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Discrete Lifetime Data

by

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ABSTRACT

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In addressing quality issues in an industrial, manufacturing or engineering context, inherently non-negative measures of quality are often used. Statistical methodology for reliability analysis is well developed, at least in the situation where the quality measure is continuous. However, the quality measure may be discrete either through the method of observation or the countable nature of the data.

There are very clear theoretical differences between continuous and discrete reliability methods. For example the simple relationship between the failure rate function and the reliability function in the continuous case does not hold in the discrete case. Also, some naïve statistical methods can be misleading in the discrete case. For example, a plot of the empirical failure rate will always suggest that the failure rate eventually increases even when the true failure rate is non-increasing.

The theoretical differences between continuous and discrete reliability methods are highlighted. The basic properties of some discrete distributions that are potentially useful for lifetime data analysis are presented and the general behaviour of the failure rate function is investigated. Two exact distributions for an empirical failure rate estimator of discrete lifetimes are proposed. The use of the two proposed distributions is illustrated by introducing a new method, the failure rate control chart, to detect departures from a constant failure rate. The properties of the proposed method and other related tests are investigated by simulation.

In dealing with failure time data it is common that there are two or more failure modes, the competing risks phenomenon. The observed data in such a situation will comprise the failure times together with the relevant failure mode for each failure time. In trying to understand the failure process it is reasonable to ask whether the failure modes are acting independently. In the case of continuous failure times, it is well known that it is not possible to answer this question with such data. This is the so-called identifiability crisis in competing risks. However, when the failure times are discrete the hypothesis of independence can sometimes be addressed. Crowder, in his Lifetime Data Analysis paper of 1997, proposed a test in such circumstances, derived its large sample null properties and illustrated its use on a medical data set.

The results presented by Crowder are summarised and corrected, and some potential practical shortcomings of his test are discussed. Also, a modified version of the Crowder test is proposed. Simplified forms of Crowder's test statistic and its modified version are proposed. Finally, by recasting Crowder's results in terms of classical contingency tables, the relationship of his test to other tests of independence is highlighted and the properties are investigated by simulation.

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Chapter 1

Review of Some Discrete Lifetime Distributions

Statistical methodology for reliability analysis is well developed, at least in situations where the quality measure is continuous. However, the quality measure may be discrete either through the method of observation or the countable nature of the data. There are very clear theoretical differences between continuous and discrete reliability methods. For example the simple relationship between the failure rate function and the reliability function in the continuous case does not hold in the discrete case. In this chapter the basic properties of some discrete distributions that are potentially useful for lifetime data analysis are discussed and the general behaviour of the failure rate function is investigated.

1.1 Introduction

The statistical analysis of lifetime data is a very important field in many areas such as engineering, biomedical, and social sciences where the time taken for an event from a fixed time origin is of interest. Examples include time to failure of a component, time to alleviation of symptoms after treatment and the time spent by a migrant worker in a foreign country. In addition, inherently non-negative measures are often used when addressing quality issues in an industrial, manufacturing or engineering context. Examples include the strength of cord, the depth of penetration of carbonation in reinforced concrete and the time to failure of an electronic component. A natural approach is to model the quality measures parametrically using distributions that are usually associated with survival analysis, even though the measures of interest may not be times.

Lifetime data have other names such as time to event, time to failure, reliability, or survival data. Usually the term reliability is used for engineering time to failure data. Reliability can be defined as the ability to preserve the quality of performance of a subject under given conditions for a given period of time. Statistically, it can be defined as the probability that a subject will perform its intended function without failure beyond a certain time (Hoyland and Rausand, 1994, Lawless, 2003).

Statistical methodology for reliability is developing rapidly, especially for continuous lifetimes even though most of the data available are in effect discrete. Discrete lifetimes arise either through the method of observation or the countable nature of the data. Thus, lifetimes may be continuous but measured discretely through grouping or rounding (e.g. inherently continuous lifetimes that are only observed hourly, or strengths obtained through testing using discretely increasing loads) or they may be genuinely discrete (e.g. where the failure time is the number of cycles to failure) (Grimshaw et al., 2005, Lai and Xie, 2006, Lawless, 2003).

The distributions of lifetimes can be characterized by three functions that are uniquely determined by each other (Hoyland and Rausand, 1994, Lee, 1992):

$f(k)$: The probability density (or mass) function, the probability of a subject failure at time k, discrete, or in the short interval $(k, k+\Delta k)$, continuous, per unit time.

$R(k)$: The reliability function, the probability of a subject survival beyond time k.

$h(k)$: The failure rate function, the conditional probability of a subject failure at time k, discrete, or the instantaneous rate of a subject failure at time k, continuous, given that it did not fail before.

Here K, the underlying lifetime random variable, which need not be measured in calendar time, is defined over the set of positive real numbers, IR^+, in the continuous case and over the set of positive integers, IN^+, in the discrete case. The relationships between the three above functions are summarized in Table 1.1.

There are clear theoretical differences between continuous and discrete reliability methods. For example, the simple relationship between the failure rate function and the reliability function in the continuous case does not hold in the discrete case (Bracquemond et al., 2001, Xie et al., 2002).

Table 1.1: Summary of the relationships between $f(k)$, $R(k)$, and $h(k)$.

Discrete / Continuous	$f(k)$	$R(k)$	$h(k)$
$f(k)$	-	$R(k-1)-R(k)$	$h(k)\prod_{i=1}^{k}[1-h(i)]$
	-	$-\frac{d}{dk}R(k)$	$h(k)e^{-\int_0^k h(u)du}$
$R(k)$	$\sum_{i=k+1}^{\infty} f(i)$	-	$\prod_{i=1}^{k}[1-h(i)]$
	$\int_k^{\infty} f(u)du$	-	$e^{-\int_0^k h(u)du}$
$h(k)$	$\frac{f(k)}{\sum_{i=k}^{\infty} f(i)}$	$\frac{R(k-1)-R(k)}{R(k-1)}$	-
	$\frac{f(k)}{\int_k^{\infty} f(u)du}$	$-\frac{d}{dk}\ln R(k)$	-

The distribution of lifetimes can be characterized by three functions as we mentioned before and in the next three sections, sections 1.2-1.4, we will discuss those functions for the cases of continuous and discrete lifetimes. The basic properties of some discrete distributions that are potentially useful for lifetime data analysis will be discussed in section 1.5. Then, a discussion of this chapter will be presented in section 1.6.

1.2 The Probability Density (or Mass) Function

1.2.1 Continuous Lifetimes

The probability density function (pdf) of a continuous random variable K, $f(k)$, is defined as

$$f(k)=\lim_{\Delta k\to 0}\frac{P(k<K\le k+\Delta k)}{\Delta k}\quad \forall k\in IR^{+}.$$

The cumulative distribution function is defined by $F(k) = P(K \le k) = \int_0^k f(u)du$.

1.2.2 Discrete Lifetimes

A random variable K is said to be discrete if it takes a finite number of different values $k_1, k_2, ..., k_t$ or a countably infinite sequence of different values $k_1, k_2, k_3, ...$. Thus, the range of K is countable. From now on we will assume that K can take positive integer values 1, 2, ... only. The probability mass function (pmf) of a discrete integer-valued random variable K, $f(k)$, is defined as

$$f(k) = P(K = k) \quad \forall k \in IN^+.$$

The cumulative distribution function in the discrete case is defined by

$$F(k) = P(K \le k) = \sum_{i=1}^{k} P(K = i).$$

1.3 The Reliability Function

The probability that a subject is still functioning beyond time k is defined as

$$R(k) = P(K > k) \quad \forall k \ge 0.$$

This function is a non-increasing function with $R(0) = 1$ and $R(\infty) = 0$.

In the case of continuous lifetimes, the reliability function, $R(k)$, is a continuous monotone decreasing function,

$$R(k) = 1 - F(k) = P(K > k) = \int_k^{\infty} f(x)dx = 1 - \int_0^k f(x)dx.$$

In the case of discrete lifetimes, $R(k)$ is a non increasing left-continuous step function,

$$R(k) = 1 - F(k) = P(K > k) = \sum_{i=k+1}^{\infty} f(i) = 1 - \sum_{i=1}^{k} f(i).$$

1.4 The Failure Rate Function

The failure rate, FR, function has other names such as the force of mortality, the age-specific failure rate, the inverse of the Mills' ratio or the hazard function. It is a non-negative function. This function is useful in describing the way in which the conditional probability of failure changes over time. It displays the risks of failure associated with each point or interval of time given survival up to that time. When studying risk in terms of failure rates, we can identify when events are more or less likely to occur: peaks mean high risk and troughs mean low risk.

Typically a FR function will comprise some combination of the following three components: (a) an infant mortality component where the FR decreases from an initial high value (b) a phase in which the FR function is flat so that failure tends to occur at random (c) a wear out phase in which the FR increases with time due to aging. Common choices of models for lifetime data have increasing (IFR), decreasing (DFR), upturned bathtub shaped and bathtub shaped FR functions. The flat, or constant, FR function model may be thought of as the boundary between the IFR and DFR models. Examples in which these models might be appropriate are: time from puberty to onset of a degenerative disease such as osteoarthritis (IFR), time to failure of a component where the main risk of failure is in a burn-in period (DFR), time from transplantation to rejection of the transplanted organ (upturned bathtub), and time to failure of system (bathtub). The bathtub shaped FR is particularly appropriate in population studies (Kemp, 2004, Lawless, 2003, Singer and Willett, 2003).

1.4.1 Continuous Lifetimes

The instantaneous rate of a subject failure at time k, given that it did not fail before is

$$h(k) = \lim_{\Delta k \to 0} \frac{P(k < K \le k + \Delta k \mid K > k)}{\Delta k} = \frac{f(k)}{R(k)} = -\frac{d \ln[R(k)]}{dk} = -\frac{R'(k)}{R(k)}.$$

The cumulative failure rate function is defined by $H(k) = \int_0^k h(u)du = -\ln R(k)$.

The FR function is additive for series systems. To see this (Tobias and Trindade, 1995), suppose that we have a system that consists of n independent components in series with $R_i(k)$ and $h_i(k)$ being respectively the reliability and the FR function of component i, then the system reliability is $R(k)=\prod_{i=1}^{n} R_i(k)$. So,

$$-\ln R(k) = -\ln \prod_{i=1}^{n} R_i(k) = \sum_{i=1}^{n} -\ln R_i(k).$$

Therefore,

$$h(k) = \frac{d}{dk}[-\ln R(k)] = \sum_{i=1}^{n} \frac{d}{dk}[-\ln R_i(k)] = \sum_{i=1}^{n} h_i(k).$$

1.4.2 Discrete Lifetimes

The probability of a subject failure at time k given that it did not fail before is

$$h(k) = P(K = k \mid K \geq k) = \frac{P(K = k)}{P(K \geq k)} = \frac{f(k)}{R(k-1)} = \frac{R(k-1) - R(k)}{R(k-1)}.$$

Thus, it is a conditional probability of a subject failure at time k given that it did not fail by $k-1$. It is bounded since it is a conditional probability. It cannot be a convex function and hence cannot grow linearly, or exponentially (Bracquemond et al., 2001, Xie et al., 2002).

The cumulative failure rate function is $H(k) = \sum_{i=1}^{k} h(i) \neq -\ln R(k)$.

The FR function is not additive for series systems. To see this, suppose that we have a system that consists of n independent components in series as in 1.4.1, then the system reliability is $R(k)=\prod_{i=1}^{n} R_i(k)$. However, the system FR function is

$$h(k) = 1 - \frac{R(k)}{R(k-1)} = 1 - \frac{\prod_{i=1}^{n} R_i(k)}{\prod_{i=1}^{n} R_i(k-1)} = 1 - \prod_{i=1}^{n} \frac{R_i(k)}{R_i(k-1)} = 1 - \prod_{i=1}^{n} [1 - h_i(k)] \neq \sum_{i=1}^{n} h_i(k).$$

1.4.3 Alternative Definition for the Discrete Failure Rate

Since the cumulative FR in the discrete case is not the negative logarithm of the reliability function, unlike the continuous case, a variant of the FR function has been defined as the sequence $\{s(k)\}_{k\geq 1}$ such that

$$H(k) = -\ln R(k) = \sum_{i=1}^{k} s(i) \;\Rightarrow\; s(k) = -\ln R(k) + \ln R(k-1) = \ln\frac{R(k-1)}{R(k)}.$$

This is known as the Second Rate of Failure, SRF. The SRF is not a probability and is unbounded. It can be convex, can grow linearly or exponentially, e.g. the SRF function for the type I discrete Weibull distribution which will be defined in 1.5.4 can grow linearly (Xie et al., 2002).

The two functions $h(k)$ and $s(k)$ determine each other uniquely. Thus,

$$s(k) = -\ln\frac{R(k)}{R(k-1)} = -\ln[1-h(k)]$$

or

$$h(k) = 1 - e^{-s(k)}.$$

Hence $s(k)$ is increasing or decreasing if and only if $h(k)$ is increasing or decreasing respectively (Xie et al., 2002, Bracquemond et al., 2001).

The SRF function is additive for series of systems, i.e. if we have a system that consists of n independent components in series as in 1.4.1 with $s_i(k)$ being the SRF function of component i, then the system SRF function is

$$s(k) = \ln\frac{R(k-1)}{R(k)} = \ln\frac{\prod_{i=1}^{n} R_i(k-1)}{\prod_{i=1}^{n} R_i(k)} = \sum_{i=1}^{n}\ln\frac{R_i(k-1)}{R_i(k)} = \sum_{i=1}^{n} s_i(k).$$

1.5 Discrete Lifetimes

Discrete lifetimes arise either through the method of observation or the countable nature of the data. Thus, lifetimes may be continuous but are measured discretely through rounding or grouping, e.g. inherently continuous lifetimes that are only observed hourly, or strengths obtained through testing using discretely increasing loads, or they may be genuinely discrete, e.g. the number of cycles to failure for a unit that operates on demand, or the number of bearing rotations to failure.

Unthinking application of a continuous probability model to discrete data is potentially misleading. Sometimes discrete models can be adequately approximated by continuous models, e.g. approximating a binomial distribution by a normal one. However, the continuous model approximation may be poor. For example, if a (continuous) gamma distribution is fitted to discrete data or data that have been heavily rounded, then, for example, the maximum likelihood estimator of the gamma shape parameter may be seriously biased because of its sensitivity to low values (Kimber, 1980). Thus, it may be advisable to use discrete models when the data available are discrete (Grimshaw et al., 2005, Hoyland and Rausand, 1994).

Of course, one can use interval censoring for grouped/rounded continuous observations. The geometric and negative binomial distributions are the discrete equivalents of the exponential and gamma distributions respectively. It is easy to show that if we round exponential observations to integer values, we obtain the geometric distribution for the rounded observations (Balakrishnan and Nevzorov, 2003). So, likelihood based inference for the geometric distribution is equivalent to that for the interval censored exponential distribution. However, we do not get this exact correspondence with, e.g., the gamma and negative binomial distributions. Hence, in general "standard" discrete models give different results from interval censored "standard" continuous models.

Bracquemond and Gaudoin (2003) give a comprehensive review of discrete lifetime distributions. They give two criteria for selecting useful discrete lifetime distributions. The first is based on simplicity, flexibility and the interpretation of parameters. The second is based on the quality of parameter estimators. Here, we present some probability distributions that are potentially useful for discrete lifetime data analysis, some of which are recommended by Bracquemond and Gaudoin (2003). The basic properties of each distribution are given and the general behaviour of the failure rate function is investigated.

1.5.1 Geometric Distribution

The random variable K has geometric distribution with parameter p $(0 < p < 1)$ if it has the following properties:

$$
\begin{aligned}
&f(k) = p(1-p)^{k-1};\ k = 1,2,3,\ldots,\\
&R(k) = (1-p)^k\\
&h(k) = p\,,\\
&s(k) = -\ln(1-p)\,.
\end{aligned}
$$

It may be thought of as the number of trials to the first failure in a sequence of independent Bernoulli trials, each with failure probability p. It is the only discrete distribution that has the lack of memory property (no ageing, no burn-in) and hence a constant failure rate.

The geometric distribution is the discrete equivalent of the exponential distribution, the only continuous distribution that has the lack of memory property (Evans et al., 1993). Thus, if we round exponential observations to integer values, we obtain the geometric distribution for the rounded observations.

Let $Y \sim \text{exponential}(\lambda)$, so that $f(y) = \lambda e^{-\lambda y}$ for $y > 0$ and $\lambda > 0$. Therefore,

$$
\begin{aligned}
&P(Y \le 1) = 1 - e^{-\lambda}\,,\\
&P(1 < Y \le 2) = e^{-\lambda}(1 - e^{-\lambda})\,,\\
&P(2 < Y \le 3) = e^{-2\lambda}(1 - e^{-\lambda})\,,\\
&\ldots\\
&P(k-1 < Y \le k) = e^{-(k-1)\lambda}(1 - e^{-\lambda})\,.
\end{aligned}
$$

Now take $p = 1 - e^{-\lambda}$, so that

$$P(k-1<Y\le k)=p(1-p)^{k-1}.$$

So, likelihood based inferences for the geometric distribution are equivalent to those for the interval censored exponential distribution in this special case.

1.5.2 Shifted Negative Binomial Distribution

The random variable X is defined as the number of successes before the z^{th} failure in a sequence of independent Bernoulli trials with probability of failure p in each trial. Thus, if $X_1, X_2, ..., X_z$ are independent random variables from a geometric distribution with probability of failure p, then $\sum_{i=1}^{z} X_i$ has a negative binomial distribution with parameters z and p. So the negative binomial is the discrete equivalent of the gamma distribution (Bracquemond and Gaudoin, 2003). More generally, the parameter z can take non-integer values and in this case the distribution has no interpretation in terms of repeated trials. However, it is useful in modelling count data. The negative binomial distribution is defined as

$$f(x)=\binom{x+z-1}{z-1}p^{z}(1-p)^{x};\ x=0,1,2,....$$

The shifted negative binomial distribution is obtained by setting $K=X+1$ where X has a negative binomial distribution with parameters z and p. The shifted negative binomial distribution is defined as

$$f(k)=\binom{k+z-2}{z-1}p^{z}(1-p)^{k-1};\ k=1,2,3,....$$

The properties of the shifted negative binomial distribution are

$$R(k)=1-\sum_{i=1}^{k}\binom{i+z-2}{z-1}p^{z}(1-p)^{i-1},$$

$$h(k) = \frac{\binom{k+z-2}{z-1} p^z (1-p)^{k-1}}{1 - \sum_{i=1}^{k-1} \binom{i+z-2}{z-1} p^z (1-p)^{i-1}},$$

$$s(k) = -\ln\left(\frac{1 - \sum_{i=1}^{k} \binom{i+z-2}{z-1} p^z (1-p)^{i-1}}{1 - \sum_{i=1}^{k-1} \binom{i+z-2}{z-1} p^z (1-p)^{i-1}} \right).$$

Note that when the parameter z is non-integer, the gamma function can be used to calculate the factorial terms, i.e.

$$\binom{k+z-2}{z-1} = \frac{\Gamma(k+z-1)}{\Gamma(k)\Gamma(z)} \text{ where } \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx .$$

Also note that the FR function is monotone decreasing if $z < 1$, monotone increasing if $z > 1$, and constant if $z = 1$, since k reduces to the geometric distribution in this case (Lawless, 2003).

The likelihood based inference for the negative binomial distribution is not equivalent to those of the interval censored gamma distribution, e.g. let $Y \sim Gamma(2, \beta)$, so that $f(y) = y\beta^2 e^{-y\beta}$ for $y > 0$ and $\beta > 0$. Therefore,

$$
\begin{aligned}
&P(Y \le 1) = e^{-\beta}[e^{\beta} - (1+\beta)], \\
&P(1 < Y \le 2) = e^{-2\beta}[e^{\beta}(1+\beta) - (1+2\beta)], \\
&P(2 < Y \le 3) = e^{-3\beta}[e^{\beta}(1+2\beta) - (1+3\beta)], \\
&\ldots \\
&P(k-1 < Y \le k) = e^{-k\beta}[e^{\beta}(1+(k-1)\beta) - (1+k\beta)],
\end{aligned}
$$

which does not give negative binomial probabilities.

The failure rate of the gamma distribution tends to a limiting value $\beta > 0$, the scale parameter, as $k \to \infty$ (Lawless, 2003). A similar result holds in the discrete case. It turns out that the FR of the negative binomial distribution tends to a limiting value $p > 0$ as $k \to \infty$. For example, for a negative binomial distribution with $z = 2$ we have $f(k) = kp^2(1-p)^{k-1}$, so that

$$\begin{aligned} R(k) &= 1 - \sum_{i=1}^{k} ip^2(1-p)^{i-1} = 1 - p^2 \sum_{i=1}^{k} i(1-p)^{i-1} \\ &= 1 + p^2 \frac{d}{dp} \sum_{i=1}^{k} (1-p)^i = 1 + p^2 \frac{d}{dp} \left\{ \frac{1-p}{p} - \frac{(1-p)^{k+1}}{p} \right\} \\ &= (1-p)^k \left[(1-p) + (k+1)p \right] = (1-p)^k (1+kp). \end{aligned}$$

Hence,

$$h(k) = \frac{f(k)}{R(k-1)} = \frac{kp^2(1-p)^{k-1}}{(1-p)^{k-1}\{1+(k-1)p\}} = \frac{kp^2}{1+(k-1)p} \to p \text{ as } k \to \infty.$$

Bracquemond and Gaudoin (2003) stated that the parameters of the shifted negative binomial distribution have no practical interpretation and the distribution is appealing only because it is the analogue of the gamma distribution. However, we can argue that the parameters of the shifted negative binomial also have an interpretation, i.e. z is the shape parameter, as with the gamma distribution, and p is the long term hazard. Alternatively, p^z is the probability of not surviving the first period or cycle.

1.5.3 Shifted Poisson Distribution

The random variable X has Poisson distribution with parameter $\lambda > 0$ if it has pmf

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}; \; x = 0,1,2,\ldots.$$

The shifted Poisson distribution is obtained by setting $K = X + 1$ where X has a Poisson distribution with parameter λ. The shifted Poisson distribution has pmf

$$f(k) = \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}; \; k = 1,2,3,\ldots.$$

The properties of the shifted Poisson distribution are

$$R(k) = 1 - \sum_{i=1}^{k} \frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!},$$

$$h(k) = \frac{e^{-\lambda}\lambda^{k-1}/(k-1)!}{1-\sum_{i=1}^{k-1}\frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!}} = \frac{\Gamma(k,\lambda)-(k-1)\Gamma(k-1,\lambda)}{\Gamma(k)-(k-1)\Gamma(k-1,\lambda)};\ \Gamma(k,\lambda)=\int_{\lambda}^{\infty} x^{k-1}e^{-x}dx,$$

$$s(k) = -\ln\left(\frac{\Gamma(k)-\Gamma(k,\lambda)}{\Gamma(k)-(k-1)\Gamma(k-1,\lambda)}\right).$$

Note that the FR function, $h(k)$, is monotone increasing (Lawless, 2003). It increases rapidly and tends to 1 as $k \to \infty$. Gardiner and Kimber (1978) investigated the tail behaviour of the Poisson distribution and have shown that for given $\lambda > 0$ and integer $x \geq \lambda - 1$,

$$\frac{\lambda^{x+1}e^{-\frac{\lambda(x+1)}{x+2}}}{(x+1)!} \leq P(X > x) \leq \frac{(x+2)\lambda^{x+1}e^{-\lambda}}{(x+2-\lambda)(x+1)!},$$

where $P(X > x) = \sum_{i=x+1}^{\infty} \frac{\lambda^{i}e^{-\lambda}}{i!}$.

Hence,

$$P(X > x) \sim \frac{(x+2)\lambda^{x+1}e^{-\lambda}}{(x+2-\lambda)(x+1)!} \quad \text{as } x \to \infty.$$

Now, $h(x) = P(X = x)/P(X \geq x) = P(X = x)/P(X > x-1)$, then for large x

$$h(x) \sim \frac{(x+2-\lambda)}{(x+2)} = 1 - \frac{\lambda}{x+2},$$

which tends to 1 as $x \to \infty$. This result may trivially be adapted for the shifted Poisson case.

The Poisson distribution is the limiting form for some sequences of binomial, negative binomial and hypergeometric random variables. Also, it is used as a probability model for the occurrence of rare events because of its many interesting properties (Balakrishnan and Nevzorov, 2003).

1.5.4 Type I Discrete Weibull Distribution

Nakagawa and Osaki (1975) defined the type I discrete Weibull distribution, $W_I(\alpha,\beta)$. It is a discrete equivalent of the continuous Weibull distribution. The type I discrete Weibull distribution, $W_I(\alpha,\beta)$, is defined as

$$
\begin{aligned}
f(k) &= \alpha^{(k-1)^\beta} - \alpha^{k^\beta};\ 0<\alpha<1, \beta>0 \text{ and } k=1,2,\ldots,\\
R(k) &= \alpha^{k^\beta},\\
h(k) &= 1-\alpha^{k^\beta-(k-1)^\beta},\\
s(k) &= \left((k-1)^\beta - k^\beta\right)\ln(\alpha).
\end{aligned}
$$

Here α and β are the probability of surviving the first period or cycle and the shape parameter respectively. The distribution reduces to the geometric distribution when $\beta=1$.

The FR function, $h(k)$, is increasing for $\beta>1$, decreasing for $0<\beta<1$ and constant for $\beta=1$. The limiting values of $h(k)$ as $k\to\infty$ are as follows:

$$
h(k)\to\begin{cases} 0 & \text{if } 0<\beta<1 \\ 1-\alpha & \text{if } \beta=1 \\ 1 & \text{if } \beta>1 \end{cases}
$$

The parameters of the type I Weibull distribution have practical meaning and the distribution is simple and flexible with respect to the shape parameter, β. The maximum likelihood estimation of the parameters is satisfactory except for very small values of α. The MLE of α is slightly biased, over estimated when $\alpha<0.9$ and under estimated otherwise. Also, the MLE of β is slightly biased as well (Bracquemond and Gaudoin, 2003).

1.5.5 Type III Discrete Weibull Distribution

Padgett and Spurrier (1985) defined another discrete Weibull distribution, the type III discrete Weibull distribution, $W_{III}(\eta, \beta)$. It is flexible with respect to the choice of the shape parameter, β, and corresponds to the continuous Weibull distribution. The type III discrete Weibull distribution, $W_{III}(\eta, \beta)$, is defined by:

$$f(k) = \left(1 - e^{-\eta k^{\beta}}\right) e^{-\eta \sum_{i=1}^{k-1} i^{\beta}}; \; -\infty < \beta < \infty, \; \eta > 0 \text{ and } k = 1, 2, \ldots,$$

$$R(k) = e^{-\eta \sum_{i=1}^{k} i^{\beta}},$$

$$h(k) = 1 - e^{-\eta k^{\beta}},$$

$$s(k) = \eta k^{\beta},$$

where β is the shape parameter and η is linked with the probability of not surviving the first period or cycle since $f(k=1) = 1 - e^{-\eta}$. The distribution reduces to the geometric distribution when $\beta = 0$.

The FR function, $h(k)$, is increasing for $\beta > 0$, decreasing for $\beta < 0$ and constant for $\beta = 0$. The limiting values of $h(k)$ as $k \to \infty$ are as follows:

$$h(k) \to \begin{cases} 0 & \text{if } \beta < 0 \\ 1 - e^{-\eta} & \text{if } \beta = 0 . \\ 1 & \text{if } \beta > 0 \end{cases}$$

The parameters of the type III Weibull distribution have practical meaning, but less obviously than the type I distribution. The distribution is simple and flexible, but the maximum likelihood estimation of the parameters is biased. The MLE of η is little biased but the MLE seems to improve for large values of η. Also, the bias in the MLE of β is low except for very small samples (Bracquemond and Gaudoin, 2003).

1.5.6 "S" Distribution

Bracquemond and Gaudoin (2003) proposed a discrete distribution, $S(p,\alpha)$, which is equivalent to the "S" distribution proposed by Soler (1996) for the continuous case. The distribution is defined by:

$$f(k) = p(1-\alpha^k)\prod_{i=0}^{k-1}(1-p+\alpha^i);\ 0 \le \alpha < 1,\ 0 < p \le 1 \text{ and } k = 1,2,\ldots,$$

$$R(k) = \prod_{i=1}^{k}(1-p+\alpha^i),$$

$$h(k) = p(1-\alpha^k),$$

$$s(k) = -\ln\left(1-p(1-\alpha^k)\right).$$

Here p and α are the probability that a failure occurs during a period or cycle and the probability of surviving the first period or cycle given that a failure has occurred respectively. The FR function, $h(k)$, is increasing since $\alpha < 1$ and its limiting value is p as $k \to \infty$.

The distribution is useful in modelling the lifetime of a device subjected to random stress. It reduces to the type III discrete Weibull distribution with $\beta = 1$ and $\eta = -\ln(\alpha)$ if a shock occurs at each period or cycle, $p = 1$. The parameters of the distribution have some identifiability and numerical estimation problems which have not been solved. However, the distribution is simple and has an appealing practical meaning (Bracquemond and Gaudoin, 2003).

1.6 Discussion

The statistical analysis of lifetimes is developing rapidly especially for continuous lifetimes even though most of the data available are discrete. Discrete lifetimes arise either through the method of observation or the countable nature of the data.

The simple relationship between the FR function and the reliability function in the discrete case is being resolved by redefining the FR as the SRF. Thus, the relationship between the SRF and the reliability function in the discrete case holds as the relationship between the FR function and the reliability function in the continuous case.

The basic properties of some discrete distributions that are potentially useful for lifetime data analysis are discussed. Also, the general behaviour of the failure rate function for the given distributions is investigated.

The use of simple graphical methods to give some insight about the underlying distribution of the data is natural in an early stage of a statistical analysis. In the next chapter the empirical failure rate plot will be investigated. The geometric, negative binomial and Poisson distributions will be used in this investigation but the Weibull type I, Weibull type III and "S" distributions will not be used further in this thesis.

Chapter 2

Discrete Empirical Failure Rate

At an early stage of a statistical analysis it is natural to use simple graphical methods to summarise the data and to give some insight about the underlying distribution of the data. In the case of discrete lifetime data a natural way to investigate the underlying failure rate, FR, function is to use a plot of the empirical failure rate, EFR. In this chapter we discuss some potentially undesirable features of the EFR plot and consider some simple transformations to try to resolve this problem. We then derive some distributional results for points on the empirical distribution function, EDF, plot. Finally, a simple smoothed version of the EFR plot is introduced and used to develop a control chart to detect departure from a constant FR.

2.1 Introduction

The failure rate function of discrete lifetimes at time k is the conditional probability of a subject failure at time k given that it did not fail at an earlier time:

$$h(k) = P(K = k \mid K \geq k) = \frac{P(K = k)}{P(K \geq k)};\ k = 1,2,3,\ldots .$$

In a particular sample we observe the number of failures $n_1, n_2, n_3, \ldots$ at times $1, 2, 3, \ldots$ and can calculate the EFR $\hat{h}_1, \hat{h}_2, \hat{h}_3, \ldots$ such that

$$\hat{h}_k = \hat{P}(K = k \mid K \geq k) = \frac{\#\ failed\ at\ time\ k}{\#\ at\ risk\ at\ time\ k}\ ;\ k = 1,2,3,\ldots.$$

We have no control over the behaviour of the number of failures. The values of the EFR are just estimated conditional probabilities that display the unique risk at each point in time within a study period and they are very sensitive in describing patterns of event occurrence (Singer and Willett, 2003).

The layout of discrete lifetimes in a particular sample is shown in Table 2.1. Here n_k is the number of subjects failed at time k , $n_k^+ = n - \sum_{i=1}^{k-1} n_i$ is the number of subjects at risk at time k and k_* is the highest observed failure time in the sample. Note that $0 \leq n_1, n_2, \ldots,\ n = n_1^+ \geq n_2^+ \geq \ldots \geq n_{k_*}^+$ and the EFR for the highest observed failure time, k_*, in the sample will equal 1 if there are no censored observations because $n_{k_*} = n_{k_*}^+$.

Table 2.1: Data layout of discrete lifetimes.

Time	Number at Risk	Number Failed	EFR
1	$n_1^+ = n$	n_1	$\hat{h}_1 = n_1/n_1^+$
2	$n_2^+ = n - n_1$	n_2	$\hat{h}_2 = n_2/n_2^+$
3	$n_3^+ = n - n_2 - n_1$	n_3	$\hat{h}_3 = n_3/n_3^+$
...	...	...	...
k_*	$n_{k_*}^+ = n - \sum_{k=1}^{k_*-1} n_k$	n_{k_*}	$\hat{h}_{k_*} = n_{k_*}/n_{k_*}^+$

In order to illustrate the EFR plot, samples of size $n = 25, 50, 100$ and 200 were simulated using SAS software version 9.1 from four discrete distributions and the EFR plotted in each case along with the true FR function. The four distributions are: Geometric with $p = 0.25$ (flat FR), shifted Negative Binomial with $p = 0.25$ and $z = 2$ (gently increasing FR), shifted Negative Binomial with $p = 0.25$ and $z = 0.2$ (decreasing FR) and shifted Poisson with mean 2 (FR that increases to 1). The EFR plots are shown in Figures 2.1 - 2.4 respectively.

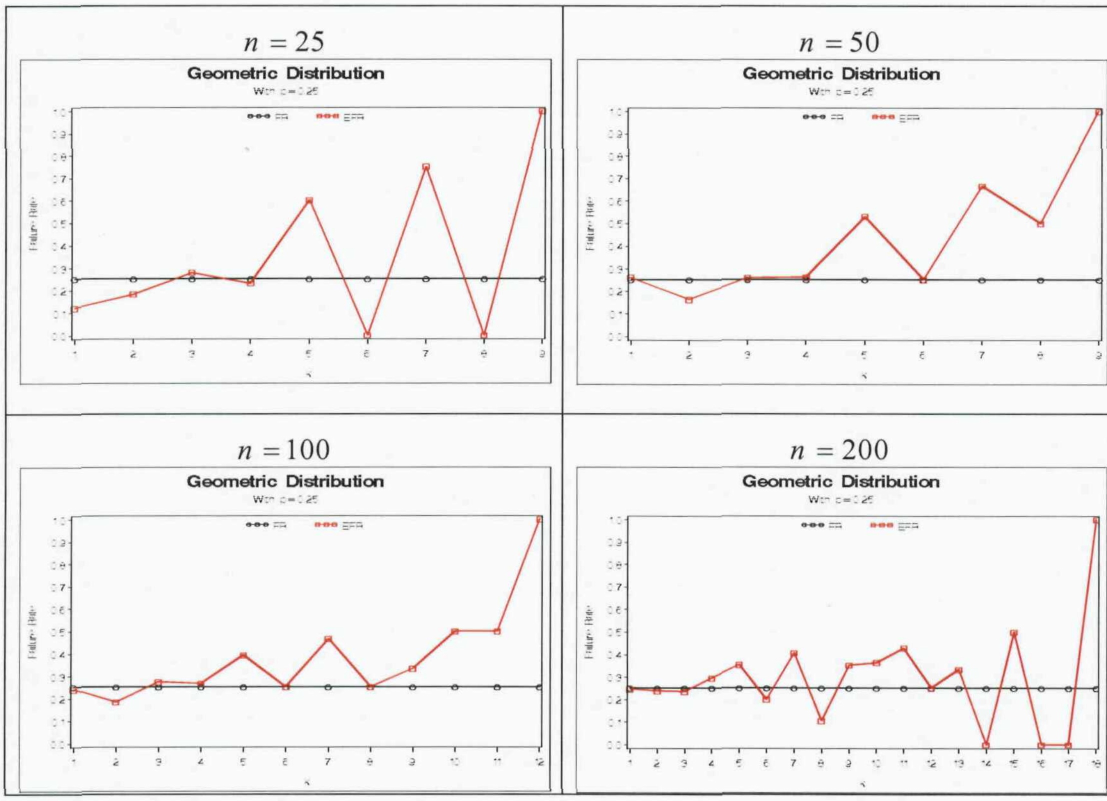


Figure 2.1: The true FR and EFR for four data sets simulated from the geometric(0.25) distribution.

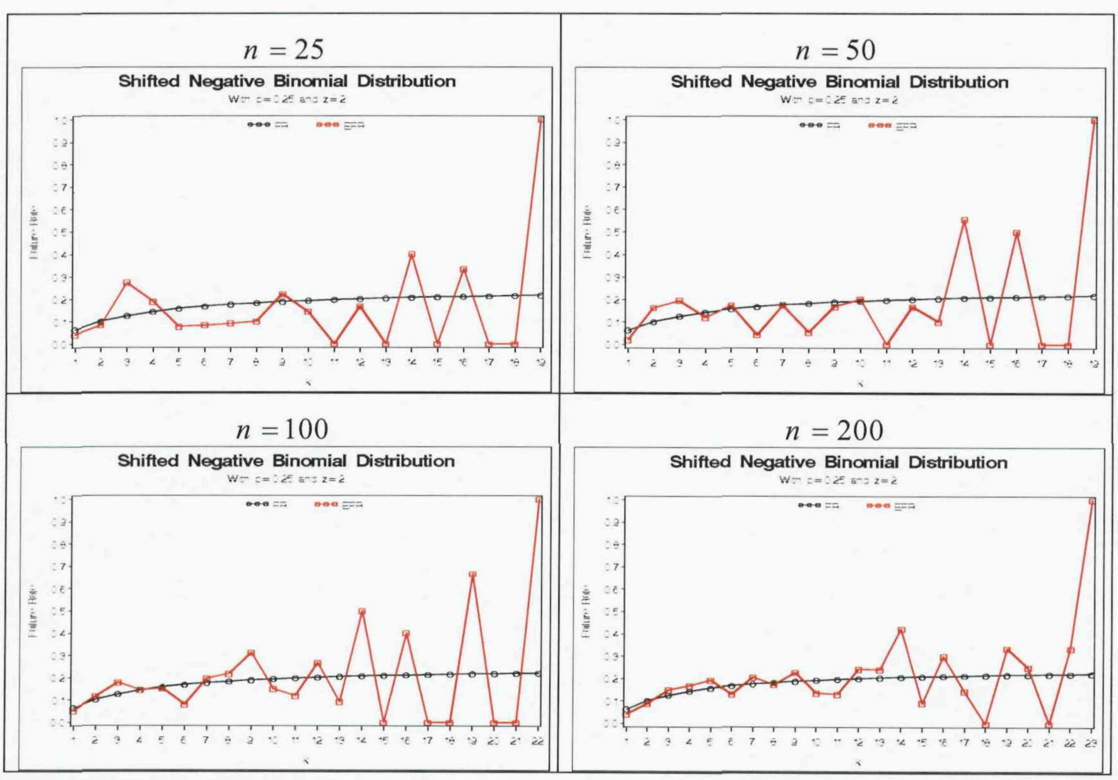


Figure 2.2: The true FR and EFR for four data sets simulated from the shifted negative-binomial(0.25,2) distribution.

These EFR plots suggest that broadly the EFR appears to follow the underlying FR function for smaller values of k. However, two situations emerge that make interpretation difficult. First consider the situation where there are no "gaps" in the data (that is, there are no zero frequencies for all $k \le k_*$). Then the fact that $\hat{h}_{k_*} = 1$ gives the impression that the EFR is broadly increasing, even though the FR is not. Examples of this behaviour are plots (1,2) and (2,1) of Figure 2.1. A second difficulty occurs for some data sets where there are "gaps" for larger values of k. Here the EFR may display a saw tooth pattern where the peaks tend to grow with k while the troughs are necessarily constrained to be zero (see plot (1,1) of Figure 2.1 and plot (2,1) of Figure 2.2), which tends to give the impression of a broad increase. Alternatively, for data with isolated points, the EFR may drift far from the true FR for a large number of consecutive k – values (see plots (1,2) and (2,1) of Figure 2.3 and plot (1,2) of Figure 2.4). The visual impact of these behaviours can make interpretation of the EFR plot difficult. This issue was also noted in Kimber and Hansford (1993) and Kimber (2007) in the context of cricket scores.

Pathiyil and Jeevanand (2007) discussed the problem of estimating the reliability measures of the geometric distribution in the presence of discordant or outlier observations using Bayes point estimators. Here, we are interested in the behaviour of the random variables $H_1, H_2, H_3, \ldots$, where $H_k = N_k / N_k^+$, N_k is the number of observations failed at time k and $N_k^+ = N_k + N_{k+1} + \ldots$ is the overall number of observations at risk at time k.

Transformations are usually used to resolve the boundary problem and/or improve distribution behaviour. In section 2.2 four transformations for the discrete EFR are discussed. The mean and standard deviation of a distribution can be used to summarise the distribution behaviour. The marginal probability function of the discrete EFR under two formulations is investigated in section 2.3. A failure rate control chart, FRCC, to detect departure from the constant FR is introduced in section 2.4. Finally, a discussion of this chapter is presented in section 2.5.

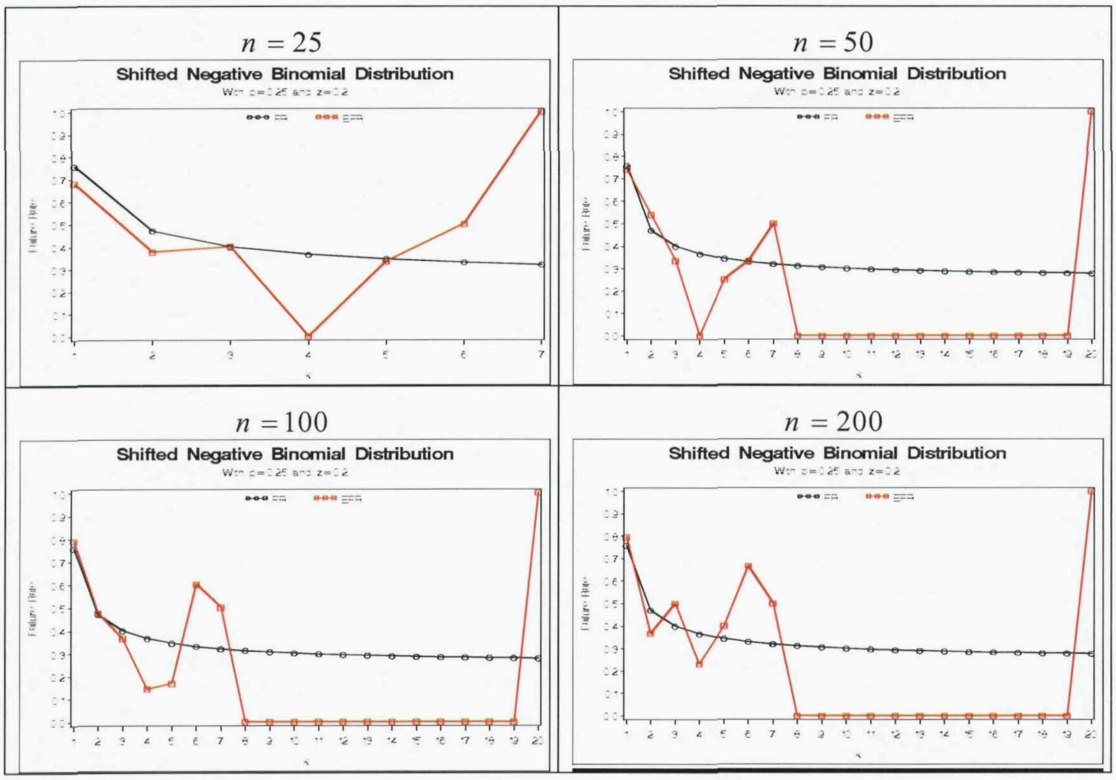


Figure 2.3: The true FR and EFR for four data sets simulated from the shifted negative-binomial(0.25,0.2) distribution.

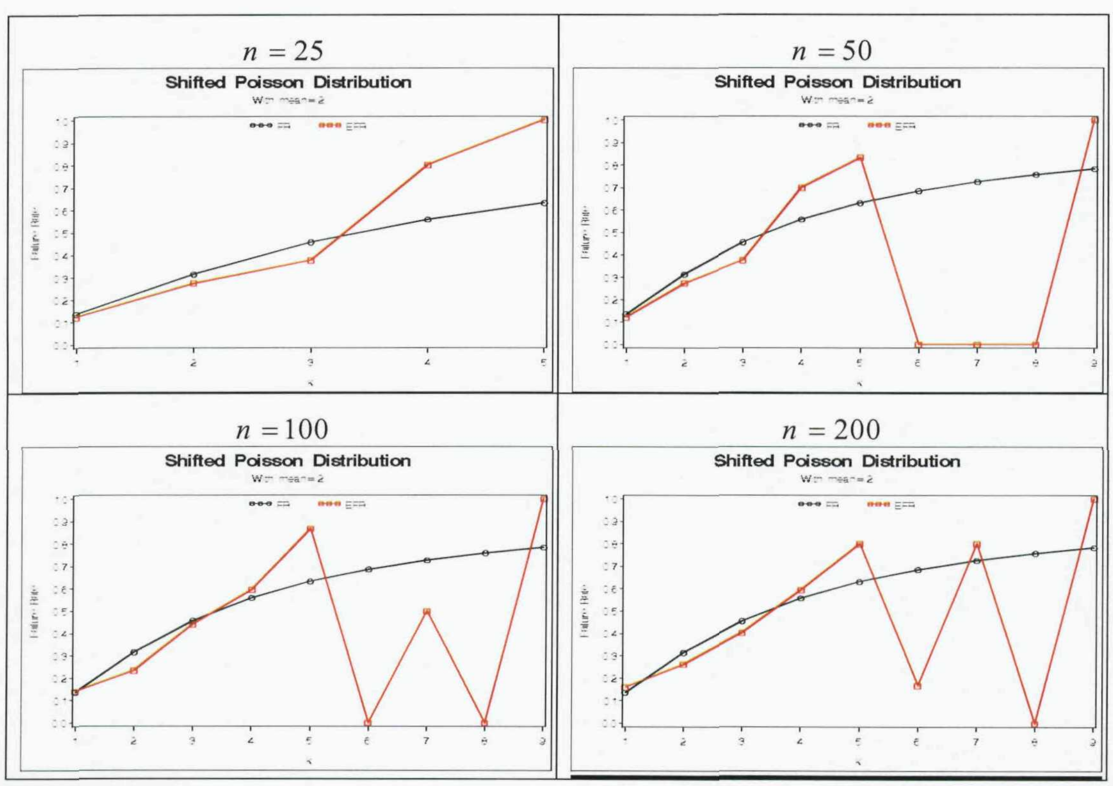


Figure 2.4: The true FR and EFR for four data sets simulated from the shifted Poisson(2) distribution.

$$u'[\hat{h}_k] = \frac{1}{2}\left[\arcsin\left(\sqrt{\frac{n_k}{n_k^+ + 1}}\right) + \arcsin\left(\sqrt{\frac{n_k + 1}{n_k^+ + 1}}\right)\right].$$

Figure 2.8 shows the transformed EFR plots for the $n = 25, 50$ and 100 cases of Figures 2.1-2.3. The plots reveal that the improved arcsine transformation has little effect on the shape of the EFR.

2.2.5 Other Transformations

Of course one could try other transformations such as the empirical logit (Haldane, 1955). However, in view of the unpromising results obtained thus far, this approach was not pursued further.

2.3 Properties of the Discrete Empirical Failure Rate

We will now study the behaviour of the EFR in more detail. In particular, we obtain the marginal probability function of the EFR for each k under two formulations. From these results the corresponding means and standard deviations are then derived.

In a particular sample of size n we observed n_k failures for $k = 1, 2, \ldots, k_*$ with $\sum_{k=1}^{k_*} n_k = n$ where k_* is the highest observed failure time in the sample. Let $n_k^+ = n - \sum_{i=1}^{k-1} n_i$ be the number of subjects at risk at time k. Then we may calculate $\hat{h}_k = n_k / n_k^+$. To obtain the probability function of the EFR, we must be careful to avoid the situation in which $N_k^+ = 0$. If we tried to find $\hat{h}_k$ when $n_k^+ = 0$, we would get $\hat{h}_k = \frac{0}{0}$ which is undefined. Hence, we need to find the probability function of $H_k \mid N_k^+ > 0$.

A slightly different approach is that we are interested in the behaviour of the EFR given that the largest observed failure time is k_*. Let $K_{\max}$ denote the largest observed failure time. Then, we find the probability function of $H_k \mid K_{\max} = k_*$.

The two approaches describe the behaviour of H_k but in two different ways. The probability function of $H_k \mid N_k^+ > 0$ tells us about the behaviour of H_k for $k = 1,2,...$ across all "allowable" samples (i.e., these for which $N_k > 0$ so that $\hat{h}_k$ is defined) but the probability function of $H_k \mid K_{\max} = k_*$ for $k = 1,2,...,k_* - 1$ tells us about the behaviour of H_k across all samples with the largest observed failure time k_*.

2.3.1 The Probability Function of $H_k \mid N_k^+ > 0$

Let H_k, N_k and N_k^+ denote the random variables corresponding to the observed $\hat{h}_k, n_k$ and n_k^+ respectively. Then we now investigate the behaviour of $H_k \mid N_k^+ > 0$. In what follows we shall assume that the sample size n (≥ 1) is fixed and that there is no censoring.

Proposition 2.1:

Let K be the random variable whose failure rate we wish to estimate.
Let N_k be the number of observations failed at time k $(k = 1,2,3,...)$.
Then, the empirical failure rate is defined as $H_k = N_k / N_k^+$ where $N_k^+ = N_k + N_{k+1} + ...$
The probability function of $H_k \mid N_k^+ > 0$ is given by

$$P\left(H_k = \frac{n_k}{n_k^+} \mid N_k^+ > 0\right) = \begin{cases} \binom{n}{n_1} p_1^{n_1}(1-p_1)^{n-n_1} & \text{if } k = 1 \\ \sum \frac{(1-p_1-p_2)^n}{1-(1-p_1-p_2)^n} \binom{n}{n_k^+}\binom{n_k^+}{n_k}\left(\frac{p_1}{p_2}\right)^{n_k}\left(\frac{p_2}{1-p_1-p_2}\right)^{n_k^+} & \text{if } k > 1 \end{cases},$$

where $p_1 = f(k) = P(K = k)$, $p_2 = R(k) = P(K > k)$,

$1 - p_1 - p_2 = F(k-1) = P(K \leq k-1), n_k = 0,1,2,...,n_k^+$, $n_k^+ = 1,2,3,...,n$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+.

* See Appendix A.1 for the proof.

2.2 Transformations for the Discrete Empirical Failure Rate

One approach to reducing the visual impact of certain aspects of a plot is to use a transformation. In this section four transformations are used.

2.2.1 The Second Rate of Failure

The second rate of failure, SRF, transformation is not a probability and is unbounded as discussed in chapter 1, section 1.4.3. The SRF is

$$\hat{s}(k) = -\ln[1 - \hat{h}(k)].$$

Figure 2.5 shows the transformed EFR plots for the $n = 25, 50$ and 100 cases of Figures 2.1-2.3. The effect of the transformation increases the visual impact and indeed when $\hat{h}_k = 1$, then $s(k) = \infty$. Infinite values have not been plotted. Clearly the SRF transformation is unhelpful.

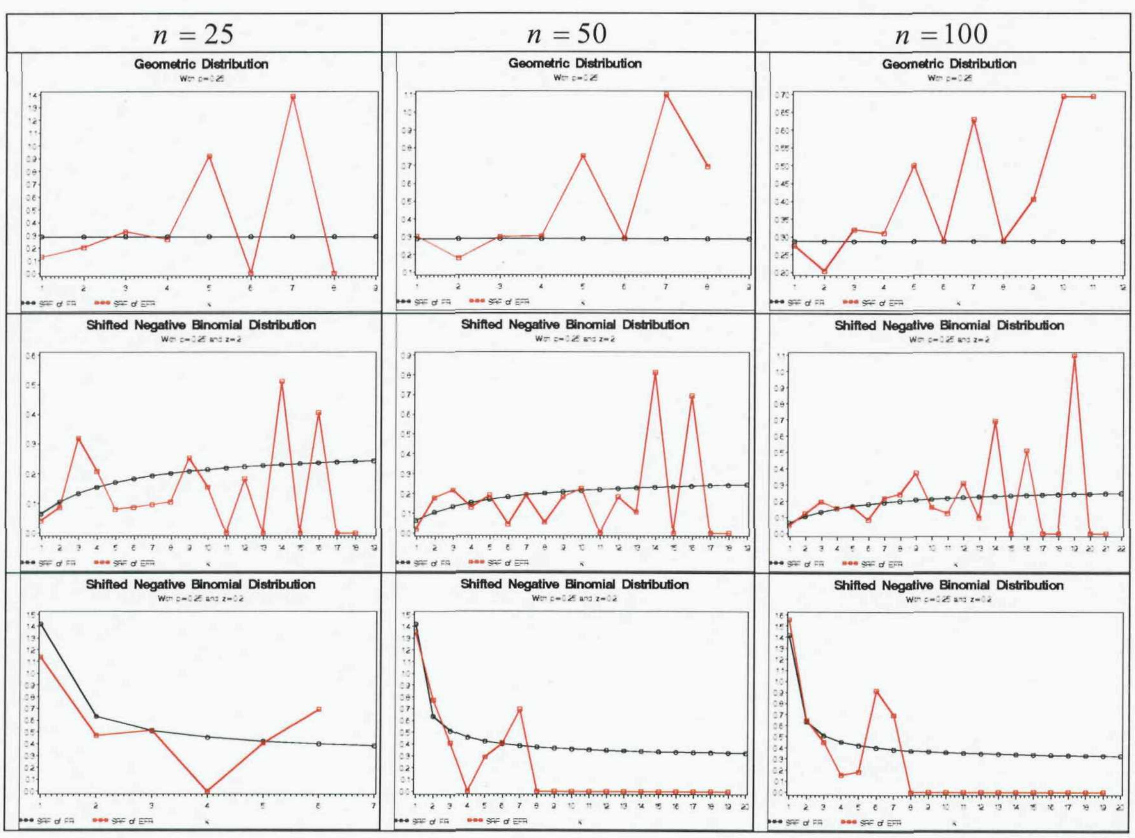


Figure 2.5: The SRF transformation of the FR and EFR for nine simulated samples.

Example 2.1:

If $n = 4$ from *Geometric*(0.25) and we would like to find the probability of $H_k = 0.5 \mid N_k^+ > 0$ for $k = 1$ and $k = 2$, then for $k = 1$, $p_1 = P(K = 1) = 0.25$ and the only possible value that H_1 can take is $2/4$. Thus,

$$P(H_1 = 0.5 \mid N_1^+ > 0) = P(H_1 = \frac{2}{4} \mid N_1^+ > 0) = \binom{4}{2} 0.25^2 (1 - 0.25)^2 = 0.211 .$$

For $k = 2$, $p_1 = P(K = 2) = 0.1875$, $p_2 = P(K > 2) = 0.5625$ and the possible values that H_2 can take are $1/2$ and $2/4$. Thus,

$$\begin{aligned} P(H_2 = 0.5 \mid N_2^+ > 0) &= \frac{0.25^4}{1 - 0.25^4} \binom{4}{4} \binom{4}{2} \left(\frac{0.1875}{0.5625} \right)^2 \left(\frac{0.5625}{0.25} \right)^4 \\ &+ \frac{0.25^4}{1 - 0.25^4} \binom{4}{2} \binom{2}{1} \left(\frac{0.1875}{0.5625} \right)^1 \left(\frac{0.5625}{0.25} \right)^2 \\ &= 0.067 + 0.079 = 0.146 . \end{aligned}$$

Proposition 2.2:

Given $N_k^+ > 0$, then H_k is an unbiased estimator of $h(k)$. Thus,

$$E(H_k \mid N_k^+ > 0) = h(k) = \frac{f(k)}{R(k-1)} ,$$

where $f(k) = P(K = k)$; the probability function of K,

$R(k-1) = P(K > k-1) = P(K \geq k)$; the reliability function of K.

* See Appendix A.2 for the proof.

The plots in Appendix B illustrate the behaviour of $P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the geometric, negative-binomial and Poisson distributions. Figure 2.9 illustrates the behaviour of $P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for a sample of size $n = 100$ from the geometric distribution with $p = 0.25$. For all these distributions, we found the shape of the distribution of $H_k \mid N_k^+ > 0$ when the sample size n is not small to be bell shaped but with the possibility of spikes for $k > 1$. Those spikes are the result of accumulating the

2.2.2 The Odds Ratio Transformation

Since the EFR is a conditional probability, the odds ratio compares the probability of failure occurrence and the probability of no occurrence in any given failure time. The odds ratio for the EFR is

$$odds[\hat{h}_k] = \frac{\hat{h}_k}{1-\hat{h}_k} = \frac{n_k}{n - \sum_{j=1}^{k} n_j}.$$

The effect of the odds ratio transformation is small for small values of the EFR (Singer and Willett, 2003). However, when $\hat{h}_k$ is not small, the effect of the transformation actually increases the visual impact and indeed when $\hat{h}_k = 1$, then $odds[\hat{h}_k] = \infty$. Figure 2.6 shows the transformed EFR plots for the $n = 25, 50$ and 100 cases of Figures 2.1-2.3. Infinite values have not been plotted. Clearly the odds transformation is behaving in the same way as the SRF and is unhelpful as well.

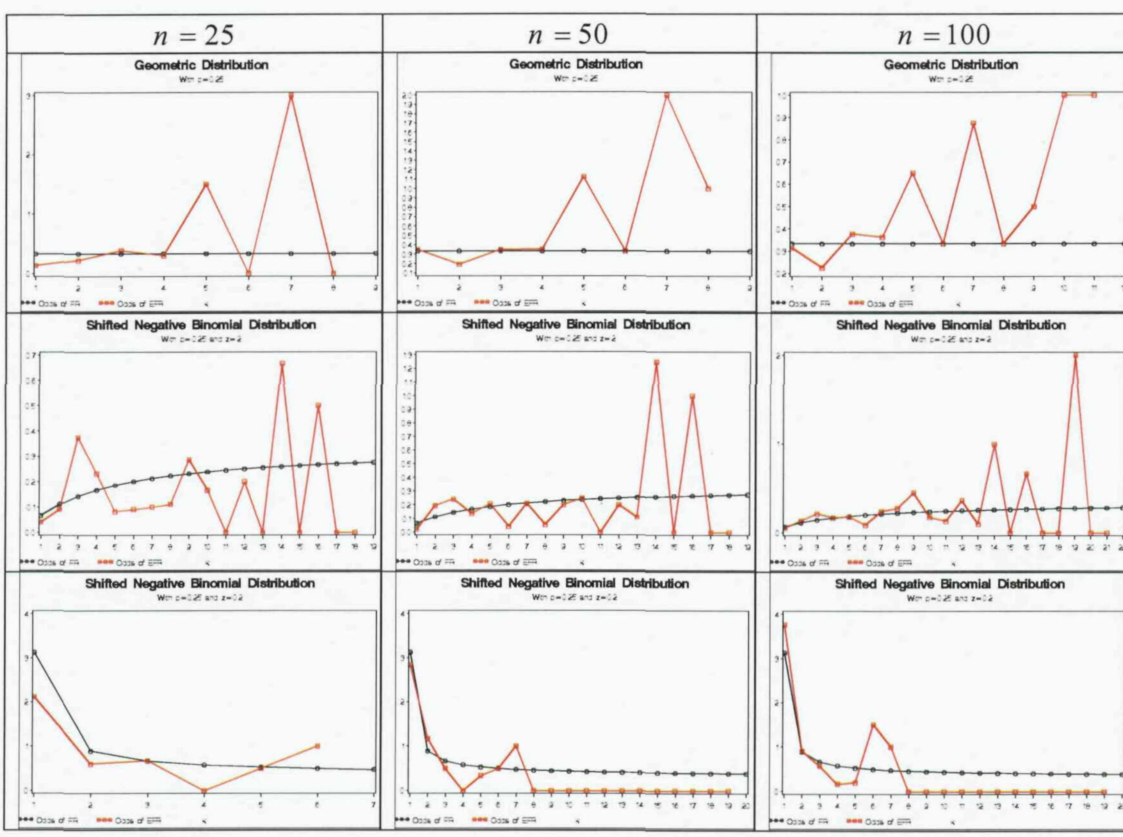


Figure 2.6: The odds ratio transformation of the FR and EFR for nine simulated samples.

We have shown in Proposition 2.2 that

$$E(H_k \mid N_k^+ > 0) = \frac{p_1}{p_1 + p_2} = h(k).$$

Now, we find $E(H_k^{\,2} \mid N_k^+ > 0)$.

$$E(H_k^{\,2} \mid N_k^+ > 0) = \frac{(1-p_1-p_2)^n}{1-(1-p_1-p_2)^n} \sum_{n_k^+=1}^{n} \sum_{n_k=0}^{n_k^+} \left(\frac{n_k}{n_k^+}\right)^2 \binom{n}{n_k^+} \binom{n_k^+}{n_k} \left(\frac{p_1}{p_2}\right)^{n_k} \left(\frac{p_2}{1-p_1-p_2}\right)^{n_k^+}$$
$$= \frac{p_1^{\,2}}{(p_1+p_2)^2} + \frac{np_1p_2(1-p_1-p_2)^{n-1}\ {}_3F_2(1,1,1-n;2,2;1-(1-p_1-p_2)^{-1})}{(1-(1-p_1-p_2)^n)(p_1+p_2)}, \quad (2.1)$$

where ${}_3F_2(1,1,1-n;2,2;1-(1-p_1-p_2)^{-1})$ is the generalized hypergeometric function (Andrews et al., 2001).

Now, for fixed n,

$$p_1 \to 0 \text{ as } k \to \infty$$
$$p_2 \to 0 \text{ as } k \to \infty$$

and

$$(1-p_1-p_2)^{n-1} \sim 1-(n-1)(p_1+p_2) \to 1 \text{ as } k \to \infty$$
$$(1-p_1-p_2)^n \sim 1-n(p_1+p_2)$$

Further (using Mathematica software version 6),

$${}_3F_2(1,1,1-n;2,2;1-(1-p_1-p_2)^{-1}) \sim 1+\frac{(n-1)}{4}(p_1+p_2)+O[p_1+p_2]^2. \quad (2.2)$$

Consequently, the first term of $E(H_k^{\,2} \mid N_k^+ > 0)$ is simply $[h(k)]^2$ and the second term is asymptotically equivalent to $\frac{np_1p_2}{n(p_1+p_2)^2} = h(k)[1-h(k)]$ for large k.

Therefore,

$$STD(H_k \mid N_k^+ > 0) \sim \sqrt{h(k)[1-h(k)]} \text{ as } k \to \infty.$$

□.

2.2.3 The Log-Odds Ratio Transformation

The number of subjects at risk decreases as k increases. Thus, the sample sizes may become moderate to small and the sampling distribution of the odds ratio is highly skewed in this case. Because of the skewness of the odds ratio and its being bounded below by 0, an equivalent measure to the odds ratio can be used and that is its natural logarithm (Agresti, 1996). The log-odds ratio or logit transformation for the EFR is

$$logit[\hat{h}_k] = log\{odds[\hat{h}_k]\} = log\left(\frac{\hat{h}_k}{1-\hat{h}_k}\right).$$

The log-odds ratios are unbounded estimates and symmetric about zero. The effect of the transformation depends on the magnitude of the EFR values.

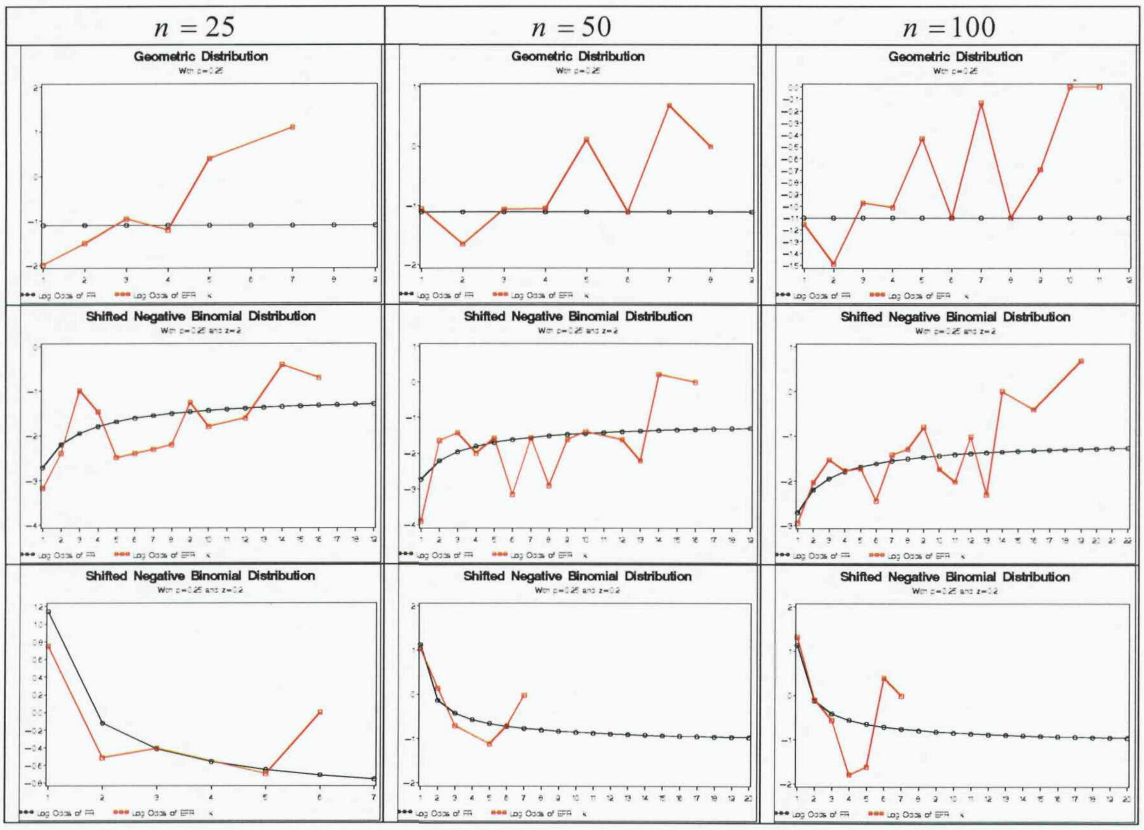


Figure 2.7: The log-odds ratio transformation of the FR and EFR for nine simulated samples.

2.3.2 The Probability Function $H_k \mid K_{\max} = k_*$

Let H_k, N_k, N_k^+ and $K_{\max}$ denote the random variables corresponding to the observed $\hat{h}_k, n_k, n_k^+$ and k_* respectively. We now investigate the behaviour of $H_k \mid K_{\max} = k_*$ for $k = 1, 2, \ldots, k_* - 1$. Of course, we know that $H_{k_*} = 1$ since there are no censored observations.

Proposition 2.4

Let K be the random variable whose hazard we wish to estimate.

Let N_k be the number of observations failed at time k $(k = 1, 2, 3, \ldots)$.

Then, the empirical failure rate is defined as $H_k = N_k / N_k^+$ where $N_k^+ = N_k + N_{k+1} + \ldots$

The probability function of $H_k \mid K_{\max} = k_*$ $\forall k = 1, 2, 3, \ldots, k_* - 1$ where $k_* > 1$ is the highest observed failure time in the sample, is given by

$$P\left(H_k = \frac{n_k}{n_k^+} \mid K_{\max} = k_*\right) = \begin{cases} \binom{n}{n_1} \dfrac{p_1^{n_1}\left((1-p_1)^{n-n_1} - (1-p_1-p_2)^{n-n_1}\right)}{1-(1-p_2)^n} & \text{if } k = 1 \\ \sum \dfrac{n!\, p_1^{n_k}\left((p_3+p_2)^{n_k^+-n_k} - p_3^{n_k^+-n_k}\right) p_4^{\,n-n_k^+}}{n_k!\,(n_k^+-n_k)!\,(n-n_k^+)!\,(1-(1-p_2)^n)} & \text{if } 1 < k < k_* - 1 \\ \sum \dfrac{n!\, p_1^{n_k}\, p_2^{n_k^+-n_k}\,(1-p_1-p_2)^{n-n_k^+}}{n_k!\,(n_k^+-n_k)!\,(n-n_k^+))!\,(1-(1-p_2)^n)} & \text{if } k = k_* - 1 \end{cases},$$

where $n_k = 0, 1, 2, \ldots, n_k^+ - 1$, $n_k^+ = 1, 2, \ldots, n$, $p_1 = P(K = k)$, $p_2 = P(K \geq k_*)$,
$p_3 = P(k < K < k_*)$, $p_4 = 1 - p_1 - p_2 - p_3$,
$\sum$ is taken over all the equivalent fractions of n_k / n_k^+.

* See Appendix A.3 for the proof.

Proposition 2.5:

Given $K_{\max} = k_*$, then H_k is not an unbiased estimator of $h(k)$. Thus, in general

$$E(H_k \mid K_{\max} = k_*) \neq h(k) = \frac{f(k)}{R(k-1)},$$

where $f(k) = P(K = k)$; the probability function of K,
$R(k-1) = P(K > k-1) = P(K \geq k)$; the reliability function of K.

* See Appendix A.4 for the proof.

Figure 2.7 shows the transformed EFR plots for the $n = 25, 50$ and 100 cases of Figures 2.1-2.3. As with the odds transformation infinite values are obtained when $\hat{h}_k = 1$. In addition negative infinite values are obtained when $\hat{h}_k = 0$. Thus this transformation is not useful.

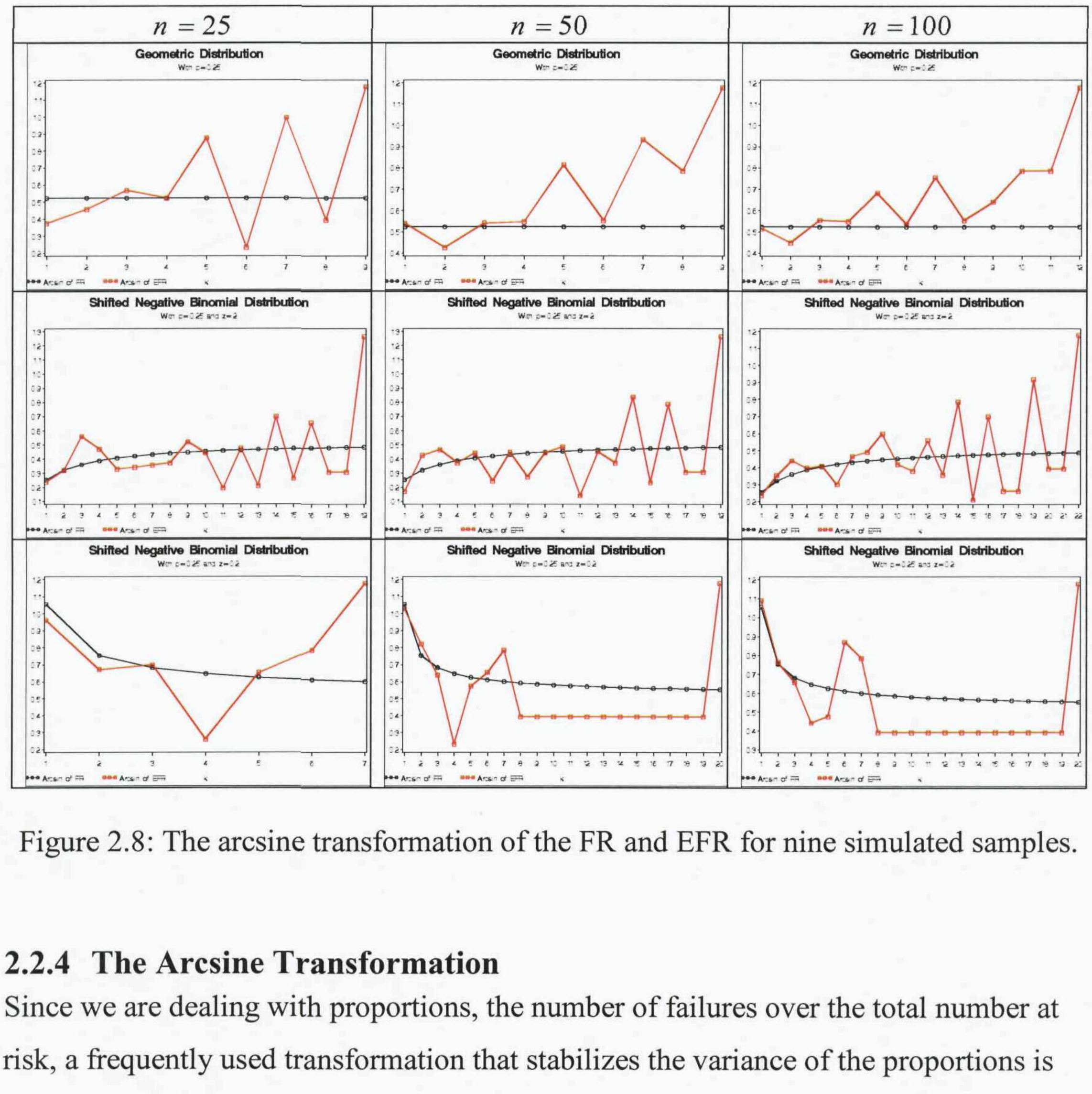


Figure 2.8: The arcsine transformation of the FR and EFR for nine simulated samples.

2.2.4 The Arcsine Transformation

Since we are dealing with proportions, the number of failures over the total number at risk, a frequently used transformation that stabilizes the variance of the proportions is the arcsine of the square root of the proportion,

$$u[\hat{h}_k] = \arcsin\left(\sqrt{\hat{h}_k}\right).$$

Since the number of subjects at risk decreases as k increases, an improved transformation that takes into account the change in the number of failures over the total number at risk has been suggested by Freeman and Tukey (Draper and Smith, 1998):

The downward divergence in the mean and standard deviation of $H_k \mid K_{\max} = k_*$ when $k < k_*$ $(k_* > 1)$ at the tail of the data is caused by conditioning on the fact that the highest observed failure time has EFR value equal to 1. We have noticed that the downward divergence at the tail of the data is larger for the data that have wider range of failure times with respect to the sample size, i.e., the standard deviations of $H_k \mid N_k^+ > 0$, $H_k \mid K_{\max} = 6$ and $H_k \mid K_{\max} = 8$ for the shifted negative binomial distribution with parameters $p = 0.25$, $z = 0.2$ and sample size $n = 25$ are plotted in Figure 2.16, the downward divergence in $STD(H_k \mid K_{\max} = 8)$ is larger.

Proposition 2.6:

When $k < k_*$ $(k_* > 1)$,

$$E(H_k \mid K_{\max} = k_*) \le E(H_k \mid N_k^+ > 0).$$

Proof:

From Proposition 2.2,

$$E(H_k \mid N_k^+ > 0) = h(k) = \begin{cases} p_1 & \text{if } k = 1 \\ \dfrac{p_1}{p_1 + p_2} & \text{if } k > 1 \end{cases},$$

where $p_1 = P(K = k)$ and $p_2 = P(K > k)$.

From Proposition 2.5,

$$E(H_k \mid K_{\max} = k_*) = \begin{cases} p_1 \dfrac{(1-(1-p_2)^{n-1})}{1-(1-p_2)^n} & \text{if } k = 1 \\ \dfrac{p_1}{p_1 + p_2 + p_3} - \dfrac{p_1 p_2 ((1-p_2)^n - (1-p_1-p_2-p_3)^n)}{(1-(1-p_2)^n)(p_1+p_3)(p_1+p_2+p_3)} & \text{if } 1 < k < k_* - 1, \\ \dfrac{p_1}{p_1+p_2} - \dfrac{p_2((1-p_2)^n - (1-p_1-p_2)^n)}{(p_1+p_2)(1-(1-p_2)^n)} & \text{if } k = k_* - 1 \end{cases}$$

where $n_k = 0,1,2,\ldots,n_k^+ - 1$, $n_k^+ = 1,2,\ldots,n$, $p_1 = P(K = k)$, $p_2 = P(K \ge k_*)$, $p_3 = P(k < K < k_*)$ and $p_4 = 1 - p_1 - p_2 - p_3$.

Note that $p_1 = P(K = k)$ is the same in the two conditional means. $p_2 = P(K > k)$ in $E(H_k \mid N_k^+ > 0)$ is same as $p_2 + p_3 = P(K \geq k_*) + P(k < K < k_*)$ in $E(H_k \mid K_{\max} = k_*)$ when $1 < k < k_* - 1$ and same as $p_2 = P(k \geq k_*)$ when $k = k_* - 1$.

Now for $k = 1$, clearly $E(H_1 \mid K_{\max} = k_*) \leq E(H_1 \mid N_1^+ > 0)$ since $(1-p_2)^{n-1} > (1-p_2)^n$. For $1 < k < k_* - 1$, since $p_1/(p_1 + p_2)$ in $E(H_k \mid N_k^+ > 0)$ is same as $p_1/(p_1 + p_2 + p_3)$ in $E(H_k \mid K_{\max} = k_*)$, clearly $E(H_k \mid K_{\max} = k_*) \leq E(H_k \mid N_k^+ > 0)$ because of the additional subtraction in $E(H_k \mid K_{\max} = k_*)$. Also, for $k = k_* - 1$, clearly $E(H_k \mid K_{\max} = k_*) \leq E(H_k \mid N_k^+ > 0)$ because of the additional subtraction in $E(H_k \mid K_{\max} = k_*)$ since $p_1/(p_1 + p_2)$ is the same in the two conditional means when $k = k_* - 1$. Thus, in general,

$$E(H_k \mid K_{\max} = k_*) \leq E(H_k \mid N_k^+ > 0).$$

$\square$.

2.4 The Failure Rate Control Chart

The idea of the control chart was developed by Dr. Walter A. Shewhart in the 1920s. It is a graphical tool that is used to distinguish between random variation and variation resulting from some special causes in a process. In its simplest form the control chart consists of a centreline and two control bounds. The centreline corresponds to the average quality of a process performance over time and the control bounds correspond to the variation from the centreline. There are two types of control charts: charts for measurements such as temperature and charts for attributes such as defective or non-defective (Hines and Montgomery, 1990, Joglekar, 2003).

In a particular sample we observe $n_1, n_2, n_3, \ldots$ failures at times $1, 2, 3, \ldots$ and can calculate the EFR, $\hat{h}_1, \hat{h}_2, \hat{h}_3, \ldots$. On the basis of the EFR, we are interested in testing if a sample has a constant failure rate. The control chart will be used to distinguish between a constant process, coming from a geometric distribution, and an increasing or decreasing one, coming from another distribution. Thus, it may be used as a goodness-of-fit test for the geometric distribution.

The mean and standard deviation of the EFR distribution for a hypothesised geometric distribution will be used to establish a failure rate control chart, FRCC. The centreline of the FRCC is the mean of the EFR distribution and the two control bounds correspond to the standard deviation of the EFR distribution under the geometric assumption. If the parameter of the geometric distribution is unknown, then it must first be estimated before the FRCC can be set up.

When studying risk in terms of failure rates, peaks, troughs and regions of sparse data are typical. Kimber and Hansford (1993) used an exponentially smoothed version of the EFR to overcome this type of problem. Here, we will use an exponentially smoothed version of the EFR and compare the results with the results of the original EFR.

2.4.1 Properties of the EFR for the Geometric Distribution

Explicit expressions for $E(H_k \mid N_k^+ > 0)$, $STD(H_k \mid N_k^+ > 0)$, $E(H_k \mid K_{\max} = k_*)$ and $STD(H_k \mid K_{\max} = k_*)$ in the geometric case with parameter p are of interest.

From Proposition 2.1

$$P\left(H_k = \frac{n_k}{n_k^+} \mid N_k^+ > 0\right) = \begin{cases} \binom{n}{n_1} p_1^{n_1} (1-p_1)^{n-n_1} & \text{if } k = 1 \\ \sum \frac{(1-p_1-p_2)^n}{1-(1-p_1-p_2)^n} \binom{n}{n_k^+} \binom{n_k^+}{n_k} \left(\frac{p_1}{p_2}\right)^{n_k} \left(\frac{p_2}{1-p_1-p_2}\right)^{n_k^+} & \text{if } k > 1 \end{cases},$$

where $p_1 = f(k) = P(K = k)$, $p_2 = R(k) = P(K > k)$,

$1 - p_1 - p_2 = F(k-1) = P(K \le k-1), n_k = 0,1,2,\dots,n_k^+ \text{ and } n_k^+ = 1,2,3,\dots,n$,

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+.

In the geometric case

$$p_1 = P(K = k) = p(1-p)^{k-1},$$

$$p_2 = P(K > k) = \sum_{i=k+1}^{\infty} p(1-p)^{i-1} = (1-p)^k .$$

So,

$$P(H_k = \frac{n_k}{n_k^+} \mid N_k^+ > 0) = \begin{cases} \binom{n}{n_1} p^{n_1}(1-p)^{n-n_1} & \text{if } k = 1 \\ \sum \frac{\binom{n}{n_k^+}\binom{n_k^+}{n_k}\left(\frac{p}{1-p}\right)^{n_k}\left(\frac{(1-p)^k}{1-(1-p)^{k-1}}\right)^{n_k^+}}{(1-(1-(1-p)^{k-1})^n)(1-(1-p)^{k-1})^{-n}} & \text{if } k > 1 \end{cases}.$$

Hence, by Proposition 2.2,

$$E(H_k \mid N_k^+ > 0) = p \quad \forall\, k = 1,2,3,\ldots.$$

Further, from (2.1) it follows that

$$E(H_k^2 \mid N_k^+ > 0) = \begin{cases} p^2 + \frac{p(1-p)}{n} & \text{if } k = 1 \\ p^2 + \frac{np(1-p)^k\, {}_3F_2[1,1,1-n;2,2;(1-(1-p)^{1-k})^{-1}]}{(1-(1-p)^{k-1})^{1-n}(1-(1-(1-p)^{k-1})^n)} & \text{if } k > 1 \end{cases}.$$

Therefore,

$$\text{Var}(H_k \mid N_k^+ > 0) =$$

$$\begin{cases} \frac{p(1-p)}{n} & \text{if } k = 1 \\ \frac{np(1-p)^k(1-(1-p)^{k-1})^{n-1}\, {}_3F_2[1,1,1-n;2,2;(1-(1-p)^{1-k})^{-1}]}{1-(1-(1-p)^{k-1})^n} & \text{if } k > 1 \end{cases}.$$

For given values of n, k and p, the generalized hypergeometric function, ${}_3F_2[1,1,1-n;2,2;(1-(1-p)^{1-k})^{-1}]$, can be evaluated using any mathematical software such as Mathematica.

Now, since the explicit form of $STD(H_k \mid N_k^+ > 0)$ is complicated when $k > 1$, because of the generalized hypergeometric function, a relatively simple linear transformation for $STD(H_k \mid N_k^+ > 0)$ versus k is of interest. We know that

$$STD(H_1 \mid N_1^+ > 0) = \sqrt{p(1-p)/n}$$

probabilities of the equivalent fractions of n_k / n_k^+ for $k > 1$. There are no spikes for $k = 1$ because there are no equivalent fractions of n_k / n_k^+ in this case.

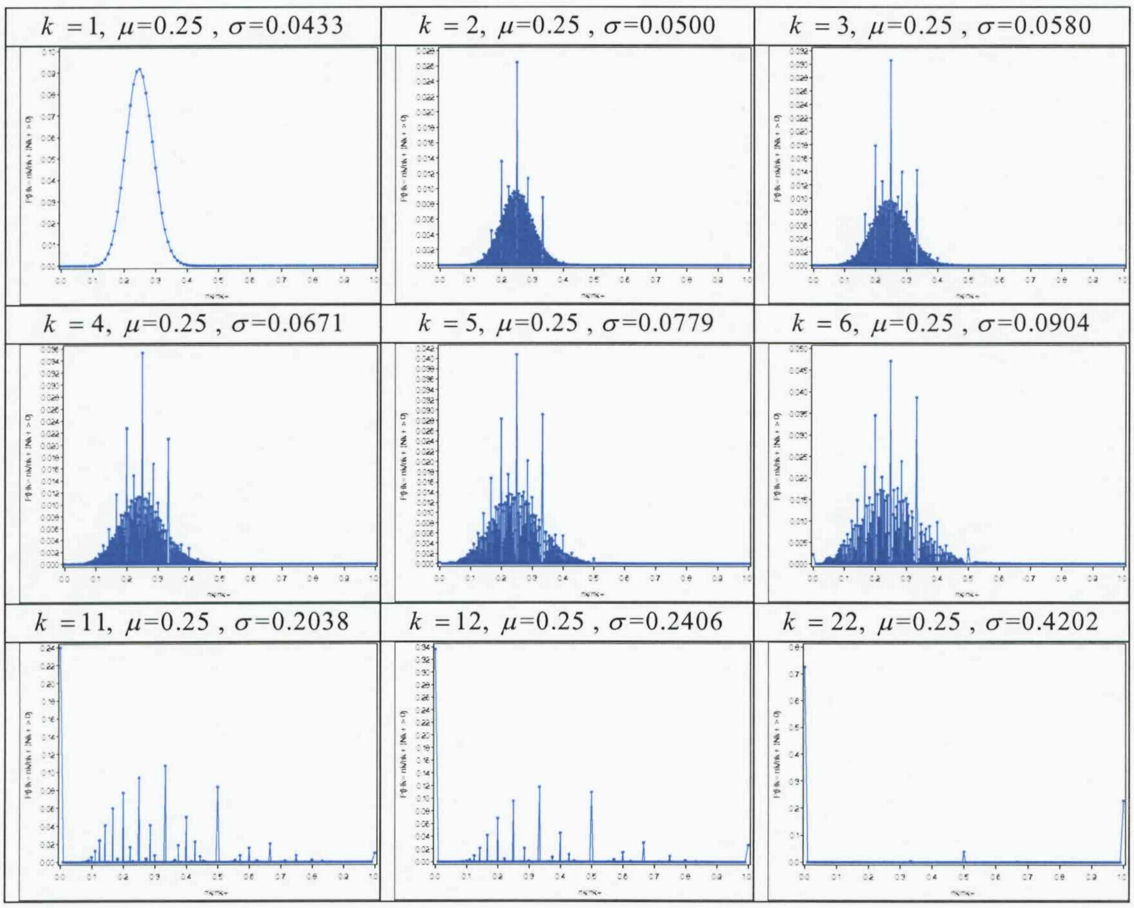


Figure 2.9: $P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the geometric(0.25) distribution with $n = 100$ where $\mu = E(H_k \mid N_k^+ > 0)$ and $\sigma = STD(H_k \mid N_k^+ > 0)$.

The standard deviation of $H_k \mid N_k^+ > 0$ is

$$STD(H_k \mid N_k^+ > 0) = \sqrt{E(H_k^2 \mid N_k^+ > 0) - [E(H_k \mid N_k^+ > 0)]^2},$$

where $E(H_k \mid N_k^+ > 0) = \sum_{0 \le n_k / n_k^+ \le 1} \frac{n_k}{n_k^+} P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$.

Now, in the geometric case Proposition 2.4 gives

$$P(H_k = \frac{n_k}{n_k^+} \mid K_{\max} = k_*) =$$

$$\begin{cases} \dbinom{n}{n_1} \dfrac{p^{n_1}(1-p)^{n-n_1}\left(1-(1-(1-p)^{k_*-2})^{n-n_1}\right)}{1-(1-(1-p)^{k_*-1})^n} & \text{if } k=1 \\ \sum \dfrac{n!\left(p(1-p)^{k-1}\right)^{n_k}\left[\left((1-p)^k\right)^{n_k^+-n_k} - \left((1-p)^k-(1-p)^{k_*-1}\right)^{n_k^+-n_k}\right]}{n_k!(n_k^+-n_k)!(n-n_k^+)!\left(1-(1-(1-p)^{k_*-1})^n\right)\left(1-(1-p)^{k-1}\right)^{n_k^+-n}} & \text{if } 1<k<k_*-1 \\ \sum \dfrac{n!\left(p(1-p)^{k_*-2}\right)^{n_k}\left((1-p)^{k_*-1}\right)^{n_k^+-n_k}\left(1-(1-p)^{k_*-2}\right)^{n-n_k^+}}{n_k!(n_k^+-n_k)!(n-n_k^+))!(1-(1-(1-p)^{k_*-1})^n)} & \text{if } k=k_*-1 \end{cases},$$

where $p_1 = P(K=k) = p(1-p)^{k-1}$, $p_2 = P(K>k) = \sum_{i=k+1}^{\infty} p(1-p)^{i-1} = (1-p)^k$,
$p_3 = P(k<K<k_*) = (1-p)^k - (1-p)^{k_*-1}$ and $p_4 = 1 - p_1 - p_2 - p_3$,
$\sum$ is taken over all the equivalent fractions of n_k/n_k^+.

Hence,

$$E(H_k \mid K_{\max}=k_*) = \begin{cases} p\,\dfrac{1-\left(1-(1-p)^{k_*-1}\right)^{n-1}}{1-\left(1-(1-p)^{k_*-1}\right)^n} & \text{if } k=1 \\ p - \dfrac{p(1-p)^{k_*}\left((1-(1-p)^{k_*-1})^n-(1-(1-p)^{k-1})^n\right)}{\left(1-(1-(1-p)^{k_*-1})^n\right)\left((1-p)^k-(1-p)^{k_*}\right)} & \text{if } 1<k<k_*-1 \\ 1-\dfrac{(1-p)\left(1-(1-(1-p)^{k_*-2})^n\right)}{1-(1-(1-p)^{k_*-1})^n} & \text{if } k=k_*-1 \end{cases}$$

and

$$E(H_k^2 \mid K_{\max}=k_*) =$$

$$\begin{cases} \dfrac{p\left(1+(n-1)p-(A_{k_*})^{n-2}(A_{k_*}+(n-1)p)\right)}{n\left(1-(A_{k_*})^n\right)} & \text{if } k=1 \\ \dfrac{p^2\left(\left(1-(A_k)^n\right)\left((1-p)^{2k_*}-2(1-p)^{k+k_*}\right)+\left(1-(A_{k_*})^n\right)(1-p)^{2k}\right)}{\left(1\text{-}(\mathrm{A}_{k_*})^n\right)\left((1-p)^k-(1-p)^{k_*}\right)^2} \\ \quad -\dfrac{np(1-p)^k(A_k)^{n-1}\,{}_3F_2[A]}{1-(A_{k_*})^n} \\ \quad -\dfrac{np(1-p)^{k-1}(A_k)^{n-1}\left((1-p)^{k+1}-(1-p)^{k_*}\right){}_3F_2[B]}{\left(1-(A_{k_*})^n\right)\left((1-p)^k-(1-p)^{k_*}\right)} & \text{if } 1<k<k_*-1 \\ 1-\dfrac{(1-p^2)(1-(A_{k2_*})^n)}{1-(A_{k_*})^n}+\dfrac{np(1-p)^{k_*-1}\left(A_{k2_*}\right)^{n-1}{}_3F_2[C]}{1-(A_{k_*})^n} & \text{if } k=k_*-1 \end{cases},$$

The mean and standard deviation of $H_k \mid N_k^+ > 0$ for the four lifetime distributions are plotted in Figure 2.10. Note that the values of the conditional standard deviation for the different sample sizes, n, tend to a limit as k increases.

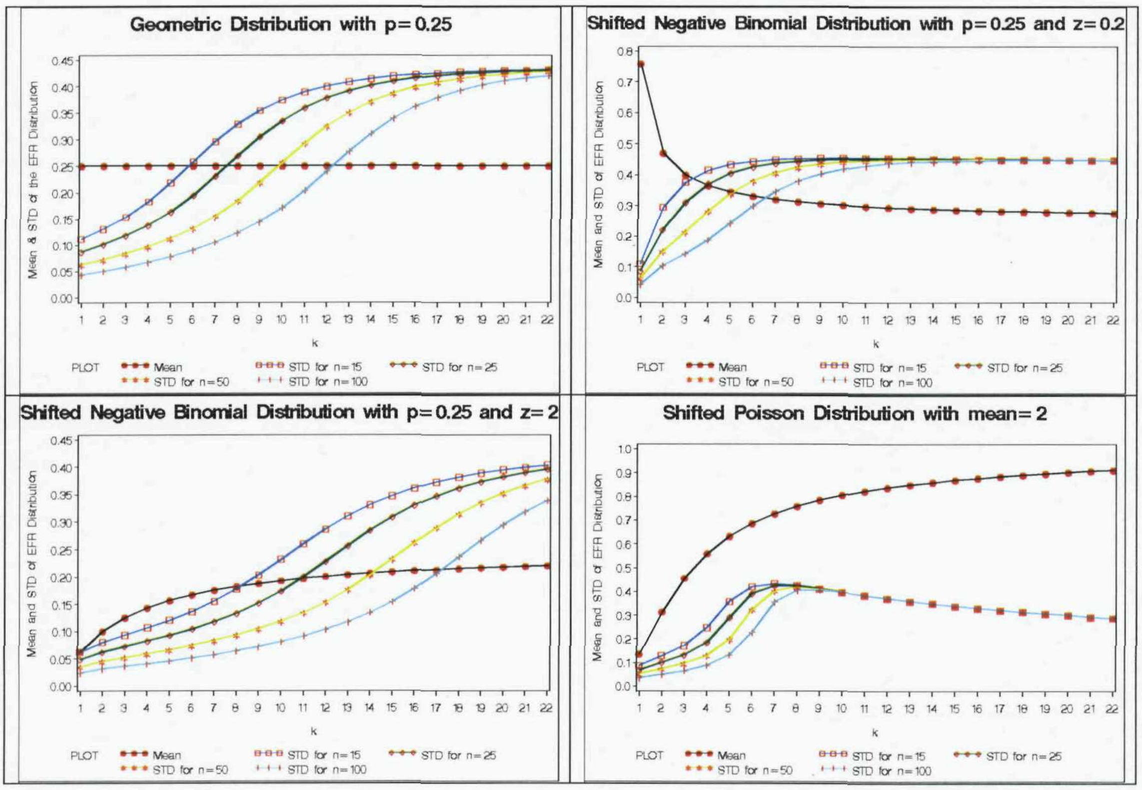


Figure 2.10: The mean and standard deviation of $H_k \mid N_k^+ > 0$ for four lifetime distributions.

Proposition 2.3:

$$STD(H_k \mid N_k^+ > 0) \sim \sqrt{h(k)[1-h(k)]} \text{ as } k \to \infty .$$

Proof:

From Proposition 2.1,

$$P\left(H_k = \frac{n_k}{n_k^+} \mid N_k^+ > 0\right) = \begin{cases} \binom{n}{n_1} p_1^{n_1} (1-p_1)^{n-n_1} & \text{if } k = 1 \\ \sum \dfrac{(1-p_1-p_2)^n}{1-(1-p_1-p_2)^n} \binom{n}{n_k^+} \binom{n_k^+}{n_k} \left(\dfrac{p_1}{p_2}\right)^{n_k} \left(\dfrac{p_2}{1-p_1-p_2}\right)^{n_k^+} & \text{if } k > 1 \end{cases},$$

where $p_1 = f(k) = P(K = k)$, $p_2 = R(k) = P(K > k)$,

$1 - p_1 - p_2 = F(k-1) = P(K \le k-1), n_k = 0,1,2,\dots,n_k^+ \text{ and } n_k^+ = 1,2,3,\dots,n$,

$\sum$ is over all the equivalent fractions of n_k / n_k^+.

where $A_k = 1-(1-p)^{k-1}, A_{k_*} = 1-(1-p)^{k_*-1}, A_{k2_*} = 1-(1-p)^{k_*-2}$,

$$ {}_3F_2[A] = {}_3F_2[1,1,1-n;2,2;\frac{1}{1-(1-p)^{1-k}}], $$

$$ {}_3F_2[B] = {}_3F_2[1,1,1-n;2,2;\frac{(1-p)^{k_*-1}-(1-p)^{k-1}}{1-(1-p)^{k-1}}], $$

$$ {}_3F_2[C] = {}_3F_2[1,1,1-n;2,2;\frac{1}{1-(1-p)^{2-k_*}}]. $$

Therefore,

$$ \mathrm{Var}(H_k \mid K_{\max} = k_*) = \begin{cases} \dfrac{p(1-p)(1-(A_{k_*})^{n-2}(1-(1-p)^{k_*-2})}{n\left(1-(A_{k_*})^n\right)} - \dfrac{p^2(1-p)^{2k_*-2}(A_{k_*})^{n-2}}{\left(1-(A_{k_*})^n\right)^2} & \text{if } k = 1 \\ \dfrac{p^2(1-p)^{2k_*}\left((A_k)^n-(A_{k_*})^n\right)(1-(A_k)^n)}{\left(1\text{-}(\mathrm{A}_{k_*})^n\right)^2\left((1-p)^k-(1-p)^{k_*}\right)^2} + \dfrac{np(1-p)^k(A_k)^{n-1}\,{}_3F_2[A]}{1-(A_{k_*})^n} \\ \qquad - \dfrac{np(1-p)^{k-1}(A_k)^{n-1}\left((1-p)^{k+1}-(1-p)^{k_*}\right){}_3F_2[B]}{\left(1-(A_{k_*})^n\right)\left((1-p)^k-(1-p)^{k_*}\right)} & \text{if } 1 < k < k_*-1 . \\ \dfrac{(1-p)^2(1-(A_{k2_*})^n)((A_{k2_*})^n-(A_{k_*})^n)}{(1-(A_{k_*})^n)^2} \\ \qquad + \dfrac{np\left(1-p\right)^{k_*-1}(A_{k2_*})^{n-1}\,{}_3F_2[C]}{1-(A_{k_*})^n} & \text{if } k = k_*-1 \end{cases} $$

Figures 2.18 – 2.20 illustrate the behaviour of $STD(H_k \mid N_k^+ > 0)$, $E(H_k \mid K_{\max} = k_*)$ and $STD(H_k \mid K_{\max} = k_*)$ respectively. As was noted before in 2.3.1, the values of $STD(H_k \mid N_k^+ > 0)$ for the different sample sizes tend to the limiting value $\sqrt{p(1-p)}$ as k increases. The values of $E(H_k \mid K_{\max} = k_*)$ for the different sample sizes start close to each other, then as k increased they diverge and then they get closer for large values of k and tend to zero. The values of $STD(H_k \mid K_{\max} = k_*)$ for the different sample sizes start apart with smaller value of $STD(H_k \mid K_{\max} = k_*)$ for the largest sample size then as k increased the order of the values of $STD(H_k \mid K_{\max} = k_*)$ reversed and then they decrease as k gets larger and tend to zero.

Now, putting $dl/dp = 0$ gives the maximum likelihood estimator since the second derivative, d^2l/dp^2, is negative which means that

$$\hat{p} = \frac{n}{\sum k_i} = \frac{1}{\bar{k}} \tag{2.5}$$

is the maximum value. Note that the MLE is equal to the moment estimator since $E(K) = \frac{1}{p}$ (Hines and Montgomery, 1990).

2.4.3 Method

Assuming that the data come from a geometric distribution with parameter p that is being estimated using the MLE, the estimated true failure rate, $h(k) = \hat{p} = 1/\bar{k}$, and the standard deviation of $H_k \mid N_k^+ > 0$ will be used to construct the FRCC.

The use of the control bounds assumes normality and the plots in Appendix B and C indicate that the EFR values are approximately normally distributed for large sample sizes apart from the spikes for $k > 1$. The centreline will be $h(k) = \hat{p}$ since the geometric distribution has constant FR. Two standard deviation control bounds $\hat{p} \pm 2\,\widehat{STD}(H_k \mid N_k^+ > 0)$ will be used to detect any departure from a constant failure rate where

$$\widehat{STD}(H_k \mid N_k^+ > 0) = \begin{cases} \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} & \text{if } k = 1 \\ \sqrt{\dfrac{n\hat{p}(1-\hat{p})^k (1-(1-\hat{p})^{k-1})^{n-1} {}_3F_2[1,1,1-n;2,2;(1-(1-\hat{p})^{1-k})^{-1}]}{1-(1-(1-\hat{p})^{k-1})^n}} & \text{if } k > 1 \end{cases}$$

Another FRCC can be constructed using the probability distribution of $H_k \mid K_{\max} = k_*$. Two standard deviation control bounds $\widehat{E}(H_k \mid K_{\max} = k_*) \pm 2\,\widehat{STD}(H_k \mid K_{\max} = k_*)$ will be used also to detect any departure from a constant failure rate.

The FRCC will be applied to the values of the EFR, $\hat{h}_1, \hat{h}_2, \hat{h}_3, ..., \hat{h}_{k_*}$, denoted FRCC1 when using the probability function of $H_k \mid N_k^+ > 0$ and FRCC2 when using the probability function of $H_k \mid K_{\max} = k_*$, and to an exponentially smoothed version of them, $\hat{h}_{s1}, \hat{h}_{s2}, \hat{h}_{s3}, ..., \hat{h}_{sk_*}$, denoted FRCC1s when using the probability function of $H_k \mid N_k^+ > 0$ and FRCC2s when using the probability function of $H_k \mid K_{\max} = k_*$, for a given data set in the same way using the mean and standard deviation of the EDF even though the smoothed version will have a different true STD which is not obtainable from our results. The exponentially smoothed version of the EFR values are given by the formulas (Kendall and Ord, 1990)

$$\hat{h}_{s1} = \hat{h}_1 ,$$
$$\hat{h}_{sk} = \alpha \hat{h}_k + (1-\alpha)\hat{h}_{k-1} \quad \forall\ 1 < k \le k_* ,$$

where $0 < \alpha < 1$ is the smoothing factor.

Here α is the value that minimizes the sum of squared errors, $\text{SSE} = \sum_k \left(\hat{h}_{sk} - \hat{p}\right)^2$, where $\hat{p}$ is the MLE of the parameter of the geometric distribution. We restrict α to lie between 0.1 and 0.9 in order to guard against very strong or very weak smoothing. We further restricted the permissible values of α to be $0.1(0.1)0.9$. On reflection, this was needlessly restrictive, but in further work it would be simple to remedy this. Indeed the choice of smoothing parameter will be re-considered in section 2.4.6.

The control bounds of the FRCC must be between 0 and 1. If a calculated upper bound is greater than 1, then it will be set to 1. Also, if a calculated lower bound is less than 0, then it will be set to 0. An EFR value at an observed failure time is considered to signal if it falls on or outside the control bounds. If more than 5% of the values of the EFR or the values of the smoothed version of the EFR signal, we can say that the data are unlikely to have a constant FR. The rejection criterion here is set by trial and error and will be justified via simulation.

The plots in Appendix C illustrate the behaviour of $P(H_k = n_k / n_k^+ \mid K_{\max} = k_*)$ for the geometric, negative-binomial and Poisson distributions. Figure 2.11 illustrates the behaviour of $P(H_k = n_k / n_k^+ \mid K_{\max} = 12)$ for a sample of size $n = 100$ from the geometric distribution with $p = 0.25$. For all these distributions, we found the shape of the distribution of $H_k \mid K_{\max} = k_*$ when the sample size n is not small to be bell shaped but with the possibility of spikes for $k > 1$. Those spikes are the result of accumulating the probabilities of the equivalent fractions of n_k / n_k^+ for $k > 1$. There are no spikes for $k = 1$ because there are no equivalent fractions of n_k / n_k^+ in this case.

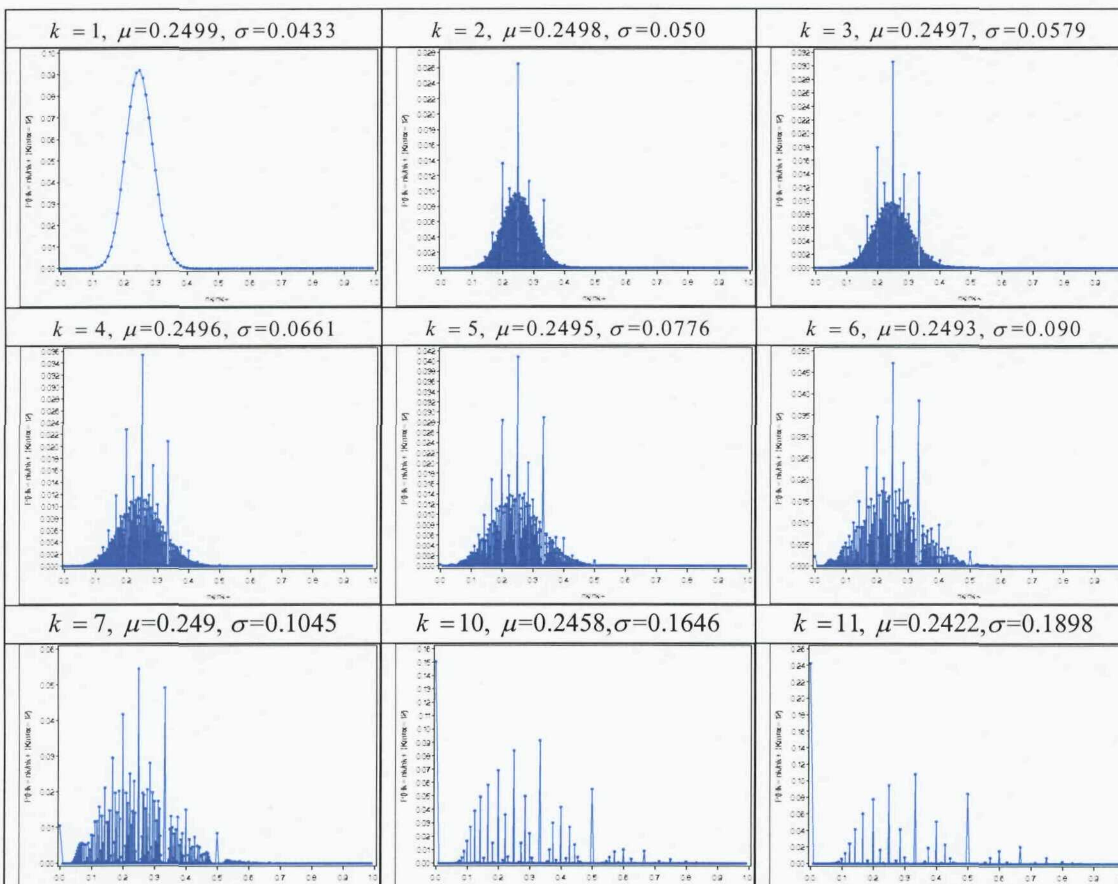


Figure 2.11: $P(H_k = n_k / n_k^+ \mid K_{\max} = 12)$ for the geometric(0.25) distribution with $n = 100$ where $\mu = E(H_k \mid K_{\max} = 12)$ and $\sigma = STD(H_k \mid K_{\max} = 12)$.

Example 2.2:

A sample with 50 observations simulated using SAS software version 9.1 from the geometric(0.25) distribution is displayed in Table 2.2, along with n_k^+ and $\hat{h}_k$.

Table 2.2: Simulated data for Example 2.2 along with n_k^+ and $\hat{h}_k$.

k	n_k	n_k^+	$\hat{h}_k$
1	13	50	0.26
2	6	37	0.16216
3	8	31	0.25806
4	6	23	0.26087
5	9	17	0.52941
6	2	8	0.25
7	4	6	0.66667
8	1	2	0.5
9	1	1	1

Now, $\widehat{E}(H_k \mid N_k^+ > 0) = \hat{p} = \frac{50}{175} = 0.28571$. $\widehat{STD}(H_k \mid N_k^+ > 0)$, $\widehat{E}(H_k \mid K_{\max} = 9)$ and $\widehat{STD}(H_k \mid K_{\max} = 9)$ are shown in Table 2.3. As an illustration, the calculations of $\widehat{STD}(H_2 \mid N_2^+ > 0)$, $\widehat{E}(H_2 \mid K_{\max} = 9)$, $\widehat{STD}(H_2 \mid K_{\max} = 9)$ and the FRCC lower bound, LB, and upper bound, UB, for $k = 2$ are as it follows:

For $k > 1$, we have

$$\widehat{STD}(H_k \mid N_k^+ > 0) = \sqrt{\frac{50\hat{p}(1-\hat{p})^k (1-(1-\hat{p})^{k-1})^{49} {}_3F_2[1,1,-49;2,2;(1-(1-\hat{p})^{1-k})^{-1}]}{1-(1-(1-\hat{p})^{k-1})^{50}}}.$$

So,

$$\widehat{STD}(H_2 \mid N_2^+ > 0) = \sqrt{\frac{50(0.28571)(0.71429)^2(0.28571)^{49} {}_3F_2[1,1,-49;2,2;\frac{1}{1-0.7149^{-1}}]}{1-0.28571^{50}}}$$

$$= 0.07591.$$

The standard deviation of $H_k \mid K_{\max} = k_*$ is

$$STD(H_k \mid K_{\max} = k_*) = \sqrt{E(H_k^2 \mid K_{\max} = k_*) - [E(H_k \mid K_{\max} = k_*)]^2},$$

where $E(H_k \mid K_{\max} = k_*) = \sum_{0 \le n_k / n_k^+ < 1} \frac{n_k}{n_k^+} P(H_k = n_k / n_k^+ \mid K_{\max} = k_*)$.

The mean and standard deviation of $H_k \mid K_{\max} = k_*$ are compared to the corresponding quantities for $H_k \mid N_k^+ > 0$ for four distributions which are plotted in Figures 2.12 - 2.15 respectively. When $k < k_*$ ($k_* > 1$), the mean and standard deviation of $H_k \mid K_{\max} = k_*$ behave similarly to those of $H_k \mid N_k^+ > 0$. They begin very close to the mean and standard deviation of $H_k \mid N_k^+ > 0$, especially for large sample sizes and then they diverge downward at the tail of the data.

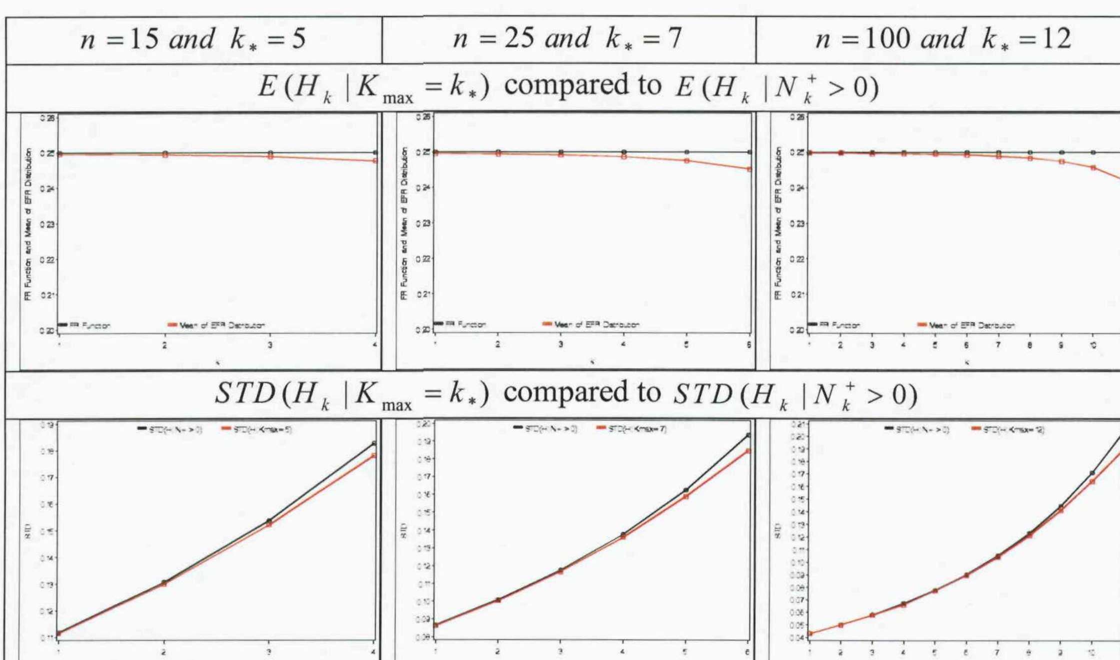


Figure 2.12: The conditional mean and standard deviation of the EFR distributions for the geometric(0.25) distribution.

Now, for $1 < k < 8$, we have

$$\widehat{E}(H_k \mid K_{\max} = 9) = \hat{p} - \frac{\hat{p}(1-\hat{p})^9\left((1-(1-\hat{p})^8)^{50} - (1-(1-\hat{p})^{k-1})^{50}\right)}{\left(1-(1-(1-\hat{p})^8)^{50}\right)\left((1-\hat{p})^k - (1-\hat{p})^9\right)}$$

and

$$\widehat{STD}(H_k \mid K_{\max} = 9) = \sqrt{\frac{\hat{\mathrm{p}}^2(1-\hat{p})^{18}\left((A_k)^{50} - (A_9)^{50}\right)(1-(A_k)^{50})}{\left(1-(\mathrm{A}_9)^{50}\right)^2\left((1-\hat{p})^k - (1-\hat{p})^9\right)^2} + \frac{n\hat{p}(1-\hat{p})^k (A_k)^{49}\,{}_3F_2[A]}{1-(A_9)^{50}} - \frac{50\hat{p}(1-\hat{p})^{k-1}(A_k)^{49}\left((1-\hat{p})^{k+1} - (1-\hat{p})^9\right){}_3F_2[B]}{\left(1-(A_9)^{50}\right)\left((1-\hat{p})^k - (1-\hat{p})^9\right)}}$$

where $A_k = 1-(1-\hat{p})^{k-1}$, $A_9 = 1-(1-\hat{p})^8$,

$${}_3F_2[A] = {}_3F_2[1,1,-49;2,2;\frac{1}{1-(1-\hat{p})^{1-k}}],$$

$${}_3F_2[B] = {}_3F_2[1,1,-49;2,2;\frac{(1-\hat{p})^8 - (1-\hat{p})^{k-1}}{1-(1-\hat{p})^{k-1}}].$$

So,

$$\widehat{E}(H_2 \mid K_{\max} = 9) = 0.28571 - \frac{0.28571(0.71429)^9\left((1-0.71429^8)^{50} - (0.28571)^{50}\right)}{\left(1-(1-(0.71429)^8)^{50}\right)\left((0.71429)^2 - (0.71429)^9\right)}$$
$$= 0.28479,$$

$A_2 = 0.28571,\ A_9 = 1-(0.71429)^8 = 0.93224$,

$${}_3F_2[A] = {}_3F_2[1,1,-49;2,2;\frac{1}{1-0.71429^{-1}}] = 3.6104\times 10^{23},$$

$${}_3F_2[B] = {}_3F_2[1,1,-49;2,2;\frac{0.71429^8 - 0.71429}{0.28571}] = 1.23124\times 10^{22}$$

and

$\widehat{STD}(H_2 \mid K_{\max} = 9) =$

$$\sqrt{\frac{0.28571^2(0.71429)^{18}\left(0.28571^{50} - 0.93224^{50}\right)(1-0.28571^{50})}{\left(1-(0.93224)^{50}\right)^2\left(0.71429^2 - 0.71429^9\right)^2} + \frac{50(0.28571)(0.71429)^2(0.28571)^{49}(3.6104\times 10^{23})}{1-(0.93224)^{50}} - \frac{50(0.28571)(0.71429)(0.28571)^{49}\left(0.71429^3 - 0.71429^9\right)(1.23124\times 10^{22})}{\left(1-0.93224^{50}\right)\left(0.71429^2 - 0.71429^9\right)}}$$

$$= 0.07561.$$

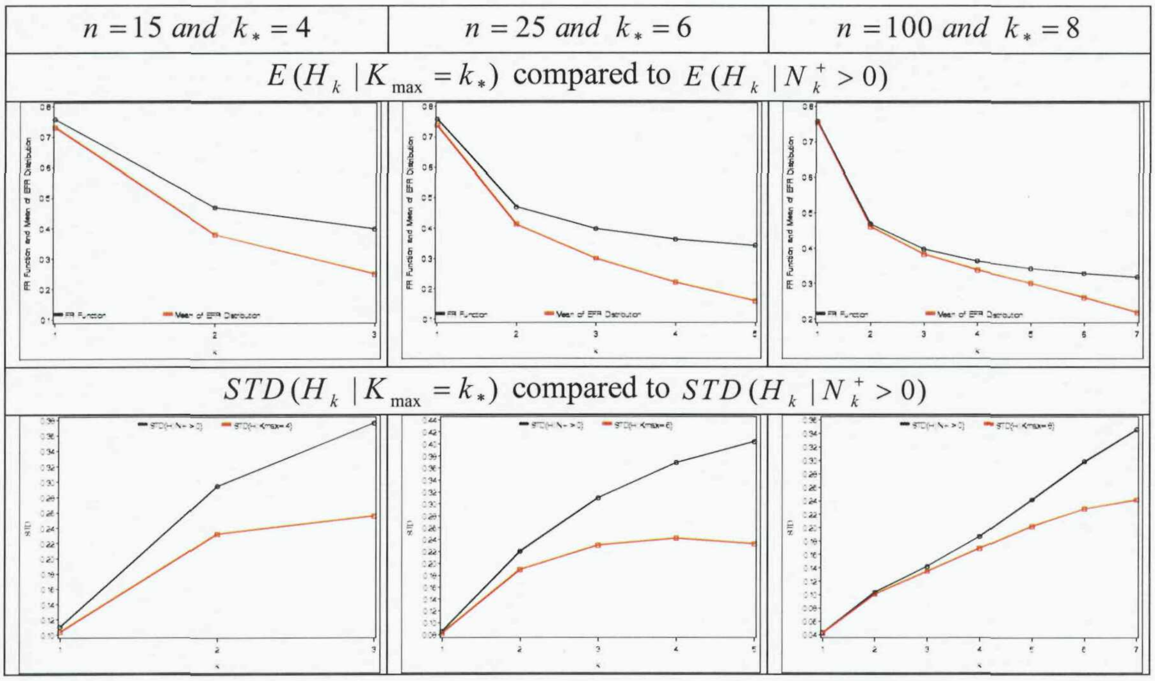


Figure 2.13: The conditional mean and standard deviation of the EFR distributions for the shifted negative-binomial(0.25, 0.2) distribution.

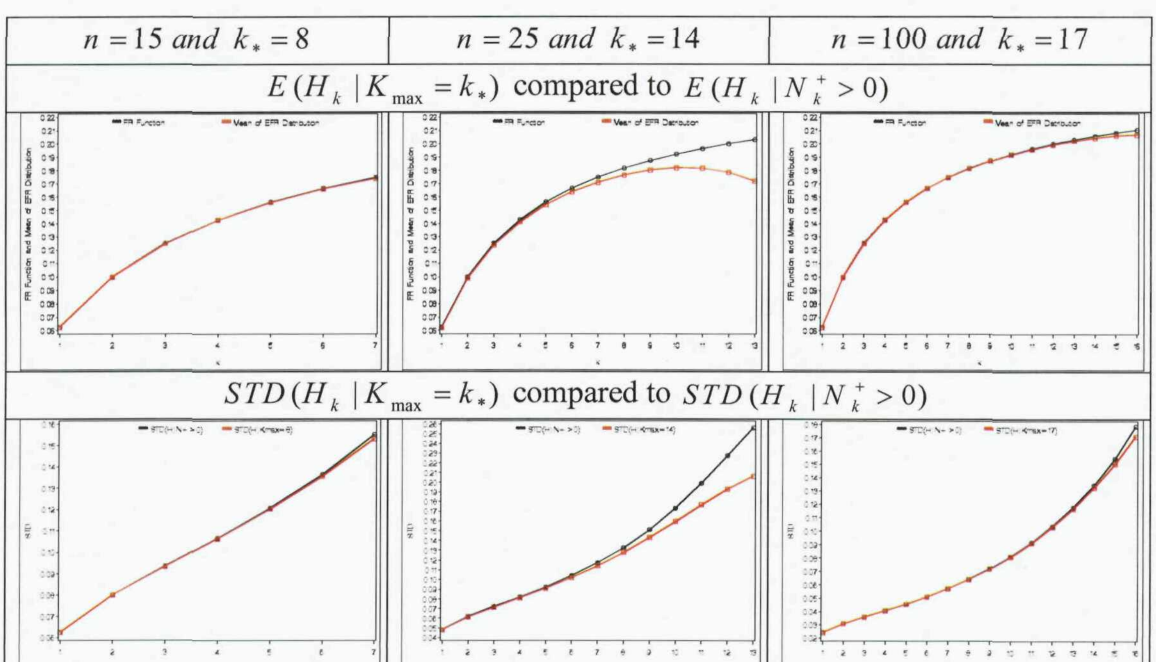


Figure 2.14: The conditional mean and standard deviation of the EFR distributions for the shifted negative-binomial(0.25, 2) distribution.

The LB for the FRCC1 using $H_2 \mid N_2^+ > 0$ is

$$\hat{p} - 2\,\widehat{STD}(H_2 \mid N_2^+ > 0) = 0.28571 - 2 \times 0.07591 = 0.13389$$

and the UB is

$$\hat{p} + 2\,\widehat{STD}(H_2 \mid N_2^+ > 0) = 0.28571 + 2 \times 0.07591 = 0.43753.$$

The LB for the FRCC2 using $H_2 \mid K_{\max} = 9$ is

$$\widehat{E}(H_2 \mid K_{\max} = 9) - 2\,\widehat{STD}(H_2 \mid K_{\max} = 9) = 0.28479 - 2 \times 0.07561 = 0.13357$$

and the UB is

$$\widehat{E}(H_2 \mid K_{\max} = 9) + 2\,\widehat{STD}(H_2 \mid K_{\max} = 9) = 0.28479 + 2 \times 0.07561 = 0.43601.$$

Table 2.3: The $\widehat{STD}(H_k \mid N_k^+ > 0)$, $\widehat{E}(H_k \mid K_{\max} = 9)$ and $\widehat{STD}(H_k \mid K_{\max} = 9)$ for Example 2.2.

k	$\widehat{STD}(H_k \mid N_k^+ > 0)$	$\widehat{E}(H_k \mid K_{\max} = 9)$	$\widehat{STD}(H_k \mid K_{\max} = 9)$
1	0.06389	0.28507	0.06374
2	0.07591	0.28479	0.07561
3	0.09035	0.28436	0.08978
4	0.10785	0.28370	0.10673
5	0.12932	0.28261	0.12705
6	0.15631	0.28066	0.15129
7	0.19112	0.27677	0.17931
8	0.23410	0.26871	0.20898
9	0.28022		

Now, in order to find the appropriate $\hat{h}_{sk}$, we need to find the value of the smoothing factor α that minimizes the sum of the squared errors, $\text{SSE} = \sum_k \left(\hat{h}_{sk} - \hat{p}\right)^2$ subject to the restrictions of α that have been imposed. Table 2.4 shows the minimum SSE is when $\alpha = 0.1$ (though of course the unrestricted minimum could occur between 0 and 0.2). Thus, $\hat{h}_{sk}$ will be calculated such that $\hat{h}_{s1} = 0.26$ and $\hat{h}_{sk} = 0.1\,\hat{h}_k + 0.9\,\hat{h}_{k-1}$ $\forall\ 1 < k \le k_*$.

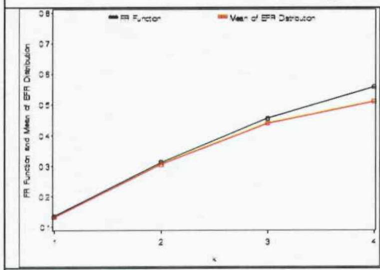
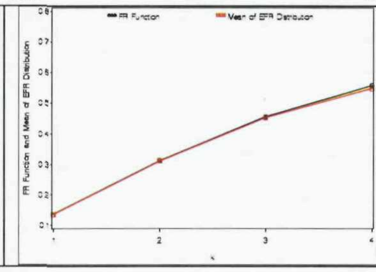
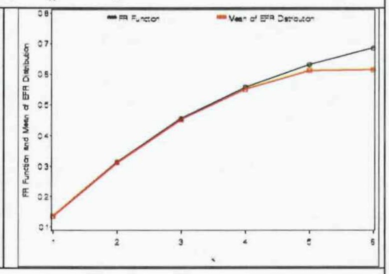
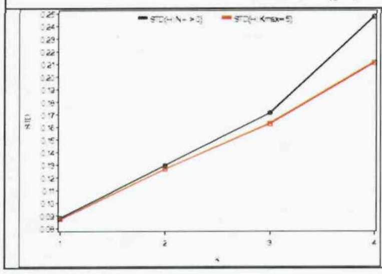
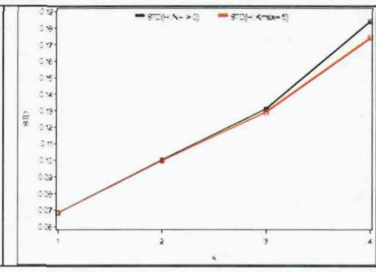
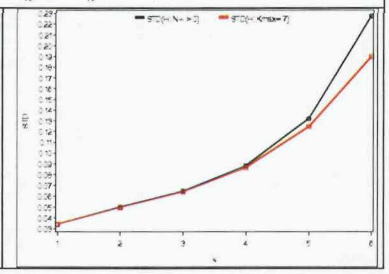
$n = 15$ *and* $k_* = 5$	$n = 25$ *and* $k_* = 5$	$n = 100$ *and* $k_* = 7$
$E(H_k \mid K_{\max} = k_*)$ compared to $E(H_k \mid N_k^+ > 0)$		
		
$STD(H_k \mid K_{\max} = k_*)$ compared to $STD(H_k \mid N_k^+ > 0)$		
		

Figure 2.15: The conditional mean and standard deviation of the EFR distributions for the shifted Poisson(2) distribution.

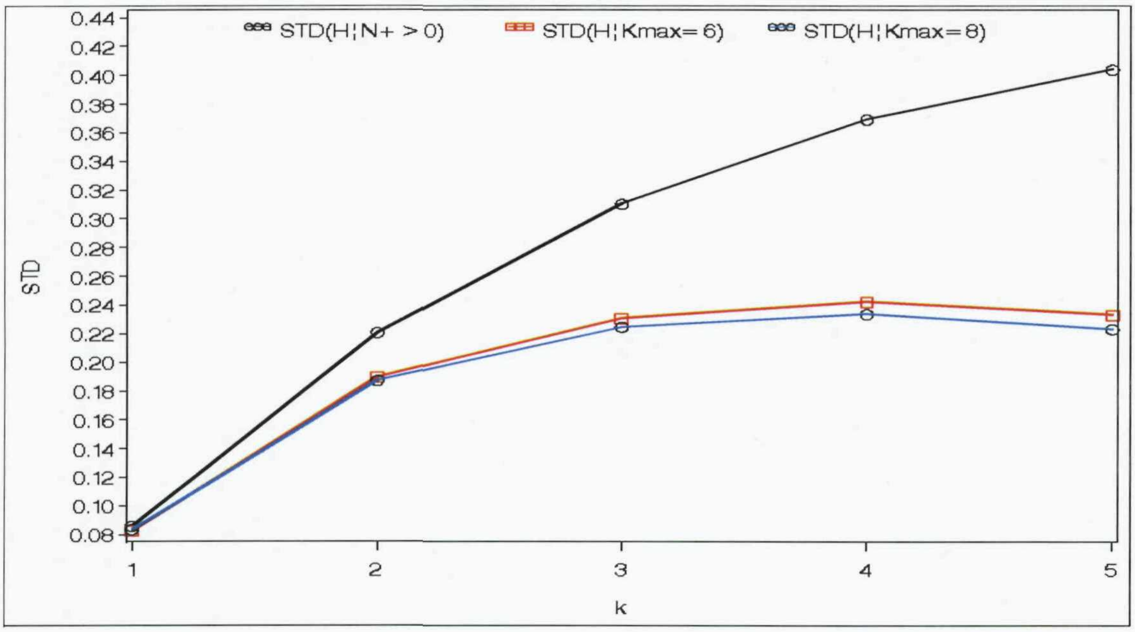


Figure 2.16: The conditional standard deviation of the EFR distributions for the shifted negative-binomial$(0.25, 0.2)$ distribution.

Table 2.4: SSE for Example 2.2.

k	1	2	3	4	5	6	7	8	9	
$\hat{h}_k$	0.26	0.162	0.258	0.261	0.529	0.25	0.667	0.5	1	
α	$\hat{h}_{sk}$									SSE
0.1	0.26	0.250	0.251	0.252	0.280	0.277	0.316	0.334	0.401	0.021
0.2	0.26	0.240	0.244	0.247	0.304	0.293	0.368	0.394	0.515	0.077
0.3	0.26	0.231	0.239	0.245	0.331	0.306	0.414	0.440	0.608	0.154
0.4	0.26	0.221	0.236	0.246	0.359	0.315	0.456	0.474	0.684	0.238
0.5	0.26	0.211	0.235	0.248	0.389	0.319	0.493	0.496	0.748	0.323
0.6	0.26	0.201	0.235	0.251	0.418	0.317	0.527	0.511	0.804	0.408
0.7	0.26	0.191	0.238	0.254	0.447	0.309	0.559	0.518	0.855	0.492
0.8	0.26	0.182	0.243	0.257	0.475	0.295	0.592	0.518	0.904	0.580
0.9	0.26	0.172	0.249	0.260	0.502	0.275	0.627	0.513	0.951	0.674

Table 2.5 shows the LB and UB of the FRCC using the probability functions of $H_k \mid N_k^+ > 0$ and $H_k \mid K_{\max} = 9$ along with the results of FRCC1, FRCC1s, FRCC2 and FRCC2s. Also, plots of the FRCC1, FRCC1s, FRCC2 and FRCC2s are presented in Figure 2.21.

Table 2.5: The results of the FRCC for Example 2.2.

k	$\hat{h}_k$	$\hat{h}_{sk}$	$H_k \mid N_k^+ > 0$		FRCC1		FRCC1s		$H_k \mid K_{\max} = 9$		FRCC2		FRCC2s	
			LB	UB	in	out	in	out	LB	UB	in	out	in	out
1	0.26	0.26	0.16	0.41	1	0	1	0	0.16	0.41	1	0	1	0
2	0.16	0.25	0.13	0.44	2	0	2	0	0.13	0.44	2	0	2	0
3	0.26	0.25	0.10	0.47	3	0	3	0	0.10	0.46	3	0	3	0
4	0.26	0.25	0.07	0.50	4	0	4	0	0.07	0.50	4	0	4	0
5	0.53	0.28	0.03	0.54	5	0	5	0	0.03	0.54	5	0	5	0
6	0.25	0.28	0	0.60	6	0	6	0	0	0.58	6	0	6	0
7	0.67	0.32	0	0.67	7	0	7	0	0	0.63	6	1	7	0
8	0.5	0.33	0	0.75	8	0	8	0	0	0.69	7	1	8	0
9	1	0.40	0	0.85	8	1	9	0						

Table 2.6: Simulated data for Example 2.3 along with n_k^+ and $\hat{h}_k$.

k	n_k	n_k^+	$\hat{h}_k$
1	6	50	0.12
2	12	44	0.27273
3	12	32	0.375
4	14	20	0.7
5	5	6	0.83333
6	0	1	0
7	0	1	0
8	0	1	0
9	1	1	1

Table 2.7: The $\widehat{STD}(H_k \mid N_k^+ > 0)$, $\widehat{E}(H_k \mid K_{\max} = 9)$ and $\widehat{STD}(H_k \mid K_{\max} = 9)$ for Example 2.3.

k	$\widehat{STD}(H_k \mid N_k^+ > 0)$	$\widehat{E}(H_k \mid K_{\max} = 9)$	$\widehat{STD}(H_k \mid K_{\max} = 9)$
1	0.0660	0.31886	0.0657
2	0.08046	0.31802	0.07980
3	0.09834	0.31672	0.09699
4	0.12071	0.31463	0.11782
5	0.14931	0.31110	0.14304
6	0.18728	0.30470	0.17272
7	0.23668	0.29244	0.20479
8	0.29130	0.27054	0.23376
9	0.34049		

In order to find the appropriate $\hat{h}_{sk}$, the value of the smoothing factor α was found in the same way as for example 2.2. Thus, for this example $\alpha = 0.2$. So, $\hat{h}_{sk}$ will be calculated such that $\hat{h}_{s1} = 0.12$ and $\hat{h}_{sk} = 0.2\,\hat{h}_k + 0.8\,\hat{h}_{k-1}$ $\forall\ 1 < k \le k_*$.

Table 2.8 shows $\hat{h}_{sk}$, the LB and UB of the FRCC using the probability functions of $H_k \mid N_k^+ > 0$ and $H_k \mid K_{\max} = k_*$ along with the results of FRCC1, FRCC1s, FRCC2 and FRCC2s. Also, plots of the FRCC1, FRCC1s, FRCC2 and FRCC2s are presented in Figure 2.22.

2.4.4 Simulation Study

In this section we investigate the performance of the proposed FRCC methods (FRCC1, FRCC2, FRCC1s and FRCC2s), the Pearson's chi-square test, χ_p^2, and the likelihood ratio test, χ_D^2 (Agresti, 2002) through a simulation study. The use of the chi-square tests with small expected frequencies is controversial (Koehler and Larntz, 1980, Roscoe and Byars, 1971). Here, the chi-square tests are applied directly without any modification to the simulated data.

To estimate the true significance levels of the tests, samples of sizes 25, 50, 100 and 200 observations have been simulated from the geometric distribution with $p = 0.1, 0.15, 0.2, 0.25, 0.3$ and 0.4 (flat FR). Also to estimate the true power of the tests, samples of sizes 25, 50, 100 and 200 observations have been simulated from the negative binomial distribution with parameters $p = 0.1, 0.15, 0.2, 0.25, 0.3, 0.4$ and $z = 0.2, 2$ (decreasing and gently increasing FR respectively) and Poisson distribution with mean $\lambda = 1, 1.5, 2, 2.5, 3, 4$ (FR the increases to 1).

The simulated samples were generated using SAS software version 9.1. At a nominal 5% significance level the reject / not reject inference was made for each test procedure. This was repeated 1000 times and the percentage of rejections was calculated for each test to estimate the true significance level.

Table 2.9 shows the percentage of samples for which the geometric hypothesis is rejected when simulating from the geometric distribution (significance) using the probability function of $H_k \mid N_k^+ > 0$ and $H_k \mid K_{\max} = k_*$. The estimated significance level of the FRCC for the EFR values, FRCC1 and FRCC2, is very large which means that this method will not be useful. The estimated significance level of the FRCC for the exponentially smoothed version of the EFR values, FRCC1s and FRCC2s, is conservative whereas χ_P^2 and χ_D^2 are liberal. Also, the estimated significance level of FRCC1s is slightly lower than that of FRCC2s. χ_P^2 is very liberal, more so than χ_D^2. This may be due to the effect of the small expected frequencies.

Table 2.9: Estimated significance of the FRCC methods, χ_p^2 and χ_D^2 at the nominal 5% significance level.

Geometric (p)	$p=0.10$				$p=0.15$				$p=0.20$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1	100	100	100	100	99.8	99.9	99.0	99.2	98.2	98.0	98.1	96.9
FRCC2	100	100	100	100	99.9	100	99.1	99.6	98.5	98.6	98.3	97.8
FRCC1s	0.9	0.4	0.5	0.2	1.2	0.9	0.5	0.2	1.4	1.1	0.9	0.3
FRCC2s	1.0	0.7	0.5	0.2	1.4	1.0	0.5	0.3	1.5	1.2	0.8	0.4
χ_p^2	43.9	42.9	39.2	39.5	34.8	32.7	32.3	34.1	28.0	27.7	27.2	29.9
χ_D^2	4.6	5.3	5.8	4.3	6.3	6.3	7.5	5.8	6.8	8.4	7.7	7.6
Geometric (p)	$p=0.25$				$p=0.30$				$p=0.40$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1	93.6	93.5	93.5	94.4	86.5	87.5	87.1	87.4	70.8	70.0	70.7	73.7
FRCC2	94.9	94.5	94.6	94.9	88.1	88.1	88.6	88.9	71.4	71.3	71.7	74.6
FRCC1s	1.4	2.3	1.9	1.3	1.7	1.5	1.2	2.0	1.3	0.8	0.6	1.1
FRCC2s	2.0	2.4	2.2	1.3	2.9	2.5	2.0	2.7	4.6	4.4	4.0	6.2
χ_p^2	23.8	23.1	25.3	23.6	20.6	19.8	21.8	21.9	15.7	15.2	17.1	20.2
χ_D^2	6.8	7.5	7.2	7.8	7.8	7.9	7.5	7.8	8.4	8.2	7.1	9.0

On the other hand, Table 2.10 shows the estimated power of the tests at the nominal 5% significance level. We first consider the tests based on FRCC1s and FRCC2s. The estimated powers were mostly rather similar except in a few situations where the estimated power for FRCC2s was considerably higher than that of FRCC1s; see for example the negative binomial cases with $z=0.2$ and $n=25$. Thus, the test based on FRCC2s may be preferable to that based on FRCC1s.

Table 2.10: Estimated power of the FRCC methods and χ_D^2 at the nominal 5% significance level.

Negative binomial $(p,0.2)$	$p=0.10$				$p=0.15$				$p=0.20$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1s	64.1	92.4	97.1	99.6	41.6	79.3	97.8	99.5	25.0	58.8	91.3	99.1
FRCC2s	85.1	97.8	97.8	99.7	75.0	93.5	99.5	99.8	63.5	85.4	97.3	99.8
χ_D^2	50.4	87.1	97.7	99.7	46.2	83.2	98.8	99.8	38.0	73.9	95.9	100
Negative binomial $(p,0.2)$	$p=0.25$				$p=0.30$				$p=0.40$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1s	14.8	40.4	77.3	98.0	8.6	24.6	58.4	90.1	3.1	8.0	24.9	52.1
FRCC2s	53.5	74.8	91.8	99.5	43.9	64.8	84.5	96.6	29.6	45.5	64.2	80.9
χ_D^2	32.8	64.0	90.8	99.8	27.4	54.7	83.2	97.6	17.8	36.2	64.9	89.8
Negative binomial $(p,2)$	$p=0.10$				$p=0.15$				$p=0.20$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1s	14.1	2.2	8.6	69.4	13.7	5.1	35.8	89.5	10.3	14.3	52.8	92.6
FRCC2s	14.1	2.2	8.5	68.9	13.7	5.0	35.6	89.4	10.3	13.8	52.3	92.4
χ_D^2	9.4	22.8	48.0	80.8	16.6	31.4	58.3	90.2	24.2	36.9	63.1	91.7
Negative binomial $(p,2)$	$p=0.25$				$p=0.30$				$p=0.40$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1s	10.7	20.9	54.8	92.8	11.4	25.9	56.4	91.5	12.9	26.6	56.5	90.3
FRCC2s	10.7	20.1	54.3	92.8	10.9	25.5	56.1	91.3	12.0	26.2	55.8	90.3
χ_D^2	25.8	37.0	62.9	92.3	26.6	36.2	60.9	91.8	23.1	31.3	54.9	84.0
Poisson (λ)	$\lambda=1$				$\lambda=1.5$				$\lambda=2$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1s	18.8	43.7	85.3	99.1	43.6	80.7	99.1	99.6	62.6	93.9	99.5	99.5
FRCC2s	17.0	43.6	84.9	99.1	41.6	80.7	99.1	99.6	61.9	93.8	99.5	99.5
χ_D^2	42.1	69.2	95.2	99.2	42.1	69.2	95.2	99.2	78.0	98.6	99.6	99.7
Poisson (λ)	$\lambda=2.5$				$\lambda=3$				$\lambda=4$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
FRCC1s	73.6	97.4	99.8	99.8	79.0	94.4	99.7	99.6	80.6	98.1	99.3	99.3
FRCC2s	73.3	97.4	99.8	99.8	78.8	94.4	99.7	99.6	79.9	98.1	99.3	99.3
χ_D^2	78.0	98.6	99.6	99.7	92.9	95.1	99.7	99.6	95.5	98.2	99.3	99.1

then the $STD(H_k \mid N_k^+ > 0)$ climbs smoothly to $\sqrt{p(1-p)}$ for large k. We postulate a straightforward approximation

$$STD(H_k \mid N_k^+ > 0) \approx n^{(k-1)/2k} \sqrt{p(1-p)/n}\,. \tag{2.3}$$

Also, we postulate another approximation using (2.2),

$$STD(H_k \mid N_k^+ > 0) \approx \begin{cases} \sqrt{\dfrac{p(1-p)}{n}} & \text{if } k = 1 \\ \sqrt{\dfrac{np(1-p)^k (1-(1-p)^{k-1})^{n-1}(1+(n-1)(1-p)^{k-1}/4)}{1-(1-(1-p)^{k-1})^n}} & \text{if } k > 1 \end{cases}. \tag{2.4}$$

Figure 2.17 illustrates the performance of the two approximations compared to the $STD(H_k \mid N_k^+ > 0)$ for the geometric distribution with $p = 0.2$ and 0.25 and sample sizes $n = 50$ and 100. The two approximations performed badly for small values of $k > 1$ and as k gets larger the performance of the two approximations improved but approximation (2.4) becomes equal to $STD(H_k \mid N_k^+ > 0)$ faster than approximation (2.3). Hence, we recommend the use of approximation (2.4) for large values of k.

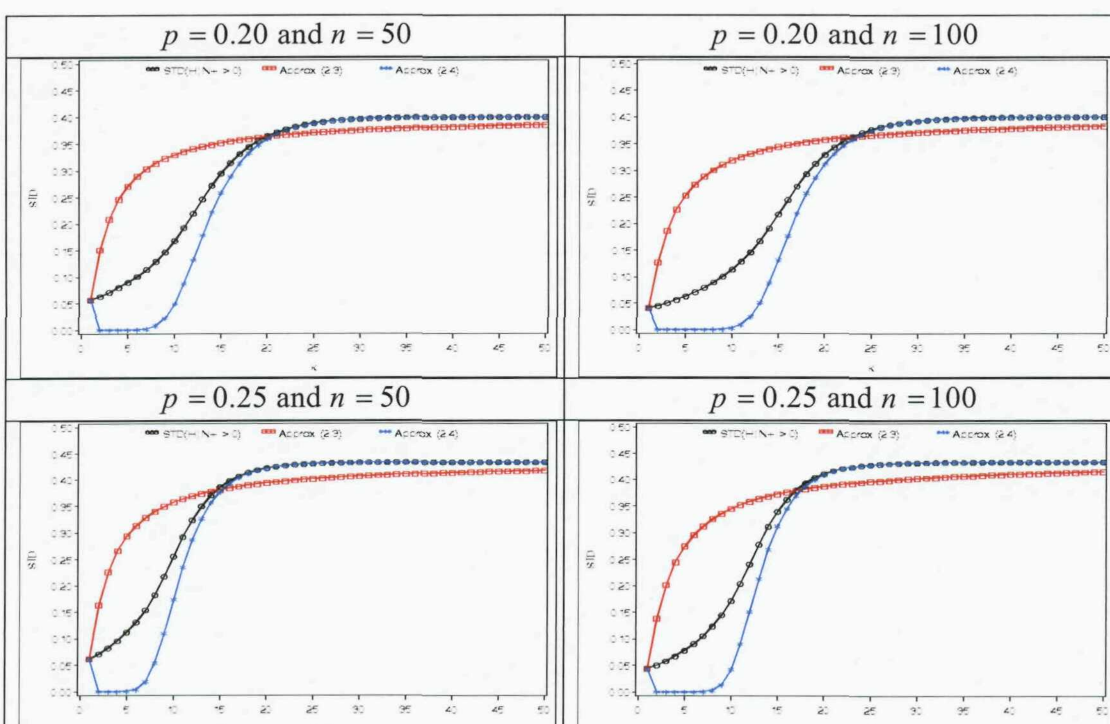


Figure 2.17: The standard deviation of $H_k \mid N_k^+ > 0$, approximation (2.3) and approximation (2.4) for the geometric distribution.

Direct comparison between χ_D^2 and the FRCC tests should be treated with caution since χ_D^2 is slightly liberal and the FRCC tests are conservative. However, in small sample sizes FRCC2s outperforms χ_D^2 in the decreasing FR case, whereas χ_D^2 outperforms the FRCC tests in the increasing FR case. The odd behaviour of FRCC1s and FRCC2s in the case of the negative binomial with $z = 2$ is due to the very slow increase in the EFR values when p and n are small. When $n = 25$, the regions of sparse data affected the power results.

The estimated significance levels of the FRCC for the exponentially smoothed version of the EFR values, FRCC1s and FRCC2s, are much lower than the nominal 5% significance level, especially for FRCC1s. Also, the estimated powers for FRCC1s and FRCC2s were low for sample sizes of 100 or less, especially for samples with very slow increase in the EFR values. Of course, the low true significance level may adversely affect the power of the test. Some further tuning of the control bounds or reduction of the 5% signal requirement is needed.

Given these results it appears that further research into the FRCC tests may be worthwhile, especially when a decreasing FR alternative is appropriate.

2.4.5 Application

The FRCC for detecting departure from a constant FR is applied to two real data sets that have been cited in Bracquemond et al. (2002). The first data set consists of numbers of thousands of demands before failure of electromechanical devices during mechanical reliability trials and is presented in Table 2.11. The second data set consists of numbers of inspections between discovery of defects in an industrial process and is presented in Table 2.12.

Table 2.11: Electromechanical devices data set.

12	15	15	15	15	17	18	18	19	20	20	
22	22	23	23	24	25	25	25	29	31	32	32

Table 2.12: Inspection data set.

13	5	2	1	2	1	9	1	3	2	1	4	1	4
1	2	29	5	18	14	7	17	3	14	3	11	26	4

Seven goodness-of-fit tests for the geometric distribution, Cramer-von Mises ($W_n^{\,2}$), Anderson-Darling (A_n^2), Kocherlakota and Kocherlakota ($\widehat{KK_n}$), Rueda, Perez-Abreu and O'Reilly ($\widehat{RPO_n}$), Baringhaus and Henze ($\widehat{BH_n}$), Neyman smooth ($\widehat{S}_4$), and the generalized Smirnov transformation ($\widehat{A}^2_{GST}$), were applied to the two data sets by Bracquemond (2002) and their results are presented here in Table 2.13.

Table 2.13: Results of the seven goodness-of-tests applied by Bracquemond et al. (2002).

Test	$W_n^{\,2}$	A_n^2	$\widehat{KK_n}$	$\widehat{RPO_n}$	$\widehat{BH_n}$	$\widehat{S}_4$	$\widehat{A}^2_{GST}$	Conclusion
Devices	0.925	4.685	-1.975	0.154	34.939	23.301	5.897	Reject
Inspection	0.164	0.835	1.168	0.038	3.571	4.390	1.084	Accept

The Pearson chi-square and likelihood ratio tests have p-values close to 0 for the electromechanical devices data set, indicating that the data are not likely to come from the geometric distribution. For the inspection data set they have p-values 0.08415 and 0.48581 respectively, indicating that the geometric hypothesis should not be rejected.

The FRCC for the two data sets are plotted in Figure 2.23. Figure 2.23 left illustrates that for the electromechanical devices data 23% of the values fall outside the control bounds for both FRCC1s and FRCC2s, indicating that the data are unlikely to come from the geometric distribution. For the inspection data set, Figure 2.23 right, all the values fall inside the control bounds of FRCC1s and FRCC2s, indicating that the geometric hypothesis should not be rejected.

Table 2.14 shows $\hat{h}_{sk}$, the LB and UB of the FRCC using the probability functions of $H_k \mid N_k^+ > 0$ and $H_k \mid K_{\max} = k_*$ along with the results of FRCC1, FRCC1s, FRCC2 and FRCC2s. Also, plots of the FRCC1, FRCC1s, FRCC2 and FRCC2s are presented in Figure 2.24.

The results of the FRCC are similar to those of example 2.2. Thus, the results of using the restricted and unrestricted smoothing factor α are the same for this example.

Example 2.5:

Using the simulated data in example 2.3, we would like to find the appropriate $\hat{h}_{sk}$ subject to the unrestricted smoothing factor α that minimizes the sum of squared errors. Thus, $\alpha = 0.18467$. So, $\hat{h}_{sk}$ will be calculated such that $\hat{h}_{s1} = 0.12$ and $\hat{h}_{sk} = 0.18467\,\hat{h}_k + 0.81533\,\hat{h}_{k-1}\ \ \forall 1 < k \le k_*$.

Table 2.15 shows $\hat{h}_{sk}$, the LB and UB of the FRCC using the probability functions of $H_k \mid N_k^+ > 0$ and $H_k \mid K_{\max} = k_*$ along with the results of FRCC1, FRCC1s, FRCC2 and FRCC2s. Also, plots of the FRCC1, FRCC1s, FRCC2 and FRCC2s are presented in Figure 2.25.

Table 2.15: The results of the FRCC for Example 2.5.

k	$\hat{h}_k$	$\hat{h}_{sk}$	$H_k \mid N_k^+ > 0$		FRCC1		FRCC1s		$H_k \mid K_{\max} = 9$		FRCC2		FRCC2s	
			LB	UB	in	out	in	out	LB	UB	in	out	in	out
1	0.12	0.12	0.19	0.45	0	1	0	1	0.19	0.45	0	1	0	1
2	0.27	0.15	0.16	0.48	1	1	0	2	0.16	0.48	1	1	0	2
3	0.38	0.19	0.12	0.52	2	1	1	2	0.12	0.51	2	1	1	2
4	0.7	0.28	0.08	0.56	2	2	2	2	0.08	0.55	2	2	2	2
5	0.83	0.39	0.02	0.62	2	3	3	2	0.02	0.60	2	3	3	2
6	0	0.31	0	0.69	2	4	4	2	0	0.65	2	4	4	2
7	0	0.26	0	0.79	2	5	5	2	0	0.70	2	5	5	2
8	0	0.21	0	0.90	2	6	6	2	0	0.74	2	6	6	2
9	1	0.36	0	1	2	7	7	2						

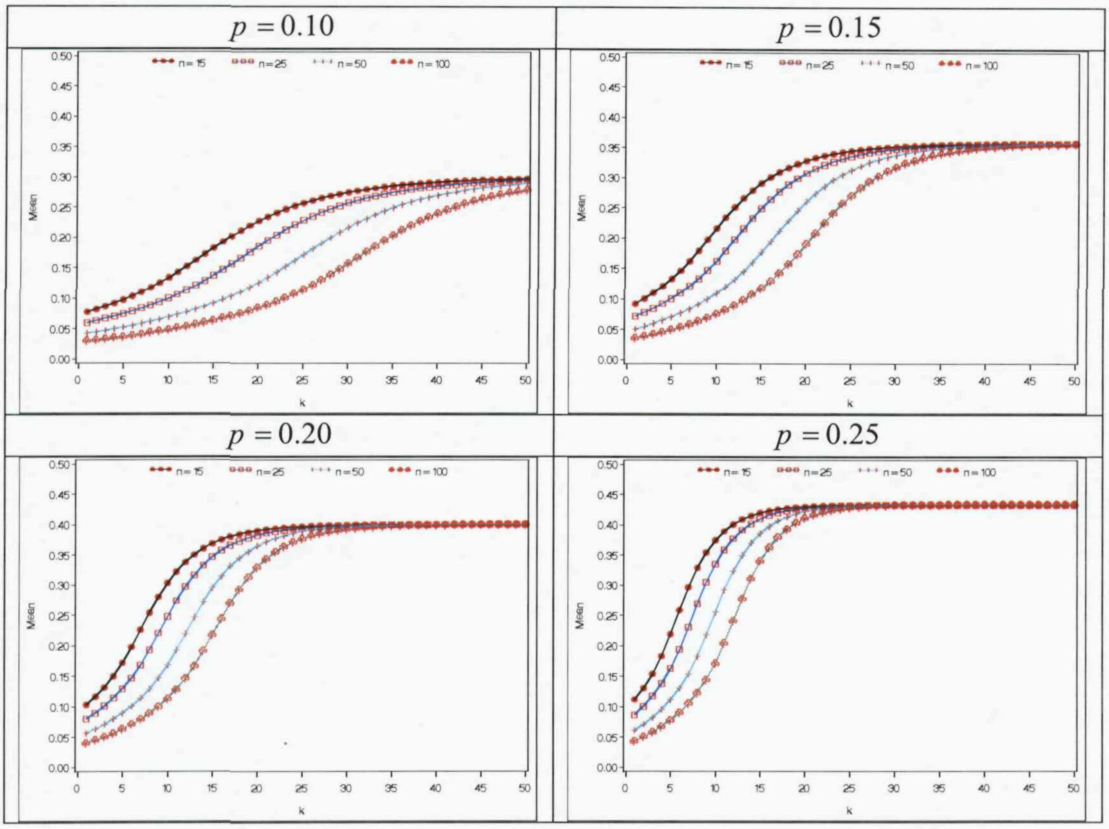


Figure 2.18: The standard deviation of $H_k \mid N_k^+ > 0$ for the geometric distribution.

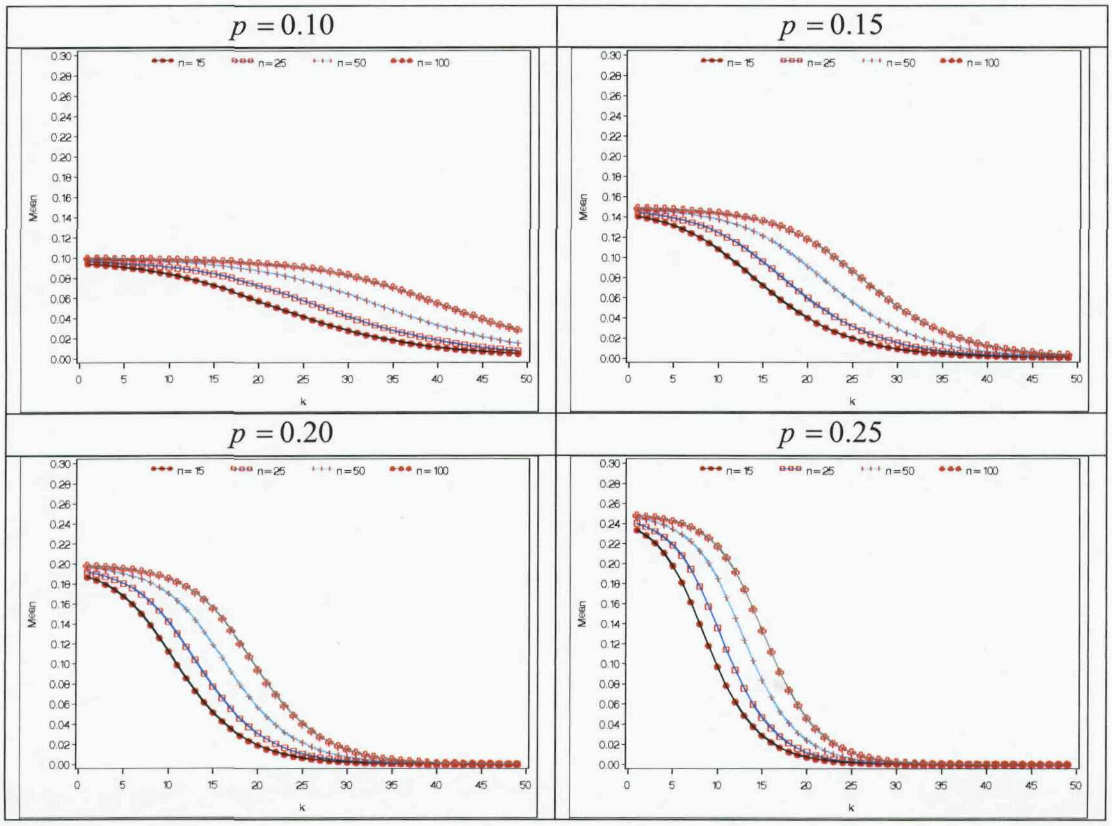


Figure 2.19: The mean of $H_k \mid K_{max} = 50$ for the geometric distribution.

To investigate the properties of the smoothed FRCC with unrestricted α a small simulation experiment was run. Tables 2.16 and 2.17 show the estimated significance and the estimated power of the smoothed FRCC methods at the nominal 5% significance level for the restricted and unrestricted α. The results for each combination of sample size and distribution are based on 1000 replications.

Table 2.17: Estimated power of the smoothed FRCC methods at the nominal 5% significance level.

Negative binomial $(p,0.2)$	$p=0.10$				$p=0.15$				$p=0.20$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
Results of the restricted α												
FRCC1s	64.1	92.4	97.1	99.6	41.6	79.3	97.8	99.5	25.0	58.8	91.3	99.1
FRCC2s	85.1	97.8	97.8	99.7	75.0	93.5	99.5	99.8	63.5	85.4	97.3	99.8
Results of the unrestricted α												
FRCC1s	64.0	92.6	25.1	25.8	42.4	80.3	98.3	88.2	24.8	46.3	64.7	99.7
FRCC2s	86.4	29.5	6.3	15.6	75.2	93.7	99.5	53.7	64.2	9.9	17.3	99.8
Poisson (λ)	$\lambda=1$				$\lambda=1.5$				$\lambda=2$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
Results of the restricted α												
FRCC1s	18.8	43.7	85.3	99.1	43.6	80.7	99.1	99.6	62.6	93.9	99.5	99.5
FRCC2s	17.0	43.6	84.9	99.1	41.6	80.7	99.1	99.6	61.9	93.8	99.5	99.5
Results of the unrestricted α												
FRCC1s	18.8	43.7	85.5	99.8	43.5	80.7	98.7	100	63.0	93.7	99.9	100
FRCC2s	17.1	43.6	85.1	99.6	41.5	80.7	98.6	100	62.3	93.6	99.7	100

Clearly, these simulation results indicate that the choice of smoothing parameter is important in determining the properties of the method and that further work is needed in this area.

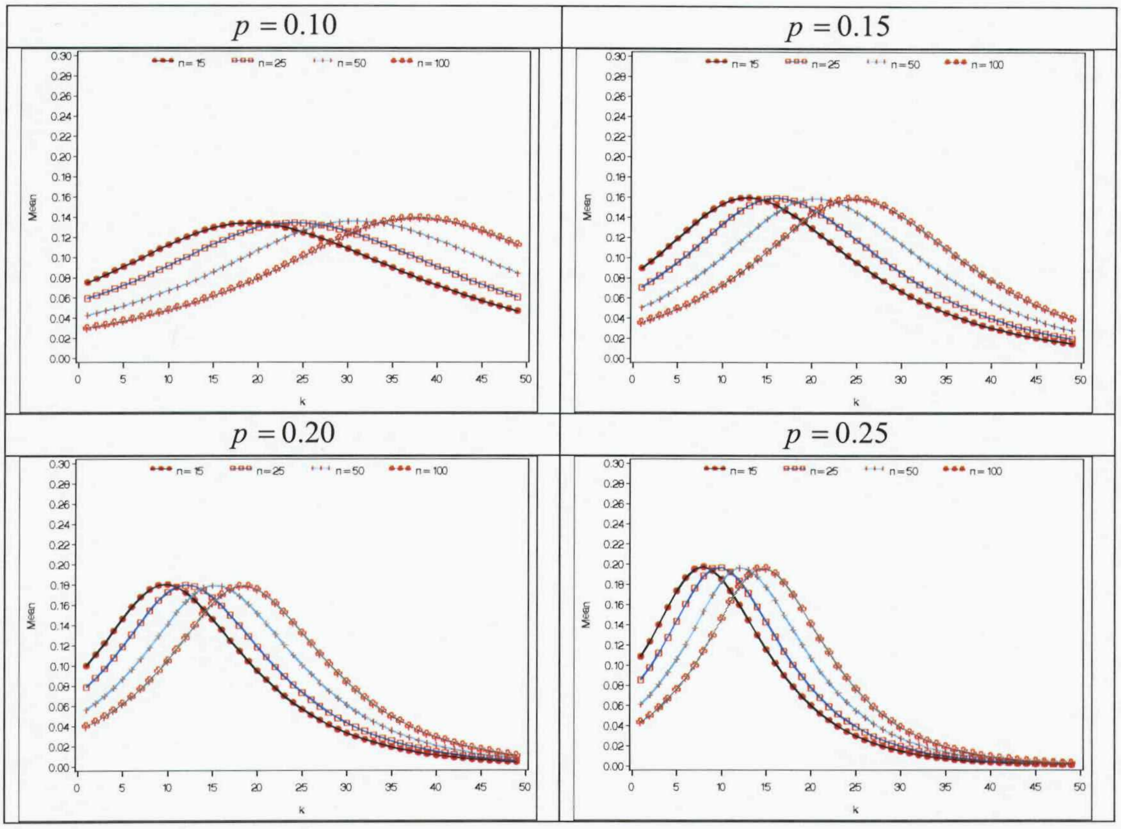


Figure 2.20: The standard deviation of $H_k \mid K_{\max} = 50$ for the geometric distribution.

2.4.2 The MLE for the Geometric Distribution

The geometric random variable K with parameter p has probability mass function

$$f(k) = p(1-p)^{k-1};\ k = 1,2,3,...$$

The likelihood for a random sample $k_1, k_2, ..., k_n$ from this distribution is

$$L(p) = \prod_{i=1}^{n} p(1-p)^{k_i - 1} = p^n (1-p)^{\sum k_i - n}.$$

The log likelihood is

$$l(p) = n\ln(p) + (\sum k_i - n)\ln(1-p)$$

and

$$\frac{dl}{dp} = \frac{n}{p} - \frac{\sum k_i - n}{1-p} \quad and \quad \frac{d^2 l}{dp^2} = -\frac{n}{p^2} - \frac{\sum k_i - n}{(1-p)^2}.$$

2.5 Discussion

In this chapter, the behaviour of the discrete EFR values is considered through its probability function. The mean and standard deviation of the probability function of the EFR are used to summarise its behaviour.

The probability function of $H_k \mid N_k^+ > 0$ is obtained and $H_k \mid N_k^+ > 0$ is shown to be an unbiased estimator of the FR function, $h(k)$. Also, we obtained $STD(H_k \mid N_k^+ > 0)$ and showed that it approaches $\sqrt{h(k)(1-h(k))}$ for large values of k.

Another approach for obtaining the probability function of the EFR is to condition on the fact that the highest observed failure time has EFR value equal to 1. The probability function of $H_k \mid K_{\max} = k_*$ is obtained and its mean found to be bounded by the mean of $H_k \mid N_k^+ > 0$. An expression for $STD(H_k \mid K_{\max} = k_*)$ was also obtained.

A FRCC is developed using the mean and standard deviation of $H_k \mid N_k^+ > 0$ and $H_k \mid K_{\max} = k_*$ to detect departure from a constant FR. If more than 5% of the exponentially smoothed EFR values fall outside the control bounds of the FRCC, then we can say that the data are unlikely to come from the geometric distribution with constant FR.

The FRCC approach seems to be promising but comprehensive investigations are needed to turn this approach into a practical goodness-of-fit test for the geometric distribution. In particular the conservative nature of the smoothed FRCC tests needs to be addressed. Issues to be resolved are: choice of smoothing parameter, whether limits other than 2 STD should be used and whether 5% of observations outside the limits is the most appropriate cut-off for rejection of the geometric model. Also, all the results here have been developed for uncensored data and need to be generalized to include censored data as well. This topic will not be discussed further in the thesis.

Chapter 3

Crowder Test for Independence of Competing Risks with Discrete Lifetime Data

In dealing with lifetime data it is common that there are two or more failure modes. The observed data in such a situation will comprise the failure times together with the relevant failure mode for each failure time. In trying to understand the failure process it is reasonable to ask whether the failure modes are acting independently. In the case of continuous failure times, it is well known that it is not possible to answer this question with such data. This is the so-called identifiability crisis in competing risks (Crowder, 2001). However, when the failure times are discrete the hypothesis of independence can sometimes be addressed. Crowder, in his Lifetime Data Analysis paper of 1997, proposed a test in such circumstances, derived its large sample null properties and illustrated its use on a medical data set. In this chapter the results presented by Crowder (1997) are summarised and corrected, and some potential practical shortcomings of his test are discussed. A modified version of his test is then proposed.

3.1 Introduction

The so-called competing risks phenomenon is a common situation in reliability and other areas involving lifetime data in which there is more than one failure mode, see Prentice et al. (1978), Crowder (2001), Bedford (2005), Pintilie (2006), Lindqvist (2007) and others. Thus, for each subject we observe the failure time and the mode or type of failure. For example, a car tyre can either wear until the tread depth is too low for safe use, or fail catastrophically by puncturing.

One way of describing a competing risks situation with g risks, is by associating a failure time K_i, $i = 1,2,...,g$ to each risk. These g failure times are called latent failure times. In the presence of all risks, the observed failure time is the smallest of the g failure times associated with the observed cause of failure and the other $g-1$ latent failure times are not observable (Crowder, 1997). It is this set up that will be used throughout the remainder of this thesis.

In understanding the failure process it is important to learn how the latent failure times for the competing failure modes interact. In statistical terms that means modelling the latent failure times and/or assessing the correlation between latent failure times. Thus, let $k_1, k_2, ..., k_g$ be the latent failure times that associated to each of the g risks then the independence hypothesis for the competing risks can be stated in terms of the joint reliability function, $R(k_1, k_2, ..., k_g) = P(K_1 > k_1, K_2 > k_2, ..., K_g > k_g)$, as

$$R(k_1, k_2, ..., k_g) = \prod_{j=1}^{g} R_j(k_j),$$

where $R_j(k_j) = P(K_j > k_j)$ is the marginal reliability function of K_j (Crowder, 2001).

For continuous lifetime data it is not possible to model the latent failure times, the so-called identifiability crisis in competing risks, see Tsiatis (1975), Crowder (1991), Crowder (1994). Thus, the joint distribution of all failure times K_i is not identifiable from the joint distribution of the observed failure time and the cause of the failure. This means that given a joint distribution, there is always a model with independent latent failure times that can be constructed to fit the same joint distribution of the observed failure time and the cause of the failure (independent risks proxy model), the joint distribution of the latent failure times is not uniquely determined by the joint distribution of the observed failure time and the cause of the failure (Ng and Watson, 2002). However, when the failure times are discrete, the hypothesis of independence can sometimes be addressed (Crowder, 1996).

In section 3.2 basic characteristics and notations for the competing risks phenomenon are presented. The Crowder test for independence of competing risks with purely discrete failure times is presented and discussed in section 3.3. A modified version of the Crowder test is proposed in section 3.4. Finally, in section 3.5 a discussion of this chapter is presented.

3.2 Basic Characteristics and Notations

A commonly applied approach to the competing risks phenomenon assumes that each risk, $1,2,3,...,g$, has a potential or latent failure time associated with it. Thus, a subject at time K can experience any one of the g failure types, call it C. The pair (K,C) is defined for each subject where $K = \min(K_1, K_2, ..., K_g)$ can be continuous or discrete and C which identifies the minimum K_i can be only discrete (Crowder, 2001).

In some cases more than one cause of failure can occur simultaneously. An obvious way to deal with that is by allowing multiple causes of failure. Thus, $K = \min(K_1, K_2, ..., K_g)$ as before but C is defined as the failure configuration. For example, if a subject fails from causes $1,2$ and 3 simultaneously, we define $K = K_1 = K_2 = K_3$ and $C = \{1,2,3\}$. Hence, the number of possible values that C can take is $2^g - 1$ (Crowder, 1997).

The joint distribution of the failure time, K, and the failure configuration (single or multiple), C, can be presented in terms of the sub-distribution functions

$$F(k,c) = P(K \leq k, C = c),$$

or equivalently by the sub-reliability functions

$$R(k,c) = P(K > k, C = c).$$

The marginal distribution of the failure configuration, C, is

$$p_c = P(C = c) = F(\infty, c) = R(0, c) = F(k, c) + R(k, c),$$
where $p_c > 0$ and $\sum_c p_c = 1$.

Note that $F(k,c)$ is not a proper distribution function since it is equal to p_c instead of 1 at $k = \infty$.

The marginal reliability function of K can be calculated such that

$$R(k) = \sum_c R(k, c).$$

For a subject of age k, the probability of failure from cause c is

$$P(C = c \mid K > k) = \frac{R(k,c)}{R(k)}.$$

3.2.1 Continuous Lifetimes

The sub-density functions are

$$f(k,c) = \frac{dF(k,c)}{dk} = \frac{-dR(k,c)}{dk}.$$

The sub-failure rate functions are

$$h(k,c) = \frac{f(k,c)}{R(k)}.$$

Note that the denominator of $h(k,c)$ is $R(k)$ instead of $R(k,c)$ to reflect its meaning as a failure rate for a subject of age k fails in the next instant in time from cause c in the presence of all risks. The marginal failure rate function of K can be calculated such that

$$h(k) = \sum_c h(k, c).$$

3.2.2 Discrete Lifetimes

The sub-density functions are

$$f(k,c) = P(K = k, C = c).$$

The sub-failure rate functions are

$$h(k,c) = \frac{f(k,c)}{R(k-1)}.$$

Also, note that the denominator of $h(k,c)$ is $R(k)$ instead of $R(k,c)$ to reflect its meaning as the probability of a subject failure at time k from cause c given that it did not fail before in the presence of all risks. The marginal failure rate function of K can be calculated such that

$$h(k) = \sum_c h(k,c).$$

3.3 The Crowder Test of Independence of Competing Risks

The following derivation of the test statistic of independence of competing risks in the case of purely discrete data was developed by Crowder (1997). The derivation closely follows Crowder's paper and is here indicated by indented text. However, the paper contains ambiguities and errors. In places where these occur comments in square brackets have been added.

> Let n_{kc} be the number of subjects failed from cause c at time k; n_{kr} be the number of subjects censored at time k; $n_{k\bullet}^{-} = \sum_c n_{kc}$ be the overall number of subjects failed at time k and let $n_{k\bullet}^{+} = \sum_{i=k}^{k_*} (n_{i\bullet}^{-} + n_{ir})$ be the number of subjects at risk at time k where k_* is the largest observed failure time.
>
> Then the nonparametric maximum likelihood estimator for the sub-failure rate functions $h_{kc} = h(k,c)$ is
>
> $$\hat{h}_{kc} = \frac{n_{kc}}{n_{k\bullet}^{+}} \quad (k = 1, \ldots, k_*),$$

and for the marginal failure rate function $h_k = h(k)$ we have

$$\hat{h}_k = \frac{n_{k\bullet}^-}{n_{k\bullet}^+}.$$

Now, the condition for independence of the competing risks is framed in terms of the quantities

$$\bar{h}_{kc} = \bar{h}(k,c) = \frac{f(k,c)}{R(k)},$$

which are not the same as the sub-failure rate functions

$$h_{kc} = \frac{f(k,c)}{R(k-1)}$$

for discrete lifetime data. Thus, in the case of discrete lifetimes

$$\bar{h}_{kc} = \frac{h_{kc}}{1-h_k},$$

which are known as the sub-conditional odds functions. The estimates of $\bar{h}_{kc}$ are

$$\hat{\bar{h}}_{kc} = \frac{\hat{h}_{kc}}{1-\hat{h}_k}.$$

Also, given survival to time k, the conditional odds for failure at time k is

$$\bar{h}_k = \bar{h}(k) = \sum_c \bar{h}_{kc} = \frac{h_k}{1-h_k} = \frac{pr(K=k \mid K \ge k)}{pr(K \ne k \mid K \ge k)}.$$

Now, let

$$\psi_{kc} = \frac{\bar{h}_{kc}}{\prod_{j \in c} \bar{h}_{kj}}.$$ [The meaning of $j \in c$ is unclear]

If $\psi_{kc} = 1$ for all multiple cause configurations, c, and $k = 1,2,3,\ldots,k_*$, then it satisfies the condition $\bar{h}_{kc} = \prod_{j \in c} \bar{h}_{kj}$ $\forall (k,c)$, a necessary and sufficient condition for the existence of an independent risks proxy model (Crowder, 1996).

Let $Y_{kc} = \log(\hat{\psi}_{kc})$ [$\hat{\psi}_{kc}$ has not been defined]. The test developed here is based on the log-transformed $\hat{\psi}_{kc}$ rather than $\hat{\psi}_{kc}$ to improve the asymptotic

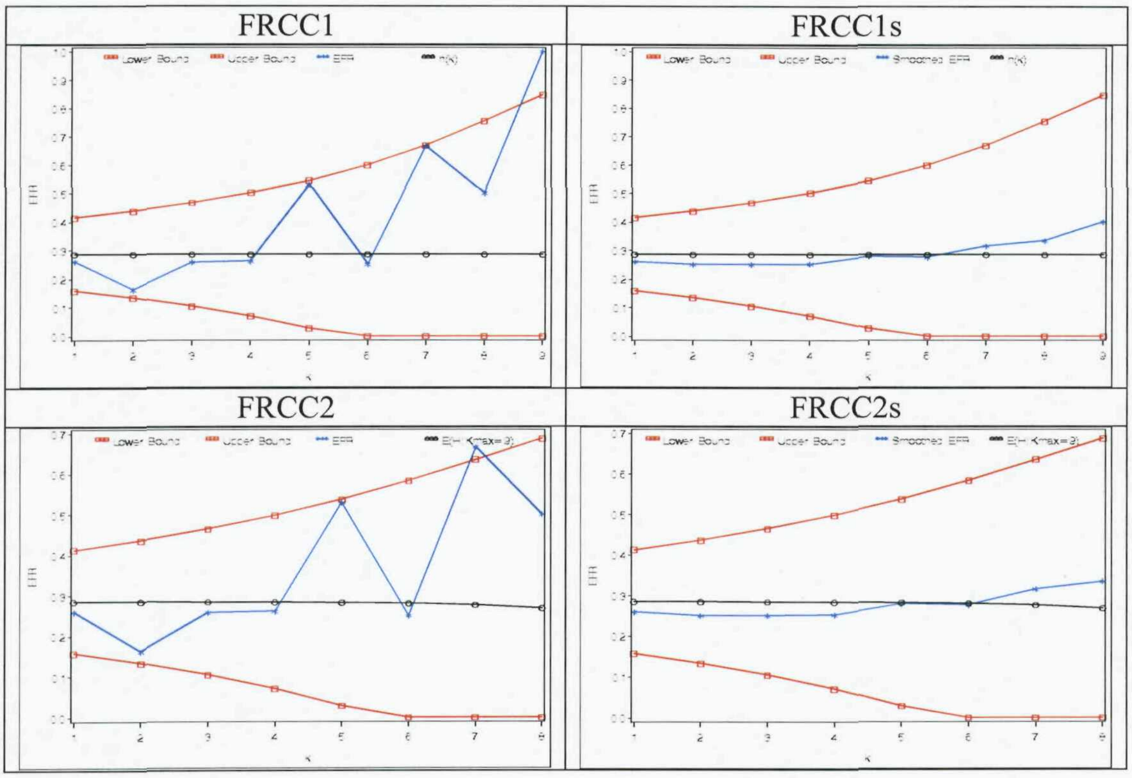


Figure 2.21: The FRCC plots for Example 2.2.

The results of FRCC1 show that more than 5%, $1/9 \times 100 = 11\%$, of the EFR values fall outside the control bounds, indicating wrongly that the data are unlikely to come from the geometric distribution. Also, the same conclusion is achieved when using FRCC2, $1/8 \times 100 = 13\%$. The results of FRCC1s show that none of the smoothed version of the EFR values fall outside the control bounds, indicating that the data are likely to come from the geometric distribution. Also, the same conclusion is achieved when using FRCC2s.

Example 2.3:

A sample with 50 observations simulated using SAS software version 9.1 from the Poisson(2) distribution is displayed in Table 2.6, along with n_k^+ and $\hat{h}_k$.

Now, $\widehat{E}(H_k \mid N_k^+ > 0) = \hat{p} = \dfrac{50}{156} = 0.32051$. The $\widehat{STD}(H_k \mid N_k^+ > 0)$, $\widehat{E}(H_k \mid K_{\max} = 9)$ and $\widehat{STD}(H_k \mid K_{\max} = 9)$ are shown in Table 2.7. The calculations of $\widehat{STD}(H_k \mid N_k^+ > 0)$, $\widehat{E}(H_k \mid K_{\max} = 9)$ and $\widehat{STD}(H_k \mid K_{\max} = 9)$ are as for example 2.2.

normality approximation. Assume that k_* is finite, $h_{kc} > 0 \ \forall (k,c)$ and $h_k < 1 \ \forall k < k_*$. Then standard asymptotic theory may be applied.

Let $Y_k = (Y_{kc_1}, Y_{kc_2}, ...)^T$ be of length $d = 2^g - g - 1$ and $c_j \ (j = 1, 2, ..., d)$ be the multiple cause configurations. Then, a Wald test statistic of the form $W = Y^T V^{-1} Y$ is used to test the independence, where $Y^T = (Y_1^T, ..., Y_{k_*}^T)$ and $V = \text{var}(Y)$. Since the Y_k are asymptotically independent, the test statistic will be

$$W = \sum_{k=1}^{k_*} Y_k^T \ V_k^{-1} \ Y_k \ .$$

Under the null hypothesis $W \to \chi^2_{dk_*}$ asymptotically.

Now, the covariance matrix V has as its elements the covariances $\text{cov}(Y_{kc}, Y_{lb})$, where c and b are multiple cause configurations and $k, l = 1, 2, ..., k_*$. Using the delta method,

$$\begin{aligned}\text{cov}(Y_{kc}, Y_{lb}) &= \text{cov}\left\{\log \hat{\bar{h}}_{kc} - \sum_{i \varepsilon c} \log \hat{\bar{h}}_{ki}, \log \hat{\bar{h}}_{lb} - \sum_{j \varepsilon b} \log \hat{\bar{h}}_{lj}\right\} \\ &= \text{cov}(\log \hat{\bar{h}}_{kc}, \log \hat{\bar{h}}_{lb}) - \sum_{j \varepsilon b} \text{cov}(\log \hat{\bar{h}}_{kc}, \log \hat{\bar{h}}_{lj}) \\ &\quad - \sum_{i \varepsilon c} \text{cov}(\log \hat{\bar{h}}_{ki}, \log \hat{\bar{h}}_{lb}) + \sum_{i \varepsilon c} \sum_{j \varepsilon b} \text{cov}(\log \hat{\bar{h}}_{ki}, \log \hat{\bar{h}}_{lj}) \\ &\approx \frac{\bar{H}_{kc,lb}}{\bar{h}_{kc} \bar{h}_{lb}} - \sum_{j \varepsilon b} \frac{\bar{H}_{kc,lj}}{\bar{h}_{kc} \bar{h}_{lj}} - \sum_{i \varepsilon c} \frac{\bar{H}_{ki,lb}}{\hat{h}_{ki} \bar{h}_{lb}} + \sum_{i \varepsilon c} \sum_{j \varepsilon b} \frac{\bar{H}_{ki,lj}}{\bar{h}_{ki} \bar{h}_{lj}} \ ,\end{aligned}$$

where $\bar{H}_{kc,lb} = \text{cov}(\hat{\bar{h}}_{kc}, \hat{\bar{h}}_{lb})$. [It is unclear whether the $\bar{h}$ terms in the last line of the formula should be $\hat{\bar{h}}$]

Let δ_{kl} denote the Kronecker delta, and let δh denote the increment. Using the delta method,

$$\begin{aligned}\bar{H}_{kc,lb} &= \text{cov}\left\{\frac{\hat{h}_{kc}}{1 - \hat{h}_k}, \frac{\hat{h}_{lb}}{1 - \hat{h}_l}\right\} \approx E\left[\left\{\frac{\delta \hat{h}_{kc}}{1 - h_k} + \frac{h_{kc} \delta \hat{h}_k}{(1 - h_k)^2}\right\}\left\{\frac{\delta \hat{h}_{lb}}{1 - h_l} + \frac{h_{lb} \delta \hat{h}_l}{(1 - h_l)^2}\right\}\right] \\ &\approx \frac{\delta_{kl}}{(1 - h_k)(1 - h_l)}\left\{H_{kc,lb} + \frac{h_{lb} \sum_b H_{kc,lb}}{1 - h_l} + \frac{h_{kc} \sum_c H_{kc,lb}}{1 - h_k} + \frac{h_{kc} h_{lb} \sum_{cb} H_{kc,lb}}{(1 - h_k)(1 - h_l)}\right\},\end{aligned}$$

where $H_{kc,lb} = \text{cov}(\hat{h}_{kc}, \hat{h}_{lb})$. [It is unclear whether the h terms should be $\hat{h}$ and the precise nature of the summations is also unclear]

Now, we need to calculate $H_{kc,lb}$. The likelihood function can be expressed as

$$L = \prod_{k=1}^{k_*} \left\{ \prod_c h_{kc}{}^{n_{kc}} (1-h_k)^{n_{k\cdot}^+ - n_{k\cdot}^-} \right\}.$$

So,

$$\frac{d \log L}{dh_{kc}} = \frac{n_{kc}}{h_{kc}} - \frac{n_{k\bullet}^+ - n_{k\bullet}^-}{1-h_k} \quad \text{and} \quad -\frac{d^2 \log L}{dh_{kc} dh_{lb}} = \delta_{kl} \left\{ \frac{\delta_{cb} n_{kc}}{h_{kc}{}^2} + \frac{n_{k\bullet}^+ - n_{k\bullet}^-}{(1-h_k)^2} \right\}.$$

The $-d^2 \log L / (dh_{kc} dh_{lb})$ can be estimated as

$$\delta_{kl} n_{k\bullet}^+ \left\{ \frac{\delta_{cb} n_{kc}}{n_{k\bullet}^+ \hat{h}_{kc}{}^2} + \frac{n_{k\bullet}^+ - n_{k\bullet}^-}{n_{k\bullet}^+ (1-\hat{h}_k)^2} \right\} = \delta_{kl} n_{k\bullet}^+ \left\{ \frac{\delta_{cb}}{\hat{h}_{kc}} + \frac{1}{(1-\hat{h}_k)} \right\}.$$

Note that the second term is absent for $k = k_*$.

Under the assumption that $h_{kc} > 0 \ \forall (k,c)$, $h_k < 1 \ \forall k < k_*$ and the parameter set $\{h_{kc}\}$ is finite, asymptotic likelihood theory is standard. Then,

$$\hat{h}_{kc} \sim AN(h_{kc}, H_{kc,lb}); \ H_{kc,lb} = \text{cov}(\hat{h}_{kc}, \hat{h}_{lb}) \approx \delta_{kl} (H_k)^{cb},$$

where H_k is the $k_* \times k_*$ [sic] matrix with (c,b) entry $E\left(-d^2 \log L / dh_{kc} dh_{kb}\right)$ and $(H_k)^{cb}$ denotes the (c,b) element of its inverse. So, we have

$$\hat{H}_{kc,lb} = \delta_{kl} (\hat{H}_k)^{cb} \quad \text{where} \quad (\hat{H}_k)_{cb} = n_{k\bullet}^+ \left\{ \frac{\delta_{cb}}{\hat{h}_{kc}} + \frac{1-\delta_{kk_*}}{1-\hat{h}_k} \right\},$$

or in a matrix form,

$$\hat{H}_k = n_{k\cdot}^+ \left\{ diag\left\{ \frac{1}{\hat{h}_{kc}} \right\} + \frac{1-\delta_{kk_*}}{1-\hat{h}_k} J_{k_* \times k_*} \right\} \text{ where } J_{k_* \times k_*} \text{ is a matrix of ones.}$$

Thus,

$$\hat{H}_k^{-1} = \frac{1}{n_{k\bullet}^+} \left\{ diag\left(\hat{h}_{kc}\right) + \frac{1-\delta_{kk_*}}{1-\hat{h}_k{}^2} diag\left(\hat{h}_{kc}\right) J \ diag\left(\hat{h}_{kc}\right) \right\}. \text{ [sic]}$$

Table 2.8: The results of the FRCC for Example 2.3.

k	$\hat{h}_k$	$\hat{h}_{sk}$	$H_k \mid N_k^+ > 0$		FRCC1		FRCC1s		$H_k \mid K_{max} = 9$		FRCC2		FRCC2s	
			LB	UB	in	out	in	out	LB	UB	in	out	in	out
1	0.12	0.12	0.19	0.45	0	1	0	1	0.19	0.45	0	1	0	1
2	0.27	0.15	0.16	0.48	1	1	0	2	0.16	0.48	1	1	0	2
3	0.38	0.19	0.12	0.52	2	1	1	2	0.12	0.51	2	1	1	2
4	0.7	0.30	0.08	0.56	2	2	2	2	0.08	0.55	2	2	2	2
5	0.83	0.40	0.02	0.62	2	3	3	2	0.02	0.60	2	3	3	2
6	0	0.32	0	0.69	2	4	4	2	0	0.65	2	4	4	2
7	0	0.25	0	0.79	2	5	5	2	0	0.70	2	5	5	2
8	0	0.21	0	0.90	2	6	6	2	0	0.74	2	6	6	2
9	1	0.36	0	1	2	7	7	2						

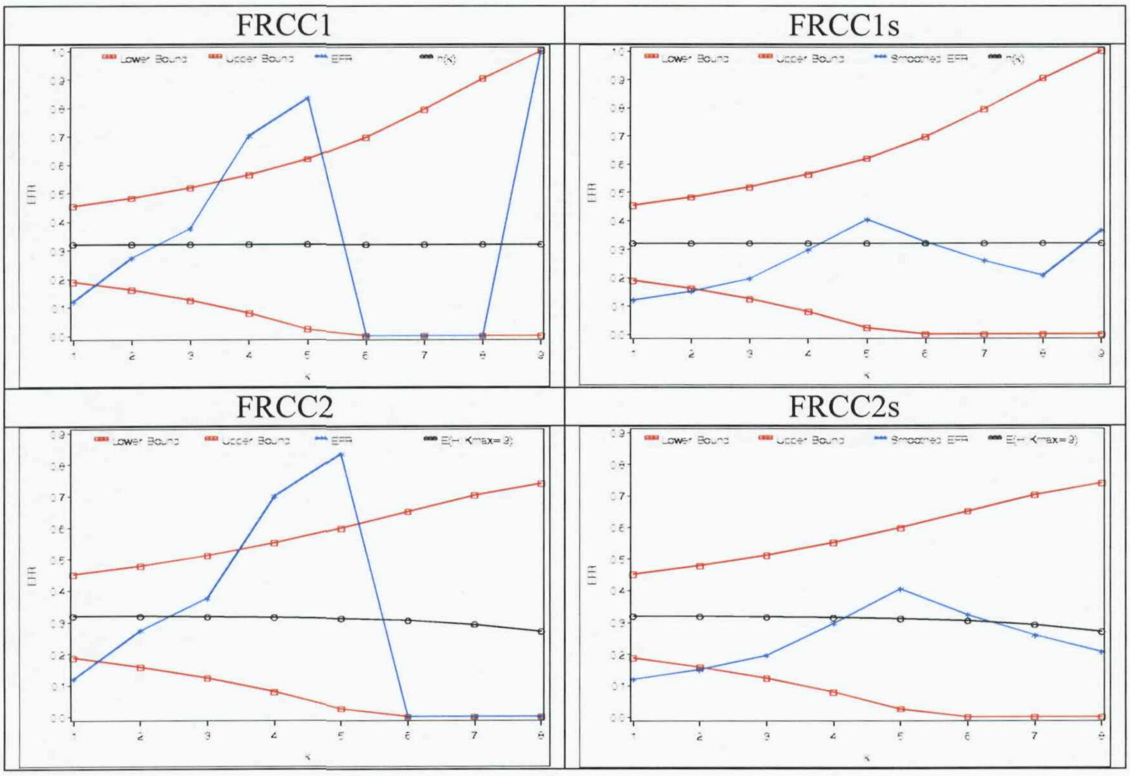


Figure 2.22: The FRCC plots for Example 2.3.

The results of FRCC1 show that more than 5%, $7/9 \times 100 = 78\%$, of the EFR values fall outside the control bounds, indicating that the data are unlikely to come from the geometric distribution. Also, the same conclusion is achieved for FRCC2, $6/8 \times 100 = 75\%$, and for the smoothed version of the EFR values, FRCC1s ($2/9 \times 100 = 22\%$) and FRCC2s ($2/8 \times 100 = 25\%$).

Hence,

$$\hat{H}_{kc,lb} = \frac{\delta_{kl\cdot}}{n_{k\cdot}^{+}}\left\{\delta_{cb}\hat{h}_{kc} - \frac{(1-\delta_{kk\cdot})}{(1-\hat{h}_k{}^2)}\hat{h}_{kc}\hat{h}_{kb}\right\}. \text{ [sic]}$$

□.

In reviewing the derivation of the covariance matrix V, the notations used in $\bar{H}_{kc,lb} = \operatorname{cov}(\hat{\bar{h}}_{kc}, \hat{\bar{h}}_{lb})$ are not clear. In particular, the summations $\sum_b H_{kc,lb}$, $\sum_c H_{kc,lb}$ and $\sum_{cb} H_{kc,lb}$ seem to be over the elements of the multiple cause configurations c and b which is not correct. Also, the dimension of H_k and its inverse are found to be incorrect.

First, we need to correct the notations used for the V matrix. Let c and b be the causes of failure, single or multiple, and a_v and a_u be the multiple causes of failure only. Now, $Y_{ka_v} = \log(\hat{\psi}_{ka_v})$ where $\hat{\psi}_{ka_v} = \hat{\bar{h}}_{ka_v} \Big/ \prod_{j \in a_v} \hat{\bar{h}}_{kj}$, the multiplication here is over the single causes, j, that form a_v. The covariance matrix V has as its elements the covariances $\operatorname{cov}(Y_{ka_v}, Y_{la_u})$, where $k, l = 1, 2, \ldots, k_*$. So,

$$\operatorname{cov}(Y_{ka_v}, Y_{la_u}) \approx \frac{\bar{H}_{ka_v,la_u}}{\hat{\bar{h}}_{ka_v}\hat{\bar{h}}_{la_u}} - \sum_{j \varepsilon a_u} \frac{\bar{H}_{ka_v,lj}}{\hat{\bar{h}}_{ka_v}\hat{\bar{h}}_{lj}} - \sum_{i \varepsilon a_v} \frac{\bar{H}_{ki,la_u}}{\hat{\bar{h}}_{ki}\hat{\bar{h}}_{la_u}} + \sum_{i \varepsilon a_v}\sum_{j \varepsilon a_u} \frac{\bar{H}_{ki,lj}}{\hat{\bar{h}}_{ki}\hat{\bar{h}}_{lj}},$$

where $\bar{H}_{kc,lb} = \operatorname{cov}(\hat{\bar{h}}_{kc}, \hat{\bar{h}}_{lb})$.

The summations here are over the single causes i and j that form a_v and a_u respectively.

Now, in Crowder's derivation $\bar{H}_{ka_v,la_u}$ has the following format

$$\bar{H}_{ka_v,la_u} = \operatorname{cov}\left\{\frac{\hat{h}_{ka_v}}{1-\hat{h}_k}, \frac{\hat{h}_{la_u}}{1-\hat{h}_l}\right\} \approx E\left[\left\{\frac{\delta\hat{h}_{ka_v}}{1\text{-}h_k} + \frac{h_{ka_v}\delta h_k}{(1-h_k)^2}\right\}\left\{\frac{\delta\hat{h}_{la_u}}{1\text{-}h_l} + \frac{h_{la_u}\delta\hat{h}_l}{(1-h_l)^2}\right\}\right]$$

$$\approx \frac{\delta_{kl}}{(1-h_k)(1-h_l)}\left\{H_{ka_v,la_u} + \frac{h_{la_u}\sum_{a_u} H_{ka_v,la_u}}{1-h_l} + \frac{h_{ka_v}\sum_{a_v} H_{ka_v,la_u}}{1-h_k} + \frac{h_{ka_v}h_{la_u}\sum_{a_v a_u} H_{ka_v,la_u}}{(1-h_k)(1-h_l)}\right\}.$$

Note that h and $\hat{h}$ are being used here but $\overline{H}_{ka_v,la_u} = \text{cov}(\hat{\overline{h}}_{ka_v}, \hat{\overline{h}}_{la_u})$ so $\hat{h}$ should be used. Also, note that $\overline{H}_{ka_v,la_u}$ is the covariance between the multiple causes only and it should be the covariance between the singles, multiples or single and multiple causes of failure. Thus, $\overline{H}_{ka_v,la_u}$ should be $\overline{H}_{kc,lb}$. Now, to find the right indices of the summations $\sum_{a_u} H_{ka_v,la_u}$, $\sum_{a_v} H_{ka_v,la_u}$ and $\sum_{a_v a_u} H_{ka_v,la_u}$, we should go back and find out where the summations came from.

The first summation, $\sum_{a_u} H_{ka_v,la_u}$, is the result of $E\{\delta\hat{h}_{ka_v}\delta\hat{h}_l\}$, but $\hat{h}_l = \sum_b \hat{h}_{lb}$ is the overall failure rate from all causes, single and multiple, at time l, e.g. $b = 1, 2, \{1,2\}$ in the case of two failure modes. The second summation, $\sum_{a_v} H_{ka_v,la_u}$, is the result of $E\{\delta\hat{h}_k\delta\hat{h}_{la_u}\}$, but $\hat{h}_k = \sum_c \hat{h}_{kc}$ is the overall failure rate from all causes, single and multiple, at time k. Also, the summations $\sum_{a_v a_u} H_{ka_v,la_u}$ are the result of $E\{\delta\hat{h}_k\delta\hat{h}_l\}$ where $\hat{h}_k = \sum_c \hat{h}_{kc}$ and $\hat{h}_l = \sum_b \hat{h}_{lb}$ are the overall failure rate from all causes, single and multiple, at time k and l respectively. Thus, the summations must be over all the causes of failure, single and multiple, at the indicated failure time and not only over the elements of the multiple cause configurations. Thus,

$$\overline{H}_{kc,lb} \approx \frac{\delta_{kl}}{(1-\hat{h}_k)(1-\hat{h}_l)}\left\{H_{kc,lb} + \frac{\hat{h}_{lb}\sum_b H_{kc,lb}}{1-\hat{h}_l} + \frac{\hat{h}_{kc}\sum_c H_{kc,lb}}{1-\hat{h}_k} + \frac{\hat{h}_{kc}\hat{h}_{lb}\sum_c\sum_b H_{kc,lb}}{(1-\hat{h}_k)(1-\hat{h}_l)}\right\},$$

where $H_{kc,lb} = \text{cov}(\hat{h}_{kc}, \hat{h}_{lb})$.

Note that the δ_{lk} term in $\overline{H}_{kc,lb}$ indicates that $\hat{h}_k$ and $\hat{h}_l$ are asymptotically independent for $k \neq l$.

Now, we will consider the simple cases of the competing risks phenomenon, two and three failure modes, to see the correct format of the covariance matrix V. In the case of two failure modes only, $g = 2$, there are three causes of failure, failure from cause 1, failure from cause 2 or failure from the multiple cause configuration $a = \{1,2\}$. We should have the following variance-covariance terms: $\text{var}(\hat{h}_{k1})$, $\text{var}(\hat{h}_{k2})$, $\text{var}(\hat{h}_{ka})$,

$\text{cov}(\hat{h}_{k1},\hat{h}_{k2})$, $\text{cov}(\hat{h}_{k1},\hat{h}_{ka})$, $\text{cov}(\hat{h}_{k2},\hat{h}_{ka})$. Thus, the dimension of the H_k matrix should be in this case 3×3. In the case of having three failure modes, the dimension of the H_k matrix should be 7×7. Hence, in general the dimension of the $\hat{H}_k$ matrix is $m\times m$ where $m = 2^g - 1$ is the maximum number of possible causes, single and multiple, of failures that can be obtained from g failure modes or types. So in the case of having two failure modes only, $g = 2$,

$$\hat{H}_k = n_{k\bullet}^{+}\left\{diag\left\{\frac{1}{\hat{h}_{kc}}\right\} + \frac{1-\delta_{kk.}}{1-\hat{h}_k}J_{3\times 3}\right\}, \text{ where } J_{3\times 3} \text{ is a matrix of ones.}$$

Thus,

$$\hat{H}_k = \left\{\begin{matrix} \frac{n_{k\bullet}^{+}}{\hat{h}_{k1}} + \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} & \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} & \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} \\ \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} & \frac{n_{k\bullet}^{+}}{\hat{h}_{k2}} + \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} & \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} \\ \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} & \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} & \frac{n_{k\bullet}^{+}}{\hat{h}_{ka}} + \frac{n_{k\bullet}^{+}}{1-\hat{h}_k} \end{matrix}\right\}.$$

Hence,

$$\hat{H}_k^{-1} = \frac{1}{n_{k\bullet}^{+}}\left\{\begin{matrix} \frac{\hat{h}_{k1}(1-\hat{h}_k+\hat{h}_{k2}+\hat{h}_{ka})}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} & -\frac{\hat{h}_{k1}\hat{h}_{k2}}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} & -\frac{\hat{h}_{k1}\hat{h}_{ka}}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} \\ -\frac{\hat{h}_{k2}\hat{h}_{k1}}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} & \frac{\hat{h}_{k2}(1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{ka})}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} & -\frac{\hat{h}_{k2}\hat{h}_{ka}}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} \\ -\frac{\hat{h}_{ka}\hat{h}_{k1}}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} & -\frac{\hat{h}_{ka}\hat{h}_{k2}}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} & \frac{\hat{h}_{ka}(1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2})}{1-\hat{h}_k+\hat{h}_{k1}+\hat{h}_{k2}+\hat{h}_{ka}} \end{matrix}\right\}.$$

But $\hat{h}_k = \sum_c \hat{h}_{kc} = \hat{h}_{k1} + \hat{h}_{k2} + \hat{h}_{ka}$ and hence

$$\hat{H}_k^{-1} = \frac{1}{n_{k\bullet}^{+}}\left\{\begin{matrix} \hat{h}_{k1} - \hat{h}_{k1}^2 & -\hat{h}_{k1}\hat{h}_{k2} & -\hat{h}_{k1}\hat{h}_{ka} \\ -\hat{h}_{k2}\hat{h}_{k1} & \hat{h}_{k2} - \hat{h}_{k2}^2 & -\hat{h}_{k2}\hat{h}_{ka} \\ -\hat{h}_{ka}\hat{h}_{k1} & -\hat{h}_{ka}\hat{h}_{k2} & \hat{h}_{ka} - \hat{h}_{ka}^2 \end{matrix}\right\}.$$

Thus, $H_{kc,lb} = \frac{\delta_{kl}}{n_{k.}^{+}}\left\{\delta_{cb}\hat{h}_{kc} - (1-\delta_{kk_*})\hat{h}_{kc}\hat{h}_{kb}\right\}$.

Now, in the case of having three failure modes, $g = 3$,

$$\hat{H}_k = n_{k\bullet}^{+} \left\{ diag\left\{ \frac{1}{\hat{h}_{kc}} \right\} + \frac{1-\delta_{kk_*}}{1-\hat{h}_k} J_{7\times7} \right\}.$$

Hence,

$$\hat{H}_k^{-1} = \frac{1}{n_{k\bullet}^{+}} \left\{ \text{diag}(\hat{h}_{kc}) - (1-\delta_{kk_*})\ \text{diag}(\hat{h}_{kc})\ J_{7\times7}\ \text{diag}(\hat{h}_{kc}) \right\}.$$

Thus, $H_{kc,lb} = \frac{\delta_{kl}}{n_{k\bullet}^{+}} \left\{ \delta_{cb}\hat{h}_{kc} - (1-\delta_{kk_*})\hat{h}_{kc}\hat{h}_{kb} \right\}$.

So, in general,

$$H_{kc,lb} = \text{cov}(\hat{h}_{kc}, \hat{h}_{lb}) = \frac{\delta_{kl}}{n_{k\bullet}^{+}} \left\{ \delta_{cb}\hat{h}_{kc} - (1-\delta_{kk_*})\hat{h}_{kc}\hat{h}_{kb} \right\}.$$

Therefore, the Crowder test for the independence of competing risks with discrete lifetimes should, after making the above corrections to Crowder's original derivation, be in the following format:

Let n_{kc} be the number of subjects failed from cause c, single or multiple, at time k; n_{kr} be the number of subjects censored at time k; $n_{k\bullet}^{-} = \sum_c n_{kc}$ be the overall number of subjects failed at time k and let $n_{k\bullet}^{+} = \sum_{i=k}^{k_*} (n_{i\bullet}^{-} + n_{ir})$ be the number of subjects at risk at time k where k_* is the largest observed failure time.

So,

$$\hat{h}_{kc} = \frac{n_{kc}}{n_{k\bullet}^{+}},\ \hat{h}_k = \frac{n_{k\bullet}^{-}}{n_{k\bullet}^{+}} \text{ and } \hat{\hat{h}}_{kc} = \frac{\hat{h}_{kc}}{1-\hat{h}_k} = \frac{n_{kc}}{n_{k\bullet}^{+} - n_{k\bullet}^{-}}.$$

Now, let $\hat{\psi}_{ka_v} = \hat{\hat{h}}_{ka_v} \Big/ \prod_{j\in a_v} \hat{\hat{h}}_{kj}$ and $Y_{ka_v} = \log(\hat{\psi}_{ka_v})$ where a_v is a multiple cause configuration and let $Y_k = (Y_{ka_1}, Y_{ka_2}, \ldots, Y_{ka_d})^T$ be of length $d = 2^g - g - 1$ where $a_v\ (v = 1, 2, \ldots, d)$ are the multiple cause configurations. So, when k_* is finite,

$h_{kc} > 0 \ \forall (k,c)$ and $h_k < 1 \ \forall k < k_*$, standard asymptotic theory can be applied. Then the test statistic will be

$$W = \sum_{k=1}^{k_*} Y_k^T V_k^{-1} Y_k ,$$

which has asymptotic null distribution $\chi^2_{dk_*}$.

The covariance matrix V_k has as its elements the covariances $\text{cov}(Y_{ka_v}, Y_{ka_u})$, where a_v and a_u are multiple cause configurations and $k = 1, 2, \dots, k_*$. Let c and b be causes of failure, single or multiple. Hence,

$$\text{cov}(Y_{ka_v}, Y_{ka_u}) \approx \frac{\bar{H}_{ka_v,ka_u}}{\hat{h}_{ka_v}\hat{h}_{ka_u}} - \sum_{j \varepsilon a_u} \frac{\bar{H}_{ka_v,kj}}{\hat{h}_{ka_v}\hat{h}_{kj}} - \sum_{i \varepsilon a_v} \frac{\bar{H}_{ki,ka_u}}{\hat{h}_{ki}\hat{h}_{ka_u}} + \sum_{i \varepsilon a_v} \sum_{j \varepsilon a_u} \frac{\bar{H}_{ki,kj}}{\hat{h}_{ki}\hat{h}_{kj}} , \tag{3.1}$$

where $\bar{H}_{kc,kb} = \text{cov}(\hat{\bar{h}}_{kc}, \hat{\bar{h}}_{kb})$,

and

$$\bar{H}_{kc,kb} \approx \frac{1}{(1-\hat{h}_k)^2} \left\{ H_{kc,kb} + \frac{\hat{h}_{kc} \sum_c H_{kc,kb} + \hat{h}_{kb} \sum_b H_{kc,kb}}{1-\hat{h}_k} + \frac{\hat{h}_{kc}\hat{h}_{kb} \sum_c \sum_b H_{kc,kb}}{(1-\hat{h}_k)^2} \right\}, \tag{3.2}$$

$$\text{where } H_{kc,kb} = \text{cov}(\hat{h}_{kc}, \hat{h}_{kb}) = \frac{1}{n_{k_*}^+} \left\{ \delta_{cb}\hat{h}_{kc} - (1-\delta_{kk_*})\hat{h}_{kc}\hat{h}_{kb} \right\}. \tag{3.3}$$

3.4 A Modified Version of the Crowder Test

Competing risks data have several failure modes or types and it is not unusual to have a zero frequency for at least one of these failure modes at one or more observed failure time. This need not be a sparse data issue or a problem that arises only in very small samples. For example, suppose $g = 2$ and the two causes of failure act in a strongly dependent way, so that the failure mode $\{1, 2\}$ is much more common than the two single failure modes. Then it is quite likely, even in a reasonably large data set, that at least one of the single failure modes will have zero frequency at one or more observed failure time.

For such data, Crowder's test statistic W is not computable. In his paper, Crowder (1997) acknowledges this problem and in the treatment of his example uses the following solution. He evaluates only those W_k terms for which $n_{kc} > 0$ for all c, calculates W on the basis of these W_k only and reduces the degrees of freedom to dq, where $q < k_*$ is the number of computable W_k terms. This may result in a loss of power or may even lead to potentially misleading results. This will be considered further in chapter 4.

A modified version of Crowder's test statistic can be applied by adding the classical continuity correction of 0.5 to each n_{kc}. This at least ensures that a suitably modified version of W_k can always be calculated for all times with at least one failure. Let $\tilde{n}_{kc} = n_{kc} + 0.5$ be the number of subjects failed from cause c, single or multiple, at time k plus the continuity correction; n_{kr} be the number of subjects censored at time k and let $n_{k\bullet}^{-} = \sum_c n_{kc}$ and $\tilde{n}_{k\bullet}^{+} = \sum_{i=k}^{k_*} (n_{i\bullet}^{-} + n_{ir}) + 0.5$.

Then, the nonparametric maximum likelihood estimator for h_{kc} is $\hat{h}'_{kc} = \tilde{n}_{kc} / \tilde{n}_{k\bullet}^{+}$, and for h_k we have $\hat{h}'_k = n_{k\bullet}^{-} / \tilde{n}_{k\bullet}^{+}$. The estimates for the sub-conditional odds functions are

$$\hat{\bar{h}}'_{kc} = \frac{\hat{h}'_{kc}}{1 - \hat{h}'_k} = \frac{\tilde{n}_{kc}}{\tilde{n}_{k\cdot}^{+} - n_{k\cdot}^{-}}.$$

Now, let $\hat{\tilde{\psi}}_{ka_v} = \hat{\bar{h}}'_{ka_v} \Big/ \prod_{j \in a_v} \hat{\bar{h}}'_{kj}$ and $\tilde{Y}_{ka_v} = \log(\hat{\tilde{\psi}}_{ka_v})$ where a_v is a multiple cause configuration and let $\tilde{Y}_k = (\tilde{Y}_{ka_1}, \tilde{Y}_{ka_2}, \ldots, \tilde{Y}_{ka_d})^T$ be of length $d = 2^p - g - 1$ where a_v $(v = 1, 2, \ldots, d)$ are the multiple cause configurations. So, when k_* is finite, $h'_{kc} > 0 \ \forall (k, c)$ and $h'_k < 1 \ \forall k \le k_*$, standard asymptotic theory can be applied. Then the modified Crowder test statistic will be

$$\tilde{W} = \sum_{k=1}^{k_*} \tilde{Y}_k^T \tilde{V}_k^{-1} \tilde{Y}_k .$$

This has $\chi^2_{dk_*}$ asymptotic null distribution.

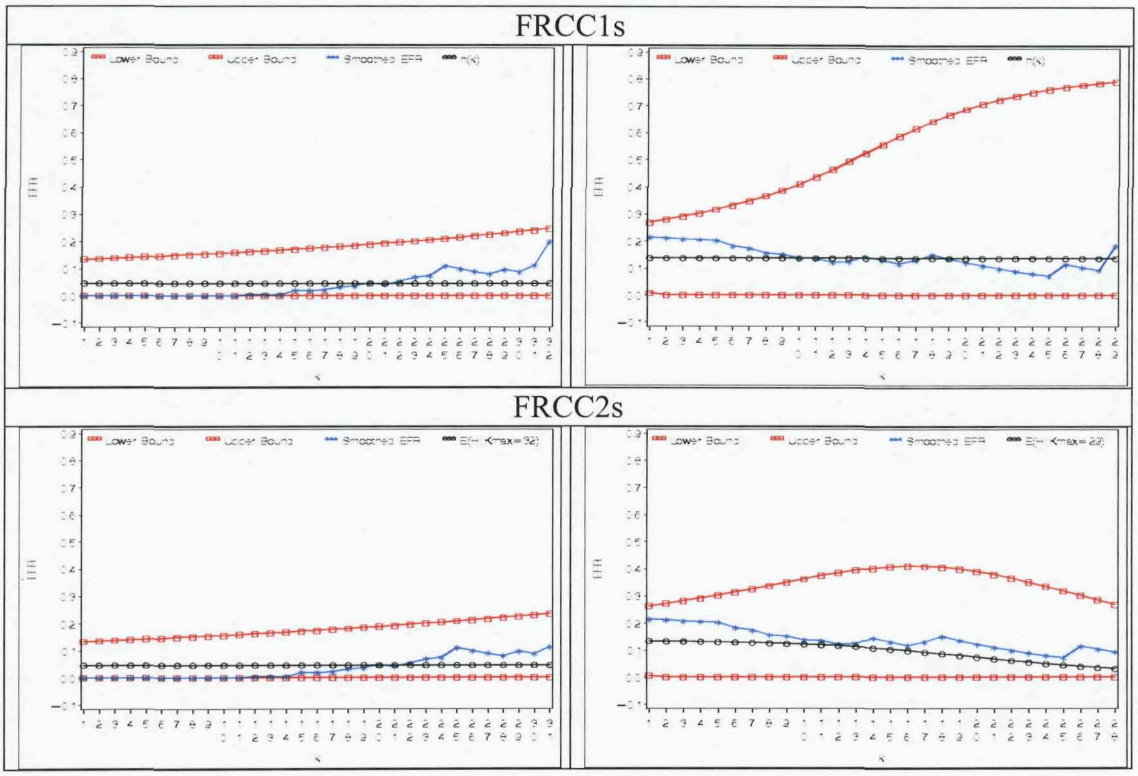


Figure 2.23: FRCC for Electromechanical devices data (left) and Inspection data (right).

Note that the results of the FRCC test for the two data sets coincide with those of Pearson chi-square and likelihood ratio tests as well as the seven goodness-of-fit tests that were applied to the data sets by Bracquemond et al. (2002) at the nominal 5% significance level.

2.4.6 A Less Restrictive Choice of Smoothing Parameter

In 2.4.3 we commented that forcing α to take one of the values 0.1(0.1)0.9 may be too restrictive. In this section we undertake an initial investigation of whether this is a factor in the over-conservatism of the smoothed FRCC by using a value of α that is unrestricted (other than lying between zero and one).

First we re-run Examples 2.2 and 2.3 with an unrestricted choice of α.

Example 2.4:

Using the simulated data in example 2.2, we would like to find the appropriate $\hat{h}_{sk}$ subject to the unrestricted smoothing factor α that minimizes the sum of squared

The covariance matrix $\tilde{V}_k$ has as its elements the covariances $\operatorname{cov}(\tilde{Y}_{ka_v}, \tilde{Y}_{ka_u})$, where a_v and a_u are multiple cause configurations, c and b are the causes of failure, single or multiple, and $k = 1, 2, ..., k_*$. Applying the delta method, we get

$$\operatorname{cov}(\tilde{Y}_{ka_v}, \tilde{Y}_{ka_u}) \approx \frac{\bar{H}'_{ka_v,ka_u}}{\hat{\bar{h}}'_{ka_v}\hat{\bar{h}}'_{ka_u}} - \sum_{j \varepsilon a_u} \frac{\bar{H}'_{ka_v,kj}}{\hat{\bar{h}}'_{ka_v}\hat{\bar{h}}'_{kj}} - \sum_{i \varepsilon a_v} \frac{\bar{H}'_{ki,ka_u}}{\hat{\bar{h}}'_{ki}\hat{\bar{h}}'_{ka_u}} + \sum_{i \varepsilon a_v}\sum_{j \varepsilon a_u} \frac{\bar{H}'_{ki,kj}}{\hat{\bar{h}}'_{ki}\hat{\bar{h}}'_{kj}}, \tag{3.4}$$

where $\bar{H}'_{kc,kb} = \operatorname{cov}(\hat{\bar{h}}'_{kc}, \hat{\bar{h}}'_{kb})$.

Then, applying the delta method again we get

$$\bar{H}'_{kc,kb} \approx \frac{1}{(1-\hat{h}'_k)^2}\left\{H'_{kc,kb} + \frac{\hat{h}'_{kc}\sum_c H'_{kc,kb} + \hat{h}'_{kb}\sum_b H'_{kc,kb}}{1-\hat{h}'_k} + \frac{\hat{h}'_{kc}\hat{h}'_{kb}\sum_c\sum_b H'_{kc,kb}}{(1-\hat{h}'_k)^2}\right\}, \tag{3.5}$$

where $H'_{kc,kb} = \operatorname{cov}(\hat{h}'_{kc}, \hat{h}'_{kb})$.

Now, we need to calculate $H'_{kc,kb}$. The likelihood function can be expressed as

$$L = \prod_{k=1}^{k_*}\left\{\prod_c h'^{\,\tilde{n}_{kc}}_{kc}(1-h'_k)^{\tilde{n}^+_{k\bullet}-n^-_{k\bullet}}\right\}.$$

So,

$$\frac{d\log L}{dh'_{kc}} = \frac{\tilde{n}_{kc}}{h'_{kc}} - \frac{\tilde{n}^+_{k\bullet}-n^-_{k\bullet}}{1-h'_k} \quad \text{and} \quad -\frac{d^2\log L}{dh'_{kc}dh'_{kb}} = \left\{\frac{\delta_{cb}\tilde{n}_{kc}}{h'^{\,2}_{kc}} + \frac{\tilde{n}^+_{k\bullet}-n^-_{k\bullet}}{(1-h'_k)^2}\right\}.$$

The $-d^2\log L/(dh'_{kc}dh'_{kb})$ can be estimated as

$$\tilde{n}^+_{k\bullet}\left\{\frac{\delta_{cb}\tilde{n}_{kc}}{\tilde{n}^+_{k\bullet}\hat{h}'^{\,2}_{kc}} + \frac{\tilde{n}^+_{k\bullet}-n^-_{k\bullet}}{\tilde{n}^+_{k\bullet}(1-\hat{h}'_k)^2}\right\} = \tilde{n}^+_{k\bullet}\left\{\frac{\delta_{cb}}{\hat{h}'_{kc}} + \frac{1}{(1-\hat{h}'_k)}\right\}.$$

Note that the second term is not absent for $k = k_*$ since $\hat{h}'_k = n^-_{k\bullet}/\tilde{n}^+_{k\bullet} < 1 \ \forall\ k \le k_*$.

Under the assumption that $h'_{kc} > 0\ \forall (k, c)$, $h'_k < 1\ \forall k \le k_*$ and the parameter set $\{h'_{kc}\}$ is finite, asymptotic likelihood theory can be applied to $\hat{h}'_{kc}$. Thus, errors. Thus, $\alpha = 0.022734211$. So, $\hat{h}_{sk}$ will be calculated such that $\hat{h}_{s1} = 0.26$ and $\hat{h}_{sk} = 0.022734211\,\hat{h}_k + 0.977265789\,\hat{h}_{k-1} \quad \forall\, 1 < k \le k_*$.

Table 2.14: The results of the FRCC for Example 2.4.

k	$\hat{h}_k$	$\hat{h}_{sk}$	$H_k \mid N_k^+ > 0$		FRCC1		FRCC1s		$H_k \mid K_{max} = 9$		FRCC2		FRCC2s	
			LB	UB	in	out	in	out	LB	UB	in	out	in	out
1	0.26	0.26	0.16	0.41	1	0	1	0	0.16	0.41	1	0	1	0
2	0.16	0.26	0.13	0.44	2	0	2	0	0.13	0.44	2	0	2	0
3	0.26	0.26	0.10	0.47	3	0	3	0	0.10	0.46	3	0	3	0
4	0.26	0.26	0.07	0.50	4	0	4	0	0.07	0.50	4	0	4	0
5	0.53	0.26	0.03	0.54	5	0	5	0	0.03	0.54	5	0	5	0
6	0.25	0.26	0	0.60	6	0	6	0	0	0.58	6	0	6	0
7	0.67	0.27	0	0.67	7	0	7	0	0	0.63	6	1	7	0
8	0.5	0.28	0	0.75	8	0	8	0	0	0.69	7	1	8	0
9	1	0.29	0	0.85	8	1	9	0						

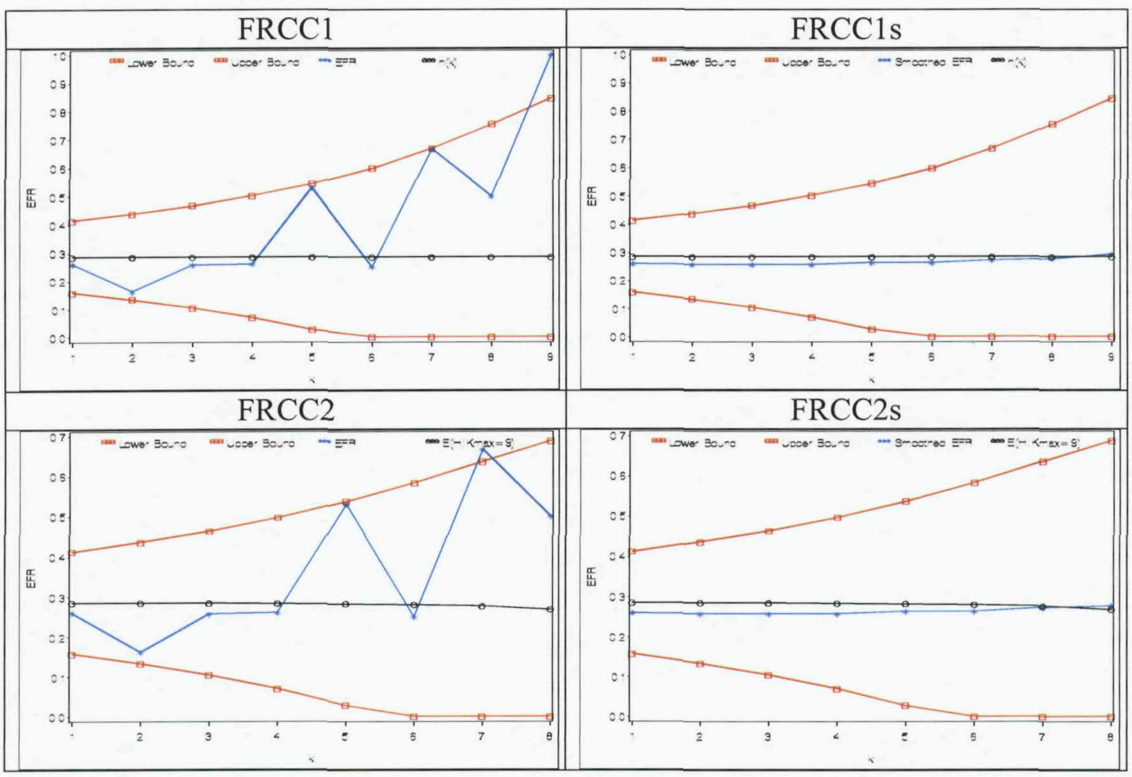


Figure 2.24: The FRCC plots for Example 2.4.

$$\hat{h}'_{kc} \sim AN\,(h'_{kc}, H'_{kc,kb});\; H'_{kc,kb} = \mathrm{cov}(\hat{h}'_{kc}, \hat{h}'_{kb}) \approx (H'_k)^{cb},$$

where H'_k is the $m \times m$ matrix where $m = 2^g - 1$ with (c,b) entry $E\left(-d^2 \log L / dh'_{kc} dh'_{kb}\right)$ and $(H'_k)^{cb}$ denotes the (c,b) element of its inverse. So, we have

$$H'_{kc,kb} = (\hat{H}'_k)^{cb} \text{ where } (\hat{H}'_k)_{cb} = \tilde{n}^+_{k\bullet} \left\{ \frac{\delta_{cb}}{\hat{h}'_{kc}} + \frac{1}{1 - \hat{h}'_k} \right\}$$

or in a matrix form,

$$\hat{H}'_k = \tilde{n}^+_{k\bullet} \left\{ diag\left\{ \frac{1}{\hat{h}'_{kc}} \right\} + \frac{1}{1-\hat{h}'_k} J_{m\times m} \right\} \text{ where } J_{m\times m} \text{ is a matrix of ones.}$$

Hence,

$$\hat{H}'^{-1}_k = \frac{1}{\tilde{n}^+_{k\bullet}} \left\{ \mathrm{diag}(\hat{h}'_{kc}) - \mathrm{diag}(\hat{h}'_{kc})\, J_{m\times m}\, \mathrm{diag}(\hat{h}'_{kc}) \right\}.$$

Thus,

$$H'_{kc,kb} = \mathrm{cov}(\hat{h}'_{kc}, \hat{h}'_{kb}) = \frac{1}{\tilde{n}^+_{k\bullet}} \left\{ \delta_{cb} \hat{h}'_{kc} - \hat{h}'_{kc} \hat{h}'_{kb} \right\}. \tag{3.6}$$

□.

3.5 Discussion

In this chapter two issues relating to the Crowder (1997) test for independence of competing risks have been considered. First, some errors and ambiguities in Crowder's results were found and have been corrected. The corrections left the basic form of the test unchanged but changed some important details. Secondly, because of a practical shortcoming of Crowder's test statistic, a modified version of the test statistic has been proposed.

The properties of the original (but corrected) Crowder test and the modified version will be discussed later, starting with the special case of two failure modes in chapter 4.

Chapter 4

Testing for Independence of Two Failure Modes with Discrete Lifetime Data

In the interests of clarity and notational simplicity, the simplest case of the competing risks phenomena, two failure modes, is considered. The relationship of the Crowder test to other tests of independence is highlighted by recasting Crowder's results in terms of classical contingency tables. The properties of the Crowder test, the modified test introduced in chapter 3 and other related tests are investigated by simulation.

4.1 Introduction

In chapter 3 we considered the test developed by Crowder (1997) and found errors in his derivation. From now on what we shall refer to as the Crowder test statistic, denoted by W, will always mean the corrected version of the test statistic.

In 4.2 we evaluate W in the special case of just two failure modes. A similar approach is used in 4.3 to evaluate the modified test statistic $\tilde{W}$ in this case. The familiar form of W in this case suggests some other classical tests for independence which are defined in 4.4. The behaviour of the tests is then investigated by simulation and the results are presented in 4.5. Notes on the behaviour of $\tilde{W}$ are presented in 4.6. The chapter concludes with a discussion in 4.7.

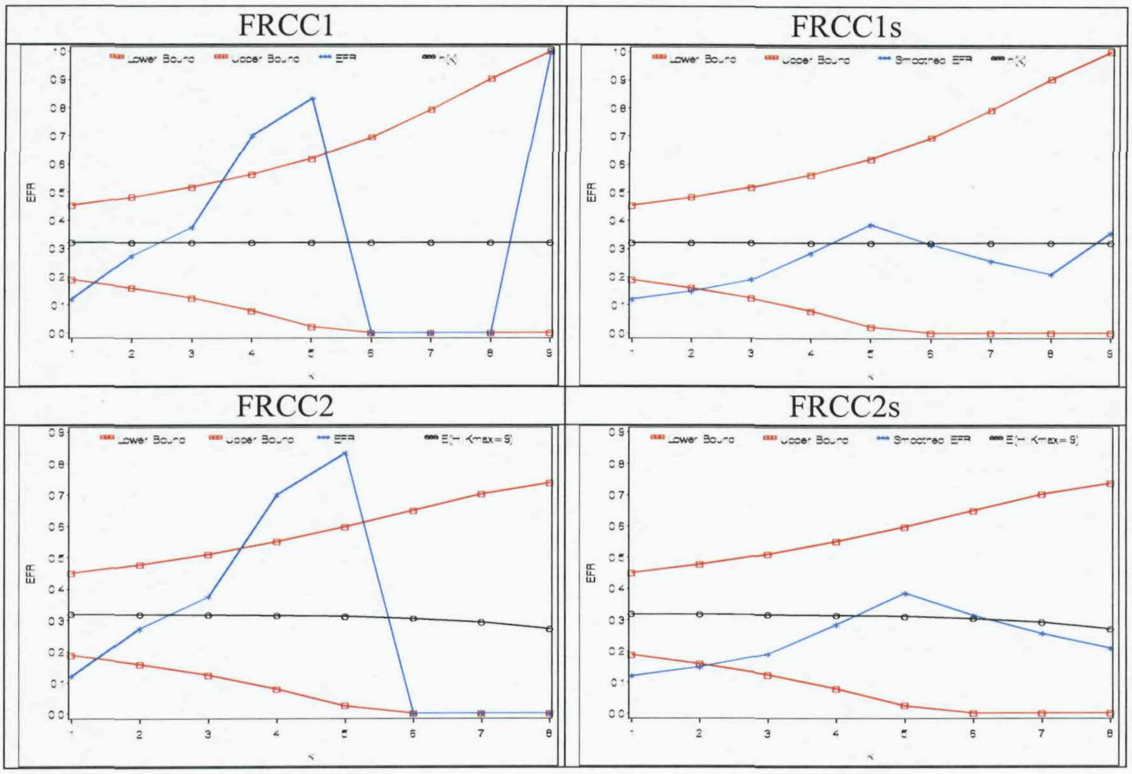


Figure 2.25: The FRCC plots for Example 2.5.

The results of the FRCC are similar to those of example 2.3. Thus, the results of using the restricted and unrestricted smoothing factor α are the same for this example too.

Table 2.16: Estimated significance of the smoothed FRCC methods at the nominal 5% significance level.

Geometric (p)	$p=0.10$				$p=0.15$				$p=0.20$			
	Sample size				Sample size				Sample size			
	25	50	100	200	25	50	100	200	25	50	100	200
Results of the restricted α												
FRCC1s	0.9	0.4	0.5	0.2	1.2	0.9	0.5	0.2	1.4	1.1	0.9	0.3
FRCC2s	1.0	0.7	0.5	0.2	1.4	1.0	0.5	0.3	1.5	1.2	0.8	0.4
Results of the unrestricted α												
FRCC1s	0.5	0.5	0.6	0.7	1.1	1.1	0.6	0.7	1.2	1.1	0.9	0.5
FRCC2s	2.5	3.1	3.8	4.7	4.6	5.6	5.6	6.9	5.9	6.1	7.6	7.1

4.2 Evaluation of W

When there are two failure modes only, the simplest case of the competing risks phenomenon, the data layout will be as shown in Table 4.1. The notation used in Table 4.1 is as follows:

n_{k1}, n_{k2}, and n_{ka} are the number of subjects failed from cause 1, cause 2 and multiple cause configuration $a=\{1,2\}$ respectively at time k ; n_{kr} is the number of subjects lost to view just after time k so their failure times are right-censored at time k , $n_{kr}=0$ when there are no censored observations at time k ; $n_{k\bullet}^{-}=\sum_{c} n_{kc}$ is the overall number of subjects failed at time k , where c is the cause of failure at time k , i.e. $c=1,2,a$ and $n_{k\bullet}^{+}=\sum_{i=k}^{k_*}(n_{i\bullet}^{-}+n_{ir})$ is the number of subjects at risk at time k .

Table 4.1: Data layout for two competing risks.

Time / Cause	1	2	3	4	5	...	k_*
1	n_{11}	n_{21}	n_{31}	n_{41}	n_{51}		n_{k_*1}
2	n_{12}	n_{22}	n_{32}	n_{42}	n_{52}		n_{k_*2}
$\{1,2\}$	n_{1a}	n_{2a}	n_{3a}	n_{4a}	n_{5a}		n_{k_*a}
Censored	n_{1r}	n_{2r}	n_{3r}	n_{4r}	n_{5r}		n_{k_*r}
Observed	$n_{1\bullet}^{-}$	$n_{2\bullet}^{-}$	$n_{3\bullet}^{-}$	$n_{4\bullet}^{-}$	$n_{5\bullet}^{-}$		$n_{k_*\bullet}^{-}$
At risk	$n_{1\bullet}^{+}$	$n_{2\bullet}^{+}$	$n_{3\bullet}^{+}$	$n_{4\bullet}^{+}$	$n_{5\bullet}^{+}$		$n_{k_*\bullet}^{+}$

We now evaluate W in this case. The nonparametric maximum likelihood estimator for h_{kc} is $\hat{h}_{kc}=n_{kc}/n_{k\bullet}^{+}$, and for h_k we have $\hat{h}_k=n_{k\bullet}^{-}/n_{k\bullet}^{+}$. The estimates for the sub-conditional odds functions are

$$\hat{\bar{h}}_{kc}=\frac{\hat{h}_{kc}}{1-\hat{h}_k}=\frac{n_{kc}}{n_{k\bullet}^{+}-n_{k\bullet}^{-}} \quad \forall k=1,\dots,k_*;\ n_{k\bullet}^{+}>n_{k\bullet}^{-}.$$

Hence,

$$\hat{\psi}_{ka} = \frac{\hat{\bar{h}}_{ka}}{\prod_{j \in a} \hat{\bar{h}}_{kj}} = \frac{n_{ka}(n_{k.}^{+} - n_{k.}^{-})}{n_{k1} n_{k2}} \text{ where } n_{k1} > 0 \text{ and } n_{k2} > 0.$$

Now, $Y_k = Y_{ka} = \log(\hat{\psi}_{ka})$ is a scalar since $d = 2^2 - 2 - 1 = 1$. So,

$$W = \sum_k \frac{Y_k^{\,2}}{\operatorname{var}(Y_k)}.$$

Now, we need to calculate $\operatorname{var}(Y_k)$. Let c and b be causes of failure, single or multiple, so using (3.3),

$$H_{kc,kb} = H_{kb,kc} = \frac{1}{n_{k.}^{+}} \left\{ \delta_{cb} \hat{h}_{kc} - (1 - \delta_{kk_*}) \hat{h}_{kc} \hat{h}_{kb} \right\}.$$

Thus, if $k < k_*$,

$$H_{ka,ka} = \operatorname{var}(\hat{h}_{ka}) = \frac{1}{n_{k.}^{+}} \left\{ \hat{h}_{ka} - \hat{h}_{ka}^{\,2} \right\} = \frac{n_{ka}(n_{k\bullet}^{+} - n_{ka})}{n_{k\bullet}^{+\,3}},$$

$$H_{k1,k1} = \operatorname{var}(\hat{h}_{k1}) = \frac{1}{n_{k.}^{+}} \left\{ \hat{h}_{k1} - \hat{h}_{k1}^{\,2} \right\} = \frac{n_{k1}(n_{k\bullet}^{+} - n_{k1})}{n_{k\bullet}^{+\,3}},$$

$$H_{k2,k2} = \operatorname{var}(\hat{h}_{k2}) = \frac{1}{n_{k.}^{+}} \left\{ \hat{h}_{k2} - \hat{h}_{k2}^{\,2} \right\} = \frac{n_{k2}(n_{k\bullet}^{+} - n_{k2})}{n_{k\bullet}^{+\,3}},$$

$$H_{ka,k1} = H_{k1,ka} = \frac{1}{n_{k.}^{+}} \left\{ -\hat{h}_{k1} \hat{h}_{ka} \right\} = -\frac{n_{k1} n_{ka}}{n_{k\bullet}^{+\,3}},$$

$$H_{ka,k2} = H_{k2,ka} = \frac{1}{n_{k.}^{+}} \left\{ -\hat{h}_{k2} \hat{h}_{ka} \right\} = -\frac{n_{k2} n_{ka}}{n_{k\bullet}^{+\,3}},$$

$$H_{k2,k1} = H_{k1,k2} = \frac{1}{n_{k.}^{+}} \left\{ -\hat{h}_{k1} \hat{h}_{k2} \right\} = -\frac{n_{k1} n_{k2}}{n_{k\bullet}^{+\,3}}.$$

Now, using (3.2),

$$\bar{H}_{kc,kb} \approx \frac{1}{(1 - \hat{h}_k)^2} \left\{ H_{kc,kb} + \frac{\hat{h}_{kc} \sum_c H_{kc,kb} + \hat{h}_{kb} \sum_b H_{kc,kb}}{1 - \hat{h}_k} + \frac{\hat{h}_{kc} \hat{h}_{kb} \sum_c \sum_b H_{kc,kb}}{(1 - \hat{h}_k)^2} \right\},$$

and using (3.1),

$$\operatorname{var}(Y_k) = \operatorname{var}(Y_{ka}) \approx \frac{\bar{H}_{ka,ka}}{\hat{h}_{ka}^{2}} - \sum_{j \in a} \frac{\bar{H}_{ka,kj}}{\hat{h}_{ka}\hat{h}_{kj}} - \sum_{i \in a} \frac{\bar{H}_{ki,ka}}{\hat{h}_{ki}\hat{h}_{ka}} + \sum_{i \in a}\sum_{j \in a} \frac{\bar{H}_{ki,kj}}{\hat{h}_{ki}\hat{h}_{kj}}$$
$$= \frac{1}{n_{k1}} + \frac{1}{n_{k2}} + \frac{1}{n_{ka}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}}.$$

Therefore,

$$W = \sum_{k=1}^{k\cdot} \frac{\left[\log\left(\frac{n_{ka}(n_{k\cdot}^{+} - n_{k\cdot}^{-})}{n_{k1}n_{k2}}\right)\right]^2}{\frac{1}{n_{k1}} + \frac{1}{n_{k2}} + \frac{1}{n_{ka}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}}},$$

where $n_{k1} > 0$, $n_{k2} > 0$ and $n_{ka} > 0$.

□.

Writing $W = \sum_{k=1}^{k\cdot} W_k$, we see that

$$W_k = \frac{\left[\log\left(\frac{n_{ka}(n_{k\cdot}^{+} - n_{k\cdot}^{-})}{n_{k1}n_{k2}}\right)\right]^2}{\frac{1}{n_{k1}} + \frac{1}{n_{k2}} + \frac{1}{n_{ka}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}}}$$

turns out to be the log odds ratio test statistic in a 2×2 contingency table (Agresti, 2002).

Example 4.1: Crowder (1997)

The data were collected in a study of the growth of organisms in catheters inserted on day 1, and then samples were taken from two sites in the equipment on day 2,3,...,7. The experiment was terminated after day 7. Thus, the nominal times to infection are 1,2,...,6 days. Some of the times are censored before the last day because the patient was removed for reasons unrelated to catheter infection. We follow Crowder (1997) in assuming that a censored value at day k means that the patient was at risk on days 1, 2,

..., k but not on days $k+1, k+2, \ldots$. The data and the W_k at each computable failure time k are presented in Table 4.2.

The computation of the Crowder statistic in its simple form for the first failure time, $k=1$, is as follows:

$$n_{1\bullet}^{-} = \sum_c n_{1c} = n_{11} + n_{12} + n_{1a} = 2 + 7 + 11 = 20,$$

$$n_{1\bullet}^{+} = \sum_{i=1}^{6} (n_{i\bullet}^{-} + n_{ir}) = (20+35) + (29+104) + (21+55) + (6+19) + (5+9) + (3+28) = 334,$$

$$W_1 = \frac{\left[\log\left(\frac{n_{1a}(n_{1\bullet}^{+} - n_{1\bullet}^{-})}{n_{11}n_{12}}\right)\right]^2}{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{1a}} + \frac{1}{n_{1\bullet}^{+} - n_{1\bullet}^{-}}} = \frac{\left[\log\left(\frac{11(334-20)}{2\times 7}\right)\right]^2}{\frac{1}{2} + \frac{1}{7} + \frac{1}{11} + \frac{1}{314}} = 41.1705.$$

Table 4.2: Times to infection for 334 patients and the W_k.

Time (days)	1	2	3	4	5	6
n_{k1}	2	2	1	0	0	0
n_{k2}	7	7	5	3	2	2
n_{ka}	11	20	15	3	3	1
n_{kr}	35	104	55	19	9	28
$n_{k\bullet}^{-}$	20	29	21	6	5	3
$n_{k\bullet}^{+}$	334	279	146	70	45	31
W_k	41.1705	49.5833	27.5589			

Note that W_k is not computable for times 4, 5 and 6 because of the zero frequencies for site 1. The resulting Crowder test statistic, $W = \sum_{k=1}^{3} W_k$, is 118.313 with 3 degrees of freedom which gives strong evidence against the independence of infection occurrence at the two sites.

Example 4.2: Simulated $NB(0.1, 2)$ data

The data comprise 35 independent pairs (k, c) that were generated using SAS software version 9.1. The observed failure time, K, and the type of failure, C, are determined as follows:

Two samples with 35 observations each were simulated independently from the negative binomial distribution, defined in 1.5.2, with parameters $p = 0.1$ and $z = 2$ denoted X_1 and X_2 respectively. There are two causes of failure, $g = 2$, and so the number of multiple cause configurations is $d = 2^2 - 2 - 1 = 1$. Then,

$$K = \min(X_1, X_2),$$

$$C = \begin{cases} 1 & \text{if } K = X_1 \text{ and } K \neq X_2 \\ 2 & \text{if } K = X_2 \text{ and } K \neq X_1 \\ \{1,2\} & \text{if } K = X_1 = X_2 \end{cases}.$$

The data are displayed in Table 4.3 along with W_k at each computable failure time k. The computation of W_7 is as follows:

$$n_{7\bullet}^{-} = \sum\nolimits_c n_{7c} = n_{71} + n_{72} + n_{7a} = 2 + 1 + 2 = 5,$$

$$n_{7\bullet}^{+} = \sum_{i=7}^{21} n_{i\bullet}^{-} = 5+2+1+3+2+1+2+1+1+1+2+1+2+1+1 = 26,$$

since there are no censored observations, $n_{ir} = 0 \ \forall \ 7 \leq i \leq 21$,

$$W_7 = \frac{\left[\log\left(\dfrac{n_{7a}(n_{7\cdot}^{+} - n_{7\cdot}^{-})}{n_{71}n_{72}}\right)\right]^2}{\dfrac{1}{n_{71}} + \dfrac{1}{n_{72}} + \dfrac{1}{n_{7a}} + \dfrac{1}{n_{7\cdot}^{+} - n_{7\cdot}^{-}}} = \frac{\left[\log\left(\dfrac{2(26-5)}{2\times 1}\right)\right]^2}{\dfrac{1}{2} + \dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{21}} = 4.52678.$$

Note that the Crowder statistic W_k is computable only when $k = 7$ because of the zero frequencies at other times. The resulting Crowder test statistic W is 4.52678 with 1 degree of freedom, which has a p-value 0.033, indicating wrongly that the independence hypothesis may be rejected at the 5% significance level.

Table 4.3: Simulated data for Example 4.2 and the W_k.

k	n_{k1}	n_{k2}	n_{ka}		$n_{k\bullet}^-$	$n_{k\bullet}^+$	W_k
1	1	0	0		1	35	
2	1	0	0		1	34	
3	1	2	0		3	33	
5	0	4	0		4	30	
7	2	1	2		5	26	4.52678
8	2	0	0		2	21	
9	1	0	0		1	19	
10	2	1	0		3	18	
11	2	0	0		2	15	
12	1	0	0		1	13	
14	2	0	0		2	12	
15	0	1	0		1	10	
17	0	1	0		1	9	
18	1	0	0		1	8	
20	1	1	0		2	7	
22	0	1	0		1	5	
25	2	0	0		2	4	
26	0	1	0		1	2	
29	0	1	0		1	1	

4.3 Evaluation of $\tilde{W}$

We now consider the modified Crowder test statistic $\tilde{W}$ in the special case of two failure modes. With $c = 1, 2, a$ where $a = \{1,2\}$, the estimate for h_{kc} is

$$\hat{h}'_{kc} = \frac{n_{kc} + 0.5}{n_{k\bullet}^+ + 0.5}$$

and for h_k is

$$\hat{h}'_k = \frac{n_{k\bullet}^-}{n_{k\bullet}^+ + 0.5}.$$

The estimates for the sub-conditional odds functions are

$$\hat{\bar{h}}'_{kc} = \frac{\hat{h}'_{kc}}{1 - \hat{h}'_k} = \frac{n_{kc} + 0.5}{n_{k\bullet}^+ - n_{k\bullet}^- + 0.5} \quad \forall k = 1, \dots, k_* .$$

Hence,

$$\hat{\tilde{\psi}}_{ka} = \frac{\hat{\tilde{h}}'_{ka}}{\prod_{j \in a} \hat{\tilde{h}}'_{kj}} = \frac{(n_{ka}+0.5)(n^+_{k\cdot} - n^-_{k\cdot} + 0.5)}{(n_{k1}+0.5)(n_{k2}+0.5)}.$$

Now, $\tilde{Y}_k = \tilde{Y}_{ka} = \log(\hat{\tilde{\psi}}_{ka})$ is a scalar since $d = 2^2 - 2 - 1 = 1$. So,

$$\tilde{W} = \sum_{k=1}^{k_\bullet} \frac{\tilde{Y}_k^{\,2}}{\operatorname{var}(\tilde{Y}_k)}.$$

Now, we need to calculate $\operatorname{var}(\tilde{Y}_k)$. Let c and b be causes of failure, single or multiple, so using (3.6),

$$H'_{kc,kb} = H'_{kb,kc} = \frac{1}{n^+_{k\cdot}+0.5}\left\{\delta_{cb}\hat{h}'_{kc} - \hat{h}'_{kc}\hat{h}'_{kb}\right\}.$$

Thus,

$$H'_{ka,ka} = \operatorname{var}(\hat{h}'_{ka}) = \frac{1}{n^+_{k\cdot}+0.5}\left\{\hat{h}'_{ka} - \hat{h}'^{\,2}_{ka}\right\} = \frac{(n_{ka}+0.5)(n^+_{k\bullet} - n_{ka})}{(n^+_{k\bullet}+0.5)^3},$$

$$H'_{k1,k1} = \operatorname{var}(\hat{h}'_{k1}) = \frac{1}{n^+_{k\cdot}+0.5}\left\{\hat{h}'_{k1} - \hat{h}'^{\,2}_{k1}\right\} = \frac{(n_{k1}+0.5)(n^+_{k\bullet} - n_{k1})}{(n^+_{k\bullet}+0.5)^3},$$

$$H'_{k2,k2} = \operatorname{var}(\hat{h}'_{k2}) = \frac{1}{n^+_{k\cdot}+0.5}\left\{\hat{h}'_{k2} - \hat{h}'^{\,2}_{k2}\right\} = \frac{(n_{k2}+0.5)(n^+_{k\bullet} - n_{k2})}{(n^+_{k\bullet}+0.5)^3},$$

$$H'_{ka,k1} = H'_{k1,ka} = \frac{1}{n^+_{k\cdot}+0.5}\left\{-\hat{h}'_{k1}\hat{h}'_{ka}\right\} = -\frac{(n_{k1}+0.5)(n_{ka}+0.5)}{(n^+_{k\bullet}+0.5)^3},$$

$$H'_{ka,k2} = H'_{k2,ka} = \frac{1}{n^+_{k\cdot}+0.5}\left\{-\hat{h}'_{k2}\hat{h}'_{ka}\right\} = -\frac{(n_{k2}+0.5)(n_{ka}+0.5)}{(n^+_{k\bullet}+0.5)^3},$$

$$H'_{k2,k1} = H'_{k1,k2} = \frac{1}{n^+_{k\cdot}+0.5}\left\{-\hat{h}'_{k1}\hat{h}'_{k2}\right\} = -\frac{(n_{k1}+0.5)(n_{k2}+0.5)}{(n^+_{k\bullet}+0.5)^3}.$$

Now, using (3.5),

$$\bar{H}'_{kc,kb} \approx \frac{1}{(1-\hat{h}'_k)^2}\left\{H'_{kc,kb} + \frac{\hat{h}'_{kc}\sum_c H'_{kc,kb} + \hat{h}'_{kb}\sum_b H'_{kc,kb}}{1-\hat{h}'_k} + \frac{\hat{h}'_{kc}\hat{h}'_{kb}\sum_c\sum_b H'_{kc,kb}}{(1-\hat{h}'_k)^2}\right\},$$

and using (3.4),

$$\begin{aligned}\operatorname{var}(\tilde{Y}_k) = \operatorname{var}(\tilde{Y}_{ka}) &\approx \frac{\bar{H}'_{ka,ka}}{\hat{h}'^{\,2}_{ka}} - \sum_{j\in a}\frac{\bar{H}'_{ka,kj}}{\hat{h}'_{ka}\hat{h}'_{kj}} - \sum_{i\in a}\frac{\bar{H}'_{ki,ka}}{\hat{h}'_{ki}\hat{h}'_{ka}} + \sum_{i\in a}\sum_{j\in a}\frac{\bar{H}'_{ki,kj}}{\hat{h}'_{ki}\hat{h}'_{kj}} \\ &= \frac{1}{n_{k1}+0.5} + \frac{1}{n_{k2}+0.5} + \frac{1}{n_{ka}+0.5} + \frac{1}{n^+_{k\bullet} - n^-_{k\bullet} + 0.5}.\end{aligned}$$

Therefore,

$$\tilde{W} = \sum_{k=1}^{k_\bullet} \frac{\left[\log\left(\dfrac{(n_{ka}+0.5)(n^+_{k\bullet} - n^-_{k\bullet}+0.5)}{(n_{k1}+0.5)(n_{k2}+0.5)}\right)\right]^2}{\dfrac{1}{n_{k1}+0.5} + \dfrac{1}{n_{k2}+0.5} + \dfrac{1}{n_{ka}+0.5} + \dfrac{1}{n^+_{k\bullet} - n^-_{k\bullet}+0.5}}.$$

□.

Example 4.3: Crowder (1997)

The modified version of Crowder's test statistic, $\tilde{W}$, is applied to the data of times to infection for 334 patients that presented in example 4.1 and the results are presented in Table 4.4.

Table 4.4: Times to infection for 334 patients and the $\tilde{W}_k$.

Time(days)	1	2	3	4	5	6
n_{k1}	2	2	1	0	0	0
n_{k2}	7	7	5	3	2	2
n_{ka}	11	20	15	3	3	1
n_{kr}	35	104	55	19	9	28
$n^-_{k\bullet}$	20	29	21	6	5	3
$n^+_{k\bullet}$	334	279	146	70	45	31
$\tilde{W}_k$	44.4129	53.7485	32.4046	9.1296	8.2577	4.0224

For example, the calculation of $\tilde{W}_2$ is as follows:

$n^-_{2\bullet} = \sum_c n_{2c} = n_{21} + n_{22} + n_{2a} = 2+7+20 = 29$,

$n^+_{2\bullet} = \sum_{i=2}^{6}(n^-_{i\bullet} + n_{ir}) = (29+104)+(21+55)+(6+19)+(5+9)+(3+28) = 279$,

$$\tilde{W}_2 = \frac{\left[\log\left(\frac{(n_{2a}+0.5)(n_{2\cdot}^{+}-n_{2\cdot}^{-}+0.5)}{(n_{21}+0.5)(n_{22}+0.5)}\right)\right]^2}{\frac{1}{n_{21}+0.5}+\frac{1}{n_{22}+0.5}+\frac{1}{n_{2a}+0.5}+\frac{1}{n_{2\cdot}^{+}-n_{2\cdot}^{-}+0.5}} = \frac{\left[\log\left(\frac{20.5(279-29+0.5)}{2.5\times 7.5}\right)\right]^2}{\frac{1}{2.5}+\frac{1}{7.5}+\frac{1}{20.5}+\frac{1}{250.5}}$$

$$= 53.7485\ .$$

Hence $\tilde{W} = \sum_{k=1}^{6}\tilde{W}_k = 151.976$ with 6 degrees of freedom, leading to the same conclusion as with W in example 4.1.

Example 4.4: Simulated $NB(0.1, 2)$ data

$\tilde{W}$ is applied to the data simulated in example 4.2 and the results are presented in Table 4.5. $\tilde{W} = \sum_{k=1}^{29}\tilde{W}_k = 17.02194$ with 19 degrees of freedom, which has a p-value 0.588, indicating that the independence hypothesis cannot be rejected at the 5% significance level. Note that the conclusions based on W and $\tilde{W}$ differ for this set of data.

Table 4.5: Simulated data for example 4.4 and the $\tilde{W}_k$.

k	n_{k1}	n_{k2}	n_{ka}		$n_{k\bullet}^{-}$	$n_{k\bullet}^{+}$	$\tilde{W}_k$
1	1	0	0		1	35	2.09371
2	1	0	0		1	34	2.05423
3	1	2	0		3	33	0.63492
5	0	4	0		4	30	0.73798
7	2	1	2		5	26	4.68509
8	2	0	0		2	21	0.94791
9	1	0	0		1	19	1.33702
10	2	1	0		3	18	0.16830
11	2	0	0		2	15	0.63565
12	1	0	0		1	13	0.94709
14	2	0	0		2	12	0.45814
15	0	1	0		1	10	0.71398
17	0	1	0		1	9	0.62889
18	1	0	0		1	8	0.53964
20	1	1	0		2	7	0.01146
22	0	1	0		1	5	0.24688
25	2	0	0		2	4	0
26	0	1	0		1	2	0
29	0	1	0		1	1	0.18104

4.4 Some Other Tests for Independence

In the case of only two risks of failure, it is clear from table 4.1 that at each observed failure time k we have a 2×2 contingency table. The layout of each table at each observed failure time k is shown in Table 4.6.

Table 4.6: 2×2 contingency table at an observed failure time k.

		Cause 1	
		Yes	No
Cause 2	Yes	n_{ka}	n_{k2}
	No	n_{k1}	$n_{k\bullet}^{+} - n_{k\bullet}^{-}$

The Crowder statistic W_k at each observable failure time k turns out to be the log odds ratio test statistic in a 2×2 contingency table. Various test statistics can be used for testing the independence in 2×2 contingency tables. Since with this type of data there may be cells with small or zero frequencies at some observed failure times, we will consider Yates' chi-square and Fisher's exact tests as comparative test statistics.

4.4.1 Yates' Chi-Square Test

Pearson's chi-square statistic is often used as a test statistic for testing independence in a 2×2 contingency table (Agresti, 2002). The use of Pearson's chi-square test with small expected frequencies is controversial. Hence, a continuity-corrected version of it was suggested by Yates (1934). In the present notation the corresponding statistic at failure time k is

$$\chi_{Yk}^2 = \frac{n_{k\bullet}^{+}\left(| n_{ka}(n_{k\bullet}^{+} - n_{k\bullet}^{-}) - n_{k1}n_{k2} | - n_{k\bullet}^{+}/2\right)^2}{(n_{ka} + n_{k1})(n_{k2} + n_{k\bullet}^{+} - n_{k\bullet}^{-})(n_{ka} + n_{k2})(n_{k1} + n_{k\bullet}^{+} - n_{k\bullet}^{-})}.$$

The suggested correction for continuity overcomes the approximation of the discrete probability by the chi-square distribution which is continuous. It has more effect when the expected frequencies are small but it is a good idea to apply it to all χ^2 tests in 2×2 contingency tables (Armitage and Berry, 1987).

In the present context we will be using $\chi_Y^2 = \sum_k \chi_{Yk}^2$ with degrees of freedom equal to the number of k − values that allow χ_{Yk}^2 to be evaluated.

4.4.2 Fisher's Exact Test

An exact test that is used to test the independence in a 2×2 contingency table is known as Fisher's exact test (Agresti, 2002). Fisher showed that conditional on the marginal totals being fixed the exact null probability of observing a table with frequencies set out as in Table 4.6 is given by the hypergeometric probability

$$p = \binom{n_{ka}+n_{k2}}{n_{ka}}\binom{n_{k1}+n_{k\bullet}^{+}-n_{k\bullet}^{-}}{n_{k1}}\Bigg/\binom{n_{k\bullet}^{+}}{n_{ka}+n_{k1}}$$

$$= \frac{(n_{ka}+n_{k2})!(n_{k1}+n_{k\bullet}^{+}-n_{k\bullet}^{-})!(n_{ka}+n_{k1})!(n_{k2}+n_{k\bullet}^{+}-n_{k\bullet}^{-})!}{n_{k\bullet}^{+}!n_{ka}!n_{k2}!n_{k1}!(n_{k\bullet}^{+}-n_{k\bullet}^{-})!}.$$

In order to calculate the p-value for the observed data at each observed failure time k, the hypergeometric probabilities of all tables with the same marginal totals as the observed one must be calculated. Then the p-value will be the sum of all the hypergeometric probabilities that are less than or equal to the hypergeometric probability of the observed table (Armitage and Berry, 1987). The calculations of Fisher's exact test using a calculator are tedious even with small sample sizes, but statistical software can calculate the significance of Fisher's exact test even with large sample sizes (Agresti, 2002). Here SAS software version 9.1 is used to calculate the p-value for Fisher's exact test.

In order to decide whether or not to reject the independence hypothesis we must combine the p-value results for all k − values. To achieve this we use a Bonferroni-type approach. Let k' be the number of k values that form non-trivial 2×2 contingency tables, i.e. tables in which none of the marginal totals is zero. Then the null hypothesis is rejected at a nominal 5% significance if any of the observed p-values is less than $0.05/k'$. Denote this test procedure by F. An alternative approach is to replace $0.05/k'$ in the above with $0.05/k_*$. Denote this test procedure by $\ddot{F}$.

4.5 Simulation Study

In this section we compare the performances of the Crowder test (unmodified, W, and modified, $\tilde{W}$), the test based on Yates' chi-square, χ_Y^2, and Fisher's exact test, F and $\ddot{F}$, using simulation in the special case of two failure modes.

4.5.1 Design to Estimate Significance

For data with no censored observations, let X_1 and X_2 be independent discrete random variables on $0,1,2,3,\ldots$. The observed failure time, K, and the type of failure, C, are determined as in Example 4.2.

For data with censored observations, let X_1, X_2 and X_3 be independent discrete random variables on $0,1,2,3,\ldots$. K and C are determined as follows:

$$K = \min(X_1, X_2),$$

$$C = \begin{cases} 1 & \text{if } K = X_1 \le X_3 \text{ and } K \ne X_2 \\ 2 & \text{if } K = X_2 \le X_3 \text{ and } K \ne X_1 \\ \{1,2\} & \text{if } K = X_1 = X_2 \le X_3 \\ r & \text{if } K > X_3;\ r: \text{removed /censored} \end{cases}$$

For a given sample size n and for specified distributions for X_1 and X_2 (no censoring) or for X_1, X_2 and X_3 (with censoring), n pairs (k,c) were generated using SAS version 9.1. At a nominal 5% significance level the reject / not reject inference was made for each test procedure. This was repeated 1000 times and the percentage of rejections was calculated for each test to estimate the true significance level.
This procedure was carried out for $n = 50, 100, 150, 200, 250$ and 300 and for four distributions with different FR shapes. The distributions used here are geometric $p = 0.1\,(0.05)\,0.5$ (flat FR), negative binomial with $p = 0.1\,(0.05)\,0.5$ and $z = 0.2$

(decreasing FR), negative binomial with $p = 0.1\,(0.05)\,0.5$ and $z = 2$ (gently increasing FR) and Poisson with mean $\lambda = 1\,(0.5)\,5$ (FR that increases to 1).

4.5.2 Significance Results

The estimated significance probabilities of the test procedures at the nominal 5% significance level for the above design are presented in Figures D.1 and D.2 of Appendix D for samples with no censored observations and with censored observations respectively. Every tick mark on the x-axes of the graphical representations corresponds to the estimated significance for samples generated from the indicated distribution with different parameter values and sample sizes in ascending order.

For samples generated from negative binomial distribution with no censored observations, e.g. Figure 4.1, tick mark 1 corresponds to the estimated significance for samples generated from $NB(0.1,2)$ for X_1 and $NB(0.1,2)$ for X_2 with $n = 50$, tick mark 2 corresponds to the estimated significance for samples generated from $NB(0.1,2)$ for X_1 and $NB(0.15,2)$ for X_2 with $n = 50$, ..., tick mark 45 corresponds to the estimated significance for samples generated from $NB(0.5,2)$ for X_1 and $NB(0.5,2)$ for X_2 with $n = 50$, tick mark 46 corresponds to the estimated significance for samples generated from $NB(0.1,2)$ for X_1 and $NB(0.1,2)$ for X_2 with $n = 100$, ... and tick mark 270 corresponds to the estimated significance for samples generated from $NB(0.5,2)$ for X_1 and $NB(0.5,2)$ for X_2 with $n = 300$. For samples with censored observations, the samples for X_3 are generated from the same sample distribution of X_2, e.g. tick mark 2 for samples generated from the negative binomial distribution with censored observations corresponds to the estimated significance for samples generated from $NB(0.1,2)$ for X_1, $NB(0.15,2)$ for X_2 and $NB(0.15,2)$ for X_3 with $n = 50$. Table 4.7 shows the values of the distributions parameters associated with each tick mark for $n = 50$. Tick marks 46 to 90 will have the same distributions as tick marks 1 to 45 but for $n = 100$ and the same will apply to $n = 150$ (91-135), $n = 200$ (136-180), $n = 250$ (181-225) and $n = 300$ (226-270).

Figures D.1 and D.2 show that $\tilde{W}$ tends to be conservative in absolute terms and relative to the other four tests. The only exception occurs with Poisson based situations with $\lambda \geq 4$. W also tends to be conservative except for $NB\,(p, z = 2)$ based situations, where p is relatively small, $p = 0.1$ for X_1 and $p = 0.1, 0.15, 0.2$ for X_2 with all the sample sizes. The test based on χ_Y^2 is liberal for situations based on increasing FR distributions, $NB\,(p, z = 2)$ and $P(\lambda)$. In particular the high estimated significances for χ_Y^2 occur with failure modes that were generated based on sampling from $NB\,(p, z = 2)$ distribution with $p = 0.1$ for X_1, $p = 0.1, 0.15, 0.2, 0.25$ for X_2 when $n = 50$, $p = 0.1, 0.15, 0.2$ for X_2 when $n = 100$, $p = 0.1, 0.15$ for X_2 when $n = 150, 200, 250$, and $p = 0.1$ for X_2 when $n = 300$. Also, the high estimated significances occur with failure modes that were generated based on sampling from the Poisson distribution with $\lambda = 3, 3.5, 4, 4.5, 5$ for X_1 and X_2 when $n = 50$, $\lambda = 3.5, 4, 4.5, 5$ for X_1 and X_2 when $n = 100, 150, 200$, and $\lambda = 4, 4.5, 5$ for X_1 and X_2 when $n = 250, 300$. Note that the number of situations for which the high estimated significance levels of χ_Y^2 emerges decreases as the sample size increases. This scenario happened for both types of data, with and without censoring. Figure 4.1 gives an illustration of the $NB\,(p, z = 2)$ based situations.

Tables 4.8 and 4.9 show that all the estimated significances of $\tilde{W}$ for geometric and negative binomial based samples, without or with censoring, are below 5%. For Poisson based samples, 98.9% of the estimated significances are below 5% with 5.8% being the maximum for samples with no censoring and 96.3% of the estimated significances are below 5% with 7.3% being the maximum for samples with censoring. The estimated significances of Fisher's exact test F are greater than those of $\ddot{F}$ for most of the situations but all are 5% or below for all the four distributions. However, the estimated significances of $\tilde{W}$ are less than those of F for most of the generated samples, as can be seen from the graphical representation, Figures D.1 and D.2 of Appendix D.

As pointed out in section 4.2, a potential problem with the Crower test based on W is that W is not always computable. This behaviour was tracked throughout the simulation experiment. Indeed, the estimated significance levels for W plotted in Figures D.1 and D.2 were calculated using those samples for which W was computable. For example, for samples with censoring, $n = 50$ and based on $NB(0.5, 0.2)$, W was computable for only 85 out of the 1000 simulated samples. Figures 4.2 and 4.3 show the percentages of samples, without and with censoring respectively, for which W was not computable. Clearly, for many situations these percentages are high.

In summary, the modified Crowder test and the tests based on Fisher's exact test tend to be conservative. The test based on χ_Y^2 is highly liberal in some circumstances and the Crowder test cannot be used routinely because of computability problems.

4.5.3 Design to Estimate Power

For data with no censored observations, let X_1, X_2 and U be independent discrete random variables on $0,1,2,3,\ldots$. A model for dependent risks that is easy to simulate and makes reasonable sense in practice is as follows:

Let

$$Y_1 = X_1 + U\ ,$$
$$Y_2 = X_2 + U\ .$$

Note that $\mathrm{cov}(Y_1, Y_2) = \mathrm{var}(U) > 0$, thereby giving positive correlation.

The observed failure time, K, and the type of failure, C, are determined as follows:

$$K = \min(Y_1, Y_2),$$

$$C = \begin{cases} 1 & \text{if } K = Y_1 \text{ and } K \neq Y_2 \\ 2 & \text{if } K = Y_2 \text{ and } K \neq Y_1 \\ \{1,2\} & \text{if } K = Y_1 = Y_2 \end{cases}.$$

For data with censored observations, let X_1, X_2, X_3 and U be independent discrete random variables on $0,1,2,3,\ldots$. K and C are determined as follows:

$$Y_1 = X_1 + U\ ,$$
$$Y_2 = X_2 + U\ ,$$
$$Y_3 = X_3 + U\ ,$$
$$K = \min(Y_1, Y_2),$$

$$C = \begin{cases} 1 & \text{if } K = Y_1 \leq Y_3 \text{ and } K \neq Y_2 \\ 2 & \text{if } K = Y_2 \leq Y_3 \text{ and } K \neq Y_1 \\ \{1,2\} & \text{if } K = Y_1 = Y_2 \leq Y_3 \\ r & \text{if } K > Y_3;\ r: \text{removed /censored} \end{cases}.$$

For a given sample size n and for specified distributions for X_1, X_2 and U (no censoring) or for X_1, X_2, X_3 and U (with censoring), n pairs (k, c) were generated using SAS version 9.1. At a nominal 5% significance level the reject / not reject inference was made for each test procedure. This was repeated 1000 times and the percentage of rejections was calculated for each test to estimate the true power level. This procedure was carried out in the same way as for the significance levels.

4.5.4 Power Results

The estimated powers of the test procedures at the nominal 5% significance level for the above design are presented in Figures E.1 and E.3 of Appendix E for samples with no censored observations and with censored observations respectively. The tick marks on the x-axes of the graphical representations correspond to the estimated powers for samples generated from the indicated distribution with different parameter values and sample sizes in ascending order as for the estimated significances. For a reminder about the tick marks see the discussion in section 4.5.2 and Table 4.7. The samples for U are generated from the same sample distributions of X_1, e.g. tick mark 2 for samples generated from the geometric distribution with censored observations corresponds to the estimated significance for samples generated from $G(0.1)$ for X_1, $G(0.15)$ for X_2, $G(0.15)$ for X_3 and $G(0.1)$ for U with $n = 50$. The percentages of samples for which W is not computable were tracked throughout the simulation experiment as for the estimated significance and shown in Figures E.2 and E.4 for samples without and with censoring respectively.

Because of the weaknesses found in the null behaviour of the tests based on W (frequently not computable) and χ_Y^2 (some situations where the test is highly liberal), the power results for these tests are of limited interest. Also, it is clear that F is more powerful than $\ddot{F}$. Hence here we shall concentrate on the power comparison between $\tilde{W}$ and F.

Figures 4.4 – 4.7 show the estimated powers of $\tilde{W}$ and F at a nominal 5% significance level for all 4×270 situations without and with censoring. As a further illustration Figures 4.8 and 4.9 show scatter diagrams of the estimated powers of $\tilde{W}$ and F for $n = 150$ for each of the 4 distributions without and with censoring. Clearly, the estimated power of $\tilde{W}$ is usually greater than that of F even though the estimated significance of $\tilde{W}$ was less than that of F for most of the situations.

4.6 Notes on $\tilde{W}$

The estimated significance levels of $\tilde{W}$ were much lower than the nominal level of 5% for most of the situations considered. One reason for this may be the use of the classical continuity correction which adds 0.5 to each cell as a way of avoiding the computational difficulties of Crowder test. This has the effect of essentially adding two observations to each 2×2 contingency table. This may smooth the data too much, which may have an influence on the test statistic (Agresti, 2002). Further we shall see in the following chapters that as the number of failure modes increases, so the size of the contingency tables increases and hence the continuity correction approach will in effect add more than two observations to the data. For example, with three competing risks tables with 8 cells will be used and consequently four extra observations will be added by using the continuity correction.

Whilst the classical continuity correction approach is a natural starting point, it should be regarded purely as a preliminary approach. In further research it would be advisable to try some other approaches with a view to making the test less conservative and potentially increasing its power. One possible approach is an empirical one that is similar to the adjustments used in plotting positions in, for example, probability and related plots; see Barnett (1975), Cunnane (1978) or Waller and Turnbull (1992). Alternatively, by thinking of these extra observations as coming from a prior distribution in a Bayesian approach, we might also consider different and possibly less informative priors; see, for example, Galindo-Garre et al. (2000). Both of these approaches in essence augment the data with additional and possibly non-integer numbers of observations.

We now consider this idea further. To fix ideas, suppose that a fixed correction $c_0 > 0$ is used on all cell frequencies in all tables ($c_0 = 0.5$ in the original modification). A natural approach is to take $c_0 < 0.5$ so that there is less data augmentation. Some care is needed here, however. For example, suppose a given cell frequency is zero. Then in effect the correction term replaces the cell frequency by c_0. If $c_0 \ll 0.5$, then the contribution to the variance term in the denominator will be $1/c_0$. This will be large and hence will dominate other terms in the denominator and lead to a small value of

$\tilde{W}_k$. This may make the test *more* conservative than with the original modification. Therefore, it may be that correction terms for the numerator and denominator should be different. A simple suggestion is first to use the c_0 correction to calculate the numerator (and here $c_0 = 0.5$ may be adequate as in Yates' correction). However, to calculate the denominator a possibly different correction $c_1 \geq 0$ could be used on non-zero frequencies but for a cell with zero frequency the contribution to the denominator could be $1 + c_2$ (with $c_2 \geq 0$) rather than $1/c_1$. Here $c_0 = 0.5 = c_1$ and $c_2 = 1$ give the original modification. In order to make the modified Crowder test less conservative, we anticipate that a good choice of $c_2 < 1$ will be particularly important.

4.7 Discussion

For clarity and notational simplicity, the Crowder test and its modified version have been applied to the special case of two failure modes. It turns out that at each observable failure time k, Crowder's test statistic is actually the log odds ratio test statistic, a well known statistic for testing independence in a 2×2 contingency table.

By recasting Crowder's results in terms of classical contingency tables, the properties of the Crowder test and its modified version have been compared to other tests of independence. The simulation studies reveal that the modified version of Crowder test has better performance compared to the other tests for this type of data, especially for large sample sizes. However, the conservative nature of the test, possibly due to the over-smoothing inherent in the use of the continuity correction, needs to be investigated further before the test is used routinely in practice. Nevertheless the modified version of the Crowder test solves the computability problem of the original Crowder test and, even with its over conservatism, is an attractive alternative to the other tests considered here.

In the next chapter these results are generalized to the case of three failure modes.

Chapter 5

Testing for Independence of Three Failure Modes with Discrete Lifetime Data

In this chapter the case of three failure modes is considered. The relationship between the Crowder test and other tests of independence is highlighted by generalizing the classical contingency table set up of chapter 4. The properties of the Crowder test, the modified test introduced in chapter 3 and other related tests are investigated by simulation.

5.1 Introduction

In chapter 4 we considered the special case of testing the independence of two failure modes using the test developed by Crowder (1997). Here we generalize the contingency table set up to the case of three failure modes.

In 5.2 and 5.3 we evaluate W and $\tilde{W}$ in the case of three failure modes. Two classical tests for independence in this special case are defined in 5.4. The behaviour of the tests is then investigated by simulation and the results are presented in 5.5. The chapter concludes with a discussion in 5.6.

5.2 Evaluation of $\hat{W}$

When there are three failure modes, the data layout will be as shown in Table 5.1. The notation used in Table 5.1 is as follows:

n_{k1}, n_{k2}, n_{k3}, n_{ka_1}, n_{ka_2}, n_{ka_3} and n_{ka_4} are the number of subjects failed from cause 1, cause 2, cause 3 and the multiple cause configurations $a_1 = \{1,2\}$, $a_2 = \{1,3\}$, $a_3 = \{2,3\}$ and $a_4 = \{1,2,3\}$ respectively at time k ; n_{kr} is the number of subjects lost to view just after time k so their failure times are right-censored at time k , $n_{kr} = 0$ when there are no censored observations at time k ; $n_{k\bullet}^{-} = \sum_c n_{kc}$ is the overall number of subjects failed at time k , where c is the cause of failure at time k , i.e. $c = 1,2,3,a_1,a_2,a_3,a_4$; $n_{k\bullet}^{+} = \sum_{i=k}^{k_*} (n_{i\bullet}^{-} + n_{ir})$ is the number of subjects at risk at time k .

Table 5.1: Data layout for three competing risks.

Cause \ Time	1	2	3	4	5	...	k_*
1	n_{11}	n_{21}	n_{31}	n_{41}	n_{51}		n_{k_*1}
2	n_{12}	n_{22}	n_{32}	n_{42}	n_{52}		n_{k_*2}
3	n_{13}	n_{23}	n_{33}	n_{43}	n_{53}		n_{k_*3}
{1, 2}	n_{1a_1}	n_{2a_1}	n_{3a_1}	n_{4a_1}	n_{4a_1}		$n_{k_*a_1}$
{1, 3}	n_{1a_2}	n_{2a_2}	n_{3a_2}	n_{4a_2}	n_{5a_2}		$n_{k_*a_2}$
{2, 3}	n_{1a_3}	n_{2a_3}	n_{3a_3}	n_{4a_3}	n_{5a_3}		$n_{k_*a_3}$
{1, 2, 3}	n_{1a_4}	n_{2a_4}	n_{3a_4}	n_{4a_4}	n_{5a_4}		$n_{k_*a_4}$
Censored	n_{1r}	n_{2r}	n_{3r}	n_{4r}	n_{5r}		n_{k_*r}
Observed	$n_{1\bullet}^{-}$	$n_{2\bullet}^{-}$	$n_{3\bullet}^{-}$	$n_{4\bullet}^{-}$	$n_{5\bullet}^{-}$		$n_{k_*\bullet}^{-}$
At risk	$n_{1\bullet}^{+}$	$n_{2\bullet}^{+}$	$n_{3\bullet}^{+}$	$n_{4\bullet}^{+}$	$n_{5\bullet}^{+}$		$n_{k_*\bullet}^{+}$

We now evaluate W in this case. The nonparametric maximum likelihood estimator for h_{kc} is $\hat{h}_{kc} = n_{kc} / n_{k\bullet}^{+}$, and for h_k we have $\hat{h}_k = n_{k\bullet}^{-} / n_{k\bullet}^{+}$. The estimates for the sub-conditional odds functions are

$$\hat{\hat{h}}_{kc} = \frac{\hat{h}_{kc}}{1-\hat{h}_k} = \frac{n_{kc}}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \forall k = 1,\ldots,k_*;\ n_{k\cdot}^{+} > n_{k\cdot}^{-}.$$

Hence,

$$\hat{\psi}_{ka_1} = \frac{\hat{\hat{h}}_{ka_1}}{\prod_{j\in a_1} \hat{\hat{h}}_{kj}} = \frac{n_{ka_1}(n_{k\cdot}^{+} - n_{k\cdot}^{-})}{n_{k1}n_{k2}} \quad \text{where } n_{k1} > 0 \text{ and } n_{k2} > 0,$$

$$\hat{\psi}_{ka_2} = \frac{\hat{\hat{h}}_{ka_2}}{\prod_{j\in a_2} \hat{\hat{h}}_{kj}} = \frac{n_{ka_2}(n_{k\cdot}^{+} - n_{k\cdot}^{-})}{n_{k1}n_{k3}} \quad \text{where } n_{k1} > 0 \text{ and } n_{k3} > 0,$$

$$\hat{\psi}_{ka_3} = \frac{\hat{\hat{h}}_{ka_3}}{\prod_{j\in a_3} \hat{\hat{h}}_{kj}} = \frac{n_{ka_3}(n_{k\cdot}^{+} - n_{k\cdot}^{-})}{n_{k2}n_{k3}} \quad \text{where } n_{k2} > 0 \text{ and } n_{k3} > 0,$$

$$\hat{\psi}_{ka_4} = \frac{\hat{\hat{h}}_{ka_4}}{\prod_{j\in a_4} \hat{\hat{h}}_{kj}} = \frac{n_{ka_4}(n_{k\cdot}^{+} - n_{k\cdot}^{-})^2}{n_{k1}n_{k2}n_{k3}} \quad \text{where } n_{k1} > 0,\ n_{k2} > 0 \text{ and } n_{k3} > 0.$$

Now, $Y_k^T = [Y_{ka_1}\ Y_{ka_2}\ Y_{ka_3}\ Y_{ka_4}] = [\log(\hat{\psi}_{ka_1})\ \log(\hat{\psi}_{ka_2})\ \log(\hat{\psi}_{ka_3})\ \log(\hat{\psi}_{ka_4})]$ is of length $d = 2^3 - 3 - 1 = 4$. So,

$$W = \sum_k Y_k^T\ V_k^{-1}\ Y_k\ .$$

Now, we need to calculate the covariance matrix V_k. Let c and b be causes of failure, single or multiple, so using (3.3),

$$H_{kc,kb} = H_{kb,kc} = \frac{1}{n_{k\cdot}^{+}}\left\{\delta_{cb}\hat{h}_{kc} - (1-\delta_{kk\cdot})\hat{h}_{kc}\hat{h}_{kb}\right\}.$$

Thus, if $k < k_*$,

$$H_{kc,kb} = \begin{cases} \dfrac{1}{n_{k\cdot}^{+}}\left\{\hat{h}_{kc} - \hat{h}_{kc}^{2}\right\} = \dfrac{n_{kc}(n_{k\bullet}^{+} - n_{kc})}{n_{k\bullet}^{+\,3}} & \text{if } c = b \\ \dfrac{1}{n_{k\cdot}^{+}}\left\{-\hat{h}_{kc}\hat{h}_{kb}\right\} = -\dfrac{n_{kc}n_{kb}}{n_{k\bullet}^{+\,3}} & \text{if } c \neq b \end{cases}.$$

Now, using (3.2),

$$\bar{H}_{kc,kb} \approx \frac{1}{(1-\hat{h}_k)^2}\left\{H_{kc,kb} + \frac{\hat{h}_{kc}\sum_c H_{kc,kb} + \hat{h}_{kb}\sum_b H_{kc,kb}}{1-\hat{h}_k} + \frac{\hat{h}_{kc}\hat{h}_{kb}\sum_c\sum_b H_{kc,kb}}{(1-\hat{h}_k)^2}\right\}.$$

Then, let a_v and a_u be the multiple causes of failure only, so using (3.1),

$$\mathrm{cov}(Y_{ka_v}, Y_{ka_u}) \approx \frac{\bar{H}_{ka_v,ka_u}}{\hat{\bar{h}}_{ka_v}\hat{\bar{h}}_{ka_u}} - \sum_{j\in a_u}\frac{\bar{H}_{ka_v,kj}}{\hat{\bar{h}}_{ka_v}\hat{\bar{h}}_{kj}} - \sum_{i\in a_v}\frac{\bar{H}_{ki,ka_u}}{\hat{\bar{h}}_{ki}\hat{\bar{h}}_{ka_u}} + \sum_{i\in a_v}\sum_{j\in a_u}\frac{\bar{H}_{ki,kj}}{\hat{\bar{h}}_{ki}\hat{\bar{h}}_{kj}}.$$

Thus,

$$\mathrm{var}(Y_{ka_1}) = \frac{1}{n_{k1}} + \frac{1}{n_{k2}} + \frac{1}{n_{ka_1}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k1} > 0,\ n_{k2} > 0 \text{ and } n_{ka_1} > 0,$$

$$\mathrm{var}(Y_{ka_2}) = \frac{1}{n_{k1}} + \frac{1}{n_{k3}} + \frac{1}{n_{ka_2}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k1} > 0,\ n_{k3} > 0 \text{ and } n_{ka_2} > 0,$$

$$\mathrm{var}(Y_{ka_3}) = \frac{1}{n_{k2}} + \frac{1}{n_{k3}} + \frac{1}{n_{ka_3}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k2} > 0,\ n_{k3} > 0 \text{ and } n_{ka_3} > 0,$$

$$\mathrm{var}(Y_{ka_4}) = \frac{1}{n_{k1}} + \frac{1}{n_{k2}} + \frac{1}{n_{k3}} + \frac{1}{n_{ka_4}} + \frac{4}{n_{k\cdot}^{+} - n_{k\cdot}^{-}},$$

where $n_{k1} > 0$, $n_{k2} > 0$, $n_{k3} > 0$ and $n_{ka_4} > 0$,

$$\mathrm{cov}(Y_{ka_1}, Y_{ka_2}) = \frac{1}{n_{k1}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k1} > 0,$$

$$\mathrm{cov}(Y_{ka_1}, Y_{ka_3}) = \frac{1}{n_{k2}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k2} > 0,$$

$$\mathrm{cov}(Y_{ka_1}, Y_{ka_4}) = \frac{1}{n_{k1}} + \frac{1}{n_{k2}} + \frac{2}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k1} > 0 \text{ and } n_{k2} > 0,$$

$$\mathrm{cov}(Y_{ka_2}, Y_{ka_3}) = \frac{1}{n_{k3}} + \frac{1}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k3} > 0,$$

$$\mathrm{cov}(Y_{ka_2}, Y_{ka_4}) = \frac{1}{n_{k1}} + \frac{1}{n_{k3}} + \frac{2}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k1} > 0 \text{ and } n_{k3} > 0,$$

$$\mathrm{cov}(Y_{ka_3}, Y_{ka_4}) = \frac{1}{n_{k2}} + \frac{1}{n_{k3}} + \frac{2}{n_{k\cdot}^{+} - n_{k\cdot}^{-}} \quad \text{where } n_{k2} > 0 \text{ and } n_{k3} > 0.$$

Therefore,

$$V_k = \begin{bmatrix} \mathrm{var}(Y_{ka_1}) & \mathrm{cov}(Y_{ka_1}, Y_{ka_2}) & \mathrm{cov}(Y_{ka_1}, Y_{ka_3}) & \mathrm{cov}(Y_{ka_1}, Y_{ka_4}) \\ \mathrm{cov}(Y_{ka_1}, Y_{ka_2}) & \mathrm{var}(Y_{ka_2}) & \mathrm{cov}(Y_{ka_2}, Y_{ka_3}) & \mathrm{cov}(Y_{ka_2}, Y_{ka_4}) \\ \mathrm{cov}(Y_{ka_1}, Y_{ka_3}) & \mathrm{cov}(Y_{ka_2}, Y_{ka_3}) & \mathrm{var}(Y_{ka_3}) & \mathrm{cov}(Y_{ka_3}, Y_{ka_4}) \\ \mathrm{cov}(Y_{ka_1}, Y_{ka_4}) & \mathrm{cov}(Y_{ka_2}, Y_{ka_4}) & \mathrm{cov}(Y_{ka_3}, Y_{ka_4}) & \mathrm{var}(Y_{ka_4}) \end{bmatrix}$$

and by calculating its inverse, we get

$$W = \sum_{k=1}^{k_*} Y_k^T \; V_k^{-1} \; Y_k \to \chi_{4q}^2 \text{ as } n \to \infty ,$$

where $q < k_*$ is the number of observed failure times for which $n_{kc} > 0 \; \forall c$.

Note that W can be written as $W = \sum_{k=1}^{k_*} W_k$ where $W_k = Y_k^T \; V_k^{-1} \; Y_k$.

□.

Example 5.1: Simulated $NB(p, z = 2)$ data

The data comprise 400 independent pairs (k, c) that were generated using SAS software version 9.1. The observed failure time, K, and the type of failure, C, are determined as follows:

Three samples with 400 observations each were simulated independently from $NB(0.25, 2)$, $NB(0.25, 2)$ and $NB(0.40, 2)$, defined in 1.5.2, denoted X_1, X_2 and X_3 respectively. There are three causes of failure, $g = 3$, and so the number of multiple cause configurations is $d = 2^3 - 3 - 1 = 4$. Then,

$$K = \min(X_1, X_2, X_3),$$

$$C = \begin{cases} 1 & \text{if } K = X_1, K \neq X_2 \text{ and } K \neq X_3 \\ 2 & \text{if } K = X_2, K \neq X_1 \text{ and } K \neq X_3 \\ 3 & \text{if } K = X_3, K \neq X_1 \text{ and } K \neq X_2 \\ \{1,2\} & \text{if } K = X_1 = X_2 \text{ and } K \neq X_3 \\ \{1,3\} & \text{if } K = X_1 = X_3 \text{ and } K \neq X_2 \\ \{2,3\} & \text{if } K = X_2 = X_3 \text{ and } K \neq X_1 \\ \{1,2,3\} & \text{if } K = X_1 = X_2 = X_3 \end{cases} .$$

The data are displayed in Table 5.2 along with W_k at each computable failure time k. The calculation of W_2 is as follows:

$$n_{2\bullet}^{-} = \sum_c n_{2c} = n_{21} + n_{22} + n_{23} + n_{2a_1} + n_{2a_2} + n_{2a_3} + n_{2a_4} = 17 + 10 + 49 + 2 + 4 + 8 + 2 = 92,$$

$$n_{2\bullet}^{+} = \sum_{i=2}^{13} n_{i\bullet}^{-} = 92 + 87 + 68 + 32 + 11 + 9 + 4 + 2 + 4 + 1 = 310,$$

since there are no censored observations, $n_{ir} = 0 \ \forall\ 2 \le i \le 13$,

$$\hat{\psi}_{2a_1} = \frac{n_{2a_1}(n_{2\bullet}^{+} - n_{2\bullet}^{-})}{n_{21}n_{22}} = \frac{2\times 218}{17\times 10} = 2.56471,\ \hat{\psi}_{2a_2} = \frac{4\times 218}{17\times 49} = 1.04682,$$

$$\hat{\psi}_{2a_3} = \frac{8\times 218}{10\times 49} = 3.55918,\ \hat{\psi}_{2a_4} = \frac{2\times 218^2}{17\times 10\times 49} = 11.41032,$$

$$\mathrm{var}(Y_{2a_1}) = \frac{1}{n_{21}} + \frac{1}{n_{22}} + \frac{1}{n_{2a_1}} + \frac{1}{n_{2\bullet}^{+} - n_{2\bullet}^{-}} = \frac{1}{17} + \frac{1}{10} + \frac{1}{2} + \frac{1}{218} = 0.66341,$$

$$\mathrm{var}(Y_{2a_2}) = \frac{1}{17} + \frac{1}{49} + \frac{1}{4} + \frac{1}{218} = 0.33382,\ \mathrm{var}(Y_{2a_3}) = \frac{1}{10} + \frac{1}{49} + \frac{1}{8} + \frac{1}{218} = 0.24999,$$

$$\mathrm{var}(Y_{2a_4}) = \frac{1}{17} + \frac{1}{10} + \frac{1}{49} + \frac{1}{2} + \frac{4}{218} = 0.69758,\ \mathrm{cov}(Y_{2a_1}, Y_{2a_2}) = \frac{1}{17} + \frac{1}{218} = 0.06341,$$

$$\mathrm{cov}(Y_{2a_1}, Y_{2a_3}) = \frac{1}{10} + \frac{1}{218} = 0.10459,\ \mathrm{cov}(Y_{2a_1}, Y_{2a_4}) = \frac{1}{17} + \frac{1}{10} + \frac{2}{218} = 0.16710,$$

$$\mathrm{cov}(Y_{2a_2}, Y_{2a_3}) = \frac{1}{49} + \frac{1}{218} = 0.02499,\ \mathrm{cov}(Y_{2a_2}, Y_{2a_4}) = \frac{1}{17} + \frac{1}{49} + \frac{2}{218} = 0.08841,$$

$$\mathrm{cov}(Y_{2a_3}, Y_{2a_4}) = \frac{1}{10} + \frac{1}{49} + \frac{2}{218} = 0.12958.$$

So,

$$Y_2^T = [\log(2.56471)\ \log(1.04682)\ \log(3.55918)\ \log(11.41032)]$$

and

$$V_2 = \begin{bmatrix} 0.66341 & 0.06341 & 0.10459 & 0.16710 \\ 0.06341 & 0.33382 & 0.02499 & 0.08841 \\ 0.10459 & 0.02499 & 0.24999 & 0.12958 \\ 0.16710 & 0.08841 & 0.12958 & 0.69758 \end{bmatrix}.$$

Hence,

$$W_2 = \begin{bmatrix} \log(2.56471) \\ \log(1.04682) \\ \log(3.55918) \\ \log(11.41032) \end{bmatrix}^T \begin{bmatrix} 1.68177 & -0.205821 & -0.539711 & -0.276516 \\ -0.205821 & 3.12798 & -0.0516119 & -0.337545 \\ -0.539711 & -0.0516119 & 4.60402 & -0.719401 \\ -0.276516 & -0.337545 & -0.719401 & 1.67618 \end{bmatrix} \begin{bmatrix} \log(2.56471) \\ \log(1.04682) \\ \log(3.55918) \\ \log(11.41032) \end{bmatrix}$$

$$=11.748 .$$

Note that W_k is computable only when $k = 2$ because of the zero frequencies at other times. The resulting Crowder test statistic W is 11.748 with 4 degrees of freedom, which has a p-value 0.0193, indicating wrongly that the independence hypothesis may be rejected at the 5% significance level.

Table 5.2: Simulated data for Example 5.1 and the W_k.

k	n_{k1}	n_{k2}	n_{k3}	n_{ka_1}	n_{ka_2}	n_{ka_3}	n_{ka_4}		$n^-_{k\bullet}$	$n^+_{k\bullet}$	W_k
1	15	11	61	2	1	0	0		90	400	
2	17	10	49	2	4	8	2		92	310	11.748
3	12	13	54	0	4	4	0		87	218	
4	7	8	44	2	1	6	0		68	131	
5	6	7	16	0	2	1	0		32	63	
6	1	1	7	0	1	1	0		11	31	
7	1	1	5	0	2	0	0		9	20	
8	1	0	3	0	0	0	0		4	11	
9	0	0	1	0	0	1	0		2	7	
10	0	0	4	0	0	0	0		4	5	
13	0	1	0	0	0	0	0		1	1	

Example 5.2: Simulated $NB(p, z = 2)$ data with censoring

The data comprise 400 independent pairs (k, c) that were generated using SAS software version 9.1 as in the previous example but with censored observations imposed. The observed failure time, K, and the type of failure, C, are determined as follows:

Four samples with 400 observations each were simulated independently from $NB(0.25,2)$, $NB(0.25,2)$, $NB(0.40,2)$ and $NB(0.40,2)$ denoted X_1, X_2, X_3 and X_4 respectively. There are three causes of failure, $g=3$, and so the number of multiple cause configurations is $d=2^3-3-1=4$. Then,

$$K=\min(X_1,X_2,X_3),$$

$$C=\begin{cases} 1 & \text{if } K=X_1\le X_4, K\ne X_2 \text{ and } K\ne X_3 \\ 2 & \text{if } K=X_2\le X_4, K\ne X_1 \text{ and } K\ne X_3 \\ 3 & \text{if } K=X_3\le X_4, K\ne X_1 \text{ and } K\ne X_2 \\ \{1,2\} & \text{if } K=X_1=X_2\le X_4 \text{ and } K\ne X_3 \\ \{1,3\} & \text{if } K=X_1=X_3\le X_4 \text{ and } K\ne X_2 \\ \{2,3\} & \text{if } K=X_2=X_3\le X_4 \text{ and } K\ne X_1 \\ \{1,2,3\} & \text{if } K=X_1=X_2=X_3\le X_4 \\ r & \text{if } K>X_4;\ r: \text{removed /censored} \end{cases}.$$

The data are displayed in Table 5.3 along with W_k at each computable failure time k. The calculation of W_k is similar to the previous example but taking into account the censored observations, $n_{k\bullet}^{+}=\sum_{i=k}^{7}(n_{i\bullet}^{-}+n_{ir})$.

Note also that W_k is computable only when $k=2$ because of the zero frequencies at other times. The resulting Crowder test statistic W is 12.930 with 4 degrees of freedom, which has a p-value 0.01176, giving the same conclusion as in example 5.1.

Table 5.3: Simulated data for Example 5.2 and the W_k.

k	n_{k1}	n_{k2}	n_{k3}	n_{ka_1}	n_{ka_2}	n_{ka_3}	n_{ka_4}	n_{kr}		$n_{k\bullet}^{-}$	$n_{k\bullet}^{+}$	W_k
1	15	11	61	2	1	0	0	43		90	400	
2	14	8	42	2	3	7	2	36		78	267	12.903
3	9	9	38	0	3	4	0	27		63	153	
4	4	6	20	0	0	4	0	7		34	63	
5	2	0	9	0	0	0	0	5		11	22	
6	1	0	2	0	0	0	0	1		3	6	
7	0	0	1	0	0	0	0	1		1	2	

5.3 Evaluation of $\tilde{W}$

We now consider the modified Crowder test statistic $\tilde{W}$ in the special case of three failure modes. With $c = 1,2,3,a_1,a_2,a_3,a_4$ where $a_1 = \{1,2\}$, $a_2 = \{1,3\}$, $a_3 = \{2,3\}$ and $a_4 = \{1,2,3\}$, the modified Crowder test statistic is

$$\tilde{W} = \sum_{k=1}^{k_*} \tilde{Y}_k^{\,T} \; \tilde{V}_k^{-1} \; \tilde{Y}_k \to \chi^2_{dk_*} \text{ as } n \to \infty,$$

where

$$\tilde{Y}_k^{\,T} = [\tilde{Y}_{ka_1} \;\; \tilde{Y}_{ka_2} \;\; \tilde{Y}_{ka_3} \;\; \tilde{Y}_{ka_4}] = [\log(\hat{\tilde{\psi}}_{ka_1}) \;\; \log(\hat{\tilde{\psi}}_{ka_2}) \;\; \log(\hat{\tilde{\psi}}_{ka_3}) \;\; \log(\hat{\tilde{\psi}}_{ka_4})],$$

$$\hat{\tilde{\psi}}_{ka_1} = \frac{(n_{ka_1}+0.5)(n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5)}{(n_{k1}+0.5)(n_{k2}+0.5)}, \quad \hat{\tilde{\psi}}_{ka_2} = \frac{(n_{ka_2}+0.5)(n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5)}{(n_{k1}+0.5)(n_{k3}+0.5)},$$

$$\hat{\tilde{\psi}}_{ka_3} = \frac{(n_{ka_3}+0.5)(n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5)}{(n_{k2}+0.5)(n_{k3}+0.5)}, \quad \hat{\tilde{\psi}}_{ka_4} = \frac{(n_{ka_4}+0.5)(n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5)^2}{(n_{k1}+0.5)(n_{k2}+0.5)(n_{k3}+0.5)}$$

and

$$\tilde{V}_k = \begin{bmatrix} \mathrm{var}(\tilde{Y}_{ka_1}) & \mathrm{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_2}) & \mathrm{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_3}) & \mathrm{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_4}) \\ \mathrm{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_2}) & \mathrm{var}(\tilde{Y}_{ka_2}) & \mathrm{cov}(\tilde{Y}_{ka_2},\tilde{Y}_{ka_3}) & \mathrm{cov}(\tilde{Y}_{ka_2},\tilde{Y}_{ka_4}) \\ \mathrm{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_3}) & \mathrm{cov}(\tilde{Y}_{ka_2},\tilde{Y}_{ka_3}) & \mathrm{var}(\tilde{Y}_{ka_3}) & \mathrm{cov}(\tilde{Y}_{ka_3},\tilde{Y}_{ka_4}) \\ \mathrm{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_4}) & \mathrm{cov}(\tilde{Y}_{ka_2},\tilde{Y}_{ka_4}) & \mathrm{cov}(\tilde{Y}_{ka_3},\tilde{Y}_{ka_4}) & \mathrm{var}(\tilde{Y}_{ka_4}) \end{bmatrix},$$

$$\mathrm{var}(\tilde{Y}_{ka_1}) = \frac{1}{n_{k1}+0.5} + \frac{1}{n_{k2}+0.5} + \frac{1}{n_{ka_1}+0.5} + \frac{1}{n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5},$$

$$\mathrm{var}(\tilde{Y}_{ka_2}) = \frac{1}{n_{k1}+0.5} + \frac{1}{n_{k3}+0.5} + \frac{1}{n_{ka_2}+0.5} + \frac{1}{n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5},$$

$$\mathrm{var}(\tilde{Y}_{ka_3}) = \frac{1}{n_{k2}+0.5} + \frac{1}{n_{k3}+0.5} + \frac{1}{n_{ka_3}+0.5} + \frac{1}{n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5},$$

$$\mathrm{var}(\tilde{Y}_{ka_4}) = \frac{1}{n_{k1}+0.5} + \frac{1}{n_{k2}+0.5} + \frac{1}{n_{k3}+0.5} + \frac{1}{n_{ka_4}+0.5} + \frac{4}{n_{k\cdot}^{+}-n_{k\cdot}^{-}+0.5},$$

$$\text{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_2})=\frac{1}{n_{k1}+0.5}+\frac{1}{n_{k\bullet}^{+}-n_{k\bullet}^{-}+0.5},\ \text{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_3})=\frac{1}{n_{k2}+0.5}+\frac{1}{n_{k\bullet}^{+}-n_{k\bullet}^{-}+0.5},$$

$$\text{cov}(\tilde{Y}_{ka_1},\tilde{Y}_{ka_4})=\frac{1}{n_{k1}+0.5}+\frac{1}{n_{k2}+0.5}+\frac{2}{n_{k\bullet}^{+}-n_{k\bullet}^{-}+0.5},$$

$$\text{cov}(\tilde{Y}_{ka_2},\tilde{Y}_{ka_3})=\frac{1}{n_{k3}+0.5}+\frac{1}{n_{k\bullet}^{+}-n_{k\bullet}^{-}+0.5},$$

$$\text{cov}(\tilde{Y}_{ka_2},\tilde{Y}_{ka_4})=\frac{1}{n_{k1}+0.5}+\frac{1}{n_{k3}+0.5}+\frac{2}{n_{k\bullet}^{+}-n_{k\bullet}^{-}+0.5},$$

$$\text{cov}(\tilde{Y}_{ka_3},\tilde{Y}_{ka_4})=\frac{1}{n_{k2}+0.5}+\frac{1}{n_{k3}+0.5}+\frac{2}{n_{k\bullet}^{+}-n_{k\bullet}^{-}+0.5}.$$

□.

Example 5.3: Simulated $NB\,(p,z=2)$ data

The modified version of Crowder's test statistic, $\tilde{W}$, is applied to the data simulated in example 5.1 and the results are presented in Table 5.4. For example, the calculation of $\tilde{W}_1$ is as follows:

$$n_{1\bullet}^{-}=\sum_{c}n_{1c}=n_{11}+n_{12}+n_{13}+n_{1a_1}+n_{1a_2}+n_{1a_3}+n_{1a_4}=15+11+61+2+1+0+0=90,$$

$$n_{1\bullet}^{+}=\sum_{i=1}^{13}n_{i\bullet}^{-}=90+92+87+68+32+11+9+4+2+4+1=400,$$

since there are no censored observations, $n_{ir}=0\ \forall\ 1\le i\le 13$,

$$\hat{\tilde{\psi}}_{1a_1}=\frac{(n_{1a_1}+0.5)(n_{1\bullet}^{+}-n_{1\bullet}^{-}+0.5)}{(n_{11}+0.5)(n_{12}+0.5)}=\frac{2.5\times 310.5}{15.5\times 11.5}=4.35484,\ \hat{\tilde{\psi}}_{1a_2}=\frac{1.5\times 310.5}{15.5\times 61.5}=0.48859,$$

$$\hat{\tilde{\psi}}_{1a_3}=\frac{0.5\times 310.5}{11.5\times 61.5}=0.21951,\ \hat{\tilde{\psi}}_{1a_4}=\frac{0.5\times 310.5^2}{15.5\times 11.5\times 61.5}=4.39732,$$

$$\text{var}(\tilde{Y}_{1a_1})=\frac{1}{n_{11}+0.5}+\frac{1}{n_{12}+0.5}+\frac{1}{n_{1a_1}+0.5}+\frac{1}{n_{1\bullet}^{+}-n_{1\bullet}^{-}+0.5}$$
$$=\frac{1}{15.5}+\frac{1}{11.5}+\frac{1}{2.5}+\frac{1}{310.5}=0.55469,$$

$$\text{var}(\tilde{Y}_{1a_2})=\frac{1}{15.5}+\frac{1}{61.5}+\frac{1}{1.5}+\frac{1}{310.5}=0.75066,$$

$$\text{var}(\tilde{Y}_{1a_3})=\frac{1}{11.5}+\frac{1}{61.5}+\frac{1}{0.5}+\frac{1}{310.5}=2.10644,$$

$$\mathrm{var}(\tilde{Y}_{1a_4}) = \frac{1}{15.5} + \frac{1}{11.5} + \frac{1}{61.5} + \frac{1}{0.5} + \frac{4}{310.5} = 2.18062,$$

$$\mathrm{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_2}) = \frac{1}{15.5} + \frac{1}{310.5} = 0.06774, \ \mathrm{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_3}) = \frac{1}{11.5} + \frac{1}{310.5} = 0.09018,$$

$$\mathrm{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_4}) = \frac{1}{15.5} + \frac{1}{11.5} + \frac{2}{310.5} = 0.15791, \ \mathrm{cov}(\tilde{Y}_{1a_2}, \tilde{Y}_{1a_3}) = \frac{1}{61.5} + \frac{1}{310.5} = 0.01948,$$

$$\mathrm{cov}(\tilde{Y}_{1a_2}, \tilde{Y}_{1a_4}) = \frac{1}{15.5} + \frac{1}{61.5} + \frac{2}{310.5} = 0.08722,$$

$$\mathrm{cov}(\tilde{Y}_{1a_3}, \tilde{Y}_{1a_4}) = \frac{1}{11.5} + \frac{1}{61.5} + \frac{2}{310.5} = 0.10966.$$

So,

$$\tilde{Y}_1^T = [\log(4.35484) \ \log(0.48859) \ \log(0.21951) \ \log(4.39732)]$$

and

$$\tilde{V}_1 = \begin{bmatrix} 0.55469 & 0.06774 & 0.09018 & 0.15791 \\ 0.06774 & 0.75066 & 0.01948 & 0.08722 \\ 0.09018 & 0.01948 & 2.10644 & 0.10966 \\ 0.15791 & 0.08722 & 0.10966 & 2.18062 \end{bmatrix}$$

Hence,

$$\tilde{W}_1 = \begin{bmatrix} \log(4.35484) \\ \log(0.48859) \\ \log(0.21951) \\ \log(4.39732) \end{bmatrix}^T \begin{bmatrix} 1.86887 & -0.15218 & -0.07206 & -0.12526 \\ -0.15218 & 1.35097 & -0.00375 & -0.04283 \\ -0.07206 & -0.00375 & 0.478828 & -0.01871 \\ -0.12526 & -0.04283 & -0.01871 & 0.470336 \end{bmatrix} \begin{bmatrix} \log(4.35484) \\ \log(0.48859) \\ \log(0.21951) \\ \log(4.39732) \end{bmatrix}$$

$$= 7.133 .$$

The resulting modified Crowder test statistic is $\tilde{W} = \sum_{k=1}^{13} \tilde{W}_k = 36.393$ with 52 degrees of freedom, which has a p-value 0.78548, indicating that the independence hypothesis cannot be rejected at the 5% significance level. Note that the conclusions based on W and $\tilde{W}$ differ for this set of data.

Table 5.4: Simulated data for Example 5.3 and the $\tilde{W}_k$.

k	n_{k1}	n_{k2}	n_{k3}	n_{ka_1}	n_{ka_2}	n_{ka_3}	n_{ka_4}		$n_{k\bullet}^-$	$n_{k\bullet}^+$	$\tilde{W}_k$
1	15	11	61	2	1	0	0		90	400	7.133
2	17	10	49	2	4	8	2		92	310	14.072
3	12	13	54	0	4	4	0		87	218	0.544
4	7	8	44	2	1	6	0		68	131	4.015
5	6	7	16	0	2	1	0		32	63	1.466
6	1	1	7	0	1	1	0		11	31	2.011
7	1	1	5	0	2	0	0		9	20	1.583
8	1	0	3	0	0	0	0		4	11	1.412
9	0	0	1	0	0	1	0		2	7	2.194
10	0	0	4	0	0	0	0		4	5	1.509
13	0	1	0	0	0	0	0		1	1	0.453

Example 5.4: Simulated $NB(p, z = 2)$ data with censoring

$\tilde{W}$ is applied to the data simulated in example 5.2 and the results are presented in Table 5.5. The computation of $\tilde{W}$ is similar to the previous example but taking into account the censored observations, $n_{k\bullet}^+ = \sum_{i=k}^{7} (n_{i\bullet}^- + n_{ir})$. $\tilde{W} = \sum_{k=1}^{7} \tilde{W}_k = 27.56127$ with 28 degrees of freedom, which has a p-value 0.48787, giving the same conclusion as in example 5.3.

Table 5.5: Simulated data for Example 5.4 and the $\tilde{W}_k$.

k	n_{k1}	n_{k2}	n_{k3}	n_{ka_1}	n_{ka_2}	n_{ka_3}	n_{ka_4}	n_{kr}		$n_{k\cdot}^-$	$n_{k\cdot}^+$	$\tilde{W}_k$
1	15	11	61	2	1	0	0	43		90	400	7.1326
2	14	8	42	2	3	7	2	36		78	267	15.336
3	9	9	38	0	3	4	0	27		63	153	0.3283
4	4	6	20	0	0	4	0	7		34	63	1.5772
5	2	0	9	0	0	0	0	5		11	22	2.1480
6	1	0	2	0	0	0	0	1		3	6	0.6772
7	0	0	1	0	0	0	0	1		1	2	0.3621

5.4 Some Other Tests for Independence

In the case of having three risks of failure, it is clear from Table 5.1 that at each observed failure time k we have a $2 \times 2 \times 2$ contingency table. The layout of each table at each observed failure time k is:

		Cause 3	
Cause 1	Cause 2	Yes	No
Yes	Yes	n_{ka_4}	n_{ka_1}
	No	n_{ka_2}	n_{k1}
No	Yes	n_{ka_3}	n_{k2}
	No	n_{k3}	$n_{k\bullet}^{+} - n_{k\bullet}^{-}$

There are many methods available to test the mutual independence in $2 \times 2 \times 2$ contingency tables (Agresti, 2002, Zelterman et al., 1995). The Pearson chi-square test and an exact test will be presented.

5.4.1 Pearson's Chi-Square Test

In the present context, Pearson's chi-square statistic at time k is

$$\chi_k^2 = \frac{(n_{ka_4} - e_{ka_4})^2}{e_{ka_4}} + \frac{(n_{ka_3} - e_{ka_3})^2}{e_{ka_3}} + \frac{(n_{ka_2} - e_{ka_2})^2}{e_{ka_2}} + \frac{(n_{ka_1} - e_{ka_1})^2}{e_{ka_1}} + \frac{(n_{k1} - e_{k1})^2}{e_{k1}} + \frac{(n_{k2} - e_{k2})^2}{e_{k2}} + \frac{(n_{k3} - e_{k3})^2}{e_{k3}} + \frac{(n_{k\bullet}^{+} - n_{k\bullet}^{-} - e_{k+-})^2}{e_{k+-}}$$

where

$$e_{ka_4} = \frac{n_{k1\bullet\bullet} n_{k\bullet1\bullet} n_{k\bullet\bullet1}}{n_{k\bullet}^{+\,2}},\ e_{ka_3} = \frac{n_{k2\bullet\bullet} n_{k\bullet1\bullet} n_{k\bullet\bullet1}}{n_{k\bullet}^{+\,2}},\ e_{ka_2} = \frac{n_{k1\bullet\bullet} n_{k\bullet2\bullet} n_{k\bullet\bullet1}}{n_{k\bullet}^{+\,2}},\ e_{ka_1} = \frac{n_{k1\bullet\bullet} n_{k\bullet1\bullet} n_{k\bullet\bullet2}}{n_{k\bullet}^{+\,2}},$$

$$e_{k1} = \frac{n_{k1\bullet\bullet} n_{k\bullet2\bullet} n_{k\bullet\bullet2}}{n_{k\bullet}^{+\,2}},\ e_{k2} = \frac{n_{k2\bullet\bullet} n_{k\bullet1\bullet} n_{k\bullet\bullet2}}{n_{k\bullet}^{+\,2}},\ e_{k3} = \frac{n_{k2\bullet\bullet} n_{k\bullet2\bullet} n_{k\bullet\bullet1}}{n_{k\bullet}^{+\,2}},\ e_{k+-} = \frac{n_{k2\bullet\bullet} n_{k\bullet2\bullet} n_{k\bullet\bullet2}}{n_{k\bullet}^{+\,2}},$$

$$n_{k1\bullet\bullet} = n_{ka_4} + n_{ka_1} + n_{ka_2} + n_{k1},\ n_{k2\bullet\bullet} = n_{ka_3} + n_{k2} + n_{k3} + n_{k\bullet}^{+} - n_{k\bullet}^{-},$$

$$n_{k\bullet1\bullet} = n_{ka_4} + n_{ka_1} + n_{ka_3} + n_{k2},\ n_{k\bullet2\bullet} = n_{ka_3} + n_{ka_2} + n_{k1} + n_{k\bullet}^{+} - n_{k\bullet}^{-},$$

$$n_{k\bullet\bullet1} = n_{ka_4} + n_{ka_3} + n_{ka_1} + n_{k3},\ n_{k\bullet\bullet2} = n_{ka_1} + n_{k2} + n_{k1} + n_{k\bullet}^{+} - n_{k\bullet}^{-}.$$

The statistic has 4 degrees of freedom and will be calculated for each failure time k. The resulting chi-square values and the 4 degrees of freedom will be accumulated over all the failure times to form the value of the test statistic

$$\chi_P^2 = \sum_{k=1}^{k_*} \chi_k^2 \to \chi_{4q}^2 \text{ as } n \to \infty,$$

where $q \le k_*$ is the number of observed failure times.

5.4.2 Exact Test of Significance

When the cells frequencies are small, there will be always uncertainty about the adequacy of Pearson's chi-square statistic. An exact test of significance is described by Zelterman et al. (1995) to test the mutual independence of $2 \times 2 \times 2$ contingency tables, following the well known Fisher's exact test in the 2×2 contingency tables.

The exact likelihood of $\{n_{kc}\}$ given the marginal totals is

$$P(n_{kc} \mid n_{k\cdot x}, n_{k\bullet}^{+}) = \frac{\prod_{x=1}^{3} n_{k\cdot x}!(n_{k\bullet}^{+} - n_{k\cdot x})!}{(n_{k\bullet}^{+}!)^2 \prod_c n_{kc}!},$$

where $n_{k\cdot 1} = n_{ka_4} + n_{ka_1} + n_{ka_2} + n_{k1}$, $n_{k\cdot 2} = n_{ka_4} + n_{ka_1} + n_{ka_3} + n_{k2}$ and $n_{k\cdot 3} = n_{ka_4} + n_{ka_3} + n_{ka_1} + n_{k3}$.

In order to calculate the significance of an observed failure time, we have to calculate the probabilities of all tables with the same marginal totals. Then the p-value for the significance can be calculated by summing the probabilities that are not larger than the probability of the observed table.

In order to decide whether or not to reject the independence hypothesis we must combine the p-value results for all k – values. To achieve this we use a Bonferroni type approach. Let k' be the number of k values that form non-trivial $2 \times 2 \times 2$ contingency tables, i.e. tables in which none of the marginal totals is zero. Then the null hypothesis is rejected at a nominal 5% significance if any of the observed p-values is less than $0.05/k'$.

Table 4.7: Values of p and λ associated with each tick mark for $n = 50$.

Tick Mark	$G(p)$				$NB(p, z = 0.2, 2)$				$P(\lambda)$			
	X_1	X_2	X_3	U	X_1	X_2	X_3	U	X_1	X_2	X_3	U
1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	1	1	1	1
2	0.1	0.15	0.15	0.1	0.1	0.15	0.15	0.1	1	1.5	1.5	1
3	0.1	0.2	0.2	0.1	0.1	0.2	0.2	0.1	1	2	2	1
4	0.1	0.25	0.25	0.1	0.1	0.25	0.25	0.1	1	2.5	2.5	1
5	0.1	0.3	0.3	0.1	0.1	0.3	0.3	0.1	1	3	3	1
6	0.1	0.35	0.35	0.1	0.1	0.35	0.35	0.1	1	3.5	3.5	1
7	0.1	0.4	0.4	0.1	0.1	0.4	0.4	0.1	1	4	4	1
8	0.1	0.45	0.45	0.1	0.1	0.45	0.45	0.1	1	4.5	4.5	1
9	0.1	0.5	0.5	0.1	0.1	0.5	0.5	0.1	1	5	5	1
10	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	1.5	1.5	1.5	1.5
11	0.15	0.2	0.2	0.15	0.15	0.2	0.2	0.15	1.5	2	2	1.5
12	0.15	0.25	0.25	0.15	0.15	0.25	0.25	0.15	1.5	2.5	2.5	1.5
13	0.15	0.3	0.3	0.15	0.15	0.3	0.3	0.15	1.5	3	3	1.5
14	0.15	0.35	0.35	0.15	0.15	0.35	0.35	0.15	1.5	3.5	3.5	1.5
15	0.15	0.4	0.4	0.15	0.15	0.4	0.4	0.15	1.5	4	4	1.5
16	0.15	0.45	0.45	0.15	0.15	0.45	0.45	0.15	1.5	4.5	4.5	1.5
17	0.15	0.5	0.5	0.15	0.15	0.5	0.5	0.15	1.5	5	5	1.5
18	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	2	2	2	2
19	0.2	0.25	0.25	0.2	0.2	0.25	0.25	0.2	2	2.5	2.5	2
20	0.2	0.3	0.3	0.2	0.2	0.3	0.3	0.2	2	3	3	2
21	0.2	0.35	0.35	0.2	0.2	0.35	0.35	0.2	2	3.5	3.5	2
22	0.2	0.4	0.4	0.2	0.2	0.4	0.4	0.2	2	4	4	2
23	0.2	0.45	0.45	0.2	0.2	0.45	0.45	0.2	2	4.5	4.5	2
24	0.2	0.5	0.5	0.2	0.2	0.5	0.5	0.2	2	5	5	2
25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	2.5	2.5	2.5	2.5
26	0.25	0.3	0.3	0.25	0.25	0.3	0.3	0.25	2.5	3	3	2.5
27	0.25	0.35	0.35	0.25	0.25	0.35	0.35	0.25	2.5	3.5	3.5	2.5
28	0.25	0.4	0.4	0.25	0.25	0.4	0.4	0.25	2.5	4	4	2.5
29	0.25	0.45	0.45	0.25	0.25	0.45	0.45	0.25	2.5	4.5	4.5	2.5
30	0.25	0.5	0.5	0.25	0.25	0.5	0.5	0.25	2.5	5	5	2.5
31	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	3	3	3	3
32	0.3	0.35	0.35	0.3	0.3	0.35	0.35	0.3	3	3.5	3.5	3
33	0.3	0.4	0.4	0.3	0.3	0.4	0.4	0.3	3	4	4	3
34	0.3	0.45	0.45	0.3	0.3	0.45	0.45	0.3	3	4.5	4.5	3
35	0.3	0.5	0.5	0.3	0.3	0.5	0.5	0.3	3	5	5	3
36	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	3.5	3.5	3.5	3.5
37	0.35	0.4	0.4	0.35	0.35	0.4	0.4	0.35	3.5	4	4	3.5
38	0.35	0.45	0.45	0.35	0.35	0.45	0.45	0.35	3.5	4.5	4.5	3.5
39	0.35	0.5	0.5	0.35	0.35	0.5	0.5	0.35	3.5	5	5	3.5
40	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	4	4	4	4
41	0.4	0.45	0.45	0.4	0.4	0.45	0.45	0.4	4	4.5	4.5	4
42	0.4	0.5	0.5	0.4	0.4	0.5	0.5	0.4	4	5	5	4
43	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	4.5	4.5	4.5	4.5
44	0.45	0.5	0.5	0.45	0.45	0.5	0.5	0.45	4.5	5	5	4.5
45	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	5	5	5	5

When the sample size is large the calculation of the significance will take a long time to finish. Hence, we should check the total number of tables in the complete enumeration which depends on the sample size and the marginal totals in the observed table. If the marginal totals are very close to 1 or to the sample size, the number of enumerations will not be very large relative to the sample size. Otherwise, the number of enumerations will be great especially if the marginal totals are close to half of the sample size (Zelterman et al., 1995). Because of the excessive computing time that would have been involved, this method was not used in the simulation study.

5.5 Simulation Study

The exact test of significance is preferable with small expected frequencies but as we mentioned in the previous section if the sample size is large then the test will take a long time to finish. Hence, in this section we compare the performances of the Crowder test (unmodified and modified) and Pearson's chi-square test using simulation in the special case of three failure modes.

5.5.1 Design to Estimate Significance

For data with no censored observations, let X_1, X_2 and X_3 be independent discrete random variables on $0,1,2,3,\ldots$. The observed failure time, K, and the type of failure, C, are determined as in Example 5.1.

For data with censored observations, let X_1, X_2, X_3 and X_4 be independent discrete random variables on $0,1,2,3,\ldots$. K and C are determined as in Example 5.2.

For a given sample size n and for specified distributions for X_1, X_2 and X_3 (no censoring) or for X_1, X_2, X_3 and X_4 (with censoring), n pairs (k,c) were generated using SAS version 9.1. At a nominal 5% significance level the reject / not reject inference was made for each test procedure. This was repeated 1000 times and the percentage of rejections was calculated for each test to estimate the true significance level.

This procedure was carried out for $n = 50, 100, 200, 400, 600$ and 900 and for four distributions with different FR shapes. The distributions used here are geometric with $p = 0.1, 0.15, 0.2, 0.25, 0.3, 0.4$ (flat FR), negative binomial with $p = 0.1, 0.15, 0.2, 0.25, 0.3, 0.4$ and $z = 0.2$ (decreasing FR), negative binomial with $p = 0.1, 0.15, 0.2, 0.25, 0.3, 0.4$ and $z = 2$ (gently increasing FR) and Poisson with mean $\lambda = 1, 1.5, 2, 2.5, 3, 4$ (FR that increases to 1).

5.5.2 Significance Results

The estimated significance probabilities of the test procedure at the nominal 5% significance level for the above design are presented in Figures F.1 and F.2 of Appendix F for samples with no censored observations and with censored observations respectively. Every tick mark on the x-axes of the graphical representations corresponds to the estimated significance for samples generated from the indicated distribution with different parameter values and sample sizes in ascending order.

For samples generated from the negative binomial distribution with no censored observations, e.g. Figure 5.1, tick mark 1 corresponds to the estimated significance for samples generated from $NB(0.1, 2)$ for X_1, $NB(0.1, 2)$ for X_2 and $NB(0.1, 2)$ for X_3 with $n = 50$, tick mark 2 corresponds to the estimated significance for samples generated from $NB(0.1, 2)$ for X_1, $NB(0.1, 2)$ for X_2 and $NB(0.15, 2)$ for X_3 with $n = 50$, ..., tick mark 56 corresponds to the estimated significance for samples generated from $NB(0.4, 2)$ for X_1, $NB(0.4, 2)$ for X_2 and $NB(0.4, 2)$ for X_3 with $n = 50$, tick mark 57 corresponds to the estimated significance for samples generated from $NB(0.1, 2)$ for X_1, $NB(0.1, 2)$ for X_2 and $NB(0.1, 2)$ for X_3 with $n = 100$, ... and tick mark 336 corresponds to the estimated significance for samples generated from $NB(0.4, 2)$ for X_1, $NB(0.4, 2)$ for X_2 and $NB(0.4, 2)$ for X_3 with $n = 900$. For samples with censored observations, the samples for X_4 are generated from the same sample distributions of X_3, e.g. tick mark 2 for samples generated from the negative binomial distribution with censored observations corresponds to the estimated significance for samples generated from $NB(0.1, 2)$ for X_1, $NB(0.1, 2)$ for X_2, $NB(0.15, 2)$ for X_3 and $NB(0.15, 2)$ for X_4 with $n = 50$. Table 5.6 shows the values of the distributions parameters associated with each tick mark for $n = 50$. Tick marks

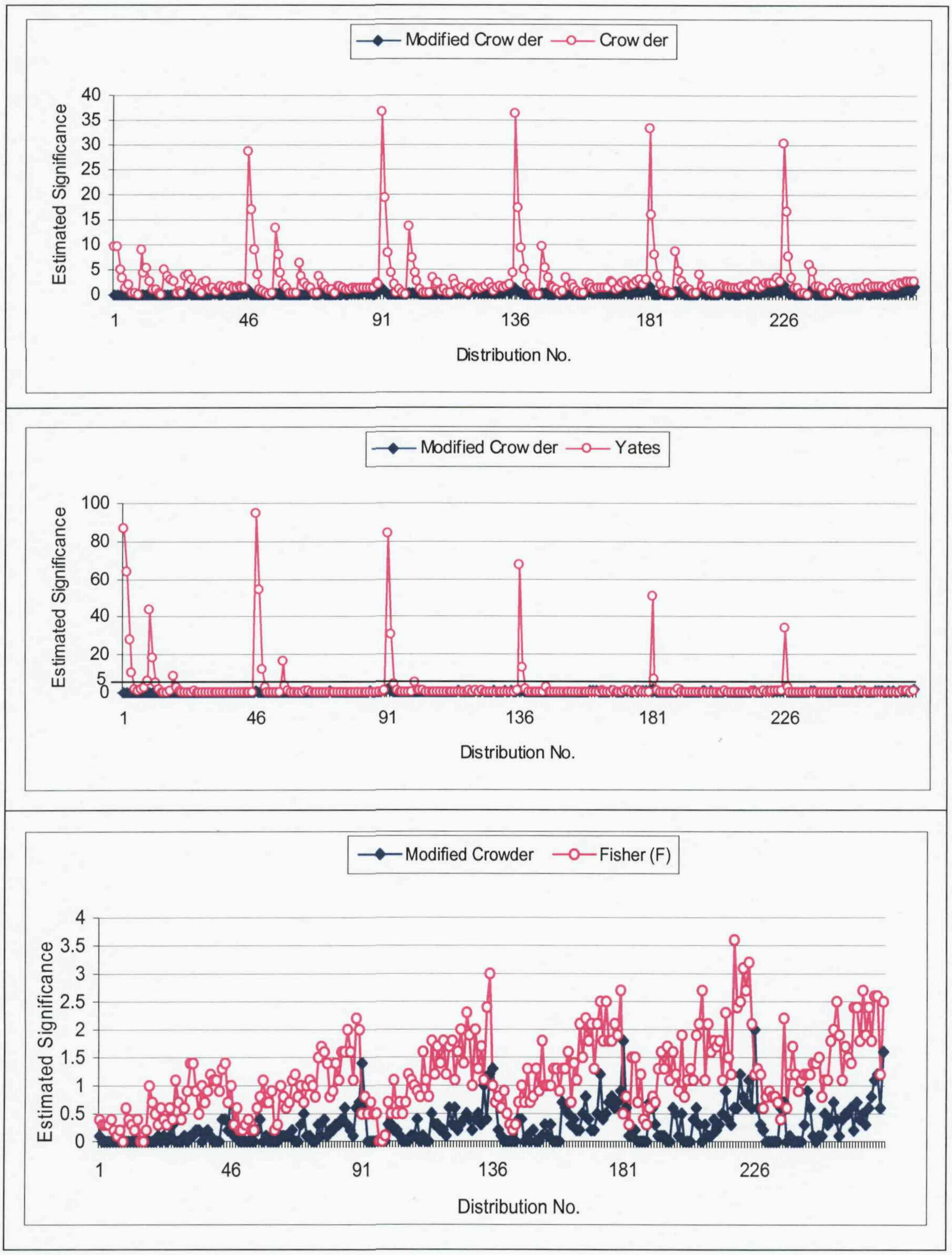


Figure 4.1: The estimated significances of $\tilde{W}$ compared to W , χ_Y^2 and F at the nominal 5% significance level for samples with no censoring from $NB(p, z = 2)$ with $p = 0.1\,(0.05)\,0.5$ and sample sizes 50 (plotted on the x-axis 1-45), 100 (46-90), 150 (91-135), 200 (136-180), 250 (181-225) and 300 (226-270).

Tables 5.7 and 5.8 show that all the estimated significances of $\tilde{W}$ for geometric and negative binomial with decreasing FR based samples, without or with censoring, are below 5%. For negative binomial with increasing FR based samples, 97.3% of the estimated significances are below 5% with 82.3% being the maximum for samples with no censoring and 94.6% of the estimated significances are below 5% with 91.7% being the maximum for samples with censoring. For Poisson based samples, 99.1% of the estimated significances are below 5% with 7.7% being the maximum for samples with no censoring and 98.8% of the estimated significances are below 5% with 10.8% being the maximum for samples with censoring. The test based on W tends to be conservative for most of the situations especially for Poisson based samples in which all the estimated significances are below 5% for samples with no censoring and 99.7% of the estimated significances are below 5% for samples with censoring with 5.1% being the maximum. However, the estimated significances of $\tilde{W}$ are less than those of W for most of the generated samples as can be seen from the graphical representations, Figures E.1 and E.2 of Appendix E.

As pointed out in 5.2, a potential problem with the Crowder test based on W is that W is not always computable. This behaviour was tracked throughout the simulation experiment. Indeed, the estimated significance levels for W presented in Figures E.1 and E.2 were calculated using those samples for which W was computable. For example, for samples with no censoring, $n = 50$ and based on $G(0.1)$, W was computable for only 3 out of the 1000 simulated samples. Figures 5.2 and 5.3 show the percentages of samples, without and with censoring respectively, for which W was not computable. Clearly, for many situations these percentages are high.

In summary, the modified Crowder test tends to be very conservative and the Crowder test tends to be conservative for most of the simulated samples but cannot be used routinely because of computability problems. The test based on χ_p^2 is liberal for most of the simulated samples.

Table 4.8: Summary for the estimated significances for samples with no censoring.

Test	$G(p)$		$NB(p,z=0.2)$		$NB(p,z=2)$		$P(\lambda)$	
	Min < 5% Max	%	Min < 5% Max	%	Min < 5% Max	%	Min < 5% Max	%
W	0.1	0.4	1.1	0.4	0	1.8	0	1.1
	< 5%	98.9	< 5%	99.6	< 5%	88.5	< 5%	99.3
	5.5	0.4	5.1	0.4	36.6	0.4	5.1	0.4
$\tilde{W}$	0	19.6	0	1.1	0	33.3	0	7.8
	< 5%	100	< 5%	100	< 5%	100	< 5%	98.9
	1.8	0.4	1.8	0.4	2	0.4	5.8	0.4
χ_Y^2	0	7.4	0	0.4	0	18.1	0	4.1
	< 5%	100	< 5%	100	< 5%	92.6	< 5%	78.1
	1.7	0.4	2.4	0.4	94.6	0.4	52.2	0.4
F	0	0.4	0.7	0.4	0	1.8	0.1	0.7
	< 5%	100	< 5%	100	< 5%	100	< 5%	100
	3.7	0.4	3.8	0.4	3.6	0.4	3.9	0.7
$\ddot{F}$	0	0.4	0.5	0.7	0	7	0	1.1
	< 5%	100	< 5%	100	< 5%	100	< 5%	100
	3	0.4	2.5	1.1	2.7	0.7	3.3	0.4

Table 4.9: Summary for the estimated significances for samples with censoring.

Test	$G(p)$		$NB(p,z=0.2)$		$NB(p,z=2)$		$P(\lambda)$	
	Min < 5% Max	%	Min < 5% Max	%	Min < 5% Max	%	Min < 5% Max	%
W	0.1	0.4	1.7	0.4	0	1.1	0	1.5
	< 5%	99.3	< 5%	73	< 5%	84.1	< 5%	99.6
	5.3	0.4	9.8	0.4	36.7	0.4	5.5	0.4
$\tilde{W}$	0	5.9	0.2	0.4	0	20.7	0	8.5
	< 5%	100	< 5%	100	< 5%	100	< 5%	96.3
	2.5	0.4	3.5	0.4	4.2	0.4	7.3	0.4
χ_Y^2	0	0.7	0.6	0.4	0	4.1	0	2.6
	< 5%	100	< 5%	100	< 5%	88.5	< 5%	76.3
	2.4	0.4	3.8	0.4	95	0.4	53	0.4
F	0.5	1.1	1.2	0.4	0	1.1	0.1	1.5
	< 5%	100	< 5%	99.6	< 5%	100	< 5%	100
	3.9	0.4	5	0.4	3.4	0.4	4.9	0.4
$\ddot{F}$	0.1	0.7	0.7	0.7	0	6.3	0	0.7
	< 5%	100	< 5%	100	< 5%	100	< 5%	100
	3.6	0.4	4	0.4	2.9	0.4	3.7	0.4

5.5.3 Design to Estimate Power

Since we have three failure modes, the power is calculated for two situations; for data with two dependent failure modes and for data with all failure modes being dependent.

For data with no censored observations, let X_1, X_2, X_3 and U be independent discrete random variables on $0,1,2,3,\ldots$. Models for dependent risks that are easy to simulate and make reasonable sense in practice are as follows:

For samples with two dependent failure modes, the observed failure time, K, and the type of failure, C, are determined as follows:

Let

$$Y_2 = X_2 + U ,$$
$$Y_3 = X_3 + U .$$

Note that $\operatorname{cov}(Y_2, Y_3) = \operatorname{var}(U) > 0$, thereby giving positive correlation.

Then,

$$K = \min(X_1, Y_2, Y_3) ,$$

$$C = \begin{cases} 1 & \text{if } K = X_1, K \neq Y_2 \text{ and } K \neq Y_3 \\ 2 & \text{if } K = Y_2, K \neq X_1 \text{ and } K \neq Y_3 \\ 3 & \text{if } K = Y_3, K \neq X_1 \text{ and } K \neq Y_2 \\ \{1,2\} & \text{if } K = X_1 = Y_2 \text{ and } K \neq Y_3 \\ \{1,3\} & \text{if } K = X_1 = Y_3 \text{ and } K \neq Y_2 \\ \{2,3\} & \text{if } K = Y_2 = Y_3 \text{ and } K \neq X_1 \\ \{1,2,3\} & \text{if } K = X_1 = Y_2 = Y_3 \end{cases} .$$

For samples with three dependent failure modes, K and C are determined as follows:

Let

$$Y_1 = X_1 + U ,$$
$$Y_2 = X_2 + U ,$$
$$Y_3 = X_3 + U .$$

Note also that $\operatorname{cov}(Y_1, Y_2, Y_3) = \operatorname{var}(U) > 0$, thereby giving positive correlation.

Then,

$$K = \min(Y_1, Y_2, Y_3),$$

$$C = \begin{cases} 1 & \text{if } K = Y_1, K \neq Y_2 \text{ and } K \neq Y_3 \\ 2 & \text{if } K = Y_2, K \neq Y_1 \text{ and } K \neq Y_3 \\ 3 & \text{if } K = Y_3, K \neq Y_1 \text{ and } K \neq Y_2 \\ \{1,2\} & \text{if } K = Y_1 = Y_2 \text{ and } K \neq Y_3 \\ \{1,3\} & \text{if } K = Y_1 = Y_3 \text{ and } K \neq Y_2 \\ \{2,3\} & \text{if } K = Y_2 = Y_3 \text{ and } K \neq Y_1 \\ \{1,2,3\} & \text{if } K = Y_1 = Y_2 = Y_3 \end{cases}.$$

For data with censored observations, let X_1, X_2, X_3, X_4 and U be independent discrete random variables on $0,1,2,3,\ldots$. The observed failure time, K, and the type of failure, C, are determined as follows:

For samples with two dependent failure modes, let

$$Y_2 = X_2 + U,$$
$$Y_3 = X_3 + U.$$

Then,

$$K = \min(X_1, Y_2, Y_3),$$

$$C = \begin{cases} 1 & \text{if } K = X_1 \leq X_4, K \neq Y_2 \text{ and } K \neq Y_3 \\ 2 & \text{if } K = Y_2 \leq X_4, K \neq X_1 \text{ and } K \neq Y_3 \\ 3 & \text{if } K = Y_3 \leq X_4, K \neq X_1 \text{ and } K \neq Y_2 \\ \{1,2\} & \text{if } K = X_1 = Y_2 \leq X_4 \text{ and } K \neq Y_3 \\ \{1,3\} & \text{if } K = X_1 = Y_3 \leq X_4 \text{and } K \neq Y_2 \\ \{2,3\} & \text{if } K = Y_2 = Y_3 \leq X_4 \text{and } K \neq X_1 \\ \{1,2,3\} & \text{if } K = X_1 = Y_2 = Y_3 \leq X_4 \\ r & \text{if } K > X_4;\ r\text{: removed /censored} \end{cases}.$$

For samples with all failure modes being dependent, let

$$Y_1 = X_1 + U,$$
$$Y_2 = X_2 + U,$$
$$Y_3 = X_3 + U,$$
$$Y_4 = X_4 + U.$$

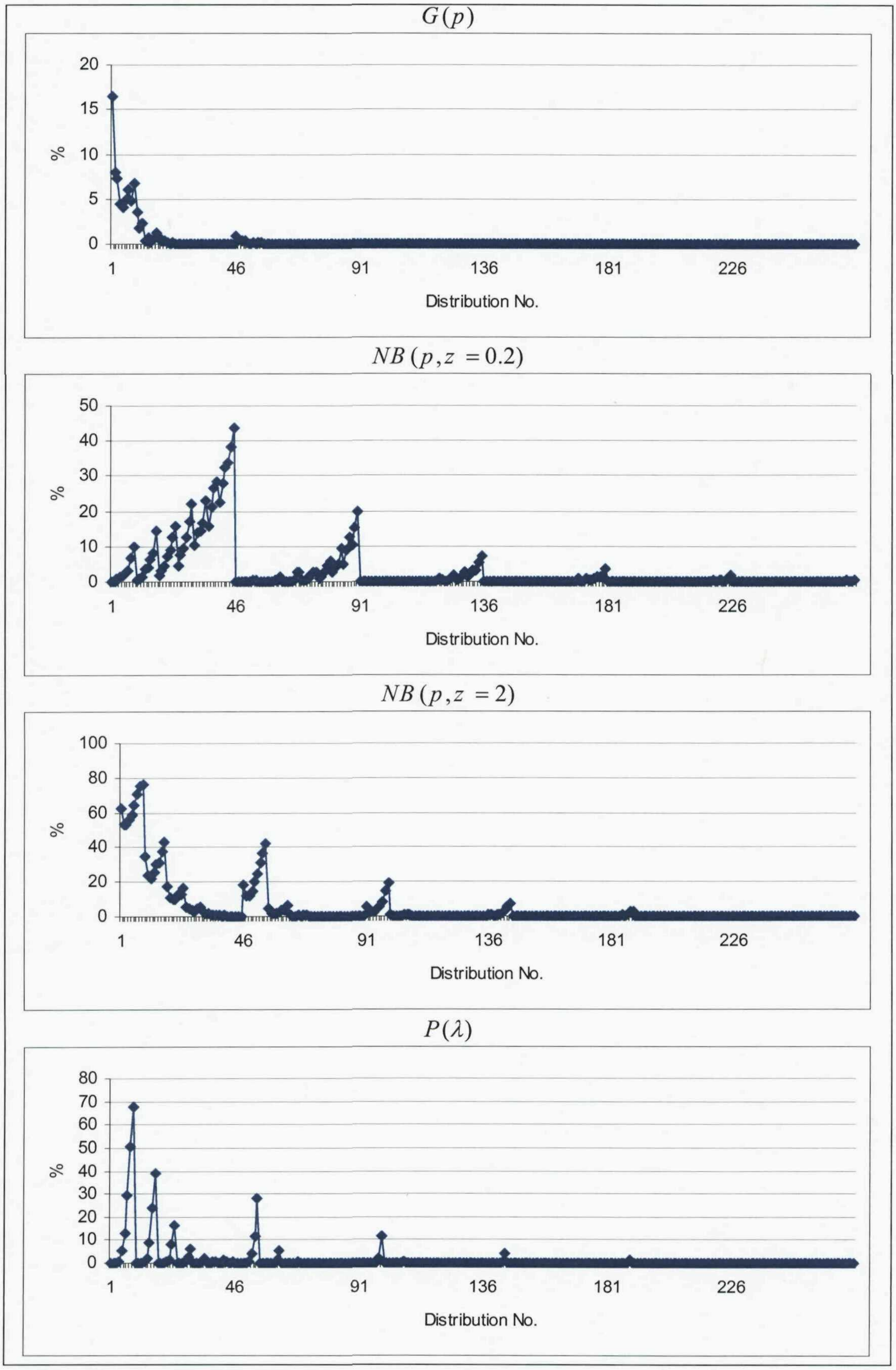


Figure 4.2: The percentages of samples with no censoring for which W is not computable for sample sizes 50 (1-45), … and 300 (226-270).

Then,

$$K = \min(Y_1, Y_2, Y_3),$$

$$C = \begin{cases} 1 & \text{if } K = Y_1 \le Y_4, K \ne Y_2 \text{ and } K \ne Y_3 \\ 2 & \text{if } K = Y_2 \le Y_4, K \ne Y_1 \text{ and } K \ne Y_3 \\ 3 & \text{if } K = Y_3 \le Y_4, K \ne Y_1 \text{ and } K \ne Y_2 \\ \{1,2\} & \text{if } K = Y_1 = Y_2 \le Y_4 \text{ and } K \ne Y_3 \\ \{1,3\} & \text{if } K = Y_1 = Y_3 \le Y_4 \text{ and } K \ne Y_2 \\ \{2,3\} & \text{if } K = Y_2 = Y_3 \le Y_4 \text{ and } K \ne Y_1 \\ \{1,2,3\} & \text{if } K = Y_1 = Y_2 = Y_3 \le Y_4 \\ r & \text{if } K > X_4;\ r\text{: removed / censored} \end{cases}.$$

For a given sample size n and for specified distributions for X_1, X_2, X_3 and U (no censoring) or for X_1, X_2, X_3, X_4 and U (with censoring), n pairs (k, c) were generated using SAS version 9.1. At a nominal 5% significance level the reject / not reject inference was made for each test procedure. This was repeated 1000 times and the percentage of rejections was calculated for each test to estimate the true power level. This procedure was carried out in the same way as for the significance levels.

5.5.4 Power Results

The estimated powers of the test procedures at the nominal 5% significance level for the above design are presented in Figures G.1 and G.5 of Appendix G for samples with no censored observations and Figures G.3 and G.7 for samples with censored observations. The tick marks on the x-axes of the graphical representations correspond to the estimated powers for samples generated from the indicated distribution with different parameter values and sample sizes in ascending order as for the estimated significance. For a reminder about the tick marks see the discussion in section 5.5.2 and Table 5.6. The samples for U are generated from the same sample distribution of X_2, e.g. tick mark 2 for samples generated from the Poisson distribution with censored observations corresponds to the estimated significance for samples generated from $P(0.1)$ for X_1, $P(0.1)$ for X_2, $P(0.15)$ for X_3, $P(0.15)$ for X_4 and $P(0.1)$ for U with $n = 50$. The percentages of samples for which W is not computable were also tracked throughout the simulation experiment and shown in Figures G.2 and G.6 for samples without censoring and Figures G.4 and G.8 for samples with censoring respectively.

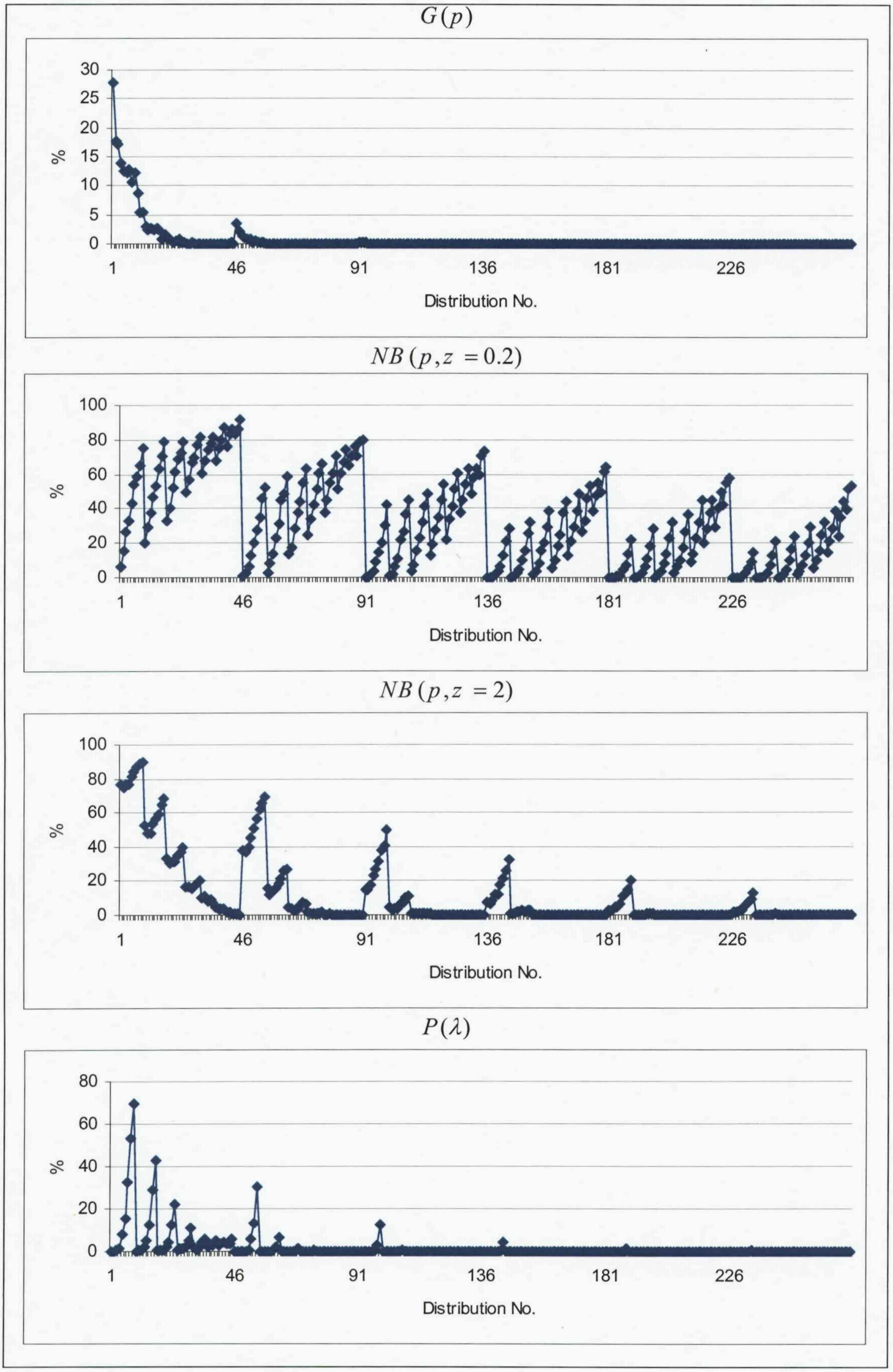


Figure 4.3: The percentages of samples with censoring for which W is not computable for sample sizes 50 (1-45), … and 300 (226-270).

Because of the weaknesses found in the null behaviour of the tests based on W (frequently not computable) and χ_P^2 (liberal in most of the situations), the power results for these tests are of limited interest. Hence here we shall concentrate on the power of $\tilde{W}$.

Figures 5.4 and 5.5 show the estimated powers of $\tilde{W}$ when only two of the three failure modes are dependent, X_1 and X_3, at a nominal 5% significance level for all 4×336 situations without and with censoring. The estimated power is low for small sample size but it improves in general as the sample size gets larger. The estimated power for samples based on increasing FR is fluctuating even for large sample size. A direct comparison between the tests used here, which should be treated with caution because of the unsatisfactory significance performance, shows that the estimated power fluctuations for the samples based on increasing FR are a general behaviour for all the tests. W and $\tilde{W}$ are acting similarly, as can be seen from Figures G.1 and G.3 of Appendix G. The estimated power of χ_P^2 also fluctuates but is generally higher.

Figures 5.6 and 5.7 show the estimated powers of $\tilde{W}$ when all the failure modes are dependent at a nominal 5% significance level for all 4×336 situations without and with censoring. The estimated power improves as the sample size gets larger. The estimated power for samples based on increasing FR fluctuates for small sample size as with the two dependent failure modes and then improves as the sample size gets larger and the fluctuation disappear. Clearly, the estimated power is larger than the estimated power for samples with only two dependent failure modes and for larger sample sizes we get 100% estimated power for all the 4×56 situations.

In summary, the estimated power for small sample sizes is low but improves as the sample size gets larger. The estimated power for samples with increasing FR and only two dependent failure modes fluctuates even with large sample sizes but this behaviour disappears for large sample sizes in the case in which all the failure modes are dependent.

5.6 Discussion

The modified Crowder test is mainly conservative with some minor exceptions for samples based on increasing FR in which the estimated significance is very large. In particular, the large estimated significance occur with samples based on the negative binomial distribution with increasing FR where p is relatively small and on the Poisson distribution where $\lambda = 4$. The issues relating to the use of the continuity correction as discussed in section 4.6 need to be addressed

Also, for large sample sizes the estimated power of $\tilde{W}$ is good when all the failure modes are dependent. This is not the case for only two dependent failure modes because of the strange behaviour for samples based on increasing FR even for large sample size. A potential solution is to make use of the contingency table structure that emerges and investigate dependence structures intermediate between complete independence and dependence between all three competing risks. For example, conditional independence and other models might be considered as in traditional log-linear modelling of contingency tables, thereby giving the possibility of learning about the dependence structure of the competing risks.

In the next chapter simpler forms of the Crowder test statistic and its modified version are considered for the general case of g competing risks.

Chapter 6

Testing for Independence of Competing Risks with Discrete Lifetime Data

The Crowder and modified tests are considered here briefly with g competing risks. Two numerical examples are presented when $g = 4$.

6.1 Introduction

As described in chapter 3, the computation of Crowder's test statistic starts by calculating the estimates of the sub-failure rate functions, $\hat{h}_{kc} = n_{kc} / n^{+}_{k\bullet}$, the estimates of the marginal hazard function, $\hat{h}_k = n^{-}_{k\bullet} / n^{+}_{k\bullet}$, and the estimates of the sub-conditional odds functions, $\hat{\hat{h}}_{kc} = \hat{h}_{kc} / (1 - \hat{h}_k)$, for all failure times $k = 1, 2, ..., k_*$. Then for each multiple cause configuration, a_v, and $k = 1, 2, ..., k_*$, we should calculate the quantities $\hat{\psi}_{ka_v} = \hat{\hat{h}}_{ka_v} \Big/ \prod_{j \in a} \hat{\hat{h}}_{kj}$ and $Y_{ka_v} = \log(\hat{\psi}_{ka_v})$. Then for every pair of failure causes, single or multiple, c and b, we should calculate the estimates of the covariances $H_{kc,kb} = \text{cov}(\hat{h}_{kc}, \hat{h}_{kb})$ such that

$$H_{kc,kb} = \frac{1}{n^{+}_{k\bullet}} \left\{ \delta_{cb} \hat{h}_{kc} - (1 - \delta_{kk_*}) \hat{h}_{kc} \hat{h}_{kb} \right\}.$$

Thus, it appears that the modified Crowder test has advantages over the other tests considered here.

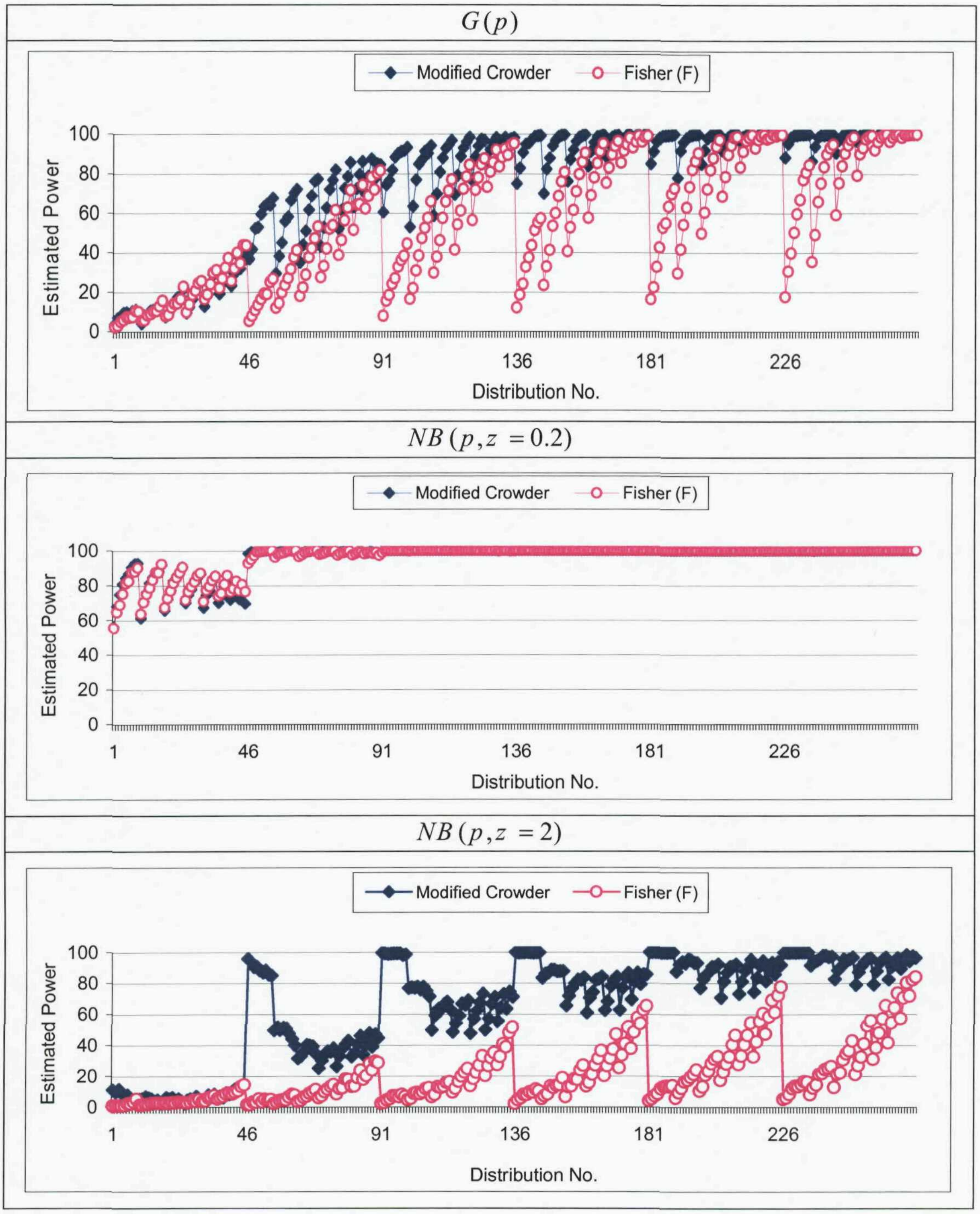


Figure 4.4: The estimated powers of $\tilde{W}$ and F for samples with no censoring from the geometric distribution with $p = 0.1\,(0.05)\,0.40$, the negative binomial distribution with $p = 0.1\,(0.05)\,0.40$ and $z = 0.2$ and with $z = 2$ for $n = 50$ (plotted on the x-axis 1-45), 100 (46-90), 150 (91-135), 200 (136-180), 250 (181-225) and 300 (226-270).

Then substitute the $H_{kc,kb}$ in

$$\bar{H}_{kc,kb} \approx \frac{1}{(1-\hat{h}_k)^2}\left\{H_{kc,kb} + \frac{\hat{h}_{kc}\sum_c H_{kc,kb} + \hat{h}_{kb}\sum_b H_{kc,kb}}{1-\hat{h}_k} + \frac{\hat{h}_{kc}\hat{h}_{kb}\sum_c\sum_b H_{kc,kb}}{(1-\hat{h}_k)^2}\right\},$$

which are then used to obtain

$$\operatorname{cov}(Y_{ka_v}, Y_{ka_u}) \approx \frac{\bar{H}_{ka_v,ka_u}}{\hat{h}_{ka_v}\hat{h}_{ka_u}} - \sum_{j\varepsilon a_u}\frac{\bar{H}_{ka_v,kj}}{\hat{h}_{ka_v}\hat{h}_{kj}} - \sum_{i\varepsilon a_v}\frac{\bar{H}_{ki,ka_u}}{\hat{h}_{ki}\hat{h}_{ka_u}} + \sum_{i\varepsilon a_v}\sum_{j\varepsilon a_u}\frac{\bar{H}_{ki,kj}}{\hat{h}_{ki}\hat{h}_{kj}}.$$

This procedure should be repeated for every multiple cause to obtain the elements of the covariance matrix V_k. Then repeat for all $k = 1, 2, \ldots, k_*$. Finally, we should substitute all these estimates in the test statistic

$$W = \sum_{k=1}^{k_*} Y_k^{\,T}\; V_k^{-1}\; Y_k \rightarrow \chi^2_{dq} \text{ as n} \rightarrow \infty,$$

where d is the number of multiple causes,

$q < k_*$ is the number of observed failure times for which $n_{kc} > 0\ \forall c$.

Hence, in the interests of efficiency, clarity and notational simplicity, simpler forms of W and $\tilde{W}$ in the general case for testing the independence of competing risks phenomenon with g risks of failure are investigated. In 6.2 we evaluate W in the general case of g risks. A similar approach is used in 6.3 to evaluate the modified test statistic $\tilde{W}$ in this case. The chapter concludes with a discussion in 6.4.

6.2 Evaluation of W

The data layout for the general case of competing risks phenomenon with g risks of failure are presented in Table 6.1. The notation used in Table 6.1 is as follows:

n_{k1}, n_{k2}, n_{k3}, ..., n_{kg}, n_{ka_1}, n_{ka_2}, n_{ka_3},..., and n_{ka_d} are the number of subjects failed from cause 1, cause 2, cause 3, ..., cause g and the multiple cause configurations $a_1 = \{1, 2\}$,

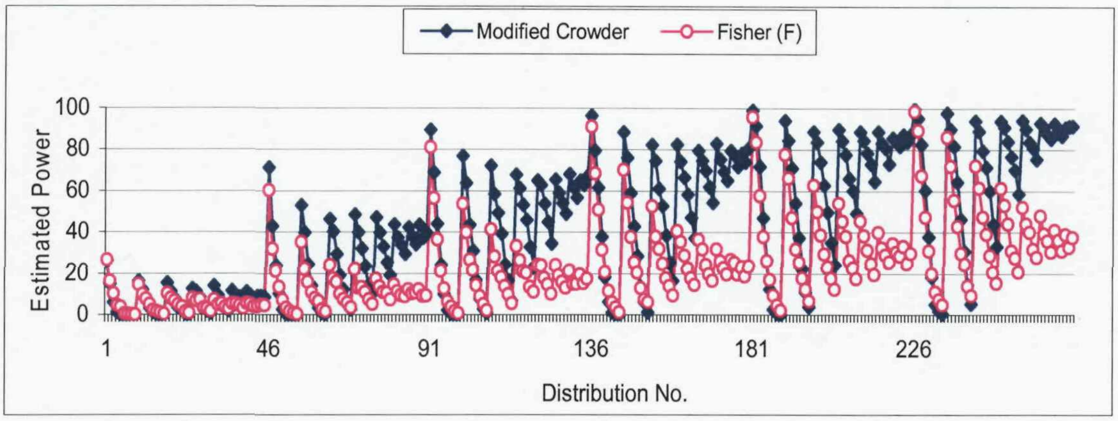


Figure 4.5: The estimated powers of $\tilde{W}$ and F for samples with no censoring from the Poisson distribution with mean $\lambda = 1\,(0.5)\,5$ for $n = 50$ (plotted on the x-axis 1-45), 100 (46-90), 150 (91-135), 200 (136-180), 250 (181-225) and 300 (226-270).

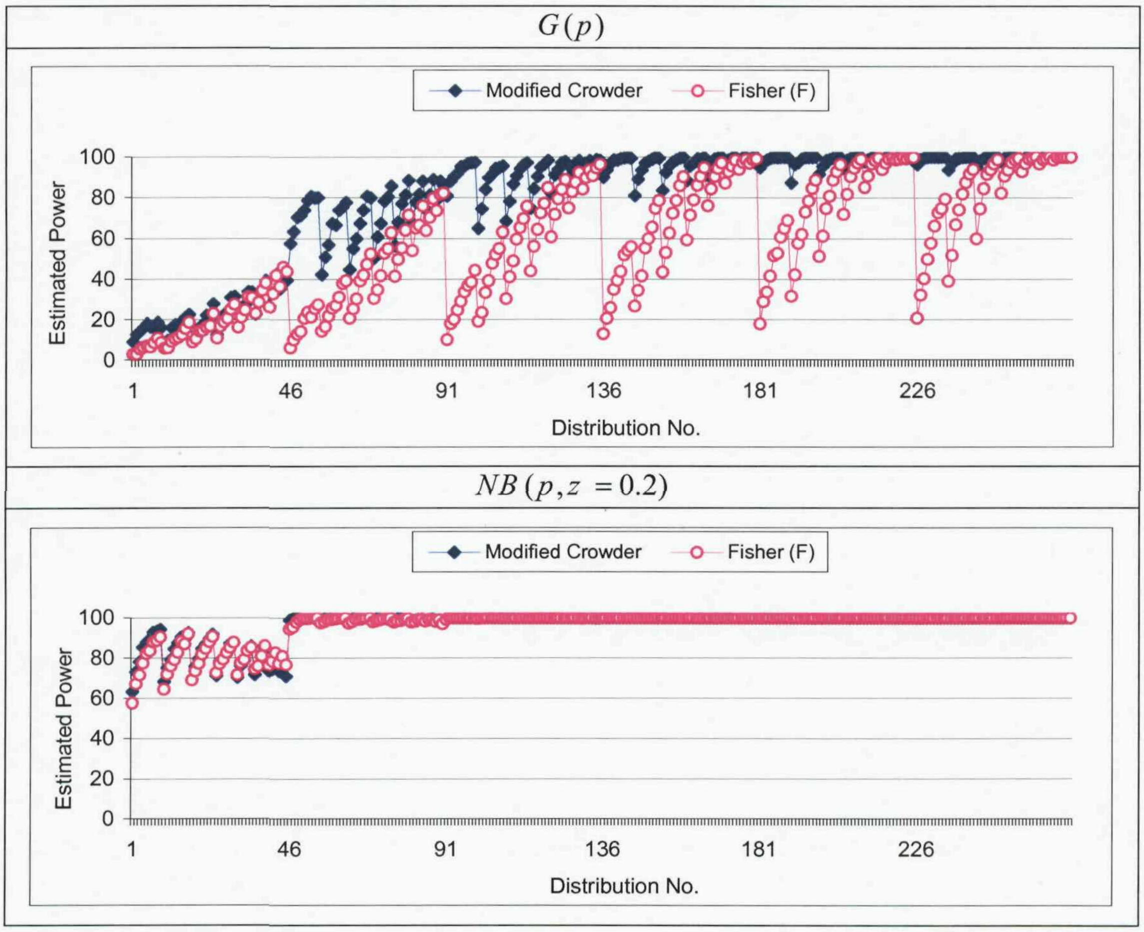


Figure 4.6: The estimated powers of $\tilde{W}$ and F for samples with censoring from the geometric with $p = 0.1\,(0.05)\,0.40$, the negative binomial with $p = 0.1\,(0.05)\,0.40$ and $z = 0.2$ for $n = 50$ (plotted on the x-axis 1-45), 100 (46-90), 150 (91-135), 200 (136-180), 250 (181-225) and 300 (226-270).

$a_2 = \{1,3\}$, $a_3 = \{2,3\}$,..., and $a_d = \{1,2,3,...,g\}$ respectively at time k, where $d = 2^g - g - 1$ is the number of possible multiple cause configurations, and as described before in the previous chapters n_{kr} is the number of subjects lost to view just after time k ; $n_{k\bullet}^- = \sum_c n_{kc}$ is the overall number of subjects failed at time k, where c is the cause of failure at time k, i.e. $c = 1,2,3,...,g,a_1,a_2,a_3,...,a_d$ and $n_{k\bullet}^+ = \sum_{i=k}^{k_\bullet} (n_{i\bullet}^- + n_{ir})$ is the number of subjects at risk at time k.

Table 6.1: Data layout for g competing risks.

Cause \ Time	1	2	3	4	5	...	k_*
1	n_{11}	n_{21}	n_{31}	n_{41}	n_{51}		$n_{k_\bullet 1}$
2	n_{12}	n_{22}	n_{32}	n_{42}	n_{52}		$n_{k_\bullet 2}$
3	n_{13}	n_{23}	n_{33}	n_{43}	n_{53}		$n_{k_\bullet 3}$
...							
g	n_{1g}	n_{2g}	n_{3g}	n_{4g}	n_{5g}		$n_{k_\bullet g}$
$\{1,2\}$	n_{1a_1}	n_{2a_1}	n_{3a_1}	n_{4a_1}	n_{4a_1}		$n_{k_\bullet a_1}$
$\{1,3\}$	n_{1a_2}	n_{2a_2}	n_{3a_2}	n_{4a_2}	n_{5a_2}		$n_{k_\bullet a_2}$
...							
$\{1,g\}$	$n_{1a_{g-1}}$	$n_{2a_{g-1}}$	$n_{3a_{g-1}}$	$n_{4a_{g-1}}$	$n_{5a_{g-1}}$		$n_{k_\bullet a_{g-1}}$
$\{2,3\}$	n_{1a_g}	n_{2a_g}	n_{3a_g}	n_{4a_g}	n_{5a_g}		$n_{k_\bullet a_g}$
...							
$\{1,2,...,g\}$	n_{1a_d}	n_{2a_d}	n_{3a_d}	n_{4a_d}	n_{5a_d}		$n_{k_\bullet a_d}$
Censored	n_{1r}	n_{2r}	n_{3r}	n_{4r}	n_{5r}		$n_{k_\bullet r}$
Observed	$n_{1\bullet}^-$	$n_{2\bullet}^-$	$n_{3\bullet}^-$	$n_{4\bullet}^-$	$n_{5\bullet}^-$		$n_{k_\bullet\bullet}^-$
At risk	$n_{1\bullet}^+$	$n_{2\bullet}^+$	$n_{3\bullet}^+$	$n_{4\bullet}^+$	$n_{5\bullet}^+$		$n_{k_\bullet\bullet}^+$

Note that at each observed failure time a 2^g contingency table can be constructed. The nonparametric maximum likelihood estimator for h_{kc} is $\hat{h}_{kc} = n_{kc}/n_{k\bullet}^+$, and for h_k we have $\hat{h}_k = n_{k\bullet}^-/n_{k\bullet}^+$. The estimates for the sub-conditional odds functions are

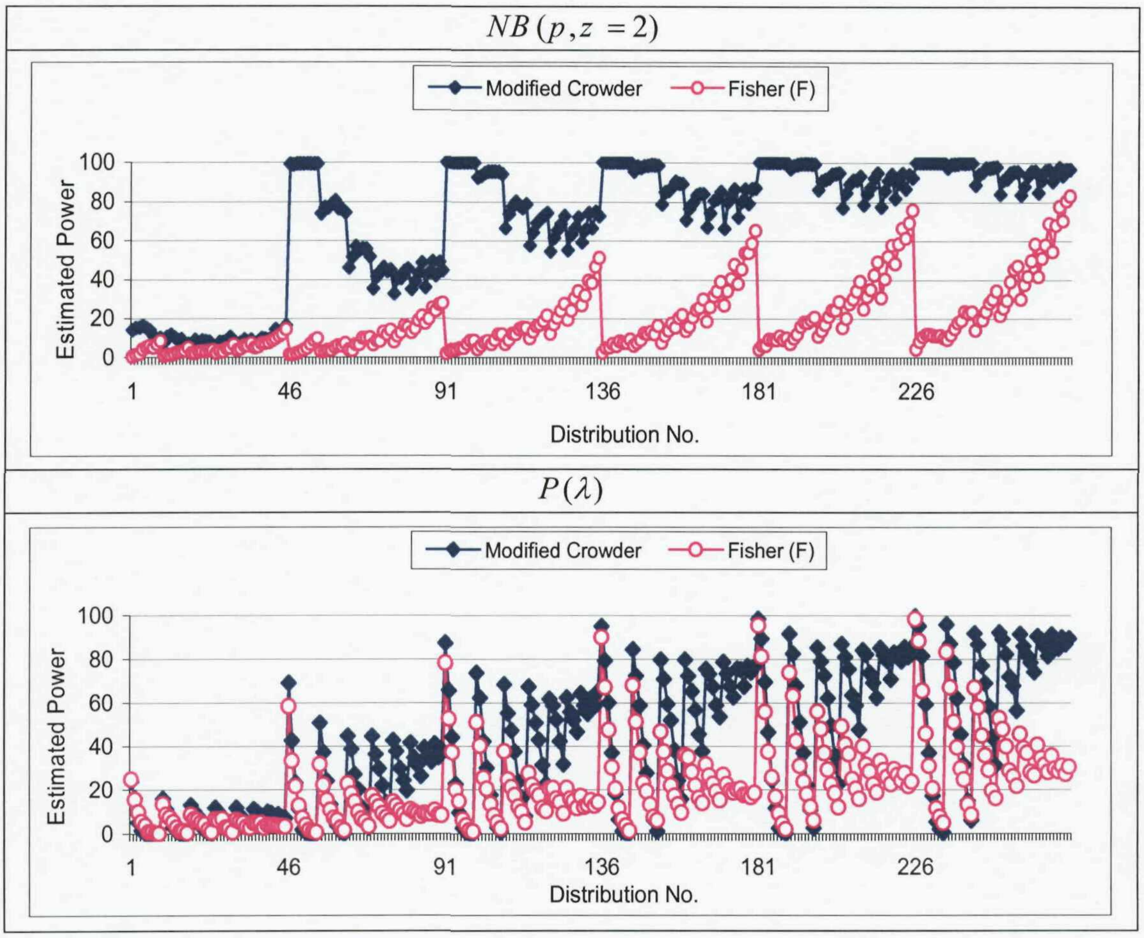


Figure 4.7: The estimated powers of $\tilde{W}$ and F for samples with censoring from the negative binomial with $p = 0.1\,(0.05)\,0.40$ and $z = 2$ and Poisson with mean $\lambda = 1\,(0.5)\,5$ for $n = 50$ (plotted on the x-axis 1-45), 100 (46-90), 150 (91-135), 200 (136-180), 250 (181-225) and 300 (226-270).

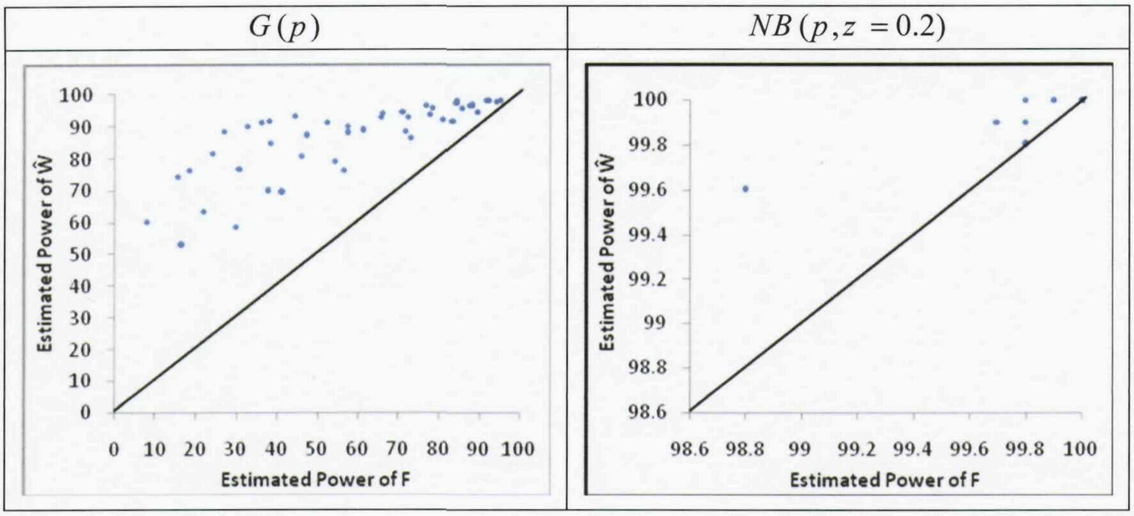


Figure 4.8: The estimated powers of $\tilde{W}$ and F for samples without censoring from the geometric and negative binomial distributions for $n = 150$.

$$\hat{\hat{h}}_{kc} = \frac{\hat{h}_{kc}}{1-\hat{h}_k} = \frac{n_{kc}}{n_{k\bullet}^{+} - n_{k\bullet}^{-}} \quad \forall k = 1,\ldots,k_*;\ n_{k\cdot}^{+} > n_{k\cdot}^{-}.$$

Let a_v be a multiple cause of failure and g_{a_v} be the number of single causes of failure that occur simultaneously to form a_v, so

$$\hat{\psi}_{ka_v} = \frac{\hat{\hat{h}}_{ka_v}}{\prod_{i \varepsilon a_v} \hat{\hat{h}}_{ki}} = \frac{n_{ka_v} (n_{k\bullet}^{+} - n_{k\bullet}^{-})^{g_{a_v}-1}}{\prod_{i \varepsilon a_v} n_{ki}}. \tag{6.1}$$

Now, $Y_k^{\ T} = [Y_{ka_1}\ Y_{ka_2}\ \ldots\ Y_{ka_d}] = [\log(\hat{\psi}_{ka_1})\ \log(\hat{\psi}_{ka_2})\ \ldots\ \log(\hat{\psi}_{ka_d})]$ is of length $d = 2^g - g - 1$. So,

$$W = \sum_{k=1}^{k_*} Y_k^{\ T}\ V_k^{-1}\ Y_k\ .$$

Now, we need to calculate the covariance matrix V_k. Let c and b be causes of failure, single or multiple, so using (3.3),

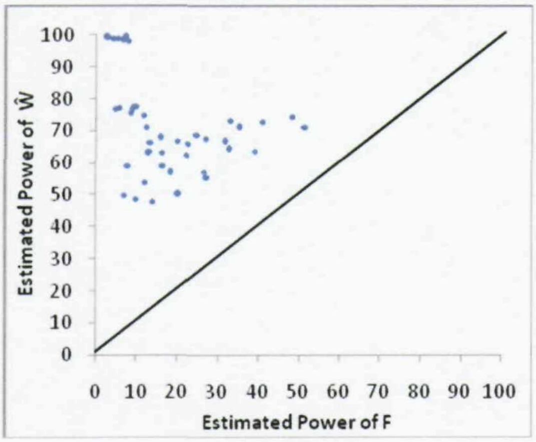
$$H_{kc,kb} = H_{kb,kc} = \frac{1}{n_{k\bullet}^{+}} \left\{ \delta_{cb} \hat{h}_{kc} - (1-\delta_{kk.}) \hat{h}_{kc} \hat{h}_{kb} \right\}.$$

Thus, if $k < k_*$,

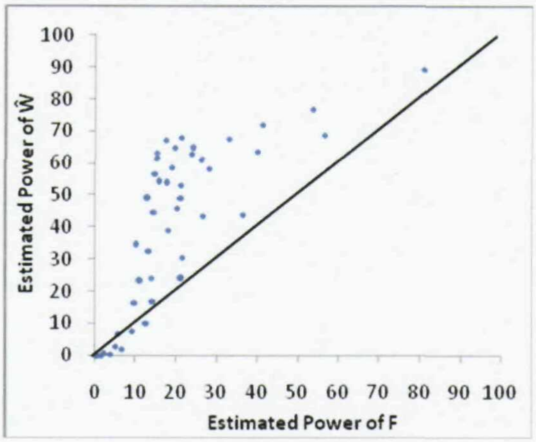
$$H_{kc,kb} = \begin{cases} \dfrac{1}{n_{k\cdot}^{+}} \left\{ \hat{h}_{kc} - \hat{h}_{kc}^{\ 2} \right\} = \dfrac{n_{kc}(n_{k\bullet}^{+} - n_{kc})}{n_{k\bullet}^{+\ 3}} & \text{if } c = b \\ \dfrac{1}{n_{k\cdot}^{+}} \left\{ -\hat{h}_{kc} \hat{h}_{kb} \right\} = -\dfrac{n_{kc} n_{kb}}{n_{k\bullet}^{+\ 3}} & \text{if } c \neq b \end{cases}.$$

Now, using (3.2),

$NB(p, z = 2)$

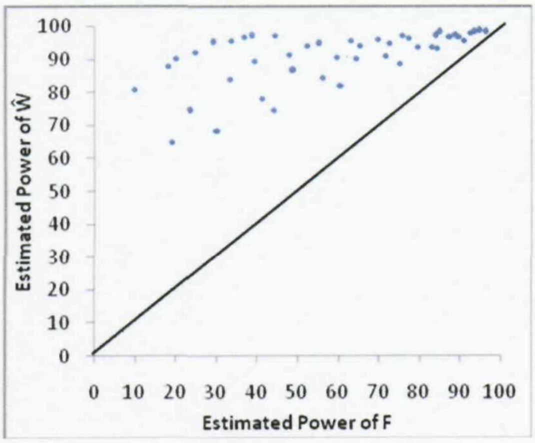


$P(\lambda)$

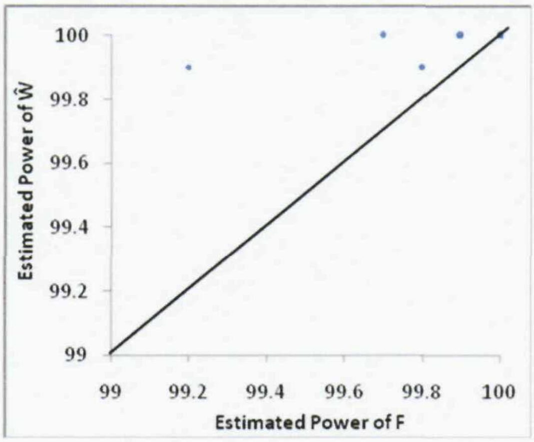


Cont. Figure 4.8: The estimated powers of $\tilde{W}$ and F for samples without censoring from the negative binomial and Poisson distributions for $n = 150$.

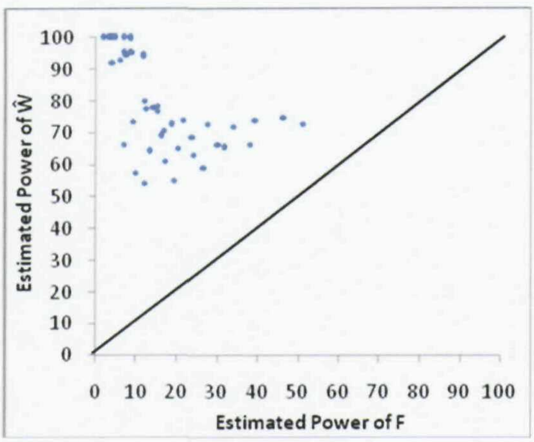
$G(p)$



$NB(p, z = 0.2)$



$NB(p, z = 2)$



$P(\lambda)$

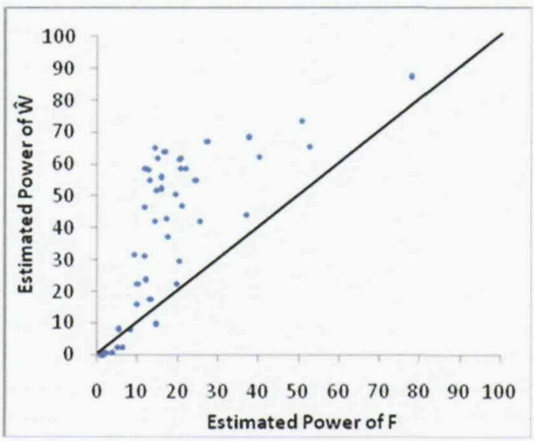


Figure 4.9: The estimated powers of $\tilde{W}$ and F for samples with censoring from the geometric, negative binomial and Poisson distributions for $n = 150$.

$$\bar{H}_{kc,kb} \approx \frac{1}{(1-\hat{h}_k)^2}\left\{ H_{kc,kb} + \frac{\hat{h}_{kc}\sum_c H_{kc,kb} + \hat{h}_{kb}\sum_b H_{kc,kb}}{1-\hat{h}_k} + \frac{\hat{h}_{kc}\hat{h}_{kb}\sum_c\sum_b H_{kc,kb}}{(1-\hat{h}_k)^2} \right\},$$

and using (3.1)

$$\text{cov}(Y_{ka_v}, Y_{ka_u}) \approx \frac{\bar{H}_{ka_v,ka_u}}{\hat{h}_{ka_v}\hat{h}_{ka_u}} - \sum_{j\in a_u} \frac{\bar{H}_{ka_v,kj}}{\hat{h}_{ka_v}\hat{h}_{kj}} - \sum_{i\in a_v} \frac{\bar{H}_{ki,ka_u}}{\hat{h}_{ki}\hat{h}_{ka_u}} + \sum_{i\in a_v}\sum_{j\in a_u} \frac{\bar{H}_{ki,kj}}{\hat{h}_{ki}\hat{h}_{kj}},$$

the covariance matrix $V_k = \left[\text{cov}(Y_{ka_v}, Y_{ka_u})\right]_{d\times d}$ has as its elements

$$\text{cov}(Y_{ka_v}, Y_{ka_u}) = \frac{\delta_{a_v a_u}}{n_{ka_v}} + \frac{(g_{a_v}-1)(g_{a_u}-1)}{n_k^+ - n_k^-} + \sum_{i\,\varepsilon(a_v\cap a_u)} \frac{1}{n_{ki}} \quad (6.2)$$

and by calculating the inverse of V_k, we get

$$W = \sum_{k=1}^{k_*} Y_k^T\ V_k^{-1}\ Y_k \rightarrow \chi^2_{dq} \text{ as } n \rightarrow \infty,$$

where d is the number of multiple causes,

$q < k_*$ is the number of observed failure times for which $n_{kc} > 0\ \forall c$.

□.

Example 6.1: Simulated $NB(0.1, 0.2)$ data

The data comprise 900 independent pairs (k, c) that were generated using SAS software version 9.1. The observed failure time, K, and the type of failure, C, are determined as in Example 5.1 but with four independent competing risks, each with latent failure time from the negative binomial distribution, defined in 1.5.2, with parameters $p = 0.1$ and $z = 0.2$.

The data are displayed in Table 6.2 along with W_k at each computable failure time k. In what follows the notation $a_1 = \{1,2\}$, $a_2 = \{1,3\}$, $a_3 = \{1,4\}$, $a_4 = \{2,3\}$, $a_5 = \{2,4\}$, $a_6 = \{3,4\}$, $a_7 = \{1,2,3\}$, $a_8 = \{1,2,4\}$, $a_9 = \{1,3,4\}$, $a_{10} = \{2,3,4\}$ and $a_{11} = \{1,2,3,4\}$ is used for multiple cause failures.

Table 6.2: Simulated data for Example 6.1 and the W_k.

Time	1	2	3	4	5
n_{k1}	29	0	0	1	1
n_{k2}	32	1	0	1	0
n_{k3}	20	2	0	0	0
n_{k4}	36	3	0	0	0
n_{ka_1}	46	1	0	0	0
n_{ka_2}	53	1	0	0	0
n_{ka_3}	45	3	0	0	0
n_{ka_4}	55	1	0	0	0
n_{ka_5}	33	1	1	0	0
n_{ka_6}	51	0	0	0	0
n_{ka_7}	72	1	1	0	0
n_{ka_8}	98	0	0	0	0
n_{ka_9}	94	0	0	0	0
$n_{ka_{10}}$	75	0	0	0	0
$n_{ka_{11}}$	142	0	0	0	0
$n_{k\bullet}^-$	881	14	2	2	1
$n_{k\bullet}^+$	900	19	5	3	1
W_k	71.405				

The computation of W_1 is as follows:

$$n_{1\bullet}^- = \sum_c n_{1c} = 29+32+20+36+46+53+45+ \\ 55+33+51+72+98+94+75+142 = 881 ,$$

$$n_{1\bullet}^+ = \sum_{i=1}^{5} n_{i\bullet}^- = 881+14+2+2+1 = 900,$$

$n_{ir} = 0 \ \forall \ 1 \le \mathrm{i} \le 13$ since there are no censored observations,

$$\hat{\psi}_{1a_1} = \frac{n_{1a_1}(n_{1\bullet}^+ - n_{1\bullet}^-)}{n_{11}n_{12}} = \frac{46\times 19}{29\times 32} = 0.94181 , \ \hat{\psi}_{1a_2} = \frac{53\times 19}{29\times 20} = 1.73621,$$

$$\hat{\psi}_{1a_3} = \frac{45\times 19}{29\times 36} = 0.81897 , \ \hat{\psi}_{1a_4} = \frac{55\times 19}{32\times 20} = 1.63281,$$

$$\hat{\psi}_{1a_5} = \frac{33\times 19}{32\times 36} = 0.54427 , \ \hat{\psi}_{1a_6} = \frac{51\times 19}{20\times 36} = 1.3458,$$

$$\hat{\psi}_{1a_7} = \frac{72\times19^2}{29\times32\times20} = 1.40043,\ \hat{\psi}_{1a_8} = \frac{98\times19^2}{29\times32\times36} = 1.05897,$$

$$\hat{\psi}_{1a_9} = \frac{94\times19^2}{29\times20\times36} = 1.62519,\ \hat{\psi}_{1a_{10}} = \frac{75\times19^2}{32\times20\times36} = 1.17513,$$

$$\hat{\psi}_{1a_{11}} = \frac{142\times19^3}{29\times32\times20\times36} = 1.45770,$$

$$\text{var}(Y_{1a_1}) = \frac{1}{n_{1a_1}} + \frac{1}{n_{1\bullet}^{+} - n_{1\bullet}^{-}} + \frac{1}{n_{11}} + \frac{1}{n_{12}} = \frac{1}{46} + \frac{1}{19} + \frac{1}{29} + \frac{1}{32} = 0.14010,$$

$\text{var}(Y_{1a_2}) = 0.15598$, $\text{var}(Y_{1a_3}) = 0.13711$, $\text{var}(Y_{1a_4}) = 0.15206$, $\text{var}(Y_{1a_5}) = 0.14196$,

$\text{var}(Y_{1a_6}) = 0.15002$, $\text{var}(Y_{1a_7}) = 0.34015$, $\text{var}(Y_{1a_8}) = 0.31424$, $\text{var}(Y_{1a_9}) = 0.33343$,

$\text{var}(Y_{1a_{10}}) = 0.33289$, $\text{var}(Y_{1a_{11}}) = 0.62424$, $\text{cov}(Y_{1a_1}, Y_{1a_2}) = 0.08711$,

$\text{cov}(Y_{1a_1}, Y_{1a_3}) = 0.08711$, $\text{cov}(Y_{1a_1}, Y_{1a_4}) = 0.08388$, $\text{cov}(Y_{1a_1}, Y_{1a_5}) = 0.08388$,

$\text{cov}(Y_{1a_1}, Y_{1a_6}) = 0.05263$, $\text{cov}(Y_{1a_1}, Y_{1a_7}) = 0.17100$, …, $\text{cov}(Y_{1a_{10}}, Y_{1a_{11}}) = 0.42482$,

So,

$$Y_1^Y = [-0.051\ \ 0.552\ \ -0.110\ \ 0.490\ \ -0.608\ \ 0.297\ \ 0.337\ \ 0.057\ \ 0.486\ \ 0.161\ \ 0.377]$$

and

$$V_1 = \begin{bmatrix} 0.140 & 0.087 & 0.087 & 0.084 & 0.084 & 0.053 & 0.171 & 0.171 & 0.140 & 0.137 & 0.224 \\ 0.087 & 0.156 & 0.087 & 0.103 & 0.053 & 0.103 & 0.190 & 0.140 & 0.190 & 0.155 & 0.242 \\ 0.087 & 0.087 & 0.137 & 0.053 & 0.080 & 0.080 & 0.140 & 0.168 & 0.168 & 0.133 & 0.220 \\ 0.084 & 0.103 & 0.053 & 0.152 & 0.084 & 0.103 & 0.187 & 0.137 & 0.137 & 0.187 & 0.239 \\ 0.084 & 0.053 & 0.080 & 0.084 & 0.142 & 0.080 & 0.137 & 0.164 & 0.133 & 0.164 & 0.217 \\ 0.053 & 0.103 & 0.080 & 0.103 & 0.080 & 0.150 & 0.155 & 0.133 & 0.183 & 0.183 & 0.236 \\ 0.171 & 0.190 & 0.140 & 0.187 & 0.137 & 0.155 & 0.340 & 0.276 & 0.295 & 0.292 & 0.432 \\ 0.171 & 0.140 & 0.168 & 0.137 & 0.164 & 0.133 & 0.276 & 0.314 & 0.273 & 0.270 & 0.409 \\ 0.140 & 0.190 & 0.168 & 0.137 & 0.133 & 0.183 & 0.295 & 0.273 & 0.333 & 0.288 & 0.428 \\ 0.137 & 0.155 & 0.133 & 0.187 & 0.164 & 0.183 & 0.292 & 0.170 & 0.288 & 0.333 & 0.425 \\ 0.224 & 0.242 & 0.220 & 0.239 & 0.217 & 0.236 & 0.432 & 0.409 & 0.428 & 0.425 & 0.624 \end{bmatrix}$$

Hence,

$$W_1 = Y_1^T\ V_1^{-1}\ Y_1 = 71.405.$$

Note that W_k is computable only when $k = 1$ because of the zero frequencies at other times. The resulting Crowder test statistic W is 71.405 with 11 degrees of freedom, which has a p-value close to 0, indicating wrongly that the independence hypothesis may be rejected at the 5% significance level.

Example 6.2: Simulated $NB(0.15, 0.2)$ data with censoring

The data comprise 1200 independent pairs (k, c) that were generated as in Example 5.2 but with four independent risks and censoring. All four latent failure times and the censoring time were drawn from the negative binomial distribution with parameters $p = 0.15$ and $z = 0.2$.

Table 6.3: Simulated data for Example 6.2 and the W_k.

Time	1	2	3
n_{k1}	28	2	0
n_{k2}	21	0	0
n_{k3}	33	0	0
n_{k4}	25	1	0
n_{ka_1}	58	0	0
n_{ka_2}	51	0	0
n_{ka_3}	66	1	0
n_{ka_4}	51	0	0
n_{ka_5}	56	1	0
n_{ka_6}	57	0	0
n_{ka_7}	115	0	0
n_{ka_8}	103	0	0
n_{ka_9}	97	0	0
$n_{ka_{10}}$	123	0	0
$n_{ka_{11}}$	295	0	0
n_{ka_r}	14	1	1
$n_{k\bullet}^-$	1179	5	0
$n_{k\bullet}^+$	1200	7	1
W_k	25.305		

The data are displayed in Table 6.3 along with W_k at each computable failure time k. The calculation of W_k is similar to the previous example but taking into account the censored observations, $n_{k\bullet}^{+} = \sum_{i=k}^{7} (n_{i\bullet}^{-} + n_{ir})$.

Note also that W_k is computable only when $k = 1$ because of the zero frequencies at other times. The resulting Crowder test statistic W is 25.3054 with 11 degrees of freedom, which has a p-value 0.0082, indicating wrongly that the independence hypothesis may be rejected at the 5% significance level.

6.3 Evaluation of $\tilde{W}$

We now consider the modified Crowder test statistic $\tilde{W}$ in the general case of g failure modes. With $c = 1,2,...,g, a_1, a_2,...,a_d$ where $a_1 = \{1,2\}$, $a_2 = \{1,3\}$, ... and $a_d = \{1,2,...,g\}$ and $d = 2^g - g - 1$, the modified Crowder test statistic is

$$\tilde{W} = \sum_{k=1}^{k_\bullet} \tilde{Y}_k^T \tilde{V}_k^{-1} \tilde{Y}_k \rightarrow \chi^2_{dk_\bullet} \text{ as } n \rightarrow \infty,$$

where

$$\tilde{Y}_k^T = [\tilde{Y}_{ka_1} \ \tilde{Y}_{ka_2} \ \dots \ \tilde{Y}_{ka_d}]_{d\times 1} = [\log(\hat{\tilde{\psi}}_{ka_1}) \ \log(\hat{\tilde{\psi}}_{ka_2}) \ \dots \ \log(\hat{\tilde{\psi}}_{ka_d})]_{d\times 1},$$

using (6.1),

$$\hat{\tilde{\psi}}_{ka_v} = \frac{(n_{ka_v} + 0.5)(n_{k\bullet}^{+} - n_{k\bullet}^{-} + 0.5)^{g_{a_v} - 1}}{\prod_{j\in a_v} (n_{kj} + 0.5)} \ ; \ v = 1,2,...,d,$$

and using (6.2),

$$\tilde{V}_k = \left[\operatorname{cov}(\tilde{Y}_{ka_v}, \tilde{Y}_{ka_u})\right]_{d\times d},$$

$$\operatorname{cov}(\tilde{Y}_{ka_v}, \tilde{Y}_{ka_u}) = \frac{\delta_{a_v a_u}}{n_{ka_v} + 0.5} + \frac{(g_{a_v} - 1)(g_{a_u} - 1)}{n_k^{+} - n_k^{-} + 0.5} + \sum_{l\varepsilon(a_v \cap a_u)} \frac{1}{n_{kl} + 0.5}.$$

Example 6.3: Simulated $NB(0.1, 0.2)$ data

The modified version of Crowder's test statistic, $\tilde{W}$, is applied to the data simulated in example 6.1 and the results are presented in Table 6.4.

Table 6.4: Simulated data for Example 6.3 and the $\tilde{W}_k$.

Time	1	2	3	4	5
n_{k1}	29	0	0	1	1
n_{k2}	32	1	0	1	0
n_{k3}	20	2	0	0	0
n_{k4}	36	3	0	0	0
n_{ka_1}	46	1	0	0	0
n_{ka_2}	53	1	0	0	0
n_{ka_3}	45	3	0	0	0
n_{ka_4}	55	1	0	0	0
n_{ka_5}	33	1	1	0	0
n_{ka_6}	51	0	0	0	0
n_{ka_7}	72	1	1	0	0
n_{ka_8}	98	0	0	0	0
n_{ka_9}	94	0	0	0	0
$n_{ka_{10}}$	75	0	0	0	0
$n_{ka_{11}}$	142	0	0	0	0
$n_{k\bullet}^{-}$	881	14	2	2	1
$n_{k\bullet}^{+}$	900	19	5	3	1
$\tilde{W}_k$	47.765	5.110	5.0817	0.958	0.766

The computation of $\tilde{W}_1$ is as follows:

$$n_{1\bullet}^{-} = \sum_c n_{1c} = 29+32+20+36+46+53+45+ \\ 55+33+51+72+98+94+75+142 = 881,$$

$$n_{1\bullet}^{+} = \sum_{i=1}^{5} n_{i\bullet}^{-} = 881+14+2+2+1 = 900,$$

$n_{ir} = 0 \ \forall \ 1 \le i \le 13$ since there are no censored observations,

$$\hat{\tilde{\psi}}_{1a_1} = \frac{(n_{1a_1}+0.5)(n_{1\bullet}^{+}-n_{1\bullet}^{-}+0.5)}{(n_{11}+0.5)(n_{12}+0.5)} = \frac{46.5\times19.5}{29.5\times32.5} = 0.94576,\ \hat{\tilde{\psi}}_{1a_2} = \frac{53.5\times19.5}{29.5\times20.5} = 1.72509,$$

$$\hat{\tilde{\psi}}_{1a_3} = \frac{45.5\times19.5}{29.5\times36.5} = 0.82401,\ \hat{\tilde{\psi}}_{1a_4} = \frac{55.5\times19.5}{32.5\times20.5} = 1.62439,\ \hat{\tilde{\psi}}_{1a_5} = \frac{33.5\times19.5}{32.5\times36.5} = 0.55068,$$

$$\hat{\tilde{\psi}}_{1a_6} = \frac{51.5\times19.5}{20.5\times36.5} = 1.34213,\ \hat{\tilde{\psi}}_{1a_7} = \frac{72.5\times19.5^2}{29.5\times32.5\times20.5} = 1.40265,$$

$$\hat{\tilde{\psi}}_{1a_8} = \frac{98.5\times19.5^2}{29.5\times32.5\times36.5} = 1.07030,\ \hat{\tilde{\psi}}_{1a_9} = \frac{94.5\times19.5^2}{29.5\times20.5\times36.5} = 1.62792,$$

$$\hat{\tilde{\psi}}_{1a_{10}} = \frac{75.5\times19.5^2}{32.5\times20.5\times36.5} = 1.18055,\ \hat{\tilde{\psi}}_{1a_{11}} = \frac{142.5\times19.5^3}{29.5\times32.5\times20.5\times36.5} = 1.47288,$$

$$\text{var}(\tilde{Y}_{1a_1}) == \frac{1}{46.5}+\frac{1}{19.5}+\frac{1}{29.5}+\frac{1}{32.5} = 0.13745,\ \text{var}(\tilde{Y}_{1a_2}) = 0.15265,$$

$\text{var}(\tilde{Y}_{1a_3}) = 0.13456$, $\text{var}(\tilde{Y}_{1a_4}) = 0.14885$, $\text{var}(\tilde{Y}_{1a_5}) = 0.13930$, $\text{var}(\tilde{Y}_{1a_6}) = 0.14688$,

$\text{var}(\tilde{Y}_{1a_7}) = 0.33237$, $\text{var}(\tilde{Y}_{1a_8}) = 0.30735$, $\text{var}(\tilde{Y}_{1a_9}) = 0.32579$, $\text{var}(\tilde{Y}_{1a_{10}}) = 0.32532$,

$\text{var}(\tilde{Y}_{1a_{11}}) = 0.60940$, $\text{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_2}) = 0.08518$, $\text{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_3}) = 0.08518$,

$\text{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_4}) = 0.08205$, $\text{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_5}) = 0.08205$, $\text{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_6}) = 0.05128$,

$\text{cov}(\tilde{Y}_{1a_1}, \tilde{Y}_{1a_7}) = 0.16723$, …, $\text{cov}(\tilde{Y}_{1a_{10}}, \tilde{Y}_{1a_{11}}) = 0.41464$,

So,

$$\tilde{Y}_1^{Y} = [-0.056\ \ 0.545\ \ -0.194\ \ 0.485\ \ -0.597\ \ 0.294\ \ 0.338\ \ 0.068\ \ 0.487\ \ 0.165\ \ 0.387]$$

and

$$\tilde{V}_1 = \begin{bmatrix} 0.137 & 0.085 & 0.085 & 0.082 & 0.082 & 0.051 & 0.167 & 0.167 & 0.136 & 0.133 & 0.219 \\ 0.085 & 0.153 & 0.085 & 0.100 & 0.051 & 0.100 & 0.185 & 0.136 & 0.185 & 0.151 & 0.237 \\ 0.085 & 0.085 & 0.135 & 0.051 & 0.080 & 0.080 & 0.136 & 0.164 & 0.164 & 0.130 & 0.215 \\ 0.082 & 0.100 & 0.051 & 0.149 & 0.082 & 0.100 & 0.182 & 0.133 & 0.133 & 0.182 & 0.233 \\ 0.082 & 0.051 & 0.080 & 0.082 & 0.139 & 0.079 & 0.133 & 0.161 & 0.130 & 0.161 & 0.212 \\ 0.051 & 0.100 & 0.080 & 0.100 & 0.079 & 0.147 & 0.151 & 0.130 & 0.179 & 0.179 & 0.230 \\ 0.167 & 0.185 & 0.136 & 0.182 & 0.133 & 0.151 & 0.332 & 0.270 & 0.288 & 0.285 & 0.421 \\ 0.167 & 0.136 & 0.164 & 0.133 & 0.161 & 0.130 & 0.270 & 0.307 & 0.266 & 0.263 & 0.400 \\ 0.136 & 0.185 & 0.164 & 0.133 & 0.130 & 0.179 & 0.288 & 0.266 & 0.326 & 0.281 & 0.418 \\ 0.133 & 0.151 & 0.130 & 0.182 & 0.161 & 0.179 & 0.285 & 0.263 & 0.281 & 0.325 & 0.415 \\ 0.219 & 0.237 & 0.215 & 0.233 & 0.212 & 0.230 & 0.421 & 0.400 & 0.418 & 0.415 & 0.609 \end{bmatrix}$$

Hence,

$\tilde{W}_1 = \tilde{Y}_1^T \ \tilde{V}_1^{-1} \ \tilde{Y}_1 = 47.765$.

The resulting modified Crowder test statistic is $\tilde{W} = \sum_{k=1}^{5} \tilde{W}_k = 59.681$ with 55 degrees of freedom, which has a p-value 0.309, indicating that the independence hypothesis cannot be rejected at the 5% significance level. Note that the conclusions based on W and $\tilde{W}$ differ for this set of data.

Example 6.4: Simulated $NB(0.15, 0.2)$ data with censoring

$\tilde{W}$ is applied to the data simulated in example 6.2 and the results are presented in Table 6.5.

Table 6.5: Simulated data for Example 6.4 and the $\tilde{W}_k$.

Time	1	2	3
n_{k1}	28	2	0
n_{k2}	21	0	0
n_{k3}	33	0	0
n_{k4}	25	1	0
n_{ka_1}	58	0	0
n_{ka_2}	51	0	0
n_{ka_3}	66	1	0
n_{ka_4}	51	0	0
n_{ka_5}	56	1	0
n_{ka_6}	57	0	0
n_{ka_7}	115	0	0
n_{ka_8}	103	0	0
n_{ka_9}	97	0	0
$n_{ka_{10}}$	123	0	0
$n_{ka_{11}}$	295	0	0
n_{ka_r}	14	1	1
$n_{k\bullet}^{-}$	1179	5	0
$n_{k\bullet}^{+}$	1200	7	1
$\tilde{W}_k$	25.242	1.754	0.766

The computation of $\tilde{W}$ is similar to the previous example but taking into account the censored observations, $n_{k\bullet}^{+} = \sum_{i=k}^{3} (n_{i\bullet}^{-} + n_{ir})$. $\tilde{W} = \sum_{k=1}^{3} \tilde{W}_k = 27.7622$ with 33 degrees of freedom, which has a p-value 0.725, giving the same conclusion as in Example 6.3.

6.4 Discussion

In this chapter we have discussed briefly the form of W and $\tilde{W}$ in the general case. It turns out that W and $\tilde{W}$ have simpler forms that are easier to work with and more efficient. The simpler forms are illustrated on two examples with $g = 4$.

In the next chapter an overview of the thesis and future work will be presented.

Chapter 7

An Overview and Future Work

In this final chapter an overview of the research findings will be presented and the possible extensions will be discussed.

7.1 An Overview

Two topics are considered in this thesis: the discrete empirical failure rate and tests of independence of competing risks with discrete lifetime data.

In chapter 1 some theoretical differences between discrete and continuous reliability methods are highlighted. The basic properties of some discrete distributions that are potentially useful for lifetime data analysis are discussed. Also, the general behaviour of the failure rate function for these distributions is presented.

In chapter 2 a discussion of some potentially undesirable features of the EFR plot is presented. Some simple transformations aimed at resolving this problem are considered but they turn out to be of little use. Some distributional results for the EFR are obtained together with results for the corresponding expected values and standard deviations. A control chart based on a suitably smoothed version of the EFR is proposed in order to test the constant FR assumption. Versions of this method are investigated by simulation and their properties compared with these of more traditional tests.

In chapter 3 a method developed by Crowder (1997) to test the independence of competing risks with discrete lifetime data is investigated. Crowder's derivation contains ambiguities and errors which have been clarified and corrected. Also, some potential practical shortcomings of the test are discussed and a modified version of it is proposed.

In chapter 4 the Crowder test and its modified version are applied to the special case of two failure modes. The Crowder test statistic has a simpler form in this case and at each observable failure time turns out to be the log odds ratio test statistic. By recasting Crowder's results in terms of classical contingency tables, the properties of the Crowder test and its modified version have been compared to other tests of independence by simulation. The simulation results suggest that the modified version of the Crowder test performs promisingly in comparison with other tests considered, especially for large sample sizes. However, it is of concern that it is conservative.

In chapter 5 the Crowder test and its modified version are applied to the special case of three failure modes. Once again by casting the problem in terms of contingency tables the form of the Crowder test and its modified version is simplified. A simulation study is undertaken, showing that the modified test performs reasonably well when the three failure modes are dependent. However, when only two of the failure modes are dependent, the test is much less impressive.

In chapter 6 the Crowder test and its modified version are considered for the general case of g competing risks. The tests are demonstrated on two numerical examples when $g = 4$.

7.2 Future Work

The distributional results for the EFR are interesting but apply only to uncensored data. Modification of these results to include censoring is a natural next step. The results presented are marginal results and it may be of interest to generalise them to account for the joint behaviour of the EFR. This might allow a sharpening of the properties of the FRCC, for example. There is future scope to investigate the FRCC to allow for censored observations. Moreover, the rather crude $\pm 2\,STD$ bounds often fall outside the allowable range. So a refinement of these bounds, possibly using the second rate of failure function transformation, may be worth considering. Further work is also needed on the proportion of observations outside the bounds that constitutes a signal to try to make the procedure less conservative. Likewise, the smoothing method used is rather restrictive and may need to be relaxed. Finally, the current FRCC requires the whole data set to be observed before it can be applied. An interesting alternative would be to develop a "real time" version that could be applied sequentially as data arrived.

The Crowder test statistic considered in chapter 3 and onward cannot be used routinely because of computability problems. The modified version of the Crowder test statistic overcomes the computability problems. It appears to have reasonable power with just two failure modes but is very conservative. However, the results of chapter 5 suggest that it performs less well in the case of $g > 2$ failure modes, especially when not all of the failure modes are dependent. It may be that the contingency table structure to the data will enable intermediate dependence structures (e.g. one failure mode independent of two dependent failure modes, two failure modes conditionally independent given a third) to be dealt with in a natural way. This is a potentially interesting area for future research.

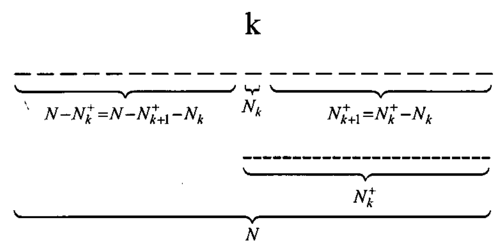
Appendix A:

A.1 Proof of Proposition 2.1:

First note that the empirical failure rate, $\hat{h}_k = \frac{n_k}{n_k^+}$, takes only rational numbers:

$\hat{h}_k = \frac{n_k}{n_k^+}$	n_k							
	0	1	2	3	4	...	n-1	n
1	0	1						
2	0	$1/2$	1					
3	0	$1/3$	$2/3$	1				
n_k^+ 4	0	$1/4$	$2/4$	$3/4$	1			
...	0	...	...	...	...	...		
n-1	0	$1/(n-1)$	$2/(n-1)$	$3/(n-1)$	$4/(n-1)$	...	1	
n	0	$1/n$	$2/n$	$3/n$	$4/n$	...	$(n-1)/n$	1

The range of failure times can be partitioned as the following:



Partition of failure times

Consider H_1. Clearly $\hat{h}_1 = \frac{n_1}{n_1^+} = \frac{n_1}{n}$, hence $nH_1 \sim Binomial(n, p_1)$; $p_1 = P(K = 1)$.

Now, consider H_k for $k > 1$.

We can find the marginal distribution of H_k via the trinomial distribution.

Let $p_1 = P(K = k)$, $p_2 = P(K \geq k+1)$ and $1 - p_1 - p_2 = P(K \leq k-1)$.

Then $(N_k, N_{k+1}^+) \sim Trinomial(n, p_1, p_2)$; $N_{k+1}^+ = N_{k+1} + N_{k+2} + \ldots$.

Thus,

$$P(N_k = n_k, N_{k+1}^+ = n_{k+1}^+) = \frac{n!}{(n - n_{k+1}^+ - n_k)! n_k ! n_{k+1}^+ !} p_1^{n_k} p_2^{n_{k+1}^+} (1 - p_1 - p_2)^{n - n_{k+1}^+ - n_k},$$

where $n_k = 0, 1, 2, \ldots, n - n_{k+1}^+$, $n_{k+1}^+ = 0, 1, 2, \ldots, n$ and $n_k + n_{k+1}^+ \leq n$

Now, let $\hat{h}_k = \dfrac{n_k}{n_k + n_{k+1}^+}$ and $n_k^+ = n_k + n_{k+1}^+$; $n_k = 0,1,...,n - n_{k+1}^+$ and $n_{k+1}^+ = 0,1,...,n$

$\Rightarrow n_k = \hat{h}_k n_k^+$ and $n_{k+1}^+ = n_k^+ - \hat{h}_k n_k^+$; $n_k = 0,1,...,n_k^+$, $n_k^+ = 0,1,...,n$ and $0 \le \hat{h}_k = \dfrac{n_k}{n_k^+} \le 1$.

Hence,

$$P(H_k = \hat{h}_k, N_k^+ = n_k^+) = \sum \frac{n!}{(n - n_k^+)!(\hat{h}_k n_k^+)!(n_k^+ - \hat{h}_k n_k^+)!} p_1^{\hat{h}_k n_k^+} p_2^{n_k^+ - \hat{h}_k n_k^+} (1 - p_1 - p_2)^{n - n_k^+} ,$$

where $0 \le \hat{h}_k = \dfrac{n_k}{n_k^+} \le 1$, $n_k = 0,1,2,...,n_k^+$, $n_k^+ = 1,2,3,...,n$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+ .

Now, we can obtain the probability function of H_k, but we must be careful to avoid the situation in which $N_k^+ = 0$. This can be done by conditioning on $N_k^+ > 0$.

Now, $N_k^+ = N_k + N_{k+1} + ...$, so if $N_k^+ = 0$, then $N_k = 0$.

Note that $N_k^+ \sim Binomial(n, p_1 + p_2)$. Therefore

$$P(N_k^+ = k) = \binom{n}{k} (p_1 + p_2)^k (1 - p_1 - p_2)^{n-k} \Rightarrow P(N_k^+ > 0) = 1 - (1 - p_1 - p_2)^n .$$

Thus,

$$P(H_k = \hat{h}_k \mid N_k^+ > 0) = \sum \frac{(1 - p_1 - p_2)^n}{1 - (1 - p_1 - p_2)^n} \binom{n}{n_k^+} \binom{n_k^+}{\hat{h}_k n_k^+} \left(\frac{p_1}{p_2}\right)^{\hat{h}_k n_k^+} \left(\frac{p_2}{1 - p_1 - p_2}\right)^{n_k^+} ,$$

where $0 \le \hat{h}_k = \dfrac{n_k}{n_k^+} \le 1$, $n_k = 0,1,2,...,n_k^+$, $n_k^+ = 1,2,3,...,n$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+ .

Hence,

$$P(H_k = \hat{h}_k \mid N_k^+ > 0) = \begin{cases} \binom{n}{n\hat{h}_1} p_1^{n\hat{h}_1} (1 - p_1)^{n - n\hat{h}_1} & \text{if } k = 1 \\ \sum \frac{(1 - p_1 - p_2)^n}{1 - (1 - p_1 - p_2)^n} \binom{n}{n_k^+} \binom{n_k^+}{\hat{h}_k n_k^+} \left(\frac{p_1}{p_2}\right)^{\hat{h}_k n_k^+} \left(\frac{p_2}{1 - p_1 - p_2}\right)^{n_k^+} & \text{if } k > 1 \end{cases} ,$$

where $0 \le \hat{h}_k = \dfrac{n_k}{n_k^+} \le 1$, $n_k = 0,1,2,...,n_k^+$, $n_k^+ = 1,2,3,...,n$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+ .

Note that $\hat{h}_k n_k^+ = n_k$, so $P(H_k = \hat{h}_k \mid N_k^+ > 0)$ can be written as

$$P\left(H_k = \frac{n_k}{n_k^+} \mid N_k^+ > 0\right) = \begin{cases} \binom{n}{n_1} p_1^{n_1}(1-p_1)^{n-n_1} & \text{if } k = 1 \\ \sum \frac{(1-p_1-p_2)^n}{1-(1-p_1-p_2)^n} \binom{n}{n_k^+}\binom{n_k^+}{n_k}\left(\frac{p_1}{p_2}\right)^{n_k}\left(\frac{p_2}{1-p_1-p_2}\right)^{n_k^+} & \text{if } k > 1 \end{cases},$$

where $p_1 = f(k) = P(K = k)$, $p_2 = R(k) = P(K > k)$,
$1 - p_1 - p_2 = F(k-1) = P(K \le k-1)$, $n_k = 0,1,2,\ldots,n_k^+$, $n_k^+ = 1,2,3,\ldots,n$ and
$\sum$ is taken over all the equivalent fractions of n_k / n_k^+ .

□.

A.2 Proof of Proposition 2.2:

$$E(H_k \mid N_k^+ > 0) = \sum_{n_k^+=1}^{n} \sum_{n_k=0}^{n_k^+} \frac{n_k}{n_k^+} P\left(H_k = \frac{n_k}{n_k^+} \mid N_k^+ > 0\right)$$

The distribution of $H_k \mid N_k^+ > 0$ is given in Proposition 2.1.

Now, for $k = 1$, clearly $nH_1 \sim Binomial(n, p_1) \Rightarrow E(H_1) = p_1$. Therefore

$$f(1) = P(H_1 = 1) = \binom{n}{n} p_1^n (1-p_1)^{n-n} = p_1 \text{ and } R(0) = 1 \Rightarrow E(H_1) = p_1 = \frac{f(1)}{R(0)}.$$

For $k > 1$,

$$\begin{aligned} E(H_k \mid N_k^+ > 0) &= \frac{(1-p_1-p_2)^n}{1-(1-p_1-p_2)^n} \sum_{n_k^+=1}^{n} \sum_{n_k=0}^{n_k^+} \frac{n_k}{n_k^+} \binom{n}{n_k^+}\binom{n_k^+}{n_k}\left(\frac{p_1}{p_2}\right)^{n_k}\left(\frac{p_2}{1-p_1-p_2}\right)^{n_k^+} \\ &= \frac{p_1(1-p_1-p_2)^n\left(\frac{1}{(1-p_1-p_2)^n} - 1\right)}{(1-(1-p_1-p_2)^n)(p_1+p_2)} = \frac{p_1}{p_1+p_2} \\ &= \frac{P(K=k)}{P(K=k)+P(K \ge k+1)} = \frac{P(K=k)}{P(K > k-1)} = \frac{f(k)}{R(k-1)}. \end{aligned}$$

Hence, $H_k \mid N_k^+ > 0$ is an unbiased estimator of $h(k)$.

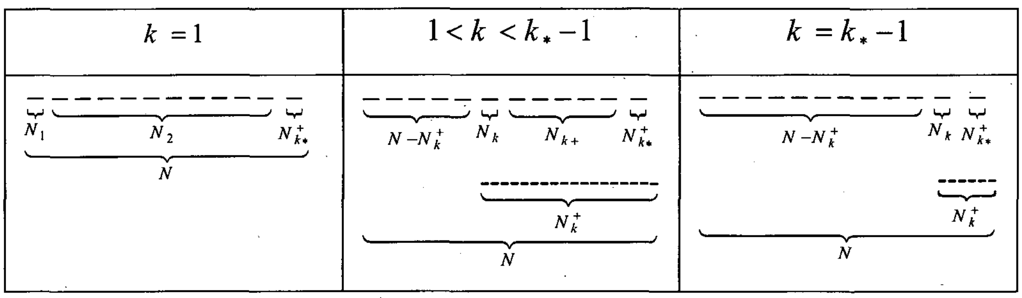
□.

A.3 Proof of Proposition 2.4:

First note that the empirical failure rate, $\hat{h}_k = \frac{n_k}{n_k^+}$, takes only rational numbers:

$\hat{h}_k = \frac{n_k}{n_k^+}$	n_k: 0	1	2	3	4	...	n-1
n_k^+: 1	0						
2	0	$1/2$					
3	0	$1/3$	$2/3$				
4	0	$1/4$	$2/4$	$3/4$			
...	0	...	...	...	...	...	
n-1	0	$1/(n-1)$	$2/(n-1)$	$3/(n-1)$	$4/(n-1)$	...	
n	0	$1/n$	$2/n$	$3/n$	$4/n$	...	$(n-1)/n$

The range of failure times can be partitioned as the following:



First, we will find the probability function of the EFR for the case of $k = 1$.

Let $p_1 = P(K = 1)$, $p_2 = P(K \geq k_*)$ and $1 - p_1 - p_2 = P(1 < K < k_*)$, where $k_* > 1$.

Then $(N_1, N_{k.}^+) \sim Trinomial(n, p_1, p_2)$ so that

$$P(N_1 = n_1, N_{k.}^+ = n_{k.}^+) = \frac{n! p_1^{n_1} p_2^{n_{k.}^+} (1 - p_1 - p_2)^{n - n_1 - n_{k.}^+}}{n_1! n_{k.}^+! (n - n_1 - n_{k.}^+)!},$$

where $n_1 + n_{k.}^+ \leq n$.

Now, let $\hat{h}_1 = \frac{n_1}{n}$ and $\hat{h}_{k.} = \frac{n_{k.}}{n_{k.}^+}$; $n_{k.}^+ = n_{k.} \Rightarrow n_1 = n\hat{h}_1$ and $n_{k.} = n_{k.}^+ \hat{h}_{k.}$, where

$0 \leq \hat{h}_1 = \frac{n_1}{n} < 1$ and $n_1 = 0, 1, 2, ..., n - 1$.

Note that $n_{k.}^+ = n_{k.}$ since there are no censored observations.

Thus,

$$P(H_1 = \hat{h}_1, H_{k.} = \hat{h}_{k.}) = \frac{n!\,p_1^{n\hat{h}_1} p_2^{n_{k.}^+ \hat{h}_{k*}} (1-p_1-p_2)^{n-n\hat{h}_1 - n_{k.}^+ \hat{h}_{k*}}}{(n\hat{h}_1)!(n_{k.}^+ \hat{h}_{k.})!(n - n\hat{h}_1 - n_{k.}^+ \hat{h}_{k.})!}.$$

Now, $\hat{h}_{k.} = \frac{n_{k.}}{n_{k.}^+}$, so $n_{k.}^+ H_{k.} \sim Binomial(n, p_2)$, where $p_2 = P(K \geq k_*)$.

$$P(H_{k.} = \hat{h}_{k.}) = \binom{n}{n_{k.}^+ \hat{h}_{k.}} p_2^{n_{k.}^+ \hat{h}_{k*}} (1-p_2)^{n - n_{k.}^+ \hat{h}_{k*}},$$

$$\Rightarrow P(H_{k.} = 1) = \sum_{n_{k*}^+ = 1}^{n} \frac{n!\,p_2^{n_{k*}^+} (1-p_2)^{n - n_{k*}^+}}{n_{k.}^+ !(n - n_{k.}^+)!} = 1 - (1-p_2)^n.$$

Thus, $P(K_{\max} = k_*) = \sum_{n_{k*}^+ = 1}^{n} \frac{n!\,p_2^{n_{k*}^+} (1-p_2)^{n - n_{k*}^+}}{n_{k.}^+ !(n - n_{k.}^+)!} = 1 - (1-p_2)^n$.

Hence,

$$P(H_1 = \hat{h}_1 \mid K_{\max} = k_*) = \frac{n!\,p_1^{n\hat{h}_1} ((1-p_1)^{n-n\hat{h}_1} - (1-p_1-p_2)^{n-n\hat{h}_1})}{(n\hat{h}_1)!(n-n\hat{h}_1)!(1-(1-p_2)^n)} \quad \text{if } k = 1,$$

where $0 \leq \hat{h}_1 = \frac{n_1}{n} < 1$ and $n_1 = 0, 1, 2, \ldots, n-1$.

Next, we will find the probability function of the EFR for the case of $1 < k < k_* - 1$.

Let $p_1 = P(K = k)$, $p_2 = P(K \geq k_*)$ and $p_3 = P(k < K < k_*)$, where $k_* > 1$.

Then $(N_k, N_{k+}, N_{k.}^+) \sim Multinomial(n, p_1, p_2, p_3)$ so that

$$P(N_k = n_k, N_{k+} = n_{k+}, N_{k.}^+ = n_{k.}^+) = \frac{n!\,p_1^{n_k} p_2^{n_{k*}^+} p_3^{n_{k+}} (1-p_1-p_2-p_3)^{n-n_k-n_{k+}-n_{k*}^+}}{n_k!\,n_{k.}^+!\,n_{k+}!\,(n-n_k-n_{k+}-n_{k.}^+)!},$$

where $n_k + n_{k+} + n_{k.}^+ \leq n$.

Now, let $\hat{h}_k = \frac{n_k}{n_k^+}$, $n_k^+ = n_k + n_{k+} + n_{k.}$ and $\hat{h}_{k.} = \frac{n_{k.}}{n_{k.}^+}$; $n_{k.}^+ = n_{k.}$

$$\Rightarrow n_k = n_k^+ \hat{h}_k,\ n_{k+} = n_k^+ - n_k^+ \hat{h}_k - n_{k.}^+ \hat{h}_{k.} \text{ and } n_{k.} = n_{k.}^+ \hat{h}_{k.},$$

where $0 \leq \hat{h}_k = \frac{n_k}{n_k^+} < 1$, $n_k^+ = 1, 2, \ldots, n$, $n_k = 0, 1, 2, \ldots, n_k^+ - 1$,

$$\hat{h}_{k.} = \frac{n_{k*}}{n_{k.}^+} = 1 \text{ and } n_{k.}^+ = n_{k.} = 1, 2, \ldots, n_k^+ - n_k.$$

Note that $n_{k*}^+ = n_{k*}$ since there are no censored observations.

So,

$$P(H_k = \hat{h}_k, N_k^+ = n_k^+, H_{k.} = \hat{h}_{k.}) = \frac{n!\,p_1^{n_k^+\hat{h}_k} p_2^{n_{k.}^+\hat{h}_{k.}} p_3^{n_k^+ - n_k^+\hat{h}_k - n_{k.}^+\hat{h}_{k.}} (1-p_1-p_2-p_3)^{n-n_k^+}}{(n_k^+\hat{h}_k)!(n_{k.}^+\hat{h}_{k.})!(n_k^+ - n_k^+\hat{h}_k - n_{k.}^+\hat{h}_{k.})!(n-n_k^+)!}.$$

Hence, if $1 < k < k_* - 1$

$$P(H_k = \hat{h}_k \mid K_{\max} = k_*) = \sum \frac{n!\,p_1^{n_k^+\hat{h}_k}((p_3+p_2)^{n_k^+ - n_k^+\hat{h}_k} - p_3^{n_k^+ - n_k^+\hat{h}_k})(1-p_1-p_2-p_3)^{n-n_k^+}}{(n_k^+\hat{h}_k)!(n_k^+ - n_k^+\hat{h}_k)!(n-n_k^+)!(1-(1-p_2)^n)},$$

where $0 \le \hat{h}_k = \dfrac{n_k}{n_k^+} < 1$, $n_k = 0,1,2,\ldots,n_k^+ - 1$, $n_k^+ = 1,2,\ldots,n$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+.

Now, we will find the probability function of the EFR for the case of $k = k_* - 1$.

Let $p_1 = P(K = k)$ and $p_2 = P(K \ge k_*)$, where $k_* > 1$.

Then $(N_k, N_{k.}^+) \sim Trinomial(n, p_1, p_2)$ so that

$$P(N_k = n_k, N_{k.}^+ = n_{k.}^+) = \frac{n!\,p_1^{n_k} p_2^{n_{k.}^+}(1-p_1-p_2)^{n-n_k-n_{k.}^+}}{n_k!\,n_{k.}^+!\,(n-n_k-n_{k.}^+)!},$$

where $n_k + n_{k.}^+ \le n$.

Now, let $\hat{h}_k = \dfrac{n_k}{n_k^+}$ and $\hat{h}_{k.} = \dfrac{n_{k.}}{n_k^+ - n_k}$; $n_{k.}^+ = n_{k.} = n_k^+ - n_k$

$\Rightarrow n_k = n_k^+\hat{h}_k$ and $n_{k.} = n_k^+\hat{h}_{k.}(1-\hat{h}_k)$,

where $0 \le \hat{h}_k = \dfrac{n_k}{n_k^+} < 1$, $n_k^+ = 1,2,\ldots,n$ and $n_k = 0,1,2,\ldots,n_k^+ - 1$.

Note that $n_{k.}^+ = n_{k.}$ since there are no censored observations.

So,

$$P(H_k = \hat{h}_k, H_{k.} = \hat{h}_{k.}) = \sum \frac{n!\,p_1^{n_k^+\hat{h}_k} p_2^{n_k^+\hat{h}_{k.}(1-\hat{h}_k)}(1-p_1-p_2)^{n-n_k^+\hat{h}_k - n_k^+\hat{h}_{k.}(1-\hat{h}_k)}}{(n_k^+\hat{h}_k)!(n_k^+\hat{h}_{k.}(1-\hat{h}_k))!(n-n_k^+\hat{h}_k - n_k^+\hat{h}_{k.}(1-\hat{h}_k))!}.$$

Hence,

$$P(H_k = \hat{h}_k \mid K_{\max} = k_*) = \sum \frac{n!\, p_1^{n_k^+ \hat{h}_k}\, p_2^{n_k^+ - n_k^+ \hat{h}_k} (1-p_1-p_2)^{n-n_k^+}}{(n_k^+ \hat{h}_k)!(n_k^+ - n_k^+ \hat{h}_k)!(n-n_k^+)!(1-(1-p_2)^n)} \quad \text{if } k = k_* - 1,$$

where $0 \le \hat{h}_k = \frac{n_k}{n_k^+} < 1$, $n_k = 0,1,2,\ldots,n_k^+ - 1$, $n_k^+ = 1,2,\ldots,n$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+ .

Therefore,

$$P(H_k = \hat{h}_k \mid K_{\max} = k_*) = \begin{cases} \dfrac{n!\, p_1^{n\hat{h}_1} ((1-p_1)^{n-n\hat{h}_1} - (1-p_1-p_2)^{n-n\hat{h}_1})}{(nh_1)!(n-nh_1)!(1-(1-p_2)^n)} & \text{if } k = 1 \\ \sum \dfrac{n!\, p_1^{n_k^+ \hat{h}_k} ((p_3+p_2)^{n_k^+ - n_k^+ \hat{h}_k} - p_3^{n_k^+ - n_k^+ \hat{h}_k}) p_4^{n-n_k^+}}{(n_k^+ \hat{h}_k)!(n_k^+ - n_k^+ \hat{h}_k)!(n-n_k^+)!(1-(1-p_2)^n)} & \text{if } 1 < k < k_* - 1 \\ \sum \dfrac{n!\, p_1^{n_k^+ \hat{h}_k}\, p_2^{n_k^+ - n_k^+ \hat{h}_k} (1-p_1-p_2)^{n-n_k^+}}{(n_k^+ \hat{h}_k)!(n_k^+ - n_k^+ \hat{h}_k)!(n-n_k^+)!(1-(1-p_2)^n)} & \text{if } k = k_* - 1 \end{cases},$$

where $n_k = 0,1,2,\ldots,n_k^+ - 1$, $n_k^+ = 1,2,\ldots,n$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+.

Note that $\hat{h}_k n_k^+ = n_k$, so $P(H_k = \hat{h}_k \mid K_{\max} = k_*)$ can be written as

$$P(H_k = \frac{n_k}{n_k^+} \mid K_{\max} = k_*) = \begin{cases} \dfrac{n!\, p_1^{n_1} ((1-p_1)^{n-n_1} - (1-p_1-p_2)^{n-n_1})}{n_1!(n-n_1)!(1-(1-p_2)^n)} & \text{if } k = 1 \\ \sum \dfrac{n!\, p_1^{n_k} ((p_3+p_2)^{n_k^+ - n_k} - p_3^{n_k^+ - n_k}) p_4^{n-n_k^+}}{n_k!(n_k^+ - n_k)!(n-n_k^+)!(1-(1-p_2)^n)} & \text{if } 1 < k < k_* - 1 \\ \sum \dfrac{n!\, p_1^{n_k}\, p_2^{n_k^+ - n_k} (1-p_1-p_2)^{n-n_k^+}}{n_k!(n_k^+ - n_k)!(n-n_k^+)!(1-(1-p_2)^n)} & \text{if } k = k_* - 1 \end{cases},$$

where $n_k = 0,1,2,\ldots,n_k^+ - 1$, $n_k^+ = 1,2,\ldots,n$, $p_1 = P(K = k)$, $p_2 = P(K \ge k_*)$,

$p_3 = P(k < K < k_*)$, $p_4 = 1 - p_1 - p_2 - p_3$ and

$\sum$ is taken over all the equivalent fractions of n_k / n_k^+ .

□.

A.4 Proof of Proposition 2.5:

$$E(H_k \mid K_{\max} = k_*) = \sum_{n_k^+ = 1}^{n} \sum_{n_k = 0}^{n_k^+} \frac{n_k}{n_k^+} P(H_k = \frac{n_k}{n_k^+} \mid K_{\max} = k_*)$$

The distribution of $H_k \mid K_{\max} = k_*$ is given in Proposition 2.4.

57 to 112 will have the same distributions as tick marks 1 to 56 but for $n = 100$ and the same will be for $n = 150$ (113-168), $n = 200$ (169-224), $n = 250$ (225-280) and $n = 300$ (281-336).

Table 5.6: Values of p and λ associated with each tick mark for $n = 50$.

Tick Mark	$G(p)$ and $NB(p, z = 0.2, 2)$					$P(\lambda)$				
	X_1	X_2	X_3	X_4	U	X_1	X_2	X_3	X_4	U
1	0.1	0.1	0.1	0.1	0.1	1	1	1	1	1
2	0.1	0.1	0.15	0.15	0.1	1	1	1.5	1.5	1
3	0.1	0.1	0.2	0.2	0.1	1	1	2	2	1
4	0.1	0.1	0.25	0.25	0.1	1	1	2.5	2.5	1
5	0.1	0.1	0.3	0.3	0.1	1	1	3	3	1
6	0.1	0.1	0.4	0.4	0.1	1	1	4	4	1
7	0.1	0.15	0.15	0.15	0.15	1	1.5	1.5	1.5	1.5
8	0.1	0.15	0.2	0.2	0.15	1	1.5	2	2	1.5
9	0.1	0.15	0.25	0.25	0.15	1	1.5	2.5	2.5	1.5
10	0.1	0.15	0.3	0.3	0.15	1	1.5	3	3	1.5
11	0.1	0.15	0.4	0.4	0.15	1	1.5	4	4	1.5
12	0.1	0.2	0.2	0.2	0.2	1	2	2	2	2
13	0.1	0.2	0.25	0.25	0.2	1	2	2.5	2.5	2
14	0.1	0.2	0.3	0.3	0.2	1	2	3	3	2
15	0.1	0.2	0.4	0.4	0.2	1	2	4	4	2
16	0.1	0.25	0.25	0.25	0.25	1	2.5	2.5	2.5	2.5
17	0.1	0.25	0.3	0.3	0.25	1	2.5	3	3	2.5
18	0.1	0.25	0.4	0.4	0.25	1	2.5	4	4	2.5
19	0.1	0.3	0.3	0.3	0.3	1	3	3	3	3
20	0.1	0.3	0.4	0.4	0.3	1	3	4	4	3
21	0.1	0.4	0.4	0.4	0.4	1	4	4	4	4
22	0.15	0.15	0.15	0.15	0.15	1.5	1.5	1.5	1.5	1.5
23	0.15	0.15	0.2	0.2	0.15	1.5	1.5	2	2	1.5
24	0.15	0.15	0.25	0.25	0.15	1.5	1.5	2.5	2.5	1.5
25	0.15	0.15	0.3	0.3	0.15	1.5	1.5	3	3	1.5
26	0.15	0.15	0.4	0.4	0.15	1.5	1.5	4	4	1.5
27	0.15	0.2	0.2	0.2	0.2	1.5	2	2	2	2
28	0.15	0.2	0.25	0.25	0.2	1.5	2	2.5	2.5	2
29	0.15	0.2	0.3	0.3	0.2	1.5	2	3	3	2
30	0.15	0.2	0.4	0.4	0.2	1.5	2	4	4	2
31	0.15	0.25	0.25	0.25	0.25	1.5	2.5	2.5	2.5	2.5
32	0.15	0.25	0.3	0.3	0.25	1.5	2.5	3	3	2.5
33	0.15	0.25	0.4	0.4	0.25	1.5	2.5	4	4	2.5
34	0.15	0.3	0.3	0.3	0.3	1.5	3	3	3	3
35	0.15	0.3	0.4	0.4	0.3	1.5	3	4	4	3
36	0.15	0.4	0.4	0.4	0.4	1.5	4	4	4	4

So,

$$E(H_k \mid K_{\max}=k_*)=\begin{cases}\displaystyle\sum_{n_1=0}^{n-1}\frac{(n-1)!\,p_1^{n_1}((1-p_1)^{n-n_1}-(1-p_1-p_2)^{n-n_1})}{(n_1-1)!(n-n_1)!(1-(1-p_2)^n)} & \text{if } k=1\\ \displaystyle\sum_{n_k^+=1}^{n}\sum_{n_k=0}^{n_k^+-1}\frac{n!\,p_1^{n_k}((p_3+p_2)^{n_k^+-n_k}-p_3^{n_k^+-n_k})p_4^{\,n-n_k^+}}{n_k^+(n_k-1)!(n_k^+-n_k)!(n-n_k^+)!(1-(1-p_2)^n)} & \text{if } 1<k<k_*-1\\ \displaystyle\sum_{n_k^+=1}^{n}\sum_{n_k=0}^{n_k^+-1}\frac{n!\,p_1^{n_k}p_2^{n_k^+-n_k}(1-p_1-p_2)^{n-n_k^+}}{n_k^+(n_k-1)!(n_k^+-n_k)!(n-n_k^+)!(1-(1-p_2)^n)} & \text{if } k=k_*-1\end{cases}$$

$$=\begin{cases}\dfrac{p_1(1-(1-p_2)^{n-1})}{1-(1-p_2)^n} & \text{if } k=1\\ \dfrac{p_1}{p_1+p_2+p_3}-\dfrac{p_1p_2((1-p_2)^n-(1-p_1-p_2-p_3)^n)}{(1-(1-p_2)^n)(p_1+p_3)(p_1+p_2+p_3)} & \text{if } 1<k<k_*-1,\\ \dfrac{p_1}{p_1+p_2}-\dfrac{p_2((1-p_2)^n-(1-p_1-p_2)^n)}{(p_1+p_2)(1-(1-p_2)^n)} & \text{if } k=k_*-1\end{cases}$$

where $n_k=0,1,2,\ldots,n_k^+\text{-}1.$, $n_k^+=1,2,\ldots,n$, $p_1=P(K=k)$, $p_2=P(K\geq k_*)$,
$p_3=P(k<K<k_*)$ and $p_4=1\text{-}p_1\text{-}p_2\text{-}p_3$.

Hence, $H_k \mid K_{\max}=k_*$ is not an unbiased estimator of $h(k)$.

□.

Cont. Table 5.6: Values of p and λ associated with each tick mark for $n = 50$.

Tick Mark	$G(p)$ and $NB(p, z = 0.2, 2)$					$P(\lambda)$				
	X_1	X_2	X_3	X_4	U	X_1	X_2	X_3	X_4	U
37	0.2	0.2	0.2	0.2	0.2	2	2	2	2	2
38	0.2	0.2	0.25	0.25	0.2	2	2	2.5	2.5	2
39	0.2	0.2	0.3	0.3	0.2	2	2	3	3	2
40	0.2	0.2	0.4	0.4	0.2	2	2	4	4	2
41	0.2	0.25	0.25	0.25	0.25	2	2.5	2.5	2.5	2.5
42	0.2	0.25	0.3	0.3	0.25	2	2.5	3	3	2.5
43	0.2	0.25	0.4	0.4	0.25	2	2.5	4	4	2.5
44	0.2	0.3	0.3	0.3	0.3	2	3	3	3	3
45	0.2	0.3	0.4	0.4	0.3	2	3	4	4	3
46	0.2	0.4	0.4	0.4	0.4	2	4	4	4	4
47	0.25	0.25	0.25	0.25	0.25	2.5	2.5	2.5	2.5	2.5
48	0.25	0.25	0.3	0.3	0.25	2.5	2.5	3	3	2.5
49	0.25	0.25	0.4	0.4	0.25	2.5	2.5	4	4	2.5
50	0.25	0.3	0.3	0.3	0.3	2.5	3	3	3	3
51	0.25	0.3	0.4	0.4	0.3	2.5	3	4	4	3
52	0.25	0.4	0.4	0.4	0.4	2.5	4	4	4	4
53	0.3	0.3	0.3	0.3	0.3	3	3	3	3	3
54	0.3	0.3	0.4	0.4	0.3	3	3	4	4	3
55	0.3	0.4	0.4	0.4	0.4	3	4	4	4	4
56	0.4	0.4	0.4	0.4	0.4	4	4	4	4	4

Figures E.1 and E.2 show that $\tilde{W}$ tends to be very conservative in absolute terms and relative to the other two tests with exceptions occurring when samples based on $NB(p, z = 2)$ where p is relatively small and on Poisson where $\lambda = 4$. The estimated significances that are a little high occur in particular with failure modes that ware generated based on sampling from $NB(p, z = 2)$ with $p = 0.1$ for X_1, $p = 0.1$ for X_2 and $p = 0.1, 0.15$ for X_3 with $n = 200, 400, 600, 900$ for data with no censoring and $p = 0.1$ for X_1, $p = 0.1$ for X_2, $p = 0.1, 0.15, 0.2$ for X_3 and $p = 0.1, 0.15, 0.2$ for X_4 with $n = 100, 200, 400, 600, 900$ for data with censoring. In those situations W was either not computable for all the 1000 simulated samples or not computable for a very large percentage with very large estimated significance. Also, for all those situations χ^2_P has a large estimated significance but of course far smaller than those of W and $\tilde{W}$. The estimated significances for $\tilde{W}$ that are a little high occur also in particular when the failure modes are generated based on sampling from the Poisson distribution

Glossary

Terms that apply throughout the thesis:

Reliability	The ability to preserve the quality of performance of a subject under given conditions for a given period of time.
$R(k)$	Reliability Function, the probability that a subject is still functioning beyond time k.
$f(k)$	The probability density (or mass) function.
FR	Failure Rate, the number of failures experienced for a subject divided by the number of subjects at risk at a given time.
$h(k)$	FR function, the instantaneous rate of a subject failure at a specific time given that it did not fail before (continuous). The probability of a subject failure at time k given that it did not fail before (discrete).
IFR	Increasing Failure Rate.
DFR	Decreasing Failure Rate.
SRF	Second Rate of Failure, an alternative definition for the discrete FR which has some properties in common with a continuous FR.
$s(k)$	SRF function at time k.
MLE	Maximum Likelihood Estimator.
n	The sample size.

with $\lambda = 4$ for X_1, X_2 and X_3 with $n = 400, 600, 900$ for which the estimated significance of χ_P^2 was also a little high. W tends to be conservative in most of the situations but generally its estimated significance is larger than those of $\tilde{W}$. The test based on χ_P^2 is liberal for most of the simulated samples. Figure 5.1 gives an illustration of the negative binomial based situations.

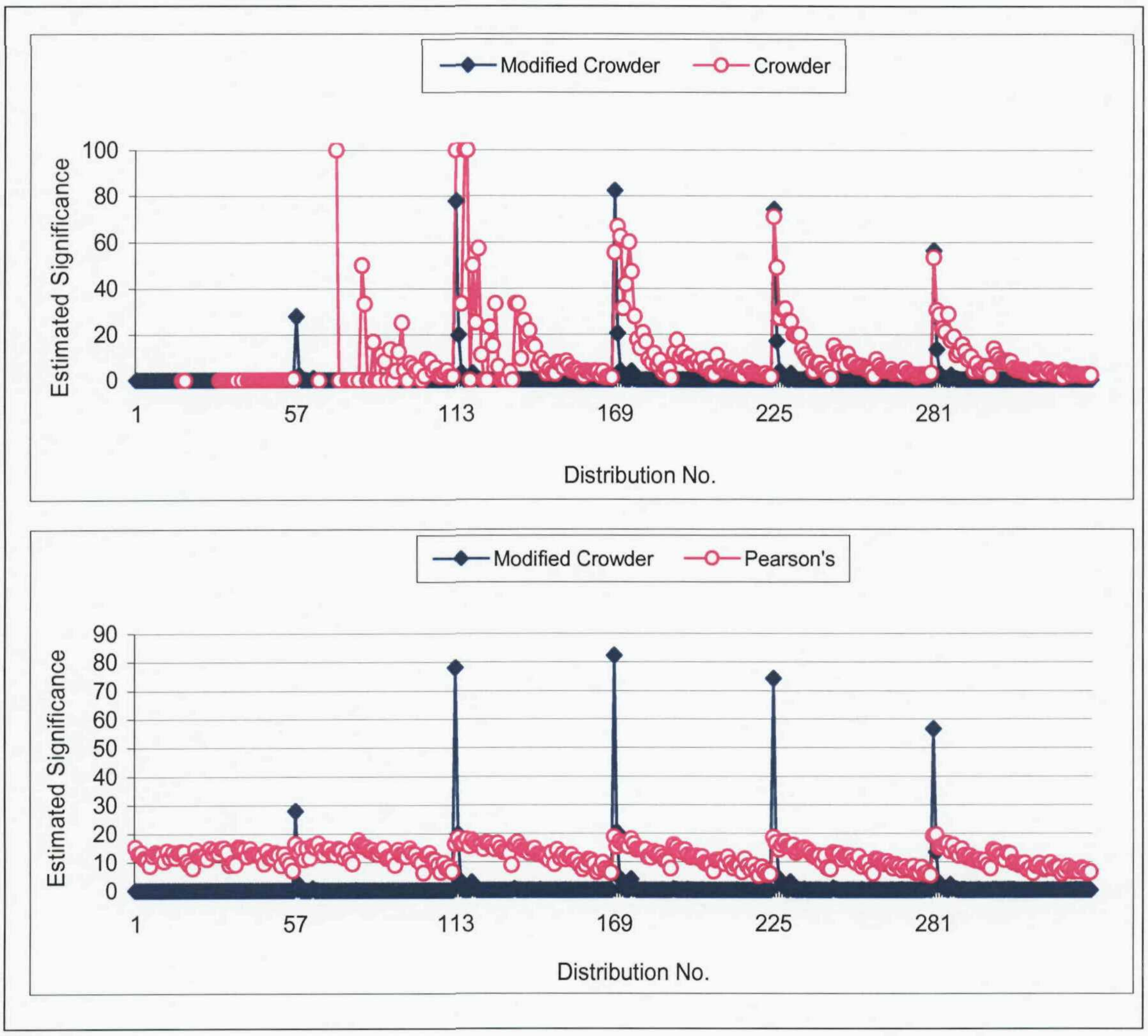


Figure 5.1: The estimated significances of $\tilde{W}$ compared to W and χ_P^2 at the nominal 5% significance level for samples with no censoring from the negative binomial distribution with parameters $p = 0.1,\ 0.15, 0.2, 0.25, 0.3, 0.4$ and $z = 2$ for sample sizes 50 (plotted on the x-axis 1-56), 100 (57-112), 200 (113-168), 400 (169-224), 600 (225-280) and 900 (281-336).

$G(p)$	Geometric distribution with parameter p.
$NB(p,z)$	Negative Binomial distribution with parameters p and z.
$P(\lambda)$	Poisson distribution with parameter λ.

Terms that apply in Chapter 2:

n_k	The number of subjects failed at time k.
n_k^+	The number of subjects at risk at time k.
$\hat{h}_k$	Empirical Failure Rate (EFR).
$\hat{h}_{sk}$	An exponentially smoothed version of the EFR.
k_*	The highest observed failure time in a sample.
FRCC	Failure Rate Control Chart, a new tool for detecting departure from a constant FR.
FRCC1	A FRCC applied to the values of the EFR using the probability function of $H_k \mid N_k^+ > 0$.
FRCC1s	A FRCC applied to the exponentially smoothed version of the EFR values using the probability function of $H_k \mid N_k^+ > 0$.
FRCC2	A FRCC applied to the values of the EFR using the probability function of $H_k \mid K_{\max} = k_*$.

FRCC2s	A FRCC applied to the exponentially smoothed version of the EFR values using the probability function of $H_k \mid K_{\max} = k_*$.
χ_P^2	Pearson's chi-square test statistic.
χ_D^2	Likelihood ratio test statistic.

Terms that apply in Chapter 3 onward:

Competing risks	A common situation in reliability and other areas involving lifetime data in which there is more than one failure mode or type.
Latent failure time	The potential failure time associated with each risk.
g	The number of risks.
g_{a_v}	The number of single causes of failure that occur simultaneously to form a_v.
d	The number of multiple cause of failure.
c , b	The cause, mode or type of failure configuration (single or multiple) unless stated otherwise.
a , a_v , a_u	The multiple cause of failure.
$f(k,c)$	The sub-density function.
$h_{kc} = h(k,c)$	The sub-failure rate function.

Table 5.7: Summary for the estimated significances for samples with no censoring.

Test	$G(p)$		$NB(p,z=0.2)$		$NB(p,z=2)$		$P(\lambda)$	
	Min < 0.05 Max	%	Min < 0.05 Max	%	Min < 0.05 Max	%	Min < 0.05 Max	%
W	0	5.1	0	1.2	0	16.2	0	11.9
	< 0.05	96.7	< 0.05	96.7	< 0.05	55.2	< 0.05	100
	10.3	0.3	6	0.3	100	1.4	4.4	0.3
$\tilde{W}$	0	86.3	0	13.1	0	66.4	0	23.5
	< 0.05	100	< 0.05	100	< 0.05	97.3	< 0.05	99.1
	0.4	0.3	0.9	0.3	82.3	0.3	7.7	0.3
χ_P^2	3.8	0.3	2.9	0.3	5.1	0.3	3.2	0.3
	< 0.05	8.3	< 0.05	42.3	< 0.05	0	< 0.05	14.3
	14.4	0.6	7.5	0.3	19.4	0.3	10.5	0.3

Table 5.8: Summary for the estimated significances for samples with censoring.

Test	$G(p)$		$NB(p,z=0.2)$		$NB(p,z=2)$		$P(\lambda)$	
	Min < 0.05 Max	%	Min < 0.05 Max	%	Min < 0.05 Max	%	Min < 0.05 Max	%
W	0	5.4	0	1.2	0	15.1	0	11.9
	< 0.05	96.7	< 0.05	87.2	< 0.05	44.3	< 0.05	99.7
	11.1	0.3	6.1	0.3	100	1.5	5.1	0.3
$\tilde{W}$	0	78.9	0	1.5	0	50.6	0	21.1
	< 0.05	100	< 0.05	100	< 0.05	94.6	< 0.05	98.8
	0.6	0.3	3	0.3	91.7	0.3	10.8	0.3
χ_P^2	3.7	0.3	3	0.3	3.3	0.3	3.5	0.6
	< 0.05	13.1	< 0.05	41.4	< 0.05	0.6	< 0.05	14
	14.4	0.3	6.9	0.9	20.5	0.3	10.8	0.3

n_{kc}	The number of subjects failed from cause c at time k.
n_{kr}	The number of subjects censored or removed at time k.
$n_{k\bullet}^{-}$	The overall number subjects failed at time k.
$n_{k\bullet}^{+}$	The number of subjects at risk at time k.
$\hat{h}_{kc}$	The nonparametric MLE for the sub-failure rate function h_{kc}.
$\hat{h}_{k}$	The nonparametric MLE for the marginal failure rate function $h(k)$.
$\bar{h}_{kc}$	The sub-conditional odds function.
$\hat{\bar{h}}_{kc}$	The nonparametric MLE for the sub-conditional odds function $\bar{h}_{kc}$.
W	Crowder's test statistic.
$\tilde{W}$	A modified version of Crowder's test statistic.
χ_{Y}^{2}	Yates' chi-square test statistic.
F	Fisher's exact test.

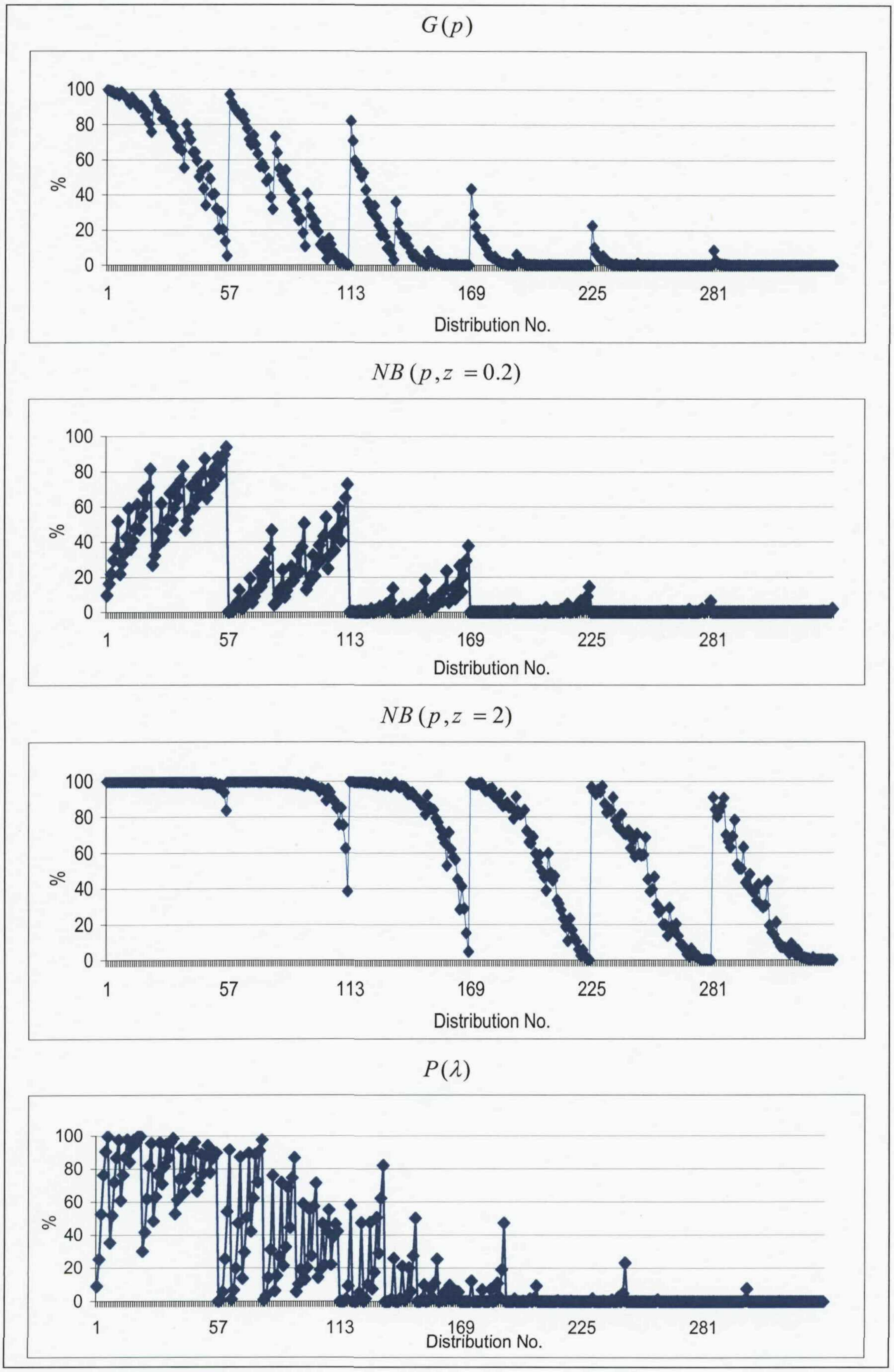


Figure 5.2: The percentages of samples with no censoring for which W is not computable for sample sizes 50 (1-56), … and 900 (281-336).

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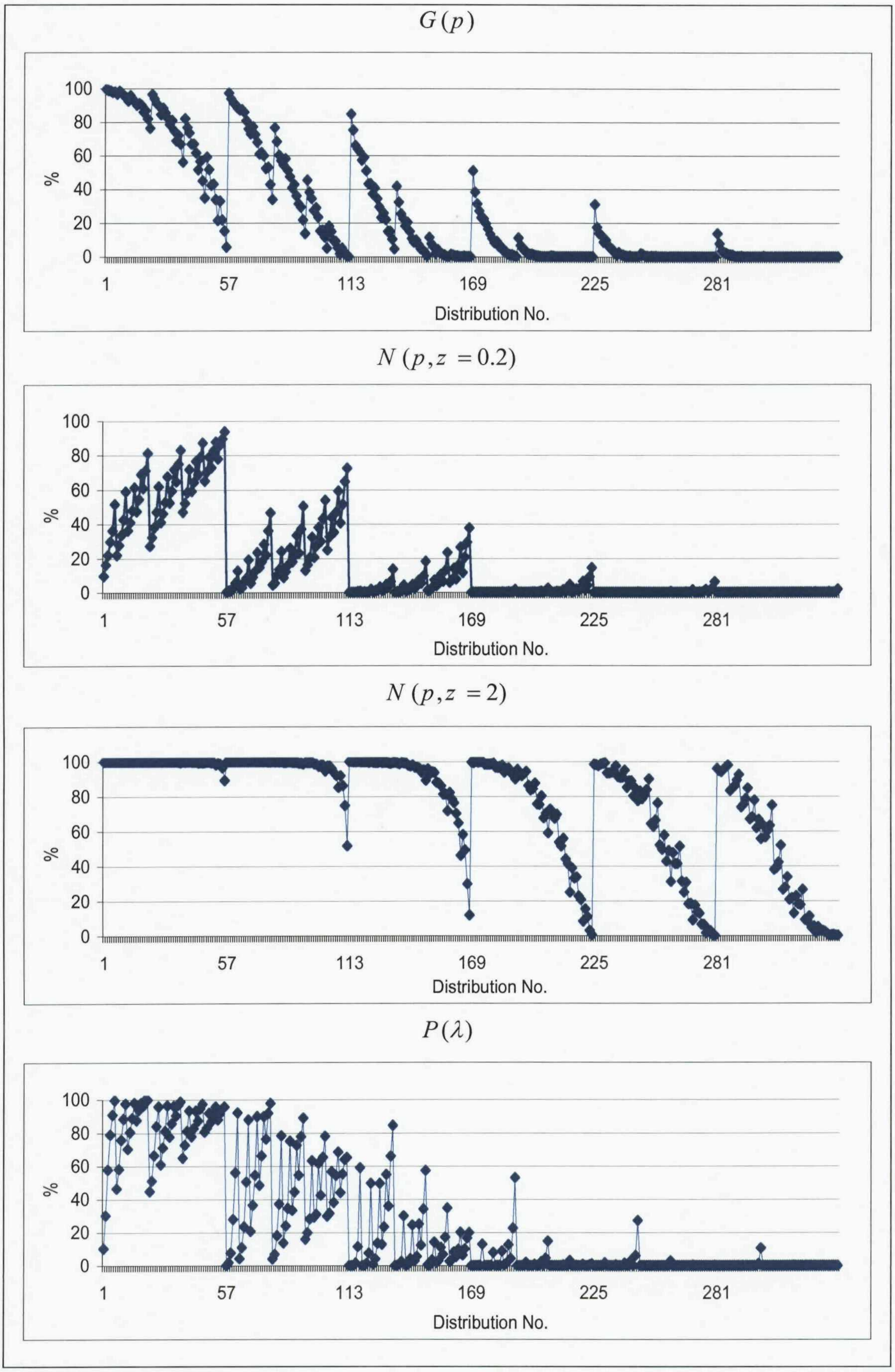

Figure 5.3: The percentages of samples with censoring for which W is not computable for sample sizes 50 (1-56), … and 900 (281-336).

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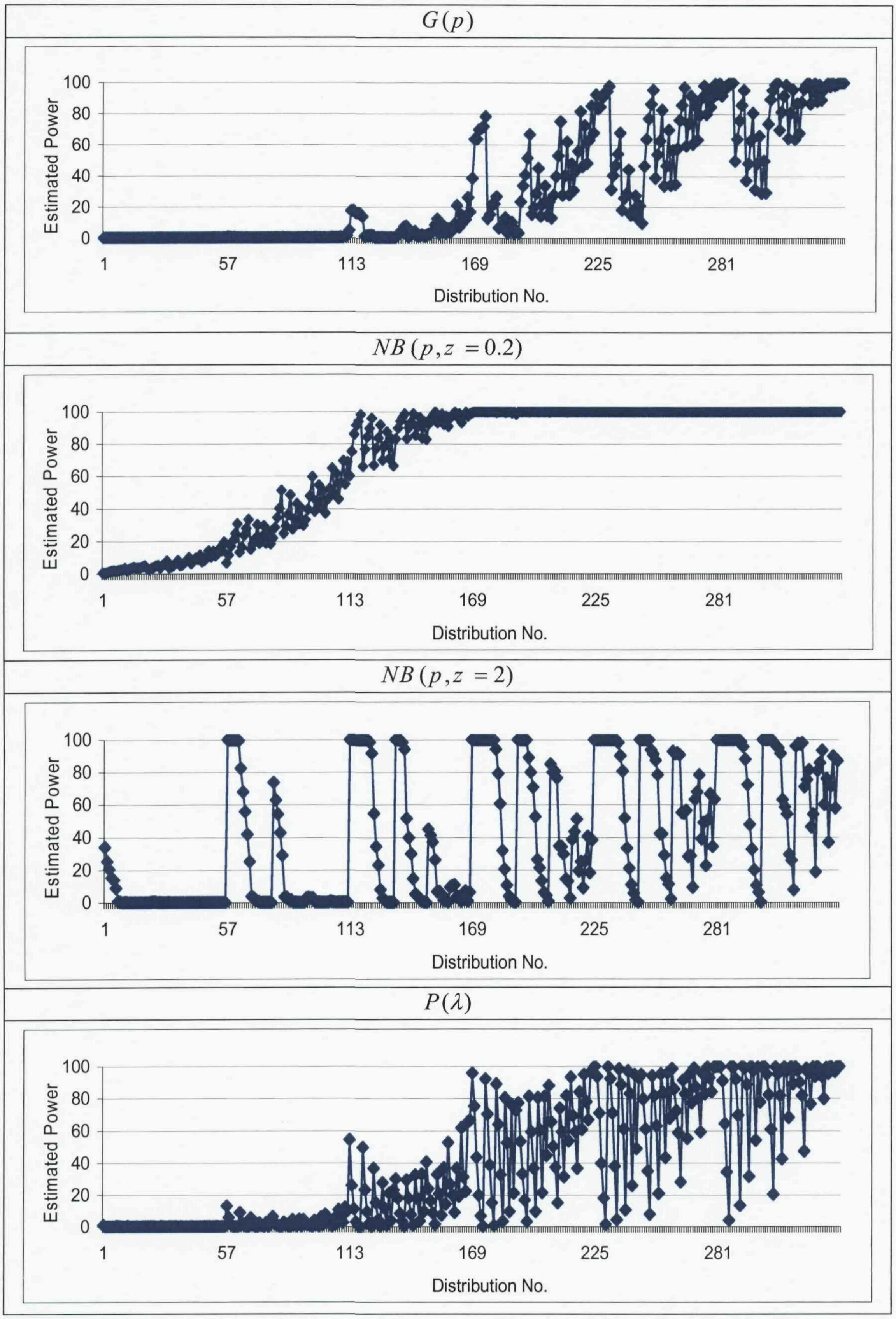


Figure 5.4: The estimated power of $\tilde{W}$ at a nominal 5% significance level for samples with no censoring and two dependent risks for $n = 50$ (plotted on the x-axis 1-56), 100 (57-112), 200 (113-168), 400 (169-224), 600 (225-280) and 900 (281-336).

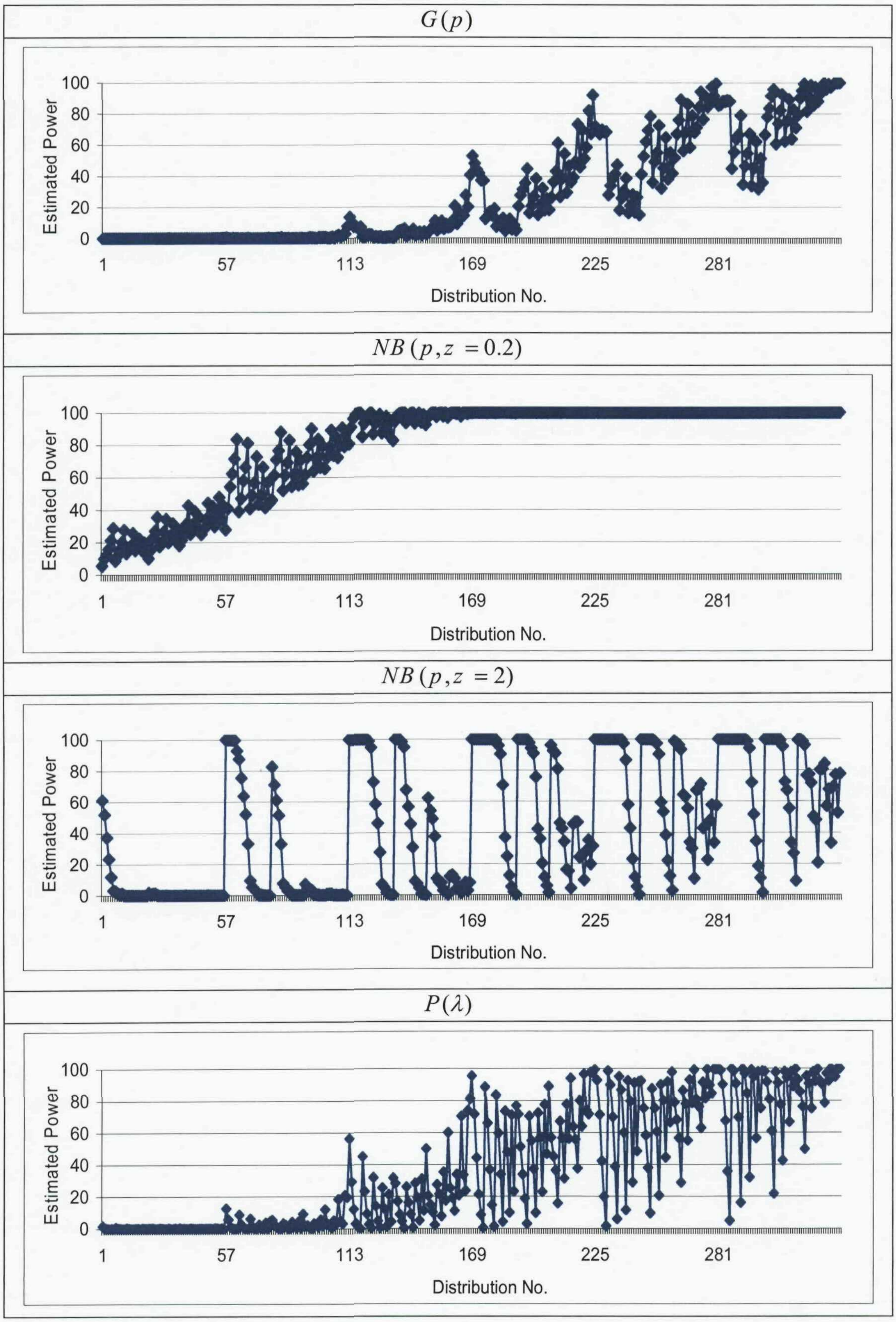


Figure 5.5: The estimated power of $\tilde{W}$ at a nominal 5% significance level for samples with censoring and two dependent risks for $n = 50$ (plotted on the x-axis 1-56), 100 (57-112), 200 (113-168), 400 (169-224), 600 (225-280) and 900 (281-336).

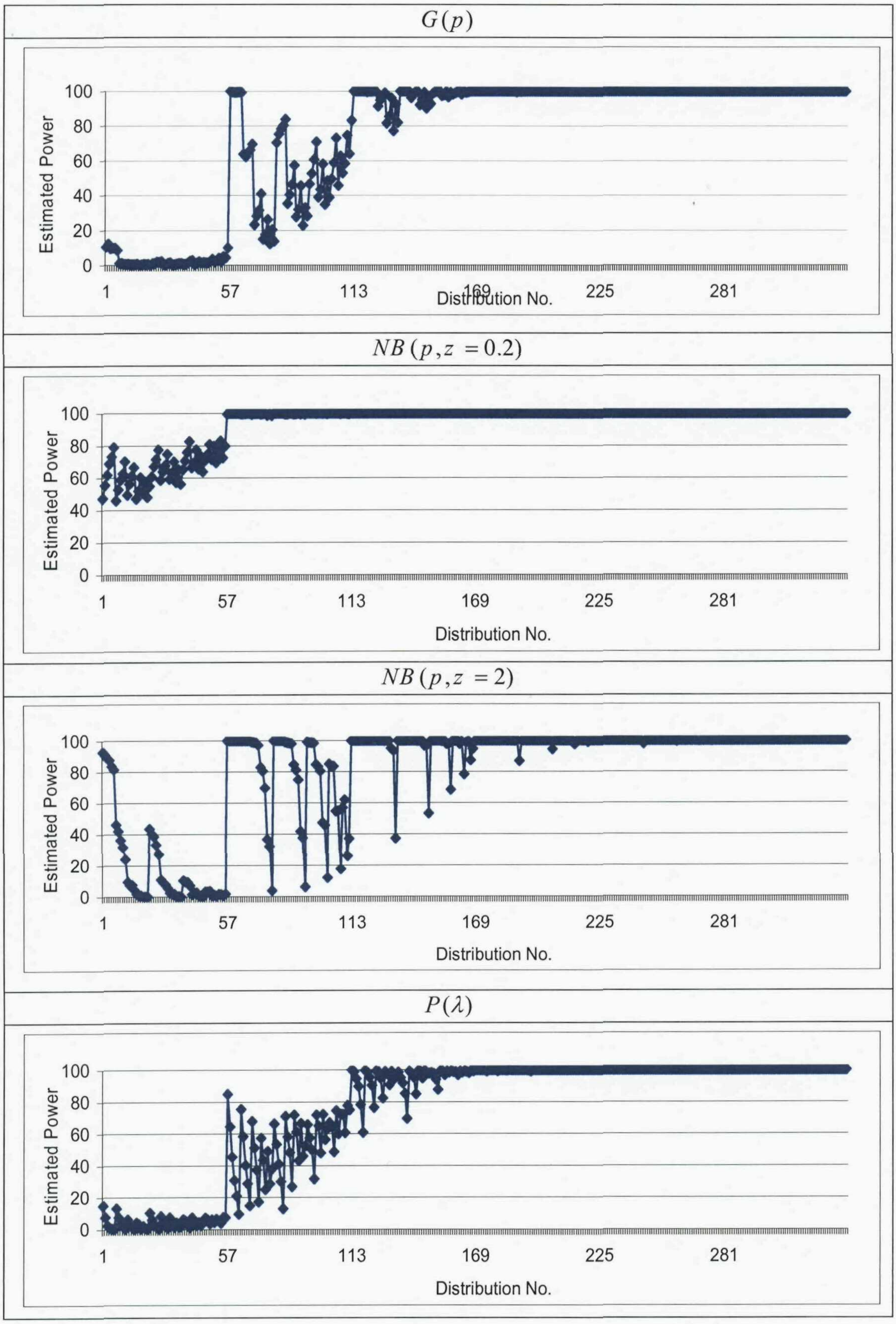


Figure 5.6: The estimated power of $\tilde{W}$ at a nominal 5% significance level for samples with no censoring and all the risks are dependent for $n = 50$ (plotted on the x-axis 1-56), 100 (57-112), 200 (113-168), 400 (169-224), 600 (225-280) and 900 (281-336).

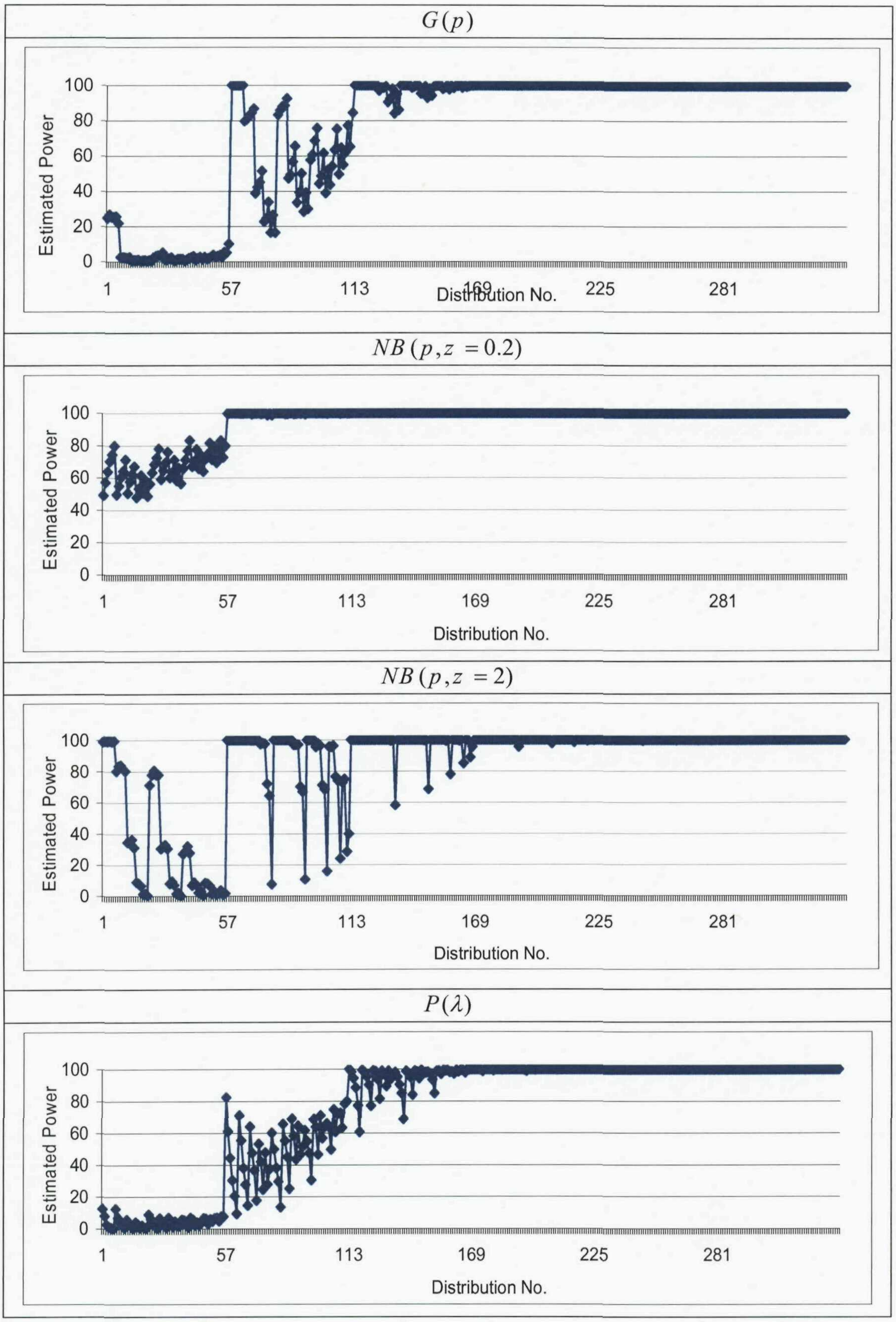
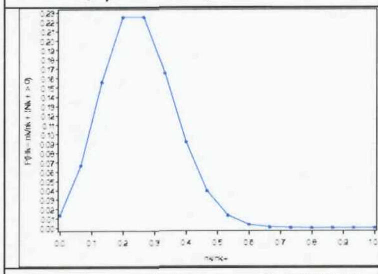
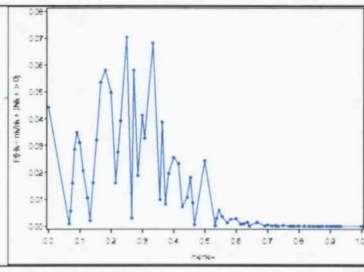
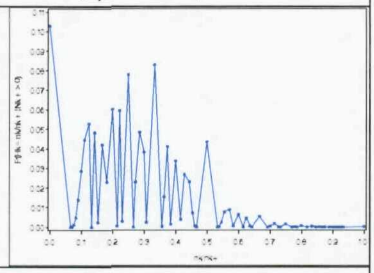
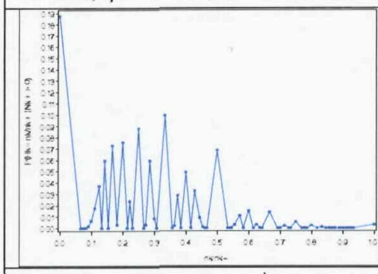
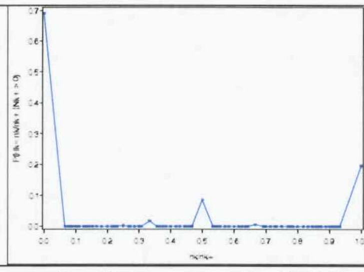
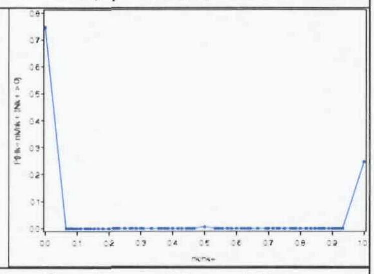


Figure 5.7: The estimated power of $\tilde{W}$ at a nominal 5% significance level for samples with censoring and all the risks are dependent for $n = 50$ (plotted on the x-axis 1-56), 100 (57-112), 200 (113-168), 400 (169-224), 600 (225-280) and 900 (281-336).

Appendix B:

Plots for the probability function of $H_k = n_k / n_k^+ \mid N_k^+ > 0$ where $\mu = E(H_k \mid N_k^+ > 0)$ and $\sigma = STD(H_k \mid N_k^+ > 0)$.

$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the geometric(0.25) distribution with n=15

$k=1$, μ=0.25 , σ=0.11180	$k=2$, μ=0.25 , σ=0.13071	$k=3$, μ=0.25 , σ=0.15377
		
$k=4$, μ=0.25 , σ=0.18276	$k=12$, μ=0.25 , σ=0.40029	$k=22$, μ=0.25 , σ=0.43121
		

$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the geometric(0.25) distribution with n=25

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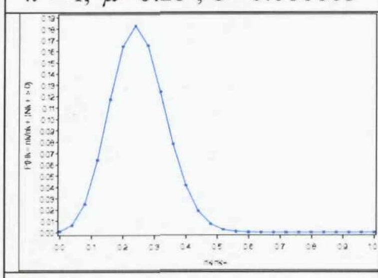	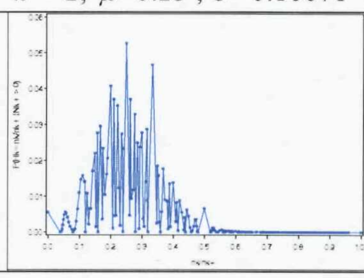	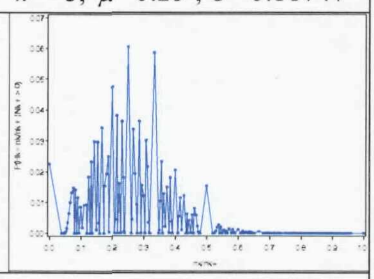
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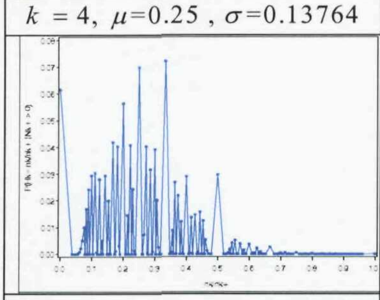
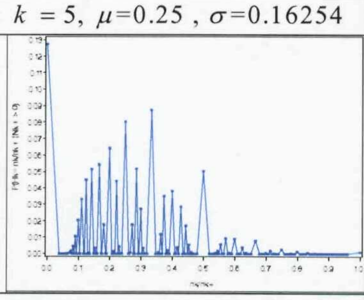
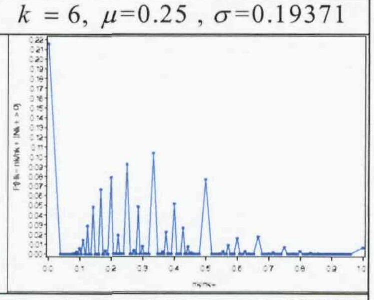
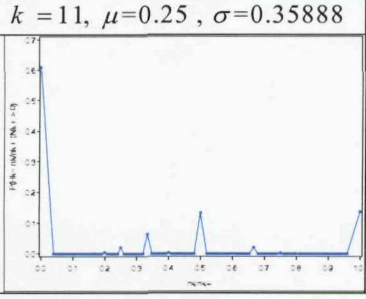
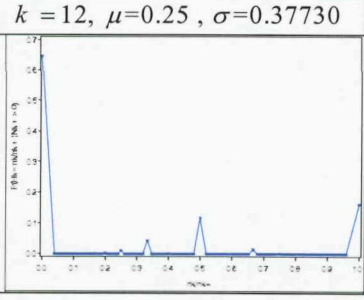
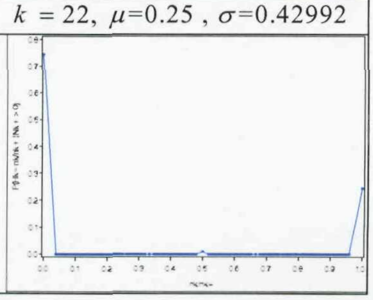
		
$k=11$, μ=0.25 , σ=0.35888	$k=12$, μ=0.25 , σ=0.37730	$k=22$, μ=0.25 , σ=0.42992
		

$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the geometric(0.25) distribution with n=50		
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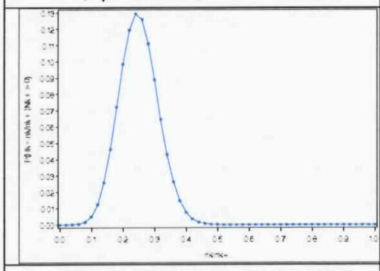
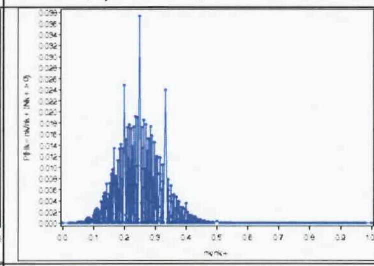
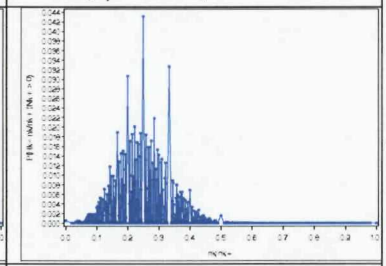
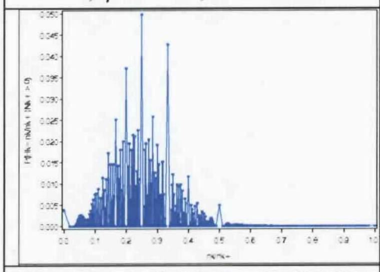
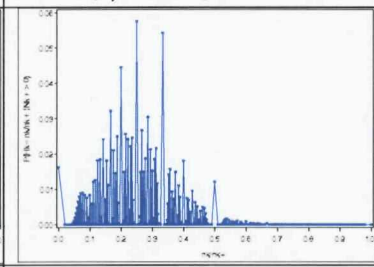
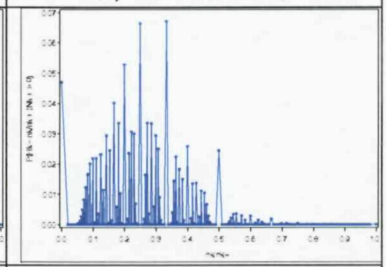
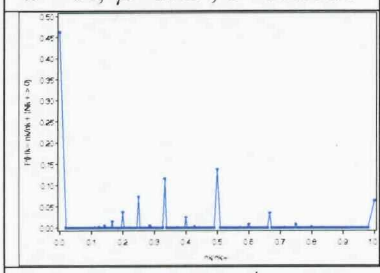
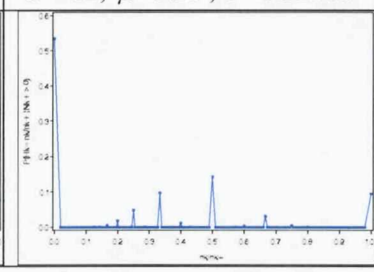
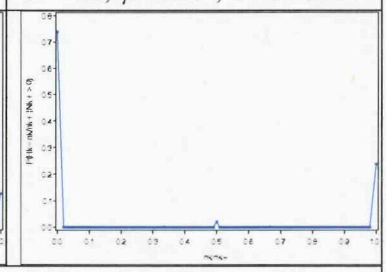
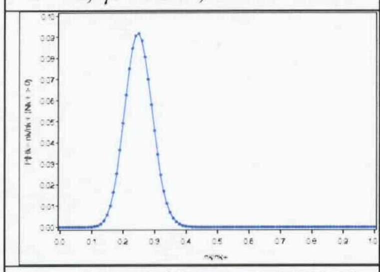
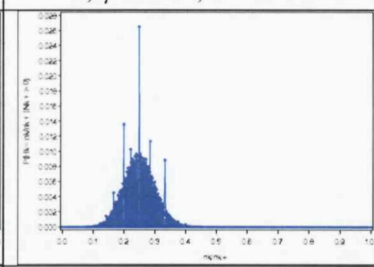
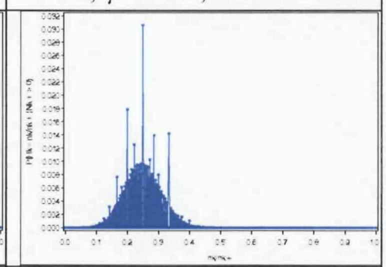
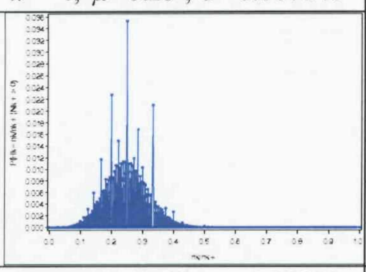
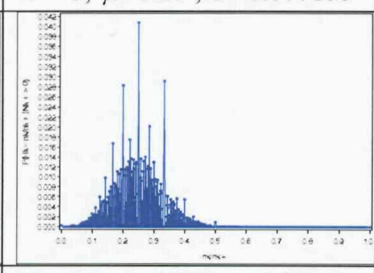
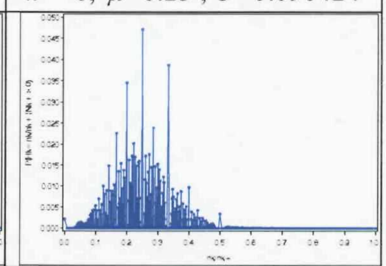
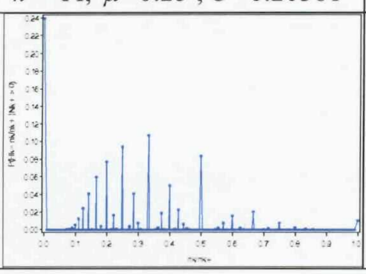
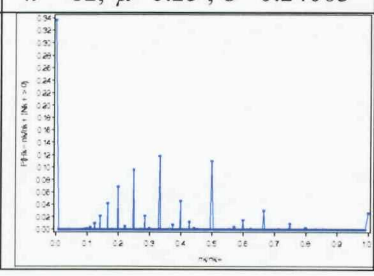
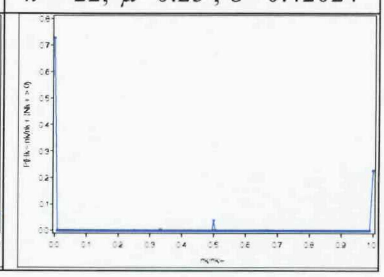
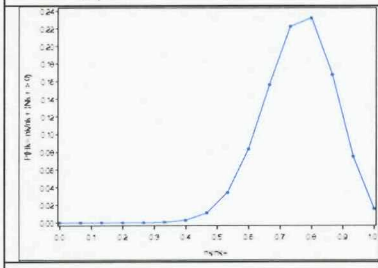
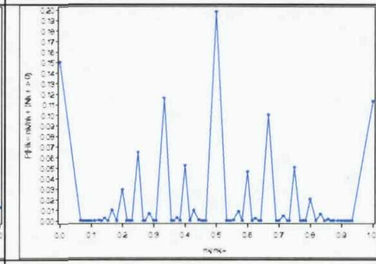
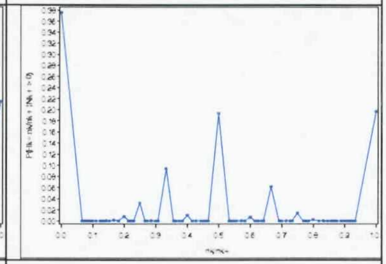
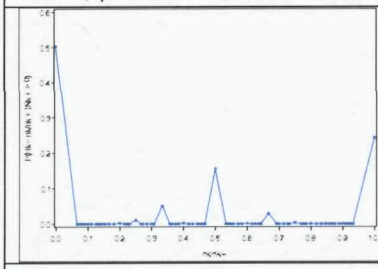
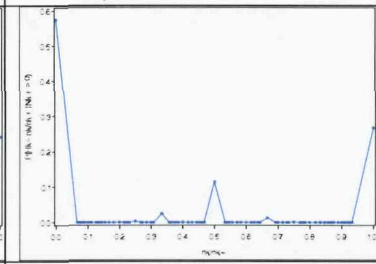
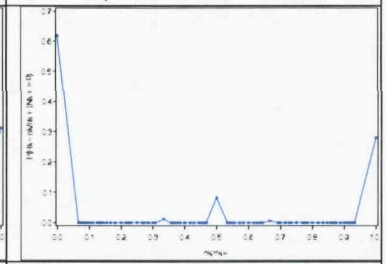
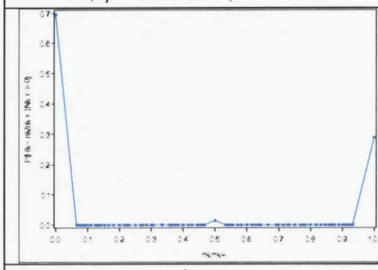
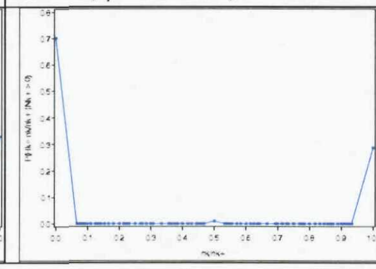
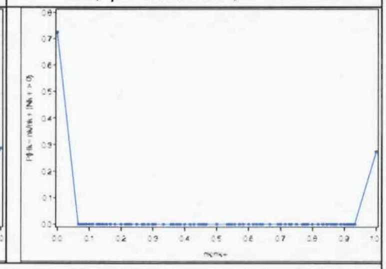
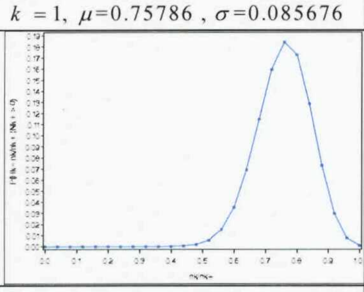
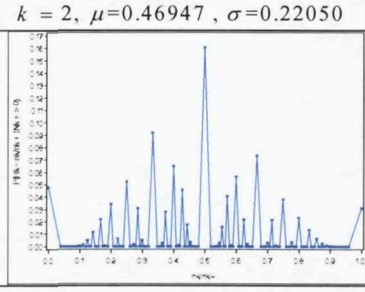
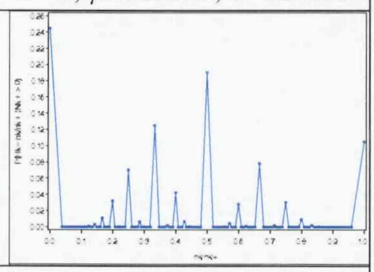
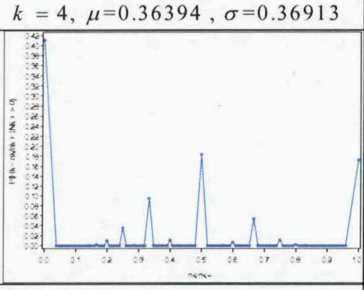
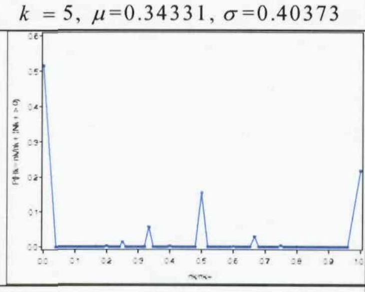
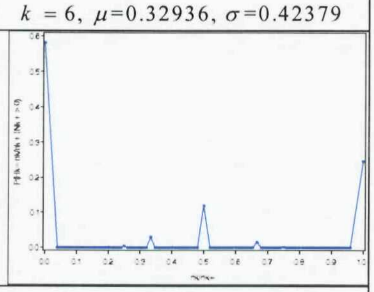
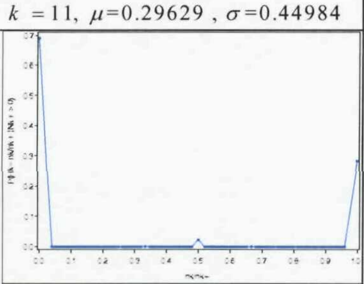
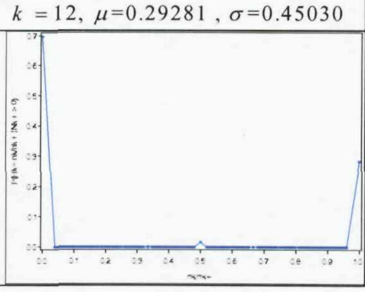
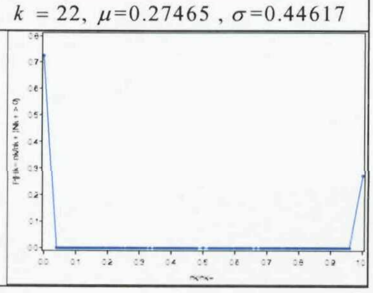
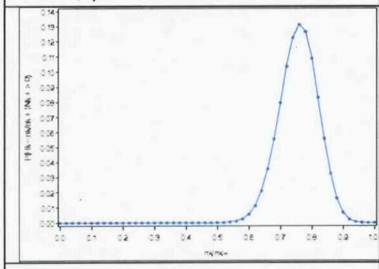
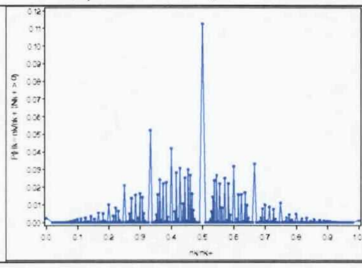
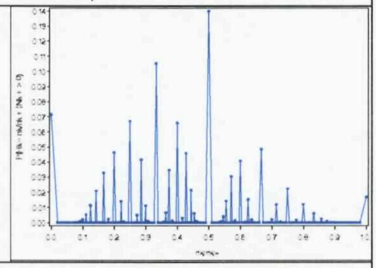
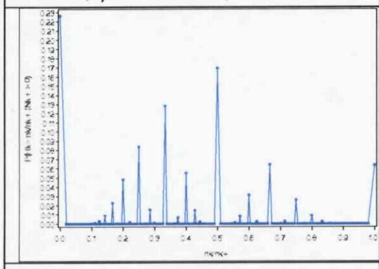
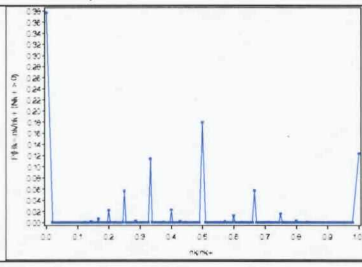
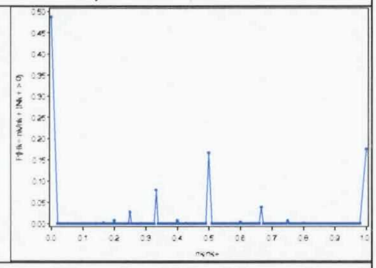
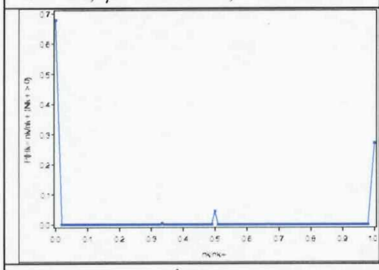
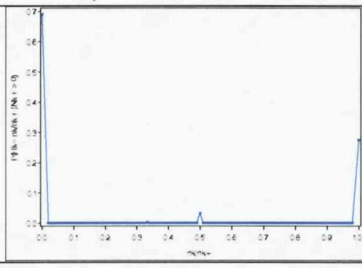
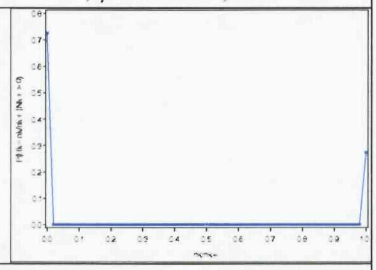
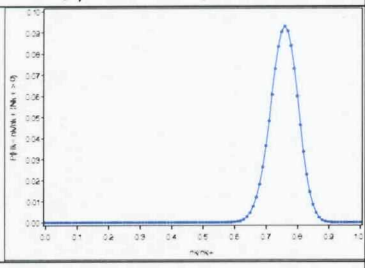
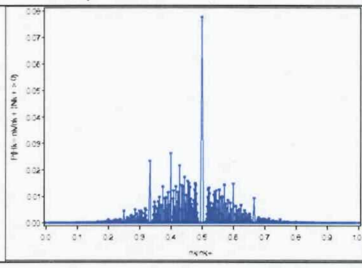
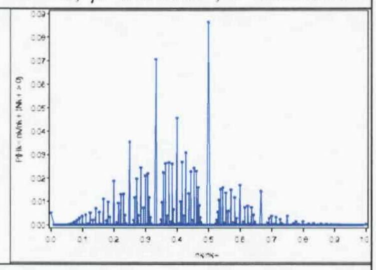
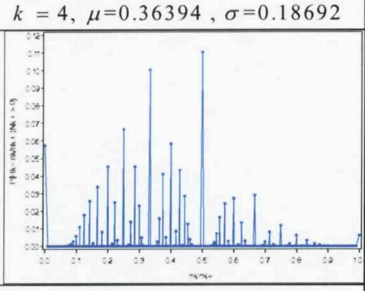
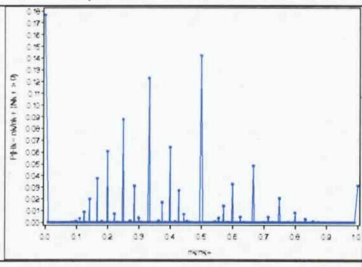
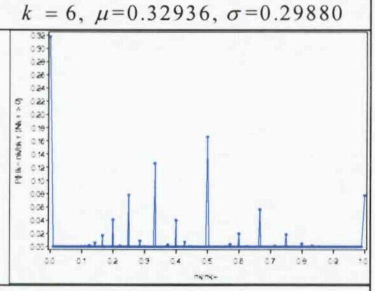
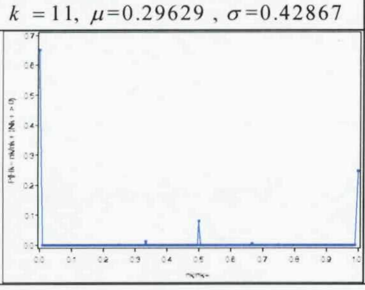
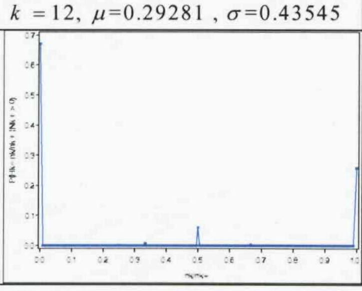
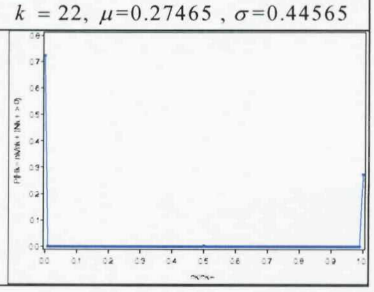
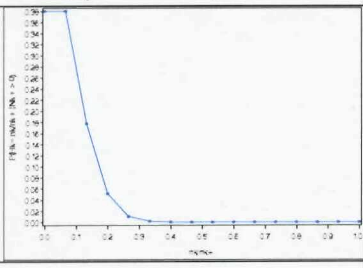
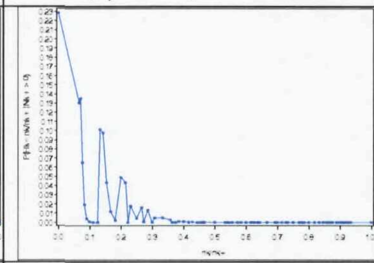
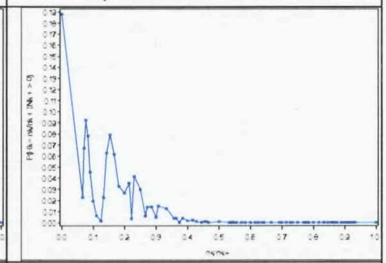
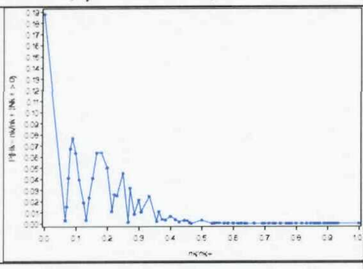
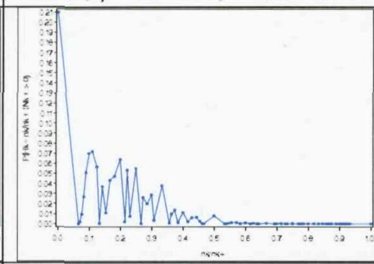
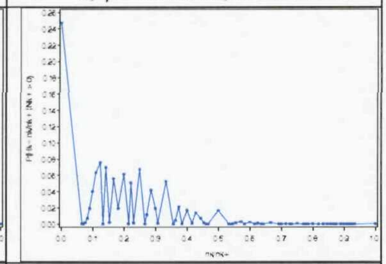
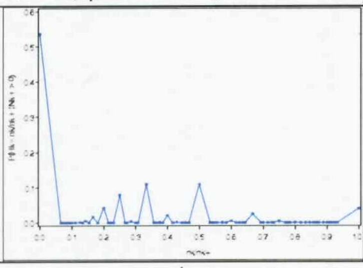
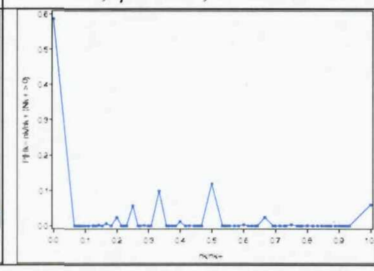
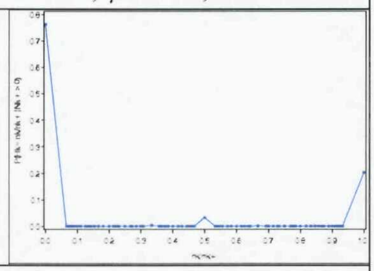
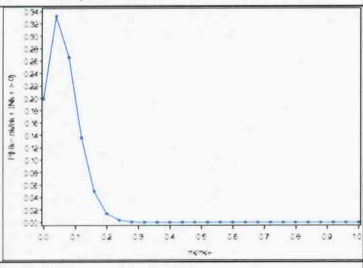
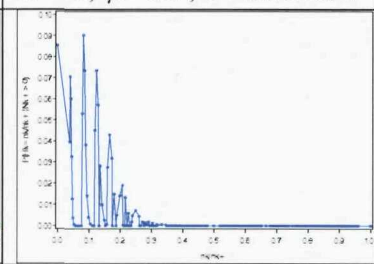
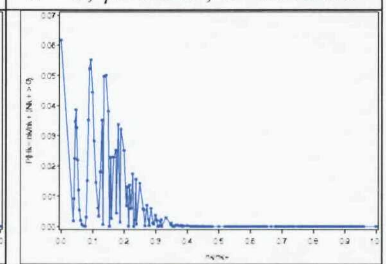
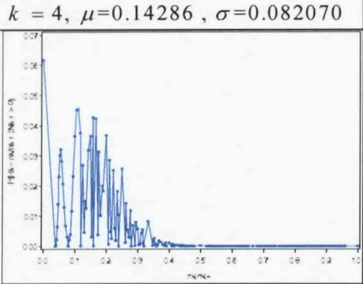
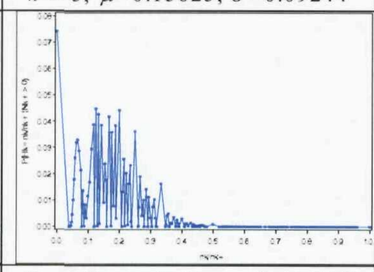
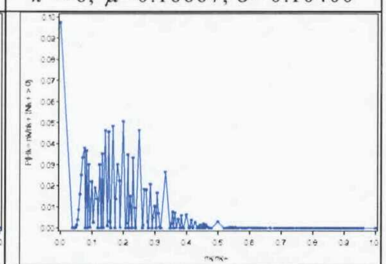
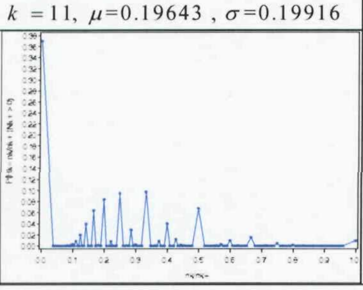
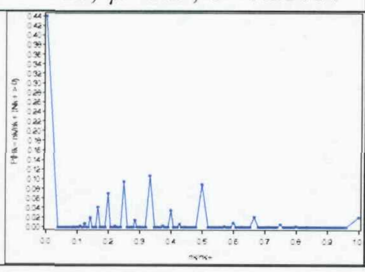
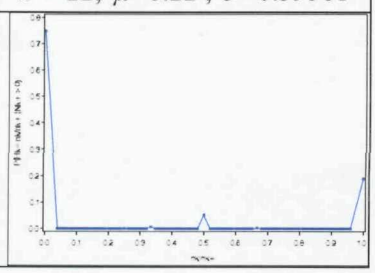
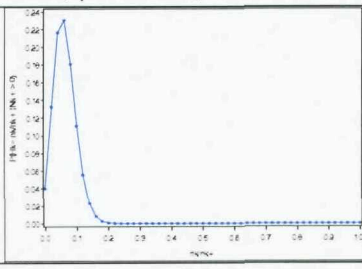
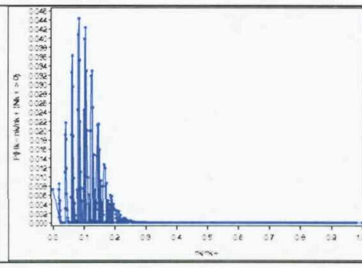
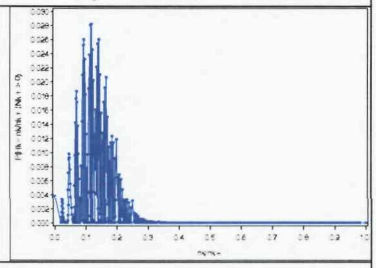
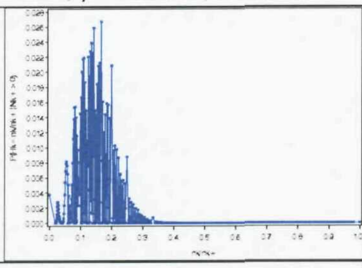
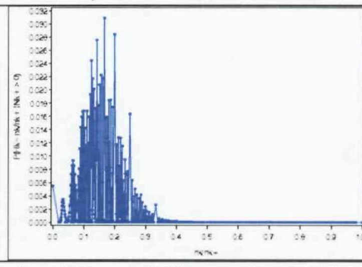
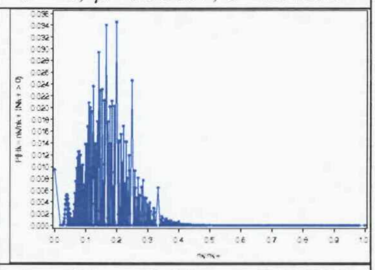
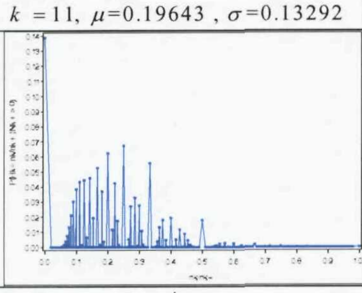
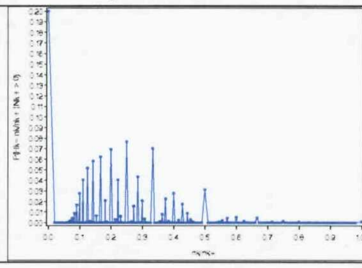
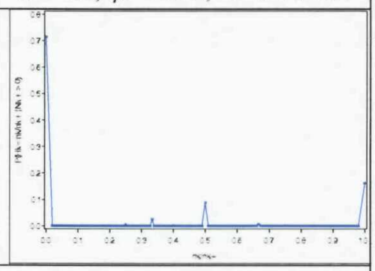
		
$k=4$, μ=0.25 , σ=0.095672	$k=5$, μ=0.25 , σ=0.11149	$k=6$, μ=0.25 , σ=0.13046
		
$k=11$, μ=0.25 , σ=0.29167	$k=12$, μ=0.25 , σ=0.32348	$k=22$, μ=0.25 , σ=0.42669
		
$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the geometric(0.25) distribution with n=100		
$k=1$, μ=0.25 , σ=0.043301	$k=2$, μ=0.25 , σ=0.050085	$k=3$, μ=0.25 , σ=0.057965
		
$k=4$, μ=0.25 , σ=0.067140	$k=5$, μ=0.25 , σ=0.077855	$k=6$, μ=0.25 , σ=0.090424
		
$k=11$, μ=0.25 , σ=0.20381	$k=12$, μ=0.25 , σ=0.24063	$k=22$, μ=0.25 , σ=0.42024
		

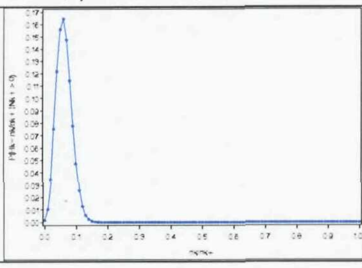
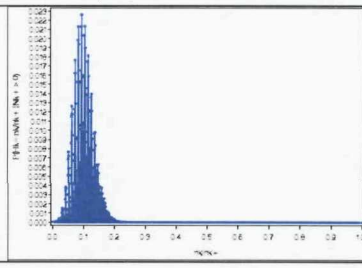
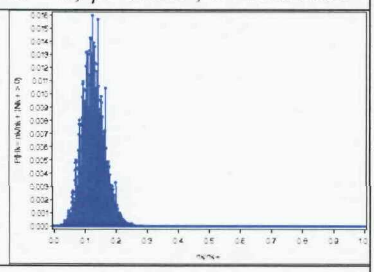
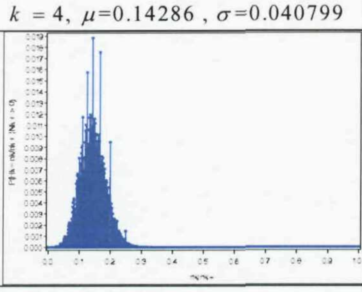
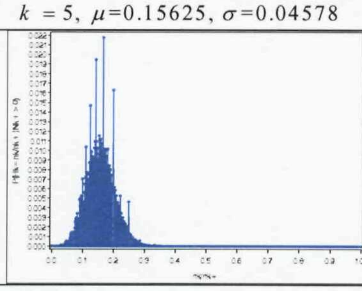
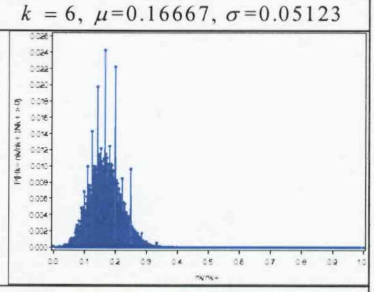
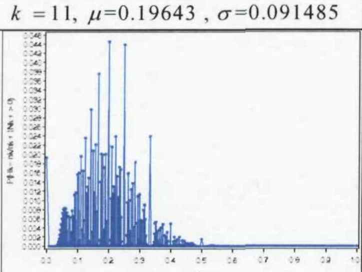
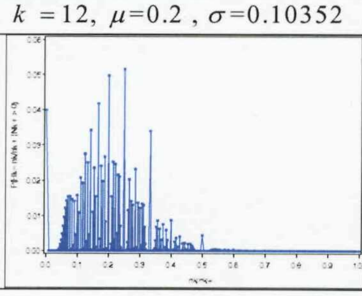
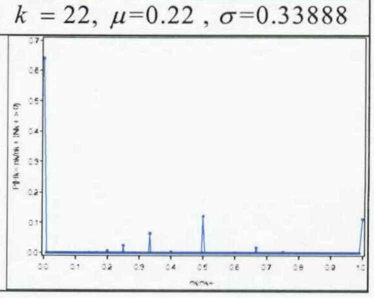
$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,0.2) distribution with n=15		
$k=1$, μ=0.75786 , σ=0.11061	$k=2$, μ=0.46947 , σ=0.29469	$k=3$, μ=0.39821 , σ=0.37601
		
$k=4$, μ=0.36394 , σ=0.41392	$k=5$, μ=0.34331, σ=0.43293	$k=6$, μ=0.32936, σ=0.44294
		
$k=11$, μ=0.29629 , σ=0.45267	$k=12$, μ=0.29281 , σ=0.45228	$k=22$, μ=0.27465 , σ=0.44624
		
$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,0.2) distribution with n=25		
$k=1$, μ=0.75786 , σ=0.085676	$k=2$, μ=0.46947 , σ=0.22050	$k=3$, μ=0.39821 , σ=0.31005
		
$k=4$, μ=0.36394 , σ=0.36913	$k=5$, μ=0.34331, σ=0.40373	$k=6$, μ=0.32936, σ=0.42379
		
$k=11$, μ=0.29629 , σ=0.44984	$k=12$, μ=0.29281 , σ=0.45030	$k=22$, μ=0.27465 , σ=0.44617
		

$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,0.2) distribution with n=50		
$k = 1,\ \mu=0.75786\ ,\ \sigma=0.060582$	$k = 2,\ \mu=0.46947\ ,\ \sigma=0.14868$	$k = 3,\ \mu=0.39821\ ,\ \sigma=0.21136$
		
$k = 4,\ \mu=0.36394\ ,\ \sigma=0.27932$	$k = 5,\ \mu=0.34331,\ \sigma=0.33703$	$k = 6,\ \mu=0.32936,\ \sigma=0.37762$
		
$k = 11,\ \mu=0.29629\ ,\ \sigma=0.44277$	$k = 12,\ \mu=0.29281\ ,\ \sigma=0.44535$	$k = 22,\ \mu=0.27465\ ,\ \sigma=0.44600$
		
$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,0.2) distribution with n=100		
$k = 1,\ \mu=0.75786\ ,\ \sigma=0.042838$	$k = 2,\ \mu=0.46947\ ,\ \sigma=0.10312$	$k = 3,\ \mu=0.39821\ ,\ \sigma=0.14200$
		
$k = 4,\ \mu=0.36394\ ,\ \sigma=0.18692$	$k = 5,\ \mu=0.34331,\ \sigma=0.24218$	$k = 6,\ \mu=0.32936,\ \sigma=0.29880$
		
$k = 11,\ \mu=0.29629\ ,\ \sigma=0.42867$	$k = 12,\ \mu=0.29281\ ,\ \sigma=0.43545$	$k = 22,\ \mu=0.27465\ ,\ \sigma=0.44565$
		

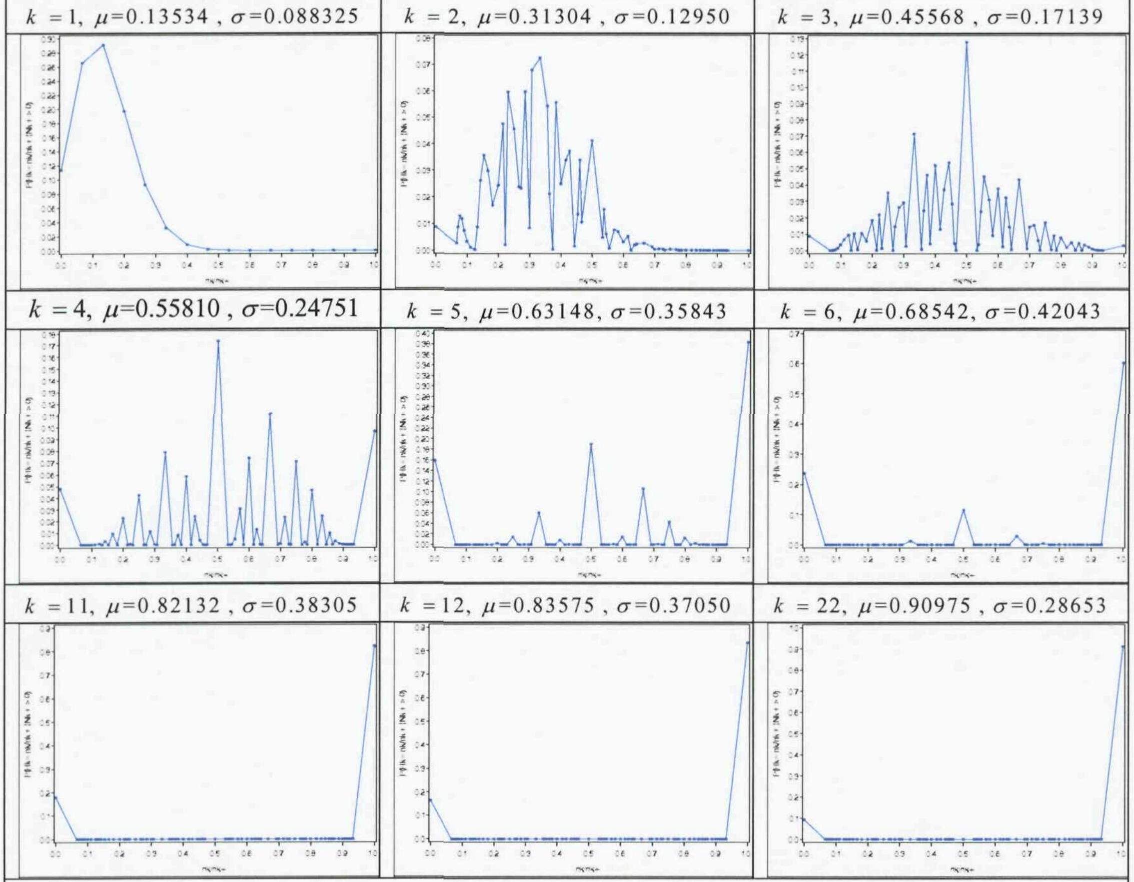
$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,2) distribution with n=15		
$k=1$, μ=0.0625 , σ=0.0625	$k=2$, μ=0.1 , σ=0.080192	$k=3$, μ=0.125 , σ=0.093593
		
$k=4$, μ=0.14286 , σ=0.10655	$k=5$, μ=0.15625, σ=0.12052	$k=6$, μ=0.16667, σ=0.13643
		
$k=11$, μ=0.19643 , σ=0.26010	$k=12$, μ=0.2 , σ=0.28668	$k=22$, μ=0.22 , σ=0.40337
		

$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,2) distribution with n=25		
$k=1$, μ=0.0625 , σ=0.048412	$k=2$, μ=0.1 , σ=0.062054	$k=3$, μ=0.125 , σ=0.072290
		
$k=4$, μ=0.14286 , σ=0.082070	$k=5$, μ=0.15625, σ=0.09244	$k=6$, μ=0.16667, σ=0.10400
		
$k=11$, μ=0.19643 , σ=0.19916	$k=12$, μ=0.2 , σ=0.22757	$k=22$, μ=0.22 , σ=0.39561
		

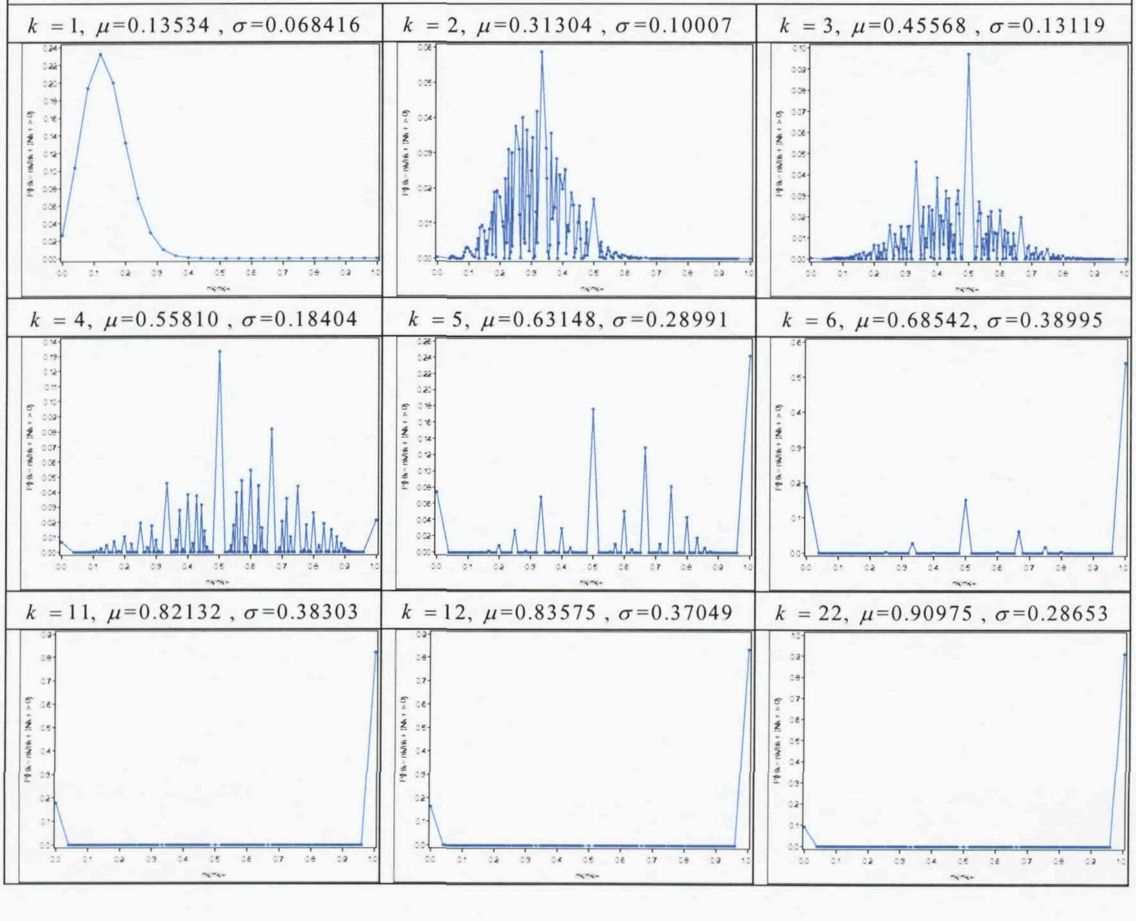
$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,2) distribution with n=50		
$k=1$, μ=0.0625 , σ=0.034233	$k=2$, μ=0.1 , σ=0.043848	$k=3$, μ=0.125 , σ=0.051014
		
$k=4$, μ=0.14286 , σ=0.057806	$k=5$, μ=0.15625, σ=0.06494	$k=6$, μ=0.16667, σ=0.07279
		
$k=11$, μ=0.19643 , σ=0.13292	$k=12$, μ=0.2 , σ=0.15212	$k=22$, μ=0.22 , σ=0.37631
		

$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for the negative binomial (0.25,2) distribution with n=100		
$k=1$, μ=0.0625 , σ=0.024206	$k=2$, μ=0.1 , σ=0.030994	$k=3$, μ=0.125 , σ=0.036038
		
$k=4$, μ=0.14286 , σ=0.040799	$k=5$, μ=0.15625, σ=0.04578	$k=6$, μ=0.16667, σ=0.05123
		
$k=11$, μ=0.19643 , σ=0.091485	$k=12$, μ=0.2 , σ=0.10352	$k=22$, μ=0.22 , σ=0.33888
		

$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for Poisson (2) distribution with n=15

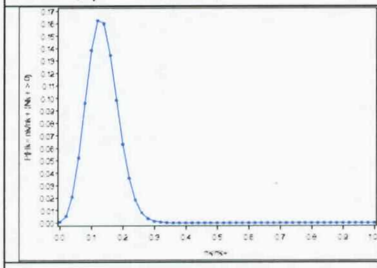


$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for Poisson (2) distribution with n=25

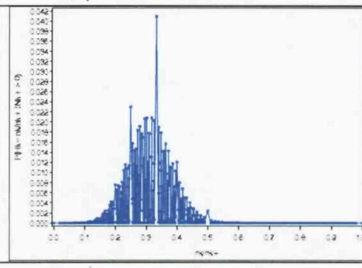


$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for Poisson (2) distribution with n=50

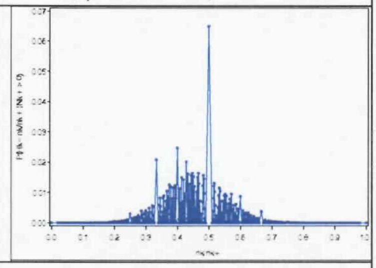
$k = 1$, μ=0.13534 , σ=0.048378



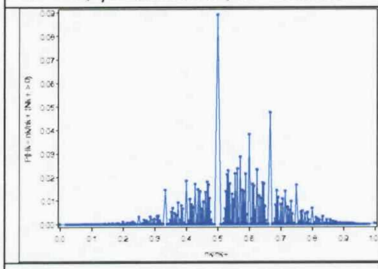
$k = 2$, μ=0.31304 , σ=0.070640



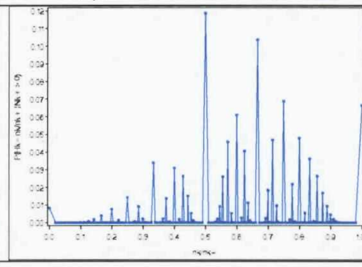
$k = 3$, μ=0.45568 , σ=0.092040



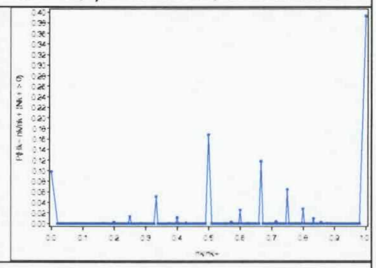
$k = 4$, μ=0.55810 , σ=0.12639



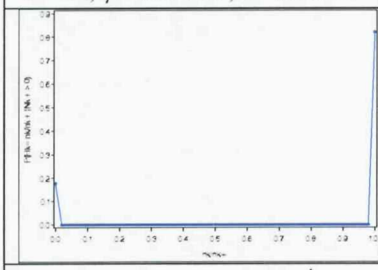
$k = 5$, μ=0.63148, σ=0.19521



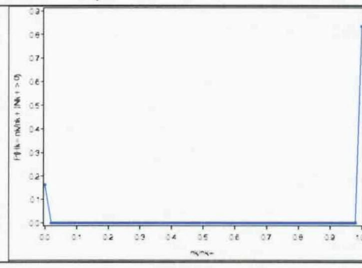
$k = 6$, μ=0.68542, σ=0.32127



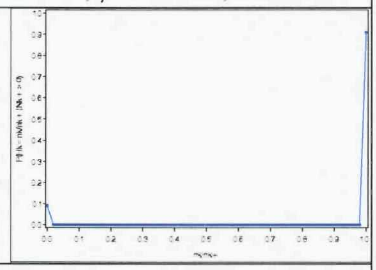
$k = 11$, μ=0.82132 , σ=0.38297



$k = 12$, μ=0.83575 , σ=0.37048

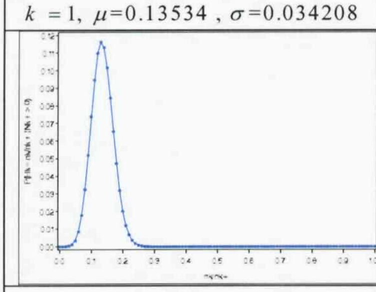


$k = 22$, μ=0.90975 , σ=0.28653

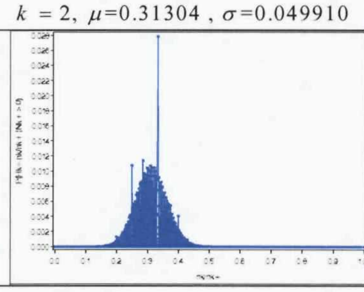


$P(H_k = n_k / n_k^+ \mid N_k^+ > 0)$ for Poisson (2) distribution with n=100

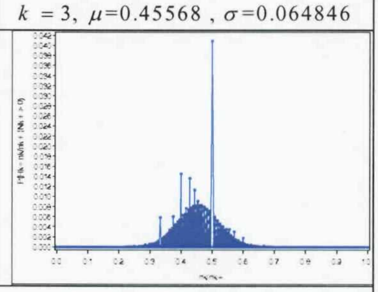
$k = 1$, μ=0.13534 , σ=0.034208



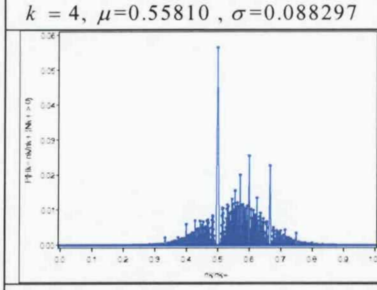
$k = 2$, μ=0.31304 , σ=0.049910



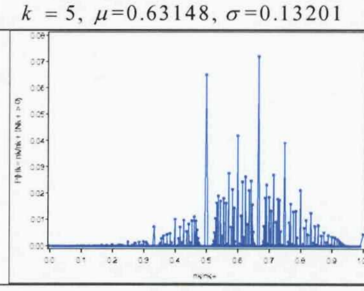
$k = 3$, μ=0.45568 , σ=0.064846



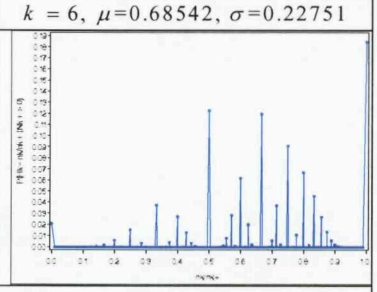
$k = 4$, μ=0.55810 , σ=0.088297



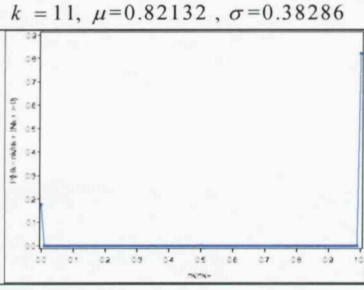
$k = 5$, μ=0.63148, σ=0.13201



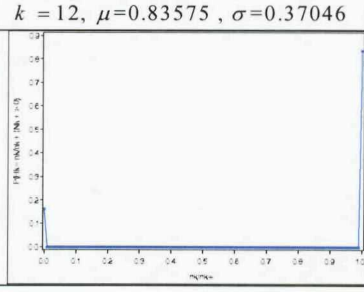
$k = 6$, μ=0.68542, σ=0.22751



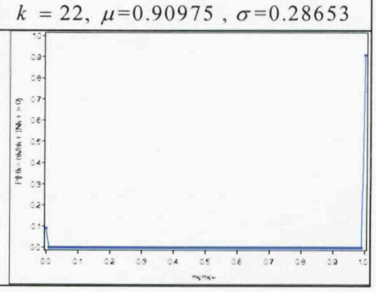
$k = 11$, μ=0.82132 , σ=0.38286



$k = 12$, μ=0.83575 , σ=0.37046



$k = 22$, μ=0.90975 , σ=0.28653

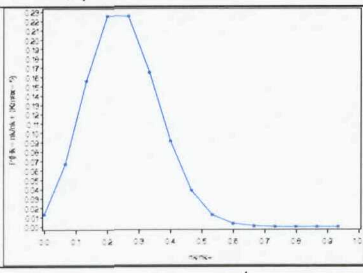


Appendix C:

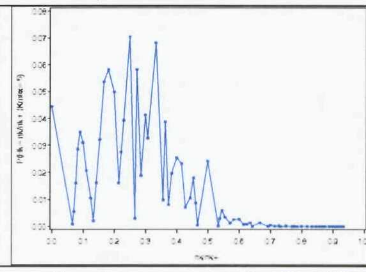
Plots for the probability function of $H_k = n_k / n_k^+ \mid K_{\max} = k_*$ where $\mu = E(H_k \mid K_{\max} = k_*)$ and $\sigma = STD(H_k \mid K_{\max} = k_*)$.

$P(H_k = n_k / n_k^+ \mid K_{\max} = 5)$ for the geometric (0.25) distribution with n=15

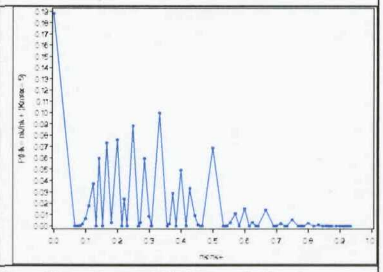
$k=1$, μ=0.24961 , σ=0.11156



$k=2$, μ=0.24939, σ=0.13026

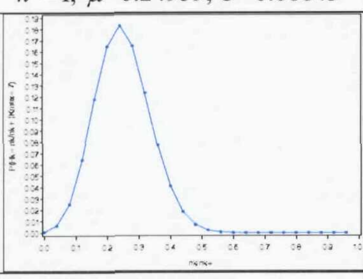


$k=4$, μ=0.24770, σ=0.17825

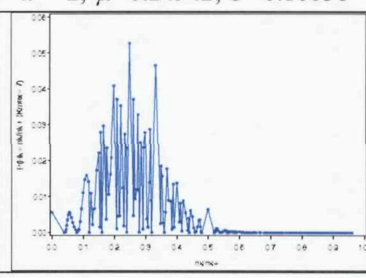


$P(H_k = n_k / n_k^+ \mid K_{\max} = 7)$ for the geometric (0.25) distribution with n=25

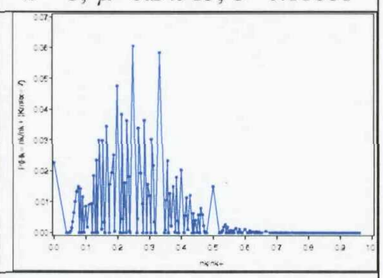
$k=1$, μ=0.24959, σ=0.08643



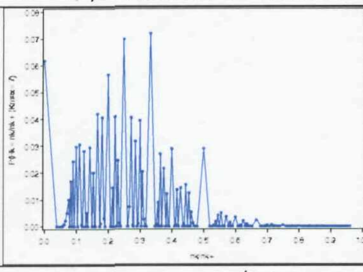
$k=2$, μ=0.24942, σ=0.10038



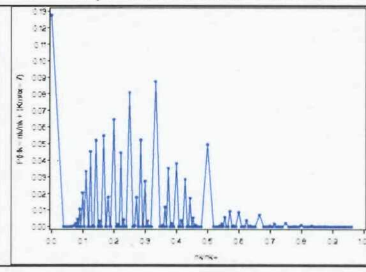
$k=3$, μ=0.24913, σ=0.11681



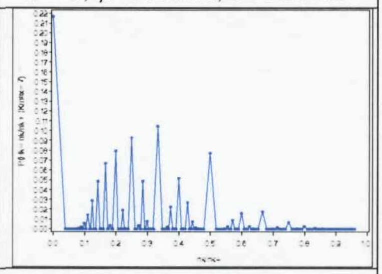
$k=4$, μ=0.24863, σ=0.13622



$k=5$, μ=0.24761, σ=0.15904

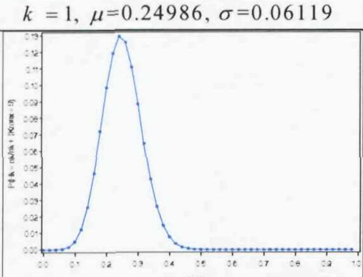


$k=6$, μ=0.24524, σ=0.18485

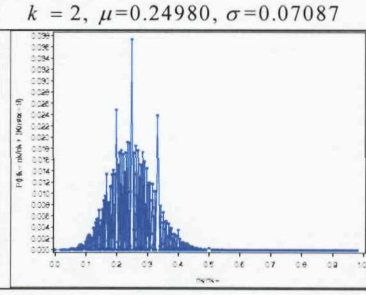


$P(H_k = n_k / n_k^+ \mid K_{\max} = 9)$ for the geometric (0.25) distribution with n=50

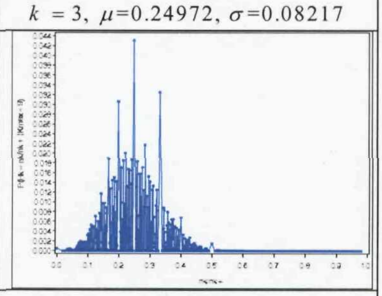
$k=1$, μ=0.24986, σ=0.06119



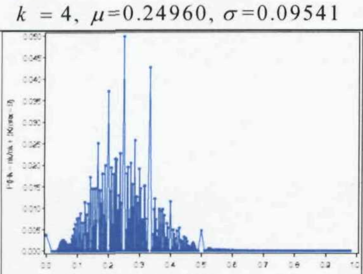
$k=2$, μ=0.24980, σ=0.07087



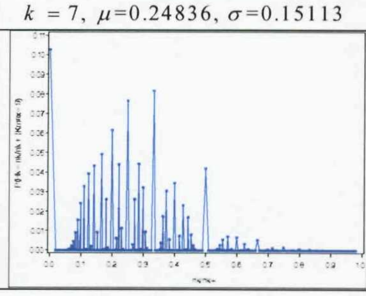
$k=3$, μ=0.24972, σ=0.08217



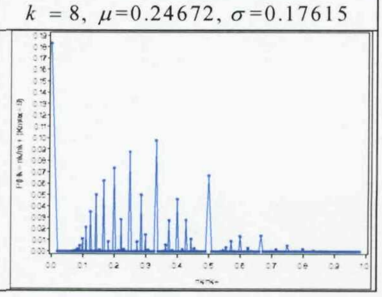
$k=4$, μ=0.24960, σ=0.09541



$k=7$, μ=0.24836, σ=0.15113

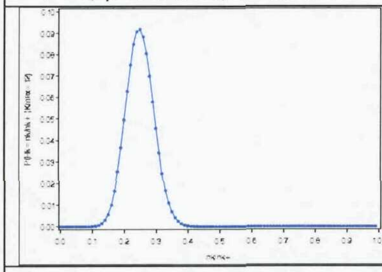


$k=8$, μ=0.24672, σ=0.17615

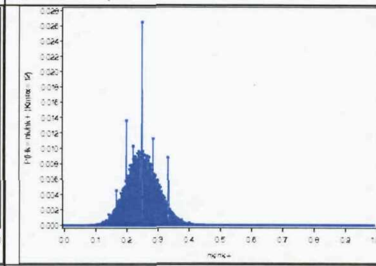


$P(H_k = n_k / n_k^+ \mid K_{\max} = 12)$ for the geometric (0.25) distribution with n=100,

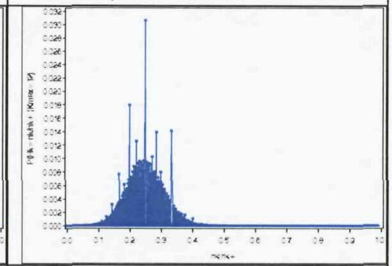
$k = 1$, μ=0.24985, σ=0.04327



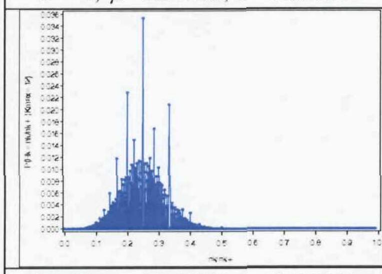
$k = 2$, μ=0.24980, σ=0.05004



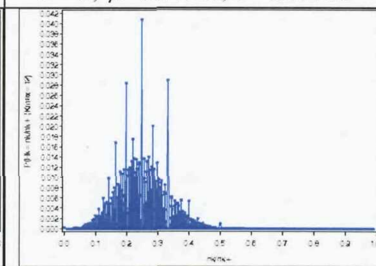
$k = 3$, μ=0.24973, σ=0.05788



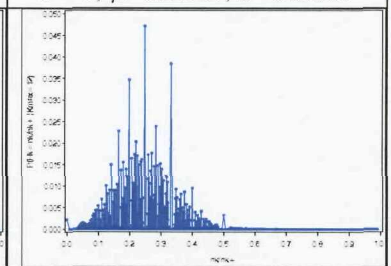
$k = 4$, μ=0.24962, σ=0.06610



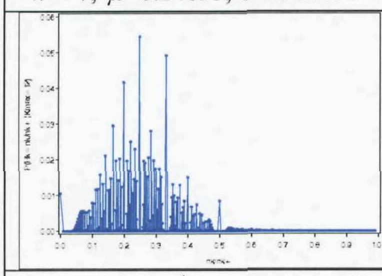
$k = 5$, μ=0.24948, σ=0.07761



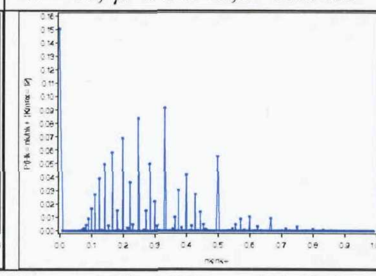
$k = 6$, μ=0.24927, σ=0.09000



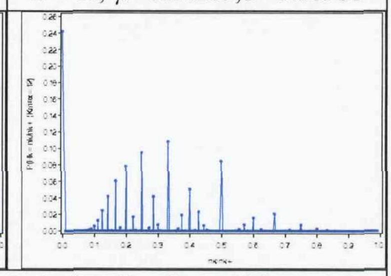
$k = 7$, μ=0.24895, σ=0.10451



$k = 10$, μ=0.24578, σ=0.16460

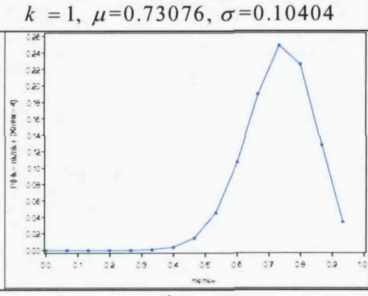


$k = 11$, μ=0.24215, σ=0.18981

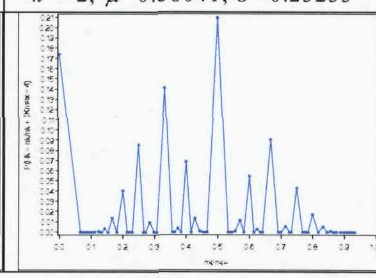


$P(H_k = n_k / n_k^+ \mid K_{\max} = 4)$ for the negative binomial (0.25, 0.2) distribution with n=15

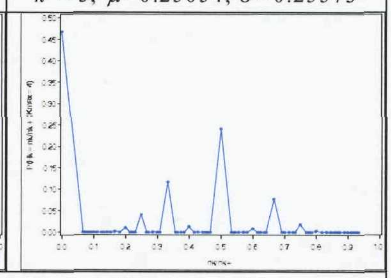
$k = 1$, μ=0.73076, σ=0.10404



$k = 2$, μ=0.38041, σ=0.23233

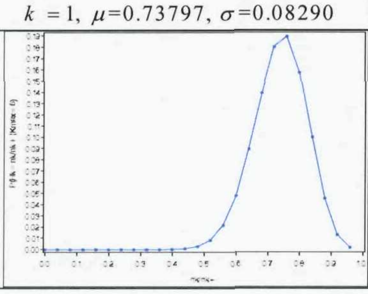


$k = 3$, μ=0.25054, σ=0.25573

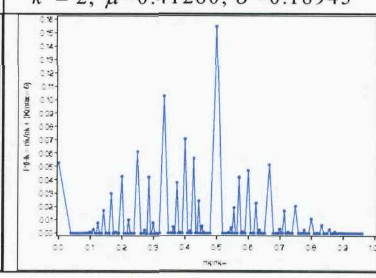


$P(H_k = n_k / n_k^+ \mid K_{\max} = 6)$ for the negative binomial (0.25, 0.2) distribution with n=25

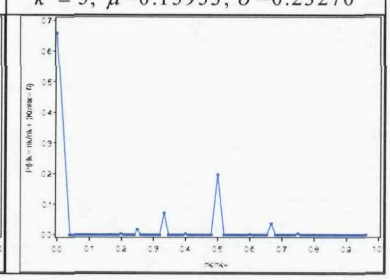
$k = 1$, μ=0.73797, σ=0.08290



$k = 2$, μ=0.41280, σ=0.18945

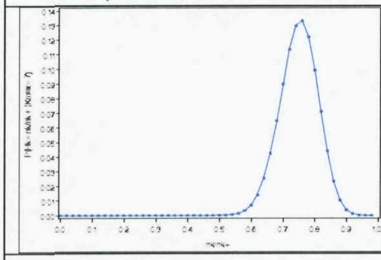


$k = 5$, μ=0.15953, σ=0.23270

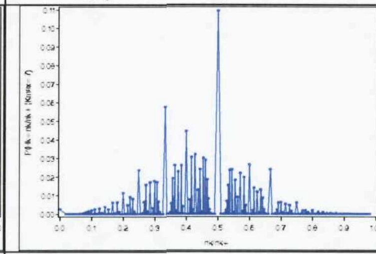


$P(H_k = n_k / n_k^+ \mid K_{max} = 7)$ for the negative binomial (0.25, 0.2) distribution with n=50

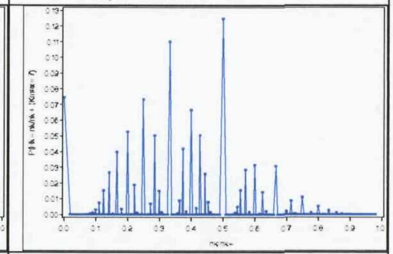
$k = 1$, μ=0.74942, σ=0.05956



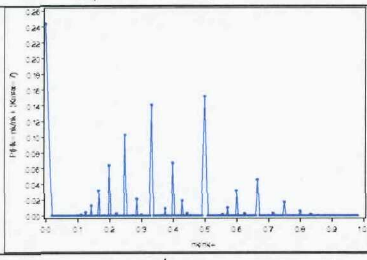
$k = 2$, μ=0.44628, σ=0.13978



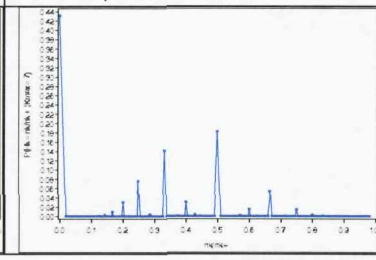
$k = 3$, μ=0.35773, σ=0.18364



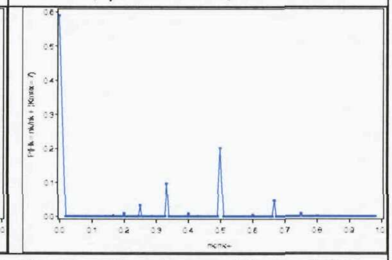
$k = 4$, μ=0.29652, σ=0.21800



$k = 5$, μ=0.24005, σ=0.23730

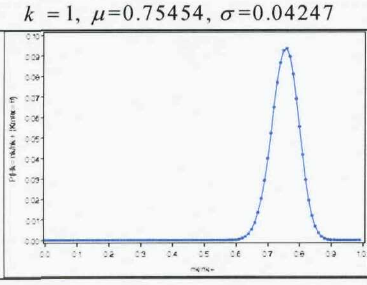


$k = 6$, μ=0.18734, σ=0.23955

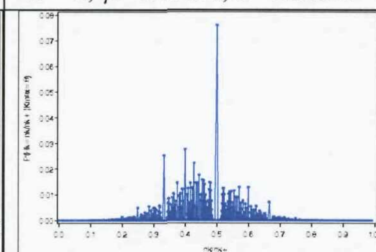


$P(H_k = n_k / n_k^+ \mid K_{max} = 8)$ for the negative binomial (0.25, 0.2) distribution with n=100

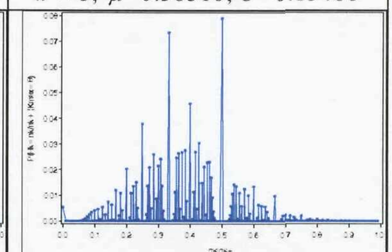
$k = 1$, μ=0.75454, σ=0.04247



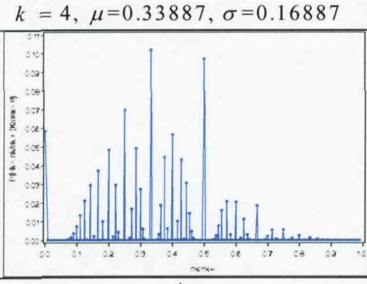
$k = 2$, μ=0.46056, σ=0.10055



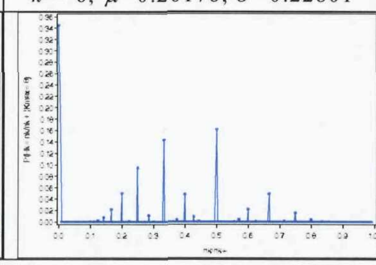
$k = 3$, μ=0.38310, σ=0.13488



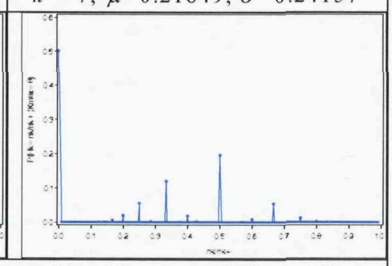
$k = 4$, μ=0.33887, σ=0.16887



$k = 6$, μ=0.26178, σ=0.22801

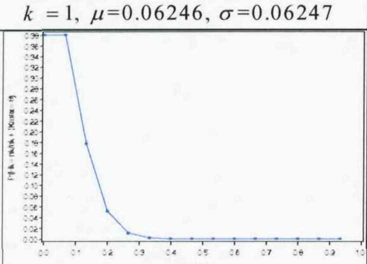


$k = 7$, μ=0.21849, σ=0.24157

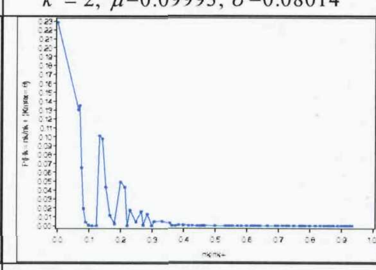


$P(H_k = n_k / n_k^+ \mid K_{max} = 8)$ for the negative binomial (0.25, 2) distribution with n=15

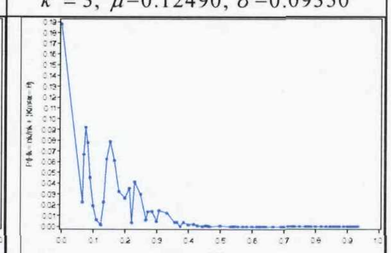
$k = 1$, μ=0.06246, σ=0.06247



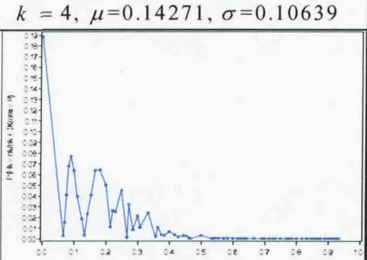
$k = 2$, μ=0.09993, σ=0.08014



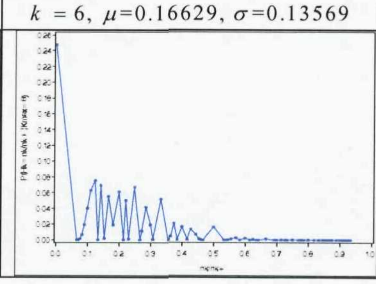
$k = 3$, μ=0.12490, σ=0.09350



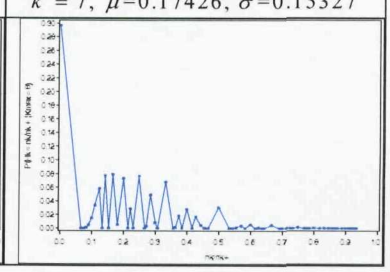
$k = 4$, μ=0.14271, σ=0.10639



$k = 6$, μ=0.16629, σ=0.13569

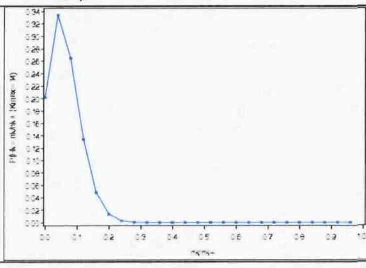


$k = 7$, μ=0.17426, σ=0.15327

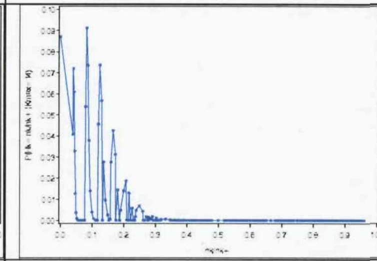


$P(H_k = n_k / n_k^+ \mid K_{max} = 14)$ for the negative binomial (0.25, 2) distribution with n=25

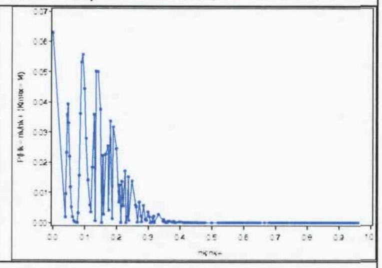
$k = 1$, μ=0.061973 , σ=0.048182



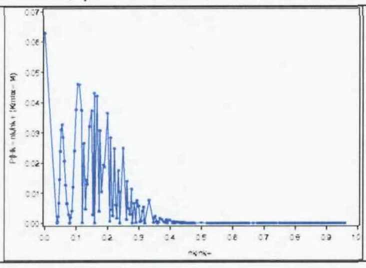
$k = 2$, μ=0.099093 , σ=0.061692



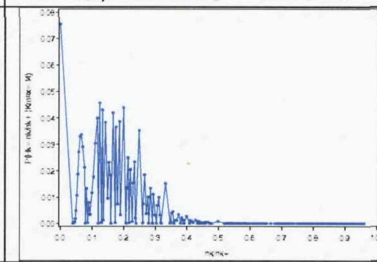
$k = 3$, μ=0.12372 , σ=0.071750



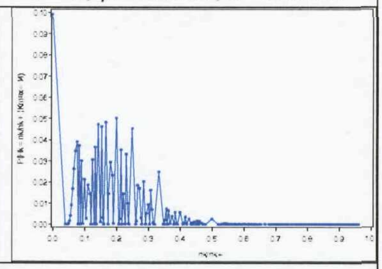
$k = 4$, μ=0.14116 , σ=0.081266



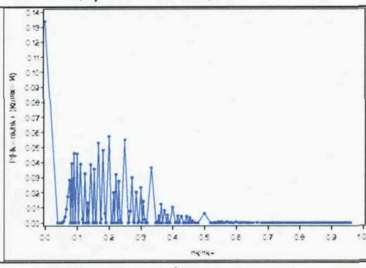
$k = 5$, μ=0.15402, σ=0.091238



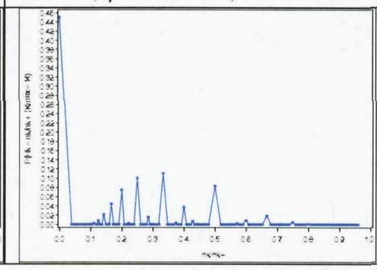
$k = 6$, μ=0.16375, σ=0.10216



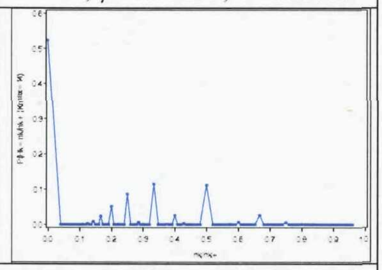
$k = 7$, μ=0.17114, σ=0.11436



$k = 12$, μ=0.17865 , σ=0.19289

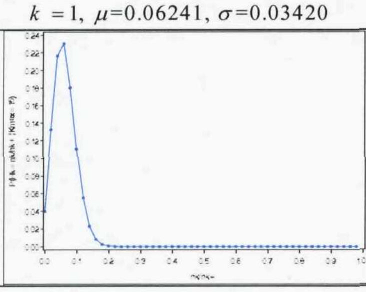


$k = 13$, μ=0.17222 , σ=0.20675

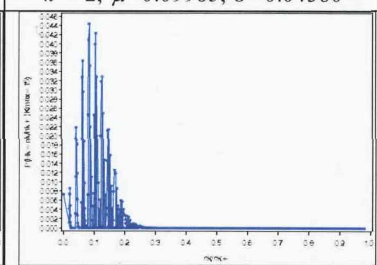


$P(H_k = n_k / n_k^+ \mid K_{max} = 15)$ for the negative binomial (0.25, 2) distribution with n=50

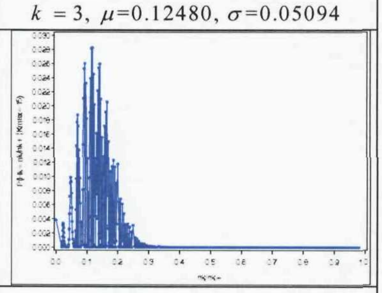
$k = 1$, μ=0.06241, σ=0.03420



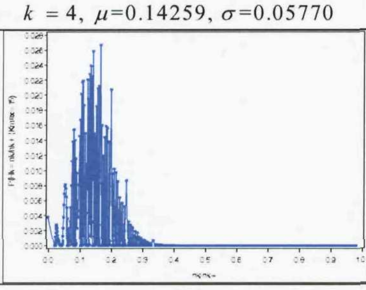
$k = 2$, μ=0.09985, σ=0.04380



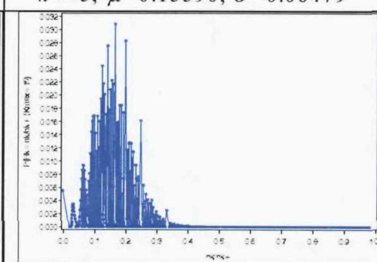
$k = 3$, μ=0.12480, σ=0.05094



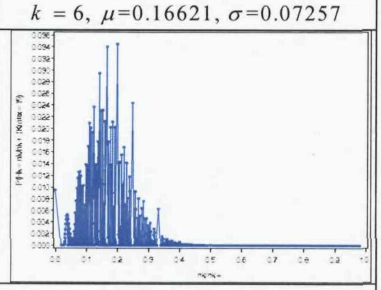
$k = 4$, μ=0.14259, σ=0.05770



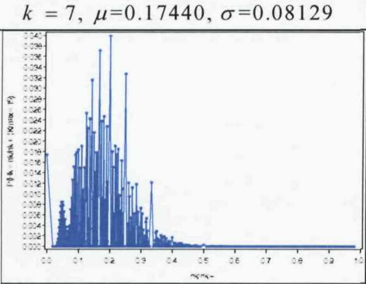
$k = 5$, μ=0.15590, σ=0.06479



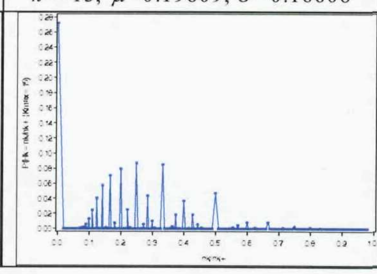
$k = 6$, μ=0.16621, σ=0.07257



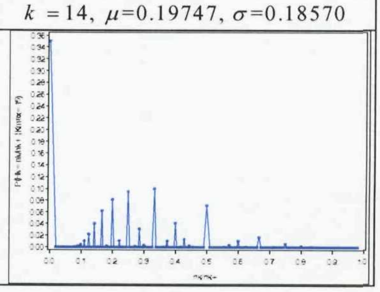
$k = 7$, μ=0.17440, σ=0.08129



$k = 13$, μ=0.19809, σ=0.16608

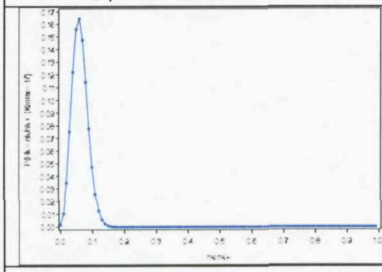


$k = 14$, μ=0.19747, σ=0.18570

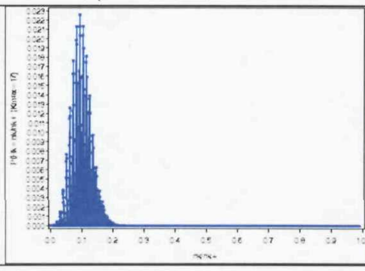


$P(H_k = n_k / n_k^+ \mid K_{max} = 17)$ for the negative binomial (0.25, 2) distribution with n=100

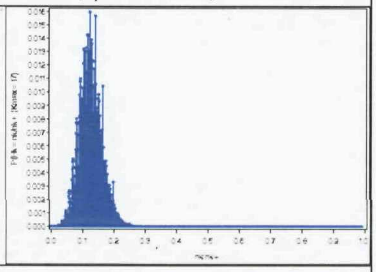
$k = 1$, μ=0.06248, σ=0.02420



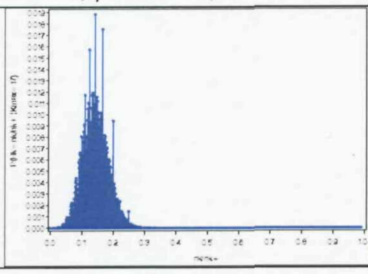
$k = 2$, μ=0.09997, σ=0.03099



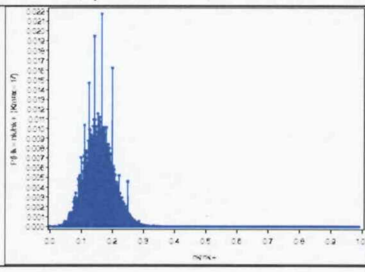
$k = 3$, μ=0.12495, σ=0.03603



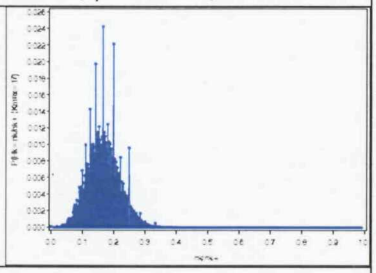
$k = 4$, μ=0.14280, σ=0.04078



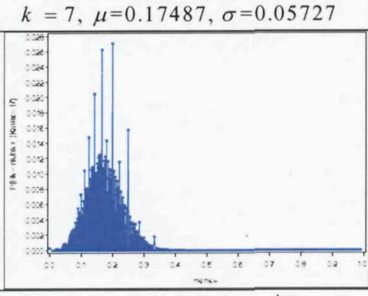
$k = 5$, μ=0.15617, σ=0.04575



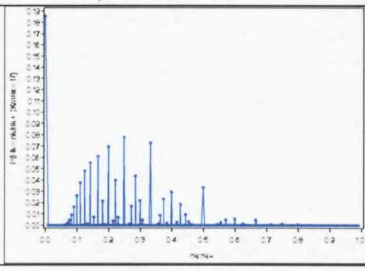
$k = 6$, μ=0.16657, σ=0.05119



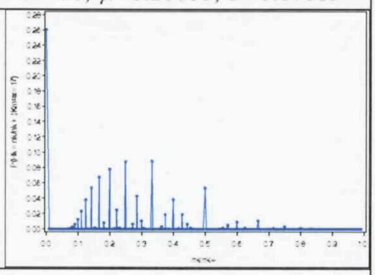
$k = 7$, μ=0.17487, σ=0.05727



$k = 15$, μ=0.20637, σ=0.15068

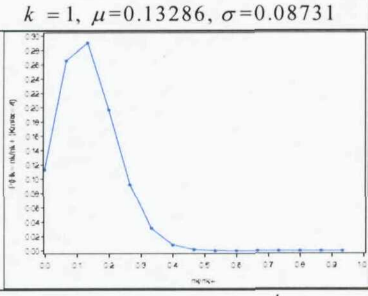


$k = 16$, μ=0.20701, σ=0.17089

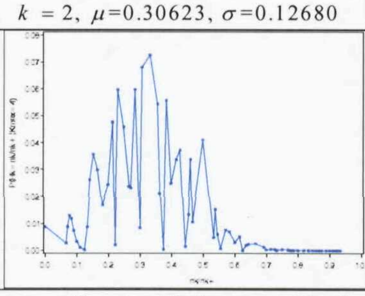


$P(H_k = n_k / n_k^+ \mid K_{max} = 4)$ for the Poisson (2) distribution with n=15

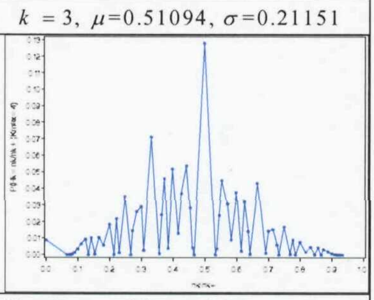
$k = 1$, μ=0.13286, σ=0.08731



$k = 2$, μ=0.30623, σ=0.12680

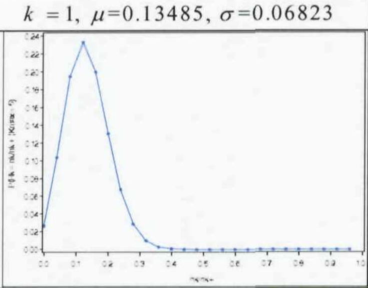


$k = 3$, μ=0.51094, σ=0.21151

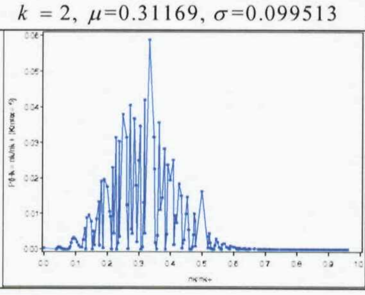


$P(H_k = n_k / n_k^+ \mid K_{max} = 5)$ for the Poisson (2) distribution with n=25

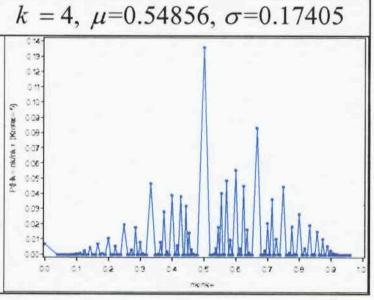
$k = 1$, μ=0.13485, σ=0.06823



$k = 2$, μ=0.31169, σ=0.099513

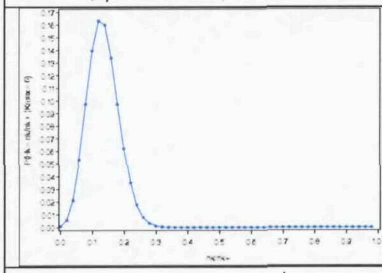


$k = 4$, μ=0.54856, σ=0.17405

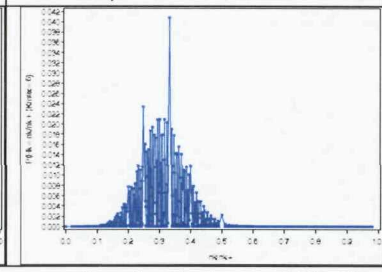


$P(H_k = n_k / n_k^+ \mid K_{\max} = 6)$ for the Poisson (2) distribution with n=50

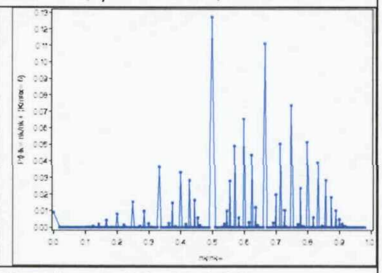
$k = 1$, μ=0.13480, σ=0.04825



$k = 2$, μ=0.31158, σ=0.07031

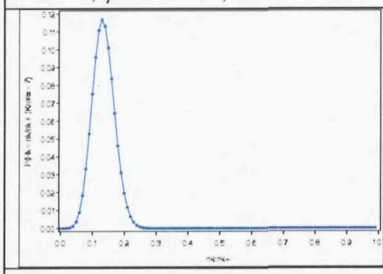


$k = 5$, μ=0.60523, σ=0.17453

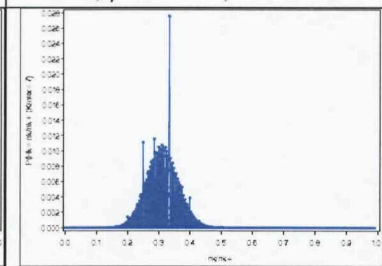


$P(H_k = n_k / n_k^+ \mid K_{\max} = 7)$ for the Poisson (2) distribution with n=100

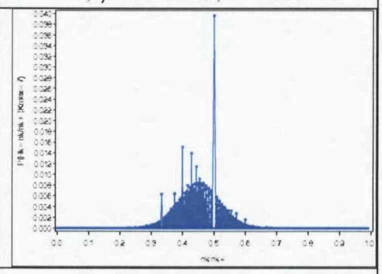
$k = 1$, μ=0.13481, σ=0.03413



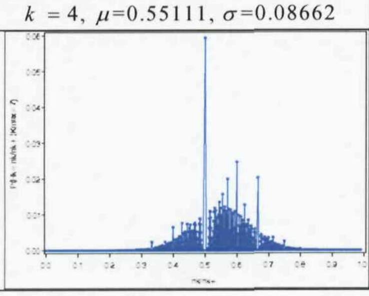
$k = 2$, μ=0.31162, σ=0.04973



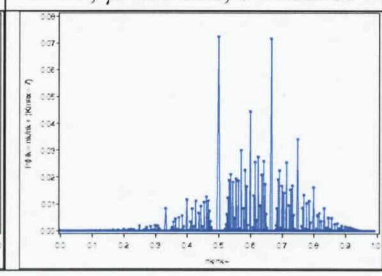
$k = 3$, μ=0.45265, σ=0.06435



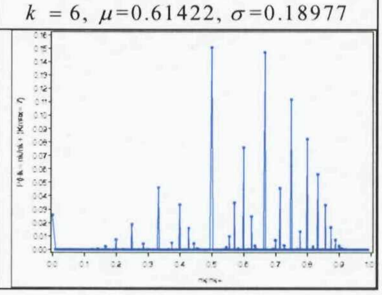
$k = 4$, μ=0.55111, σ=0.08662



$k = 5$, μ=0.61228, σ=0.12461



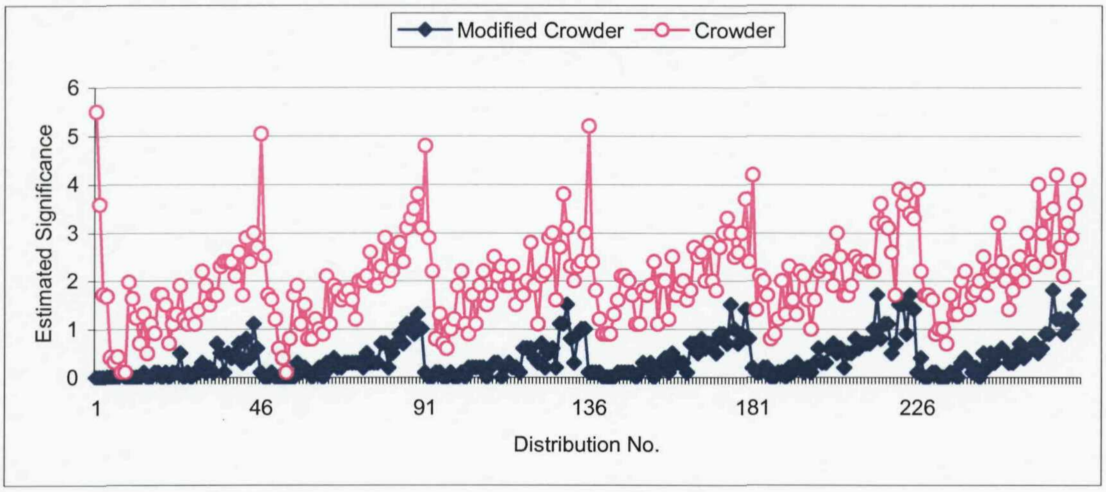
$k = 6$, μ=0.61422, σ=0.18977

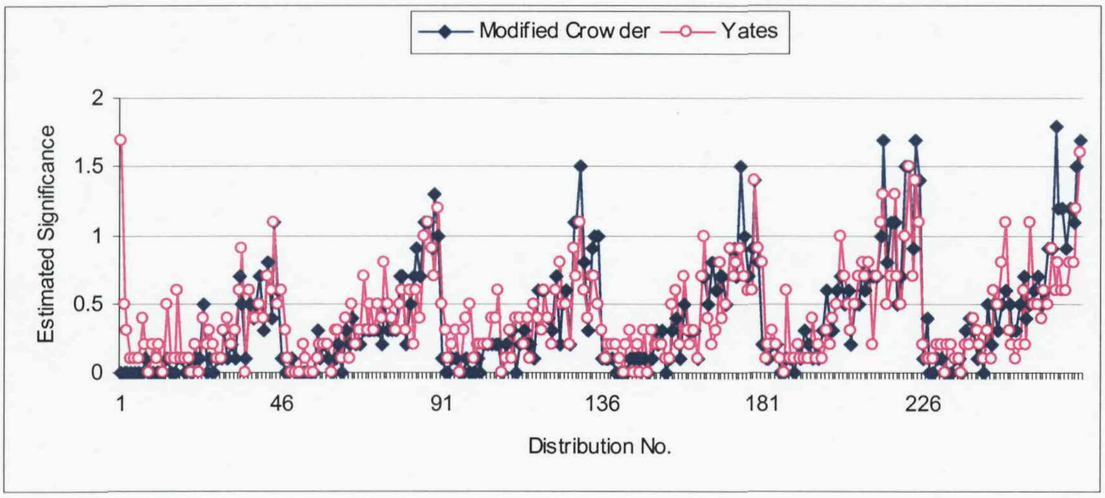


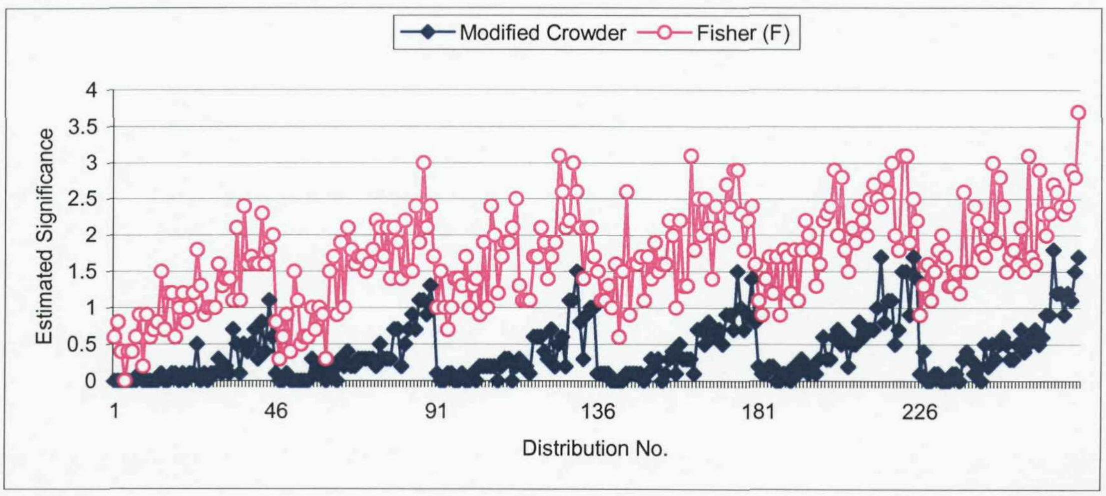
Appendix D:

Figure D.1: Plots representation for the estimated significance for samples with no censoring at the nominal 5% significance level.

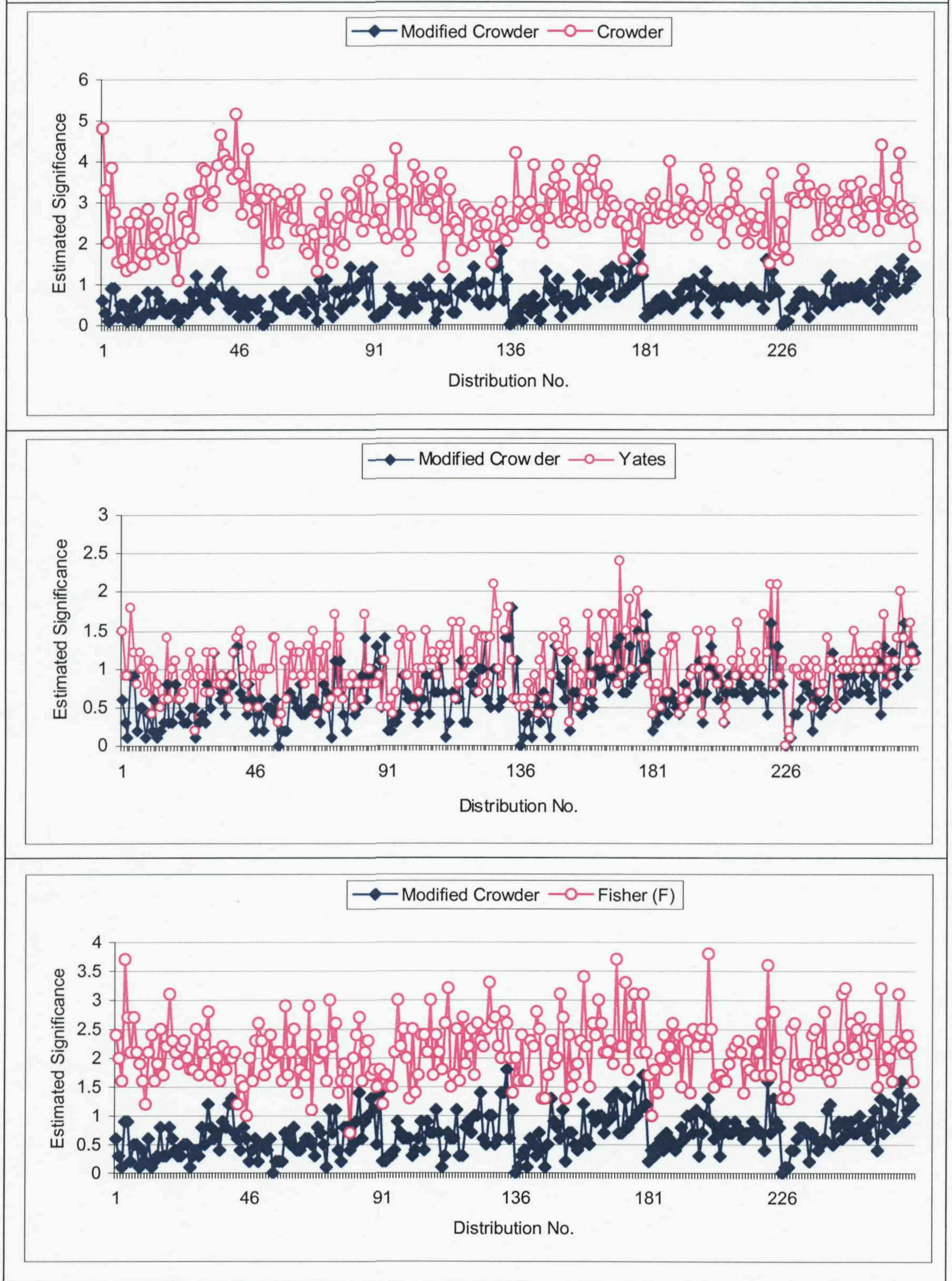
$G(p)$





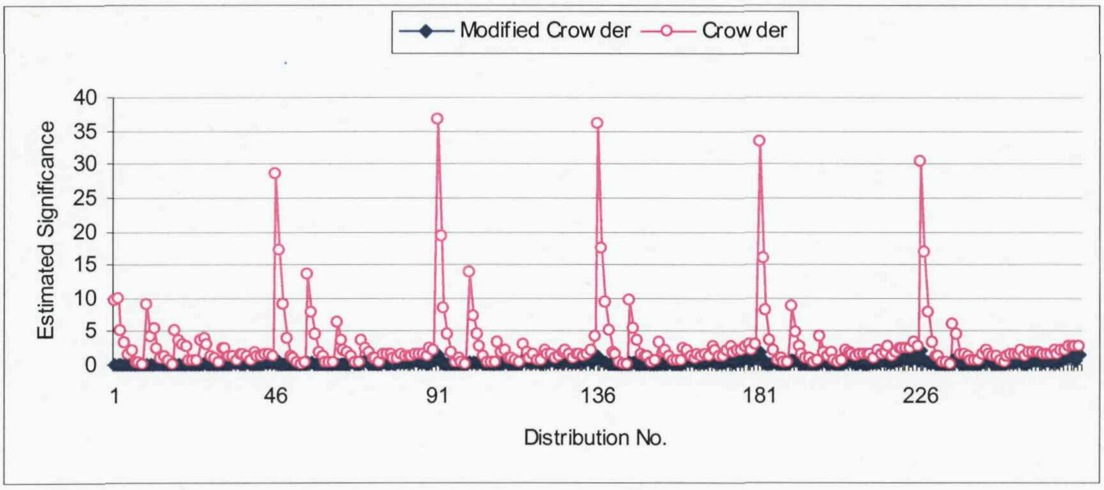


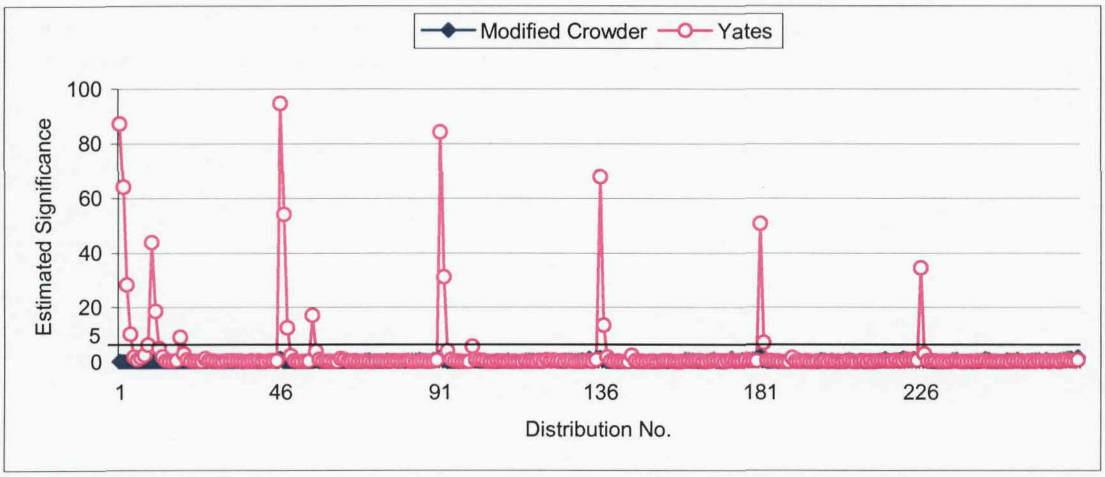
$N(p, 0.2)$

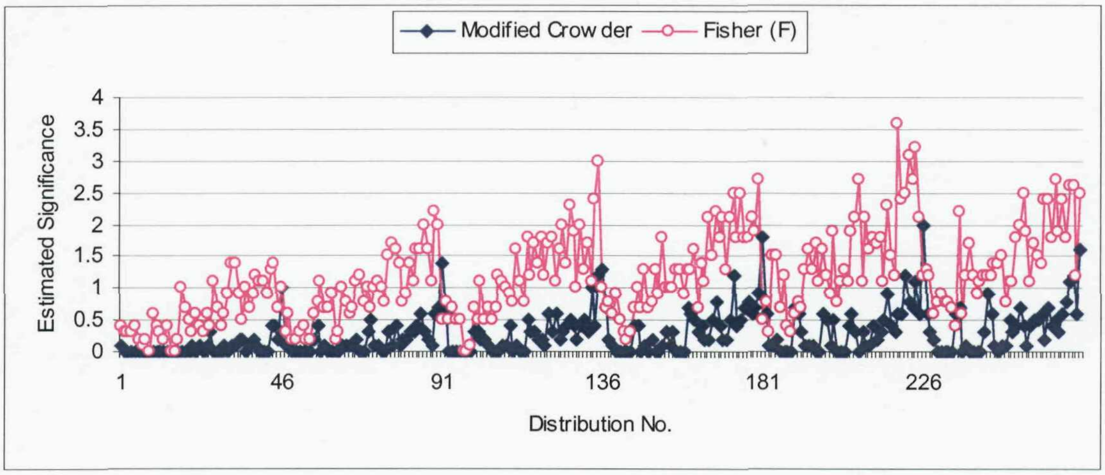


Modified Crowder
Crowder
Estimated Significance
0
1
2
3
4
5
6
1
46
91
136
181
226
Distribution No.
Modified Crow der
Yates
Estimated Significance
0
0.5
1
1.5
2
2.5
3
1
46
91
136
181
226
Distribution No.
Modified Crowder
Fisher (F)
Estimated Significance
0
0.5
1
1.5
2
2.5
3
3.5
4
1
46
91
136
181
226
Distribution No.

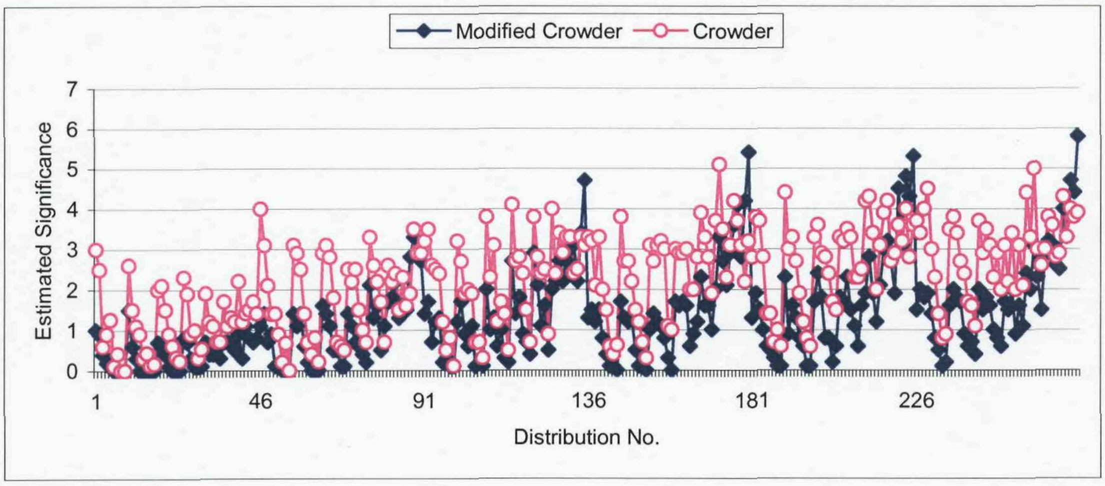
$N(p,2)$


Modified Crow der
Crow der
Estimated Significance
40
35
30
25
20
15
10
5
0
1
46
91
136
181
226
Distribution No.

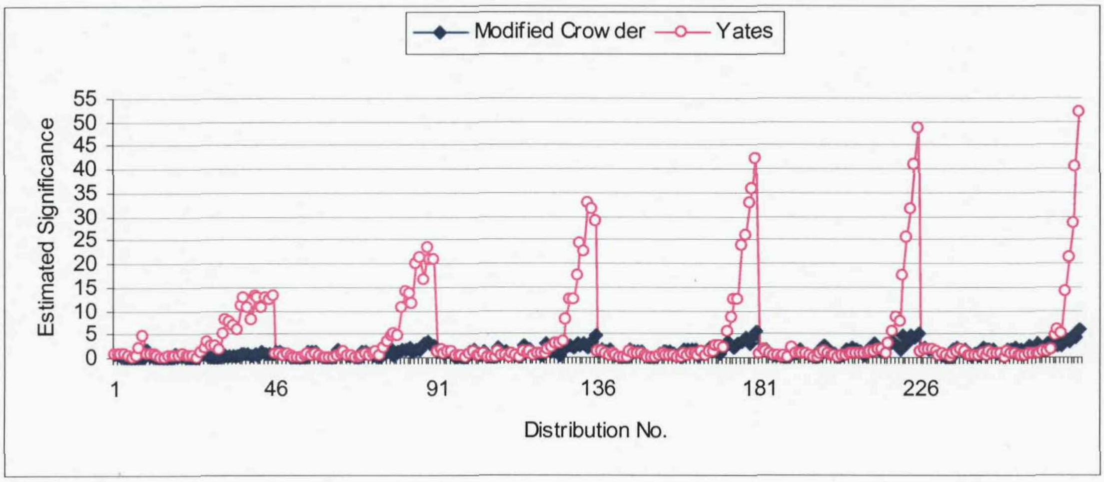

Modified Crowder
Yates
Estimated Significance
100
80
60
40
20
5
0
1
46
91
136
181
226
Distribution No.


Modified Crow der
Fisher (F)
Estimated Significance
4
3.5
3
2.5
2
1.5
1
0.5
0
1
46
91
136
181
226
Distribution No.

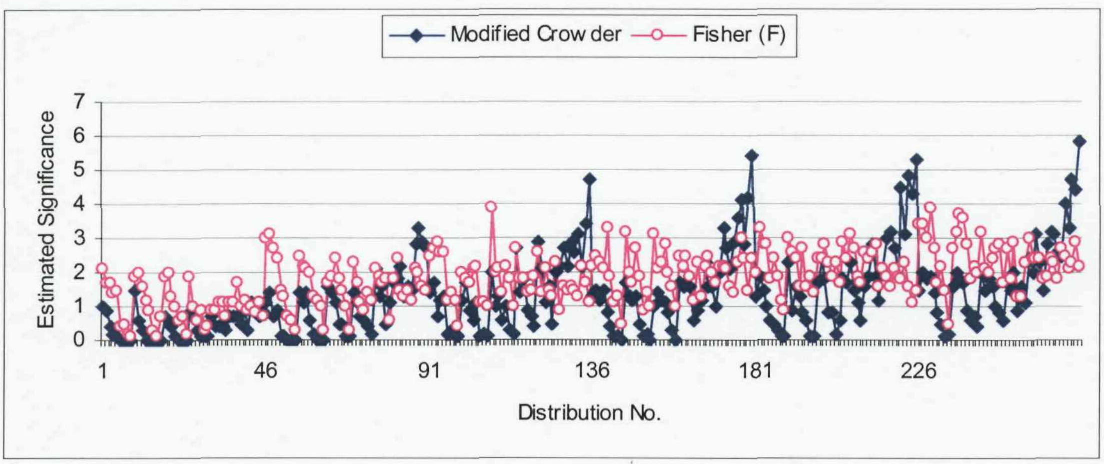
$P(\lambda)$



Modified Crowder
Crowder
Estimated Significance
0
1
2
3
4
5
6
7
1
46
91
136
181
226
Distribution No.



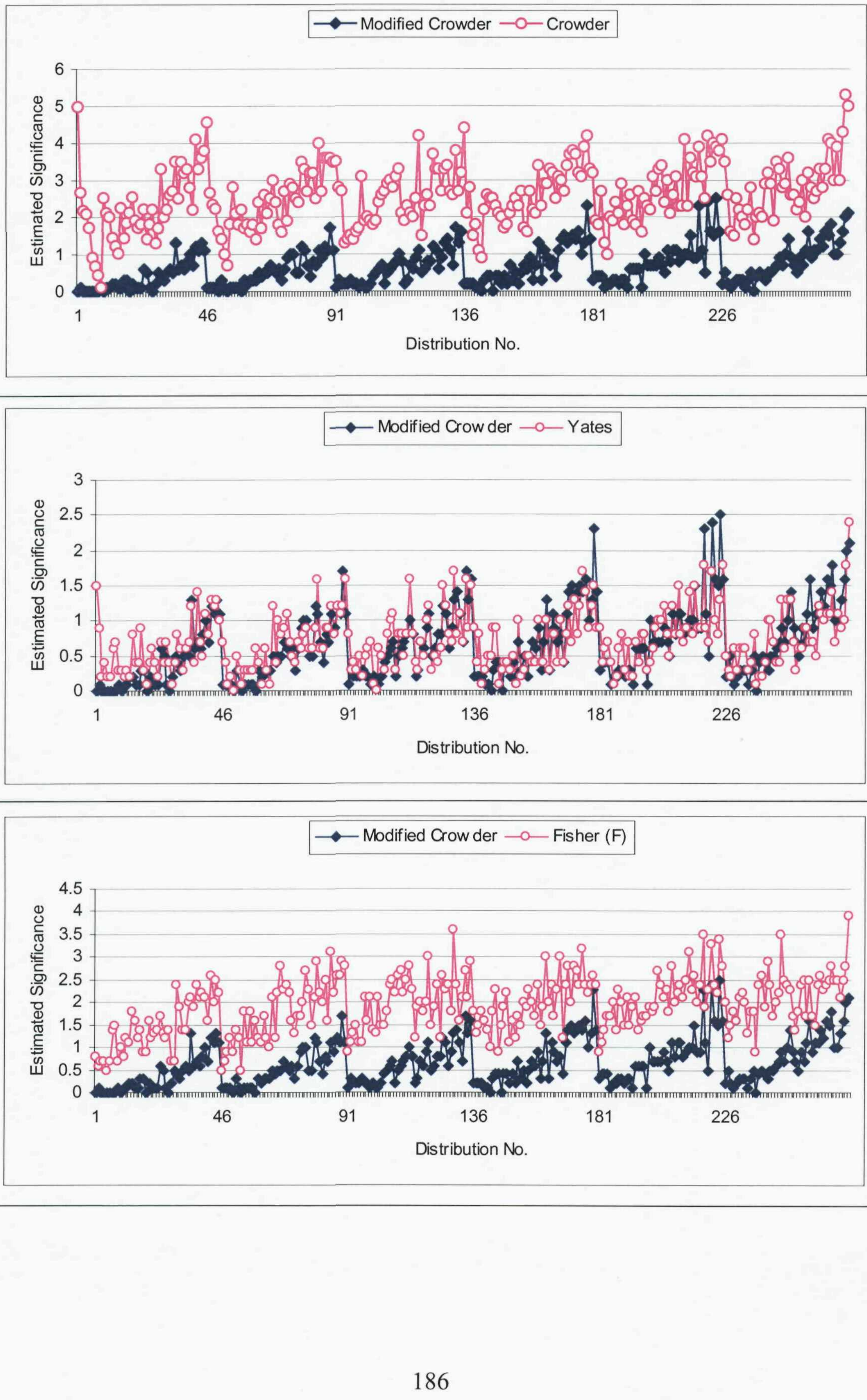
Modified Crow der
Yates
Estimated Significance
0
5
10
15
20
25
30
35
40
45
50
55
1
46
91
136
181
226
Distribution No.



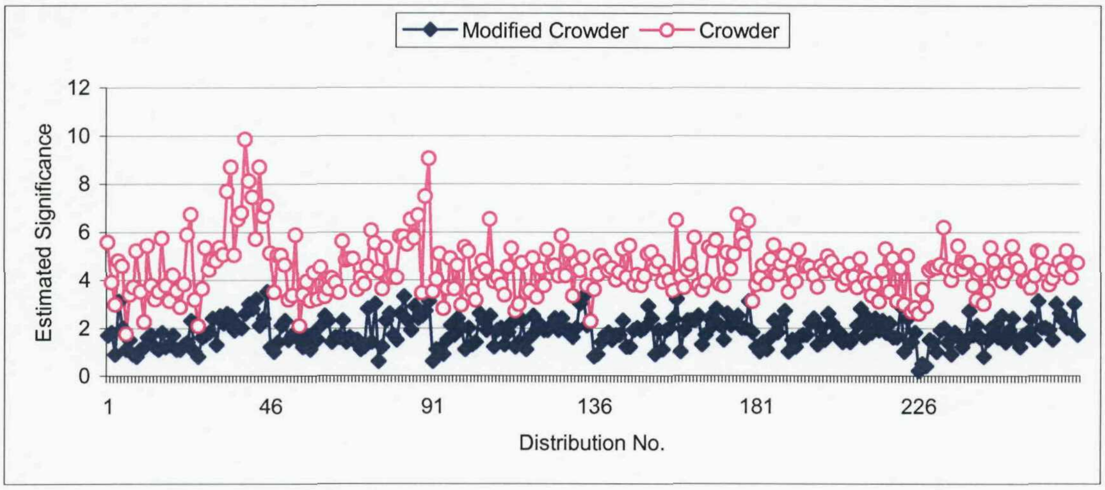
Modified Crow der
Fisher (F)
Estimated Significance
0
1
2
3
4
5
6
7
1
46
91
136
181
226
Distribution No.

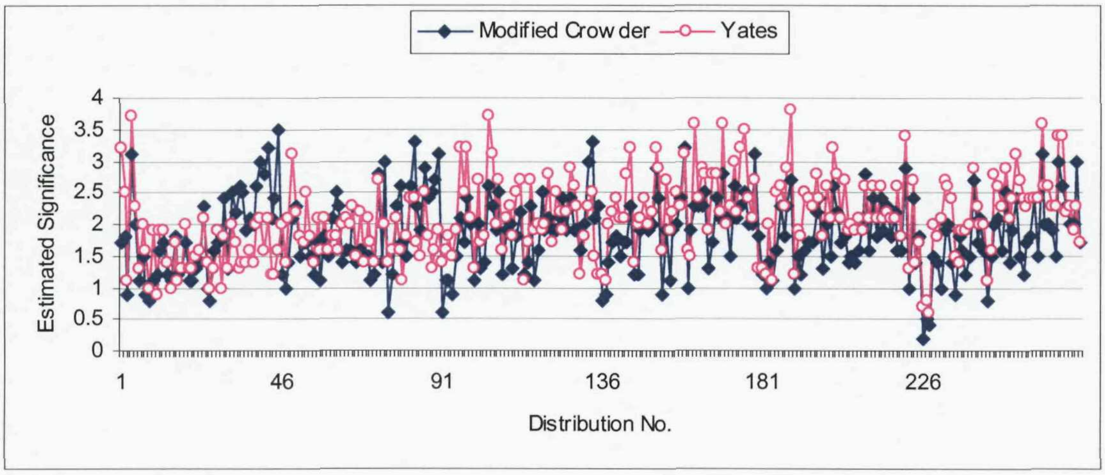
Figure D.2: Plots representation for the estimated significance for samples with censoring at the nominal 5% significance level.

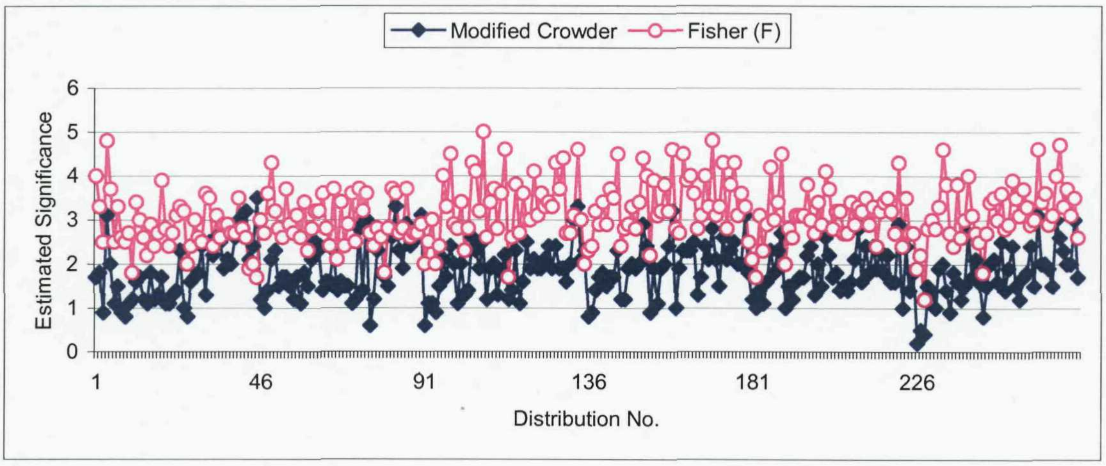
$G(p)$



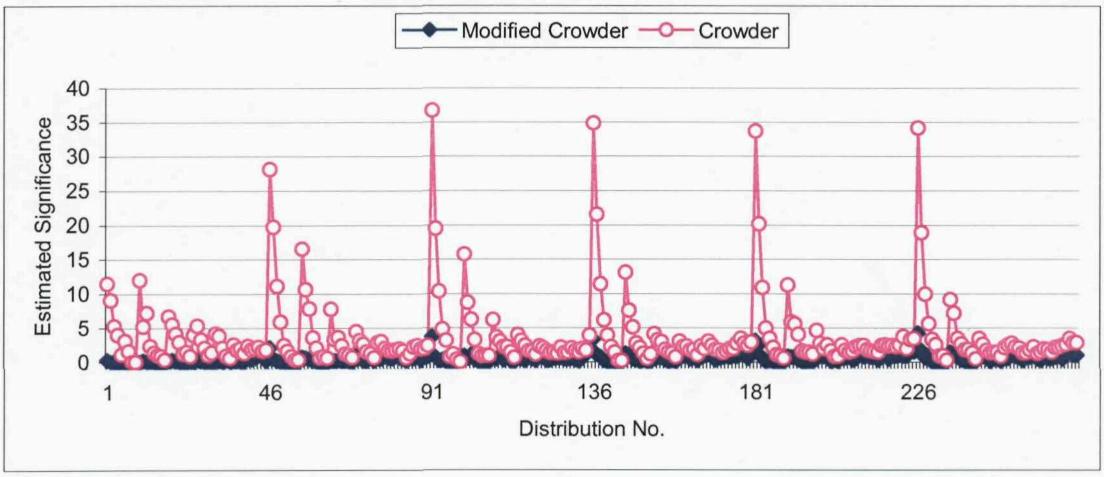
$N(p, 0.2)$


Modified Crowder
Crowder
Estimated Significance
12
10
8
6
4
2
0
1
46
91
136
181
226
Distribution No.

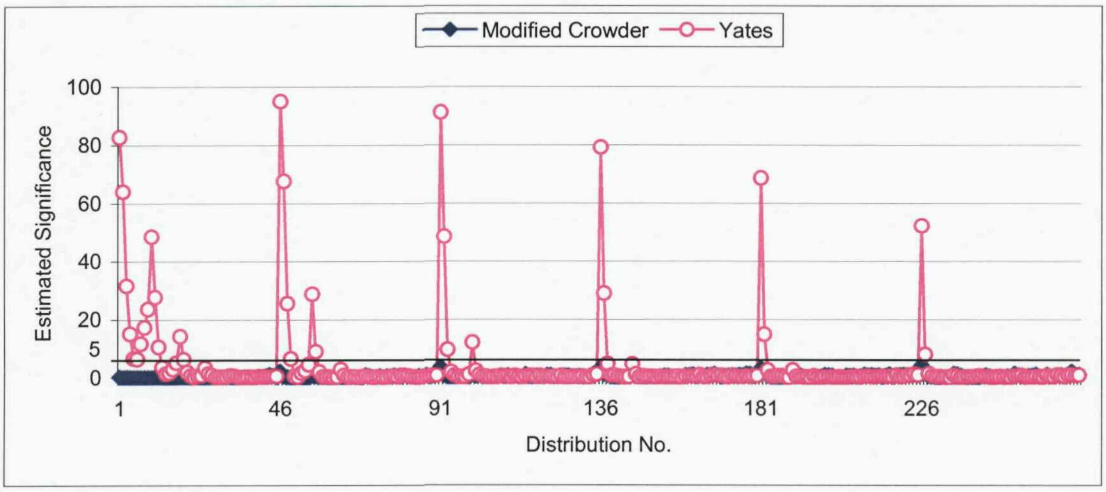

Modified Crowder
Yates
Estimated Significance
4
3.5
3
2.5
2
1.5
1
0.5
0
1
46
91
136
181
226
Distribution No.


Modified Crowder
Fisher (F)
Estimated Significance
6
5
4
3
2
1
0
1
46
91
136
181
226
Distribution No.

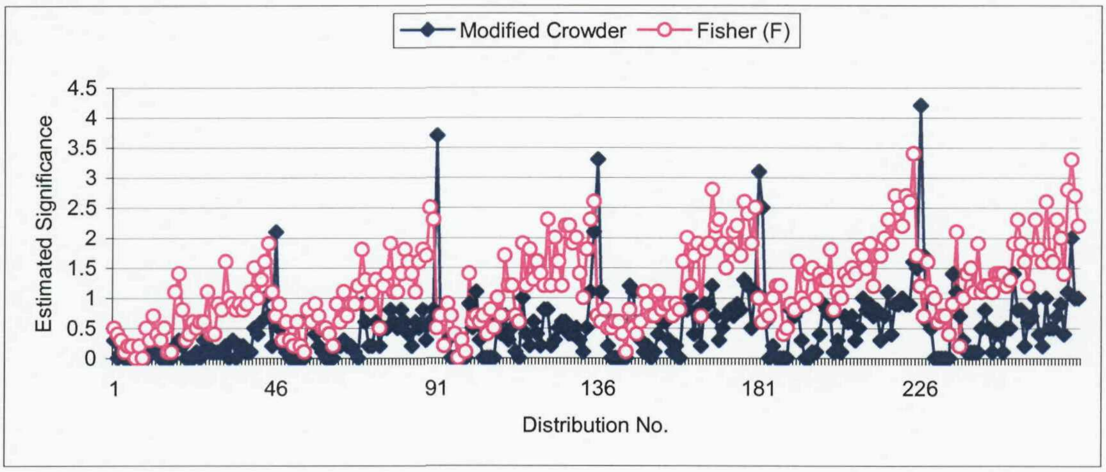
$N(p,2)$



Modified Crowder
Crowder
Estimated Significance
0
5
10
15
20
25
30
35
40
1
46
91
136
181
226
Distribution No.

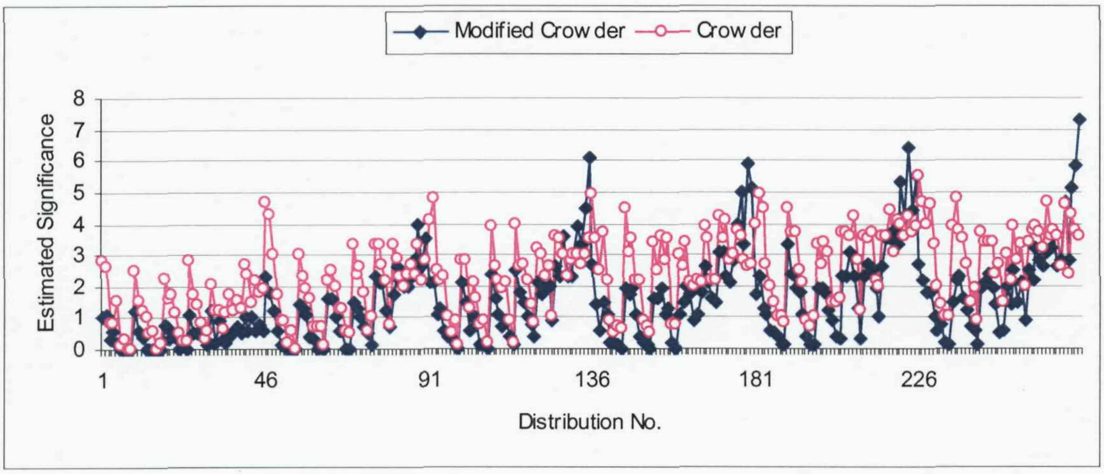


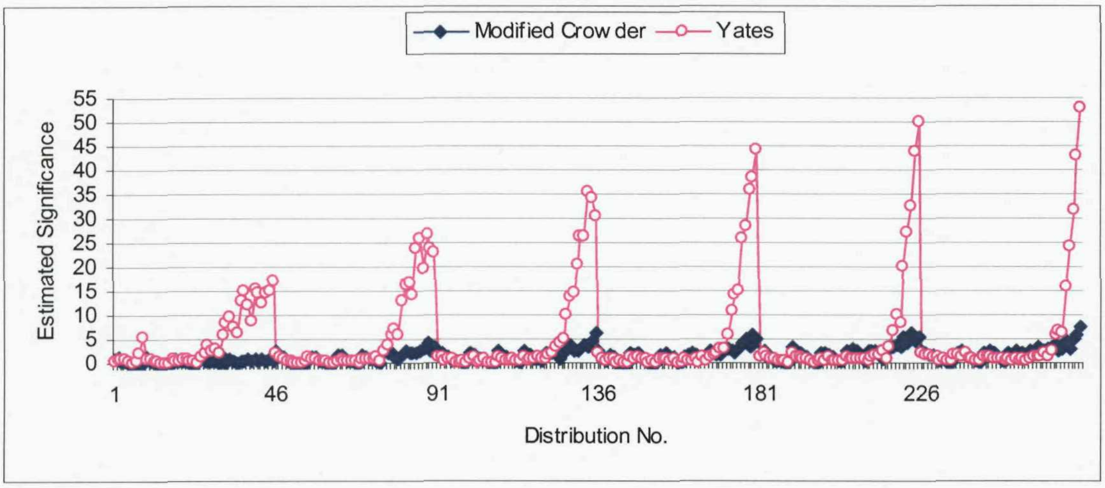
Modified Crowder
Yates
Estimated Significance
0
5
20
40
60
80
100
1
46
91
136
181
226
Distribution No.

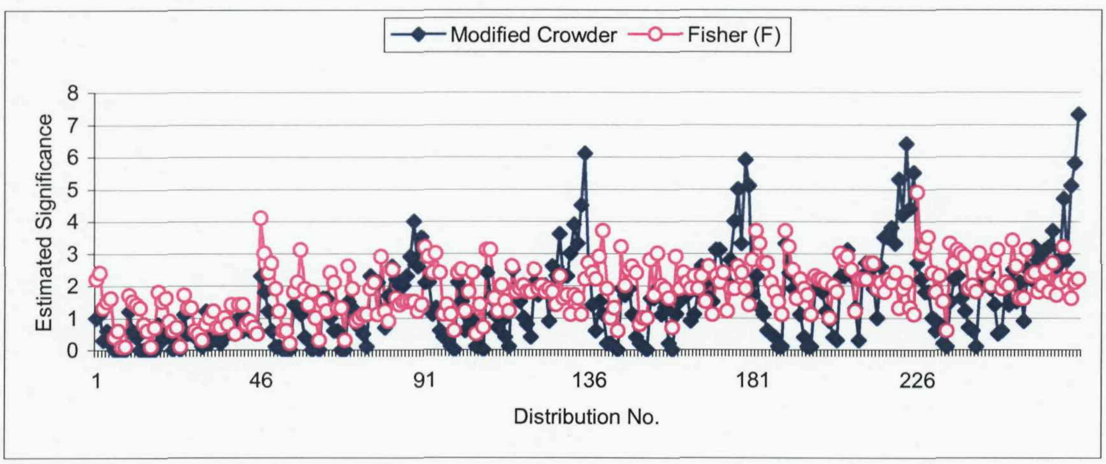


Modified Crowder
Fisher (F)
Estimated Significance
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
1
46
91
136
181
226
Distribution No.

$P(\lambda)$

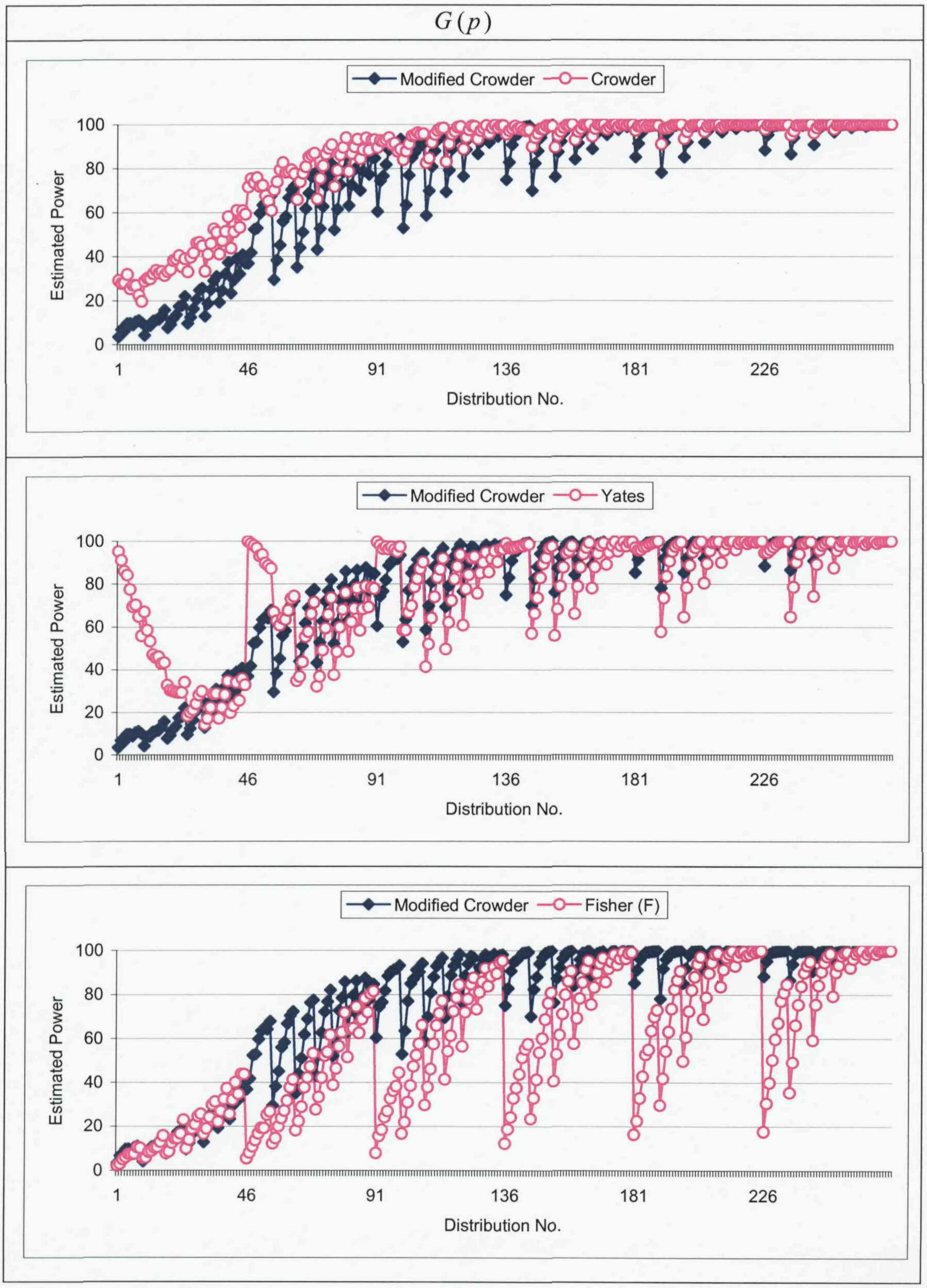

Modified Crow der
Crow der
Estimated Significance
0
1
2
3
4
5
6
7
8
1
46
91
136
181
226
Distribution No.


Modified Crow der
Yates
Estimated Significance
0
5
10
15
20
25
30
35
40
45
50
55
1
46
91
136
181
226
Distribution No.

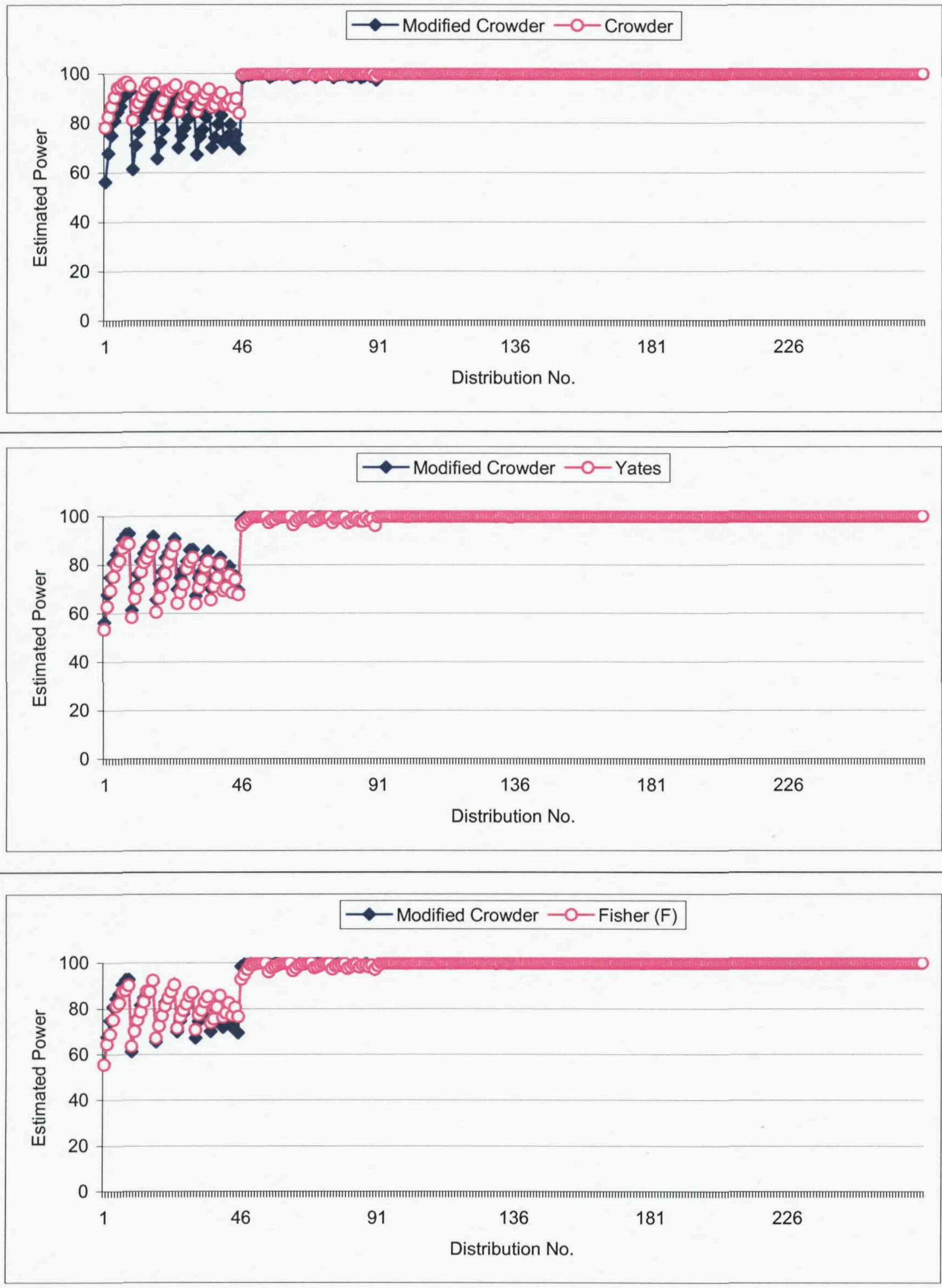

Modified Crowder
Fisher (F)
Estimated Significance
0
1
2
3
4
5
6
7
8
1
46
91
136
181
226
Distribution No.

Appendix E:

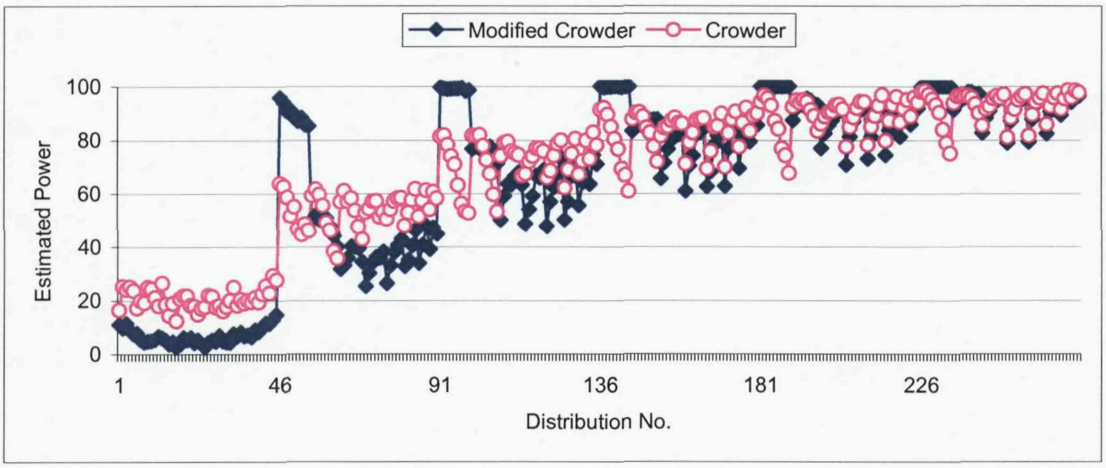
Figure E.1: Plots representation for the estimated power for samples with no censoring at the nominal 5% significance level.

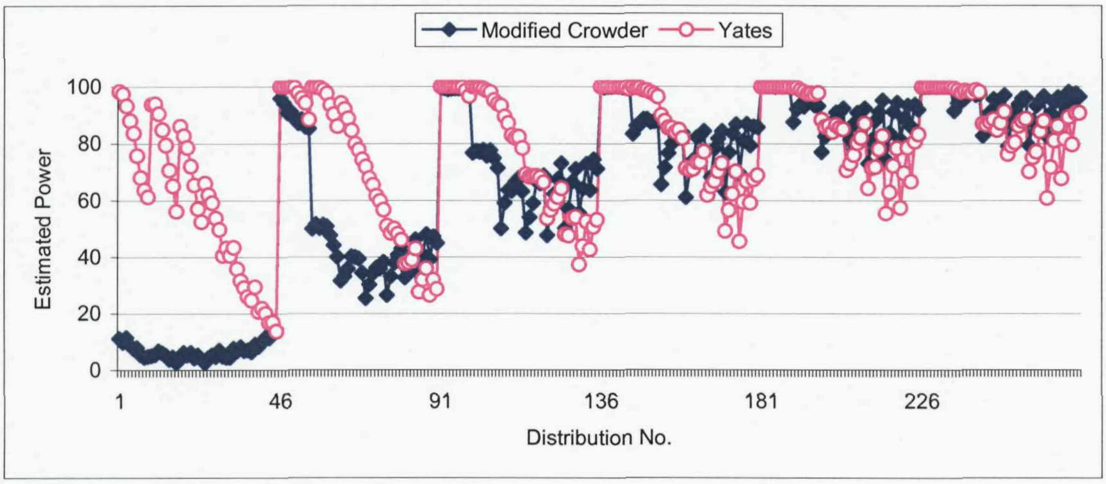


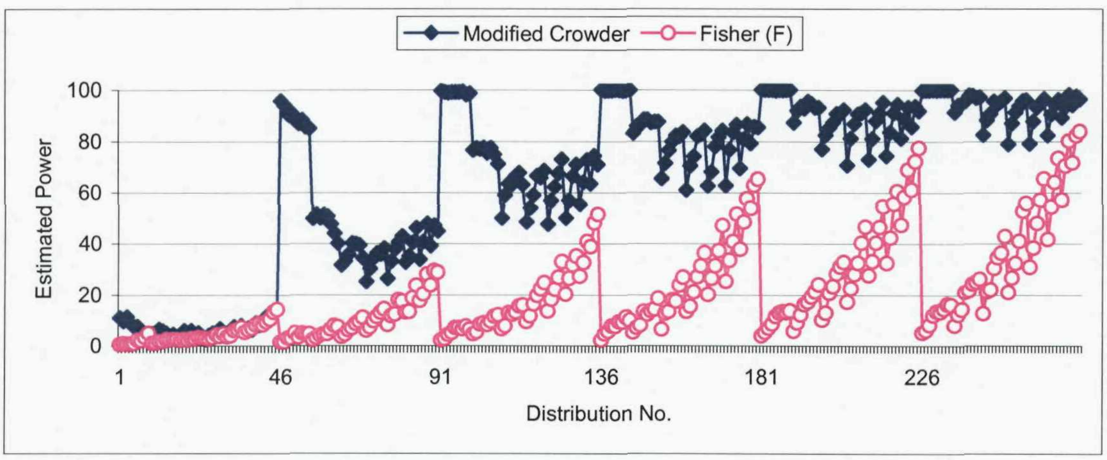
$N(p, 0.2)$


Modified Crowder
Crowder
Estimated Power
0
20
40
60
80
100
1
46
91
136
181
226
Distribution No.
Modified Crowder
Yates
Estimated Power
0
20
40
60
80
100
1
46
91
136
181
226
Distribution No.
Modified Crowder
Fisher (F)
Estimated Power
0
20
40
60
80
100
1
46
91
136
181
226
Distribution No.

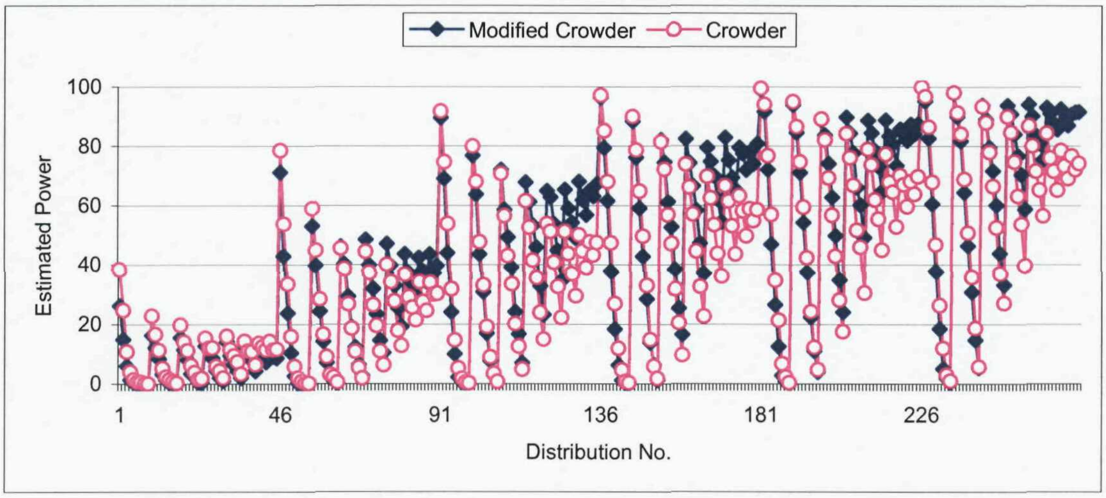
$N(p,2)$

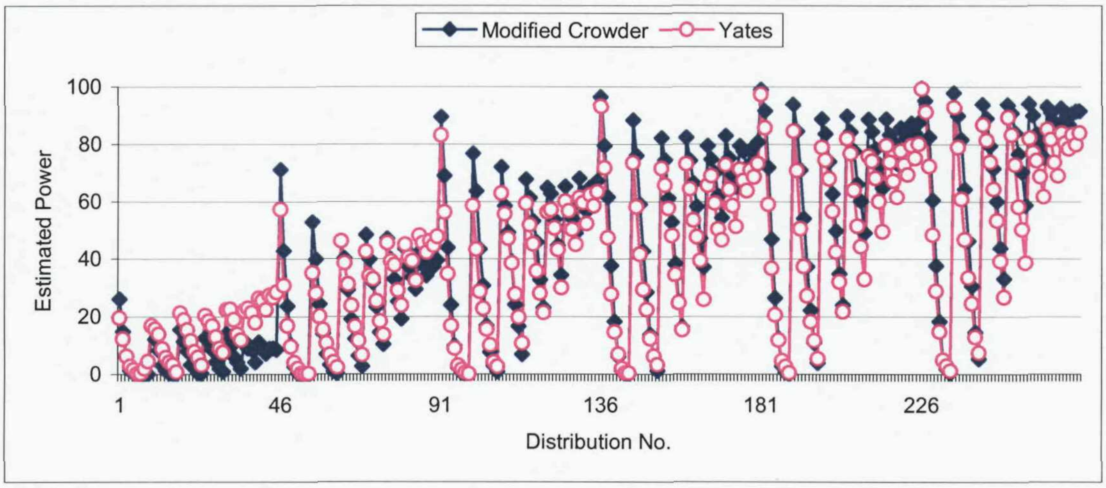

Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.


Modified Crowder
Yates
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.


Modified Crowder
Fisher (F)
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

$P(\lambda)$


Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.


Modified Crowder
Yates
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

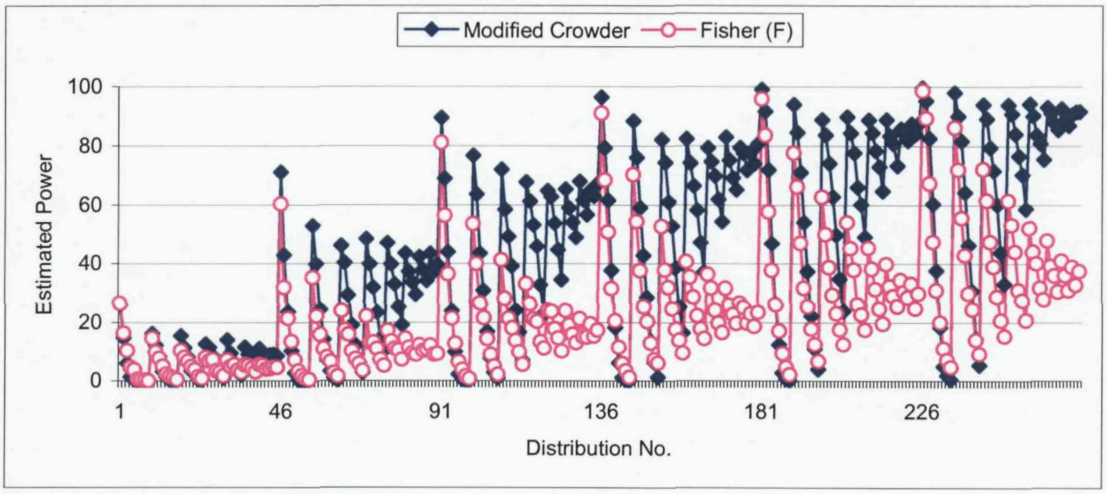
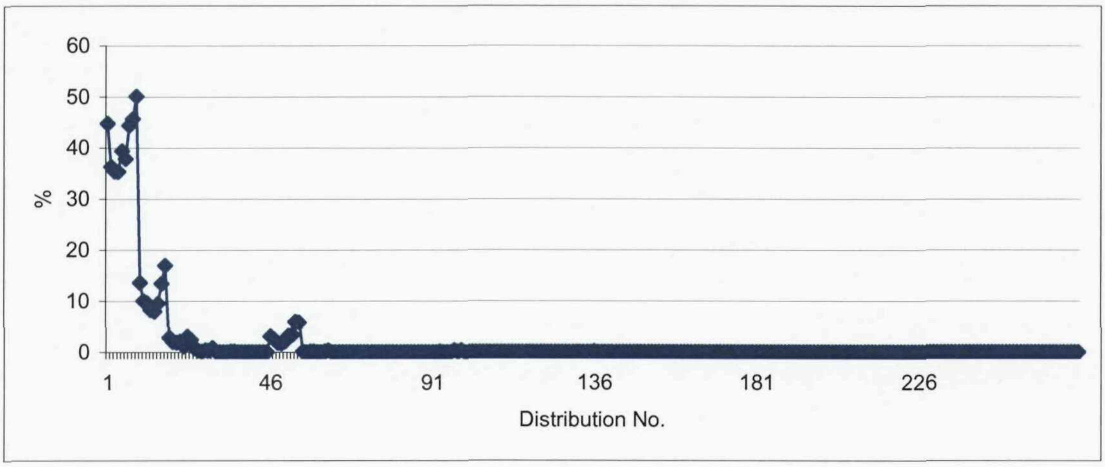
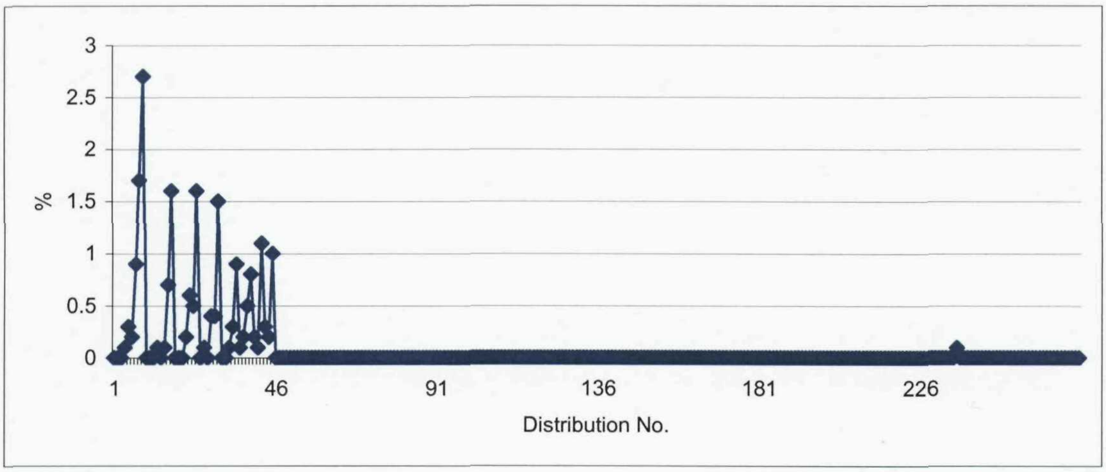

Modified Crowder
Fisher (F)
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

Figure E.2: Plots representation for the percentage of samples with no censoring for which W is not computable in Figure E.1.

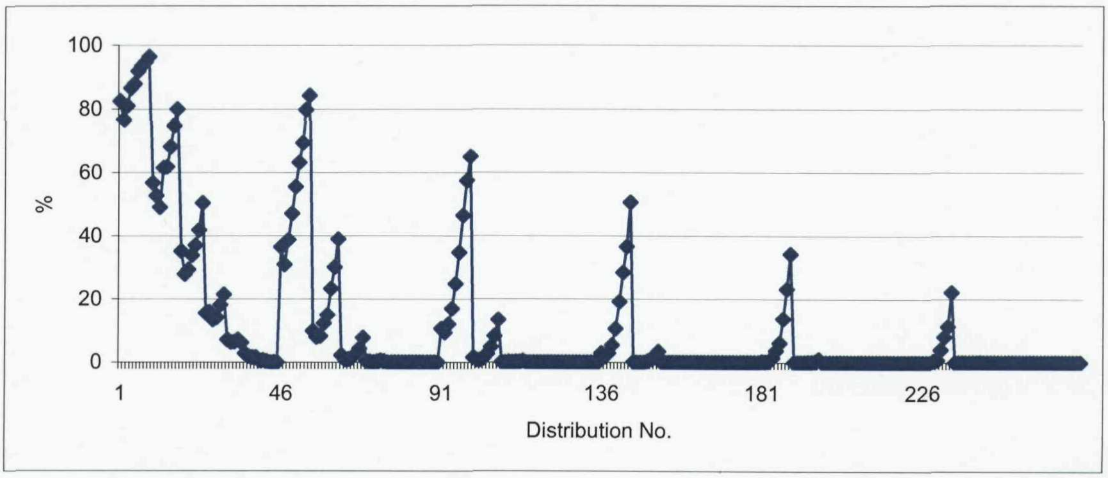
$G(p)$



$N(p, z = 0.2)$



$N(p, z = 2)$



$P(\lambda)$

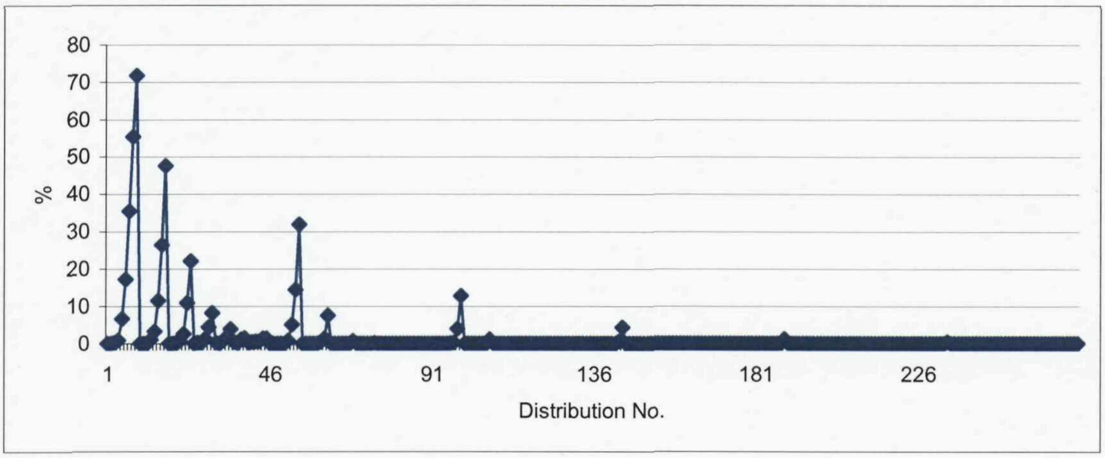
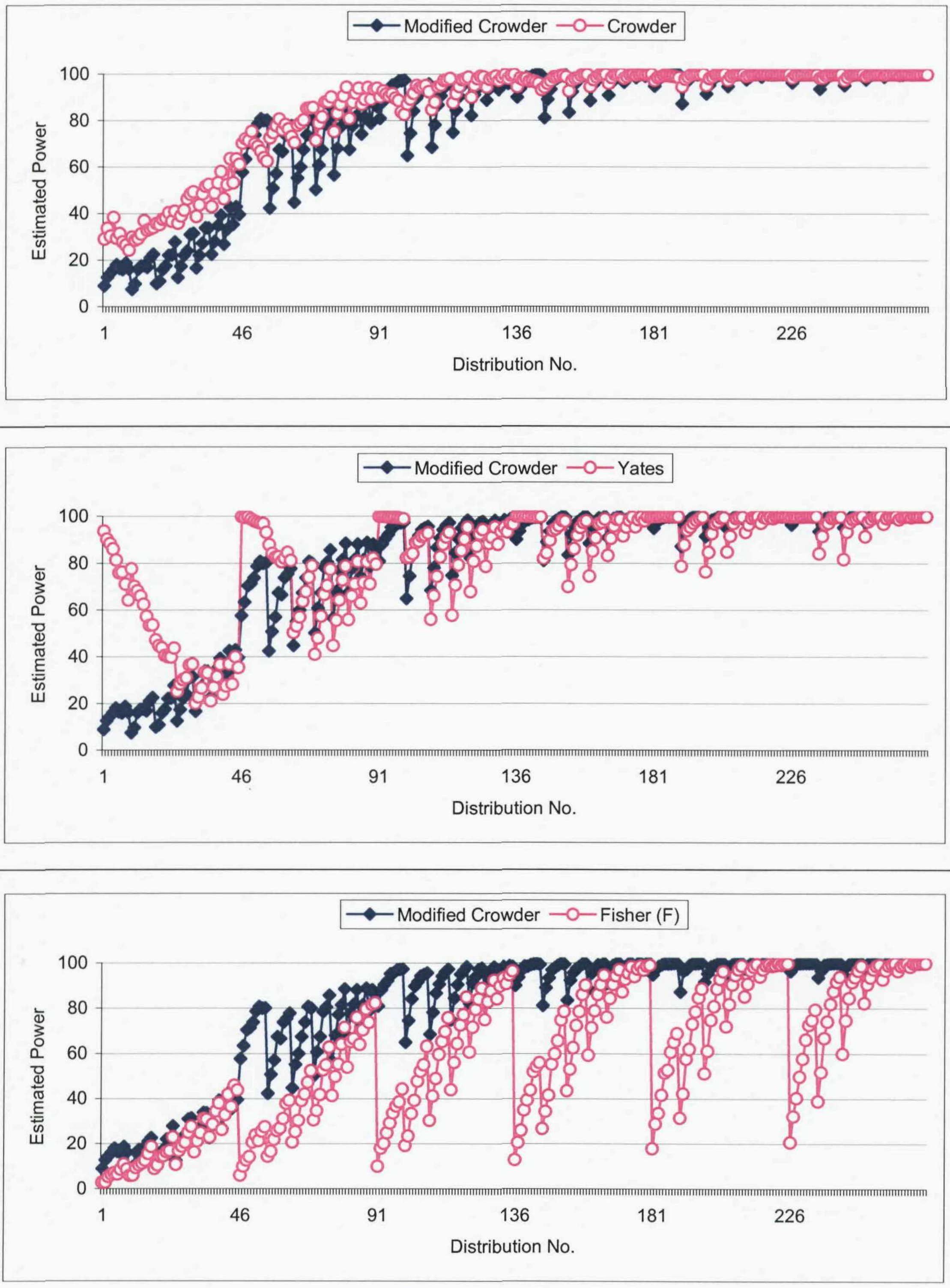
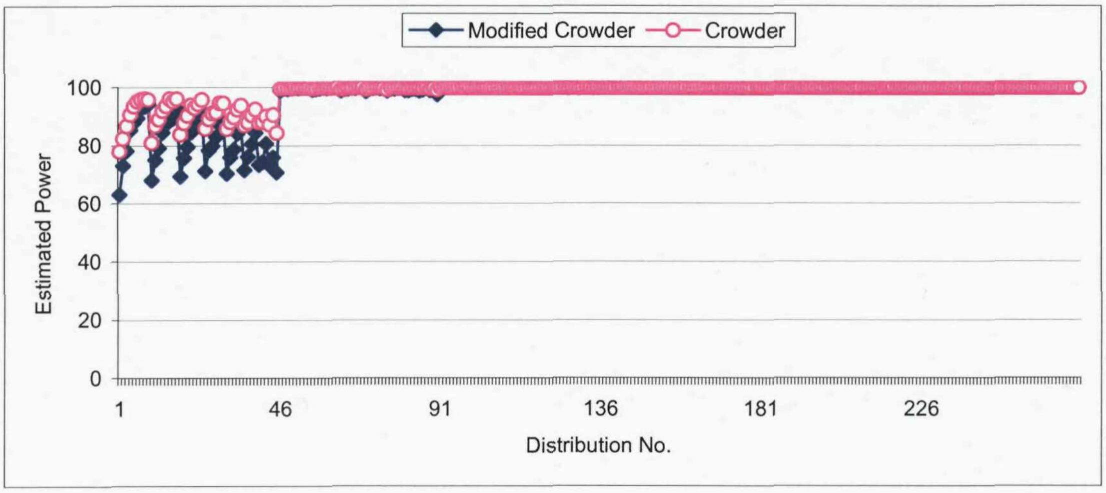


Figure E.3: Plots representation for the estimated power for samples with censoring at the nominal 5% significance level.

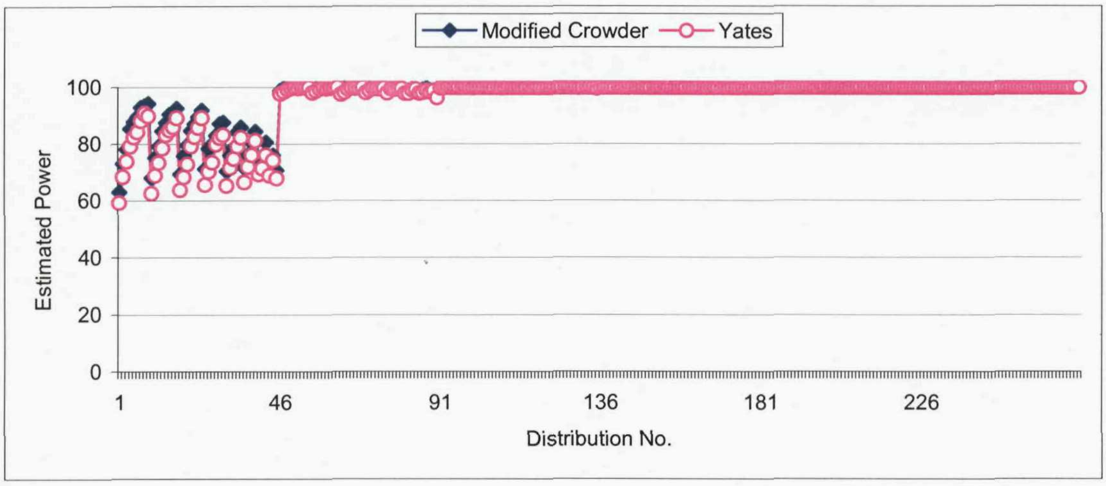
$G(p)$



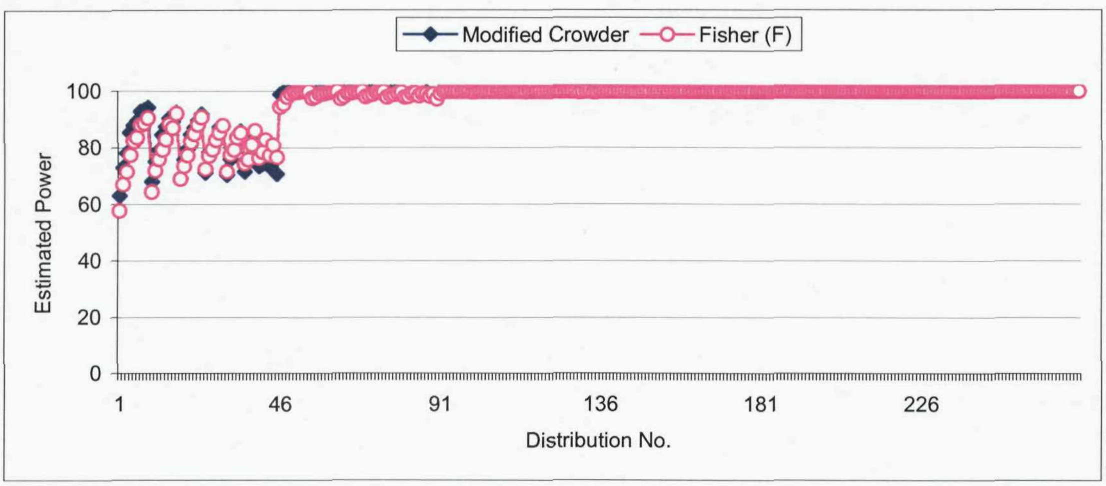
$N\,(p, 0.2)$



Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

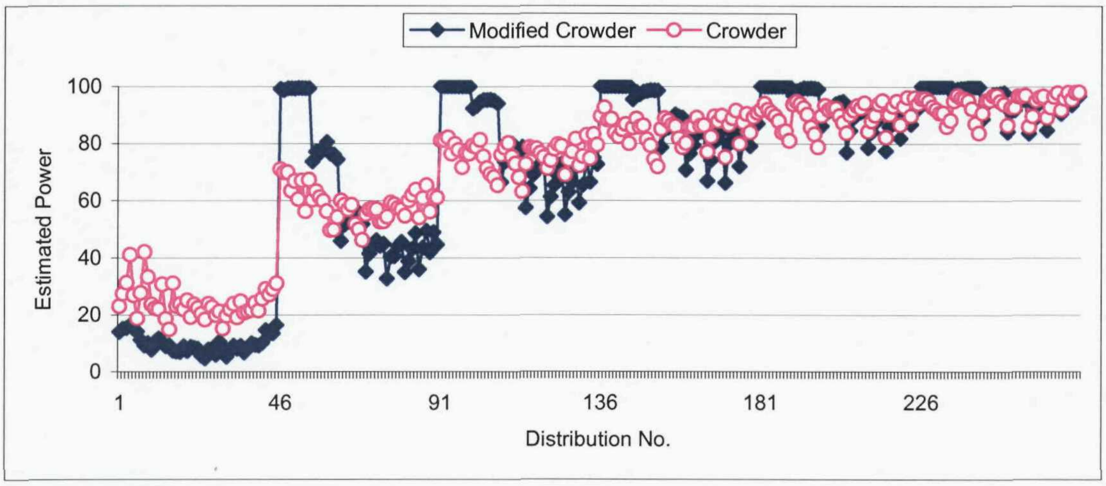


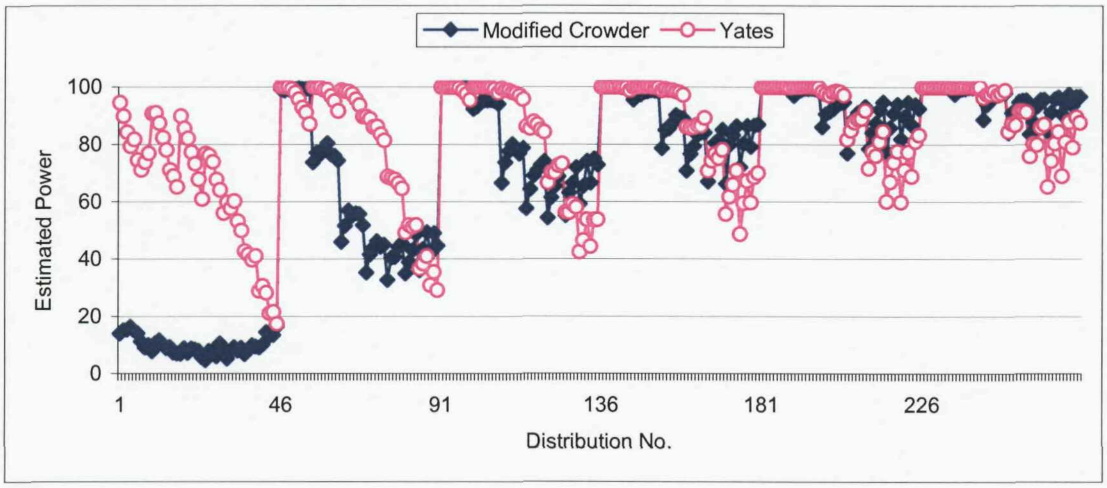
Modified Crowder
Yates
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

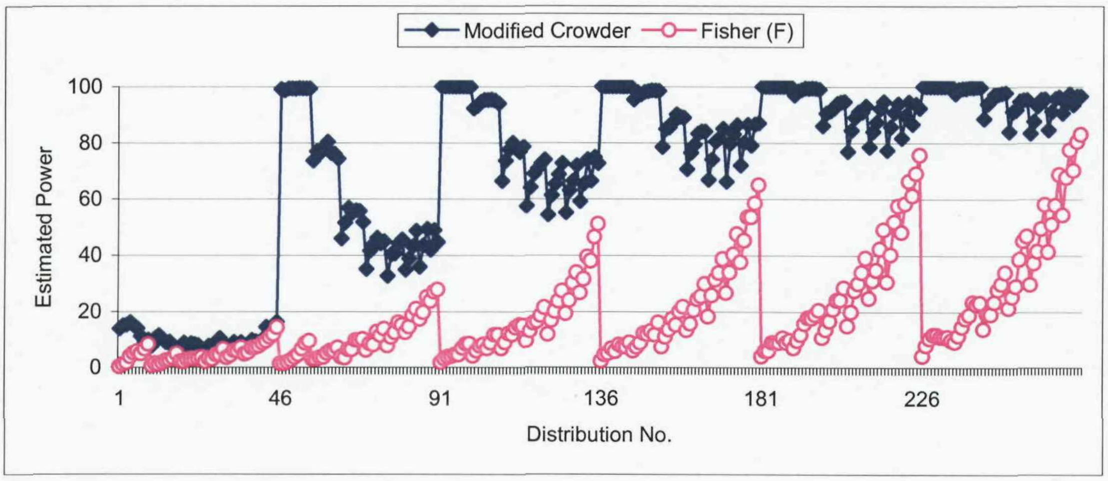


Modified Crowder
Fisher (F)
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

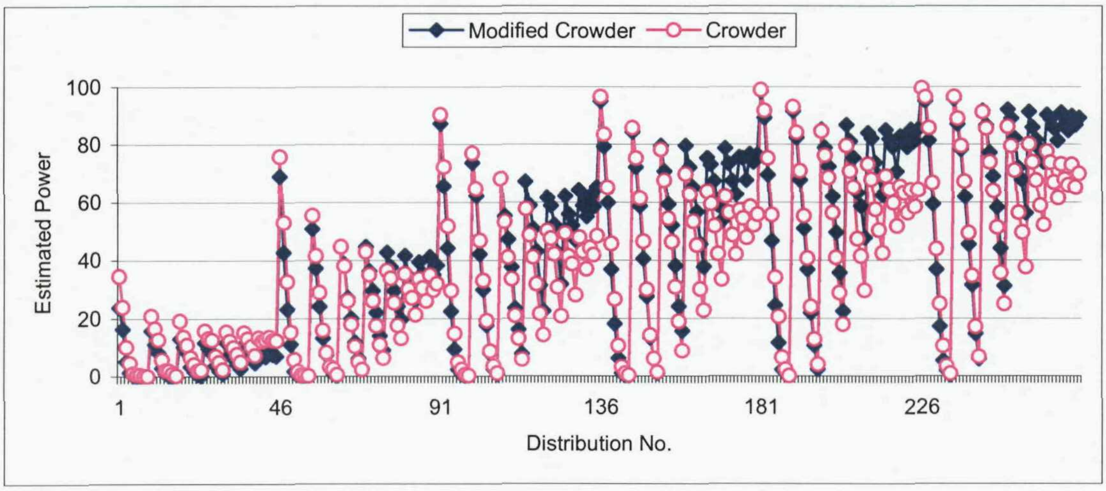
$N(p,2)$


Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

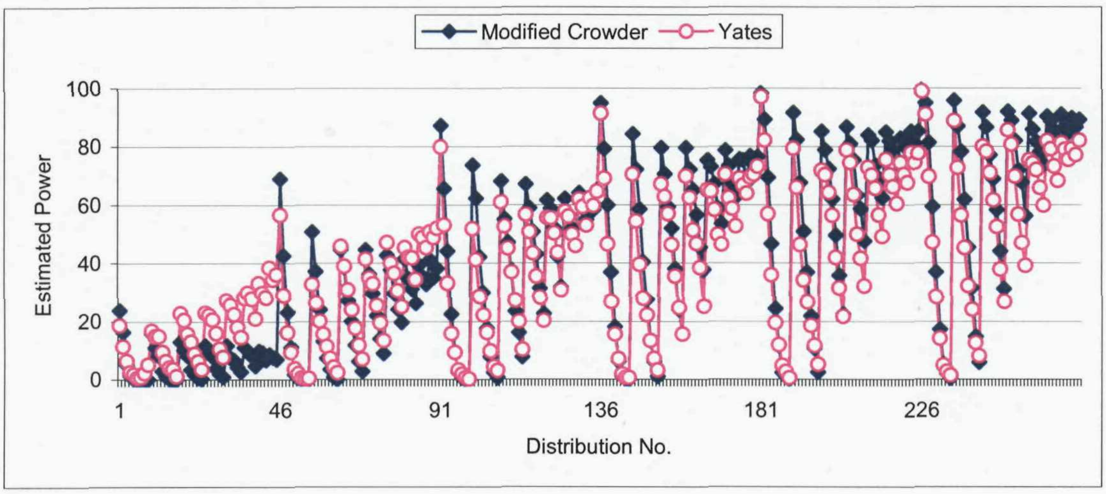

Modified Crowder
Yates
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.


Modified Crowder
Fisher (F)
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

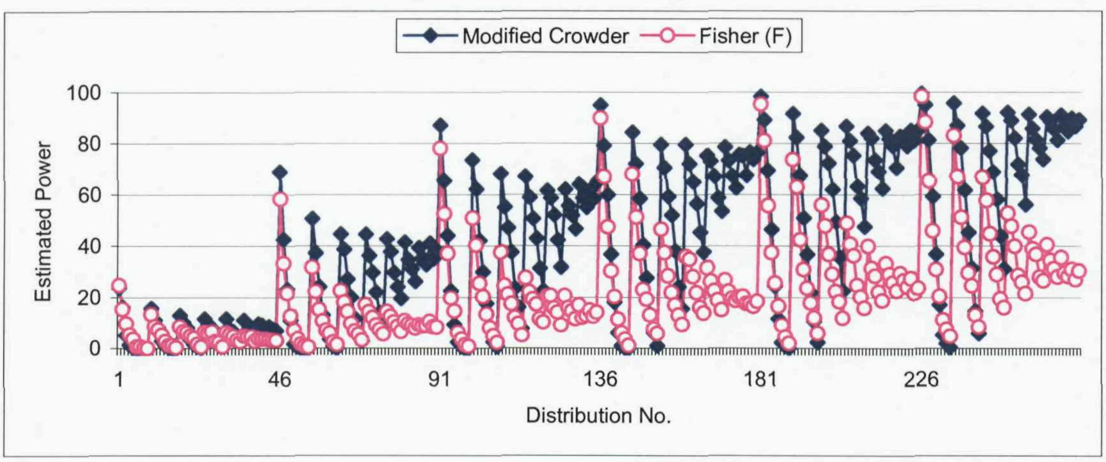
$P(\lambda)$



Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.



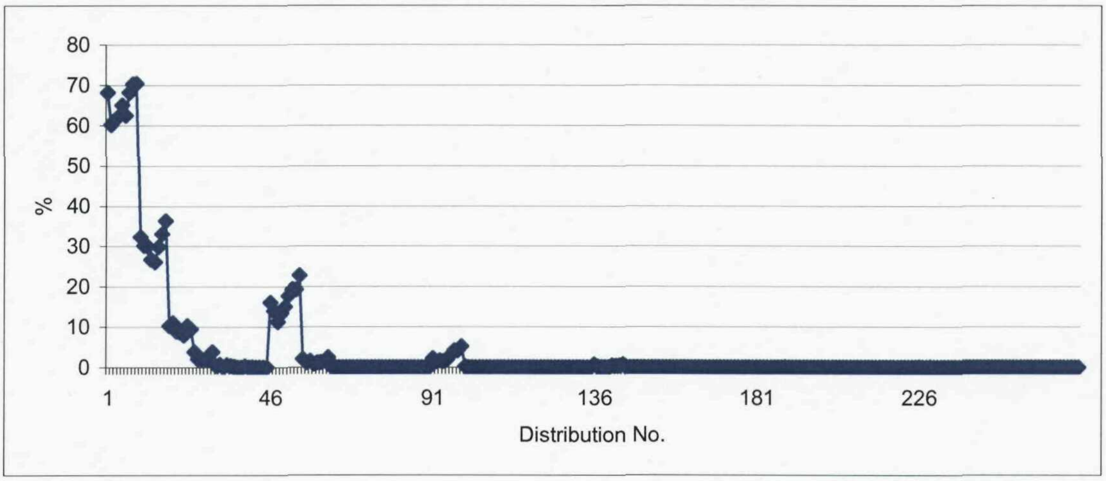
Modified Crowder
Yates
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.



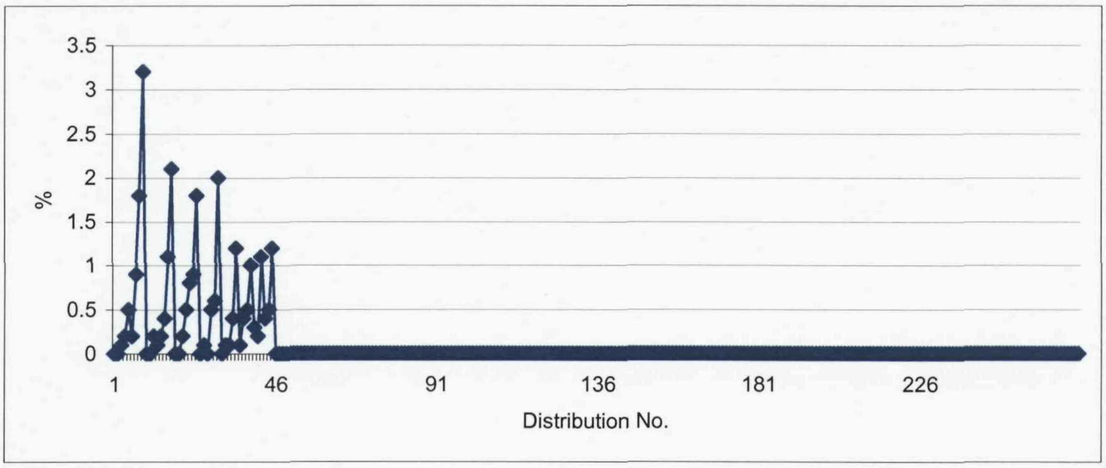
Modified Crowder
Fisher (F)
Estimated Power
100
80
60
40
20
0
1
46
91
136
181
226
Distribution No.

Figure E.4: Plots representation for the percentage of samples with censoring for which W is not computable in Figure E.3.

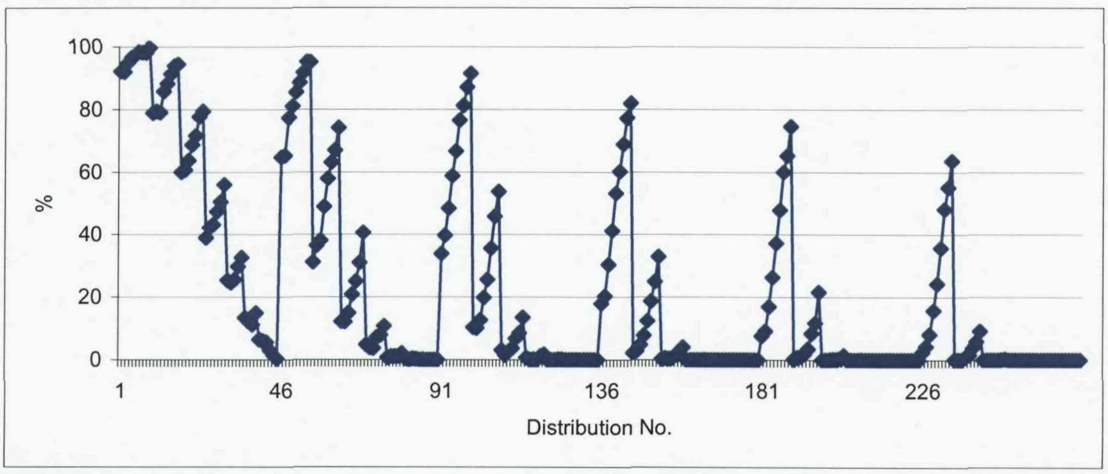
$G(p)$



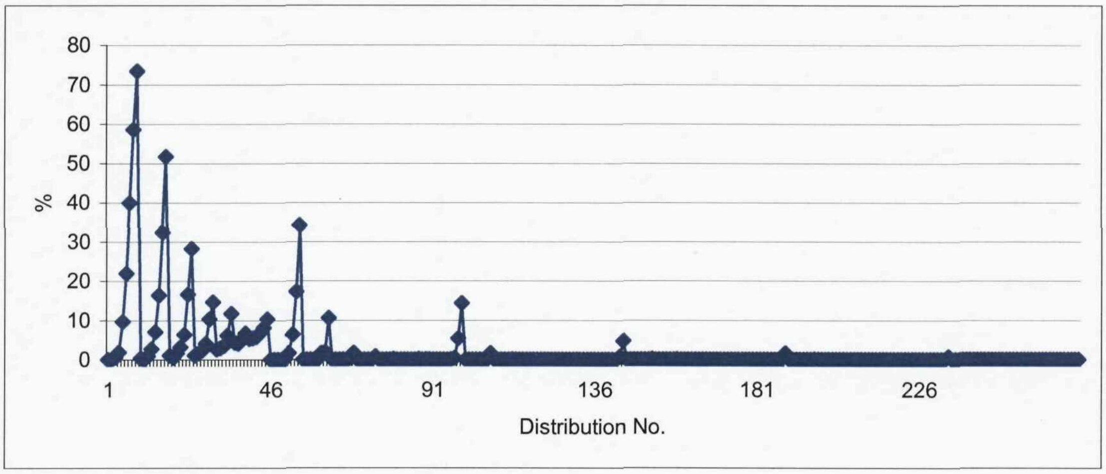
$N(p, z = 0.2)$



$N(p, z = 2)$



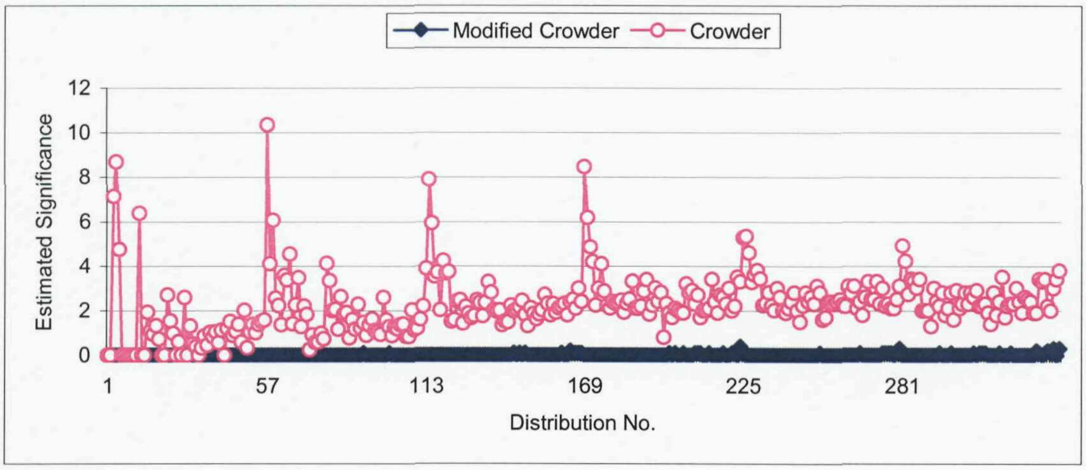
$P(\lambda)$

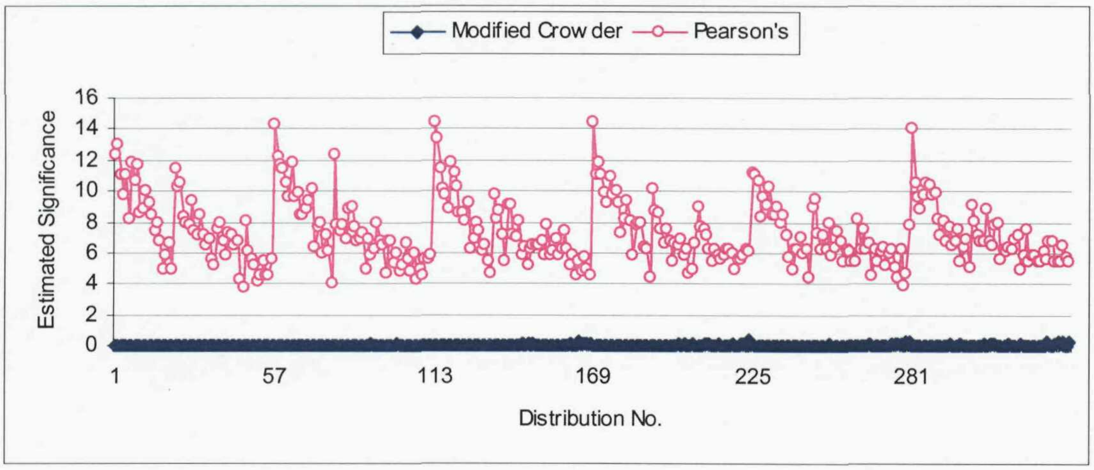


Appendix F

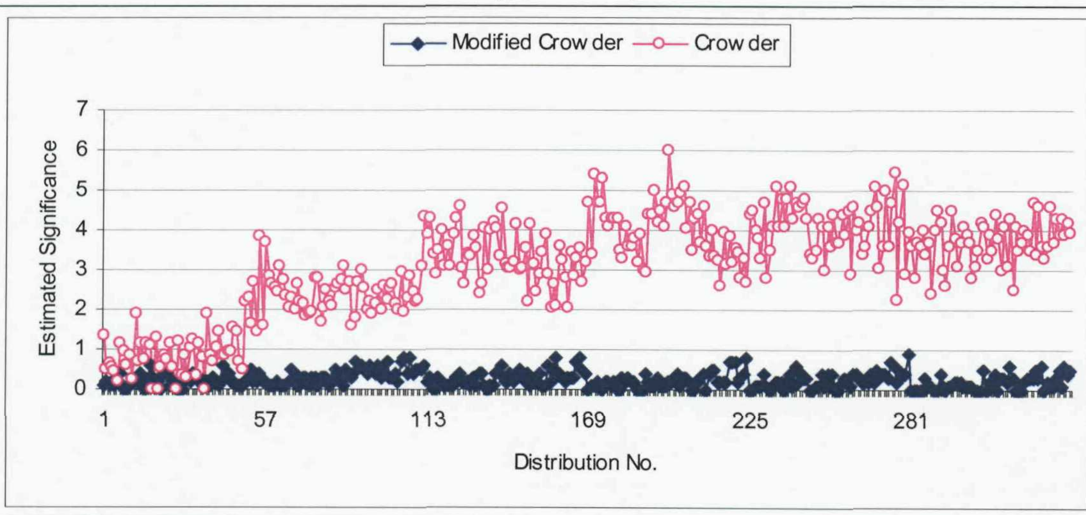
Figure F.1: Plots representation for the estimated significance for samples with no censoring at the nominal 5% significance level.

$G(p)$

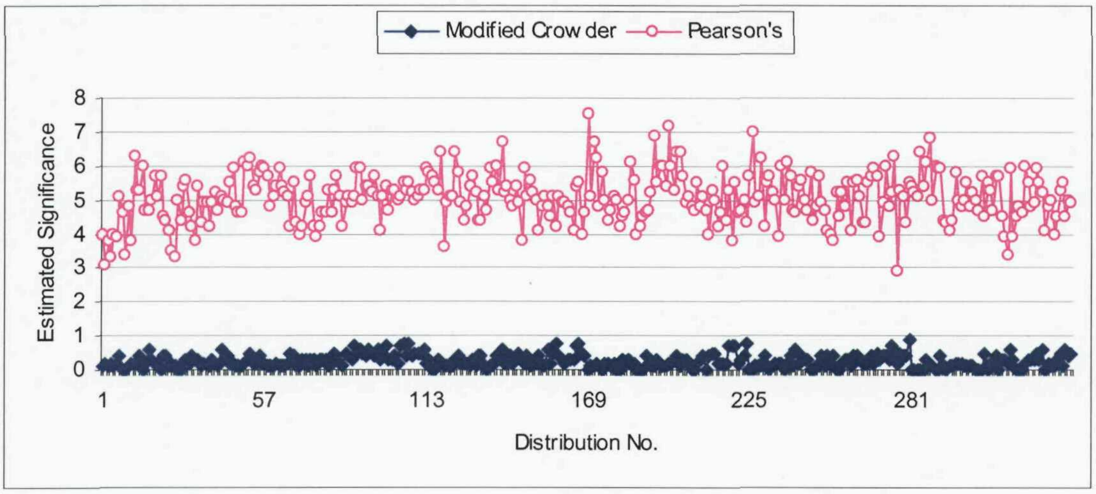




$N(p, z = 0.2)$

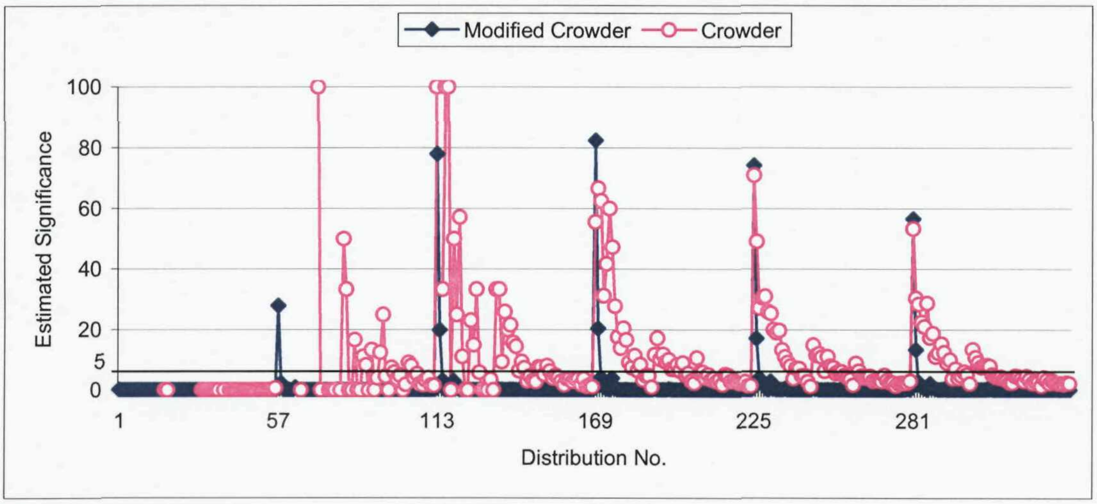


$N\,(p, z = 0.2)$

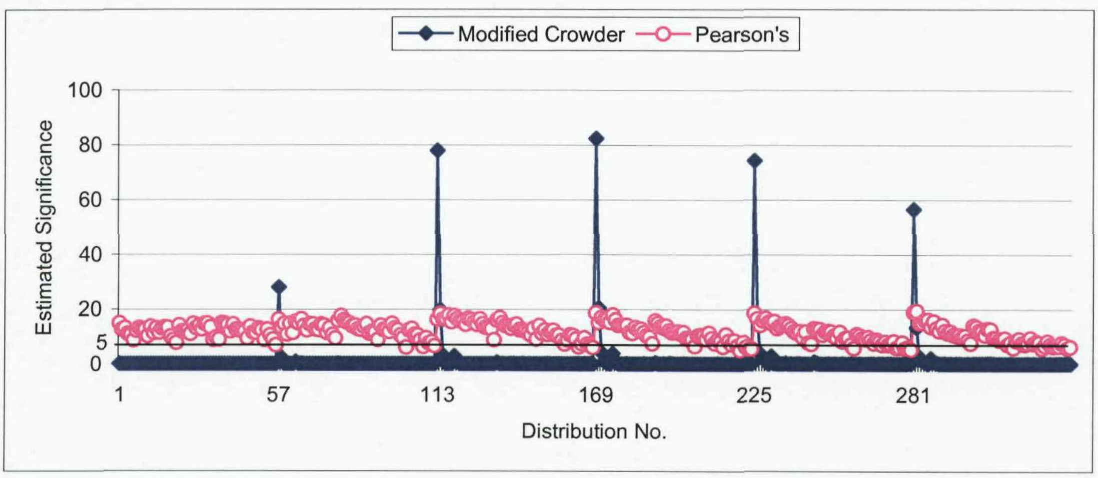


Modified Crow der
Pearson's
Estimated Significance
8
7
6
5
4
3
2
1
0
1
57
113
169
225
281
Distribution No.

$N\,(p, z = 2)$

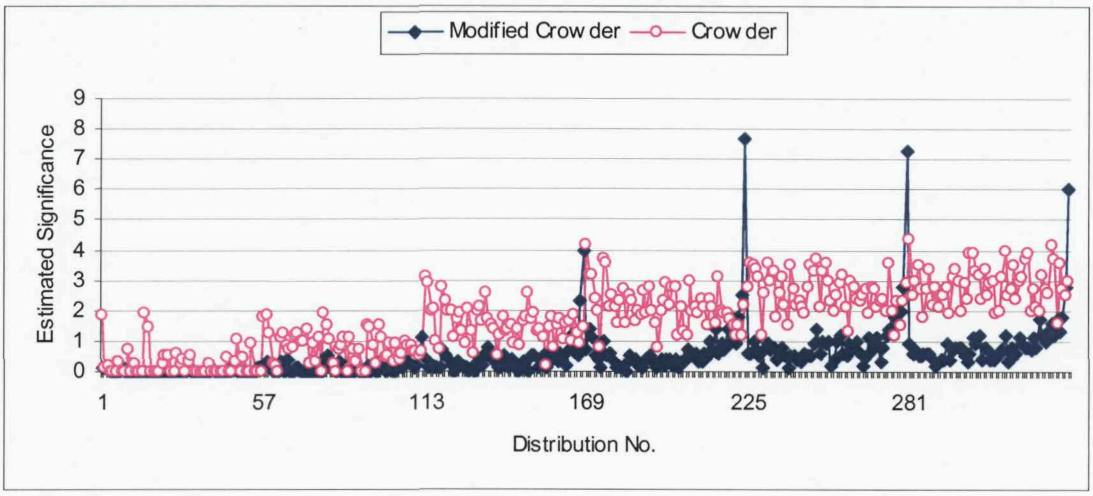


Modified Crowder
Crowder
Estimated Significance
100
80
60
40
20
5
0
1
57
113
169
225
281
Distribution No.

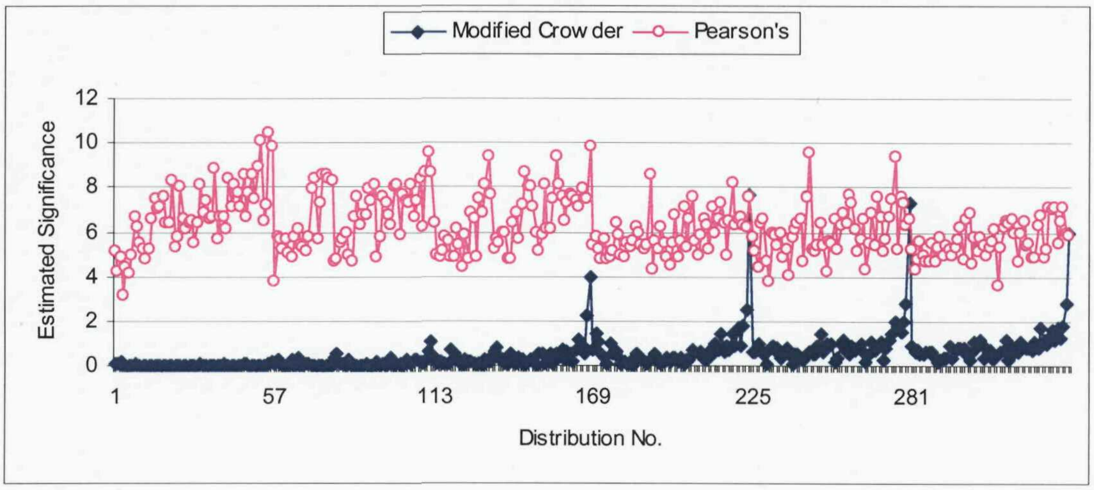


Modified Crowder
Pearson's
Estimated Significance
100
80
60
40
20
5
0
1
57
113
169
225
281
Distribution No.

$P(\lambda)$



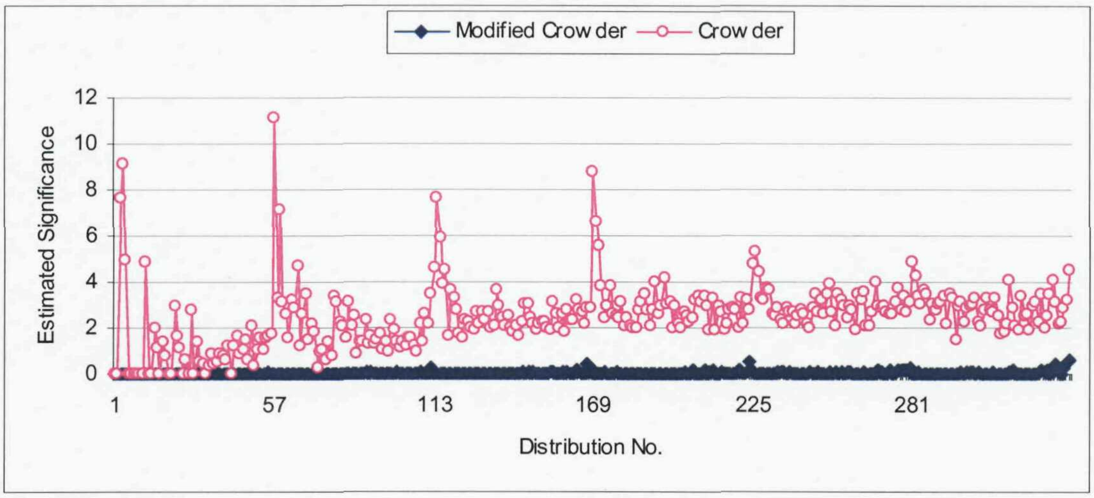
Modified Crow der
Crow der
Estimated Significance
9
8
7
6
5
4
3
2
1
0
1
57
113
169
225
281
Distribution No.

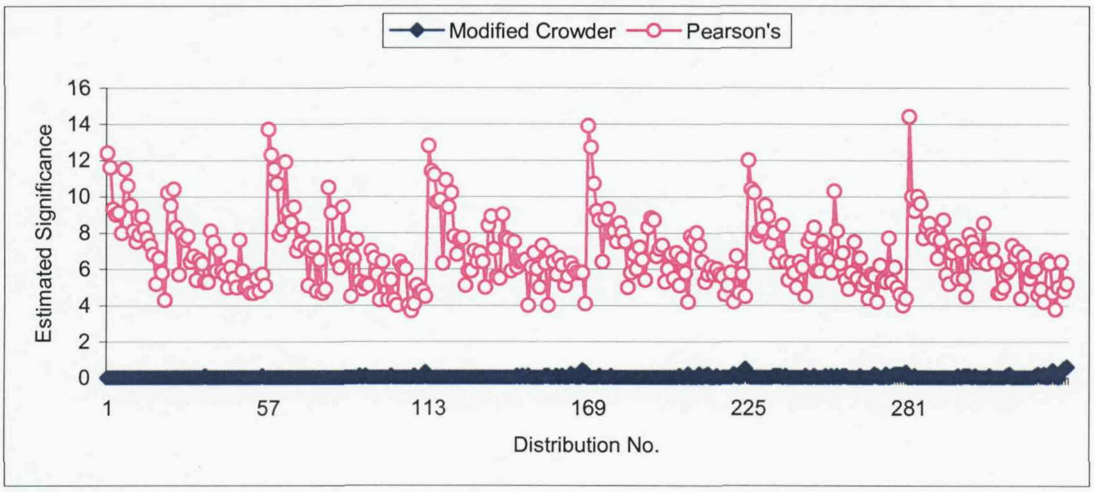


Modified Crow der
Pearson's
Estimated Significance
12
10
8
6
4
2
0
1
57
113
169
225
281
Distribution No.

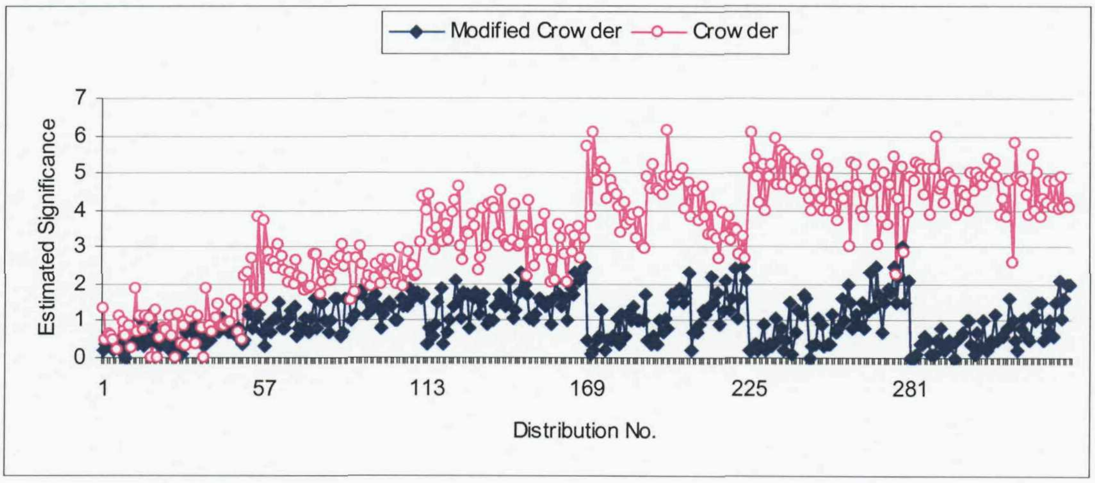
Figure F.2: Plots representation for the estimated significance for samples with censoring at the nominal 5% significance level.

$G(p)$

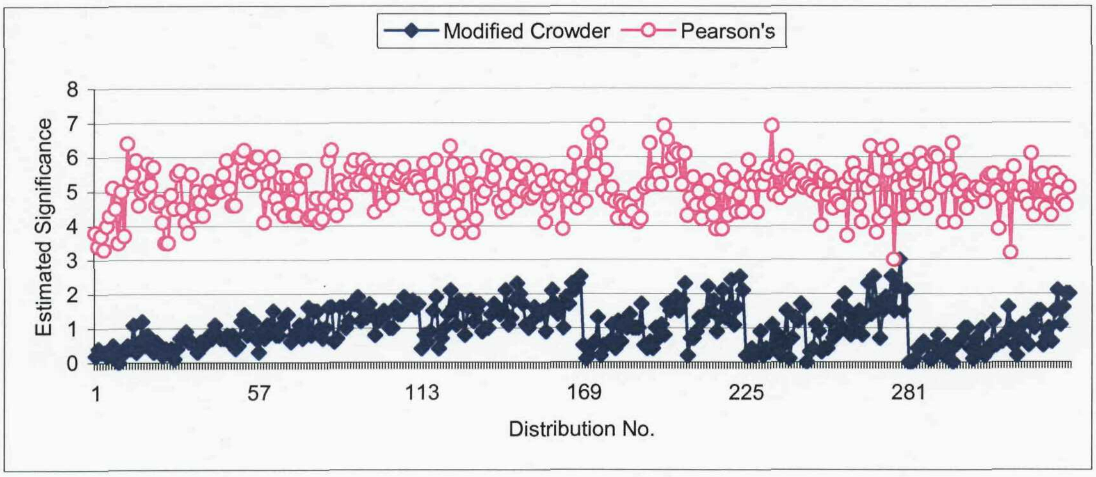




$N(p, z = 0.2)$

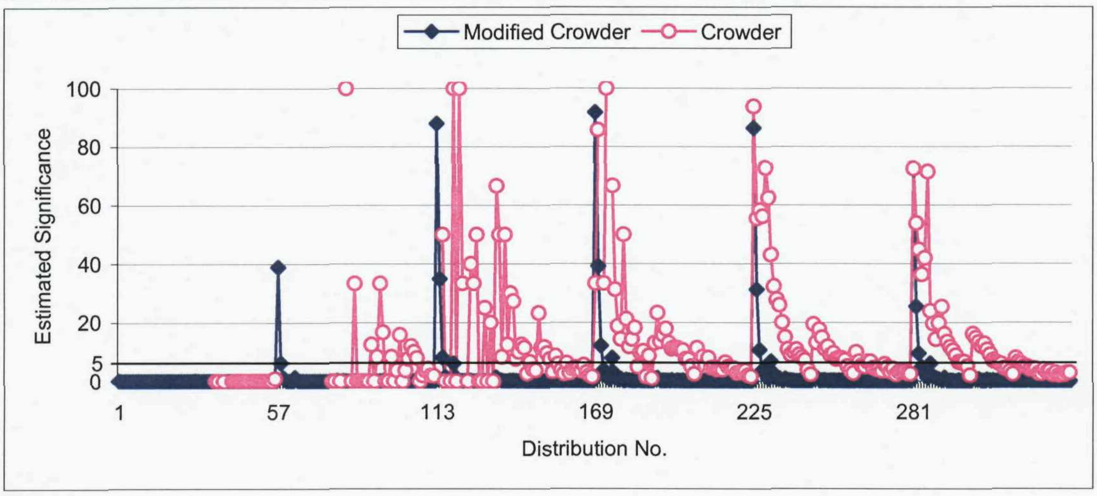


$N\,(p, z = 0.2)$

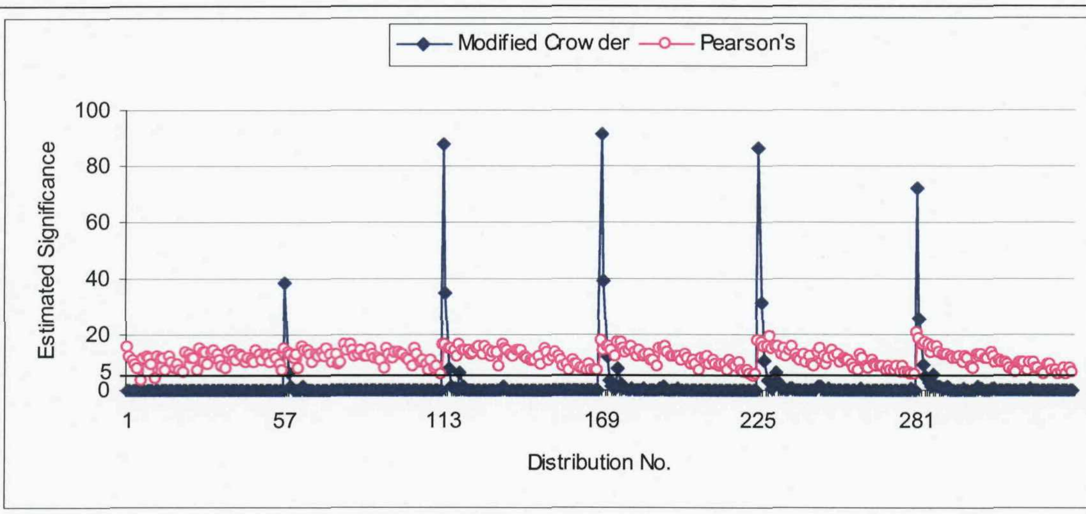


Modified Crowder
Pearson's
Estimated Significance
0
1
2
3
4
5
6
7
8
1
57
113
169
225
281
Distribution No.

$N\,(p, z = 2)$

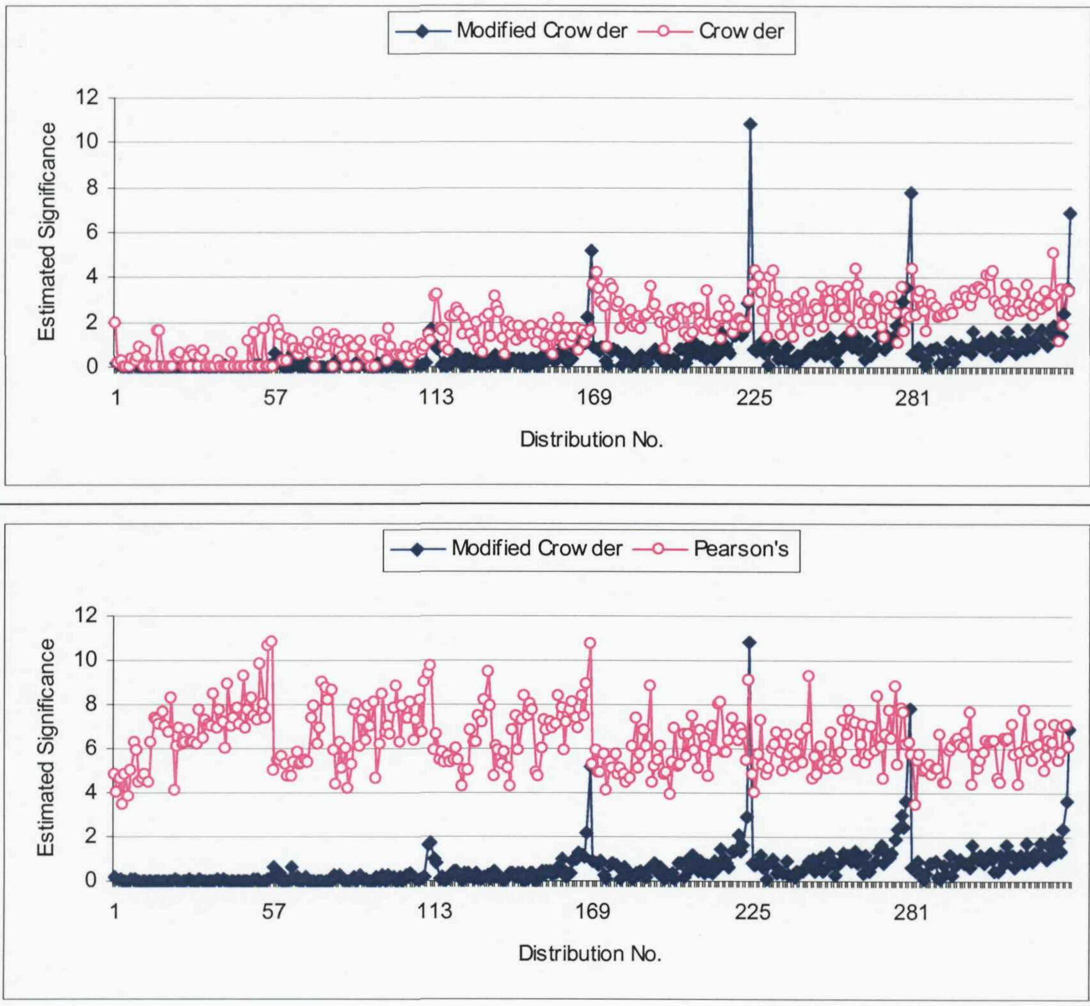


Modified Crowder
Crowder
Estimated Significance
0
5
20
40
60
80
100
1
57
113
169
225
281
Distribution No.



Modified Crow der
Pearson's
Estimated Significance
0
5
20
40
60
80
100
1
57
113
169
225
281
Distribution No.

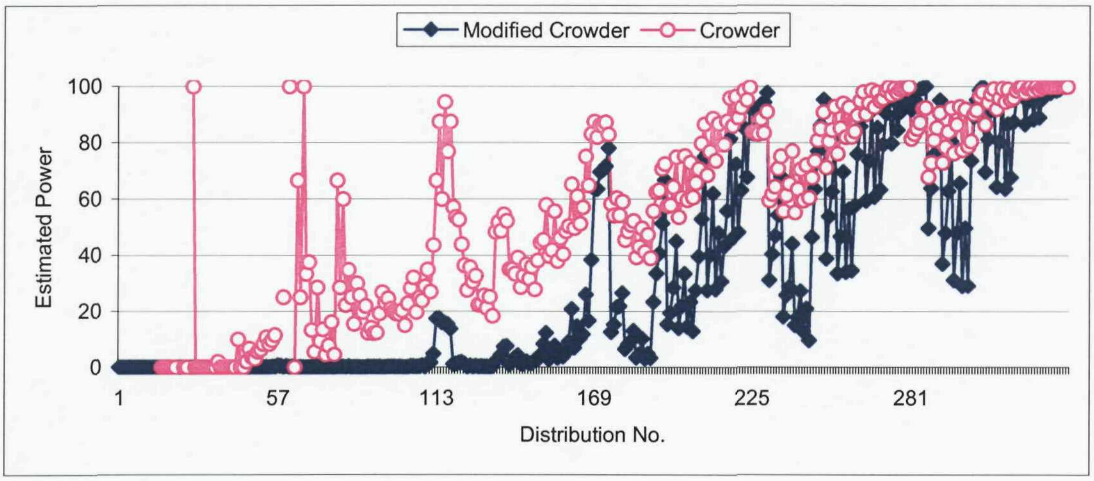
$P(\lambda)$

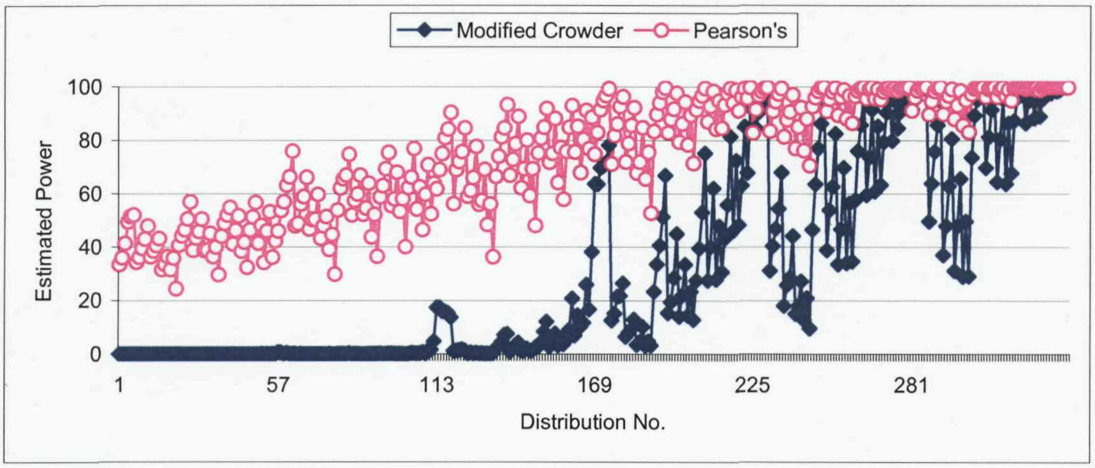

Modified Crow der
Crow der
Estimated Significance
12
10
8
6
4
2
0
1
57
113
169
225
281
Distribution No.
Modified Crow der
Pearson's
Estimated Significance
12
10
8
6
4
2
0
1
57
113
169
225
281
Distribution No.

Appendix G

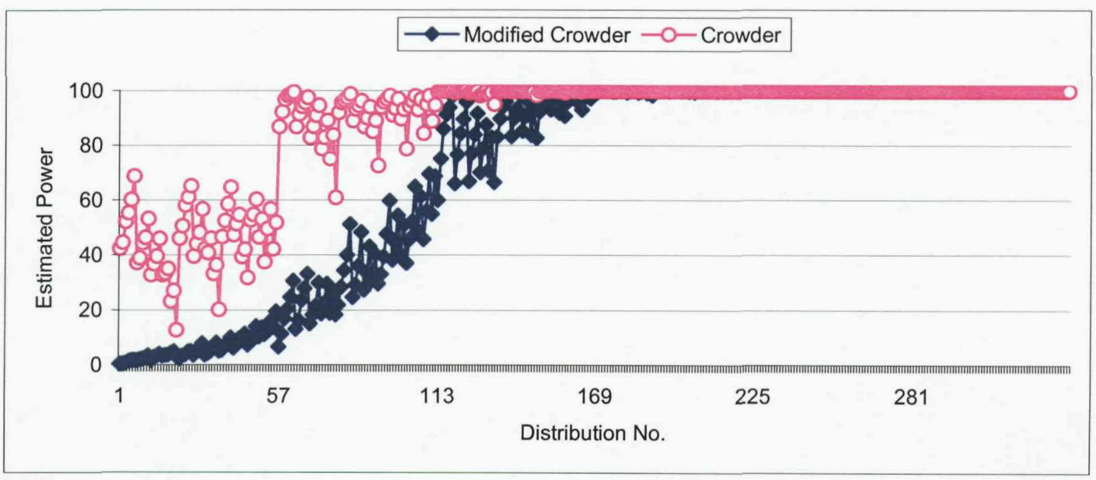
Figure G.1: Plots representation for the estimated power for samples with no censoring and two dependent risks at the nominal 5% significance level.

$G(p)$

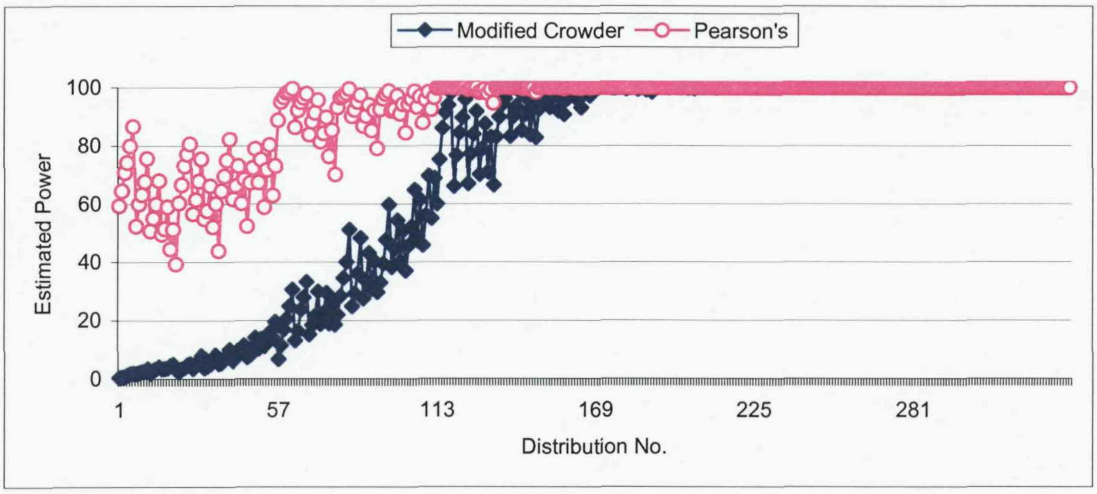




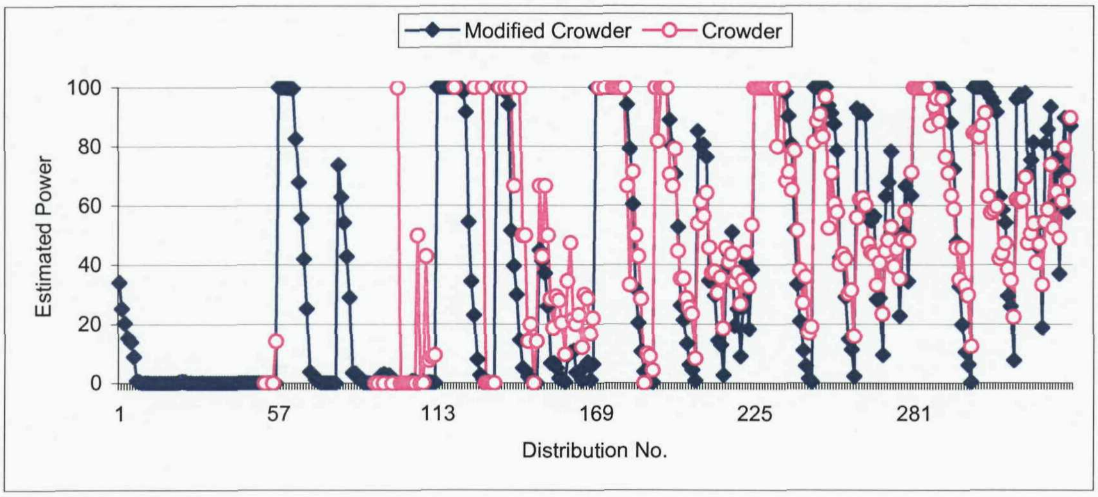
$N(p, z = 0.2)$

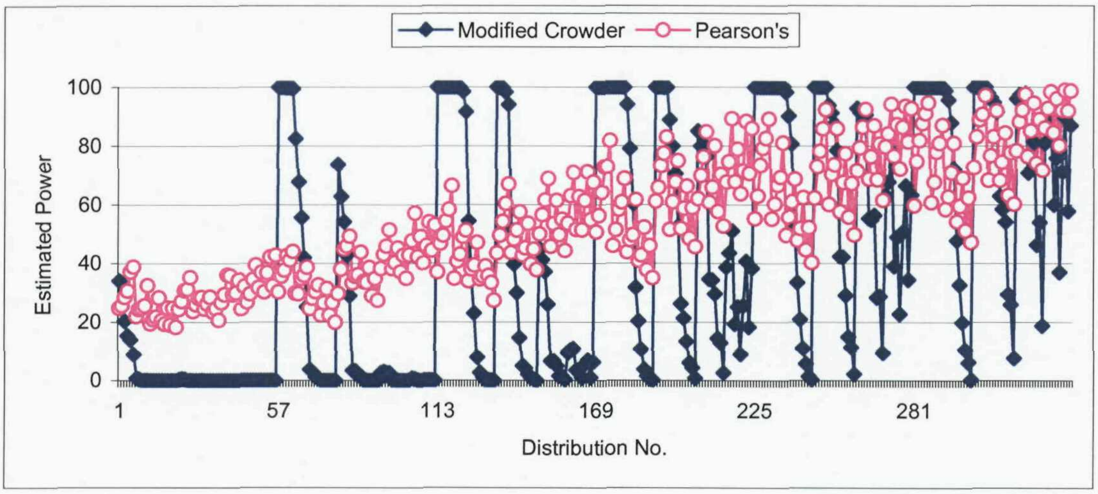


$N(p, z = 0.2)$

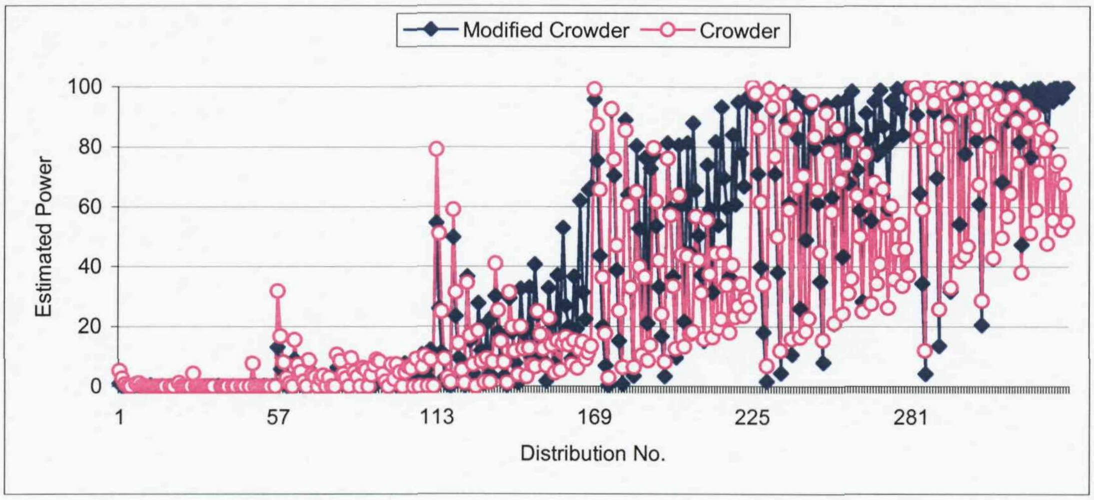

Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$N(p, z = 2)$


Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.


Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$P(\lambda)$


Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

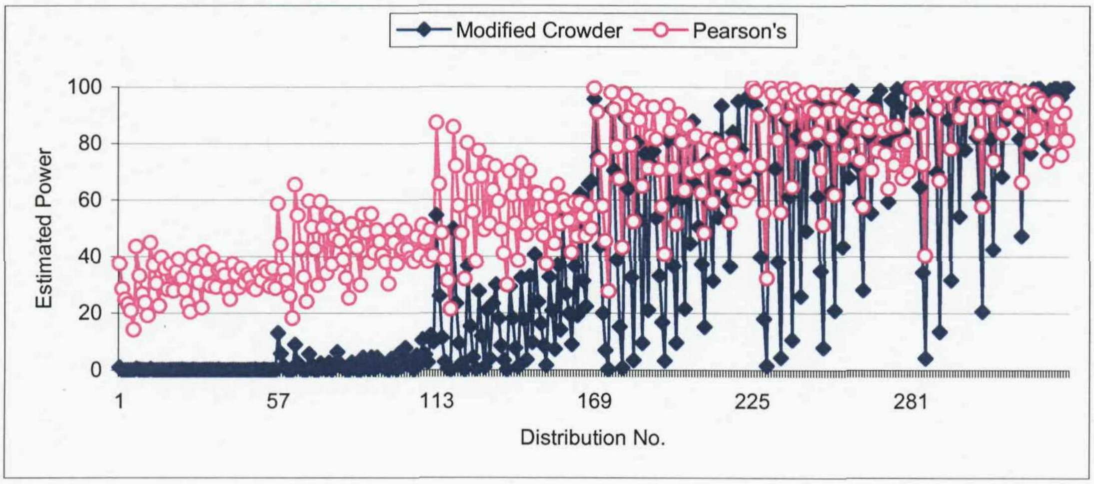
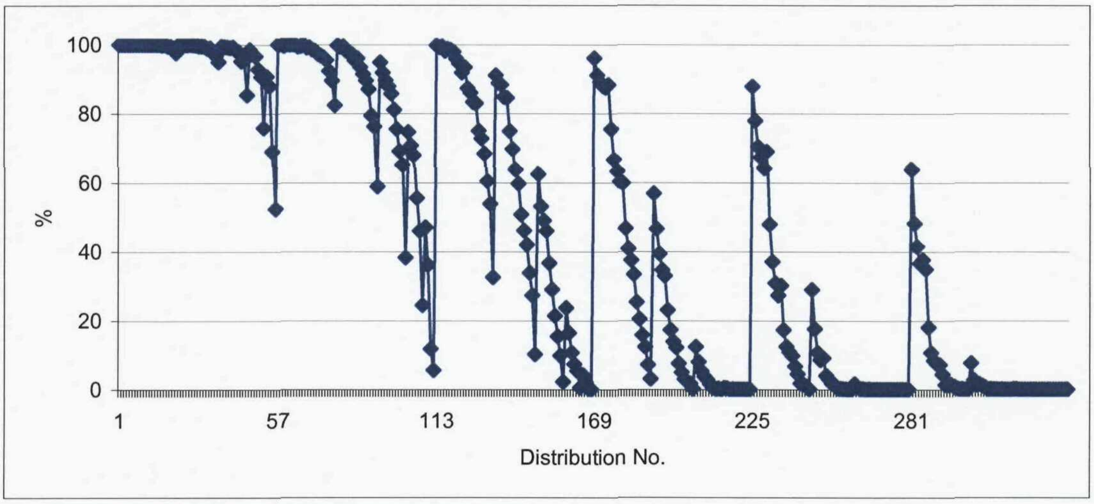
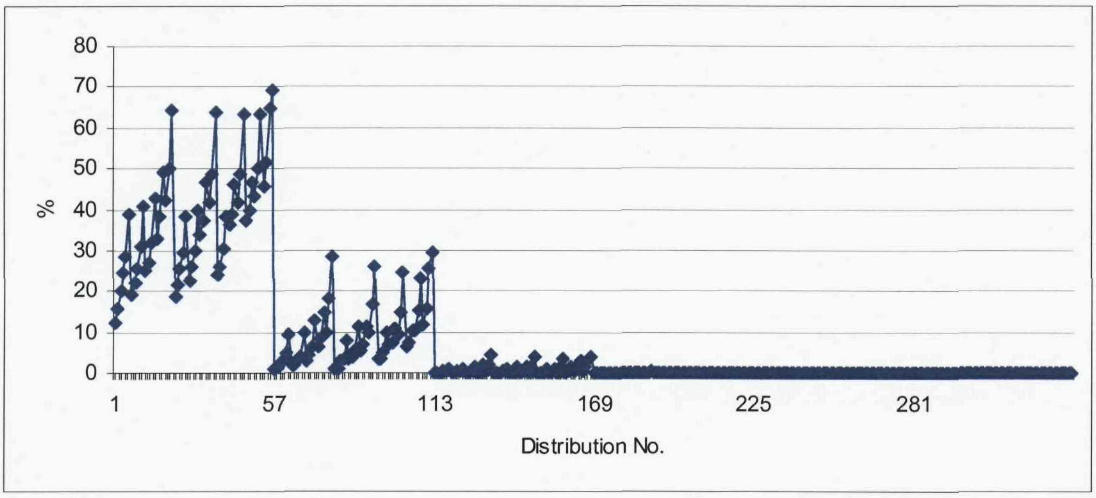

Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

Figure G.2: Plots representation for the percentage of samples for which W is not computable in Figure G.1.

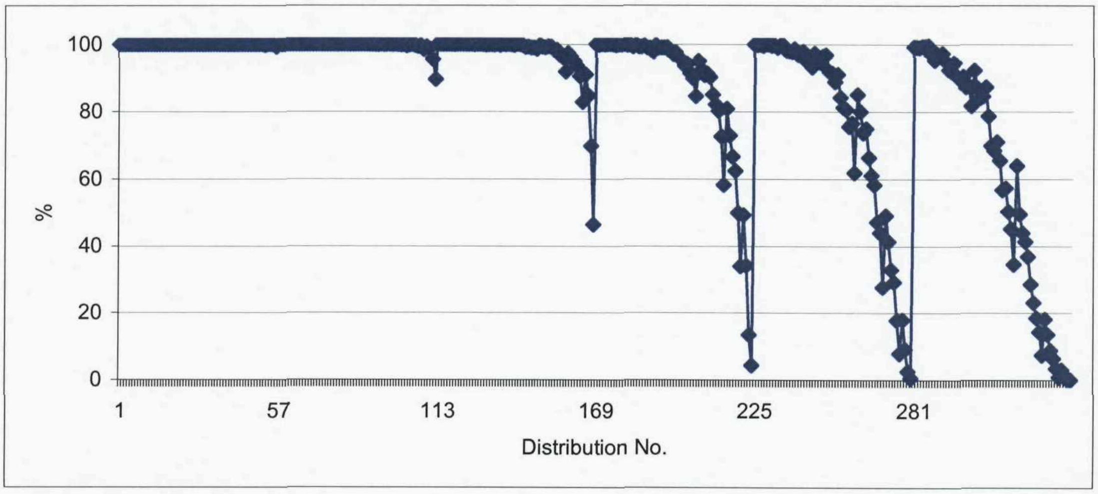
$G(p)$



$N(p, z = 0.2)$



$N(p, z = 2)$



$P(\lambda)$

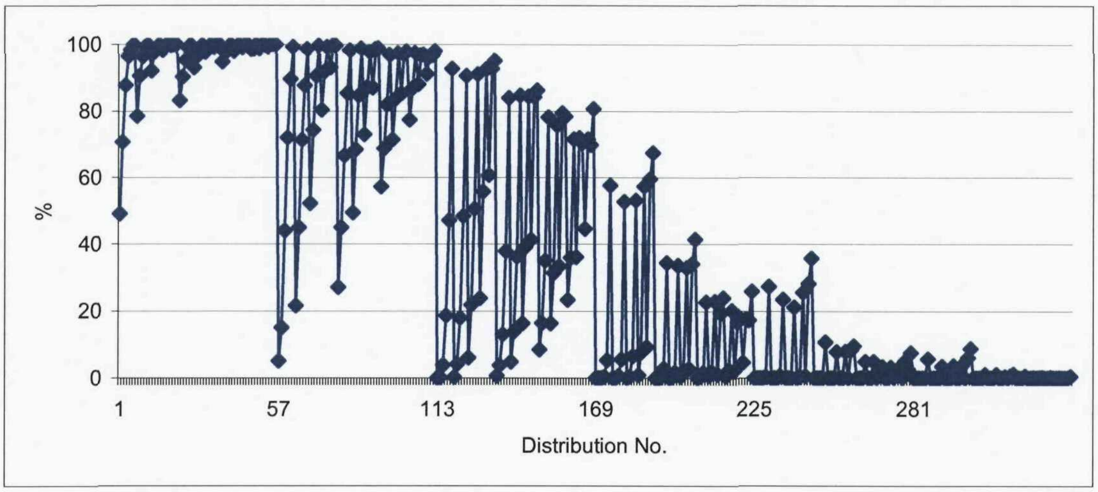
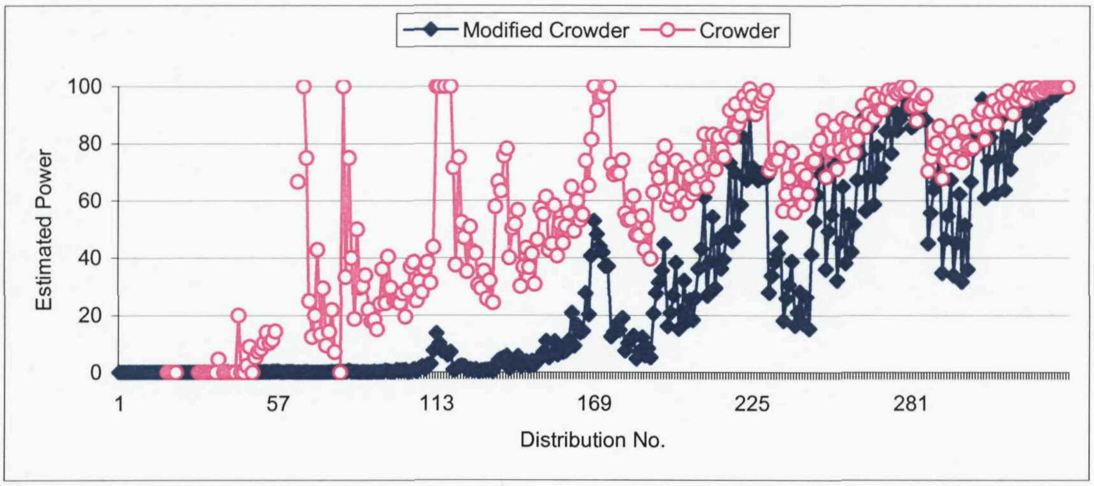
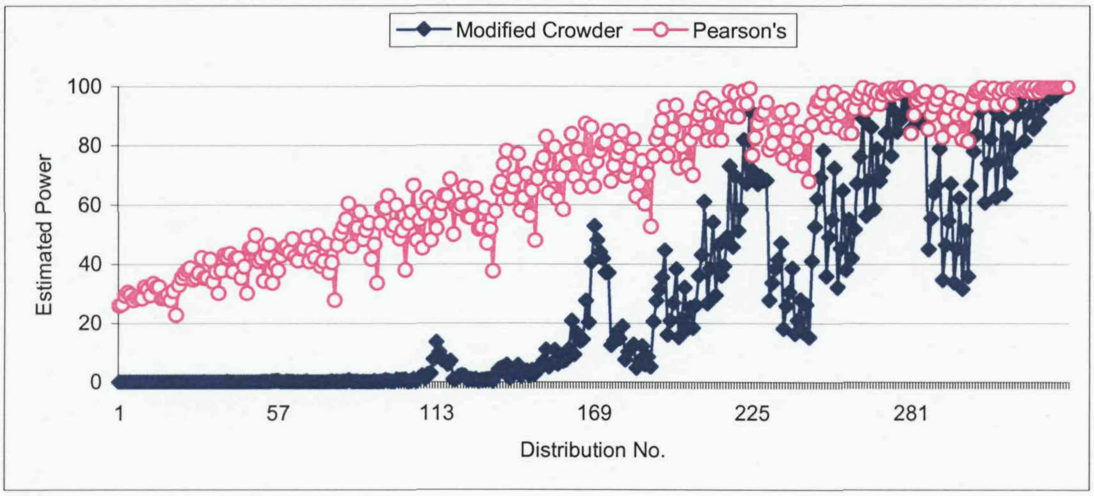


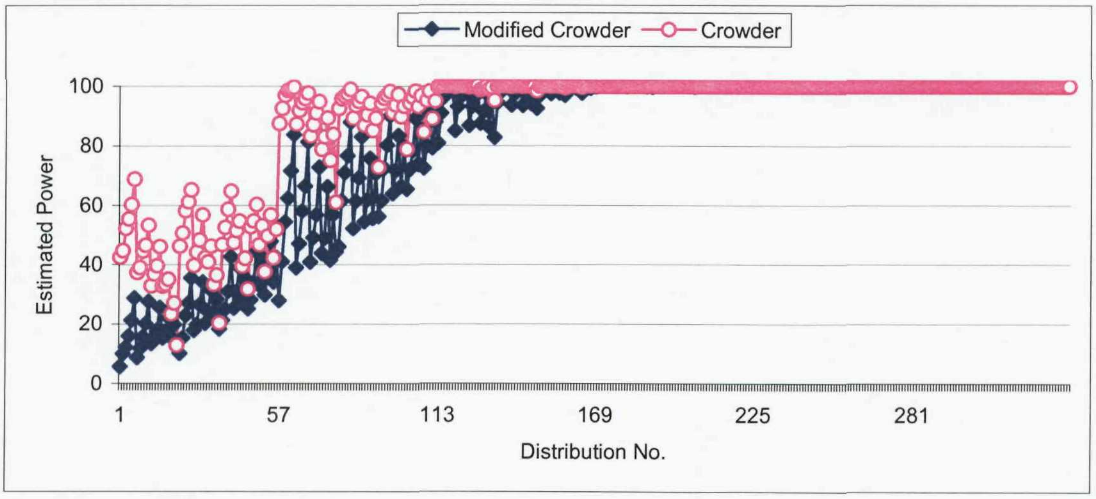
Figure G.3: Plots representation for the estimated power for samples with censoring and two dependent risks at the nominal 5% significance level.

$G(p)$

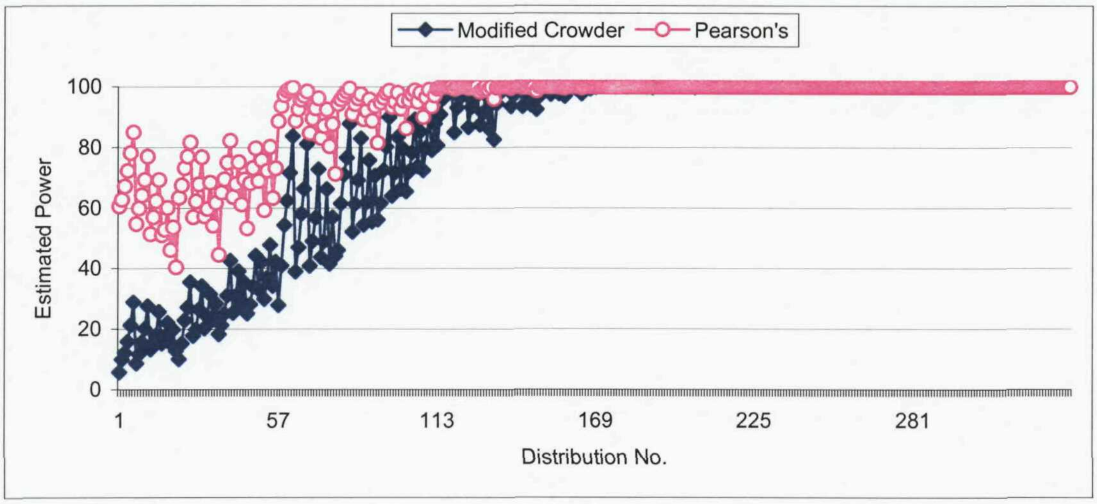




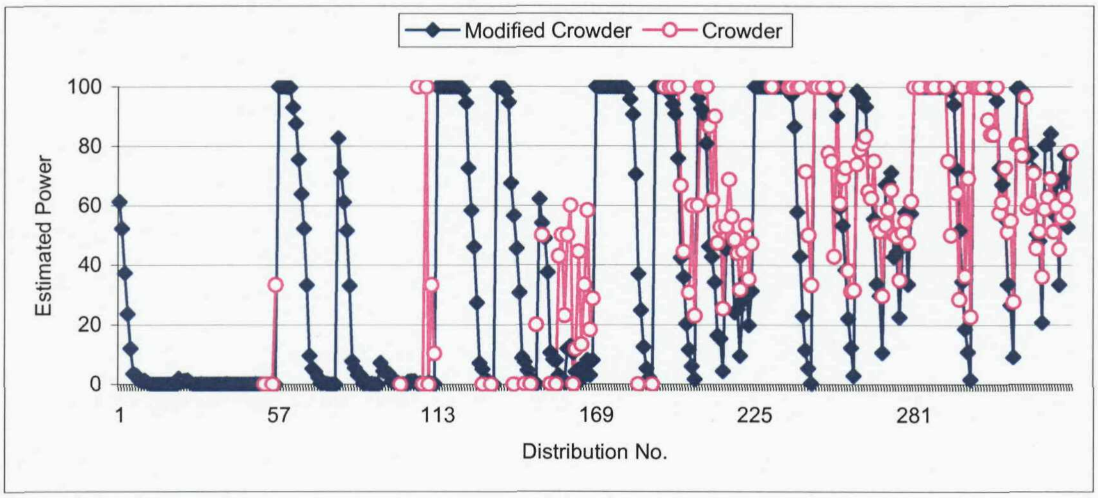
$N(p, z = 0.2)$

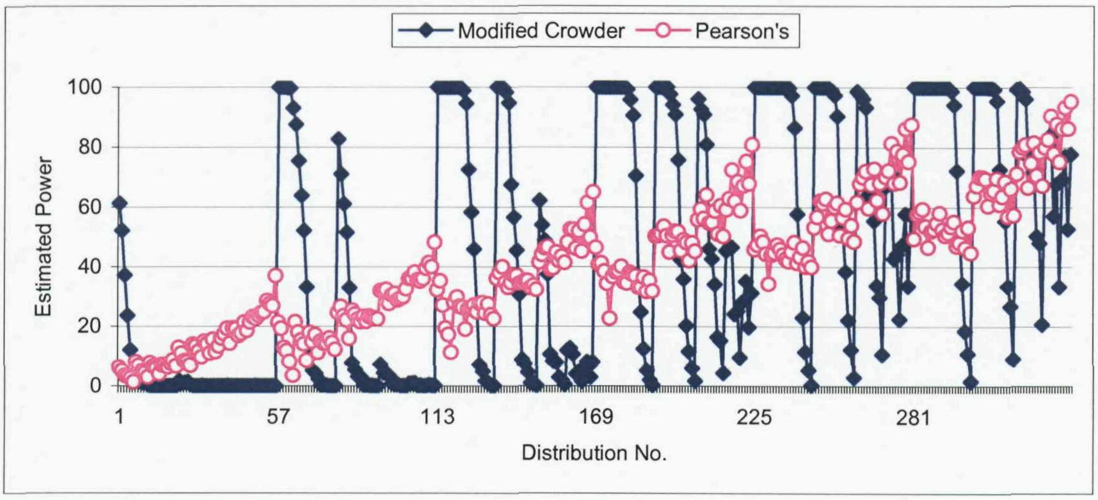


$N\,(p, z = 0.2)$

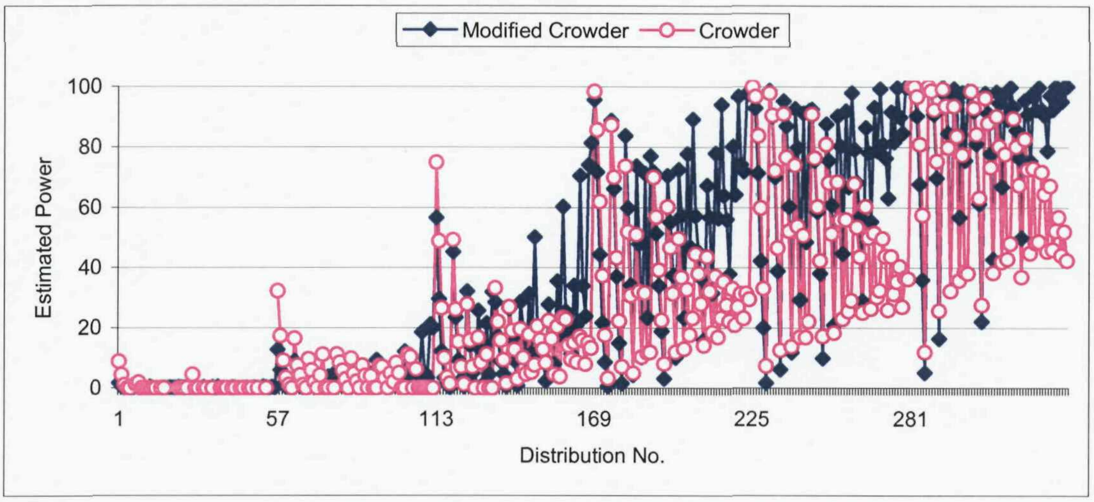

Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$N\,(p, z = 2)$

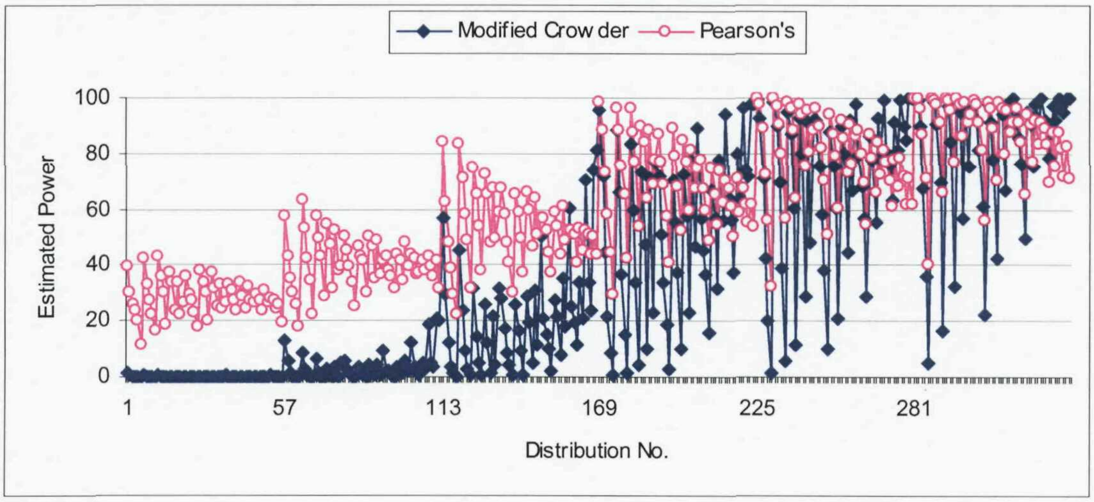

Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.


Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$P(\lambda)$



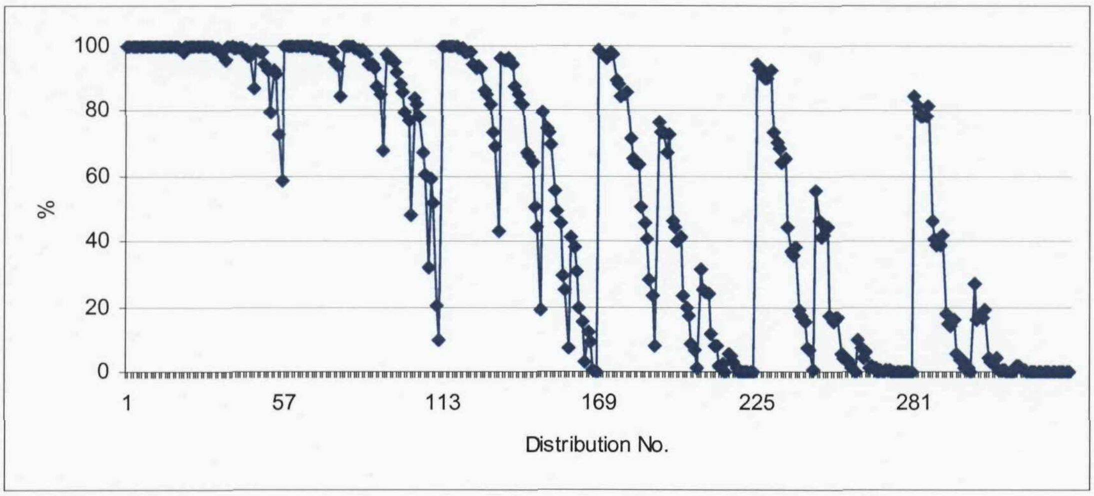
Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.



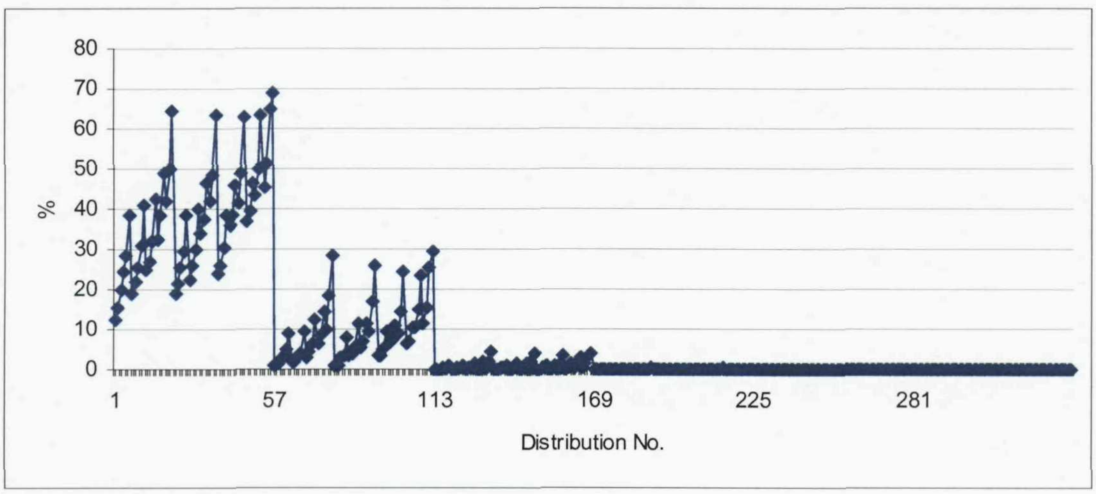
Modified Crow der
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

Figure G.4: Plots representation for the percentage of samples for which W is not computable in Figure G.3.

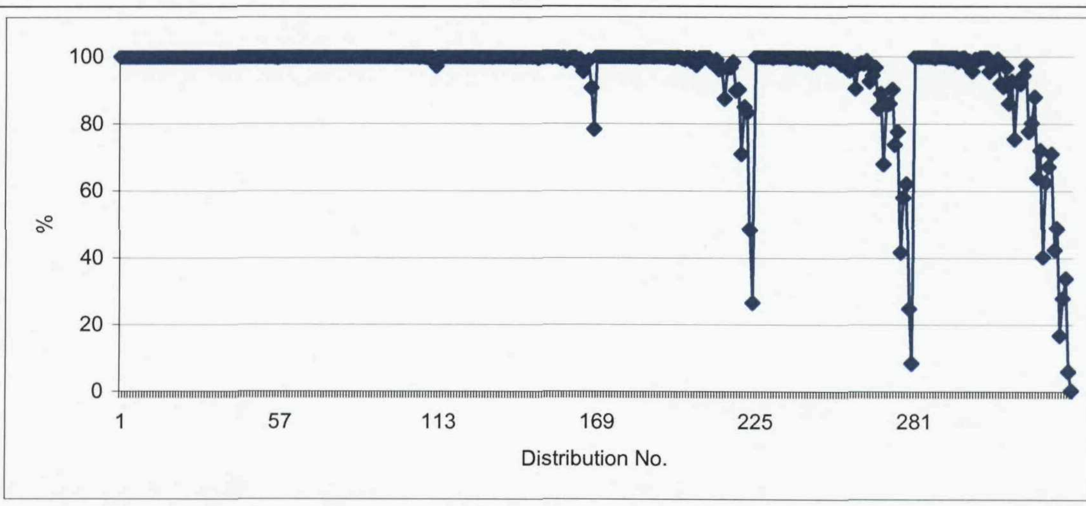
$G(p)$

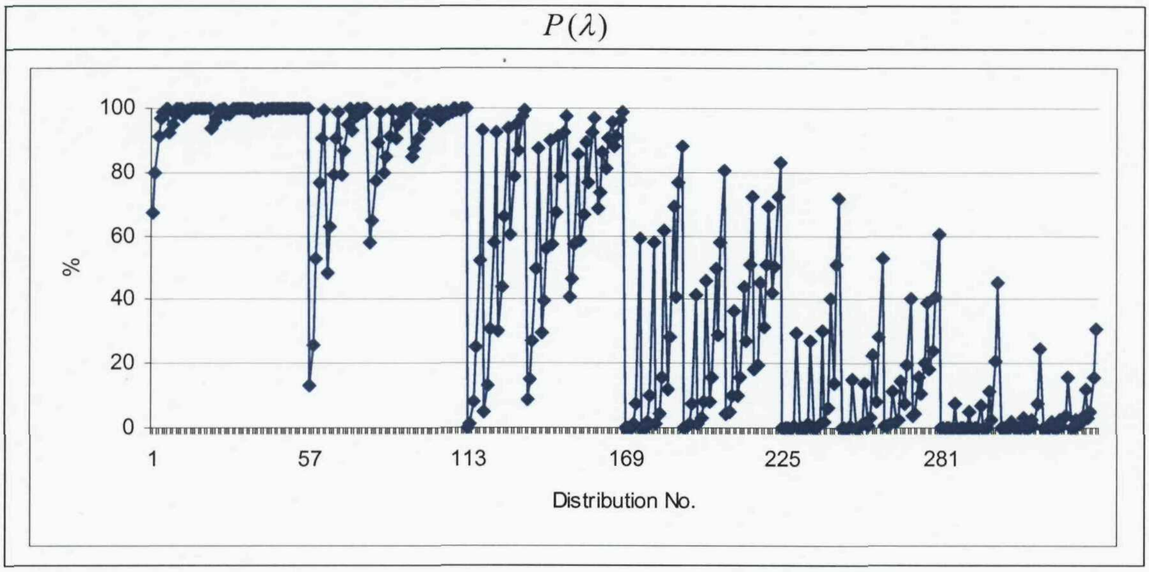


$N(p, z = 0.2)$



$N(p, z = 2)$

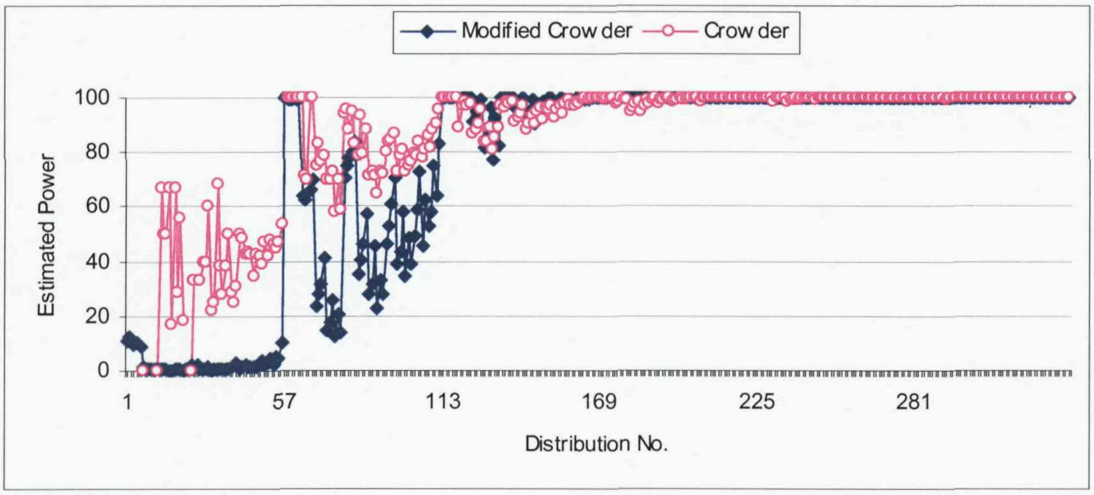


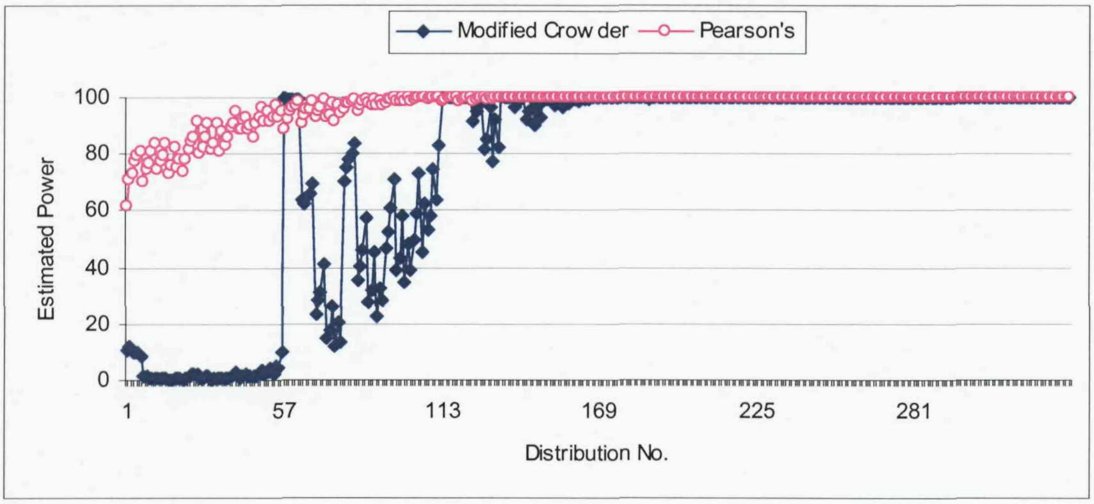


$P(\lambda)$
100
80
60
40
20
0
%
1
57
113
169
225
281
Distribution No.

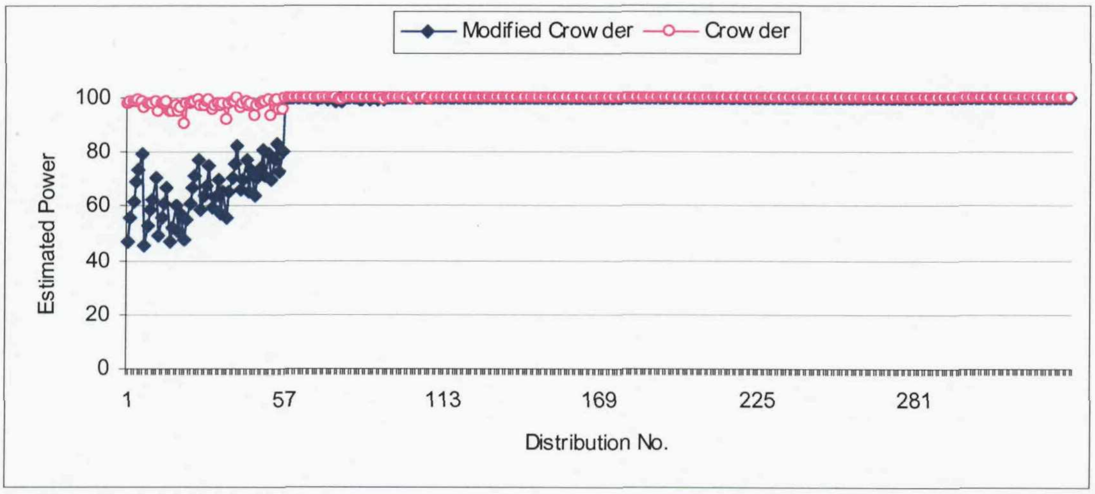
Figure G.5: Plots representation for the estimated power for samples with no censoring and all the risks are dependent at the nominal 5% significance level.

$G(p)$

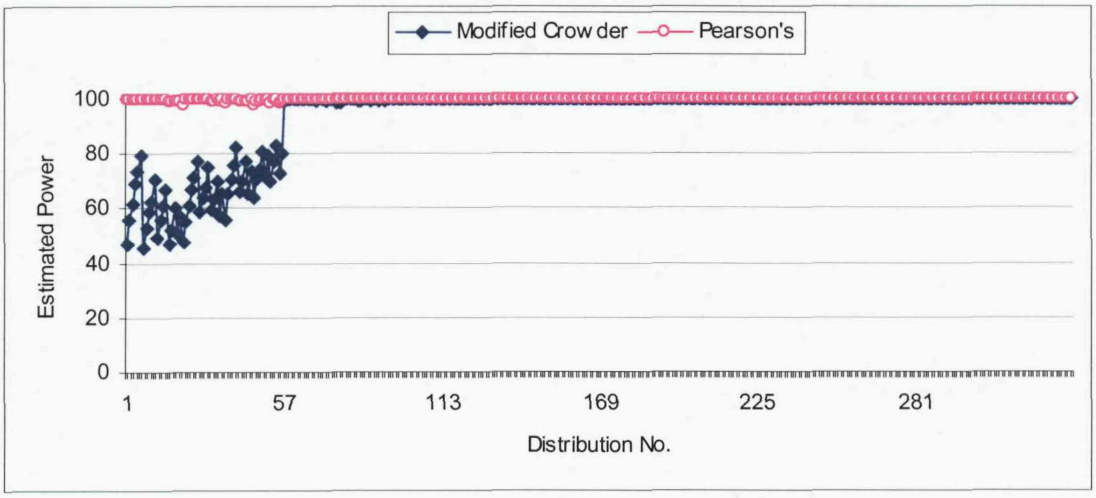




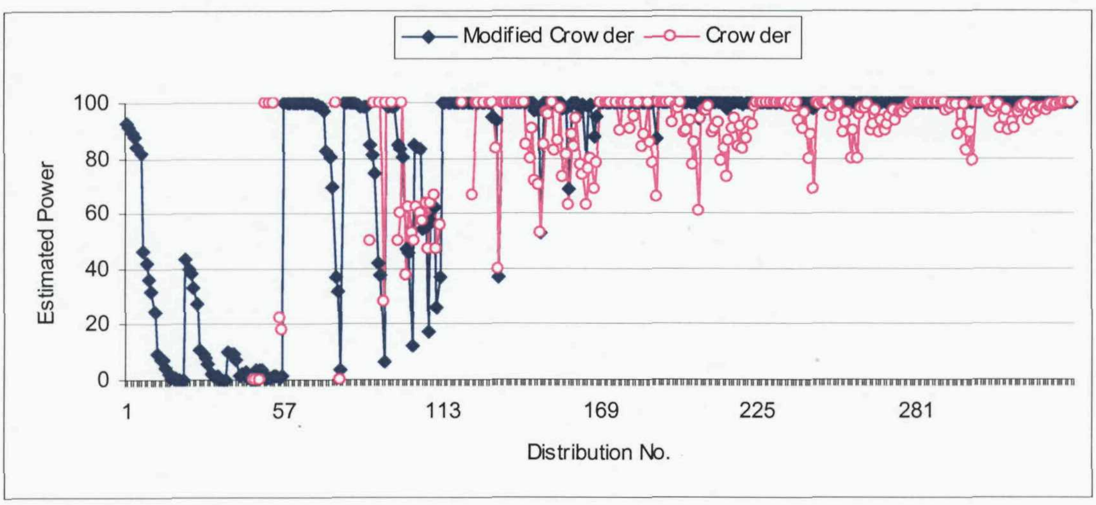
$N(p, z = 0.2)$

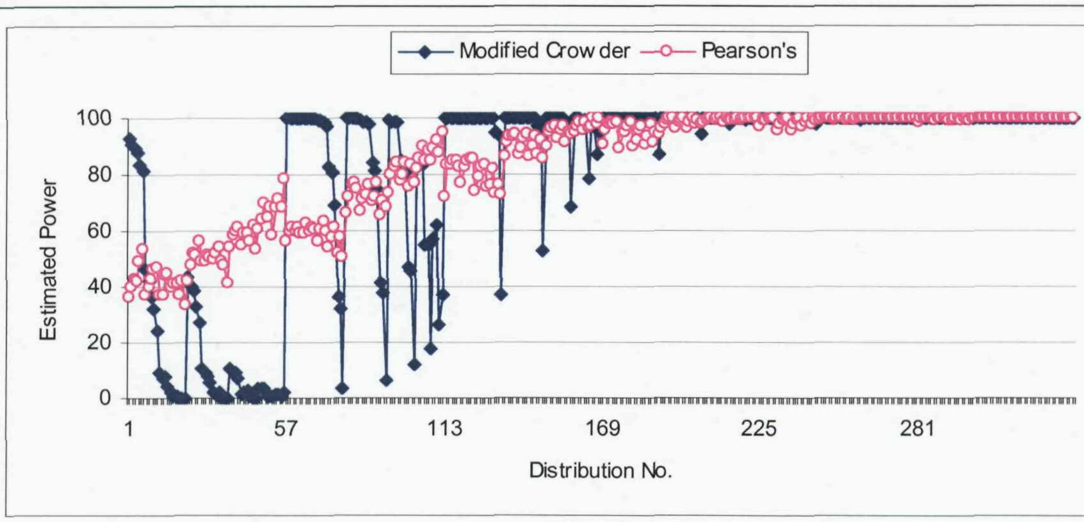


$N(p, z = 0.2)$

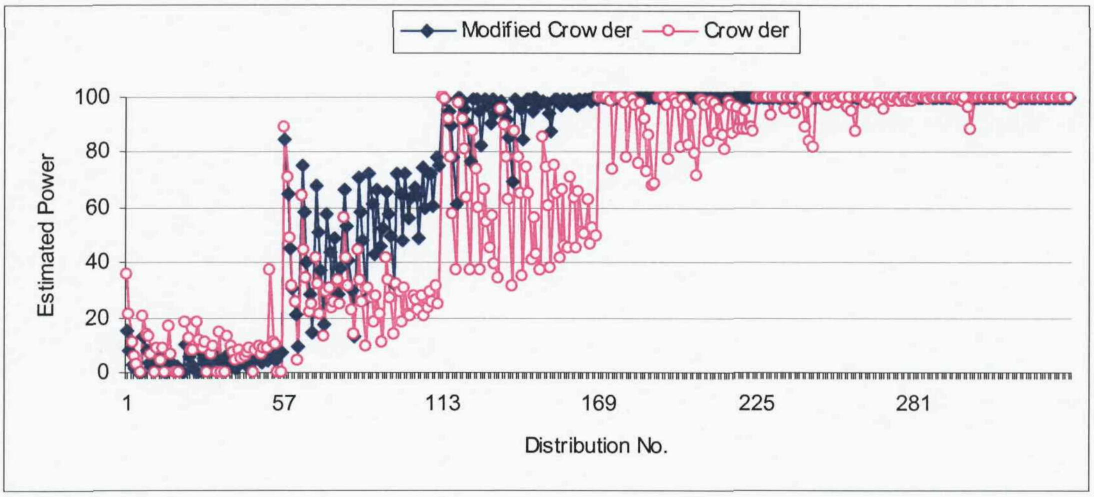

Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$N(p, z = 2)$


Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.


Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$P(\lambda)$


Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

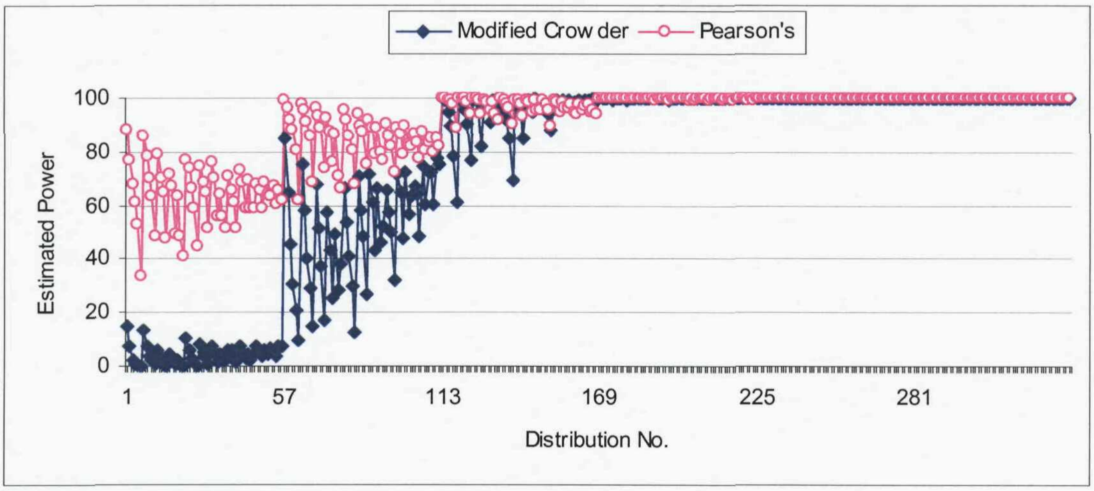
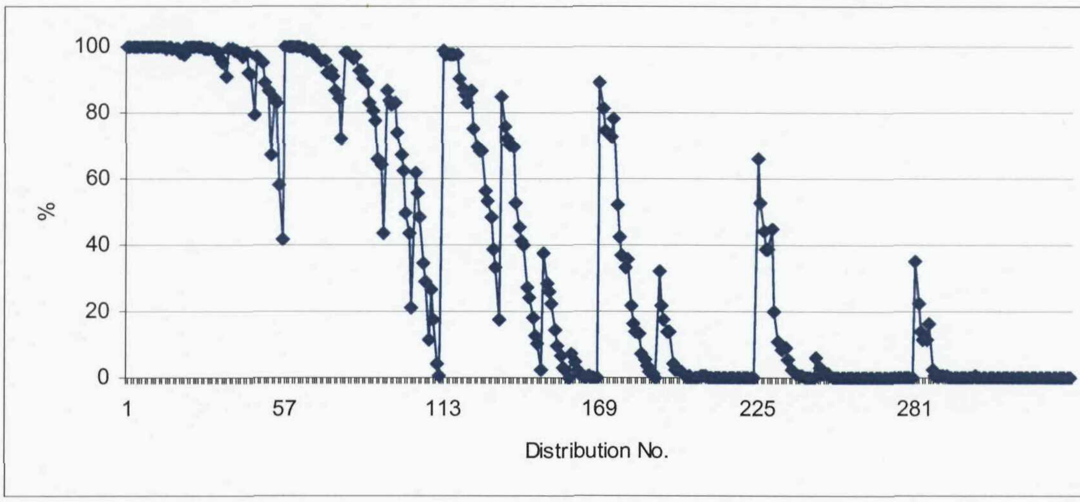
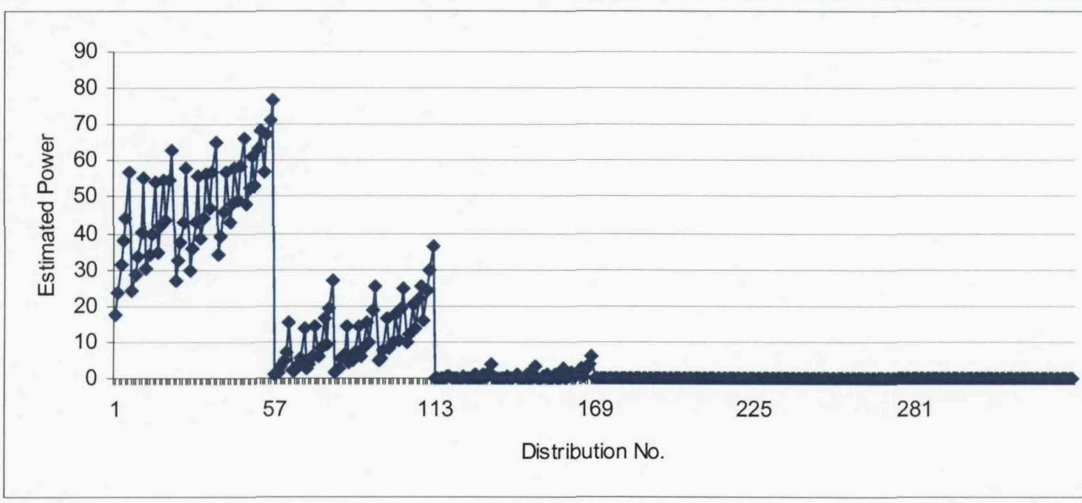

Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

Figure G.6: Plots representation for the percentage of samples for which W is not computable in Figure G.5.

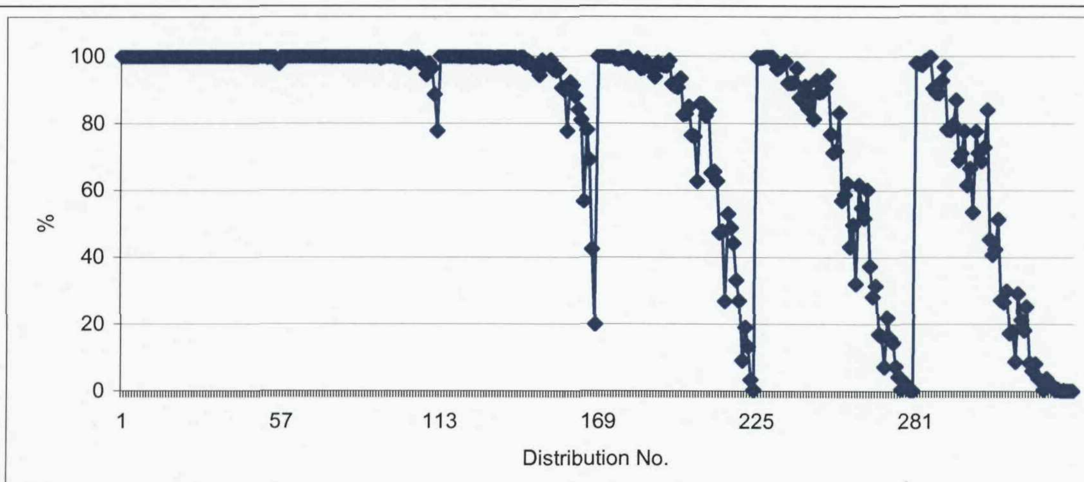
$G(p)$

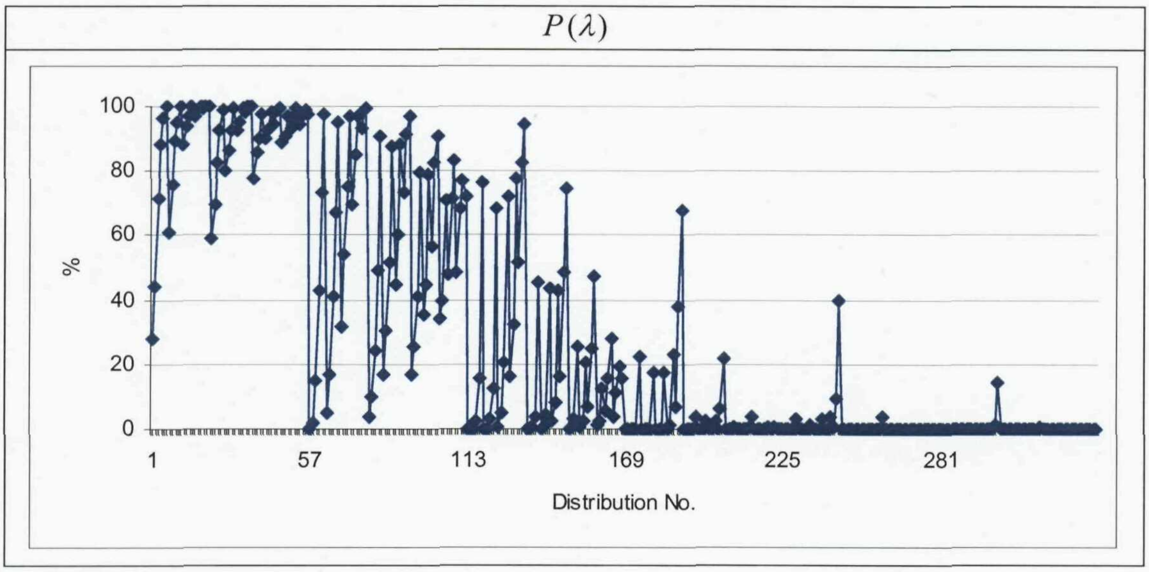


$N(p, z = 0.2)$



$N(p, z = 2)$

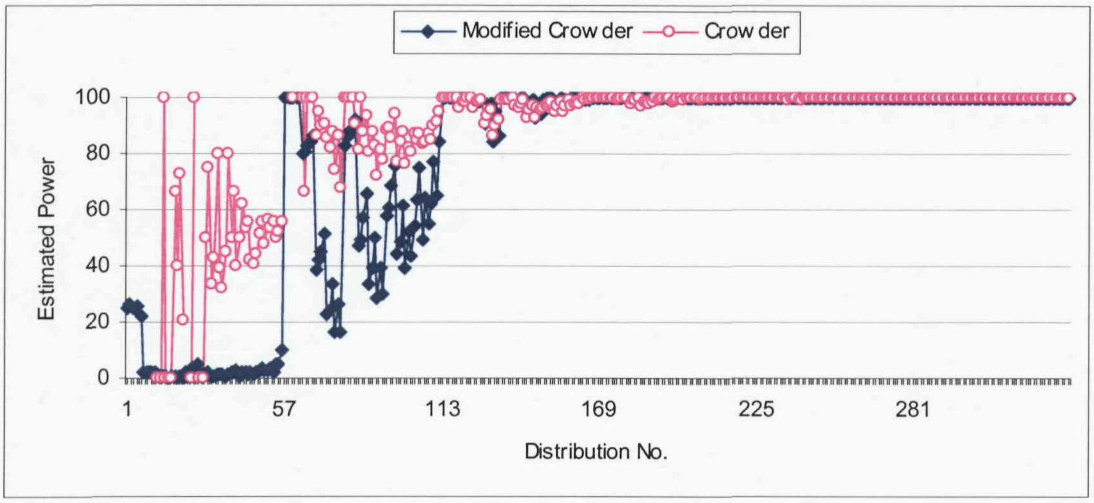


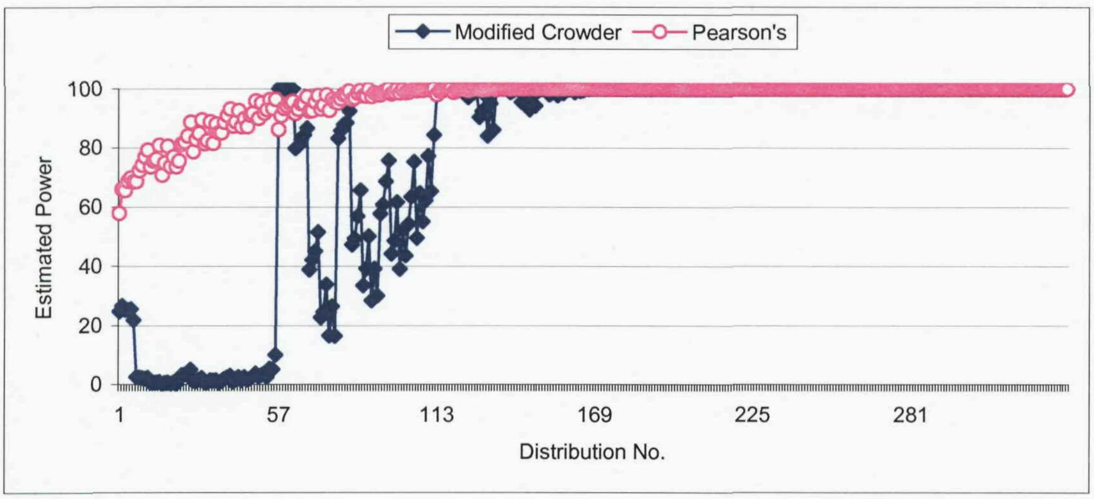


$P(\lambda)$
100
80
60
40
20
0
%
1
57
113
169
225
281
Distribution No.

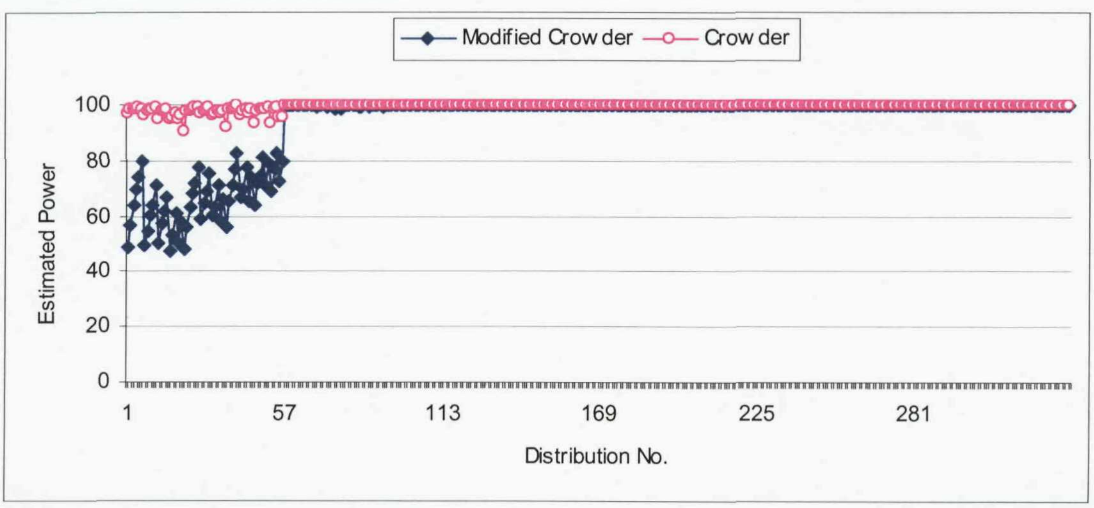
Figure G.7: Plots representation for the estimated power for samples with censoring and all the risks are dependent at the nominal 5% significance level.

$G(p)$

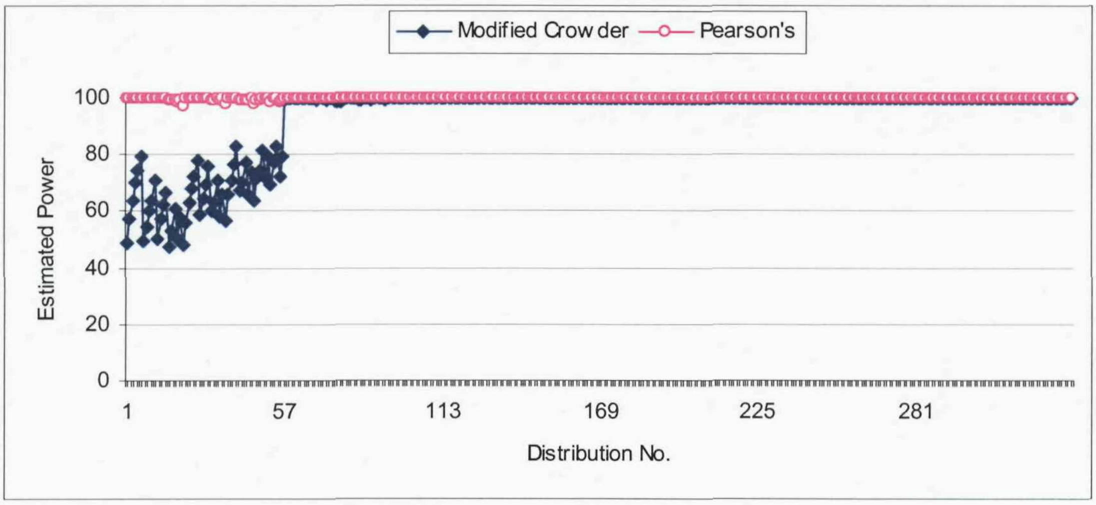




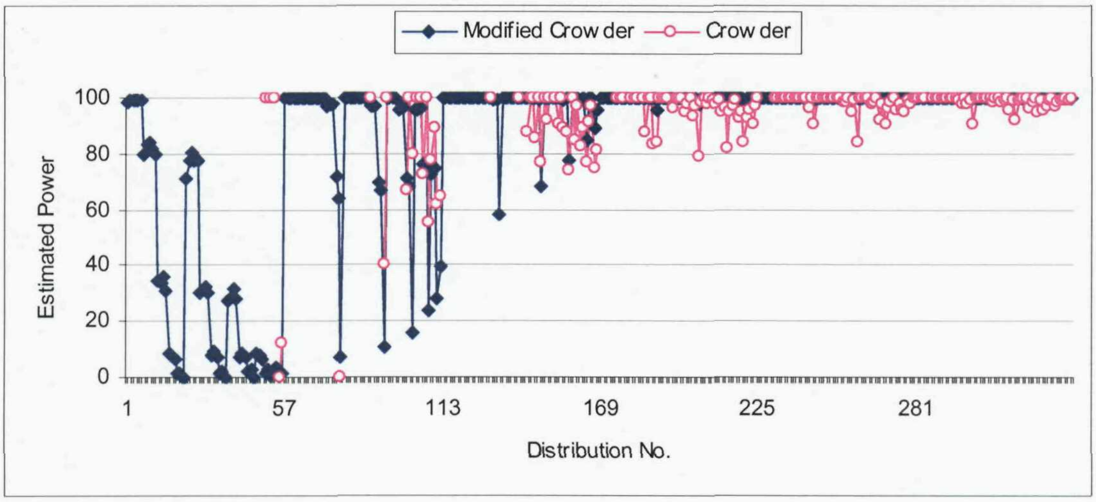
$N(p, z = 0.2)$

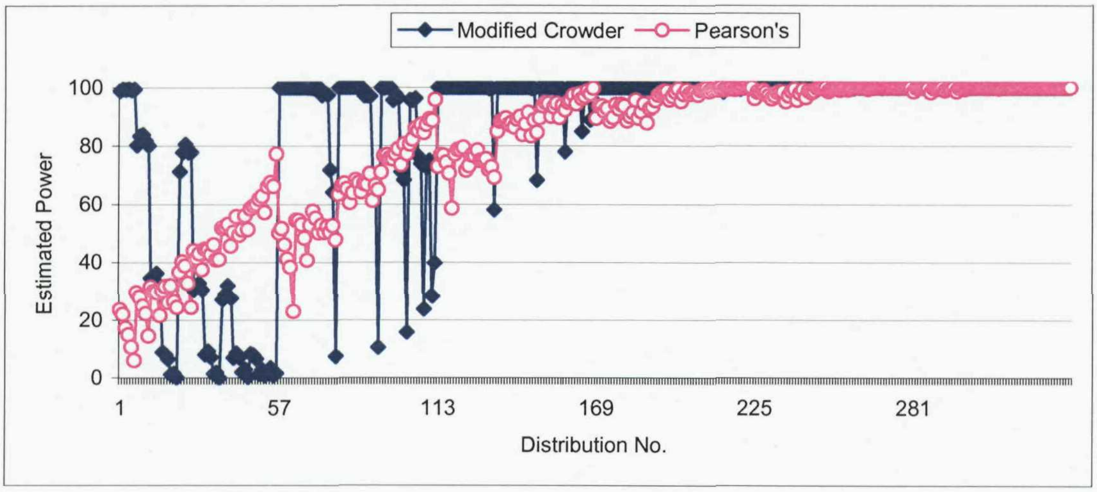


$N\,(p, z = 0.2)$

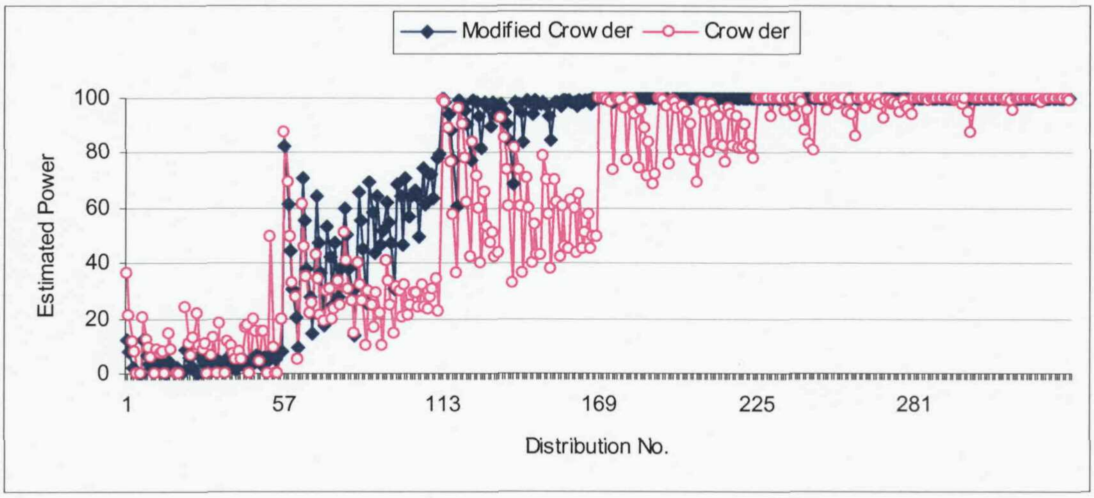

Modified Crow der
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$N\,(p, z = 2)$

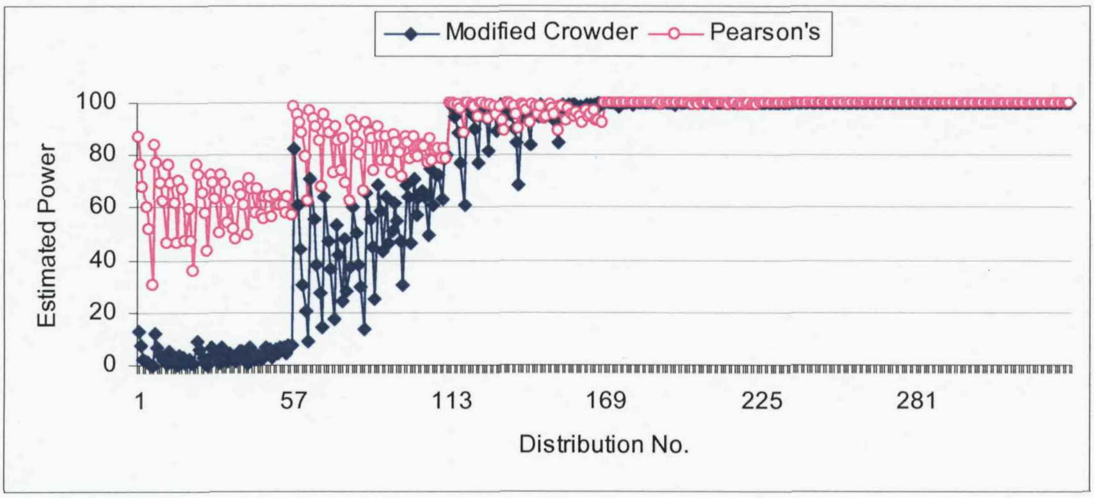

Modified Crow der
Crow der
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.


Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

$P(\lambda)$



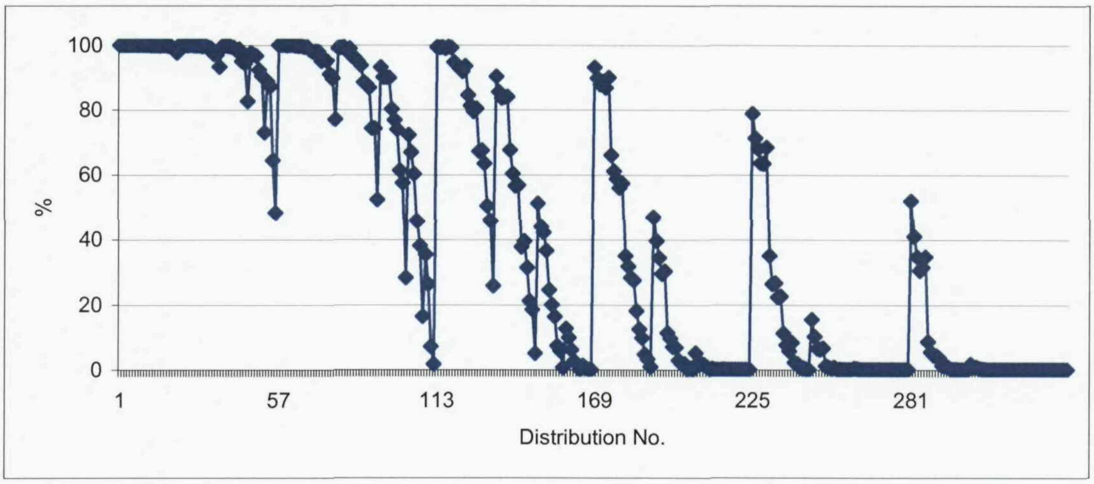
Modified Crowder
Crowder
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.



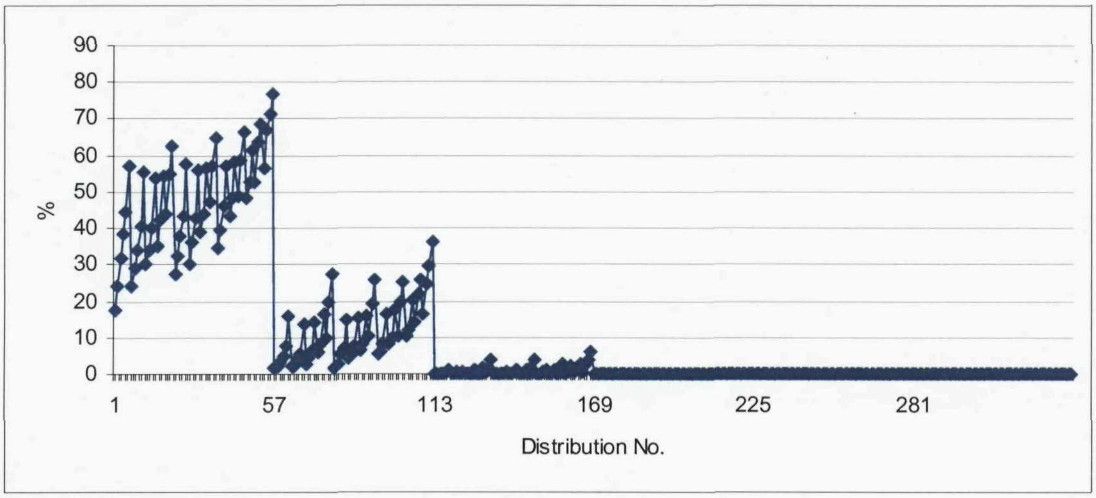
Modified Crowder
Pearson's
Estimated Power
100
80
60
40
20
0
1
57
113
169
225
281
Distribution No.

Figure G.8: Plots representation for the percentage of samples for which W is not computable in Figure G.7.

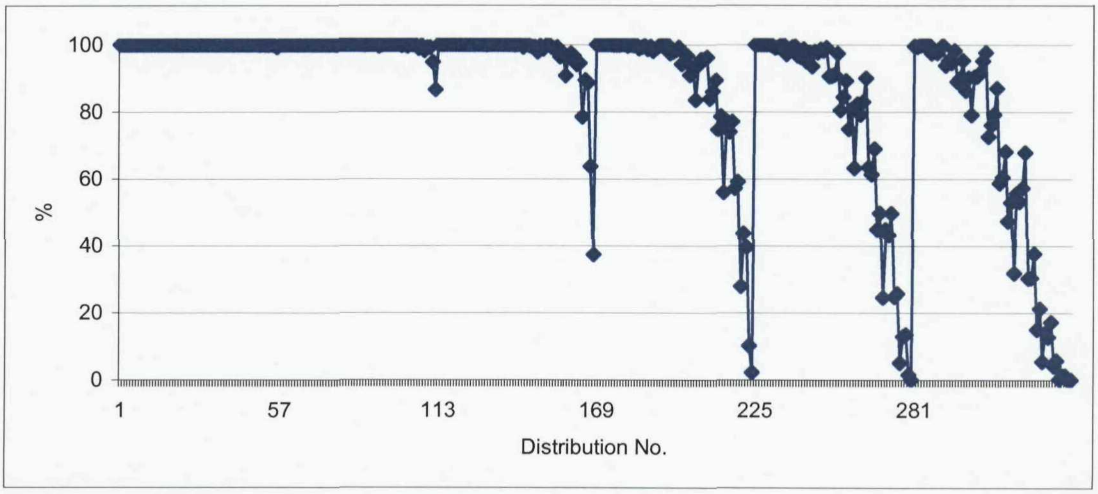
$G(p)$

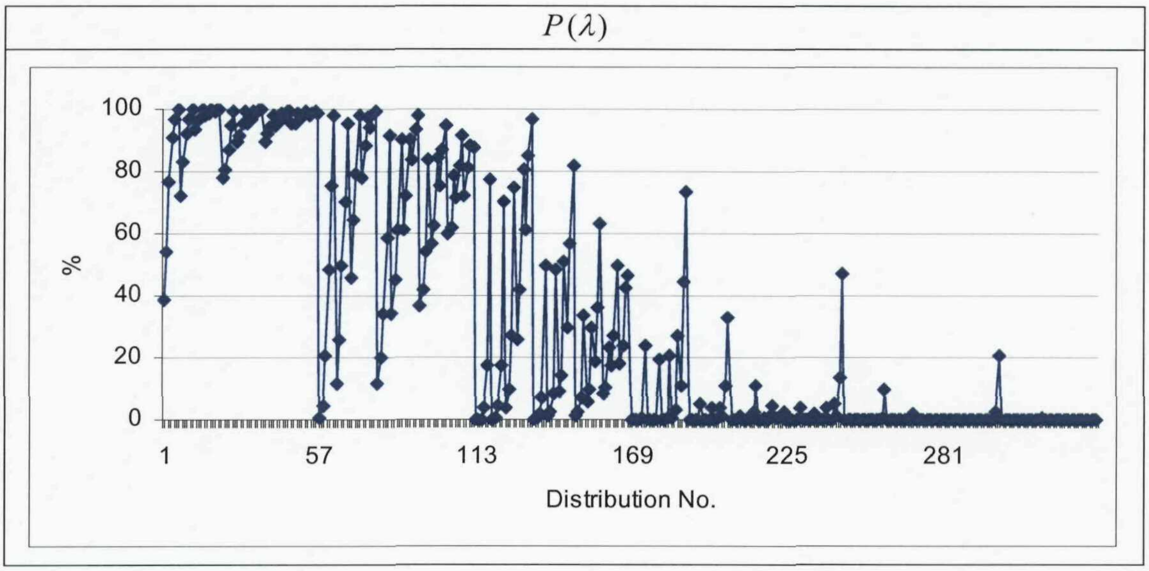


$N(p, z = 0.2)$



$N(p, z = 2)$





$P(\lambda)$
100
80
60
40
20
0
%
1
57
113
169
225
281
Distribution No.