, \ldots, T_m)$ be a formal table and let $i, j \in \{1, \ldots, m\}$. If whenever $i < j$ we have either $|T_i| > |T_j|$ or $T_i \cup \{n\} < T_j \cup \{n\}$ (in the $^{\triangle}$ ordering) then we say that the table T is in *normal form.*

It is easy to see that every table represents some index set $\alpha \subset \amalg_{k=0}^{n} \mathrm{Sur}([n], [k])$, moreover for any index set we can easily form a formal table that represents it as by putting *some* ordering on the set $\{f^{\triangle} \setminus \{n\} : f \in \alpha\}$. We can then reorder the rows of the formal table so that they satisfy the conditions of being in normal form. Hence we see that for each index set there is a unique table in normal form that represents it.

Given an arbitrary honourable index set it is not always obvious whether it is a *minimal* honourable index set. We are now interested in determining which tables represent minimal honourable index sets and which don't.

Lemma 1.4.9. *Honourable index sets correspond to tables with an entry in each column.*

Proof. If $(T_1, \ldots, T_m)$ has an entry in each column, then this is the same as saying for each $i \in \{0, \ldots, n-1\}$ there is some row in T containing i. This means $\cup_{i=0}^{m} T_i = \{0, \ldots, n-1\}$, and hence $\cup_{i=0}^{m} (T_i \cup \{n\}) = \{0, \ldots, n\}$. $\square$

Let $T = (T_1, \ldots, T_m)$ be a formal table and T_j be one of its rows. If $\cup_{i=0, i\neq j}^{m} T_i = \cup_{i=0}^{m} T_i$ then the surjection that T_j represents is superfluous in the index set that T represents; because of this when we have this situation we say that T_j is superfluous in T.

We now describe an algorithm that checks whether a table has any superfluous rows.

This algorithm works by using two different kinds of mark when we draw the table of $(T_1, \ldots, T_m)$. As before if $i \in T_r$ then we put a mark in the $(r, i-1)^{th}$ cell of the diagram, but here that cell will either contain a $\diagdown$ mark or a $\times$ mark. The rows of the diagram will be constructed one by one. When putting a new mark in the diagram the $\diagdown$ mark will be used in a cell when the column it is in has no other marks in it (yet), otherwise a $\times$ mark will be used. If a $\times$ mark is put in the diagram then any $\diagdown$ marks in that same column will be changed to $\times$ marks.

If the only marks in a row are $\times$ marks (i.e. there are no $\diagdown$ marks) then that means that each of the columns it has entries in also have entries in other rows, i.e. the row with only $\times$ marks in it is superfluous.

The program **Super** (below) should be given a formal table $(T_1, \ldots, T_m)$ and a value for n. If the table contains no superfluous rows it will return 0, otherwise it will return some value $i \in \{1, \ldots, m\}$ such that T_i is superfluous.

Example 1.4.10. The following examples shows how the algorithm **Super** works with different inputs. The diagrams show the different states of the table at different stages in the running time of the program.

(a) Input: $n = 4, T = (\{0,1\}, \{0,3\}, \{0\})$.

$$\begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \diagdown & \diagdown & & \\ & & & \\ & & & \end{array} \quad \to \quad \begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \times & \diagdown & & \\ \times & & & \diagdown \\ & & & \end{array} \quad \to \quad \begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \times & \diagdown & & \\ \times & & & \diagdown \\ \times & & & \end{array}$$

The algorithm terminates after the last table and returns the value 3, because the third row consists wholly of $\times$ marks.

Algorithm 1 Super: Check $(T_1, \dots, T_m)$ contains no superfluous rows

```
for r = 1 to m do
  for all i ∈ T_r do
    s ← 0
    repeat
      s ← s + 1
    until i ∈ T_s or s = r
    if s = r then
      Put a \ mark in the i^th column of the r^th row
    else if i ∈ T_s then
      Put an × mark in the i^th column of the s^th row and also in the
      i^th column of the r^th row
      if row s of the table consists wholly of ×s then
        return s
        end algorithm
      end if
    end if
  end for
  if row r of the table consists wholly of ×s then
    return r
    end algorithm
  end if
end for
return 0
```

(b) Input: $n = 4, T = (\{0,1\},\{1,2\},\{2,3\})$.

$$
\begin{array}{c|c|c|c}
0 & 1 & 2 & 3 \\
\backslash & \backslash & & \\
 & & & \\
 & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{c|c|c|c}
0 & 1 & 2 & 3 \\
\backslash & \times & & \\
 & \times & \backslash & \\
 & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{c|c|c|c}
0 & 1 & 2 & 3 \\
\backslash & \times & & \\
 & \times & \times & \\
 & & \times &
\end{array}
$$

The program stops there (midway through constructing the third row) and returns 2.

(c) Input: $n = 4, T = (\{0,1,2\},\{1,3\})$.

$$
\begin{array}{c|c|c|c}
0 & 1 & 2 & 3 \\
\backslash & \backslash & \backslash & \\
 & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{c|c|c|c|}
0 & 1 & 2 & 3 \\
\backslash & \times & \backslash & \\
 & \times & & \backslash
\end{array}
$$

The program returns 0.

We now describe an algorithm designed to work out all of the tables in normal form that represent a minimal honourable index set.

This algorithm will use two procedures **Increment** and **Complete** that we consider to be simple enough to only describe what they do without detailing their workings. **Increment** will take a set $I \subset \{0, \ldots, n-1\}$ as input and returns a set $O \subset \{0, \ldots, n-1\}$ as output. If $I \cup \{n\}$ is not the largest element of $\{x \cup \{n\} : x \subset \{0, \ldots, n-1\} \text{ and } |x| = |I|\}$ under the $\triangle$ ordering then $O \cup \{n\}$ will be the element directly after $I \cup \{n\}$ in the $\triangle$ ordering, otherwise $O \cup \{n\}$ will be $\{0, \ldots, |I|-2, n\}$ (i.e. the smallest element of $\{x \cup \{n\} : x \subset \{0, \ldots, n-1\} \text{ and } |x| = |I|-1\}$ under the $\triangle$ ordering). **Complete** will take a table as its input, it will return the value **TRUE** if the table has a mark in each column, otherwise it will return the value **FALSE**.

This program will take two positive integers as input l and n where $l \leq n$. It will produce all tables with n columns whose rows each have at most l marks in them, that correspond to minimal honourable index sets.

The main object in this algorithm is a formal table $(T_1, \ldots, T_r)$ that is in normal form. As the algorithm progresses all tables of interest will be constructed by gradually making changes to this table. The changes will either be adjoining new sets to the table or by modifying the existing rows. It will return as output any table which represents a minimal honourable set.

At first this table will have just one row in it, and as we work through the algorithm new rows will be adjoined to this table one by one in such a way that this table is always a table in normal form and never contains superfluous rows. We ensure that it is always in normal form by making sure that the row added to the table, say T_{r+1}, will either satisfy $T_r \cup \{n\} < T_{r+1} \cup \{n\}$ in the $^{\triangle}$ ordering or $|T_r| > |T_{r+1}|$. We use the algorithm **Super** to ensure that whenever we add a row to the table it does not cause the new table to have superfluous rows. If adding a particular row to the table would cause the table to contain superfluous rows then we try adjoining a different row, again we test using the algorithm **Super** and if that row doesn't work we keep trying new rows and testing with **Super** until we find an appropriate row (if it turns out that the only appropriate row is the empty set then we replace T_r with a different row, see below).

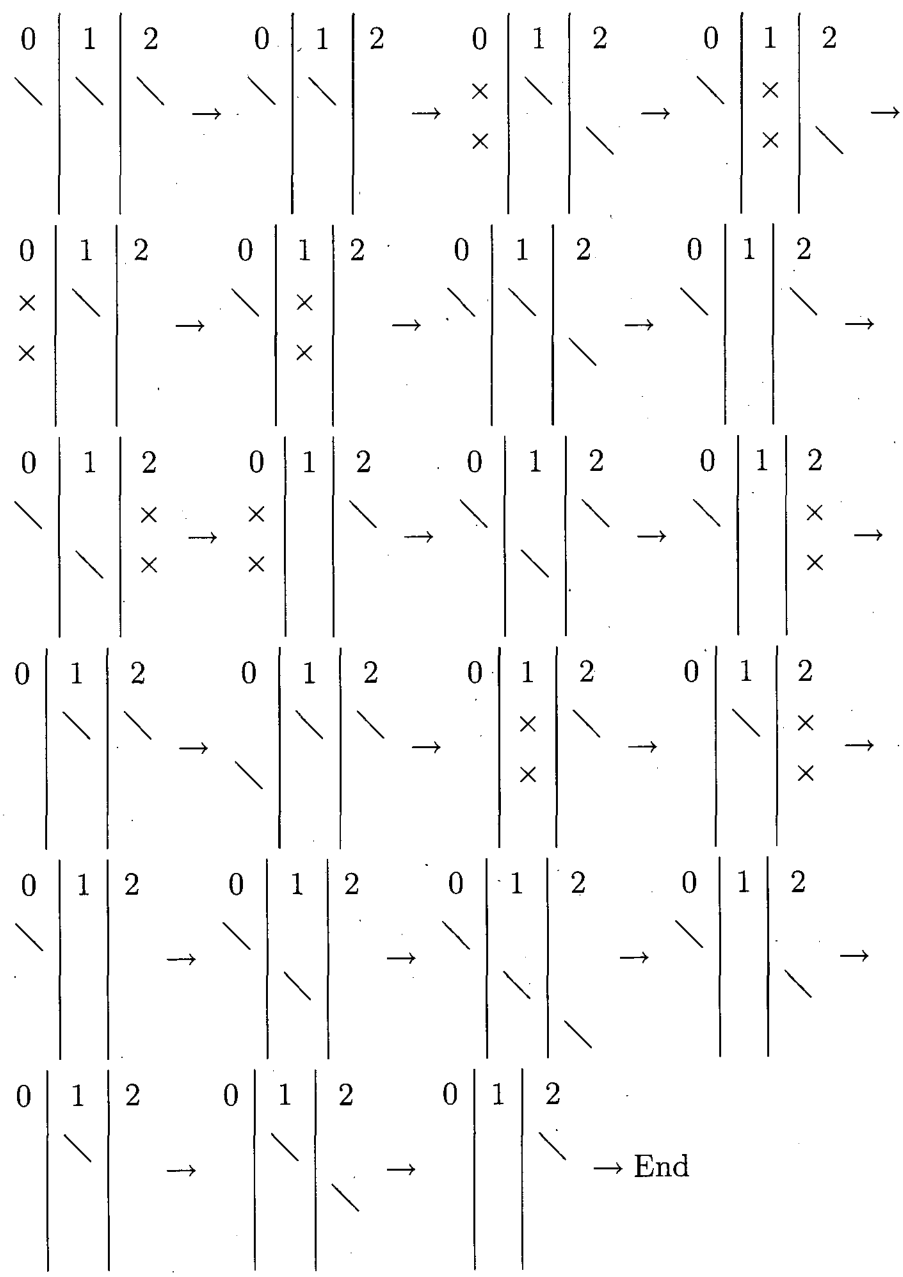
We use the algorithm **Complete** to determine whether the table represents an honourable set (the set will also be minimal, because as mentioned above we ensure the table has no superfluous rows) and if it does then the table is returned as part of the output of the program.

When $(T_1, \ldots, T_r)$ represents a minimal honourable set (or when in our attempts to adjoin a new row to the table we find that the only suitable row is the empty set) we stop adjoining new rows and instead replace T_r with a different row. We change the final row T_r to, say T_r', so that either $T_r \cup \{n\} < T_r' \cup \{n\}$ or $|T_r| > |T_r'|$ and also ensuring that the new table has no superfluous rows. If the only row that we could choose T_r' to be is the empty set then we remove T_r from the table and try to replace T_{r-1} with an appropriate row, if the only row we could choose T_{r-1} is the empty set then we remove T_{r-1} from the table and try to replace T_{r-2} with an appropriate row, etc, etc. If we are ever in the position where we will have to change T_1 to the empty set then the algorithm ends.

Algorithm 2 Find min Find all tables with n-columns which correspond to a minimal honourable index set

```
r ← 1
T_1 ← {0, ..., l − 1}
repeat
  if Complete(T_1, ..., T_r) = TRUE then
    return (T_1, ..., T_r)
    repeat
      if Increment(T_r) = ∅ then
        r ← r − 1
      end if
      T_r ← Increment(T_r)
    until Super(T_1, ..., T_r) = 0
  else if Complete(T_1, ..., T_r) = FALSE then
    if Increment(T_r) ≠ ∅ then
      r ← r + 1
      T_r ← Increment(T_{r−1})
    else if Increment(T_r) = ∅ then
      r ← r − 1
      T_r ← Increment(T − r)
    end if
    while Super(T_1, ..., T_r) > 0 do
      if Increment(T_r) = ∅ then
        r ← r − 1
      end if
      T_r ← Increment(T_r)
    end while
  end if
until r = 1 and Increment(T_1) = ∅
```

Example 1.4.11. The following example shows how **Find min** works when given the input $n = 3$ and $l = 3$. The diagrams show the different states of the table at different stages in the running time of the program. Note that if **Find min** was instead given the input $n = 3$ and $l = 2$ or the input $n = 3$ and $l = 1$, the first table it would produce would be the second or seventeenth (respectively) of the following tables, then it would continue in the exactly the same way.



The $1^{st}, 3^{rd}, 4^{th}, 7^{th}, 9^{th}, 11^{th}, 14^{th}$ and 19^{th} tables are all outputted.

If **Find min** is given input $n = n$ and $l = 1$ then it will output only one table:

$$\begin{array}{c|c|c|c}
0 & 1 & \cdots & n-1 \\
\diagdown & & & \\
 & \diagdown & & \\
 & & \ddots & \\
 & & & \diagdown
\end{array}$$

This means the only minimal honourable subset of $\mathrm{Sur}([n],[1]) \cup \mathrm{Sur}([n],[0])$ is $\mathrm{Sur}([n],[1])$. So the honourable subsets of $\mathrm{Sur}([n],[1]) \cup \mathrm{Sur}([n],[0])$ are $\mathrm{Sur}([n],[1])$ and $\mathrm{Sur}([n],[1]) \cup \mathrm{Sur}([n],[0])$. So if $B \to A$ is a chain complex, and recalling that $\Gamma(B \to A)_n = B^n \oplus A$, this means that $NF\Gamma(B \to A)_n = \mathrm{cr}_n(F)(B,\ldots,B) \oplus \mathrm{cr}_{n+1}(F)(B,\ldots,B,A)$. This recovers the description of each part of the chain complex $NF\Gamma(B \to A)$ given in Lemma 2.2 of [Kö].

1.5 The Dold-Puppe complex $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)$

The results from the previous sections can be used to calculate the Dold-Puppe complex of a chain complex of arbitrary length for any functor which we know the various cross-effects of. To illustrate this we now give an example calculation.

Let R be a commutative ring and let $\mathcal{A}$ be R-**mod** the category of modules over this ring. Let $\mathrm{Sym}^2 : \mathcal{A} \to \mathcal{A}$ be the second symmetric power functor. Let $C \to B \to A$ be a chain complex sitting in degrees 0, 1 and 2 with differential ∂. We will denote the differential of $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)$ by Δ.

The functor Sym^2 is of degree two (and $\mathrm{cr}_2(\mathrm{Sym}^2)(C,C) \neq 0$) so Theorem 1.4.7 says that $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)$ will be of length $2 \times 2 = 4$.

To work out what each place of $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)$ looks like we need to find all the minimal honourable sets in $\amalg_{k=0}^{n} \mathrm{Sur}([n][k])$ for each value of $n \in \{1,\ldots,4\}$. To do this we use **Find min**.

With input $n = 1, l = 1$ (we set l to be 1 here since **Find min** does not accept values of l larger than n) **Find min** gives us the following output:

$$\begin{array}{c} 0 \\ \diagdown \end{array}$$

With input $n = 2, l = 2$ **Find min** gives us the following output:

$$\begin{array}{c|cc|c} 0 & 1 & 0 & 1 \\ \diagdown & \diagdown & \diagdown & \\ & & & \diagdown \end{array}$$

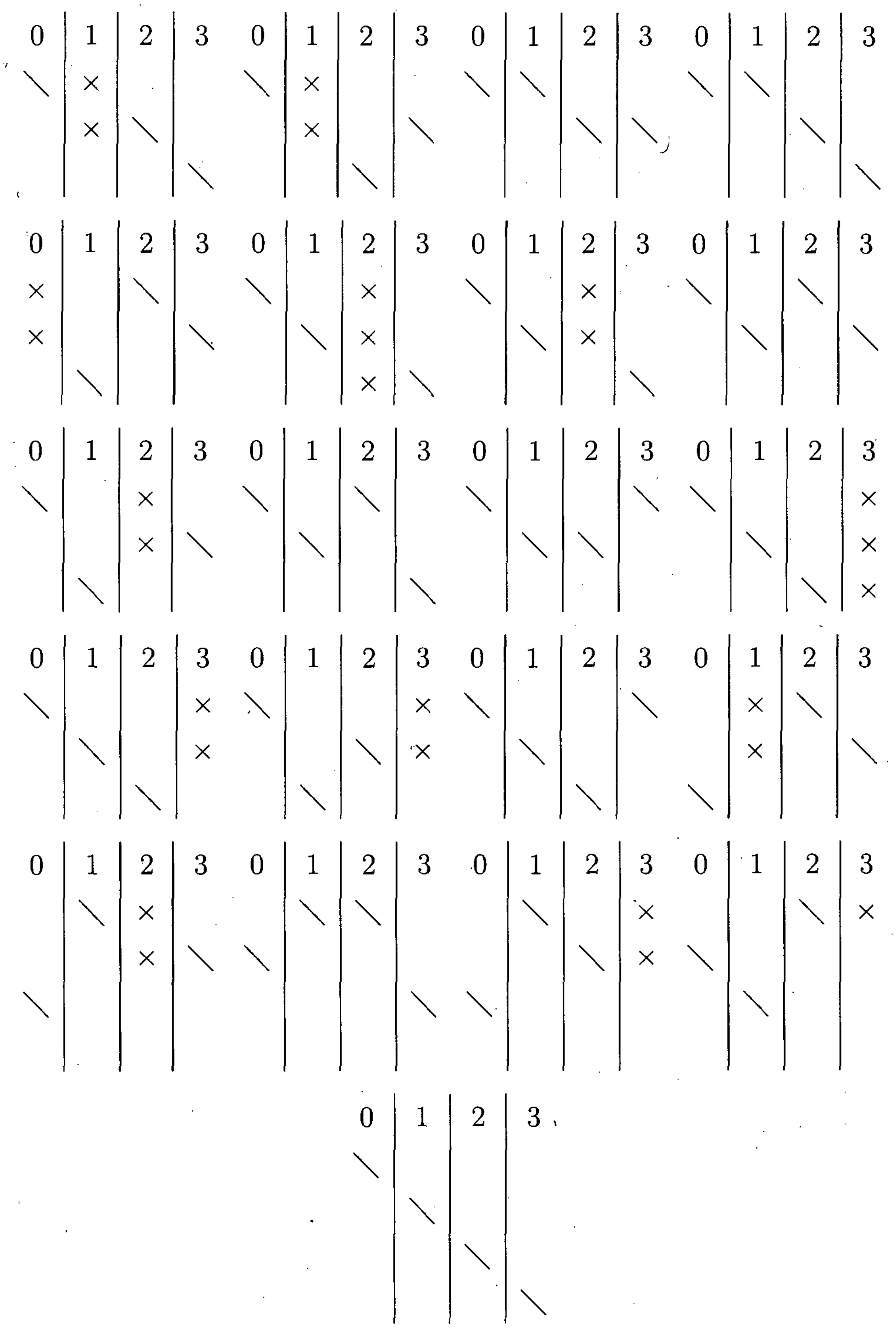
With input $n = 3, l = 2$ **Find min** gives us the following output (as seen before in Example 1.4.11):

$$\begin{array}{c|c|cc|c|cc|c|cc|c|c} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ \times & \diagdown & & \diagdown & \times & & \diagdown & \diagdown & & \diagdown & & \times \\ \times & & \diagdown & & \times & \diagdown & & & \diagdown & & \diagdown & \times \\ & & & & & & & & & & & \end{array}$$

$$\begin{array}{c|c|cc|c|cc|c|c} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ \diagdown & & \diagdown & & \diagdown & \diagdown & \diagdown & & \\ & \diagdown & & \diagdown & & & & \diagdown & \\ & & & & & & & & \diagdown \end{array}$$

With input $n = 4, l = 2$ **Find min** gives us the following output:

$$\begin{array}{c|c|c|cc|c|c|cc|c|c|cc|c|c|c} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ \times & \diagdown & & & \times & \diagdown & & & \times & \diagdown & & & \diagdown & \times & & \\ \times & & \diagdown & & \times & & \diagdown & & \times & & & \diagdown & & \times & \diagdown & \\ \times & & & \diagdown & & & & \diagdown & & & \diagdown & & & \times & & \diagdown \end{array}$$



The functor Sym^2 is of degree 2, i.e. anything past the second cross-effect of it vanishes, so we are only interested in cross-effect modules with 2 or less modules as arguments. Hence we are only interested in tables with 2 or fewer rows. So now we discard all tables with with more than 2 rows. For $n = 1$ or 2 we keep all the tables. For $n = 3$ we discard only the last table. For $n = 4$ we are left with the following 3 tables:

0	1	2	3
$\diagdown$	$\diagdown$		
		$\diagdown$	$\diagdown$

0	1	2	3
$\diagdown$		$\diagdown$	
	$\diagdown$		$\diagdown$

0	1	2	3
$\diagdown$			$\diagdown$
	$\diagdown$	$\diagdown$	

From these minimal honorable index sets we now find *all* of the honourable index sets. We do this by finding all subsets of $\coprod_{k=0}^{n} \mathrm{Sur}([n],[k])$ that have some minimal honourable set as a subset. As explained above we are only interested in honourable sets with 2 or fewer elements so for $n = 3$ or 4 we already have all the honourable sets that are relevant.

From the tables for $n = 1$ we see that the only minimal index set is the set represented by the table $(\{0\})$. The only other subset of $\coprod_{k=0}^{1} \mathrm{Sur}([n],[k])$ that contains this is as a subset is represented by the table $(\{0\}, \emptyset)$.

From the tables for $n = 2$ we see that the minimal index sets are represented by the tables $(\{0,1\})$ and $(\{0\},\{1\})$. Obviously there are no sets of cardinality 2 or less that contain the latter as a proper subset. The subsets of $\coprod_{k=0}^{2} \mathrm{Sur}([n],[k])$ that contains the former as a subset are represented by the tables $(\{0,1\},\emptyset), (\{0,1\},\{0\}), (\{0,1\},\{1\})$.

Applying Lemma 1.4.4, and again indexing in terms of ordinals rather than sets, we see that:

$$N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_0 = \mathrm{Sym}^2 A$$

$$N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_1 = \mathrm{Sym}^2 B_1 \oplus B_1 \otimes A$$

$$
\begin{aligned}
N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_2 = &\,\mathrm{Sym}^2 C_1 \oplus C_1 \otimes B_1 \oplus C_1 \otimes B_2 \\
&\oplus C_1 \otimes A \oplus B_1 \otimes B_2
\end{aligned}
$$

$$
\begin{aligned}
N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_3 = &\,C_1 \otimes C_2 \oplus C_1 \otimes C_3 \oplus C_1 \otimes B_3 \\
&\oplus C_2 \otimes C_3 \oplus C_2 \otimes B_2 \oplus C_3 \otimes B_1
\end{aligned}
$$

$$
N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_4 = C_1 \otimes C_6 \oplus C_2 \otimes C_5 \oplus C_3 \otimes C_4
$$

Now we show how the differential $\Delta = \sum_{i=0}^{n}(-1)^i d_i$ of $NF\Gamma(C \to B \to A)$ acts on each of the direct summands of each part $NF\Gamma(C \to B \to A)$. We use the face maps we calculated for $\Gamma(C \to B \to A)$ in Example 1.2.3 then use the theory of cross-effects described in Section 1.3 to see how they act on each cross-effect module in each degree of $NF\Gamma(C \to B \to A)$.

In the following $+_2$ acts on the tensor square of a module M by $+_2 : M \otimes M \to \mathrm{Sym}^2 M, m_1 \otimes m_2 \mapsto m_1 m_2$.

The action of Δ on each summand of $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_4$ is as follows.

The face map d_1 acts on the C_1 as the zero map, so it acts on $C_1 \otimes C_6$ as the zero map. Similarly since d_3 and d_4 acts on C_6 as the zero map they both act on $C_1 \otimes C_6$ as the zero map. In future any parts of ∂ that act as the zero map will be suppressed.

$$
\begin{tikzcd}
C_1 \otimes C_6 \arrow[r, "d_0=\partial\otimes\mathrm{id}_C"] \arrow[ddr, "d_2=\mathrm{id}_C\otimes\mathrm{id}_C"] & B_1 \otimes C_3 \\
& \oplus \\
& C_1 \otimes C_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_2 \otimes C_5 \arrow[r, "d_0=\partial\otimes\mathrm{id}_C"] \arrow[rdd, "-d_1=-\,\mathrm{id}_C\otimes\mathrm{id}_C"] \arrow[rdddd, "d_2=\mathrm{id}_C\otimes\mathrm{id}_C"] \arrow[rdddddd, "-d_3=-\,\mathrm{id}_C\otimes\mathrm{id}_C"'] & B_2 \otimes C_2 \\
& \oplus \\
& C_1 \otimes C_2 \\
& \oplus \\
& C_1 \otimes C_3 \\
& \oplus \\
& C_2 \otimes C_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_3 \otimes C_4 \arrow[r, "d_0=\partial\otimes\mathrm{id}_C"] \arrow[rdd, "-d_1=-\,\mathrm{id}_C\otimes\mathrm{id}_C"] \arrow[rdddd, "-d_3=-\,\mathrm{id}_C\otimes\mathrm{id}_C"] & B_3 \otimes C_1 \\
& \oplus \\
& C_2 \otimes C_1 \\
& \oplus \\
& C_2 \otimes C_3
\end{tikzcd}
$$

The action of Δ on each summand of $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_3$ is as follows.

$$
\begin{tikzcd}
C_1 \otimes C_2 \arrow[r, "d_0=\partial\otimes\partial"] \arrow[rdd, "d_2=+2"] & B_1 \otimes B_2 \\
& \oplus \\
& \mathrm{Sym}^2 C_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_1 \otimes C_3 \arrow[r, "d_0=\partial\otimes\mathrm{id}_C"] & B_1 \otimes C_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_1 \otimes B_3 \arrow[r, "d_0=\partial\otimes\mathrm{id}_B"] \arrow[rdd, "d_2=\mathrm{id}_C\otimes\mathrm{id}_B"] & B_1 \otimes B_2 \\
& \oplus \\
& C_1 \otimes B_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_2 \otimes C_3 \arrow[r, "d_0=\partial\otimes\mathrm{id}_C"] \arrow[rdd, "-d_1=-+2"] & B_2 \otimes C_1 \\
& \oplus \\
& \mathrm{Sym}^2 C_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_2 \otimes B_2 \arrow[r, "d_0=\partial\otimes\mathrm{id}_B"] \arrow[dr, "d_1=-\,\mathrm{id}_C\otimes\mathrm{id}_B"] \arrow[dddr, "d_2=\mathrm{id}_C\otimes\mathrm{id}_B"'] & B_2 \otimes B_1 \\
 & \oplus \\
 & C_1 \otimes B_1 \\
 & \oplus \\
 & C_1 \otimes B_2 .
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_3 \otimes B_1 \arrow[r, "d_0=\partial\otimes\partial"] \arrow[ddr, "d_1=-\,\mathrm{id}_C\otimes\mathrm{id}_B"] & C_1 \otimes A \\
 & \oplus \\
 & C_1 \otimes B_1
\end{tikzcd}
$$

The action of Δ on each summand of $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_2$ is as follows.

$$
\mathrm{Sym}^2 C \xrightarrow{d_0=\mathrm{Sym}^2(\partial)} \mathrm{Sym}^2 B_1
$$

$$
C_1 \otimes B_1 \xrightarrow{d_0=\partial\otimes\partial} B_1 \otimes A
$$

$$
C_1 \otimes B_2 \xrightarrow{d_0=+2(\partial\otimes\mathrm{id}_B)} \mathrm{Sym}^2 B_1
$$

$$
C_1 \otimes A \xrightarrow{d_0=\partial\otimes\mathrm{id}_A} B_1 \otimes A
$$

$$
\begin{tikzcd}
B_1 \otimes B_2 \arrow[r, "d_0=\partial\otimes\mathrm{id}_B"] \arrow[ddr, "-d_1=-+_2"] & A \otimes B_1 \\
 & \oplus \\
 & \mathrm{Sym}^2 B
\end{tikzcd}
$$

The action of Δ on each summand of $N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)_1$ is as follows.

$$
\begin{tikzcd}
\mathrm{Sym}^2 B_1 \arrow[r, "d_0=\mathrm{Sym}^2(\partial)"] & \mathrm{Sym}^2 A
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_1 \otimes A \arrow[r, "d_0=+_2(\partial\otimes\mathrm{id}_A)"] & \mathrm{Sym}^2 A
\end{tikzcd}
$$

Note were we now to calculate the Dold-Puppe complex $NF\Gamma(C \to B \to A)$ for a different functor of degree 2 practically all of the above calculation could be reused unchanged. In the description of each part of the complex $NF\Gamma(C \to B \to A)$ the cross-effect modules would be different, because the cross-effect functors would be different, but the arguments they take would be the same. Similarly the action of the face operators on the cross-effect modules would be different, because the cross-effect modules would be different, but the actual face operators would be the same.

Chapter 2

Calculation of the derived functors of the third symmetric power functor

2.1 Spectral sequences

In this section we recall a couple of spectral sequences taken from section 5 of [W]. We will not be dealing with spectral sequences in any depth, only applying some results to aid calculations in later sections.

Definition 2.1.1. Let a be an integer. A *homology spectral sequence (starting with E^a)* in an abelian category $\mathcal{A}$ consists of the following data:

1. a family $\{E^r_{p,q}\}$ of objects of $\mathcal{A}$ defined for all integers p, q and $r \geq a$;
2. maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d_r \circ d_r = 0$, so that the 'lines of slope $-(r+1)/r$' in the lattice $E^r_{*,*}$ form chain complexes (we say the differentials go 'to the left');
3. isomorphisms between $E^{r+1}_{p,q}$ and the homology of $E^r_{*,*}$ at the spot $E^r_{p,q}$:

$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q}) / \operatorname{Im}(d^r_{p+r,q-r+1}).$$

Note that $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$. The *total degree* of the term $E^r_{p,q}$ is $n = p+q$; the terms of total degree n lie on a line of slope -1, and each differential $d^r_{p,q}$ decreases the total degree by one.

There is a category of homology spectral sequences; a morphism $f : E' \longrightarrow E$ is a family of maps $f^r_{p,q} : E'^r_{p,q} \longrightarrow E^r_{p,q}$ is $\mathcal{A}$ (for suitably large r) with $d^r f^r = f^r d^r$ such that each $f^{r+1}_{p,q}$ is the map induced by $f^r_{p,q}$ on homology.

Definition 2.1.2. A homology spectral sequence $\{E^r_{p,q}\}_{r\geq a}$ is said to be *bounded* if for each n there are only finitely many nonzero terms of total degree n in $E^a_{*,*}$. If so, then for each p and q there is an r_0 such that $E^r_{p,q} = E^{r+1}_{p,q}$ for all $r \geq r_0$. We write $E^\infty_{p,q}$ for the stable value of $E^r_{p,q}$.

We say that a bounded spectral sequence *converges to* H_* if we are given a family of objects H_n of $\mathcal{A}$, each having a *finite* filtration

$$0 = F_s H_n \subseteq \ldots \subseteq F_{p-1}H_n \subseteq F_p H_n \subseteq F_{p+1}H_n \subseteq \ldots \subseteq F_t H_n = H_n$$

and we are given isomorphisms $E^\infty_{p,q} \cong F_p H_{p+q}/F_{p-1}H_{p+q}$. The traditional symbolic way of describing such a bounded convergence is like this:

$$E^a_{p,q} \Rightarrow H_{p+q}.$$

Lemma 2.1.3. *Let $\{E^r_{p,q}\}_{r\geq a}$ and $\{E'^r_{p,q}\}_{r\geq a'}$ be spectral sequences that converge to H_* and H'_*. Let $g : H_* \longrightarrow H'_*$ be a morphism compatible with the filtrations. Let $f : \{E^r_{p,q}\}_{r\geq a} \longrightarrow \{E'^r_{p,q}\}_{r\geq a'}$ be a morphism of spectral sequences such that for some fixed r, $f^r_{p,q} : E^r_{p,q} \cong E'^r_{p,q}$ is an isomorphism for all p and q. And finally let f and g be compatible with the isomorphisms between the successive quotients of the filtrations and the E^∞ terms. Then for all $s \geq r$ we have $f^s_{p,q} : E^s_{p,q} \cong E'^s_{p,q}$ and all p and q and $H_i \cong H'_i$ for every i.*

Proof. This is a consequence of the 5-Lemma. □

Let R be a commutative ring.

Definition 2.1.4. Let A_* be a bounded below chain complex consisting only of projective R-modules. For an R-module B we define the *Hypertor* functor $\mathbf{Tor}(A_*, B)$ by

$$\mathbf{Tor}_i^R(A_*, B) = H_i(A_* \otimes B).$$

If instead B_* is a bounded below chain complex consisting only of projective modules we define the *Hypertor* functor $\mathbf{Tor}(A_*, B_*)$ by

$$\mathbf{Tor}_i^R(A_*, B_*) = H_i \operatorname{Tot}(A_* \otimes B_*).$$

Note: We use these more restricted and simpler definitions rather than the more general definition given on pp148-149 of [W] to avoid the necessity of introducing Cartan-Eilenberg resolutions. Our definition of **Tor** matches that of [W] (for the modules and chain complexes we have defined it for) because of the last spectral sequence given in Exercise 5.7.5 on page 149 of [W].

Proposition 2.1.5. *Let A_* be a bounded below chain complex consisting only of projective R-modules. For an R-module B we have the following spectral sequence:*

$${}^{II}E^2_{p,q} = \operatorname{Tor}_p(H_q(A), B) \Rightarrow \mathbf{Tor}(A_*, B).$$

If B_* is instead an R-chain complex, we have the following spectral sequence

$${}^{II}E^2_{p,q} = \bigoplus_{q=q'+q''} \operatorname{Tor}_p(H_{q'}(A_*), H_{q''}(B_*)) \Rightarrow \mathbf{Tor}(A_*, B_*).$$

Proof. See Exercise 5.7.5 of [W]. □

Corollary 2.1.6. *Let A_*, A'_*, B_* and B'_* be bounded chain complexes, consisting only of projective R-modules, with A_* quasi-isomorphic to A'_*, and B_* quasi-isomorphic to B'_*, then $\operatorname{Tot}(A_* \otimes B_*)$ is quasi-isomorphic to $\operatorname{Tot}(A'_* \otimes B'_*)$.*

Proof. Proposition 2.1.5 gives us spectral sequences for $\mathrm{Tot}(A_* \otimes B_*)$ and $\mathrm{Tot}(A'_* \otimes B'_*)$ in terms of the homologies of A_*, A'_*, B_* and B'_*. The quasi-isomorphisms between A_* and A'_*, and between B_* and B'_* give us a morphism between the double complexes that give rise to the hypertor spectral sequences for $\mathbf{Tor}(A_*, B_*)$ and $\mathbf{Tor}(A'_*, B'_*)$ and also between their respective homologies that is compatible with their filtrations. We know that the second sheet of these spectral sequences is given by ${}^{II}E^2_{p,q} = \bigoplus_{q=q'+q''} \mathrm{Tor}_p(H_{q'}(A_*), H_{q''}(B_*))$, so the morphism between the spectral sequences is an isomorphism between the E^2-sheets therefore, by Lemma 2.1.3, it is also an isomorphism between all the higher sheets and also the homology. Hence $\mathrm{Tot}(A_* \otimes B_*)$ is quasi-isomorphic to $\mathrm{Tot}(A'_* \otimes B'_*)$. □

2.2 Koszul complexes

In this section we introduce Koszul complexes, which are very useful for constructing projective resolutions, and in the construction of Schur functors of hook type in section 2.3. We also recall some results regarding the homology of complexes related to Koszul complexes which will be useful in the last section of this chapter.

Let R be a commutative ring, $x_1, x_2, \ldots, x_n$ be elements of R, and I be the ideal generated by $x_1, x_2, \ldots, x_n$.

Definition 2.2.1. We say that $x_1, x_2, \ldots, x_n$ are a *regular sequence in* R, if $R \neq (x_1, \ldots, x_n)R$ and, for each $i \in \{1, \ldots n\}$, the element x_i is not a zero divisor in $R/(x_1, \ldots, x_{i-1})R$.

Proposition 2.2.2. *If R is a Noetherian local ring, and $x_1, \ldots, x_n$ are in the maximal ideal of R then the following are equivalent:*

(a) $x_1, \ldots, x_n$ form a regular sequence in R;

(b) the homomorphism of R/I-algebras $\alpha : R/I[X_1, \ldots, X_n] \to gr_I(R)$ given by $\alpha(X_i) = x_i + I^2$ is an isomorphism, where $gr_I(R)$ denotes the graded ring of R with respect to the ideal I.

Proof. See page 154 of [EK] Corollary 5.13. □

Definition 2.2.3. For $i \in \{1, \ldots, n\}$ let the *Koszul complex* $\mathrm{Kos}(x_i)$ be the following complex (concentrated in degrees 1 and 0)

$$R \xrightarrow{x_i} R.$$

And let the *Koszul complex* $\mathrm{Kos}(x_1, x_2, \ldots, x_n)$ be the total complex:

$$\mathrm{Tot}(\mathrm{Kos}(x_1) \otimes \ldots \otimes \mathrm{Kos}(x_n)).$$

The degree k part of the complex $\mathrm{Kos}(x_1, x_2, \ldots, x_n)$, which we denote by $\mathrm{Kos}_k(x_1, x_2, \ldots, x_n)$, is isomorphic to $\Lambda^k R^n$, and we consider the elements $\{e_{i_1} \wedge \ldots \wedge e_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n\}$ to be a basis for $\mathrm{Kos}_k(x_1, x_2, \ldots, x_n)$ and for $k \in \{1, \ldots, n\}$ we define the differential $d_k : \mathrm{Kos}_k(x_1, x_2, \ldots, x_n) \to \mathrm{Kos}_{k-1}(x_1, x_2, \ldots, x_n)$ to act by

$$e_{i_1} \wedge \ldots \wedge e_{i_k} \mapsto \Sigma_{p=1}^{k} (-1)^{p-1} x_p e_{i_1} \wedge \ldots \wedge \hat{e}_{i_p} \wedge \ldots \wedge e_{i_k}.$$

Proposition 2.2.4. *If* $x_1, \ldots, x_n$ *is a regular sequence then the complex* $\mathrm{Kos}(x_1, x_2, \ldots, x_n)$ *forms an* R*-projective resolution of* R/I.

Proof. See pp 113-114 of [W] Corollary 4.5.4 and 4.5.5. □

Proposition 2.2.5. *If* R *is a Noetherian ring,* I *an ideal of* R *which is locally generated by a regular sequence and* m *be an integer greater than* 1. *Then for* $P.(R/I)$, *a projective resolution of* R/I, *we have the following isomorphism*

$$H_k(P.(R/I)^{\otimes m}) \cong \Lambda^k((I/I^2)^{m-1}),$$

and in particular we have

$$\mathrm{Tor}_k^R(R/I, R/I) \cong \Lambda^k(I/I^2).$$

Proof. First we assume that I is globally generated by a regular sequence

$x_1, \dots, x_n$. Our choice of projective resolution will not affect the homology as two different resolutions will be quasi-isomorphic so by Corollary 2.1.6 the homologies of the total complexes will also be quasi-isomorphic.

Clearly we have

$$H_k(P.(R/I)^{\otimes m}) \cong H_k(P.(R/I)^{\otimes(m-1)} \otimes P.(R/I));$$

by the first spectral sequence in Proposition 2.1.5 we have

$$H_k(P.(R/I)^{\otimes(m-1)} \otimes P.(R/I)) \cong H_k(P.(R/I)^{\otimes(m-1)} \otimes R/I).$$

If $m = 2$ this coincides with the definition of $\mathrm{Tor}_k^R(R/I, R/I)$ and we calculate this value of Tor by calculating the homology of this complex.

By Proposition 2.2.4 we may take $\mathrm{Kos}(x_1, \dots, x_n)$ as our resolution of R/I. The k^{th} place of the Koszul complex $\mathrm{Kos}(x_1, x_2, \dots, x_n)$ is $\Lambda^k(R^n)$, so the k^{th} place of $\mathrm{Kos}(x_1, x_2, \dots, x_n)^{\otimes(m-1)}$ is equal to

$$\bigoplus_{k_1,\dots,k_{m-1} \in \{0,\dots,n\}, \Sigma_{i=1}^{m-1} k_i = k} \Lambda^{k_1}(R^n) \otimes \dots \otimes \Lambda^{k_{m-1}}(R^n).$$

So

$$\begin{aligned}
\mathrm{Tot}_k(P.(R/I)^{\otimes(m-1)} \otimes R/I) &= \mathrm{Tot}_k(\mathrm{Kos}(x_1, x_2, \dots, x_n)^{\otimes(m-1)} \otimes R/I) \\
&= \Big(\bigoplus_{k_1,\dots,k_{m-1} \in \{0,\dots,n\}, \Sigma_{i=1}^{m-1} k_i = k} \Lambda^{k_1}(R^n) \otimes \dots \otimes \Lambda^{k_{m-1}}(R^n) \Big) \otimes R/I \\
&= \bigoplus_{k_1,\dots,k_{m-1} \in \{0,\dots,n\}, \Sigma_{i=1}^{m-1} k_i = k} \Lambda^{k_1}((R/I)^n) \otimes \dots \otimes \Lambda^{k_{m-1}}((R/I)^n).
\end{aligned}$$

The differential of the total complex $\mathrm{Tot}_k(\mathrm{Kos}(x_1, x_2, \dots, x_n)^{\otimes(m-1)} \otimes R/I)$ is the sum of the $m-1$ maps, each of which reduces one of the $m-1$ indices by 1 and leaves all the other $m-2$ indices unchanged; on the components whose indices are unchanged these maps act as the identity; on the remaining component these maps acts by $e_{i_1} \wedge \dots \wedge e_{i_{k_l}} \mapsto \Sigma_{p=1}^{k_l} (-1)^{p-1} x_p e_{i_1} \wedge \dots \wedge$

$\hat{e}_{i_p} \wedge \ldots \wedge e_{i_{k_l}}$, but this is just the zero map since $x_p \in I$ for each p. So each component of the differential is the zero map, and hence the differential is itself the zero map. Therefore

$$H_k(\operatorname{Kos}(x_1, x_2, \ldots, x_n)^{\otimes(m-1)} \otimes R/I)$$
$$\cong \bigoplus_{k_1,\ldots,k_{m-1} \in \{0,\ldots,n\}, \Sigma_{i=1}^{m-1} k_i = k} \Lambda^{k_1}((R/I)^n) \otimes \ldots \otimes \Lambda^{k_{m-1}}((R/I)^n)$$

but this is just the canonical decomposition

$$\bigoplus \Lambda^{k_1}(R/I)^n \otimes \ldots \otimes \Lambda^{k_{m-1}}(R/I)^n \cong \Lambda^l\big((R/I)^n \oplus \ldots \oplus (R/I)^n\big).$$

where the sum ranges over all $k_1, \ldots, k_{m-1} \in \{0, \ldots, n\}$ such that $\Sigma_{i=1}^{m-1} k_i = k$

We have an isomorphism $a : (R/I)^n \to I/I^2$, which is the composition of two isomorphisms, the first being the isomorphism $(R/I)^n \to R/I[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^2, e_i \mapsto X_i + (X_1 + \ldots X_n)^2$ and the second being $R/I[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^2 \to I/I^2, X_i \mapsto x_i + I^2$ (as in Lemma 2.2.2). So we get the isomorphism

$$\Lambda^l\big((R/I)^n \oplus \ldots \oplus (R/I)^n\big) \cong \Lambda^l\big((I/I^2) \oplus \ldots \oplus (I/I^2)\big)$$

and hence we get finally get the desired isomorphism

$$H_k(P.(R/I)^{\otimes m}) \cong \Lambda^k((I/I^2)^{m-1}).$$

We now need to show that this isomorphism does not depend on the choice of regular sequence $x_1, \ldots, x_n$. Let $y_1, \ldots, y_n$ be another regular sequence which generates I. For appropriate $f_{j,i} \in R$ we have $x_j = \Sigma_i f_{j,i} y_i$. The matrix $(f_{j,i})$ defines a homomorphism $F : R^n \to R^n$ such that the following diagram commutes:

$$\begin{tikzcd} R^n \arrow[r, "(x_1,\ldots,x_n)"] \arrow[d, "F"] & R \arrow[d, equal] \\ R^n \arrow[r, "(y_1,\ldots,y_n)"] & R. \end{tikzcd}$$

This homomorphism $F : R^n \to R^n$ induces a map of complexes $\mathrm{Kos}(\overline{F}) : \mathrm{Kos}(x_1, x_2, \ldots, x_n) \to \mathrm{Kos}(y_1, \ldots, y_n)$ which extends the identity map on R/I. Tensoring $\mathrm{Kos}(y_1, \ldots, y_n)$ and $\mathrm{Kos}(x_1, x_2, \ldots, x_n)$ by R/I we get homomorphisms $\Lambda^k(F) : \Lambda^k((R/I)^n) \to \Lambda^k((R/I)^n)$ for $k \geq 0$ such that the following diagrams commute:

$$\begin{tikzcd} \Lambda^k((R/I)^n) \arrow[r, "\Lambda^k(b)"] \arrow[d, "\Lambda^k(\overline{F})"] & \Lambda^k(I/I^2) \arrow[d, equal] \\ \Lambda^k((R/I)^n) \arrow[r, "\Lambda^k(a)"] & \Lambda^k(I/I^2). \end{tikzcd}$$

This in turn can be extended to a homomorphism of the tensor powers of the Koszul complexes.

Therefore the isomorphism constructed above does not depend on the choice of regular sequence generating I. If I is only generated locally by a regular sequence we can therefore glue the local isomorphisms constructed to get the wanted global isomorphism. □

Lemma 2.2.6. *Let P be a free R-module. Then the dual of the k^{th} exterior power of the dual of P is canonically isomorphic to the k^{th} exterior power of P, i.e.*

$$(\Lambda^k(P^*))^* \cong \Lambda^k(P).$$

Proof. Let S_k denote the group of permuations on k elements. Let $\{e_1, \ldots, e_n\}$ be a basis for P, and let $\{e_1^*, \ldots, e_n^*\}$ be the corresponding dual basis. The module $\Lambda^i(P^*)$ consists of sums of elements of the form $f_1 \wedge \ldots \wedge f_k$ with each $f_i \in P^*$. We can take $\{e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* | 1 \leq i_1 < \ldots < i_k \leq n\}$ as a basis of $\Lambda^k(P^*)$.

We want to show that the following map

$$\begin{aligned}\Lambda^k(P) &\to (\Lambda^k(P^*))^* \\ x_1 \wedge \ldots \wedge x_k &\mapsto \left(f_1 \wedge \ldots \wedge f_k \mapsto \Sigma_{\sigma \in S_k} \operatorname{sgn}(\sigma) \Pi_{i=1}^k f_i(x_{\sigma(i)})\right)\end{aligned}$$

is an isomorphism that sends the basis $\{e_{i_1} \wedge \ldots \wedge e_{i_k} | 1 \le i_1 < \ldots < i_k \le n\}$ to the dual of the basis $\{e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} | 1 \le i_1 < \ldots < i_k \le n\}$, i.e.

$$(e^*_{j_1} \wedge \ldots \wedge e^*_{j_k})(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \begin{cases} 0 & \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \\ 1 & (j_1, \ldots, j_k) = (i_1, \ldots, i_k). \end{cases}$$

This map takes the element $e_{i_1} \wedge \ldots \wedge e_{i_k}$ of the basis of $\Lambda^k(P)$ to the map $\Lambda^k(P^*) \to R, f_1 \wedge \ldots \wedge f_k \mapsto \Sigma_{\sigma \in S_k} \operatorname{sgn}(\sigma) \Pi_{l=1}^k f_l(e_{i_{\sigma(l)}})$. And this latter map takes $e^*_{j_1} \wedge \ldots \wedge e^*_{j_k}$, an element of the basis of $\Lambda^k(P^*)$, to the value $\Sigma_{\sigma \in S_k} \operatorname{sgn}(\sigma) \Pi_{l=1}^k e^*_{j_l}(e_{i_{\sigma(l)}})$.

If $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$ then for some m we have $j_m \notin \{i_1, \ldots, i_k\}$. For this m we have $e^*_{j_m}(e_{i_{\sigma(l)}}) = 0$ for every $l \in \{1, \ldots, k\}$ and every $\sigma \in S_k$. Therefore $\Pi_{l=1}^k e^*_{j_l}(e_{i_{\sigma(l)}})$ can only be non-zero if $\{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\}$.

If $\{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\}$ then if $\sigma = \mathrm{id} \in S_k$ we have $\Pi_{l=1}^k e^*_{j_l}(e_{i_{\sigma(l)}}) = 1$, for $\alpha \neq \mathrm{id}$ we have $\Pi_{l=1}^k e^*_{j_l}(e_{i_{\alpha(l)}}) = 0$. Hence our result is shown. □

It is not true in general that the analogously defined map

$$\begin{aligned}\mathrm{Sym}^k(P) &\to (\mathrm{Sym}^k(P^*))^* \\ x_1 \ldots x_k &\mapsto \left(f_1 \ldots f_k \mapsto \Sigma_{\sigma \in S_k} \Pi_{i=1}^k f_i(x_{\sigma(i)})\right)\end{aligned}$$

is an isomorphism because the sum would have more than one non-zero term, since the indices of the elements of the basis of $\mathrm{Sym}^k(P)$ are not *strictly* increasing just increasing. This motivates the following definition.

Definition 2.2.7. Let P be a finitely generated projective R-module.. We define the i^{th} *divided power functor of* P to be dual of the i^{th} symmetric power of P, i.e. $D^i(P) := (\mathrm{Sym}^i(P^*))^*$. For more information the reader is directed to p214-216 of [ABW].

Definition 2.2.8. Let $f : P \to Q$ be a homomorphism between two projective R-modules, and $n \in \mathbb{N}$. Let $\mathrm{Kos}^n(f)$ be the *Koszul complex*

$$0 \to \Lambda^n(P) \xrightarrow{d_n} \Lambda^{n-1}(P) \otimes Q \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} P \otimes \mathrm{Sym}^{n-1}(Q) \xrightarrow{d_0} \mathrm{Sym}^n(Q) \to 0$$

where, for $k \in \{0, 1, \dots, n-1\}$, the differential $d_{k+1} : \Lambda^{k+1}(P) \otimes \mathrm{Sym}^{n-k-1}(Q) \to \Lambda^k(P) \otimes \mathrm{Sym}^{n-k}(Q)$ acts by

$$p_1 \wedge \dots \wedge p_{k+1} \otimes q_{k+2} \cdots q_n \mapsto \Sigma_{i=1}^{k+1} (-1)^{k+1-i} p_1 \wedge \dots \wedge \hat{p}_i \wedge \dots \wedge p_{k+1} \otimes f(p_i) q_{k+2} \cdots q_n.$$

If we take f to be a map between P^* and Q^*, the duals of P and Q, then the part of the Koszul complex $\mathrm{Kos}^n(f)$ in the k^{th} degree is $\Lambda^k(P^*) \otimes \mathrm{Sym}^{n-k}(Q^*)$. The dual of this chain complex is a co-chain complex with the part in the k^{th} degree being $\left(\Lambda^k(P^*) \otimes \mathrm{Sym}^{n-k}(Q^*)\right)^* \cong \Lambda^k(P) \otimes D^{n-k}(Q)$, i.e.

$$0 \leftarrow \Lambda^n(P) \xleftarrow{(d_n)^*} \Lambda^{n-1}(P) \otimes Q \xleftarrow{(d_{n-1})^*} \dots \xleftarrow{(d_1)^*} P \otimes D^{n-1}(Q) \xleftarrow{(d_0)^*} D^n(Q) \leftarrow 0.$$

We define the *co-Koszul complex*, $\tilde{\mathrm{Kos}}^n(f)$, to be the chain complex with $\Lambda^k(P) \otimes D^{n-k}(Q)$ the part in the $(k-n)^{\text{th}}$ degree, and with differential d_k equal to $(d_{n-k})^*$ in the above diagram, i.e. $\tilde{\mathrm{Kos}}^n(f)$ is the complex

$$0 \to D^n(P) \xrightarrow{d_n} D^{n-1}(P) \otimes Q \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} P \otimes \Lambda^{n-1}(Q) \xrightarrow{d_0} \Lambda^n(Q) \to 0.$$

Remark 2.2.9. When $P = R^n, Q = R$ and $f : R^n \to R, (e_1, \dots, e_n) \mapsto e_1x_1 + \dots + e_nx_n$ then this definition coincides with the above definition of *Koszul complexes* i.e. $\mathrm{Kos}^n(f) = \mathrm{Kos}(x_1, x_2, \dots, x_n)$.

It is well known that the complexes $\mathrm{Kos}(f)$ and $\tilde{\mathrm{Kos}}(f)$ are exact if f is an isomorphism.

The two following isomorphisms will be useful in later calculations.

Proposition 2.2.10. *Let $f : P \to Q$ be a homomorphism between two projective R-modules. If we consider consider $P \to Q$ to be a chain complex concentrated in degrees* 1 *and* 0 *then we have quasi-isomorphisms*

$$\mathrm{Kos}^n(f) \cong N\,\mathrm{Sym}^n\,\Gamma(P \to Q)$$

and

$$\tilde{\mathrm{Kos}}^n(f) \cong N\Lambda^n\Gamma(P \to Q),$$

where Γ and N are the functors of the Dold-Kan correspondence that we introduced in Chapter 1.

Proof. See Proposition 2.4 and Remark 3.6 of [Kö]. □

2.3 Schur Functors

In this section we describe the construction of Schur functors, both by the method described in Chapter I of [ABW] and as described in Chapter 2 of [Kö]. Schur functors will be important because Köck's predictions describe the derived functors of Sym^3 in terms of Schur functors of hook type, and also because they will be used in the Cauchy decompostion of $\mathrm{Sym}^3(P \otimes Q)$ described in section 2.5. Schur functors are functors of modules that are generalisations of the functors $\mathrm{Sym}^n, \Lambda^n$ and D^n.

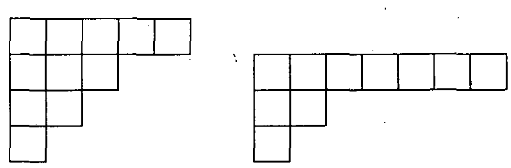
Definition 2.3.1. By $\mathbb{N}^\infty$ we mean sequences of elements of $\mathbb{N}$ with finite support (i.e. only finitely many elements of the sequence are non-zero). We identify $\mathbb{N}^p$ to a subset of $\mathbb{N}^\infty$ by adding zeroes to the end of the tuple. So $\mathbb{N}^\infty = \cup_{p\geq 0}\mathbb{N}^p$ and we consider $(\lambda_1, \ldots, \lambda_p) \in \mathbb{N}^p$ and $(\lambda_1, \ldots, \lambda_p, 0, \ldots) \in \mathbb{N}^\infty$ to be the same.

We call $\lambda \in \mathbb{N}^\infty$ a *partition* if $\lambda_1 \geq \lambda_2 \geq \ldots$. If $\Sigma_i\lambda_i = n$ we say that λ *is of weight* n, or that λ *is a partition of* n. We denote the weight of λ by $|\lambda|$. We call the number of non-zero elements of λ *the length of* λ. (Note this definition of partitions does not quite match the Definition 1.1.1; from here on when we refer to partitions we will mean this sense of partition and not definition 1.1.1.)

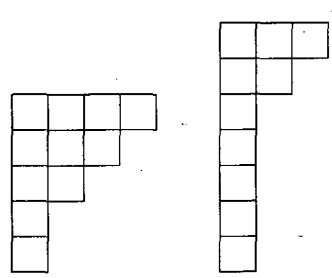
We define the *conjugate* (or *transpose*) of $\lambda \in \mathbb{N}^\infty$ as $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots) \in \mathbb{N}^\infty$, where $\tilde{\lambda}_i$ is the number of elements of λ which are less than or equal to i. Clearly for any $\lambda \in \mathbb{N}^\infty$ the conjugate of λ has the property $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \ldots$. Also $\tilde{\tilde{\lambda}}$ is the the sequence λ with the entries rearranged so they are in decreasing order. So conjugation in an involution on the set of partitions.

Definition 2.3.2. Let λ be a partition. The *Young diagram* associated with λ is the set of ordered pairs $(i, j) \in \mathbb{N}$ with $i \geq 1$ and $i \geq j \geq \lambda_i$ and is denoted Δ_λ. We use the convention that is used for matrices, i.e. that i is the row index and j is the column index (see below for a couple of examples). It is easy see that the diagram for λ contains $\tilde{\lambda}_i$ entries in the i^{th} row. Therefore it is clear that $|\lambda| = |\tilde{\lambda}|$ (i.e. conjugation preserves weight).

For example, the diagrams for $(5, 3, 2, 1)$ and $(7, 2, 1)$ look like



respectively. The conjugates of $(5, 3, 2, 1)$ and $(7, 2, 1)$ are $(4, 3, 2, 1, 1)$ and $(3, 2, 1, 1, 1, 1, 1)$ respectively, and their diagrams look like



respectively.

Notation 2.3.3. Let F be a projective module over a commutative ring R, and $\lambda = (\lambda_1, \dots, \lambda_k)$ a partition. We use the following notation

$$\begin{aligned}\Lambda_\lambda F &= \Lambda^{\lambda_1} F \otimes \dots \otimes \Lambda^{\lambda_k} F \\ \operatorname{Sym}_\lambda F &= \operatorname{Sym}^{\lambda_1} F \otimes \dots \otimes \operatorname{Sym}^{\lambda_k} F \\ D_\lambda F &= D^{\lambda_1} F \otimes \dots \otimes D^{\lambda_k} F.\end{aligned}$$

Definition 2.3.4. Let $\lambda = (\lambda_1, \dots, \lambda_q)$ be a partition, and F be a projective R-module. The *Schur functor of F of shape λ* will be the image of the function $d_\lambda : \Lambda_\lambda F \to \operatorname{Sym}_{\tilde{\lambda}} F$ that we will describe below. We denote this functor by L_λ.

We assume that L_μ has been defined for all μ of weight less than the weight of λ and define L_λ recursively. Let λ' be the partitiion $(\lambda_1 - 1, \dots, \lambda_q - 1)$, which is the partition associated with the Young diagram formed by stripping away the first column of λ.

Let δ_λ be the following composition

$$\begin{aligned}\Lambda_\lambda F = \Lambda^{\lambda_1} F \otimes \dots \otimes \Lambda^{\lambda_q} F &\xrightarrow{\Delta \otimes \dots \otimes \Delta} F \otimes \Lambda^{\lambda_1 - 1} F \otimes \dots \otimes F \otimes \Lambda^{\lambda_q - 1} F \\ &\xrightarrow{\sigma} F \otimes F \otimes \dots \otimes F \otimes \Lambda^{\lambda_1 - 1} F \otimes \dots \otimes \Lambda^{\lambda_q - 1} F \\ &\xrightarrow{m} \operatorname{Sym}^{\tilde{\lambda}_1} F \otimes \Lambda^{\lambda_1 - 1} F \otimes \dots \otimes \Lambda^{\lambda_q - 1} F = \operatorname{Sym}^{\tilde{\lambda}_1} F \otimes \Lambda_{(\lambda_1 - 1, \dots, \lambda_q - 1)} F \\ &\qquad = \operatorname{Sym}^{\tilde{\lambda}_1} F \otimes \Lambda_{\lambda'} F;\end{aligned}$$

here $\Delta : \Lambda^{\lambda_i}(F) \to \Lambda^{\lambda_i - 1}(F) \otimes F$ is the first differential of the Koszul complex $\operatorname{Kos}^{\lambda_i}(\mathrm{id}_F)$ see Definition 2.2.8 ; the function σ permutes the copies of F past the $\Lambda^{\lambda_i - 1} F$ terms; and m is induced by the canonical projection $F \otimes \dots \otimes F \to \operatorname{Sym}^{\tilde{\lambda}_1} F$ that acts by $f_1 \otimes \dots \otimes f_n \mapsto f_1 \dots f_n$. Since $d_{\lambda'} : \Lambda_{\lambda'} F \to \operatorname{Sym}_{\tilde{\lambda}'} F$ has already been defined we define d_λ as the composition $(\mathrm{id}_{\operatorname{Sym}^{\tilde{\lambda}_1} F} \otimes d_{\lambda'}) \circ \delta_\lambda$.

In an analogous way we can define $d'_\lambda : D_\lambda F \to \Lambda_{\tilde{\lambda}} F$ and we define $\tilde{L}_\lambda$, *the co-Schur functor of shape* λ, to be the image of this map.

Definition 2.3.5. We say that λ partition is of *hook type* if $\lambda_i \leq 1$ for all $i \geq 2$. We say this because the Young diagram of such a partition looks like a hook.

Definition 2.3.6. Let $k \in \{0, \dots, n\}$ and F be a finitely generated projective R-module. Let $d_{k+1} : \Lambda^{k+1}(F) \otimes \mathrm{Sym}^{n-k-1}(F) \to \Lambda^k(F) \otimes \mathrm{Sym}^{n-k}(F)$ be the Koszul differential in the Koszul complex $\mathrm{Kos}^n(\mathrm{id}_F)$ (see Definition 2.2.8). Then

$$L_k^n(F) := \mathrm{Im}(d_{k+1}) \subseteq \Lambda^k(F) \otimes \mathrm{Sym}^{n-k}(F)$$

is called the *Schur functor of hook type* $(k+1, \underbrace{1, \dots, 1}_{n-k-1})$. Similarly we define $\tilde{L}_k^n(F)$ the *co-Schur functor of hook type* $(k+1, \underbrace{1, \dots, 1}_{n-k-1})$ as the image of the Koszul differential $d_{k+1} : D^{k+1}(F) \otimes \Lambda^{n-k-1}(F) \to D^k(F) \otimes \Lambda^{n-k}(F)$ in the co-Koszul complex $\tilde{\mathrm{Kos}}(\mathrm{id}_F)$.

Since $\mathrm{Kos}^n(\mathrm{id}_F)$ is exact the R-module $L_k^n(F)$ is a finitely generated projective module for all $k \in \{0, \dots, n\}$.

Definition 2.3.6 and Definition 2.3.4 are compatible because of the following Lemma.

Lemma 2.3.7. *Let* $\lambda = (k+1, 1, \dots, 1)$ *be a partition of* n *of hook type. Then the Schur functor* L_k^n *definined in Definition 2.3.6 coincides with the Schur functor* L_λ *defined in Definition 2.3.4.*

Proof. The compatibility of the two definitions comes from the commutativity of the following diagram:

$$\begin{tikzcd}
\Lambda^{k+1}F \otimes \mathrm{Sym}^{n-k-1} F \arrow[r, "d_{k+1}"] & \Lambda^k F \otimes \mathrm{Sym}^{n-k} F \arrow[d] \\
 & \mathrm{Sym}^{n-k} F \otimes \Lambda^k F \arrow[d] \\
\Lambda^{k+1}F \otimes F^{\otimes n-k-1} \arrow[uu, "\mathrm{id}_{\Lambda^{k+1}F} \otimes m"] \arrow[r, "d_{(k+1,1,\dots,1)}"] & \mathrm{Sym}^{n-k} F \otimes F^{\otimes k}
\end{tikzcd}$$

On the left hand side of the square the map m is just the canonical projection $F^{\otimes n-k-1} \twoheadrightarrow \mathrm{Sym}^{n-k-1} F$. The map on the right hand side of the square between $\Lambda^k F \otimes \mathrm{Sym}^{n-k} F$ and $\mathrm{Sym}^{n-k} F \otimes \Lambda^k F$ is the obvious isomorphism. The map $\Lambda^k(F) \hookrightarrow F^{\otimes k}$ on the right hand side is the antisymmetrisation map

$$f_1 \wedge f_2 \wedge \ldots \wedge f_k \mapsto \Sigma_\sigma \,\mathrm{sgn}(\sigma) f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \ldots \otimes f_{\sigma(k)}$$

where the sum ranges over all elements σ of the symmetric group of k elements.

The map on the bottom of the square $\Lambda^{k+1} F \otimes F \to \mathrm{Sym}^{n-k} F \otimes F^{\otimes k}$ is given by definition 2.3.4; it decomposes into

$$\delta_{(k+1,1,\ldots,1)} : \Lambda^{k+1} F \otimes \Lambda^1 F \otimes \ldots \otimes \Lambda^1 F \to \mathrm{Sym}^{n-k} F \otimes \Lambda^k F$$

followed by $\mathrm{id}_{\mathrm{Sym}^{n-k} F} \circ \delta_{(1,\ldots,1)}$. A simple calculation shows us that the map $\delta_{(1,\ldots,1)} : \Lambda^k(F) \to F^{\otimes n}$ is just the the antisymmetrisation map described above. $\square$

2.4 The iterated Eilenberg-Zilber theorem

The Eilenberg-Zilber theorem gives a suprising homotopy equivalence between the diagonal of a bisimplicial complex and the total complex of the associated double complex. Several times in section 2.7 we will need to calculate the homology of a simplicial complex which can be viewed as the diagonal of a bisimplicial complex, more precisely of the diagonal of the tensor product of two simpler simplicial complexes. The Eilenberg-Zilber theorem will let us turn information about these simpler complexes into information about the complex we are interested in.

Let R be a commutative ring.

Lemma 2.4.1. *Let $C.$, $C.'$, $D.$ and $D.'$ be chain complexes in* $\mathrm{Ch}(R\text{-}\mathbf{mod})$ *with $C.$ and $C.'$ chain homotopy equivalent, and also $D.$ and $D.'$ chain* *homotopy equivalent. Then the complexes* $\mathrm{Tot}(C_. \otimes D_.)$ *and* $\mathrm{Tot}(C'_. \otimes D'_.)$ *are also chain homotopy equivalent.*

Proof. First we will prove our result in the case where $D'_. = D_.$, and then later show that this entails our desired result. We know that $C_.$ and $C'_.$ are chain homotopic, so we have chain maps $f_C : C_. \to C'_.$, $g_C : C'_. \to C_.$ and a chain homotopy $s_C = \{s_{C_n} : C_n \to C_{n+1}\}$ such that $s_C d_C + d_C s_C = g_C f_C - \mathrm{id}_C$.

The differential of $\mathrm{Tot}(C_. \otimes D_.)$ at $C_x \otimes D_y$ is $d_v + d_h$ where $d_h = d_C \otimes \mathrm{id}_D$ and $d_v = \mathrm{id}_C \otimes (-1)^x d_D$. We now consider the map $s : \mathrm{Tot}(C_. \otimes D_.)_n \to \mathrm{Tot}(C_. \otimes D_.)_{n+1}$ that acts as $s_C \otimes \mathrm{id}_D : C_x \otimes D_y \to C_{x+1} \otimes D_y$ for all $x, y \in \mathbb{Z}$. Look at the following square in $\mathrm{Tot}(C_. \otimes D_.)$:

$$\begin{tikzcd}
C_{x+1} \otimes D_y \arrow[d, "d_v = \mathrm{id}_C \otimes (-1)^{x+1} d_D"'] & & C_x \otimes D_y \arrow[ll, "s_C \otimes \mathrm{id}_D"'] \arrow[d, "d_v = \mathrm{id}_C \otimes (-1)^{x} d_D"] \\
C_{x+1} \otimes D_{y-1} & & C_x \otimes D_{y-1} \arrow[ll, "s_C \otimes \mathrm{id}_D"']
\end{tikzcd}$$

since the signs on the left and right are different we see that the square anti-commutes, i.e. that $sd_v + d_v s = 0$.

Now we see that:

$$\begin{aligned}
sd_{\mathrm{Tot}(C_. \otimes D_.)} + d_{\mathrm{Tot}(C_. \otimes D_.)} s &= s(d_v + d_h) + (d_v + d_h)s = sd_v + sd_h + d_v s + d_h s \\
&= sd_v + d_v s + d_h s + sd_h = d_h s + sd_h \\
&= (d_C \otimes \mathrm{id}_D)(s_C \otimes \mathrm{id}_D) + (s_C \otimes \mathrm{id}_D)(d_C \otimes \mathrm{id}_D) \\
&= d_C s_C \otimes \mathrm{id}_D + s_C d_C \otimes \mathrm{id}_D \\
&= (d_C s_C + s_C d_C) \otimes \mathrm{id}_D \\
&= (g_C f_C - \mathrm{id}_C) \otimes \mathrm{id}_D = g_C f_C \otimes \mathrm{id}_D - \mathrm{id}_C \otimes \mathrm{id}_D
\end{aligned}$$

Similarly given the chain homotopy $s_{C'} = \{s_{C'_n} : C'_n \to C'_{n+1}\}$ that satisfies the relation $f_C g_C - \mathrm{id}_C = s_{C'} d_{C'} + d_{C'} s_{C'}$ we can find a map s' : $\mathrm{Tot}(C'. \otimes D.)_n \to \mathrm{Tot}(C'. \otimes D.)_{n+1}$ such that:

$$s d_{\mathrm{Tot}(C'.\otimes D.)} + d_{\mathrm{Tot}(C'.\otimes D.)} s = g_C f_C \otimes \mathrm{id}_D - \mathrm{id}_C \otimes \mathrm{id}_D .$$

Hence we conclude that $\mathrm{Tot}(C.\otimes D.)$ is chain homotopic to $\mathrm{Tot}(C'.\otimes D.)$.

Now the twisting homomorphism $\mathrm{Tot}(C. \otimes D) \to \mathrm{Tot}(D. \otimes C.), c_x \otimes d_y \mapsto (-1)^{xy} d_y \otimes d_x$ tells us that $\mathrm{Tot}(C. \otimes D.) \cong \mathrm{Tot}(D. \otimes C.)$. So the above argument can be used to show that $\mathrm{Tot}(C. \otimes D.)$ is chain homotopic to $\mathrm{Tot}(C. \otimes D.')$. Combining this and the previous chain homotopy we get the desired result. □

Theorem 2.4.2. The Iterated Eilberg-Zilber Theorem. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $A^1, \ldots, A^n$ be simplicial complexes. Then the complexes $N\Delta(A^1 \otimes \ldots \otimes A^n)$ and $\mathrm{Tot}(NA^1 \otimes \ldots \otimes NA^n)$ are chain homotopic and (consequently) they are quasi-isomorphic:*

$$H_k(N\Delta(A^1 \otimes \ldots \otimes A^n)) \cong H_k(\mathrm{Tot}(NA^1 \otimes \ldots \otimes NA^n))$$

Note when $n = 2$ this is simply the normal Eilenberg-Zilber Theorem.

Proof. We will prove this result by induction on n. The Eilenberg-Zilber Theorem (see §28 of [M], specifically Corollary 29.6 on p132) tells us that $N\Delta(A^1 \otimes A^2)$ is chain homotopic to $\mathrm{Tot}(NA^1 \otimes NA^2)$; this serves as our inductive base and will be used in the inductive step.

The inductive step that we need to show is: if the chain complexes $N\Delta(A^1 \otimes \ldots \otimes A^{n-1})$ and $\mathrm{Tot}(NA^1 \otimes \ldots \otimes NA^{n-1})$ are chain homotopic then the chain complexes $N\Delta(A^1 \otimes \ldots \otimes A^{n-1} \otimes A^n)$ and $\mathrm{Tot}(NA^1 \otimes \ldots \otimes NA^{n-1} \otimes NA^n)$ are also chain homotopic.

Now it is easy to see that

$$N\Delta(A^1 \otimes \ldots \otimes A^{n-1} \otimes A^n) = N\Delta(\Delta(A^1 \otimes \ldots \otimes A^{n-1}) \otimes A^n)$$

and by the Eilenberg-Zilber Theorem we know that $N\Delta(\Delta(A^1\otimes\ldots\otimes A^{n-1})\otimes A^n)$ and $\mathrm{Tot}(N\Delta(A^1\otimes\ldots\otimes A^{n-1})\otimes NA^n)$ are chain homotopic. By our inductive assumption and by Lemma 2.4.1 we see that $\mathrm{Tot}(N\Delta(A^1\otimes\ldots\otimes A^{n-1})\otimes NA^n)$ and $\mathrm{Tot}(\mathrm{Tot}(NA^1\otimes\ldots\otimes NA^{n-1})\otimes NA^n)$ are chain homotopic. But $\mathrm{Tot}(\mathrm{Tot}(NA^1\otimes\ldots\otimes NA^{n-1})\otimes NA^n)$ and $\mathrm{Tot}(NA^1\otimes\ldots\otimes NA^{n-1}\otimes NA^n)$ are (by definition) equal, so our inductive step, and hence the desried result is shown. $\square$

2.5 Cauchy decomposition of $\mathrm{Sym}^3(F\otimes G)$

Let R be a ring. Let F and G be finitely generated projective R-modules. The following is a summary of the Cauchy decomposition given in chapter III of [ABW] as it applies to the third symmetric power. This decomposition of $\mathrm{Sym}^3(F\otimes G)$ will be essential in our proof of Theorem 2.7.2.

A three step filtration is put on $\mathrm{Sym}^3(F\otimes G)$

$$\begin{aligned}0\subset M_{(3)}(\mathrm{Sym}^3(F\otimes G))\subset M_{(2,1)}(\mathrm{Sym}^3(F\otimes G))&\\ \subset M_{(1,1,1)}(\mathrm{Sym}^3(F\otimes G))=\mathrm{Sym}^3(F\otimes G).&\end{aligned}$$

The $M_{(3)}(\mathrm{Sym}^3(F\otimes G))$ part is defined to be the image of the determinant map

$$\Lambda^3F\otimes\Lambda^3G\to\mathrm{Sym}^3(F\otimes G)$$

$$f_1\wedge f_2\wedge f_3\otimes g_1\wedge g_2\wedge g_3\mapsto\begin{vmatrix}f_1\otimes g_1 & f_1\otimes g_2 & f_1\otimes g_3\\ f_1\otimes g_1 & f_2\otimes g_2 & f_2\otimes g_3\\ f_1\otimes g_1 & f_3\otimes g_2 & f_3\otimes g_3\end{vmatrix}.$$

(note this is simply isomorphic to $\Lambda^3F\otimes\Lambda^3G$). The $M_{(2,1)}(\mathrm{Sym}^3(F\otimes G))$ part is defined to be equal to the previous part $M_{(3)}(\mathrm{Sym}^3(F\otimes G))$ part plus the image of the following homomorphism:

$$\Lambda_{(2,1)}F \otimes \Lambda_{(2,1)}G \to \mathrm{Sym}^3(F \otimes G)$$

$$f_1 \wedge f_2 \otimes f_3 \otimes g_1 \wedge g_2 \otimes g_3 \mapsto \begin{vmatrix} f_1 \otimes g_1 & f_1 \otimes g_2 \\ f_1 \otimes g_1 & f_2 \otimes g_2 \end{vmatrix} (f_3 \otimes g_3).$$

The quotients of this filtration are isomorphic to tensor products of Schur functors as follows:

$$M_{(2,1)}(\mathrm{Sym}^3(F \otimes G))/M_{(3)}(\mathrm{Sym}^3(F \otimes G)) \cong L_{(2,1)}F \otimes L_{(2,1)}G$$

and

$$\begin{aligned} M_{(1,1,1)}(\mathrm{Sym}^3(F \otimes G))/M_{(2,1)}(\mathrm{Sym}^3(F \otimes G)) &\cong L_{(1,1,1)}F \otimes L_{(1,1,1)}G \\ &= \mathrm{Sym}^3(F) \otimes \mathrm{Sym}^3(G). \end{aligned}$$

Or equivalently we have the two short exact sequences that follow:

$$0 \to \Lambda^3 F \otimes \Lambda^3 G \to M_{(2,1)}(\mathrm{Sym}^3(F \otimes G)) \to L_{(2,1)}F \otimes L_{(2,1)}G \to 0$$

and

$$0 \to M_{(2,1)}(\mathrm{Sym}^3(F \otimes G)) \to \mathrm{Sym}^3(F \otimes G) \to \mathrm{Sym}^3(F) \otimes \mathrm{Sym}^3(G) \to 0.$$

2.6 Characterising functors in terms of their cross-effects

In [Kö] Köck proved Theorem 2.6.2, a result which shows that two functors are isomorphic if they agree on certain data given by their cross-effect functors. In this section we introduce this theorem and apply it to show that the Schur functor L_1^3 and the co-Schur functor $\tilde{L}_1^3$ are isomorphic. Furthermore in section 2.7 our partial proof of the predictions made in [Kö] for the derived functors of Sym^3 is obtained by satisfying half the preconditions of Theorem 2.6.2.

Let A be a ring.

Definition 2.6.1. Let $l \geq k \geq 1, V_1, \ldots, V_k \in \mathcal{P}_A$, where $\mathcal{P}_A$ is the category of projective A-modules. and $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{1, \ldots, l\}^k$ with $\sum_{i=1}^k \epsilon_i = l$. The composition

$$\begin{aligned}\Delta_\epsilon : \mathrm{cr}_k(F)(V_1, \ldots, V_k) &\xrightarrow{\mathrm{cr}_k(F)(\Delta,\ldots,\Delta)} \mathrm{cr}_k(F)(V_1^{\epsilon_1}, \ldots, V_k^{\epsilon_k}) \\ &\overset{\pi}{\twoheadrightarrow} \mathrm{cr}_l(F)(V_1, \ldots, V_1, \ldots, V_k, \ldots, V_k)\end{aligned}$$

of the map $\mathrm{cr}_k(F)(\Delta, \ldots, \Delta)$ (induced by the diagonal maps $\Delta : V_i \to V_i^{\epsilon_i}, i = 1, \ldots, k$) with the canonical projection π (according to Theorem 1.3.6) is called the *diagonal map associated with* ϵ. The analogous composition

$$\begin{aligned}+_\epsilon : \mathrm{cr}_l(F)(V_1, \ldots, V_1, \ldots, V_k, \ldots, V_k) &\hookrightarrow \mathrm{cr}_k(F)(V_1^{\epsilon_1}, \ldots, V_k^{\epsilon_k}) \\ &\xrightarrow{\mathrm{cr}_k(F)(+,\ldots,+)} \mathrm{cr}_k(F)(V_1, \ldots, V_k)\end{aligned}$$

is called *plus map associated with* ϵ.

The maps Δ_ϵ and $+_\epsilon$ form natural transformations between the functors $\mathrm{cr}_k(F)$ and $\mathrm{cr}_l(F) \circ (\Delta_{\epsilon_1}, \ldots, \Delta_{\epsilon_k})$ from $\mathcal{P}_A^k$ to $\mathcal{M}$. One easily sees that the map Δ_ϵ can be decomposed into a composition of maps Δ_δ with $\delta \in \{1, 2\}$ such that $|\delta| = j + 1$ and $j \in \{k, \ldots, l-1\}$. The same holds for $+_\epsilon$.

Theorem 2.6.2. *Let A be a ring, $\mathcal{M}$ an abelian category, $d \in \mathbb{N}_+$, and*

$$F, G : (\text{f.g. projective } A\text{-modules}) \to \mathcal{M}$$

be two functors of degree $\leq d$ with $F(0) = 0 = G(0)$. Suppose that there exist isomorphisms

$$\alpha_i(A, \ldots, A) : \mathrm{cr}_i(F)(A, \ldots, A) \xrightarrow{\sim} \mathrm{cr}_i(G)(A, \ldots, A), \qquad i = 1, \ldots, d,$$

which are compatible with the action of A in each component and which make the following diagrams commute for $i \in \{1, \ldots, d-1\}$ and $\epsilon \in \{1,2\}^i$ with $|\epsilon| = i+1$:

$$\begin{array}{ccc}
\mathrm{cr}_i(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_i(G)(A,\ldots,A) \\
\downarrow{\scriptstyle \Delta_\epsilon} & & \downarrow{\scriptstyle \Delta_\epsilon} \\
\mathrm{cr}_{i+1}(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_{i+1}(G)(A,\ldots,A)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{cr}_{i+1}(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_{i+1}(G)(A,\ldots,A) \\
\downarrow{\scriptstyle +_\epsilon} & & \downarrow{\scriptstyle +_\epsilon} \\
\mathrm{cr}_i(F)(A,\ldots,A) & \longrightarrow & \mathrm{cr}_i(G)(A,\ldots,A).
\end{array}$$

Then the functors are isomorphic.

Proof. See Theorem 1.5 of [Kö]. □

We now apply Theorem 2.6.2 to compute the Schur functor L_1^3 and the co-Schur functor $\tilde{L}_1^3$.

Proposition 2.6.3.

$$\mathrm{cr}_k(L_1^3)(A,\ldots,A) \cong \begin{cases} 0 & k=1 \\ A \oplus A & k=2 \\ A \oplus A & k=3 \\ 0 & k \geq 4. \end{cases}$$

The maps $\Delta_{(2)} : \mathrm{cr}_1(L_1^3)(A) \to \mathrm{cr}_2(L_1^3)(A,A)$ and $+_{(2)} : \mathrm{cr}_2(L_1^3)(A,A) \to \mathrm{cr}_1(L_1^3)(A)$ are zero maps. The other associated maps between the above *cross-effect modules are as follows:*

$$\begin{aligned}
\Delta_{(2,1)} : \mathrm{cr}_2(L_1^3)(A, A) &\to \mathrm{cr}_3(L_1^3)(A, A, A) \\
(1,0) &\mapsto (1,1) \\
(0,1) &\mapsto (0,0) \\
\Delta_{(1,2)} : \mathrm{cr}_2(L_1^3)(A, A) &\to \mathrm{cr}_3(L_1^3)(A, A, A) \\
(1,0) &\mapsto (0,0) \\
(0,1) &\mapsto (1,1) \\
+_{(2,1)} : \mathrm{cr}_3(L_1^3)(A, A, A) &\to \mathrm{cr}_3(L_1^3)(A, A) \\
(1,0) &\mapsto (1,0) \\
(0,1) &\mapsto (1,0) \\
+_{(1,2)} : \mathrm{cr}_3(L_1^3)(A, A, A) &\to \mathrm{cr}_3(L_1^3)(A, A) \\
(1,0) &\mapsto (0,1) \\
(0,1) &\mapsto (0,1).
\end{aligned}$$

Proof. Let V, W and X be finitely generated A modules. Definition 2.3.6 tells us the Schur functor $L_1^3(V)$ is the image of d_2 in the complex $\mathrm{Kos}^3(\mathrm{id}_V)$, since this complex is exact the image of d_2 is the same as the kernel of d_1.

$$\begin{tikzcd}
\Lambda^3(V) \arrow[r] & \Lambda^2(V) \otimes V \arrow[rr] \arrow[dr] & & V \otimes \mathrm{Sym}^2(V) \arrow[r] & \mathrm{Sym}^3(V) \\
& & L_1^3(V) \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

Hence we have the short exact sequence $0 \to L_1^3(V) \to V \otimes \mathrm{Sym}^2(V) \to \mathrm{Sym}^3(V) \to 0$. In particular this tells us that $\mathrm{cr}_1(L_1^3)(A) = L_1^3(A) = 0$.

Next we want to compute $\mathrm{cr}_2(L_1^3)(A, A)$. We have the short exact sequence

$$0 \to L_1^3(V \oplus W) \to (V \oplus W) \otimes \mathrm{Sym}^2(V \oplus W) \to \mathrm{Sym}^3(V \oplus W) \to 0.$$

Using the canonical decomposition of Sym^n we see that

$$\begin{aligned}(V \oplus W) \otimes \mathrm{Sym}^2(V \oplus W) \cong & V \otimes \mathrm{Sym}^2(V) \oplus V \otimes (V \otimes W) \\ & \oplus V \otimes \mathrm{Sym}^2(W) \oplus W \otimes \mathrm{Sym}^2(V) \\ & \oplus W \otimes (V \otimes W) \oplus W \otimes \mathrm{Sym}^2(W)\end{aligned}$$

and

$$\mathrm{Sym}^3(V \oplus W) \cong \mathrm{Sym}^3(V) \oplus \mathrm{Sym}^2(V) \otimes W \oplus V \otimes \mathrm{Sym}^2(W) \oplus \mathrm{Sym}^3(W).$$

So we get the following short exact sequence:

$$\begin{aligned}0 \to \mathrm{cr}_2(L_1^3)(V, W) \to & \\ & V \otimes (V \otimes W) \oplus V \otimes \mathrm{Sym}^2(W) \oplus W \otimes \mathrm{Sym}^2(V) \oplus W \otimes (V \otimes W) \\ & \to \mathrm{Sym}^2(V) \otimes W \oplus V \otimes \mathrm{Sym}^2(W) \to 0.\end{aligned}$$

Therefore $\mathrm{cr}_2(L_1^3)(V, W) \cong V^{\otimes 2} \otimes W \oplus V \otimes W^{\otimes 2}$, and in particular

$$\mathrm{cr}_2(L_1^3)(A, A) \cong A \oplus A.$$

Next we want to compute $\mathrm{cr}_3(L_1^3)(A,A,A)$.

$$\begin{aligned}
\mathrm{cr}_2(L_1^3)(V, W\oplus X) \cong & V^{\otimes 2}\otimes (W\oplus X)\oplus V\otimes (W\oplus X)^{\otimes 2}\\
\cong & V^{\otimes 2}\otimes W\oplus V^{\otimes 2}\otimes X\\
& \oplus V\otimes W^{\otimes 2}\oplus V\otimes W\otimes X\\
& \oplus V\otimes X\otimes W\oplus V\otimes X^{\otimes 2}\\
\cong & V^{\otimes 2}\otimes W\oplus V\otimes W^{\otimes 2}\\
& \oplus V^{\otimes 2}\otimes X\oplus V\otimes X^{\otimes 2}\\
& \oplus V\otimes W\otimes X\oplus V\otimes X\otimes W.
\end{aligned}$$

Hence $\mathrm{cr}_3(L_1^3)(V,W,X)\cong V\otimes W\otimes X\oplus V\otimes X\otimes W$, and in particular

$$\mathrm{cr}_3(L_1^3)(A,A,A)\cong A\oplus A.$$

Since $\mathrm{cr}_1(L_1^3)(A)$ is the zero module it is clear that $\Delta_{(2)}:\mathrm{cr}_1(L_1^3)(A)\to \mathrm{cr}_2(L_1^3)(A,A)$ and $+_{(2)}:\mathrm{cr}_2(L_1^3)(A,A)\to \mathrm{cr}_1(L_1^3)(A)$ are zero maps.

We now calculate the relevant diagonal maps. Referring back to definition 2.6.1, $\mathrm{cr}_2(L_1^3)(\Delta_{(2,1)})$ is the composition

$$\mathrm{cr}_2(L_1^3)(V,W)\xrightarrow{\mathrm{cr}_2(L_1^3)(\Delta_V,\mathrm{id}_W)}\mathrm{cr}_2(L_1^3)(V\oplus V,W)\xrightarrow{\pi}\mathrm{cr}_3(L_1^3)(V,V,W)$$

We know that $\mathrm{cr}_2(L_1^3)(V,W)\cong V^{\otimes 2}\otimes W\oplus V\otimes W^{\otimes 2}$, $\mathrm{cr}_2(L_1^3)(V\oplus V,W)\cong (V\oplus V)^{\otimes 2}\otimes W\oplus (V\oplus V)\otimes W$ and $\mathrm{cr}_3(L_1^3)(V_1,V_2,W)\cong V_1\otimes V_2\otimes W\oplus V_1\otimes W\otimes V_2$. The first part of the composition, $\mathrm{cr}_2(L_1^3)(\Delta_V,\mathrm{id}_W)$, acts as follows

$$\begin{gathered}
V^{\otimes 2}\otimes W\oplus V\otimes W^{\otimes 2}\to (V\oplus V)^{\otimes 2}\otimes W\oplus (V\oplus V)\otimes W^{\otimes 2}\\
(v_1\otimes v_2\otimes w_1, v_3\otimes w_2\otimes w_3)\mapsto ((v_1,v_1)\otimes(v_2,v_2)\otimes w_1,(v_3,v_3)\otimes w_2\otimes w_3).
\end{gathered}$$

and the projection, π, acts by

$$(V \oplus V)^{\otimes 2} \otimes W \oplus (V \oplus V) \otimes W^{\otimes 2} \to V \otimes V \otimes W \oplus V \otimes W \otimes V$$
$$((v_1, v_1) \otimes (v_2, v_2) \otimes w_1, (v_3, v_3) \otimes w_2 \otimes w_3) \mapsto (v_1 \otimes v_2 \otimes w_1, v_1 \otimes w_1 \otimes v_2)$$

Now $\mathrm{cr}_2(L_1^3)(A, A) \cong A \oplus A$ and $\mathrm{cr}_3(L_1^3)(A, A, A) \cong A \oplus A$ and applying the above we see that:

$$\begin{aligned} \Delta_{(2,1)} : \mathrm{cr}_2(L_1^3)(R, R) &\to \mathrm{cr}_3(L_1^3)(R, R, R) \\ (1, 0) &\mapsto (1, 1) \\ (0, 1) &\mapsto (0, 0) \end{aligned}$$

and by symmetry we get our results for $\Delta_{(1,2)}$.

We now calculate the relevant plus maps. Referring back to definition 2.6.1 we see the map $\mathrm{cr}_2(L_1^3)(+_{(2,1)})$ is the composition

$$\mathrm{cr}_3(L_1^3)(V, V, W) \hookrightarrow \mathrm{cr}_2(L_1^3)(V \oplus V, W) \xrightarrow{\mathrm{cr}_2(L_1^3)(+_2, \mathrm{id}_W)} \mathrm{cr}_2(L_1^3)(V, W)$$

The first piece of the compostion acts by

$$\begin{aligned} V \otimes V \otimes W \oplus V \otimes W \otimes V &\to (V \oplus V)^{\otimes 2} \otimes W \oplus (V \oplus V) \otimes W^{\otimes 2} \\ (v_1 \otimes v_2 \otimes w_1, v_3 \otimes w_2 \otimes v_4) &\mapsto ((v_1, 0) \otimes (0, v_2) \otimes w_1, 0) \\ &\quad + ((0, v_3) \otimes (v_4, 0) \otimes w_2, 0) \end{aligned}$$

and $\mathrm{cr}_2(L_1^3)(+_2, \mathrm{id}_W)$, the second piece, acts by

$$\begin{aligned} (V \oplus V)^{\otimes 2} \otimes W \oplus (V \oplus V) \otimes W^{\otimes 2} &\to V^{\otimes 2} \otimes W \oplus V \otimes W^{\otimes 2} \\ ((v_1, 0) \otimes (0, v_2) \otimes w_1, 0) &\mapsto (v_1 \otimes v_2 \otimes w_1, 0) \\ ((0, v_3) \otimes (v_4, 0) \otimes w_2, 0) &\mapsto (v_3 \otimes v_4 \otimes w_2, 0). \end{aligned}$$

Applying this to $\mathrm{cr}_3(L_1^3)(A, A, A) \cong A \oplus A$ and $\mathrm{cr}_2(L_1^3)(A, A) \cong A \oplus A$ we see that:

$$\begin{aligned} +_{(2,1)} : \mathrm{cr}_3(L^3_1)(A, A, A) &\to \mathrm{cr}_3(L^3_1)(A, A) \\ (1,0) &\mapsto (1,0) \\ (0,1) &\mapsto (1,0) \end{aligned}$$

and by symmetry we get our result for $+_{(1,2)}$. □

Corollary 2.6.4. *The co-Schur functor* $\tilde{L}^1_3$ *is isomorphic to the Schur functor* L^1_3.

Proof. Let V and W finitely generated projective A-modules. Definition 2.3.6 tells us the co-Schur functor $\tilde{L}^3_1(V)$ is the image of d_2 in the complex $\mathrm{Kos}^3(\mathrm{id}_V)$, since this complex is exact the image of d_2 is the same as the kernel of d_1.

$$\begin{tikzcd} D^3(V) \arrow[r] & D^2(V) \otimes V \arrow[rr] \arrow[dr] & & V \otimes \Lambda^2(V) \arrow[r] & \Lambda^3(V) \\ & & \tilde{L}^3_1(V) \arrow[ur] \arrow[dr] & & \\ & 0 \arrow[ur] & & 0 & \end{tikzcd}$$

So we have the short exact sequence $0 \to \tilde{L}^3_1(V) \to V \otimes \Lambda^2(V) \to \Lambda^3(V) \to 0$. In particular this tells us that $\mathrm{cr}_1(\tilde{L}^3_1)(A) = L^3_1(A) = 0$.

Next we want to compute $\mathrm{cr}_2(\tilde{L}^3_1)(A, A)$. We have the short exact sequence $0 \to L^3_1(V \oplus W) \to (V \oplus W) \otimes \Lambda^2(V \oplus W) \to \Lambda^3(V \oplus W) \to 0$. Using the canonical decomposition of Λ^n we see that

$$\begin{aligned} (V \oplus W) \otimes \Lambda^2(V \oplus W) \cong& V \otimes \Lambda^2(V) \oplus V \otimes (V \otimes W) \oplus V \otimes \Lambda^2(W) \\ &\oplus W \otimes \Lambda^2(V) \oplus W \otimes (V \otimes W) \oplus W \otimes \Lambda^2(W) \end{aligned}$$

and

$$\Lambda^3(V \oplus W) \cong \Lambda^3(V) \oplus \Lambda^2(V) \otimes W \oplus V \otimes \Lambda^2(W) \otimes \Lambda^3(W).$$

Hence we get the following short exact sequence:

$$\begin{aligned} 0 \to &\operatorname{cr}_2(\tilde{L}^3_1)(V,W) \to \\ &V \otimes (V \otimes W) \oplus V \otimes \Lambda^2(W) \oplus W \otimes \Lambda^2(V) \oplus W \otimes (V \otimes W) \\ &\to \Lambda^2(V) \otimes W \oplus V \otimes \Lambda^2(W) \to 0. \end{aligned}$$

Therefore $\operatorname{cr}_2(\tilde{L}^3_1)(V,W) \cong V^{\otimes 2} \otimes W \oplus V \otimes W^{\otimes 2}$. From our previous calculation we know that $\operatorname{cr}_2(L^3_1)(V,W) \cong V^{\otimes 2} \otimes W \oplus V \otimes W^{\otimes 2}$, so $\operatorname{cr}_2(\tilde{L}^3_1)$ and $\operatorname{cr}_2(L^3_1)$ are isomorphic as bi-functors. Because of the way that higher cross-effects are calculated from lower cross-effects we see that for $k \geq 2$ $\operatorname{cr}_k(\tilde{L}^3_1)$ and $\operatorname{cr}(L^3_1)$ will be isomorphic as k-functors. We have shown that $\operatorname{cr}_1(\tilde{L}^3_1)(A) \cong \operatorname{cr}_1(L^3_1)(A) \cong 0$. From all these isomorphisms it is clear that we can construct all the maps neccessary to use Theorem 2.6.2, hence $\tilde{L}^3_1 \cong L^3_1$. □

2.7 The derived functors of the third symmetric power functor

Let R be a Noetherian commutative ring, let I be an ideal in R which is locally generated by a regular sequence of length 2 and let $\mathcal{P}_{R/I}$ be the category of projective R/I-modules. Let G_k be the functor defined as follows:

$$\begin{aligned} G_k : \mathcal{P}_{R/I} &\to R\textbf{-mod} \\ V &\mapsto H_k N \operatorname{Sym}^3 \Gamma P.(V) \end{aligned}$$

where $P.(V)$ is an R-projective resolution of V.

In Example 6.6 of [Kö] Köck made the following prediction about the derived functor G_k

$$G_k(V) \cong \begin{cases} \mathrm{Sym}^3(V) & k=0 \\ L_1^3(V) \otimes I/I^2 & k=1 \\ \tilde{L}_1^3(V) \otimes I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ D^3(V) \otimes \Lambda^2(I/I^2)^{\otimes 2} & k=4 \\ 0 & k \geq 5 \end{cases}$$

and for the case when $k=2$ he suggests that there exists an exact sequence:

$$0 \to D^2(V) \otimes V \otimes \Lambda^2(I/I^2) \to H_2 N \,\mathrm{Sym}^3 \Gamma(P.(V)) \\ \to \Lambda^3(V) \otimes \mathrm{Sym}^2(I/I^2) \to 0.$$

For any non-negative integer k that is not equal to 2 we let $F_k : \mathcal{P}_{R/I} \to R$-**mod** be the functor that Köck predicted for G_k to be. We let $F_2 : \mathcal{P}_{R/I} \to R$-**mod** be any functor that fits in a short exact sequence

$$0 \to D^2(V) \otimes V \otimes \Lambda^2(I/I^2) \to F_2(V) \to \Lambda^3(V) \otimes \mathrm{Sym}^2(I/I^2) \to 0.$$

Provided that I is globally generated by a regular sequence we prove that $G_k(R/I) \cong F_k(R/I)$ i.e. that these predictions hold if $V = R/I$. Moreover, regardless of whether I is globally generated or not, we prove a similar statement for the higher cross-effects of G_k and F_k namely that for all k and $l > 1$ we have

$$\mathrm{cr}_l(G_k)(R/I, \ldots, R/I) \cong \mathrm{cr}_l(F_k)(R/I, \ldots, R/I).$$

These results are a major step toward proving the predictions in general. What remains to be shown is that the diagrams described in Theorem 2.6.2 are commutative.

We first calculate the cross-effects of the predictions made by Köck.

Proposition 2.7.1. *We have the following* R/I *-module isomorphisms:*

$$F_k(R/I) \cong \begin{cases} R/I & k=0 \\ 0 & k=1 \\ \Lambda^2(I/I^2) & k=2 \\ 0 & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} & k=4 \end{cases}$$

$$\mathrm{cr}_2(F_k)(R/I,R/I)$$
$$\cong \begin{cases} R/I \oplus R/I & k=0 \\ I/I^2 \oplus I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2)^{\otimes 2} & k=4 \end{cases}$$

$$\mathrm{cr}_3(F_k)(R/I,R/I,R/I)$$
$$\cong \begin{cases} R/I & k=0 \\ I/I^2 \oplus I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus (I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2) \oplus \mathrm{Sym}^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} & k=4. \end{cases}$$

Proof. The functor F_0 is Sym^3. The canonical decomposition of Sym^3 gives us that $\mathrm{cr}_2(\mathrm{Sym}^3)(V,W) \cong \mathrm{Sym}^2(V) \otimes W \oplus V \otimes \mathrm{Sym}^2(W)$ and $\mathrm{cr}_3(\mathrm{Sym}^3)(V,W,X) \cong V \otimes W \otimes X$. So $\mathrm{Sym}^3(R/I) \cong R/I$, $\mathrm{cr}_2(F_0)(R/I,R/I) \cong R/I \oplus R/I$, and $\mathrm{cr}_3(F_0)(R/I,R/I,R/I) \cong R/I$.

The functor $F_1(-)$ is $L_1^3(-) \otimes I/I^2$. Proposition 2.6.3 tells us that: $F_1(R/I) = L_1^3(R/I) \otimes I/I^2 \cong 0$, $\mathrm{cr}_2(F_1)(R/I,R/I) = \mathrm{cr}_2(L_1^3)(R/I,R/I) \otimes I/I^2 \cong I/I^2 \oplus I/I^2$ and $\mathrm{cr}_3(F_1)(R/I,R/I,R/I) = \mathrm{cr}_3(L_1^3)(R/I,R/I,R/I) \otimes I/I^2 \cong I/I^2 \oplus I/I^2$.

The functor $F_3(-)$ is $\tilde{L}_1^3(-) \otimes I/I^2 \otimes \Lambda^2(I/I^2)$. Proposition 2.6.3 and Corollary 2.6.4 tell us that $F_3(R/I) = \tilde{L}_1^3(R/I) \otimes I/I^2 \otimes \Lambda^2(I/I^2) \cong 0$, $\mathrm{cr}_2(F_3)(R/I, R/I) = \mathrm{cr}_2(\tilde{L}_1^3)(R/I, R/I) \otimes I/I^2 \otimes \Lambda^2(I/I^2) \cong I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2)$ and that $\mathrm{cr}_3(\tilde{L}_1^3)(R/I, R/I, R/I) \otimes I/I^2 \otimes \Lambda^2(I/I^2) \cong I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2)$.

The functor $F_4(-)$ is $D^3(-) \otimes \Lambda^2(I/I^2)^{\otimes 2}$. Using the canonical decomposition of D^3 we get $\mathrm{cr}_2(D^3)(V, W) \cong D^2(V) \otimes W \oplus V \otimes D^2(W)$ and $\mathrm{cr}_3(D^3)(V, W, X) \cong V \otimes W \otimes X$. Hence $F_4(R/I) \cong \Lambda^2(I/I^2)^{\otimes 2}$, $\mathrm{cr}_2(F_4)(R/I, R/I) = (R/I \oplus R/I) \otimes \Lambda^2(I/I^2)^{\otimes 2} \cong \Lambda^2(I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2)^{\otimes 2}$ and $\mathrm{cr}_3(F_3)(R/I, R/I, R/I) \cong R/I \otimes \Lambda^2(I/I^2)^{\otimes 2} \cong \Lambda^2(I/I^2)^{\otimes 2}$.

Now for F_2 we expect a short exact sequence

$$0 \to D^2(V) \otimes V \otimes \Lambda^2(I/I^2) \to F_2(V) \to \Lambda^3(V) \otimes \mathrm{Sym}^2(I/I^2) \to 0$$

We let H stand for the functor $D^2(-) \otimes -$ and H' stand for the functor on the right hand side. The canonical decomposition of exterior powers gives us the cross-effects of H' as follows:

$$\begin{aligned} \mathrm{cr}_2(H')(V, W) &= (\Lambda^2(V) \otimes W \oplus V \otimes \Lambda^2(W)) \otimes \mathrm{Sym}^2(I/I^2) \\ \mathrm{cr}_3(H')(V, W, X) &= (V \otimes W \otimes X) \otimes \mathrm{Sym}^2(I/I^2). \end{aligned}$$

In particular we note that $\mathrm{cr}_2(H')(R/I, R/I)$ is the zero module and $\mathrm{cr}_3(H')(R/I, R/I, R/I) \cong \mathrm{Sym}^2(I/I^2)$.

We now calculate $H(V \oplus W)$ so we can calculate the $\mathrm{cr}_2(H)(V, W)$,

$$\begin{aligned} D^2(V \oplus W) \otimes (V \oplus W) \cong & D^2(V) \otimes V \oplus D^2(V) \otimes W \\ & \oplus V \otimes W \otimes V \oplus V \otimes W \otimes W \\ & \oplus D^2(W) \otimes V \oplus D^2(W) \otimes W \end{aligned}$$

hence

$$\mathrm{cr}_2(H)(V, W) \cong D^2(V) \otimes W \oplus V \otimes W \otimes V \oplus V \otimes W \otimes W \oplus D^2(W) \otimes V.$$

And similarly

$$\begin{aligned}\mathrm{cr}_2(H)(V, W \oplus X) \cong\, & D^2(V) \otimes X \oplus D^2(V) \otimes W \\ & \oplus V \otimes W \otimes V \oplus V \otimes X \otimes V \\ & \oplus V \otimes W \otimes W \oplus V \otimes X \otimes W \\ & \oplus V \otimes W \otimes X \oplus V \otimes X \otimes X \\ & \oplus D^2(W) \otimes V \oplus W \otimes X \otimes V \\ & \oplus D^2(X) \otimes W\end{aligned}$$

therefore

$$\mathrm{cr}_3(H)(V, W, X) \cong V \otimes X \otimes W \oplus V \otimes W \otimes X \oplus W \otimes X \otimes V.$$

Since $\mathrm{cr}_2(H')(R/I, R/I) \cong 0$ we get

$$\begin{aligned}\mathrm{cr}_2(F_2)(R/I, R/I) &\cong \mathrm{cr}_2(H)(R/I, R/I) \otimes \Lambda^2(I/I^2) \\ &\cong \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2).\end{aligned}$$

We also find that

$$\begin{aligned}\mathrm{cr}_3(F_2)(R/I, R/I, R/I) &\cong \mathrm{cr}_3(H)(R/I, R/I, R/I) \oplus \mathrm{cr}_3(H')(R/I, R/I, R/I) \\ &\cong \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \mathrm{Sym}^2(I/I^2).\end{aligned}$$

□

The following theorem shows that if I is globally generated by a regular sequence then the derived functors of Sym^3 evaluated on R/I matches Köck's predictions.

Theorem 2.7.2. *If I is globally generated by a regular sequence of length 2 then the module $G_k(R/I)$ is a free R/I-module of rank 1, for $k = 0, 2$ or 4 and otherwise of rank 0.*

Proof. Let f, g be a regular sequence in R, and let I be globally generated by it. Also we let $K.$ denote the Koszul complex $\ldots \to 0 \to R \xrightarrow{f} R$ and $L.$ denote the Koszul complex $\ldots \to 0 \to R \xrightarrow{g} R$.

By Proposition 2.2.4 we know that the Koszul complex

$$\operatorname{Tot}(K. \otimes L.) \cong \operatorname{Kos}(f, g) = \operatorname{Kos}^2(R \oplus R \overset{(f,g)}{\to} R)$$

is a resolution of R/I.

We see that:

$$\begin{aligned} G_k(R/I) :=& H_k N \operatorname{Sym}^3 \Gamma \operatorname{Tot}(K. \otimes L.) \\ \cong& H_k N \operatorname{Sym}^3 \Gamma \operatorname{Tot}(N\Gamma K. \otimes N\Gamma L.). \end{aligned}$$

Theorem 2.4.2 tells us that $\operatorname{Tot}(N\Gamma K. \otimes N\Gamma L.)$ is chain homotopic to $N\Delta(\Gamma K. \otimes \Gamma L.)$. Applying Γ turns the notion of chain homotopy into simplicial homotopy, all functors preserve homotopy in the simplicial world and N changes the notion of simplicial homotopy into the notion of chain homotopy. So $N \operatorname{Sym}^3 \Gamma$ turns the chain homotopy between $\operatorname{Tot}(N\Gamma K. \otimes N\Gamma L.)$ and $N\Delta(\Gamma K. \otimes \Gamma L.)$ into a chain homotopy between $N \operatorname{Sym}^3 \Gamma \operatorname{Tot}(N\Gamma K. \otimes N\Gamma L.)$ and $N \operatorname{Sym}^3 \Gamma N\Delta(\Gamma K. \otimes \Gamma L.)$. Chain homotopic complexes are quasi-isomorphic, so continuing our calculation of $G_k(R/I)$ where we left off we get:

$$\begin{aligned} G_k(R/I) &\cong H_k N \operatorname{Sym}^3 \Gamma N\Delta(\Gamma K. \otimes \Gamma L.) \cong H_k N \operatorname{Sym}^3 \Delta(\Gamma K. \otimes \Gamma L.) \\ &\cong H_k N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.). \end{aligned}$$

Now we calculate $G_k(R/I)$ by calculating $H_k N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.)$. We cannot calculate this directly, so instead we will employ the short exact sequences detailed in section 2.5 which will allow us to get information about these homologies from easier to calculate homologies. For any two finitely generated R-projective modules P, Q we have the following short exact sequences:

$$0 \to \Lambda^3 P \otimes \Lambda^3 Q \to M_{(2,1)}(\mathrm{Sym}^3(P \otimes Q)) \to L_1^3 P \otimes L_1^3 Q \to 0$$

$$0 \to M_{(2,1)}(\mathrm{Sym}^3(P \otimes Q)) \to \mathrm{Sym}^3(P \otimes Q) \to \mathrm{Sym}^3(P) \otimes \mathrm{Sym}^3(Q) \to 0.$$

(Note we are using L_1^3 rather than $L_{(2,1)}$; this is justified by Lemma 2.3.7.) Hence we get the following short exact sequences of bisimplicial modules

$$0 \to \Lambda^3 \Gamma K. \otimes \Lambda^3 \Gamma L. \to M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to L_1^3 \Gamma K. \otimes L_1^3 \Gamma L. \to 0$$

$$\begin{aligned} 0 \to M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) &\to \mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.) \\ &\to \mathrm{Sym}^3(\Gamma K.) \otimes \mathrm{Sym}^3(\Gamma L.) \to 0, \end{aligned}$$

applying $N\Delta$ to this gives us a short exact sequence of chain complexes. We can turn the homologies of these into two long exact sequences, this will allow us to get information about the homologies of $M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.))$ from the easier to calculate homologies of $\Lambda^3 \Gamma K. \otimes \Lambda^3 \Gamma L.$ and $L_1^3 \Gamma K. \otimes L_1^3 \Gamma L.$. This information about the homologies of $M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.))$ together with the homologies of the easier to calculate homologies of $\mathrm{Sym}^3(\Gamma K.) \otimes \mathrm{Sym}^3(\Gamma L.)$ will tell us the ranks of the homologies of $\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)$.

First we calculate the homologies of $L_1^3 \Gamma K$. The definition of L_1^3 gives us the following short exact sequence for any finitely generated projective module P

$$0 \to L_1^3 P \to P \otimes \mathrm{Sym}^2 P \to \mathrm{Sym}^3 P \to 0,$$

which gives us the short exact sequence of simplicial complexes

$$0 \to L_1^3 \Gamma K. \to \Gamma K. \otimes \mathrm{Sym}^2 \Gamma K. \to \mathrm{Sym}^3 \Gamma K. \to 0,$$

the middle term of this short exact sequence is the simplicial complex whose k^{th} term is $\Gamma K_k \otimes \mathrm{Sym}^2 \Gamma K_k$. We think of this middle term instead as the diagonal of a bisimplicial complex whose $(k,l)^{\text{th}}$ is $\Gamma K_k \otimes \mathrm{Sym}^2 \Gamma K_l$. Now applying the functor N turns this into a short exact sequence of chain complexes

$$0 \to NL_{(2,1)}\Gamma K. \to N\Delta(\Gamma K. \otimes \mathrm{Sym}^2 \Gamma K.) \to N \mathrm{Sym}^3 \Gamma K. \to 0,$$

and from this we can create a long exact sequence that gives us information about $H_k N L_1^3 \Gamma K.$.

Applying the Eilenberg-Zilber Theorem, then Proposition 2.2.10 and then Corollary 2.1.6 we get

$$\begin{aligned} H_k N\Delta(\Gamma K. \otimes \mathrm{Sym}^2 \Gamma K.) &\cong H_k \operatorname{Tot}(N\Gamma K. \otimes N \mathrm{Sym}^2 \Gamma K.) \\ &\cong H_k \operatorname{Tot}(K. \otimes \mathrm{Kos}^2(f)). \end{aligned}$$

Now

$$\begin{aligned} \mathrm{Kos}^2(f) &= (\Lambda^2(R) \to \Lambda^1(R) \otimes \mathrm{Sym}^1(R) \to \mathrm{Sym}^2(R)) \\ &= (0 \to R \xrightarrow{f} R) = K., \end{aligned}$$

and therefore

$$\begin{aligned} H_k N\Delta(\Gamma K. \otimes \mathrm{Sym}^2 \Gamma K.) &\cong H_k \operatorname{Tot}(K. \otimes K.) = H_k(P.(R/(f))^{\otimes 2}) \\ &= \Lambda^k((f)/(f)^2) \cong \begin{cases} R/(f) & k = 0, 1 \\ 0 & k > 1 \end{cases} \end{aligned}$$

with the last step given by Proposition 2.2.5.

Using Proposition 2.2.10 we get

$$H_k N \mathrm{Sym}^3 \Gamma K. \cong H_k(\mathrm{Kos}^3(f))$$

now

$$\begin{aligned}\mathrm{Kos}^3(f) &= (\Lambda^3(R) \to \Lambda^2(R) \otimes R \to \Lambda^1(R) \otimes \mathrm{Sym}^2(R) \to \mathrm{Sym}^3(R)) \\ &= (0 \to 0 \to R \xrightarrow{f} R) = K.,\end{aligned}$$

hence

$$H_k N \,\mathrm{Sym}^3 \,\Gamma K. \cong \begin{cases} R/(f) & k = 0 \\ 0 & k \neq 0. \end{cases}$$

So the long exact sequence of homologies that we get from the short exact sequence

$$0 \to NL_1^3\Gamma K. \to N\Delta(\Gamma K. \otimes \mathrm{Sym}^2 \Gamma K.) \to N \,\mathrm{Sym}^3 \,\Gamma K. \to 0$$

is

$$\begin{tikzcd}
{} \arrow[r, dotted] & H_3NL_1^3\Gamma K. \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
& H_2NL_1^3\Gamma K. \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
& H_1NL_1^3\Gamma K. \arrow[r] & R/(f) \arrow[r] & 0 \arrow[dll] \\
& H_0NL_1^3\Gamma K. \arrow[r] & R/(f) \arrow[r] & R/(f) \arrow[r] & 0
\end{tikzcd}$$

and so we get

$$H_k NL_1^3\Gamma K. \cong \begin{cases} 0 & k \neq 1 \\ R/(f) & k = 1. \end{cases}$$

Similarly we get

$$H_k NL_1^3\Gamma L. \cong \begin{cases} 0 & k \neq 1 \\ R/(g) & k = 1. \end{cases}$$

Now we work with the short exact sequence of simplicial modules

$$0 \to \Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L. \to M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to L_1^3\Gamma K. \otimes L_1^3\Gamma L. \to 0$$

the left and right terms are simplicial modules whose k^{th} terms are $\Lambda^3\Gamma K_k \otimes \Lambda^3\Gamma L_k$ and $L_1^3\Gamma K_k \otimes L_1^3\Gamma L_k$ respectively, but it is more useful for us to think of them as the diagonals of bisimplicial complexes whose $(k,l)^{\text{th}}$ term are $\Lambda^3\Gamma K_k \otimes \Lambda^3\Gamma L_l$ and $L_1^3\Gamma K_k \otimes L_1^3\Gamma L_l$ respectively. Now when we apply the functor N to get a short exact sequence of chain complexes we get

$$\begin{aligned} 0 \to N\Delta(\Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L.) &\to NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \\ &\to N\Delta(L_1^3\Gamma K. \otimes L_1^3\Gamma L.) \to 0. \end{aligned}$$

The Eilenberg-Zilber Theorem tells us that

$$H_k N\Delta(\Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L.) \cong H_k \operatorname{Tot}(N\Lambda^3\Gamma K. \otimes N\Lambda^3\Gamma L.)$$

and

$$H_k N\Delta(L_1^3\Gamma K. \otimes L_1^3\Gamma L.) \cong H_k \operatorname{Tot}(NL_1^3\Gamma K. \otimes NL_1^3\Gamma L.).$$

Now by Proposition 2.2.10 and Corollary 2.1.6 we get

$$H_k \operatorname{Tot}(N\Lambda^3\Gamma K. \otimes N\Lambda^3\Gamma L.) \cong H_k \operatorname{Tot}(\tilde{\mathrm{Kos}}^3(f) \otimes \tilde{\mathrm{Kos}}^3(g)).$$

Now

$$\begin{aligned} \tilde{\mathrm{Kos}}^3(f) &= (D^3(R) \to D^2(R) \otimes R \to R \otimes \Lambda^2(R) \to \Lambda^3(R)) \\ &= (R \xrightarrow{f} R \to 0 \to 0) = K.[-2], \end{aligned}$$

and similarly $\tilde{\mathrm{Kos}}^3(g) = L.[-2]$. So

$$\begin{aligned} H_k \operatorname{Tot}(N\Lambda^3\Gamma K. \otimes N\Lambda^3\Gamma L.) \cong & H_k \operatorname{Tot}(K.[-2] \otimes L.[-2]) \\ \cong & H_k(\operatorname{Tot}(K. \otimes L.)[-4]) \\ \cong & H_{k-4}(P.(R/I)) \cong \begin{cases} 0 & k \neq 4 \\ R/I & k = 4. \end{cases} \end{aligned}$$

Now to calculate $H_k \operatorname{Tot}(NL_1^3\Gamma K. \otimes NL_1^3\Gamma L.)$ we note that it is the hypertor functor $\mathbf{Tor}_k(NL_1^3\Gamma K., NL_1^3\Gamma L.)$ and use the following spectral sequence given in Proposition 2.1.5

$$^{II}E^2_{p,q} = \bigoplus_{q=q'+q''} \operatorname{Tor}_p(H_{q'}(A_*), H_{q''}(B_*)) \Rightarrow \mathbf{Tor}(A_*, B_*)$$

taking $A_* = NL_1^3\Gamma K_*$ and $B_* = NL_1^3\Gamma L_*$. But since $H_k NL_1^3\Gamma K.$ and $H_k NL_1^3\Gamma L.$ are 0 unless $k=1$ (see above) this spectral sequence collapses, with the only (potentially) non-zero terms being when $q' = q'' = 1$, i.e. when $q = 2$. These (potentially) non-zero terms are $\operatorname{Tor}_p(R/(f), R/(g))$. Taking $L.$ as a projective resolution of $R/(f)$ then tensoring throughout by $R/(g)$ we get the chain complex

$$\big(0 \to R \otimes R/(g) \xrightarrow{f} R \otimes R/(g)\big) = \big(0 \to R/(g) \xrightarrow{f} R/(g)\big)$$

which has homology R/I at the 0^{th} place and 0 everywhere else. And so

$$H_k\big(\operatorname{Tot}(NL_1^3\Gamma K. \otimes NL_1^3\Gamma L.)\big) \cong \operatorname{Tor}_{k-2}(R/(f), R/(g)) \cong \begin{cases} 0 & k \neq 2 \\ R/I & k = 2. \end{cases}$$

So the short exact sequence of chain complexes

$$\begin{aligned} 0 \to N\Delta(\Lambda^3\Gamma K. \otimes \Lambda^3\Gamma L.) \to & NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \\ & \to N\Delta(L_1^3\Gamma K. \otimes L_1^3\Gamma L.) \to 0 \end{aligned}$$

gives rise to the following long exact sequence of homologies

$$
\begin{tikzcd}
\cdots \arrow[r, dotted] & 0 \arrow[r] & H_5NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \arrow[r] & 0 \arrow[dll, rounded corners] & \\
 & R/I \arrow[r] & H_4NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \arrow[r] & 0 \arrow[dll, rounded corners] & \\
 & 0 \arrow[r] & H_3NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \arrow[r] & 0 \arrow[dll, rounded corners] & \\
 & 0 \arrow[r] & H_2NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \arrow[r] & R/I \arrow[dll, rounded corners] & \\
 & 0 \arrow[r] & H_1NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \arrow[r] & 0 \arrow[dll, rounded corners] & \\
 & 0 \arrow[r] & H_0NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

and therefore we get

$$
H_kNM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \cong \begin{cases} R/I & k = 2, 4 \\ 0 & \text{otherwise.} \end{cases}
$$

Now we work with the short exact sequence of simplicial modules

$$
\begin{aligned}
0 \to M_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to \mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)& \\
\to \mathrm{Sym}^3(\Gamma K.) \otimes \mathrm{Sym}^3(\Gamma L.) \to 0&
\end{aligned}
$$

the term $\mathrm{Sym}^3(\Gamma K.) \otimes \mathrm{Sym}^3(\Gamma L.)$ is a simplicial module whose k^{th} place is $\mathrm{Sym}^3(\Gamma K_k) \otimes \mathrm{Sym}^3(\Gamma L_k)$, but as above it is more useful to think of it as the diagonal of the bisimplicial complex whose $(k, l)^{\text{th}}$ place is $\mathrm{Sym}^3(\Gamma K_k) \otimes \mathrm{Sym}^3(\Gamma L_l)$. Applying the functor N we get the following short exact sequence of chain complexes

$$0 \to NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to N\,\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.) \\ \to N\Delta(\mathrm{Sym}^3(\Gamma K.) \otimes \mathrm{Sym}^3(\Gamma L.)) \to 0.$$

Applying the Eilenberg-Zilber Theorem, Proposition 2.2.10 and Corollary 2.1.6 we see

$$\begin{aligned} H_k N\Delta(\mathrm{Sym}^3(\Gamma K.) \otimes \mathrm{Sym}^3(\Gamma L.)) &\cong H_k \operatorname{Tot}(N\,\mathrm{Sym}^3(\Gamma K.) \otimes N\,\mathrm{Sym}^3(\Gamma L.)) \\ &\cong H_k \operatorname{Tot}(\mathrm{Kos}^3(f) \otimes \mathrm{Kos}^3(g)). \end{aligned}$$

Earlier in this proof we showed that $\mathrm{Kos}^3(f) = K.$ and similarly $\mathrm{Kos}^3(g) = L.$, so

$$H_k \operatorname{Tot}(\mathrm{Kos}^3(f) \otimes \mathrm{Kos}^3(g)) \cong H_k(\operatorname{Tot}(K. \otimes L.)) = H_k(P.(R/I)).$$

Hence

$$H_k N\Delta(\mathrm{Sym}^3\Gamma(K.) \otimes \mathrm{Sym}^3\Gamma(L.)) \cong \begin{cases} R/I & k = 0 \\ 0 & k \neq 0. \end{cases}$$

And so the short exact sequence of chain complexes

$$0 \to NM_{(2,1)}(\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.)) \to N\,\mathrm{Sym}^3(\Gamma K. \otimes \Gamma L.) \\ \to N\Delta(\mathrm{Sym}^3(\Gamma K.) \otimes \mathrm{Sym}^3(\Gamma L.)) \to 0,$$

gives rise to the following long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & 0 \arrow[r] & H_5 N\Delta \operatorname{Sym}^3(\Gamma(K.) \otimes \Gamma(L.)) \arrow[r] & 0 \arrow[dll] & & \\
& R/I \arrow[r] & H_4 N\Delta \operatorname{Sym}^3(\Gamma(K.) \otimes \Gamma(L.)) \arrow[r] & 0 \arrow[dll] & & \\
& 0 & H_3 N\Delta \operatorname{Sym}^3(\Gamma(K.) \otimes \Gamma(L.)) \arrow[r] & 0 \arrow[dll] & & \\
& R/I \arrow[r] & H_2 N\Delta \operatorname{Sym}^3(\Gamma(K.) \otimes \Gamma(L.)) \arrow[r] & 0 \arrow[dll] & & \\
& 0 \arrow[r] & H_1 N\Delta \operatorname{Sym}^3(\Gamma(K.) \otimes \Gamma(L.)) \arrow[r] & 0 \arrow[dll] & & \\
& 0 \arrow[r] & H_0 N\Delta \operatorname{Sym}^3(\Gamma(K.) \otimes \Gamma(L.)) \arrow[r] & R/I \arrow[r] & 0. &
\end{tikzcd}$$

And hence (as we know $G_k(R/I) \cong H_k N\Delta \operatorname{Sym}^3(\Gamma K. \otimes \Gamma L.))$ we see that:

$$G_k(R/I) \cong \begin{cases} R/I & k = 0, 2, 4 \\ 0 & \text{otherwise,} \end{cases}$$

as desired. □

The following theorem shows that then the second cross-effect functor of the derived functors of Sym^3 evaluated on $(R/I, R/I)$ matches Köck's predictions.

Theorem 2.7.3.

$$\mathrm{cr}_2(G_k)(R/I, R/I) \cong \begin{cases} R/I \oplus R/I & k = 0 \\ I/I^2 \oplus I/I^2 & k = 1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k = 2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k = 3 \\ \Lambda^2(I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2)^{\otimes 2} & k = 4 \\ 0 & k \geq 5. \end{cases}$$

Proof. First we calculate $G_k(V \oplus W)$, for R-modules V, W to give us an expression for $\mathrm{cr}_2(G_k)(V, W)$. To do this we use the fact that $P., \Gamma, N$ and H_k are linear functors and also the canonical decomposition $\mathrm{Sym}^n(V \oplus W) \cong \bigoplus_{l=0}^n \mathrm{Sym}^{n-l}(V) \otimes \mathrm{Sym}^l(W)$.

$$\begin{aligned} G_k(V \oplus W) =& H_k N \,\mathrm{Sym}^3 \Gamma P.(V \oplus W) \\ \cong& H_k N \,\mathrm{Sym}^3(\Gamma P.(V) \oplus \Gamma P.(W)) \\ \cong& H_k N\Big(\mathrm{Sym}^3 \Gamma P.(V) \oplus \mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W) \\ & \oplus \Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W) \oplus \mathrm{Sym}^3 \Gamma P.(W)\Big) \\ \cong& H_k N(\mathrm{Sym}^3 \Gamma P.(V)) \oplus H_k N(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)) \\ & \oplus H_k N(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W)) \oplus H_k N \,\mathrm{Sym}^3 \Gamma P.(W) \\ \cong& G_k(V) \oplus H_k N(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)) \\ & \oplus H_k N(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W)) \oplus G_k(W). \end{aligned}$$

And hence

$$\begin{aligned} \mathrm{cr}_2(G_k)(V, W) \cong& H_k N(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)) \\ & \oplus H_k N(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W)). \end{aligned}$$

In the above when we wrote $\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)$ this denoted the simplicial modules whose n^{th} place is $\mathrm{Sym}^2 \Gamma P_n(V) \otimes \Gamma P_n(W)$, we can consider this simplicial module to be the diagonal of the bisimplicial modules whose $(n,m)^{\text{th}}$ place is $\mathrm{Sym}^2\Gamma P_n(V)\otimes\Gamma P_m(W)$. For the rest of the calculation we write $\mathrm{Sym}^2\Gamma P.(V)\otimes\Gamma P.(W)$ and $\Gamma P.(W)\otimes\mathrm{Sym}^2\Gamma P.(W)$ for the bisimplical modules and consider $\mathrm{cr}_2(G_k)(V,W)$ to be

$$H_kN\Delta\big(\mathrm{Sym}^2\Gamma P.(V)\otimes\Gamma P.(W)\big)\oplus H_kN\Delta\big(\Gamma P.(V)\otimes\mathrm{Sym}^2\Gamma P.(W)\big).$$

We will now calculate $H_kN\Delta\big(\mathrm{Sym}^2\Gamma P.(V)\otimes\Gamma P.(W)\big)$. Using the Eilenberg-Zilber Theorem (Theorem 2.4.2) we see that

$$\begin{aligned} H_kN\Delta(\mathrm{Sym}^2(\Gamma P.(V)\otimes\Gamma P.(W)) &\cong H_k\,\mathrm{Tot}(N\,\mathrm{Sym}^2\Gamma P.(V)\otimes N\Gamma P.(W))\\ &\cong H_k\,\mathrm{Tot}(N\,\mathrm{Sym}^2\Gamma P.(V)\otimes P.(W)). \end{aligned}$$

So we want to calculate $H_k\,\mathrm{Tot}(N\,\mathrm{Sym}^2\Gamma P.(V)\otimes P.(W))$, but this is just the definition of the hypertor $\mathbf{Tor}_i^R(N\,\mathrm{Sym}^2\Gamma P.(V),W)$. Proposition 2.1.5 gives us a spectral sequence to calculate hypertor

$${}^{II}E^2_{pq}=\mathrm{Tor}_p(H_q(A),B)\Rightarrow\mathbf{Tor}^R_{p+q}(A_*,B).$$

Theorem 6.4 of [Kö] tells us that

$$H_kN\,\mathrm{Sym}^2\Gamma(P.(V))\cong\begin{cases}\mathrm{Sym}^2(V) & k=0\\ \Lambda^2(V)\otimes I/I^2 & k=1\\ D^2(V)\otimes\Lambda^2(I/I^2) & k=2\\ 0 & k\geq 3.\end{cases}$$

Now $\mathrm{Sym}^2(R/I)\cong R/I, D^2(R/I)\cong R/I$ and $\Lambda^2(R/I)\cong 0$. Hence the 2^{nd}

level of our spectral sequence for $\mathbf{Tor}_i^R(N\,\mathrm{Sym}^2\,\Gamma P.(R/I), R/I)$ is as follows.

$$\begin{array}{ccccc}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & \mathrm{Tor}_2(\Lambda^2(I/I^2), R/I) & 0 & \mathrm{Tor}_2(R/I, R/I) \\
\cdots & 0 & \mathrm{Tor}_1(\Lambda^2(I/I^2), R/I) & 0 & \mathrm{Tor}_1(R/I, R/I) \\
\cdots & 0 & \mathrm{Tor}_0(\Lambda^2(I/I^2), R/I) & 0 & \mathrm{Tor}_0(R/I, R/I).
\end{array}$$

Now from Proposition 2.2.5 we know that $\mathrm{Tor}_k(R/I, R/I) \cong \Lambda^k(I/I^2)$ and hence have $\mathrm{Tor}_k(V, W) \cong V \otimes W \otimes \Lambda^k(I/I^2)$. So the 2nd level of the spectral sequence looks like this:

$$\begin{array}{ccccc}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & \Lambda^2(I/I^2)^{\otimes 2} & 0 & \Lambda^2(I/I^2) \\
\cdots & 0 & I/I^2 \otimes \Lambda^2(I/I^2) & 0 & I/I^2 \\
\cdots & 0 & \Lambda^2(I/I^2) & 0 & R/I.
\end{array}$$

The differentials on this level of the spectral sequence are -2 in the p-direction and $+1$ in the q-direction, so each differential either comes from or goes to a zero module. Hence the differentials are all the zero map i.e. the spectral sequence has already converged on the second level.

Therefore

$$H_k N\Delta\big(\operatorname{Sym}^2 \Gamma P.(R/I) \otimes \Gamma P.(R/I)\big) \cong \begin{cases} R/I & k=0 \\ I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} & k=4 \\ 0 & k \geq 5. \end{cases}$$

By symmetry $H_k N\Delta\big(\operatorname{Sym}^2 \Gamma P.(R/I) \otimes \Gamma P.(R/I)\big) \cong H_k N\Delta\big(\Gamma P.(R/I) \otimes \operatorname{Sym}^2 \Gamma P.(R/I)\big)$. Hence

$$\mathrm{cr}_2(G_k)(R/I, R/I) \cong \begin{cases} R/I \oplus R/I & k=0 \\ I/I^2 \oplus I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) \oplus \Lambda^2(I/I^2) & k=2 \\ I/I^2 \otimes \Lambda^2(I/I^2) \oplus I/I^2 \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2)^{\otimes 2} \oplus \Lambda^2(I/I^2)^{\otimes 2} & k=4 \\ 0 & k \geq 5. \end{cases}$$

□

The following theorem shows that the third cross-effect functor of the derived functors of Sym^3 evaluated on $(R/I, R/I, R/I)$ matches Köck's predictions.

Theorem 2.7.4.

$$\mathrm{cr}_3(G_k)(R/I, R/I, R/I) \cong \Lambda^k(I/I^2 \oplus I/I^2)$$

Proof. We first calculate $\mathrm{cr}_2(G_k)(V, W \oplus X)$, for R-modules V, W, X to give us an expression for $\mathrm{cr}_3(G_k)(V, W, X)$ (compare the following with the above calculation of $\mathrm{cr}_2(G_k)(V, W)$ in Theorem 2.7.3). To simplify our calculation of $\mathrm{cr}_2(G_k)(V, W\oplus X)$ we split up our expression for $\mathrm{cr}_2(G_k)(V, W\oplus X)$ into $H_kN\Delta\big(\mathrm{Sym}^2\,\Gamma P.(V)\otimes\Gamma P.(W\oplus X)\big)$ and $H_kN\Delta\big(\Gamma P.(W)\otimes\mathrm{Sym}^2\,\Gamma P.(W\oplus X)\big)$, calculate each part separately then add them together afterwards.

$$\begin{aligned}
&H_kN\Delta\Big(\mathrm{Sym}^2\,\Gamma P.(V)\otimes\Gamma P.(W\oplus X)\Big)\\
&\qquad\cong H_kN\Delta\Big(\mathrm{Sym}^2\,\Gamma P.(V)\otimes(\Gamma P.(W)\oplus\Gamma P.(X))\Big)\\
&\qquad\cong H_kN\Delta\Big(\mathrm{Sym}^2\,\Gamma P.(V)\otimes\Gamma P.(W)\Big)\\
&\qquad\quad\oplus H_kN\Delta\Big(\mathrm{Sym}^2\,\Gamma P.(V)\otimes\Gamma P.(X)\Big).
\end{aligned}$$

$$\begin{aligned}
&H_kN\Delta\Big(\Gamma P.(V)\otimes\mathrm{Sym}^2\,\Gamma P.(W\oplus X)\Big)\\
&\qquad\cong H_kN\Delta\Big(\Gamma P.(V)\otimes\mathrm{Sym}^2\,\big(\Gamma P.(W)\oplus\Gamma P.(X)\big)\Big),
\end{aligned}$$

the canonical decomposition of Sym^2 tells us

$$\mathrm{Sym}^2(\Gamma P.W\oplus\Gamma P.X)\cong\mathrm{Sym}^2\,\Gamma P.(W)\oplus\big(\Gamma P.(W)\otimes\Gamma P.(X)\big)\oplus\mathrm{Sym}^2\,\Gamma P.(X)$$

(where $\Gamma P.(W)\otimes\Gamma P.(X)$ is the simplicial module whose n^{th} place is $\Gamma P_n(W)\otimes\Gamma P_n(X)$) and so we see

$$\begin{aligned}
&H_kN\Delta\Big(\Gamma P.(V)\otimes\mathrm{Sym}^2\,\big(\Gamma P.(W)\oplus\Gamma P.(X)\big)\Big)\\
&\qquad\cong H_kN\Delta\Big(\Gamma P.(V)\otimes\mathrm{Sym}^2\,\Gamma P.(W))\Big)\\
&\qquad\quad\oplus H_kN\Delta\Big(\Gamma P.(V)\otimes\big(\Gamma P.(W)\otimes\Gamma P.(X)\big)\Big)\\
&\qquad\quad\oplus H_kN\Delta\Big(\Gamma P.(V)\otimes\mathrm{Sym}^2\,\Gamma P.(X)\Big).
\end{aligned}$$

This gives us the following expression for $\mathrm{cr}_2(G_k)(V, W \oplus X)$

$$\begin{aligned}
\mathrm{cr}_2(G_k)(V, W \oplus X) \cong & H_k N\Delta\Big(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)\Big) \\
& \oplus H_k N\Delta\Big(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(X)\Big) \\
& \oplus H_k N\Delta\Big(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W)) \\
& \oplus H_k N\Delta\Big(\Gamma P.(V) \otimes \big(\Gamma P.(W) \otimes \Gamma P.(X)\big)\Big) \\
& \oplus H_k N\Delta\Big(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(X)\Big) \\
\cong & H_k N\Delta\Big(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(W)\Big) \\
& \oplus H_k N\Delta\Big(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(W)\Big) \\
& \oplus H_k N\Delta\Big(\mathrm{Sym}^2 \Gamma P.(V) \otimes \Gamma P.(X)\Big) \\
& \oplus H_k N\Delta\Big(\Gamma P.(V) \otimes \mathrm{Sym}^2 \Gamma P.(X)\Big) \\
& \oplus H_k N\Delta\Big(\Gamma P.(V) \otimes \big(\Gamma P.(W) \otimes \Gamma P.(X)\big)\Big) \\
\cong & \,\mathrm{cr}_2(G_k)(V, W) \oplus \mathrm{cr}_2(G_k)(V, X) \\
& \oplus H_k N\Delta\Big(\Gamma P.(V) \otimes \big(\Gamma P.(W) \otimes \Gamma P.(X)\big)\Big).
\end{aligned}$$

Therefore $\mathrm{cr}_3(G_k)(V, W, X) \cong H_k N\Delta\Big(\Gamma P.(V) \otimes \big(\Gamma P.(W) \otimes \Gamma P.(X)\big)\Big)$. However we may consider $\Delta\Big(\Gamma P.(V) \otimes \big(\Gamma P.(W) \otimes \Gamma P.(X)\big)\Big)$ to be the diagonal of the trisimplicial complex $\Gamma P.(V) \otimes \Gamma P.(W) \otimes \Gamma P.(X)$, so we write

$$\mathrm{cr}_3(G_k)(V, W, X) \cong H_k N\Delta\big(\Gamma P.(V) \otimes \Gamma P.(W) \otimes \Gamma P.(X)\big).$$

The Iterated Eilenberg-Zilber (Theorem 2.4.2) tells us that

$$H_k N\Delta\big(\Gamma P.(R/I) \otimes \Gamma P.(R/I) \otimes \Gamma P.(R/I)\big) \cong H_k \,\mathrm{Tot}\,\big(P.(R/I) \otimes P.(R/I) \otimes P.(R/I)\big)$$

Proposition 2.2.5 tells us that

$$H_k \,\mathrm{Tot}\,\big(P.(R/I) \otimes P.(R/I) \otimes P.(R/I)\big) \cong \Lambda^k(I/I^2 \oplus I/I^2),$$

hence we see that

$$\mathrm{cr}_3(G_k)(R/I, R/I, R/I) \cong \begin{cases} R/I & k=0 \\ I/I^2 \oplus I/I^2 & k=1 \\ \Lambda^2(I/I^2) \oplus \big((I/I^2) \otimes (I/I^2)\big) \oplus \Lambda^2(I/I^2) & k=2 \\ \Lambda^2(I/I^2) \otimes (I/I^2) \oplus (I/I^2) \otimes \Lambda^2(I/I^2) & k=3 \\ \Lambda^2(I/I^2) \otimes \Lambda^2(I/I^2) & k=4 \\ 0 & k \geq 5. \end{cases}$$

□

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