UNIVERSITY OF SOUTHAMPTON

Two topics in geometric group theory

by

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Submitted for the degree of
Doctor of Philosophy

Faculty of Mathematical Studies

August 1998

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

Doctor of Philosophy

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Niblo and Reeves [**NR2**] constructed a cubing for each Coxeter group using the hyperplanes of the Coxeter complex. In Part I **Coxeter groups and cubings** the natural action of the Coxeter group on this cubing is investigated. In particular the cocompactness or not of this action is studied. Using the geometry of the Moussong complex (another complex for a Coxeter group introduced by Gabor Moussong in [**Mou**]) it is shown that hyperbolic and right-angled Coxeter groups act cocompactly and Euclidean Coxeter groups act non-cocompactly and that the action is non-cocompact if and only if there exists an infinite family of non-conjugate isomorphic triangle subgroups.

In Part II **Engulfing and subgroup separability for word-hyperbolic groups** theorems of Darren Long [**L**] concerning fundamental groups of closed hyperbolic manifolds are generalised to word-hyperbolic groups. The main result is that if a torsion-free word-hyperbolic group has a certain engulfing property then every quasiconvex subgroup is contained as a finite index subgroup in a separable subgroup.

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To the glory of God,

Maker of heaven and earth,

of all that is - seen and unseen.

Acknowledgements

I would like to thank my supervisor, Dr Graham Niblo, for his support and enthusiasm during the preparation of this thesis and for getting me interested in the first place. I am also very grateful to my adviser, Professor Martin Dunwoody, especially for his help in my final year. I gratefully acknowledge the Engineering and Physical Sciences Research Council for their financial support.

I am very blessed in having supportive family and friends who have given me much encouragement throughout my studies. My wife, Hannah, deserves a special mention for her constant encouragement and moral support.

I have enjoyed my time in the department of Mathematics at Southampton and would like to thank both the academic and clerical staff for providing such a supportive and friendly environment in which to carry out my studies.

Part I

Coxeter groups and cubings

Chapter 1

Introduction

My research has taken me in two distinct directions, the first being the development of the area of Coxeter group actions on non-positively curved cube-complexes. The second is my work on the residual properties of word-hyperbolic groups which is dealt with in Part II.

A *Coxeter group* is a finitely presented group with a presentation in which every generator is an involution and the relations give the order of the product of two generators. The study of these groups was begun by Tits [**T1**] who named them after Harold Scott Macdonald Coxeter who studied reflection groups of Euclidean spaces. (See for example [**C1**], [**C2**], [**C3**].) The book [**DGS**] is an excellent place to look for a wide range of applications of Coxeter's work.

Coxeter groups are a beautiful class of groups that are particularly suited to geometric interpretation, as groups generated by reflections. There are several good introductions to the subject including [**Br**], [**dlH**], and [**H**].

In Chapter 2 we introduce Coxeter groups and define two complexes. The *Coxeter complex* Σ of a Coxeter group is a simplicial complex introduced by Tits [**T1**] in which the maximal simplices all have dimension equal to the number of generators. The action on Σ is generated by reflections in *walls* that are built out of codimension-1 faces of the maximal simplices. Secondly we define the *Moussong complex* M, a CAT(0) piecewise Euclidean cell complex introduced by Moussong in [**Mou**] and on which a Coxeter group acts cocompactly with an action generated by reflections in *hyperplanes* that each separate the complex into two halfspaces. We then define a third way of picturing Coxeter groups via a geometric representation as a linear group acting on a vector space with a given bilinear form.

Niblo and Reeves [**NR2**] define a CAT(0), finite dimensional, locally finite cube-complex X, called the *Coxeter cubing*, associated to a Coxeter group, constructed from the intersection relations of hyperplanes in the Coxeter complex (or in fact the Moussong complex whose intersection relations are the same). The group acts on X by permuting halfspaces, the action being induced by the action on halfspaces of the Moussong complex. We define this cubing and the action in Chapter 3. The remainder of the work is dedicated to studying the geometry of X and the action of the group on it. In particular we seek to answer the question 'For which Coxeter groups is the action on X cocompact?'.

There is much interest in the theory of cubings which have been referred to as higher dimensional analogues to trees (suggesting comparisons with the theory of groups acting on trees as in [**Se**]). Important work has been done in this area by: Sageev [**S1**], [**S2**] on codimension-1 subgroups and splittings of groups; Niblo and Reeves [**NR1**] where they prove that cocompact actions on cubings imply a biautomatic structure on the group; Roller [**R**] on poc sets (partially ordered sets with a complement), median algebras and cubings; and Mosher [**Mos**] on cubulated 3-manifolds.

Right-angled Coxeter groups are a special family of Coxeter groups for which hyperplanes in the Moussong complex are either parallel or intersect at right angles and in Chapter 4 we prove that in this case X is identical to the Moussong complex with the same action and hence the action is cocompact.

Moussong [**Mou**] proved the following theorem.

Theorem 5.1. *$G = \langle S \rangle$ is hyperbolic $\iff$ neither of the following hold*

(i) There exists $T = T_1 \sqcup T_2 \subseteq S$ so that $G_T = G_{T_1} \times G_{T_2}$ with both factors infinite.

(ii) There exists $T \subsetneq S$ so that G_T is a Euclidean Coxeter group with $\mid T \mid \geq 3$.

where a Euclidean Coxeter group is an irreducible affine reflection group on $\mathbb{E}^n$.

In Chapter 5 we introduce the notion of *phantom vertices* in X and show that X consists of levels each acted on cocompactly by the group and hence the action is cocompact if and only if there are finitely many of these levels. We show that the action is non-cocompact if and only if there is particular configuration of hyperplanes called an *infinite ladder* in M. These ladders are used to show that in the case of hyperbolic Coxeter groups the action must be cocompact. It is also shown that direct product Coxeter groups act cocompactly if and only if each factor does so on its associated cubing. We then show that Coxeter groups that contain a Euclidean Coxeter group as a *special subgroup* (a subgroup generated by a subset of the generators) with three or more generators do not act cocompactly on their cubings. Finally using infinite ladders we prove that the action is non-cocompact if and only if the group contains infinitely many non-conjugate isomorphic infinite triangle subgroups.

Chapter 2

Preliminaries

In this chapter we introduce Coxeter groups and define the Coxeter complex Σ and the Moussong complex M. We also mention a third way of picturing Coxeter groups via a geometric representation as a linear group acting on a vector space with a given bilinear form. Other introductions to Coxeter groups may be found in [**Br**], [**dlH**] and [**H**].

2.1 Coxeter groups

A Coxeter group is a group with a presentation of the form $\langle s_1, s_2, \ldots \mid (s_i s_j)^{m_{ij}} = 1\rangle$ where the $m_{ij} \in \mathbb{N} \cup \{\infty\}$ satisfy

(i) $m_{ii} = 1$

(ii) $m_{ij} = m_{ji}$

and we interpret $(s_i s_j)^\infty$ as the empty relation. For the purpose of this thesis all Coxeter groups will be finitely generated (and hence finitely presented).

The m_{ij} then can be thought of as entries in a symmetric $n \times n$ matrix (where n is the number of generators) with all diagonal entries equal to 1 and entries in $\mathbb{N} \cup \{\infty\}$. This matrix is called the *Coxeter matrix.* Throughout this work we will denote the Coxeter group under consideration, G and the (finite) generating set, S.

Coxeter groups are often represented as labelled graphs called *Coxeter graphs* which are defined as follows: The vertex set consists of a vertex v_i for each generator $s_i \in S$. Two vertices v_i, v_j are joined by an edge if and only if $m_{ij} \geq 3$ and this edge is labelled m_{ij}. When $m_{ij} = 3$, the label is usually omitted.

Given $T \subseteq S$, the *special subgroup* G_T of a Coxeter group G is the subgroup generated by T; a *special coset* is a left coset of such a subgroup. The Coxeter graph of G_T is obtained by deleting vertices corresponding to generators not in T and the edges that are incident with them. (See for example [**H**].)

A Coxeter group is a direct product of the special subgroups generated by the generators corresponding to vertices in each connected component of its Coxeter graph. To see this take two connected components of the Coxeter graph with vertex sets T_1 and T_2. These two components correspond to two special subgroups G_1 and G_2 of G. Any two generators $s_i \in T_1$ and $s_j \in T_2$ are not connected by an edge so $(s_i s_j)^2 = 1$ and since s_i and s_j are involutions we have $s_i s_j s_i s_j = [s_i, s_j] = 1$ and hence every element of G_1 commutes with every element of G_2. Also $G_1 \cap G_2 = \{e\}$ and hence $G = G_1 \times G_2$.

A Coxeter group is *irreducible* if it does not split as a direct product in this way, i.e. it has a connected Coxeter graph.

A *right-angled* Coxeter group is one for which $m_{ij} = 2$ or ∞ for all i, j.

Examples

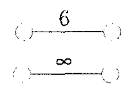
1. The Klein Viergruppe has presentation

$$V_4 = \langle a, b \mid a^2 = b^2 = (ab)^2 = 1 \rangle$$

matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, and graph

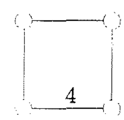
2. The group $D_6 \times D_\infty$ has presentation

$$\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^6 = (ac)^2 = (ad)^2 = (bc)^2 = (bd)^2 = 1 \rangle,$$

matrix $\begin{pmatrix} 1 & 6 & 2 & 2 \\ 6 & 1 & 2 & 2 \\ 2 & 2 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$, and graph 

3. The group

$$\langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = (ab)^4 = (ac)^2 = (ad)^3 = (bc)^3 = (bd)^2 = (cd)^3 = 1 \rangle$$

has matrix $\begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 3 & 2 & 3 & 1 \end{pmatrix}$ and graph 

A very natural way to think of Coxeter groups is as reflection groups. We define two complexes on which a Coxeter group acts by reflections.

2.2 The Coxeter complex

Firstly we give a geometric 'hands on' definition. The *Coxeter complex*, denoted Σ, is a simplicial complex defined as follows: The top dimensional simplices are (n-1)-dimensional where G has n-generators and we call them *chambers*. We begin with one such simplex T and consider building up the complex 'by reflection' in codimension-1 faces of simplices. Assign bijectively the generators of G to the faces of T. The action of each generator on T is to reflect in its face carrying with it the labelling of faces with generators. We get n new chambers s_iT, $i = 1, \ldots, n$ glued isometrically along codimension-1 faces to the first chamber. We repeat this process by reflecting in faces of these new chambers and so on to get chambers $s_{i_r} \ldots s_{i_2} s_{i_1} T$. Clearly we get a chamber for each group element g, by using a word in the generators representing g and carrying out the reflections corresponding to the generators in order. By definition, as we are building the complex out of chambers that we can reach by reflections, the action is transitive on chambers. Loops of chambers correspond to relations.

Note. To know when to glue up loops of chambers in this construction it is necessary to know when a word represents the identity element, that is, a solution of the word problem is required. It is well known that the word problem for Coxeter groups is solvable. See for example [**T2**] or [**Br**].

More formally Σ is the poset (partially ordered set) of special left cosets ordered by the opposite of inclusion. That is, if A and B are special cosets of G such that $A \subseteq B$ then in Σ we have $B \leq A$.

We note the following facts.

(i) The maximal simplices correspond to the cosets of minimal special subgroups, that is cosets of the trivial group $\{e\}$. Hence we have a maximal simplex C_g for each group element g. For each g we can obtain a sequence $gH_0 < gH_1 < \ldots < gH_{n-1}$ of cosets of special subgroups with strictly increasing numbers of generators each containing g (for example $g\{e\} < g\langle s_1 \rangle < g\langle s_1, s_2 \rangle < \ldots < g\langle s_1, s_2, \ldots, s_{n-1} \rangle$). This gives a sequence of simplices of strictly decreasing dimension contained in C_g of length n where n is the number of generators. Hence all maximal simplices have dimension $n-1$ and as above are called *chambers*.

(ii) Codimension-1 faces correspond to cosets of special subgroups generated by a single generator. Two chambers are joined via a codimension-1 simplex if and only if the elements represented by the chambers are the two elements in the coset represented by the codimension-1 face.

(iii) Codimension-2 faces correspond to cosets of special subgroups generated by two generators. Their links are $2m_{ij}$-gons where the special subgroup is $\langle s_i, s_j \rangle$ and $m_{ij} < \infty$, and $\mathbb{R}$ if $m_{ij} = \infty$. See section 3.2 The CAT(0) property for the definition of the link.

We define an action of the group on Σ by left multiplication on cosets. It is clear that the action preserves inclusion of cosets and therefore also the inclusion of simplices and therefore the poset structure.

A *wall* in Σ is a codimension-1 subcomplex built out of codimension-1 faces of chambers glued together as follows. We define a relation, $\sim$ 'belongs to the same wall as', on neighbouring codimension-1 faces of chambers. Two faces with a codimension-1 (codimension-2 in Σ) face in common are related if and only if they correspond to opposite points in the link (definition in section 3.2) of that face. In the case of $\mathbb{R}$ links, the wall does not continue beyond the codimension-2 simplex with that link. We then take the transitive closure of $\sim$. Walls are then equivalence classes of codimension-1 faces.

Brown (See [**Br**] III §4.) shows by way of *'folding maps'*, that each wall divides Σ into two half spaces and that the action is generated by automorphisms fixing a wall and interchanging its two halfspaces, that is reflections in walls of Σ. To see this, consider the chambers 1 and s_i where $s_i \in S$. (We will label chambers by their corresponding group elements.) We will study the wall W containing their common face f. First note that s_i acting on Σ fixes f and interchanges the chambers 1 and s_i. Now consider a cell C containing a simplex f' of the wall W which is glued to f. By (i) above, G preserves adjacency of chambers and we know that s_i fixes $f \cap f'$. We work our way around the (polygonal) link (definition in section 3.2) of $f \cap f'$ pairing up chambers that are exchanged by s_i until the final pair is C and the chamber adjacent to it via f'. (See Fig. 2.1 below where this pairing is suggested by the shading of chambers.) Hence f' is also fixed by s_i and exchanges the two chambers containing it. In this way we see that the whole wall is fixed by s_i which acts as reflection in it.

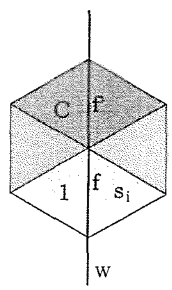


Fig. 2.1 Extending walls in the Coxeter complex

Examples.

1. The *dihedral group* of order 2n $D_{2n} \cong \langle s, t | s^2 = t^2 = (st)^n = 1 \rangle$. Since D_{2n} has two generators its Coxeter complex Σ is a 1-dimensional simplicial complex (i.e. a graph). We have a chamber (an edge) for each element of D_{2n}; each chamber has two faces (vertices) that correspond to a coset of $\langle s \rangle$ and a coset of $\langle t \rangle$. Two edges are joined if and only if the elements they represent are the two elements in the coset represented by their common vertex. Each vertex clearly has valency 2 and hence Σ is a 2n-gon.

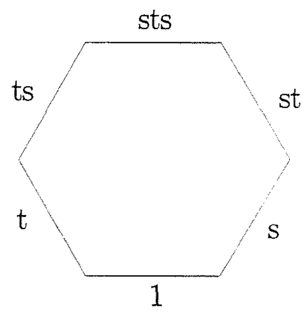


Fig. 2.2 The Coxeter complex for $D_6 \cong \langle s, t | s^2 = t^2 = (st)^3 = 1 \rangle$

2. The (3,3,3) triangle group $\widetilde{A}_2 \cong \langle s_1, s_2, s_3 | s_i^2 = (s_i s_j)^3 = 1, \forall i, i \neq j \rangle$ is one of the three so called *Euclidean triangle groups* ($\widetilde{A}_2$, $\widetilde{B}_2$, $\widetilde{G}_2$ in Fig. 5.13 on page 74) that act as reflection groups of the plane $\mathbb{E}^2$. The Coxeter complex is built out of triangles. The link of each vertex (see definition in section 3.2) is a hexagon and hence the Coxeter complex is the tiling of the Euclidean plane by equilateral triangles. Note that by a triangle group we mean a Coxeter group with three generators. This differs from the definition of triangle groups given in other literature in which they are the (index two) orientation preserving subgroups of our triangle groups.

2.3 The Moussong complex

The Moussong complex M is a piecewise Euclidean complex introduced by Moussong in 1988 in his thesis [**Mou**]. We follow the definition given by Davis in [**D**] and define it first for finite groups and then for infinite groups where it is built out of copies of Moussong complexes for its finite special subgroups.

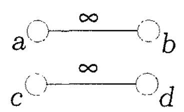
So we start by defining the Moussong complex for a finite Coxeter group in which case it is actually a single cell and hence is called a *Coxeter cell.* In 1935 Coxeter proved that every finite Coxeter group is isomorphic to some reflection group in $\mathbb{E}^r$ whose elements have a common fixed point [**C2**]. (See Fig. 2.8 at the end of this chapter for a list of the Coxeter graphs for all irreducible finite Coxeter groups. See also [**H**] for the classification of the finite groups.) Hence a finite Coxeter group acts on a Euclidean space $\mathbb{E}^r$ for some r, by reflections in codimension-1 hyperplanes $h_1, h_2, \ldots, h_n$ through the origin. The connected components of $\mathbb{E}^r - \cup h_i$, are called *chambers* and are simplicial cones. The hyperplanes contributing to the boundary of a chamber are called its *supporting hyperplanes.*

We construct the Coxeter cell as follows: Choose a point $v \in \mathbb{E}^r - \cup_i h_i$ and label it by the identity. Then translate v by the group (by reflecting in hyperplanes) labelling each translate by the appropriate group element. Then M is the convex hull of the finite set of translates of v. Since the connected component of $\mathbb{E}^r - \cup_i h_i$ in which v sits is a simplicial cone, by a simple geometric argument it is possible to ensure that v is a distance $1/2$ from all of the supporting hyperplanes of this chamber and this implies that all of the edges of M have length 1.

Now for infinite groups we take a Coxeter cell for each finite special subgroup G_T labelled as above and a cell for each finite special coset gG_T with vertices labelled by elements of gG_T in the obvious way. Coxeter cells for special subgroups are called *special cells.* M is defined as the union of these cells glued isometrically along faces which have the same group labellings. We will usually use the edge-path metric on $M^{(1)}$ although it is worth mentioning that the *intrinsic metric* defined by taking the infimum over all piecewise Euclidean paths is well defined and well behaved which is proved by Bridson in [**B**]. In fact, as Moussong showed in [**Mou**], with this geometry M is CAT(0). For the definition of CAT(0) see section 3.2. (See also Theorem 2.2.) There are no identifications between vertices of individual cells and each cell is a convex Euclidean polytope so each cell is isometrically embedded in the intrinsic metric on M.

Examples.

1. $D_\infty \times D_\infty$ generated by a, b, c and d as the following Coxeter graph suggests.



The maximal finite special subgroups are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and hence have squares as Coxeter cells. These fit together to form the Moussong complex which is the tiling of $\mathbb{E}^2$ by squares part of which is shown below.

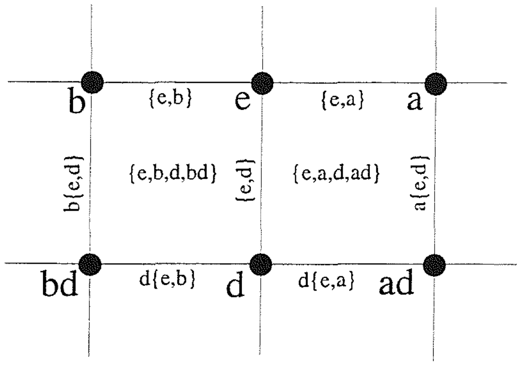


Fig. 2.3 Part of the Moussong complex for $D_\infty \times D_\infty$

2. The (3,3,3) triangle group. The maximal finite special subgroups are isomorphic to D_6 for which the Coxeter cell is a solid hexagon. These fit together to give the tiling of $\mathbb{E}^2$ by hexagons as shown below. Vertices are labelled by the group elements that they represent and two edges and a hexagon are labelled with the finite special cosets that they represent.

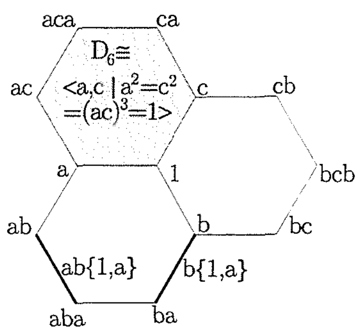


Fig. 2.4 Part of the Moussong complex for the (3,3,3) triangle group

As in the construction of the Coxeter complex we require a notion of *hyperplanes* (or *walls*). These are obtained by gluing hyperplanes of cells to hyperplanes of neighbouring cells when their intersection with the common codimension-1 face is the same. It is clear that these gluings are local isometries since hyperplanes exit cells 'at right-angles'. By an argument similar to Lemma 2.8 it can be shown that hyperplanes are isometrically embedded in M.

We have the following result about how hyperplanes sit in M.

Lemma 2.1. *Every hyperplane in M separates M into exactly two connected components.*

Proof: The proof uses cohomology theory and is similar to that of Lemma 2.3 of [**NR1**]. Let h be a hyperplane of M. Define a function $f : M^{(1)} \to \mathbb{Z}_2 = \{0, 1\}$ by $f(e) = 1$ if $e \cap h \neq \emptyset$ and $f(e) = 0$ otherwise. Then f is a cellular 1-cochain. Every 2-cell is a 2n-polygon and contains either no edges labelled by 1 or exactly two edges on opposite sides labelled with 1s. This is by the definition of hyperplanes. Hence summing around 2-cells to obtain the coboundary map we see that $\partial(f) = 0$ and hence f is a (non-zero) cocycle. Since M is contractible f is the coboundary of a 0-cochain. Let $g : M^{(0)} \to \mathbb{Z}_2$ be such a cochain defined by $f(e) = g(\iota_e) - g(\tau_e)$ where ι_e and τ_e are the initial and terminal vertices of the edge e. Then $g(x) = g(y)$ unless every path in M from x to y crosses h. Since g is non-zero, h must cut M into at least two pieces.

Now we show that $M - h$ has exactly two components. We have already noted that h is isometrically embedded in M and since every cell that h intersects is cut in half by h and is Euclidean with edge lengths all 1, there is an isometrically embedded neighbourhood $N = h \times I$ with a natural I-bundle structure. Since h is isometrically embedded in M and M is contractible it is a trivial bundle and hence N has two boundary components $h \times 1/2$ and

$h \times -1/2$. Every point of $M - h$ can be joined to one of these boundary components by a path contained in $M - h$ hence $M - h$ has exactly two components. □

There are clearly finitely many cells incident with the identity vertex. These are precisely the cells of finite special subgroups. Each of these finite cells intersects non-trivially only finitely many hyperplanes of M. We call the finitely many hyperplanes of these that are a distance $1/2$ from the identity vertex *elementary hyperplanes* of M. Note that when regarded as hyperplanes of the Coxeter complex these are the supporting hyperplanes of the identity chamber. Accordingly, we call the hyperplanes in M at a distance $1/2$ from a vertex v the *supporting* hyperplanes of v. We call a hyperplane h *minimal* with respect to v if there is no other hyperplane parallel to h and lying between h and v. It is important to note the difference between minimal and supporting hyperplanes of a vertex. In Fig. 2.5 below all of the labelled hyperplanes are minimal with respect to the vertex v but h_4, h_5, and h_6 are minimal but not supporting with respect v.

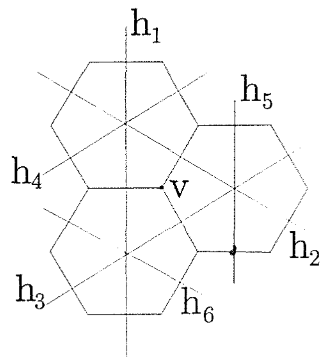


Fig. 2.5 Minimal and non-supporting hyperplanes of a vertex

We have the following theorem which can be found in [**D**].

Theorem 2.2. *(Gromov, Moussong) M is a CAT(0) piecewise Euclidean cell complex on which the group acts isometrically and cocompactly by reflections in hyperplanes.* □

See section 3.2 for the definition of CAT(0).

We define an action of G on the vertex set of M and then show that it extends to an action on M. Let v_k denote the vertex of M labelled by the group element k. Then define the action by $g(v_k) = v_{gk}$. Now let C be a cell of M. By construction of M the vertices of C are elements of a finite special coset $g_1K = g_1\{k_1, k_2 \ldots, k_n\}$ say, of G. Then $g(v_{g_1k_i}) = v_{gg_1k_i}$. Hence the labels of the vertices are the elements of the finite special coset gg_1K and hence are the vertices of a cell of M. Hence G takes cells to cells.

To see that the action is generated by reflections in hyperplanes we consider the action of generators of the group on M. Let s_i be a generator and G_T be a finite special subgroup containing s_i with corresponding Coxeter cell C_T in M. By the definition of Coxeter cells s_i acts on C_T by reflection in a hyperplane, h say, of C_T. Now consider a cell C glued to C_T along a codimension-1 face f intersecting h. Now s_i preserves f and hence must take C either to itself or to C_T. But s_i preserves C_T so it must also preserve C. The only isometry preserving C and f in this way is the reflection through the hyperplane that is the continuation of h as defined above. A simple induction argument now shows that s_i acts by reflection through the hyperplane of M containing h on the cells containing it. Since G acts freely on the vertex set of M there is a unique isometry that acts in this way, that is the reflection through this hyperplane. Note that the generating elements of G act as reflections in the elementary hyperplanes of M.

Note: By definition of M there are bijections between the elements of G and the vertices of M, and between the finite special cosets of G and the cells of M. We will tend to abuse notation by referring to vertices as group elements or vice versa or referring to cells as finite special cosets or vice versa.

We now prove some simple results about the Moussong complex.

Lemma 2.3. *The 1-skeleton $M^{(1)}$ of the Moussong complex M is a Cayley graph for G. Moreover the action of G on M by reflections in hyperplanes is the same as the usual action of G on its Cayley graph. Hence the action is free and transitive on the vertex set of M.*

Proof: The vertices of M are in bijective correspondence with the elements of G as are the vertices of any Cayley graph for G. Two vertices g, g' are joined in a Cayley graph if and only if they differ on the right by a generator s_i, i.e., $g = g's_i$. Let e be an edge of M with vertices g and h. Translating e by g^{-1} we see that $g^{-1}e$ has endpoints 1 and $g^{-1}h$. But endpoints of edges at the identity are generators of G. Hence $g^{-1}h = s_i$ for some i and so $g = hs_i^{-1} = hs_i$ as required. □

Corollary 2.4. *Let 1 be the identity vertex in M and $g \in G$ with shortest word representative of length n. Then $d_1(1, g) = n$ where d_1 is the edge path metric.* □

Corollary 2.5. *The star of the identity vertex in M, i.e., the union of all Coxeter cells in M incident with the identity vertex is a fundamental region* *for the action.*

Proof: Each cell in the star of the identity vertex is a finite special subgroup. Any other cell in M is a left coset of one of these subgroups and is hence a translate of one of them. □

Lemma 2.6. *Every hyperplane is a translate of an elementary hyperplane. Hence there are finitely many orbits of hyperplanes in M.*

Proof: By Lemma 2.3 every vertex is a translate of the identity vertex which is a distance $1/2$ from precisely the elementary hyperplanes by construction of M. Since G acts by isometries, the hyperplanes a distance $1/2$ from any vertex are translates of some elementary hyperplane. Clearly every hyperplane is distance $1/2$ from some vertex. □

Lemma 2.7.

(i) Hyperplanes are fixed pointwise by a unique element of G.

(ii) The stabiliser of a hyperplane h is the centraliser of the group element that acts by reflection in h.

Proof: (i) We show that this is true for elementary hyperplanes h_i. Then g' fixes gh_i pointwise if and only if $gg'g^{-1}$ fixes h_i pointwise.

Existence: The generator s_i fixes h_i pointwise by definition.

Uniqueness: Consider the action of the pointwise stabiliser of h_i on the edge with endpoints 1 and s_i. Either $g1 = s_i \Rightarrow g = s_i$ or $g1 = 1 \Rightarrow g = 1$.

(ii) By Lemma 2.6 every hyperplane is a translate of an elementary hyperplane. Let gh_i be a general hyperplane such that h_i is fixed pointwise by the generator s_i. Then it is clear that gh_i is fixed pointwise by gs_ig^{-1}. For any $g_1 \in G$ the element $g_1 g s_i g^{-1} g_1^{-1}$ fixes pointwise the hyperplane $g_1 g h_i$. Then $g_1 \in C_{gs_ig^{-1}}(G)$ (the centraliser of gs_ig^{-1} in G) $\iff$ $g_1 g s_i g^{-1} g_1^{-1} = gs_ig^{-1}$ which fixes $g_1 g h_i$ $\iff$ $g_1 g h_i = gh_i$ by part (i). □

Lemma 2.8. *For any subset T of the generating set S of G the Moussong complex M_T is isometrically embedded in M.*

Proof: First note that finite special subgroups of $\langle T \rangle$ are also finite special subgroups of G and finite special cosets of $\langle T \rangle$ are also finite special cosets of G so each cell of M_T isometrically embeds in a cell of M. The gluings of all these cells are the same whether considered in M or M_T. If T generates a finite subgroup then M_T embeds isometrically as a single Coxeter cell of M as remarked at the beginning of this section. Thus we can assume that T generates an infinite subgroup.

Let $f : M_T \to M$ be the embedding map sending cells of M_T to their corresponding cells in M. We require that f is an isometry but given that M is CAT(0) the Cartan-Hadamard Theorem (Theorem 3.3) for such spaces, proved by Bridson and Haefliger in [**BHa**], means that it is only required to show that f is a local isometry, that is any point of M_T lies in a neighbourhood N such that f restricted to N is an isometry.

Let p be a point in M_T and N be the open star of p in M_T, that is the union of the interiors of open cells in M_T containing p. Let x and y be points of N. It is required that the geodesic $[x, y]$ in M is contained in N. We suppose, for a contradiction, that this is not the case. Then there exist two cells, c_1 and c_2 say, of M_T which intersect (in the smallest dimension cell containing p) and a geodesic $[x, y]$ joining the two which is not contained in

N.

Suppose first that c_1, c_2 are contained in the same cell of M. Now c_1 and c_2 'generate' a (convex) Coxeter cell C in M_T which is isometrically embedded in M.

Now suppose that $c_1 \subseteq C_1$ and $c_2 \subseteq C_2$ where C_1 and C_2 are distinct maximal Coxeter cells of M. Let $[x,y]_T$ be the geodesic in M_T joining x to y be considered as a path in M. We have assumed that $[x,y] \not\subset M_T$. First suppose that the geodesic $[x,y]$ in M intersects exactly one other maximal cell C_3 of M. Since C_1, C_2, C_3 have a vertex in common we can translate them by an element of G so that they are all special cells, i.e. Coxeter cells of special subgroups. Then there is a cell $c_3 \subset M_T$ in C_3 generated by generators in c_1 and c_2 containing $[x,y]$ and hence $[x,y]$ is a shorter path in M_T from x to y contradicting that $[x,y]_T$ is a geodesic in M_T. Now suppose that p intersects maximal cells $c'_1, c'_2, \ldots c'_n$ where $n \geq 2$. All these c'_i lie in the 1-neighbourhood of $c_1 \cap c_2$ and hence all intersect $c_1 \cap c_2$ in a $(n-2)$-cell. The smallest curvature around this cell is achieved when $n = 2$ and both cells are cubes. Homotoping $[x,y]_T$ into the boundary of these cubes and hence into the boundary of $c_1 \cap c_2$ clearly reduces its length and hence it was not a geodesic in M. Other cells and more of them lead to more negative curvature which enables more reduction in the length of $[x,y]_T$ and hence leads to the same result. This completes the proof. □

2.4 Examples

In this section we give some more examples of Coxeter groups and compare their Coxeter and Moussong complexes.

1. $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$

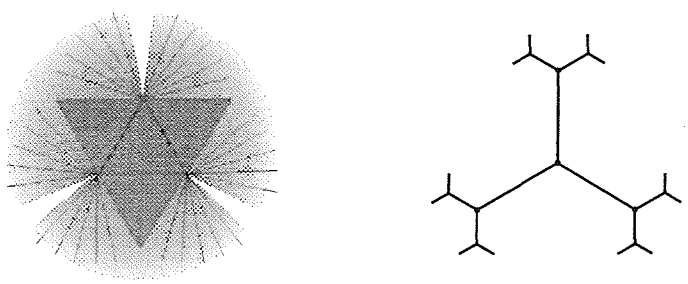


Fig. 2.6

Part of the Coxeter complex and Moussong complex for $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.

2. $D_\infty \times D_\infty$

$\cong \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ac)^2 = (ad)^2 = (bc)^2 = (bd)^2 = 1 \rangle$

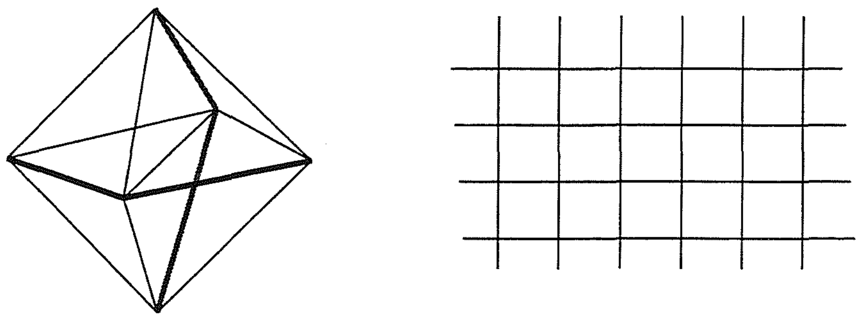


Fig. 2.7

Part of the Coxeter complex and Moussong complex for $D_\infty \times D_\infty$.

In the Coxeter complex diagram four tetrahedra are glued along a common edge, thick edges have links isomorphic to $\mathbb{R}$ and thin edges have links isomorphic to a square as with the interior edge. See section 3.2 for the definition of link.

The link of each vertex in the Coxeter complex is infinite.

One of the advantages of the Moussong complex is that it is locally finite. This is due to the fact that ∞s in the Coxeter matrix, which signify that the product of two generators has infinite order, open up the Moussong complex but lead to the existence of infinite links in the Coxeter complex. In this work we will mainly be using the Moussong complex although sometimes the Coxeter complex is more helpful (notably in the case of Euclidean reflection groups).

2.5 Geometric Representation

In this section we introduce a third way of picturing Coxeter groups. It is not possible to represent a general Coxeter group by affine reflections in hyperplanes through the origin in $\mathbb{E}^n$ but we can get close by considering the following action on a vector space and a notion of reflection similar to Euclidean reflection.

Consider the vector space V over $\mathbb{R}$ with basis $\{\alpha_s | s \in S\}$, that is a basis vector for each generator of the group. We define a bilinear form $B : V \times V \to \mathbb{R}$ on V as follows:

$$B(\alpha_s, \alpha_t) = -cos\left(\frac{\pi}{m_{ij}}\right) \text{ for } m_{ij} < \infty \text{ and } -1 \text{ if } m_{ij} = \infty$$

Now we define a reflection R_s for each $s \in S$ by

$$R_s(v) = v - 2B(\alpha_s, v)\alpha_s$$

Note the similarity between this definition and the definition for reflection in a hyperplane in Euclidean space.

Things to note

(1) $R_s(\alpha_s) = \alpha_s - 2B(\alpha_s, \alpha_s)\alpha_s = \alpha_s - 2(-cos(\pi))\alpha_s = \alpha_s - 2\alpha_s = -\alpha_s$ so α_s is normal to the reflection plane of R_s.

(2) Reflection in hyperplanes leaves orthogonal vectors invariant i.e., if $B(\alpha_s, \alpha_t) = 0$ then $R_s(\alpha_t) = \alpha_t - 2B(\alpha_s, \alpha_t)\alpha_s = \alpha_t$.

(3) This representation is faithful. In particular, the order of R_sR_t is m_{st} whenever $s \neq t$. (See [**H**].)

This representation has been particularly useful in establishing results about the geometry of Coxeter and Moussong complexes. For example see [**BHo**] and Lemmas 3.3 and 3.4 of Niblo and Reeves in the next chapter.

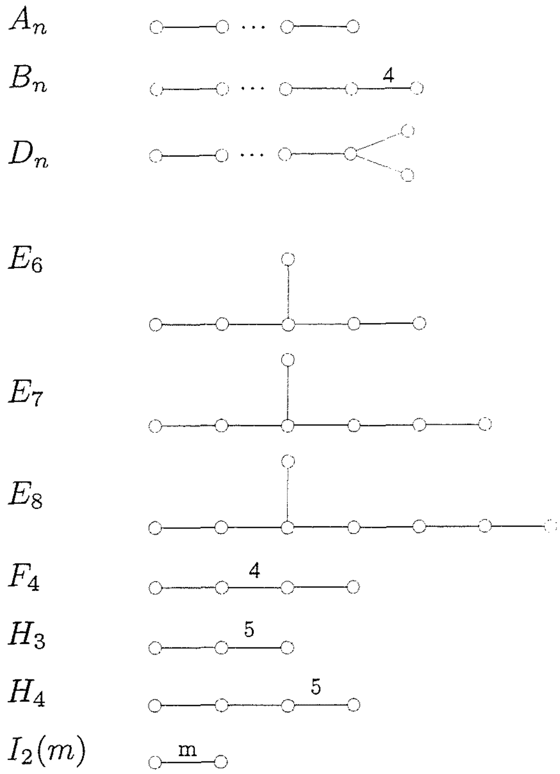


Fig. 2.8 The irreducible finite Coxeter groups

Chapter 3

Cubings for Coxeter groups

In this chapter we define a space for each Coxeter group built out of Euclidean cubes on which the group has a natural action. The construction is due to Niblo and Reeves [**NR2**] and is inspired by the cubings of Sageev introduced in [**S1**] as a tool to study codimension-1 subgroups.

3.1 Cube-complexes

Let H be a set with a partial order $\leq$ and an involution $* : H \longrightarrow H$, $h \mapsto h^*$ satisfying the following:

(1) Given any $a, b \in H$ there are finitely many $c \in H$ such that $a \leq c \leq b$.

(2) Given $a, b \in H$ at most one of the following holds: $a \leq b$, $a \leq b^*$, $a^* \leq b$, or $a^* \leq b^*$. If one does hold we say that a and b are *nested* and are

non-nested otherwise.

We think of H as a set of halfspaces of a space arising from taking complementary components of codimension-1 hyperplanes, the partial order $\leq$ as inclusion and the involution $*$ as taking the complementary halfspace. We define a *hyperplane* as an unordered pair of halfspaces $h = \{a, a^*\}$ and an *oriented hyperplane* by specifying a side of h and denote the two orientations of h by h^+ and h^-. Oriented hyperplanes are in bijective correspondence with halfspaces and both are visualised by adding an arrow on one side of and normal to a hyperplane as in Fig. 3.1.

We can build a cube-complex (that is a cell-complex in which each cell is a Euclidean cube and cells are glued along faces by isometries) $\hat{X} = (H, \leq, *)$ as follows. A vertex v in $\hat{X}$ is defined to be a subset of H such that

(i) $a \in v \iff a^* \notin v$ for all $a \in H$.

(ii) Whenever $a_i \leq a_j$, then $a_i \in v \Rightarrow a_j \in v$.

This abstract definition of a vertex is also known as an *ultra-filter*. Property (i) says that we choose exactly one halfspace from each pair defined by each hyperplane. Property (ii) says that the intersection of nested halfspaces in v must be non-empty. The allowable and not-allowable choices in this respect are given in the following diagram.

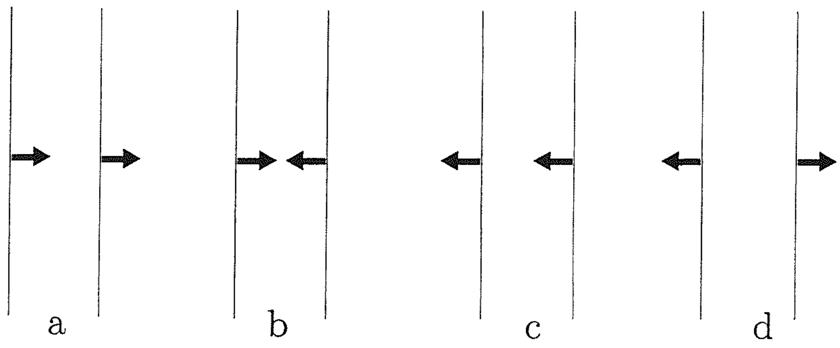


Fig. 3.1 Allowed (a,b,c) and not-allowed (d) choices of nested halfspaces

We say that a halfspace a is *minimal* with respect to a vertex v if there is no $a' \in H$ such that $a' \leq a$ and $a' \in v$. We call a hyperplane minimal with respect to v if it is the boundary of a minimal halfspace of v.

Two vertices are joined by an edge in $\hat{X}$ if and only if they differ as sets by exactly one element, i.e. exactly one minimal (in the sense of (ii) above) halfspace is swapped with its complementary halfspace.

The k-skeleton for $k \geq 2$ is defined inductively by gluing in a k-cube if and only if its boundary appears in the $(k-1)$-skeleton. This completes the definition of the cube-complex $\hat{X}$ arising from the triple $(H, \leq, *)$.

An alternative way of viewing vertices in $\hat{X}$ is as a choice of orientation of hyperplanes so that each hyperplane 'points to' the vertex. In this way we will sometimes refer to vertices as *configurations of oriented hyperplanes.*

We describe the action $*(a) = a^*$ as *switching* the halfspace a or when using the oriented hyperplane terminology switching the oriented hyperplane.

It can be shown (e.g. see Theorem 4.14 of [**S1**]) that for any set $\{a_1, a_2, \ldots, a_n\}$ of n pairwise non-nested elements of H we have $\cap_{i=1}^{n} a_i \neq \emptyset$ and hence the set defines a (not necessarily unique) n-cube in $\hat{X}$.

Usually we will be interested in a particular connected component X of $\hat{X}$ defined by choosing a vertex v in $\hat{X}$ and letting the vertex set of X be the vertices of $\hat{X}$ a finite distance from v. The higher dimension cubes are added as before. A connected cube-complex then is defined by a quadruple $(H, \leq, *, v)$.

Definition. *A midplane of a cube c is a codimension-1 cube parallel to one of the faces of c and passing through its barycentre. A hyperplane of X is a cube-complex built from midplanes of cubes which are glued isometrically as follows. Two neighbouring midplanes belong to the same hyperplane if and only if their union is convex in X.*

3.2 The CAT(0) property

We now give a brief introduction to the CAT(0) property which will be of great importance later. A comprehensive introduction is the book [**BHa**] by Bridson and Haefliger.

The CAT in "CAT(0)" stands for Cartan Alexandrov and Toponogov who each contributed to the idea of curvature in metric spaces. The CAT(0) condition is a very elegant way of expressing non-positive sectional curvature in Riemannian geometry that applies to the more general setting of geodesic metric spaces. It is very useful in the study of groups from a geometrical viewpoint.

Definition. *A comparison-triangle for a geodesic triangle $\triangle$ in a metric space X is the Euclidean triangle $\overline{\triangle}$ with side lengths equal to those of $\triangle$.*

($\overline{\triangle}$ *is unique up to isometries.)* *Given two points* $p, q \in \triangle$ *we can define comparison points* $\overline{p}, \overline{q} \in \overline{\triangle}$ *in the obvious way.*

Fig. 3.2 A geodesic triangle $\triangle(x, y, z)$ *and its comparison triangle* $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$.

Definition. *A geodesic triangle satisfies the CAT(0)-inequality if for all* $p, q \in \triangle$ *we have*

$$d(p, q) \leq \; d(\bar{p}, \bar{q})$$

Definition. *A geodesic metric space is CAT(0) if all geodesic triangles satisfy the CAT(0)-inequality.*

Examples.

Examples of CAT(0) spaces are $\mathbb{E}^n$ (in fact any convex subset of $\mathbb{E}^n$), $\mathbb{R}$-trees, $\mathbb{E}^2 - \{x, y : x > 0, y > 0\}$ considered as a length space, hyperbolic n-space $\mathbb{H}^n$.

CAT(0) spaces have many nice properties including contractibility, unique geodesics (for every pair of points x, y in the space there is a unique geodesic joining x and y), and convexity of balls (for every point x in the space and $\epsilon \geq 0$ and for any two points $p, q \in B_\epsilon(x)$ the unique geodesic $[p, q] \subseteq B_\epsilon(x)$). The following theorem and its corollary are two examples of many powerful theorems derived from the CAT(0) property. The idea of Theorem 3.1 is due to Serre and a proof may be found in [**BHa**].

Theorem 3.1. *(Unique centre of a bounded set) Let $A \subseteq X$ be a subset of a complete CAT(0) space. Then there exists a unique centre, c_A of A such that $A \subseteq B(c_A, r_A)$ where r_A is the radius of A defined as $inf\{r \mid A \subseteq B_r(x), x \in X\}$.* □

Corollary 3.2. *If X is a complete CAT(0) space and G is a group of isometries acting on X with a bounded orbit, then the fixed point set of G is a non-empty convex subset of X.* □

The following theorem is a version of the Cartan-Hadamard Theorem, proved by Bridson and Haefliger in [**BHa**], and tells us when a locally $CAT(0)$ space is globally $CAT(0)$. The original theorem was first proved in the context of surfaces by Hadamard and in the context of Riemannian manifolds by Cartan.

Theorem 3.3. *Let X be a complete, simply connected, locally-CAT(0) length space. Then X is CAT(0).* □

In certain circumstances one can glue CAT(0) spaces together to get other CAT(0) spaces and there are sufficient conditions for simplicial- and (more generally) polyhedral-complexes to be CAT(0) which are easy to verify.

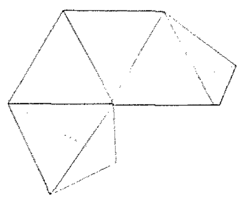
Definition. *For each n-cell σ of a polyhedral-complex X we define a simplicial-complex called the link complex $L(\sigma)$ of σ as follows. $L(\sigma)$ has a vertex for each $n+1$-cell of which σ is a face and more generally a k-simplex for each*

$n + k + 1$-cell of which σ is a face. The gluing maps are induced from the gluing maps of the space.

We regard the link complex as a spherical simplicial complex (that is a simplicial complex whose simplices are spherical)

A natural metric on $L(\sigma)$ is the shortest path metric starting from the angle metric on each simplex. (The angle metric is the natural metric on spherical simplices defined by embedding the unit n-sphere in $\mathbb{E}^{n+1}$ with its centre at the origin and taking the distance between points on a simplex to be the angle between their position vectors.)

(i)



(ii)

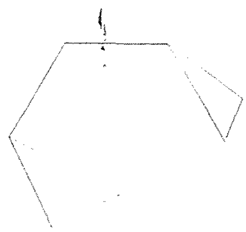


Fig. 3.3 A simplicial complex and the link complex of its centre vertex.

Definition. *A geodesic metric space is CAT(1) if all geodesic triangles with perimeter less than 2π are not 'fatter' (see definition of CAT(0) inequality) than comparison triangles taken from spherical space of constant curvature 1 (that is the unit 2-sphere).*

Definition. *A polyhedral complex X satisfies the link condition if the link complex at every vertex is a CAT(1) space.*

Theorem 3.4. [B] *Let X be a Euclidean polyhedral complex with finitely many isometry types of cells. Then K satisfies the link condition if and only if it is locally CAT(0).*

Theorem 3.5. [**B**] *Let X be a Euclidean polyhedral complex with X locally CAT(0) and with finitely many isometry types of cells. Then the following are equivalent:*

(i) X is CAT(0).

(ii) X is uniquely geodesic.

(iii) X satisfies the link condition and contains no isometrically embedded circles.

(iv) X is simply-connected and satisfies the link condition.

In the case of cube-complexes the link condition is equivalent to the following two properties:

(i) There are no bigons in the link of a vertex.

(ii) Any triangle in a link of a vertex bounds a 2-simplex.

This is for the following reasons. A consequence of the CAT(1) condition is that any geodesic loops have length $> 2\pi$ (II Theorem 7.4 [**BHa**]). Since all cells in a cube-complex are cubes, all edges in the link are of length $\pi/2$. Hence any geodesic loops shorter than this must have length $3\pi/2$ or π. Such loops do not exist if and only if properties (i) and (ii) hold.

Properties (i) and (ii) are collectively known as the *flag condition* which holds if and only if the link complex is a *flag complex*, that is a simplicial complex in which the presence of a boundary of a simplex implies that the simplex itself is present.

Hence a cube-complex is CAT(0) if and only if it is simply-connected and satisfies the link condition above. CAT(0) cube-complexes are called *cubings*.

3.3 The Coxeter cubing

The set of half-spaces in the Moussong complex or Coxeter complex of a Coxeter group is a set with a partial order given by inclusion and an involution defined by reflection in the appropriate hyperplane, and satisfies (1) and (2) in the definition of the cube-complex in section 3.1. Property (1) follows from the fact that M and Σ are connected. To see this, let $a \leq b$ be two halfspaces and take two vertices $v \in a$ and $w \in b^*$. Since the space is connected there is a finite edge path from v to w. The boundary hyperplane of any halfspace c such that $a \leq c \leq b$ is crossed by an edge of this path and hence there are finitely many such halfspaces. Property (2) is clear as a, b, a^*, b^* are all halfspaces.

Hence given a Coxeter group we can define two cube-complexes X_M and X_Σ by the construction above using the Moussong complex and Coxeter complex respectively.

We now give an illustration of how to construct the cube-complex in the case of the Moussong complex M. Each vertex in the cube-complex X_M is defined as a choice of halfspace for each hyperplane. An edge joins two vertices if their halfspace choices disagree on only one minimal hyperplane (Recall that a halfspace h is minimal with respect to a vertex v if there is no $h' \in v$ such that $h' \leq h$.) We then glue in n-cubes if and when their boundaries appear in the $n-1$ skeleta.

Every vertex v of M corresponds to a vertex of X since the set of hyperplanes of M and X are the same by definition and we take the halfspaces of M that contain the vertex v.

The particular connected components of the cube-complexes that we are interested in are the ones containing the vertex corresponding to the identity vertex in M and the identity chamber in Σ.

We now give a sufficient condition for cube-complexes to be isomorphic.

Let $X_1 = (H_1, \leq_1, *_1, v_1)$ and $X_2 = (H_2, \leq_2, *_2, v_2)$ be two connected cube-complexes arising as described above and let $\phi : H_1 \to H_2$ be a bijection preserving the partial order in both directions (i.e. $a_1 \leq_1 a_2$ if and only if $\phi(a_1) \leq_2 \phi(a_2)$), and the involution (so that $\phi(a^{*_1}) = \phi(a)^{*_2}$ and $\phi^{-1}(b^{*_1}) = \phi^{-1}(b)^{*_2}$) where $a \in H_1$ and $b \in H_2$.

Lemma 3.6. *Let X_1, X_2 and ϕ be as above. If $d(\phi(v_1), v_2)$ is finite then X_1 is isometric (with respect to the edge path metric) to X_2.*

Proof: For all $a_1 \in H_1$ and $a_2 \in H_2$ we have $a_1 \leq a_2 \iff \phi(a_1) \leq \phi(a_2)$ and $\phi(a_1^{*_1}) = \phi(a_1)^{*_2}$. We build up the cube-complexes $\hat{X}_1 = (H_1, \leq_1, *_1)$ and $\hat{X}_2 = (H_2, \leq_2, *_2)$ ignoring for the moment the v_is.

Vertices: Clearly ϕ preserves (i) and (ii) in the definition of vertices on page 28 and so sends vertices to vertices.

Edges: Two vertices v, w in X_i are joined by an edge if they differ by switching one minimal halfspace. Clearly ϕ preserves minimality and hence vertices in X_1 are joined by an edge if and only if their corresponding vertices in X_2 are joined by an edge.

Cubes: We glue in cubes in X_i when their boundary appears. This is an inductive process and depends only on the 1-skeleton which we have seen is preserved by ϕ.

ϕ is clearly bijective on the vertex sets of X_1 and X_2 and so we have a cellular bijection from X_1 to X_2. Two vertices are a distance n apart in the edge-path metric if as sets they differ by n elements. This is preserved by ϕ and hence ϕ is an isometry.

Finally we note that the condition that $d(\phi(v_1), v_2)$ is finite ensures that X_1 and X_2 are the same connected component of $\hat{X}_1 \cong \hat{X}_2$. □

Theorem 3.7. *The cube-complex X_M arising from the Moussong complex M is isomorphic to the cube-complex X_Σ arising from the Coxeter complex Σ.*

Proof: By Lemma 3.6 it is enough to show that there is a bijection ϕ between halfspaces of M and halfspaces of Σ preserving nesting and the involution on halfspaces.

By Lemma 2.7 (and a similar result for Coxeter complexes - see e.g. [**Br**]) each hyperplane is fixed pointwise by a unique (non-trivial) element of G and no other hyperplane is fixed by that element. Hence we define a bijection ϕ by mapping the halfspace $h \in M$ for which $gh = h^*$ and such that the identity vertex $1 \in h$, to the unique halfspace in Σ containing the identity chamber and for which $gh = h^*$, and by mapping h^* to the complementary halfspace of $\phi(h)$.

Two hyperplanes h_i, h_j cross in the Moussong complex $\iff$ (by Lemma 2.6) we can translate their intersection by a group element $g \in G$ so that that $gh_i \cap gh_j$ intersects a special cell. Now there exist elements, g_i and g_j say, whose action is to reflect in gh_i and gh_j respectively and whose product has finite order. The group elements that reflect in h_i and h_j are then gg_ig^{-1} and gg_jg^{-1} whose product clearly also has order $m_{ij} = n < \infty$. Hence nesting is preserved by ϕ and since we have ensured that $1 \in h \iff 1 \in \phi(h)$, it is clear that the partial order is preserved.

Similarly in Σ we can translate transversely intersecting hyperplanes back to hyperplanes intersecting the fundamental chamber where their link is a 2n-gon in the Coxeter complex if and only if the product of the corresponding elements is finite order.

The involution $*$ is preserved by the bijection by definition. □

Hence from now on we will refer only to one cube-complex X and usually assume that it is built from the Moussong complex.

Theorem 3.8. *X is CAT(0).*

Proof: We need to show that X is simply-connected and satisfies the link condition.

Simply-connected: We follow the proof that Sageev's cube-complexes are simply-connected. This can be found in [**S1**]. Let $\alpha = (v_0, v_1, \ldots, v_n = v_0)$ be a shortest non-contractible edge-loop in X. Let $(h_0, h_1, \ldots, h_n = h_0)$ be the list of halfspaces swapped to move along α so that v_{i+1} is the vertex obtained after introducing h_i. Since vertices are a choice of halfspace for each pair (h, h^*) all swaps of halfspaces must be swapped back again. Consider the earliest swap back, that is the earliest occurrence of $h_1, \ldots, h_i, \ldots, h_j = h_i^*$ in the list of halfspaces. Now h_i^* is minimal in v_i so $h_{i+1}^* \not\leq h_i^*$ and hence $h_i \not\leq h_{i+1}$. Hence h_{i+1} is minimal in v_{i-1}. Since h_i and h_{i+1} are both minimal in v_{i-1} there is a square in X with vertices $v_{i-1}, v_i, v_{i+1}, v_i'$. We can define a new loop in X with vertices $(v_0, v_1, \ldots, v_i', \ldots, v_n = v_0)$ and halfspace sequence $(h_0, h_1, \ldots, h_{i-1}, h_{i+1}, h_i, h_{i+2}, \ldots, h_n = h_0)$. We can repeat this process until the sequence of halfspaces is $(h_0, h_1, \ldots, h_j, h_j^*)$. Now the loop contains an edge traversed one way and then back again immediately. This loop can clearly be shortened contradicting the fact that α was a shortest loop.

Link condition: Suppose there exists an n-cube σ so that the link of σ contains a bigon. This means that there exist two $n+1$-cubes σ_1 and σ_2 that are glued along two adjacent faces (which intersect in σ). This leads to two hyperplanes intersecting once in a codimension-2 hyperplane of σ_1 and σ_2 and hence in the Moussong complex we have a similar configuration of hyperplanes in neighbouring cells. This contradicts the fact that Coxeter cells are glued along at most one face.

Suppose that there is a triangle (v_1, v_2, v_3) in the link of an n-cube σ in X. The three vertices correspond to three $n+1$-cubes each with σ as a face and the three edges correspond to three $n+2$-cubes. There are n pairwise intersecting hyperplanes in σ and the three $n+2$-cubes contribute three more hyperplanes which pairwise intersect and intersect all hyperplanes of σ so we have a set of $n+3$ pairwise intersecting hyperplanes and hence an $n+3$-cube which appears as the solid triangle (v_1, v_2, v_3) in the link of σ. □

Recall that CAT(0) cube-complexes are known as *cubings*. We call X the *Coxeter cubing* for G. Sageev proves in [**S1**] that each hyperplane is itself an isometrically embedded cubing and cuts X into exactly two halfspaces.

Recall that $M^{(1)}$ is the Cayley graph for $G = \langle S \rangle$.

Lemma 3.9. *$M^{(1)}$ can be isometrically embedded in X.*

Proof: We construct an embedding $\rho : M^{(1)} \to X$. Recall that a vertex in X is defined by a choice of halfspace for each hyperplane. If v is a vertex in M define $\rho(v)$ to be the vertex in X defined by taking $h \in \rho(v)$ if $v \in h$ in M. Two vertices in M are adjacent if and only if they are separated by exactly one hyperplane in M which is true if and only the corresponding vertices in X are separated by exactly one hyperplane and hence if and only if they are joined by an edge in X. □

Lemma 3.10. *(Niblo, Reeves) X is finite dimensional.* [**NR2**]

Lemma 3.11. *(Niblo, Reeves) X is locally finite.* [**NR2**]

We end this section with some more comments on the geometry of X.

Recall that traversing an edge in X corresponds to switching a hyperplane, that is changing the orientation of one minimal hyperplane.

Any geodesic edge path in X crosses each hyperplane at most once. Otherwise it can be shown that the path can be shortened by bringing a section of the path to one side of the double-crossed hyperplane. See [**S1**].

3.4 The group action on the cubing

The group action on X is defined as follows: G acts on the set of halfspaces H of M or Σ. H is also the set of halfspaces of X and so the action on X is defined by the action on H.

We need to prove that this action is well defined. First we show that G takes vertices to vertices. Let v be a vertex. Then v satisfies conditions (i) and (ii) for vertices in the definition of X on page 28. (i) A vertex contains exactly one from each pair of halfspaces and since G acts by isometries on

M this property is preserved. Similarly condition (ii) is clearly preserved by isometries and hence G takes vertices to vertices.

Two vertices v and v' are joined by an edge if they differ as sets by exactly one element. Let $h \in v$ and $h^* \in v'$. Now $v - \{h\} = v' - \{h^*\}$ so $g(v - \{h\}) = g(v' - \{h^*\})$. Then since $g(h^*) = (g(h))^*$, we have that gv and gv' also differ by exactly one element and hence G takes edges to edges.

Since the n-skeleta for $n \geq 2$ are completely determined by the one skeleton we see that G also takes cubes to cubes.

At this stage we prove that the action is by isometries with respect to the edge path metric. Let $x, y \in X^{(0)}$, $d(x, y) = m$ and $v_0 = x, v_1, v_2, \ldots, v_m = y$ be a sequence of vertices in a shortest edge path from x to y. Now consider $gv_0, gv_1, gv_2, \ldots, gv_m$. This is also an edge path from gx to gy of length m since G takes edges to edges. Hence $d(gx, gy) \leq m = d(x, y)$. Now let $w_0 = gx, w_1, w_2, \ldots, w_{m'} = gy$ be a shortest length edge path from gx to gy. Then $g^{-1}w_0, g^{-1}w_1, g^{-1}w_2, \ldots g^{-1}w_{m'}$ is also an edge path of length m' from x to y so $d(x, y) \leq m' = d(gx, gy)$ and hence $d(x, y) = d(gx, gy)$ as required.

Lemma 3.12. *(Niblo, Reeves) The action of G on $\hat{X}$ is properly discontinuous.* [**NR2**]

Proof: It is sufficient to prove that the stabiliser, $stab(v)$, of any vertex v is finite. Let v be a vertex of $\hat{X}$. By definition $d(v, v_0)$ is finite where v_0 is the image of the identity vertex of M. The orbit $stab(v) \cdot v_0$ is finite since X is locally finite by Lemma 3.11 so $stab(v) \cap stab(v_0)$ is a finite index subgroup of $stab(v)$. But since G acts freely on vertices of M we have $stab(v_0)$ is trivial and hence $stab(v)$ is finite as required. □

Lemma 3.13. *A Coxeter group is finite if and only if its cubing is finite.*

Proof: A Coxeter group is finite if and only if there are finitely many hyperplanes in M if and only if there are finitely many hyperplanes in X if and only if X is finite. □

Lemma 3.14. *The embedding ρ of $M^{(1)}$ (i.e. the Cayley graph of G) in X as in Lemma 3.9 is equivariant with respect to the group action on X.*

Proof: First consider the action of G on $\hat{X}$. G acts on M by reflection in hyperplanes. This induces an action on halfspaces of M. Let v be a vertex of M and $g \in G$. Now gv is another vertex of M. It is clear that g takes the halfspaces defining v to those defining gv. Now we consider the same action on halfspaces but now considered as halfspaces of $\hat{X}$. By the above, g sends $\rho(v)$ in $\hat{X}$ to $\rho(gv)$. □

To show that we do indeed have an action on X it remains to prove that G preserves the connected component X. It suffices to show that any element of G moves vertices by only a finite distance. Let v be a vertex in X and $g \in G$. By Lemma 3.14 and Corollary 2.4 g takes the identity vertex v_0 to a vertex a distance equal to the length of a minimal word for g, n say. v is a finite distance, m say, from v_0. Then $d(v, gv) \leq d(v, gv_0) + d(gv_0, gv) \leq d(v, v_0) + d(v_0, gv_0) + d(gv_0, gv) = 2m + n < \infty$ as required.

Note Since G acts without fixed points on a finite dimensional cubing, by the main theorem of [**S1**] we have that each hyperplane stabiliser quotiented out by the group generated by reflection in it, i.e. an index 2 subgroup of it, is a codimension-1 subgroup.

3.5 Examples

1. The cubing for D_6 is the 3-cube since the set of hyperplanes of the Moussong complex consists of three pairwise intersecting hyperplanes. By Lemma 3.9 $M^{(1)}$, which is a hexagon, is isometrically embedded in $X^{(1)}$. This embedding is shown in Figure 3.4 below.

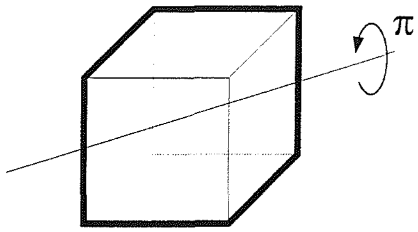


Fig. 3.4 X_{D_6} *showing the embedding of* M *and the action of a generator*

The action of a generator of G on X is to 'reflect' the hexagon in X and swap the two opposite non-hexagon vertices. The isometry of X realising this action is a rotation of π through a line joining midpoints of a pair of opposite (parallel) edges as shown in the diagram.

2. The Moussong complex for S_4 is the *3-permutahedron* shown below.

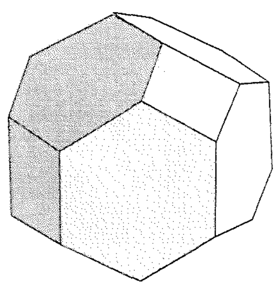


Fig. 3.5 The 3-permutahedron

This has six (pairwise intersecting) hyperplanes and hence the cubing is a 6-cube. M is embedded in the cube as shown below where the edges of the permutahedron are labelled by the index of the coordinate axis with which they are parallel.

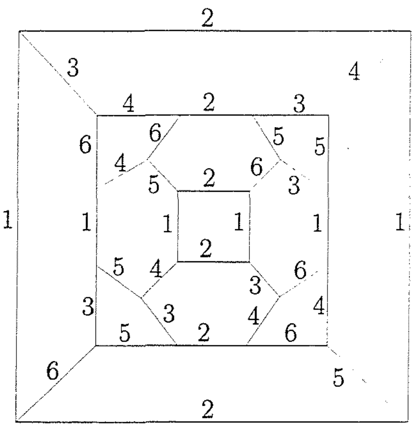


Fig. 3.6 The embedding of M_{S_4} in the 6-cube

Note that all finite groups have an n-cube for their cubing since the Moussong complex for every finite group is a cell with all hyperplanes meeting at the centre.

3. $D_\infty \cong \circ \overset{\infty}{\text{——}} \circ$

In this case there are infinitely many hyperplanes none of which intersect and the Coxeter cubing is the same as the Moussong complex. In fact this is true for all right-angled Coxeter groups which will be proved in chapter 4.

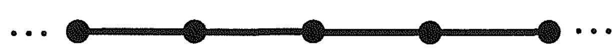
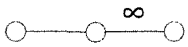


Fig. 3.7 The cubing for D_∞

4. $PGL_2(\mathbb{Z}) \cong$ 

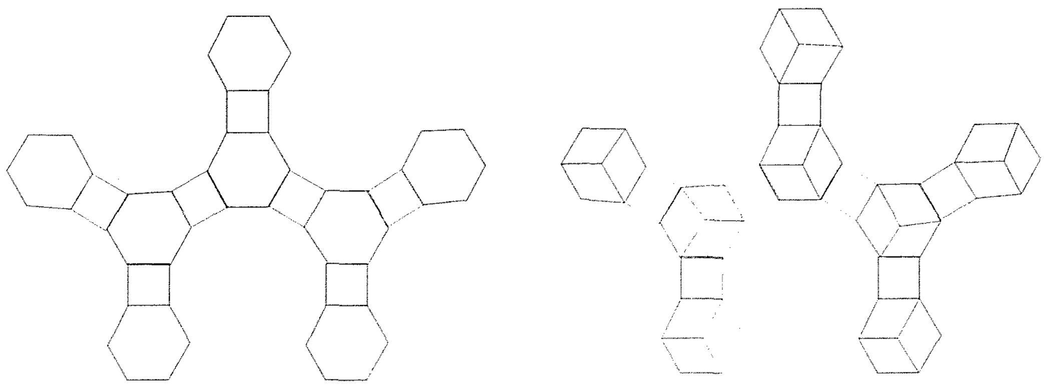
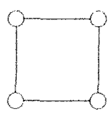


Fig. 3.8 The Moussong complex and Coxeter cubing for $PGL_2(\mathbb{Z})$

5. $\widetilde{A}_3 \cong$ 

The 'maximal' finite special subgroups in this case are all S_4 and hence the maximal cells in the Moussong complex are all 3-permutahedra. (See Fig 3.4 above.) Four of these permutahedra fit around each vertex in M. Each permutahedron is embedded in a 6-cube as described in example 3 and the cubing is 6-dimensional.

Having described the main objects of our study we now turn to our main task, that is to decide which Coxeter groups act cocompactly on their Coxeter cubings.

Chapter 4

Right-angled Coxeter groups

In our search for cocompact actions the easiest case to deal with is when the Coxeter group G is right-angled. We begin this chapter with a few comments about the geometry of the Moussong complex M in this case.

(1) Finite special subgroups of G are isomorphic to $(\mathbb{Z}_2)^r$ for which the Coxeter cell is an r-cube. Hence by Theorem 2.2 the Moussong complex is a CAT(0) cube complex.

(2) Elementary hyperplanes h_i, h_j fixed pointwise by generators s_i and s_j cross if and only if $m_{ij} < \infty$, i.e. $m_{ij} = 2$.

(3) In the star of the identity vertex there is an edge (incident with 1) for each generator, and a k-cube for each set of k pairwise commuting generators.

The main theorem for this section is the following.

Theorem 4.1. *Let G be a right-angled Coxeter group. Then the Moussong complex M is isometric to the cubing X and the G-action is the same. Hence G acts cocompactly on X.*

Let $\hat{h}_i$ and $\hat{h}_j$ be hyperplanes of X. By Lemma 2.6 $\hat{h}_i = g_1 h_i$ and $\hat{h}_j = g_2 h_j$ for some group elements g_1, g_2 and elementary hyperplanes h_i and h_j in which s_i and s_j (respectively) are the reflections. Then $r_i = g_1 s_i g_1^{-1}$ and $r_j = g_1 s_j g_1^{-1}$ are the reflections in $\hat{h}_i$ and $\hat{h}_j$ respectively.

Lemma 4.2. *With the above notation $r_i r_j$ has infinite order if and only if $\hat{h}_i$ and $\hat{h}_j$ do not intersect.*

Proof: $r_i r_j$ has infinite order $\iff$ $\hat{h}_i \cap \hat{h}_j = \emptyset$ otherwise by Lemma 2.6 we could translate a cube containing $\hat{h}_i$ and $\hat{h}_j$ back to a cube in the star of 1, i.e. a special subgroup, by a group element g say. Then by remark (2) $g r_i g^{-1}$ and $g r_j g^{-1}$ are generators that commute, i.e. $g r_i g^{-1} g r_j g^{-1} = g r_j g^{-1} g r_i g^{-1}$ and hence $r_i r_j = r_j r_i$, a contradiction. □

Proof of Theorem 4.1: By Lemma 4.2 hyperplanes intersect in M if and only if they intersect in X so we have a bijection from hyperplanes of M to hyperplanes of X preserving intersections and so by Lemma 3.6 M and X are isometric. The action on X was defined by the action on its hyperplanes induced from the action on the hyperplanes of M and so is clearly the same. By Theorem 2.2 G acts cocompactly on M and hence on X. □

Examples.

1. $m_{ij} = 2$ for all i, j. In this case G is isomorphic to the finite group $\mathbb{Z}_2^n$ and $M = X$ is an n-cube.

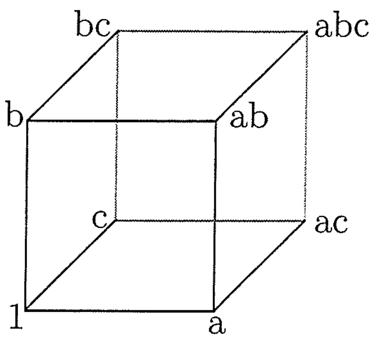


Fig. 4.1 The cubing for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

2. $m_{ij} = \infty$ for all i, j. $M = X$ is an n-valent tree where n is the number of generators of G. See Fig. 2.6.

3. $G \cong (D_\infty)^n$. $M = X \cong \mathbb{E}^n$ cubed in the normal way. See Fig. 2.3.

Chapter 5

Cocompact actions

In this chapter we begin to build up a picture of when Coxeter groups act cocompactly on their Coxeter cubings. The following theorem of Moussong suggests a starting point.

Theorem 5.1. *(Moussong) $G = \langle S \rangle$ is hyperbolic if and only if neither of the following hold*

(i) There exists $T = T_1 \sqcup T_2 \subset S$ so that $G_T = G_{T_1} \times G_{T_2}$ with both factors infinite

(ii) There exists $T \subseteq S$ such that G_T is a Euclidean Coxeter group with $|T| \geq 3$

where a Euclidean Coxeter group is an irreducible affine reflection group on $\mathbb{E}^n$. (There is a well known classification of Euclidean Coxeter groups. A table of their graphs is included at the end of this chapter for the convenience of the reader.) In this context we use the term *hyperbolic Coxeter group* to mean a Coxeter group that is word-hyperbolic in the sense of Gromov. (See for example [**A**] for an introduction to word-hyperbolic groups.)

In sections 5.3, 5.4 and 5.5 we prove results about hyperbolic Coxeter groups, direct products and Euclidean Coxeter groups but first we derive results from the pattern of hyperplanes in the Moussong complex.

5.1 Phantom Vertices

In this section we define the notion of *phantom vertices* in X.

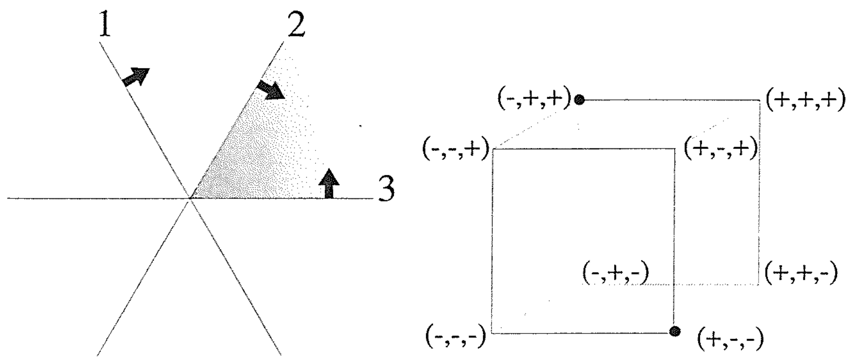


Fig. 5.1 Two phantom vertices

Figure 5.1 shows a section of a Moussong complex with three hyperplanes 1,2,3 and the corresponding 3-cube in the cubing. Hyperplanes 1, 2 and 3 oriented $(+,+,+)$ as above, considered as hyperplanes in a Moussong complex define a vertex in the complex (shown by the shaded region) whereas the orientation $(-,+,+)$ does not define a vertex in M. However, all orientations define vertices in the cube by choosing the appropriate halfspaces.

Definition. *Vertices in X corresponding to vertices in M will be called chamber vertices. Vertices defined by orientations of hyperplanes in M that do not define a vertex in M will be called phantom vertices.*

The two phantom vertices in the cube in Fig. 5.1 are shown by heavy dots.

Note that, in general, not every orientation of the hyperplanes of M defines a vertex in X, for example given two non-intersecting hyperplanes h_1 and h_2, the half spaces must be chosen to intersect as shown in Fig. 3.1 on page 29. This is a consequence of rule (ii) for vertices in the construction of the cubing in Chapter 3.

Definition. *(Levels of phantomness) A vertex v in X is said to be phantom of level n denoted $p(v) = n$ if n orientations of hyperplanes must be reversed to obtain a chamber (non-phantom) vertex. Equivalently the level of a phantom vertex is equal to its distance in the 1-skeleton from the set of chamber vertices, i.e. M in X.*

Lemma 5.2. *There are no phantom vertices if and only if G is right-angled.*

Proof: First suppose that G is right-angled. Then by Theorem 4.1 $M = X$ and hence $M^{(0)} = X^{(0)}$ and there are no phantom vertices.

Now suppose that there are no phantom vertices and assume, for a contradiction, that G is not right-angled. Then there are two generators, s_i, s_j say, such that the order of $s_i s_j$, m_{ij} is not 2 or ∞. Then there is a $2n$-polygon in M with $n > 2$. Let h_i and h_j be the hyperplanes corresponding to s_i and s_j and let h be another hyperplane intersecting the polygon. Starting with the vertex in X corresponding to the identity vertex in M and switching h gives a vertex in X which does not correspond to a vertex in M, that is a phantom vertex, which is a contradiction. □

We subdivide the vertex set of X by the level of phantomness of vertices. Let $V_i \subseteq X^{(0)}$ be the set of vertices of phantom level i. Then we have $X^{(0)} = V_0 \sqcup V_1 \sqcup V_2 \sqcup \cdots$. Note that $V_0 = M^{(0)}$ and that V_i may be empty. The following Lemma shows that the levels of phantom vertices form layers in X.

Lemma 5.3. $V_{i-1} \sqcup V_{i+1} = \{v \in X | d(v, V_i)\} = 1\}$.

Proof: For the purpose of the proof we define $V_i = \emptyset$ for $i < 0$.

($\subseteq$) Each phantom vertex of level $i+1$ and $i-1$ is obtained by switching one minimal hyperplane from a vertex of level i.

($\supseteq$) Let $v \in \{v \in X | d(v, V_i) = 1\}$ and suppose $v' \in V_i$ such that $d(v, v') = 1$. Moving from v' to v must increase or decrease the level of phantomness otherwise $v \in V_i$. Also the increase or decrease must be exactly one since we only switch one hyperplane. □

Note that since X is connected if $V_i = \emptyset$ then for all $j \geq i$ we have $V_j = \emptyset$.

Now we show that each G acts equivariantly and cocompactly on V_i in X for all i

Lemma 5.4. *G acts equivariantly on V_i in X for all i.*

Proof: The proof is very similar to that of Lemma 3.14. The phantomness of a vertex of X is the minimum number of hyperplanes that need to be switched to get to a vertex of V_0 in X. We can think of any vertex of X as a set of choices of halfspaces in M. Then the phantomness can be thought of as the minimum number of halfspaces needed to be switched before the set defines a vertex in M. This number is clearly not affected by cellular isometries of M. □

Lemma 5.5. *G acts cocompactly on V_i for all i.*

Proof: We use induction on i. The Lemma is true for $i = 0$ since $V_0 = \rho(M^{(0)})$. Now suppose that G acts cocompactly on V_i.

Each vertex v of level i has finitely many minimal hyperplanes since X is locally finite by Lemma 3.11, and therefore neighbours finitely many vertices of phantom level $i+1$. Let $\tau(v,h)$ denote the vertex obtained by switching the orientation of h, where h is a minimal hyperplane of v. We first note that if h is minimal in v then gh is minimal in gv, moreover

$$g(\tau(v,h)) = \tau(gv, gh) \qquad (*)$$

Assume that there are finitely many orbits of vertices of phantom level i. Choose a representative from each orbit, $v_1, v_2, \ldots, v_n$ say. Let $v_j^1, v_j^2, \ldots, v_j^{r_j}$ be the finitely (by local finiteness Lemma 3.11) many vertices of level $i+1$ distance 1 from v_j. Now given w_j', a vertex of phantom level $i+1$, neighbouring a vertex w_j in the orbit of v_j, there is an element $g \in G$ taking w_j to v_j and w_j' to a v_j^k (given by $(*)$). Hence the finite set $\{v_j^k\}$ is a set of representatives of orbits of $i+1$ level phantom vertices and hence G acts cocompactly on V_{i+1}. □

Corollary 5.6. *G acts cocompactly on X if and only if the level of phantom vertices is bounded above.*

Proof: Vertices of phantom level 0 are vertices of the Moussong complex $\rho(M) \subseteq X$ on which G acts cocompactly.

($\Rightarrow$) Suppose that the level of phantom vertices is unbounded. Then we can find a sequence of phantom vertices of strictly increasing level. Level of phantomness is preserved by the action of G hence there must be infinitely many orbits of vertices and so G does not act cocompactly on X.

($\Leftarrow$) By Lemma 5.3 each V_i/G is finite and if there are finitely many of them then $X^{(0)}/G$ is finite. □

5.2 Phantom rays and Ladders

Definition. *A phantom ray of length n in X is a geodesic (in $X^{(1)}$) edge path with vertices $x_0, x_1, \ldots, x_n$ where x_i has phantomness i. In particular x_0 is the image of a vertex of the Moussong complex M in X.*

Lemma 5.7. *The following are equivalent.*

(i) G acts non-cocompactly on X.

(ii) There is no bound on the length of phantom rays in X.

(iii) There is an infinite phantom ray in X.

Proof: Taking a vertex in X of phantomness n we can find a shortest edge path joining it to a vertex of V_0 (the image of $M^{(0)}$ in X). Hence by Corollary 5.6 we have $(i) \iff (ii)$. The implication $(iii) \Rightarrow (ii)$ is clear.

Now suppose that there is no bound on the length of phantom rays. Let $\{r_i\}$ be a sequence of phantom rays such that r_i has length i. Since G acts transitively on the vertices of V_0, preserves phantomness and acts by isometries on X we can assume that each ray begins at the same vertex x_0. We define an infinite phantom ray, that is a map $r : \mathbb{N} \to X^{(0)}$ such that $r(i)$ has phantomness i and $r(i)$ and $r(i+1)$ are the endpoints of an edge in X for all i. Define $r(0) = x_0$ and $r(i+1)$ to be a vertex which is the next vertex for infinitely many of the r_i. Such a vertex exists since X is locally finite. $\square$

Definition. *Given a set of halfspaces H of a space, let $\mathcal{H}$ be the corresponding set of oriented hyperplanes. We define a ladder of size n to be a subset L of $\mathcal{H}$ consisting of two hyperplanes k_1, k_2 called uprights and a finite sequence of pairwise non-intersecting hyperplanes $\{h_1, h_2, \ldots, h_n\}$ called rungs such that*

(i) $k_1 \cap k_2 \cap h_1 \neq \emptyset$

(ii) if a, b and c are the open halfspaces corresponding to the oriented hyperplanes k_1, k_2 and h_1 respectively, then $a \cap b \cap c = \emptyset$

(iii) h_i intersects k_1 and k_2 for $1 \leq i \leq n$

(iv) h_{i+1} separates h_i from h_{i+2} for $1 \leq i \leq n-2$

(v) there are no hyperplanes in M parallel to and between h_i and h_{i+1} for $1 \leq i \leq n-1$

Definition. *An infinite ladder in M is a ladder with a countable infinite set of rungs $\{h_1, h_2, h_3, \ldots\}$ satisfying (i)-(v) for all $i \geq 1$.*

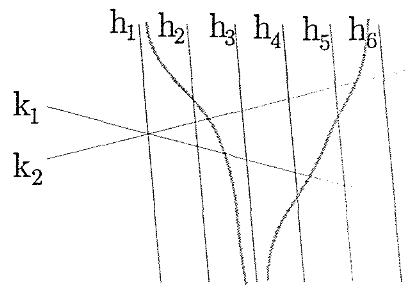


Fig. 5.2 A ladder of hyperplanes of size 6

Note that there may be hyperplanes between h_i and h_j for $|j - i| > 1$ (as shown in grey in Fig. 5.2).

Lemma 5.8. *G acts non-cocompactly on X if and only if there is an infinite ladder contained in the set of hyperplanes of M.*

Proof: ($\Leftarrow$) Let $\{k_1, k_2, h_1, h_2, \ldots\}$ be an infinite ladder in M. We will show that there are infinitely many orbits of 3-cubes in X. Recall that a 3-cube in X uniquely determines a set of 3 pairwise intersecting hyperplanes in M. We define a *triangle of hyperplanes* to be a set of three pairwise intersecting oriented hyperplanes $\{h_0, h_1, h_2\}$ such that the corresponding halfspaces of M have the following property $h_0^+ \cap h_1^+ \cap h_2^+ = \emptyset$. We define the *diameter* of such a triangle to be $max_i\{d(h_i \cap h_{i+1}, h_{i+2}\}$ for $i = 1, 2, 3$ where subscripts are taken modulo 3. Consider the triangles $\{k_1, k_2, h_i\}$. The distances $d(h_i^-, k_1^+ \cap k_2^+)$ in M strictly increase with i so there is no bound on the diameter of triangles of hyperplanes in M. Since G acts by isometries on M the diameter of triangles is constant within an orbit. Therefore there are infinitely many orbits of such triangles and hence infinitely many orbits of 3-cubes in X.

($\Rightarrow$) Suppose that G acts non-cocompactly on X. Then by Lemma 5.7 there is an infinite phantom ray $x_0, x_1, x_2, \ldots$ in X. We show that such a ray implies the existence of an infinite ladder. Let h_i be the hyperplane crossed to get from x_{i-1} to x_i. It is clear that h_1 is minimal but non-supporting for x_0 in M. Let $k_1, \ldots, k_r$ be the supporting hyperplanes of x_0 in the same Coxeter cell as h_1 in M. First we show that there are no i, j for which $h_i = k_j$. Suppose for a contradiction that this is not the case and let l be the smallest number for which h_l is a supporting hyperplane of x_0. Let y be the vertex obtained by switching h_l from x_0 and consider the edge path $(y, x_0, \ldots, x_l)$. Since $(x_0, \ldots, x_l)$ is geodesic, the hyperplanes h_i are distinct and so must cross h_l. We can now switch the order of the h_i bringing h_l to the front which gives an edge path $x_0, \ldots, x_l$ whose second vertex is a chamber vertex and hence x_l has phantomness less than l, a contradiction.

By the Infinite Ramsey Theorem (see for example [**Gr**]), infinitely many of the h_i must be pairwise parallel otherwise there are infinitely many that pairwise intersect contradicting the finite dimensionality of X. We denote these hyperplanes $h_{n_1}, h_{n_2}, \ldots$ ordered so that h_{n_i} is closer to x_0 than h_{n_j} whenever $i < j$. We claim that any two of $k_1, \ldots, k_r$ together with the h_{n_i} form an infinite ladder. We check the desired properties:

(i) $k_1 \cap k_2 \cap h_{n_1} \neq \emptyset$. Clearly $k_1 \cap k_2 \cap h_1 \neq \emptyset$ since they are all hyperplanes intersecting a cell in M. If $h_1 = h_{n_1}$ then we are done. Otherwise let i be the smallest number such that h_1 is parallel to h_{n_i}. Then the set $k_1, k_2, h_1, h_{n_i}, h_{n_{i+1}}, \ldots$ is an infinite ladder with the required property.

If h_1 intersects every h_{n_i} then we claim that h_{n_1} is minimal but non-supporting for x_0 and consider the ladder $\{k_1, k_2, h_{n_1}, h_{n_2}, \ldots\}$ which has the required property.

Proof of claim: First we note that if (x, y, z) is an edge path in a cubing and the hyperplanes crossing edges (x, y) and (y, z) intersect, then x, y and z are three vertices of a square in X. Now we can assume that for all $i < n_1$ we have $h_i \cap h_{n_1} \neq \emptyset$. Using the observation above repeatedly we deduce that there are squares all along $(x_0, x_1, \ldots, x_n)$ as the following diagram suggests so that we see that h_{n_1} is minimal with respect to x_0. We have already noted that all of the h_i are non-supporting.

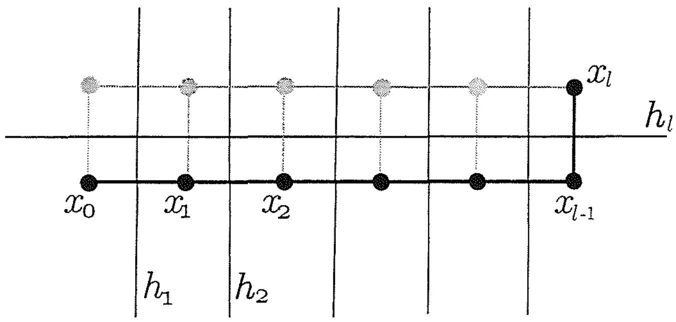


Fig. 5.3 Proving h_l is minimal

(ii) $k_1^+ \cap k_2^+ \cap h_{n_1}^- = \emptyset$ otherwise h_1 would be a supporting hyperplane of x_0.

(iii) Each h_i intersects k_1 and k_2 otherwise switching h_{n_i} would break rule (ii) for vertices (page 27) in X.

(iv) Each h_{n_i} separates $h_{n_{i-1}}$ from $h_{n_{i+1}}$ for $2 \leq i \leq n - 1$ by the definition of the h_{n_i}.

(v) There are no hyperplanes parallel to and between h_i and h_{i+1} for $1 \leq i \leq n - 1$ by definition. □

In the Moussong complex of each of the three Euclidean triangle groups it is easy to spot ladders of hyperplanes. Is it true that every infinite ladder contains an infinite subladder whose hyperplanes belong to a triangle group? If so, then this triangle group must be Euclidean as hyperbolic triangle groups do not contain infinite ladders as proved in section 5.3. These observations lead to the following conjecture for which an outline of a proof is given.

Conjecture 5.9. *Let G be a Coxeter group and X its cubing. G acts non-cocompactly on X if and only if G contains a Euclidean triangle group generated by reflections in three hyperplanes of X.*

The following is an outline of a suggested attack on the conjecture.

($\Leftarrow$) Consider the set of hyperplanes H in M generated by reflections in the three hyperplanes. The intersection pattern of the hyperplanes of H must essentially be the same as the standard pattern in $\mathbb{E}^2$, i.e. there are three infinite families of parallel hyperplanes so that each intersects every hyperplane from another family. Choosing one hyperplane from each of two of these families together with infinitely many of the third family gives an infinite ladder in M and hence by Lemma 5.8 the action is non-cocompact.

($\Rightarrow$) By Lemma 5.8 there is an infinite ladder in M. Since k_1 and k_2 are any two supporting hyperplanes of x_0 we can assume that h_1 is generated by k_1 and k_2. We now define a new ladder with k_1, k_2 and h_1 the same and the sequence of parallel hyperplanes defined as $g_i h_1$ where g is the infinite order element of G which is the reflection in h_1 followed by the reflection in h_2. Now consider the group generated by reflections in k_1, k_2 and h_2. This is a triangle group. The problem is to prove that its hyperplanes contain an infinite ladder. Infinite triangle groups are either hyperbolic or Euclidean and hyperbolic groups act cocompactly on their cubings and hence do not contain an infinite ladder. Hence G must be a Euclidean triangle group.

5.3 Hyperbolic Coxeter groups

In this section we prove the following theorem.

Theorem 5.10. *Hyperbolic Coxeter groups act cocompactly on their Coxeter cubings.*

Proof: The proof is a corollary of Lemma 5.8. Let G be a hyperbolic Coxeter group with Moussong complex M and Coxeter cubing X. Suppose, for a contradiction, that G acts non-cocompactly on X. By Lemma 5.8 there is an infinite ladder of hyperplanes $\{k_1, k_2, h_1, h_2, h_3, \ldots\}$ in M as in the diagram below. By a theorem of Moussong (Theorem 8.1 in [**D**]) M can be given a CAT(-1) metric by replacing all cells by cells in $\mathbb{H}^n$.

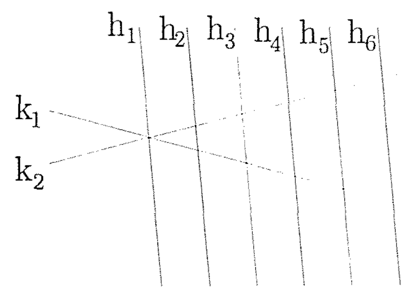


Fig. 5.4 The ladder $\{k_1, k_2, h_1, h_2, h_3, \ldots\}$

Let α_1 and α_2 be two geodesic rays in the hyperplanes k_1 and k_2 respectively starting at a common point x_0 of $k_1 \cap k_2$ so that each α_i crosses each

h_i. Consider the geodesic triangles (x_0, x_i^1, x_i^2) where x_i^j is the point where α_j crosses the hyperplane h_i.

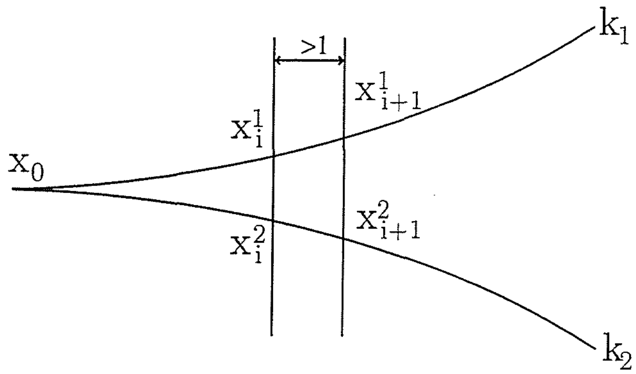
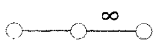


Fig. 5.5

The hyperplanes resulting in the triangles (x_0, x_i^1, x_i^2) *and* $(x_0, x_{i+1}^1, x_{i+1}^2)$

Since these triangles all have the same angle at x_0 this is also true for all of the comparison triangles in $\mathbb{H}^2$. Since the distance between any non-intersecting hyperplanes is at least 1 then the distance of x_0 to the side $[x_i^1, x_i^2]$ increases by at least 1 for each increase in i whereas this distance in the comparison triangles in $\mathbb{H}^2$ reaches an upper limit. Hence there are geodesic triangles that don't satisfy the CAT(-1) triangle condition, a contradiction. □

Example.

1. For the Moussong complex and Coxeter cubing of $PGL_2(\mathbb{Z}) \cong$ 

see Fig. 3.8 on page 43.

5.4 Direct products

In this section we will use the edge path metric on the Coxeter cubing X, that is $d(x, y)$ is equal to the minimum number of hyperplanes crossed in a path in $X^{(1)}$ from x to y. The direct product of two cubings X_1 and X_2 with edge path metrics d_1 and d_2 is then defined as the cubing X with $X^{(0)} = X_1^{(0)} \times X_2^{(0)}$ and edge path metric $d(x, y) = d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$.

Lemma 5.11. *Let $X = (H, \leq, *, v)$ be a cubing. Then X is a direct product of the two cubings $X_1 = (H_1, \leq|_{H_1}, *|_{H_1}, v_1)$ and $X_2 = (H_2, \leq|_{H_2}, *|_{H_2}, v_2)$ if and only if $H = H_1 \sqcup H_2$ and $h_1 \not\leq h_2$ for all $h_1 \in H_1$ and $h_2 \in H_2$.*

Proof: ($\Rightarrow$) Suppose, for a contradiction, that there exist $h_1 \in H_1$ and $h_2 \in H_2$ such that $h_1 \leq h_2$. It is possible to choose vertices $v = (v_1, v_2), w = (w_1, w_2) \in X_1 \times X_2$ so that h_1 is minimal with respect to v and h_2^* is minimal with respect to w and so that v and w are separated by both hyperplanes. This is because each hyperplane separates X into two non-empty halfspaces. Let $v' = (v_1', v_2')$ and $w' = (w_1', w_2')$ be the vertices obtained from v and w respectively by switching these minimal halfspaces h_1 and h_2^* as in *Fig. 5.6* below. Note that $v_2 = v_2'$ and $w_1 = w_1'$.

Let d_1, d_2, d be the edge path metrics in X_1, X_2, and X respectively. Then

$$\begin{aligned} d(v, w) &= d(v', w') + 2 \\ &= d_1(v_1', w_1') + d_2(v_2', w_2') + 2 \end{aligned}$$

And using the triangle inequality for the triangle (v, w, v') in *Fig 5.6* we have

$$\begin{array}{rcccccccc} d(v,w) & \leq & d(v,v') & & + & & d(v',w) & & \\ & = & d_1(v_1,v_1') & + & d_2(v_2,v_2') & + & d_1(v_1',w_1) & + & d_2(v_2',w_2) \\ & = & 1 & + & 0 & + & d_1(v_1',w_1') & + & d_2(v_2',w_2) \end{array}$$

The $\leq$ comes from the fact that there may be other hyperplanes separating v, w and v'. Therefore from the above we have

$1 + d_2(v_2', w_2') \leq d_2(v_2', w_2)$ but this is an equality by the definition of w and w'. Hence there are no hyperplanes in X between h_1 and h_2. Since this is true for any nested h_1, h_2 there can be no non-nested halfspaces in either X_1 or X_2 and so X_1 and X_2 are finite dimensional cubes so X is a finite dimensional cube and hence $h_1 \not\leq h_2$, a contradiction.

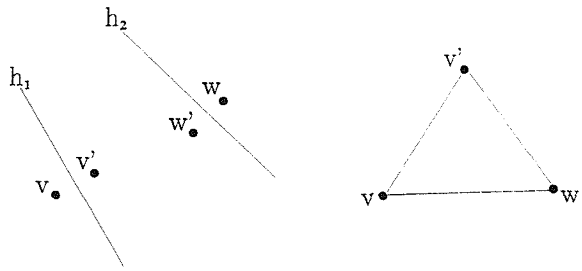


Fig. 5.6 $h_1 \leq h_2$ *and the triangle* (v, w, v')

($\Leftarrow$) $d(x,y) = min$(number of hyperplanes of H_1 crossed + number of hyperplanes of H_2 crossed to get from x to y) $= min$(number of hyperplanes of H_1 crossed in X_1 to get from x_1 to y_1) + min(number of hyperplanes of H_2 crossed in X_2 to get from x_2 to y_2) $= d_1(x_1, y_1) + d_2(x_2, y_2)$. □

Theorem 5.12. *Let* $G = G_1 \times G_2$ *be a Coxeter group such that* G_1 *and* G_2 *are infinite. Then the Coxeter cubing* X *of* G *is the direct product* $X_1 \times X_2$

of the Coxeter cubings of G_1 and G_2. Moreover G acts cocompactly on X if and only if G_1 and G_2 act cocompactly on X_1 and X_2.

Proof: Let M_1 and M_2 be the Moussong complexes of G_1 and G_2 respectively. First we show that every hyperplane in M_1 crosses every hyperplane in M_2. Let h_1, h_2 be hyperplanes in M_1, M_2 respectively. The reflections s_1 and s_2 in h_1 and h_2 satisfy $(s_1 s_2)^2 = 1$ so in M there is a square σ with vertices $1, s_1, s_2, s_1 s_2$, hence h_1 and h_2 intersect in σ.

The cubing X is obtained from the list of hyperplanes in M and their intersection information along with a choice of fundamental vertex. Therefore by Lemma 5.11 $X = X_1 \times X_2$.

Now $G = G_1 \times G_2$ has the diagonal action on X, i.e. the action on hyperplanes is defined by $(g_1, g_2)h_j = g_j h_j$ where h_j is a hyperplane of X_j. To see this we look at the action of the generators $(s_i, 1)$ for $s_i \in T_1$ and $(1, s_i)$ for $s_i \in T_2$. Since every hyperplane in H_2 intersects every hyperplane of H_1 and moreover since X is a cubing does so orthogonally, we have $(s_i, 1)h_2 = h_2$ for all $h_2 \in H_2$. It is clear that $(s_i, 1)h_1 = s_i h_1$ for all $h_1 \in H_1$. For generators of G_2 the argument is the same.

To show that this action is cocompact it is enough to show that there are finitely many orbits of *maximal cubes* in X where a maximal cube is defined as a cube which is not the face of any higher dimensional cube. This is equivalent to showing that there are finitely many orbits of maximal pairwise intersecting sets of hyperplanes. Each such set is a union of such a maximal set of hyperplanes from X_1 with such a maximal set from X_2 and the size of maximal sets in both is bounded above. Hence the size of maximal sets in X is bounded. Since the action of G_i on X_i is cocompact, the number of orbits of maximal sets in X_i is finite, n_i say, so the number of orbits of maximal sets in X is $n_1 n_2$.

Conversely suppose that G acts cocompactly on X and assume, for a contradiction, that there are infinitely many G_1-orbits of k-cubes in X_1 for some k. In X, G_1 preserves the set of hyperplanes of X_1 and G_2 fixes pointwise all of these hyperplanes. Since G_1 and G_2 together generate G, the number of G-orbits of k-cubes in X is also infinite, a contradiction. Hence G_1 acts cocompactly on X_1. The proof is identical for G_2 acting on X_2. □

5.5 Euclidean Coxeter groups

A Euclidean Coxeter group is an irreducible affine reflection group on $\mathbb{E}^n$ for some n. Such Coxeter groups were classified by Coxeter in 1934 [**C1**]. A list of all their graphs is included at the end of this chapter for the convenience of the reader. In this section we show that Coxeter groups containing a Euclidean special subgroup with at least three generators do not act cocompactly on their cubings.

First we remark that Euclidean Coxeter groups with n generators act on $\mathbb{E}^{n-1}$ and the reflection hyperplanes in $\mathbb{E}^{n-1}$ divide the space into $(n-1)$-simplices in such a way that the resulting simplicial complex is isomorphic to the Coxeter complex. There are finitely many classes of hyperplanes under the relation of parallelism. See [**Br**] for the details. Also the Moussong complex for every Euclidean Coxeter group is a cell decomposition of $\mathbb{E}^{n-1}$.

This can be seen as follows. Consider the action of a special subgroup. This is a finite group and therefore has a fixed point by Corollary 3.2. The orbit under this subgroup of any point not fixed defines a Coxeter cell for the subgroup. The Coxeter cells fit together as in the construction of the Moussong complex but from this point of view, within Euclidean space. Since a fundamental region for the action of the group is contained within the special cells, all of the space is filled. See [**C3**] Chapter XI. This leads to some interesting tesselations of space, for example, the Moussong complexes of the three-dimensional Euclidean Coxeter groups on pages 69 and 70.

Lemma 5.13. *Euclidean Coxeter groups with at least three generators act non-cocompactly on their cubings.*

Proof: Let G be a Euclidean Coxeter group with n generators. Then G acts on $\mathbb{E}^{n-1}$ by reflections in codimension-1 hyperplanes. By the remark in the previous paragraph, $\mathbb{E}^{n-1}$ is divided up into simplices by the reflection planes and is isomorphic to the Coxeter complex when considered as a simplicial complex. Each chamber has n supporting hyperplanes, one from each infinite parallel family of hyperplanes.

Given a set $\{h_1, h_2, \ldots, h_n\}$ of pairwise intersecting codimension-1 hyperplanes in $\mathbb{E}^{n-1}$, then any set $\{h'_1, h_2, \ldots, h_n\}$ where h'_1 is parallel to h_1 also has the property that every pair of hyperplanes intersect.

By Lemma 3.10 there is a bound on the number of hyperplanes that can pairwise intersect. Hence choose hyperplanes h_1, h'_1 that do not intersect (i.e. are parallel in $\mathbb{E}^r$) and a chamber for which $h_1, h_2, \ldots, h_n$ are the supporting hyperplanes with $h'_1 \neq h_i$, and let $\{h^i_1\}$ be an infinite sequence of distinct hyperplanes parallel to h_1 generated by reflection in h_1 and h'_1. Hence the set

$Y = \{\{h_1^i, h_2, \ldots, h_n\}\}$ is an infinite set of collections of pairwise intersecting hyperplanes. We need to show that Y intersects infinitely many orbits of collections of pairwise intersecting hyperplanes.

We claim that each set of hyperplanes $\{h_1^i, h_2, \ldots, h_n\}$ encloses a finite region in $\mathbb{E}^{n-1}$, called R_i. The hyperplanes $h_2, \ldots, h_k$ bound a biinfinite simplicial double cone and any h_1^i intersects each h_j and hence defines a finite region. Assign a number m_i to each collection $\{h_1^i, h_2, \ldots, h_n\}$ equal to the maximum number of hyperplanes parallel to one of the supporting walls intersecting R_i non-trivially. Passing to a subsequence and reordering if necessary we get a sequence $m_0 < m_1 < m_2 < \cdots$. Clearly within an orbit m_i is preserved hence for $i \neq j$ we have $\{h_1^i, h_2, \ldots, h_n\}$ and $\{h_1^j, h_2, \ldots, h_n\}$ lie in different orbits. Hence there are infinitely many orbits of collections of pairwise intersecting hyperplanes and so the action is not cocompact. □

Corollary 5.14. *Any Coxeter group G containing a Euclidean Coxeter group G_T with at least three generators as a special subgroup acts non-cocompactly on its cubing.*

Proof: As in the proof of Lemma 5.13 there exists an infinite order element $g \in G_T$ and hyperplane h in Σ_T so that $\{\langle g \rangle h\}$ is an infinite family of pairwise non-intersecting hyperplanes in M_T. (To see this choose any two non-intersecting hyperplanes h, h' in M_T and let a, a' be the group elements that act by reflection in h and h' respectively. Then let $g = aa'$.) By Lemma 2.8 the Moussong complex of a special subgroup is isometrically (and invariantly) embedded in M. Let h' be a hyperplane in M such that $h' \cap M_T = h$ with respect to the embedding constructed in Lemma 3.9.

We wish to show that $\{\langle g\rangle h'\}$ contains an infinite family of pairwise non-intersecting hyperplanes.

First suppose that there exists $n \in \mathbb{N}$ so that $g^n h' \cap h' = \emptyset$. Let $g_n = g^n$ and consider the hyperplanes $\{\langle g_n\rangle h'\}$. Now $g_n h' \cap h' = \emptyset \Rightarrow g_n^{m+1} h' \cap g_n^m h' = \emptyset$. We will now show that for any r we have $g_n^r h' \cap g_n^{r+2} h' = \emptyset$. We denote h_0, h_1, h_2 the hyperplanes $g_n^r h', g_n^{r+1} h', g_n^{r+2} h'$ respectively. $h_0 \cap h_1 = \emptyset$ and $h_1 \cap h_2 = \emptyset$ but restricting to M_T we see that h_0 and h_2 lie on different sides of h_1 hence they do not intersect in M. The set $\{\langle g_n^2\rangle h'\}$ is then an infinite set of pairwise non-intersecting hyperplanes.

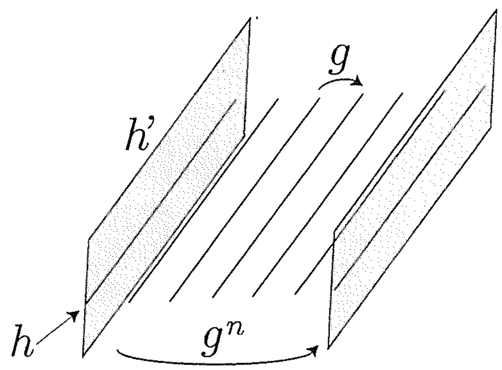


Fig. 5.7 $g^n h' \cap h' = \emptyset$

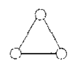
Now suppose that for all n we have $g^n h' \cap h' \neq \emptyset$. Now since G acts by isometries $g^n h' \cap h' \neq \emptyset \Rightarrow g^{n+m} h' \cap g^m h' \neq \emptyset$. But from this, for any r the set of hyperplanes $\{h', gh', g^2 h', \ldots, g^r h'\}$ is pairwise intersecting contradicting the local finiteness of the cubing (Lemma 3.11).

So we now have an infinite family of pairwise non-intersecting hyperplanes in M. Relabel and orientate to get a family of halfspaces $\{h_1, h_2, \ldots\}$ with $h_i \leq h_j$ whenever $i > j$.

Now by Theorems 5.8 and 5.13 there is an infinite ladder in M_T. We can assume that the family $\{\langle g\rangle h\}$ of hyperplanes defined at the beginning of the proof is a subset of the rungs of this ladder. (We can do this because M_T is isometric to a Euclidean space and so any two uprights in M_T crossing one of this set of parallel hyperplanes will cross them all.) The previous paragraphs of this proof have established a family $\{h_1, h_2, \ldots\}$ of pairwise nested hyperplanes of M each containing a $g^n h$ for some n. Choose two hyperplanes of M containing the one each of the two uprights of the ladder in M_T. Then filling in more rungs, if necessary, we get an infinite ladder of hyperplanes in M and hence by Theorem 5.8 the action is not cocompact. □

Examples

1. The $(3,3,3)$ triangle group: $\widetilde{A}_2 = \langle s_1, s_2, s_3 | s_i^2 = (s_i s_j)^3 = 1\rangle$

   Coxeter graph: 

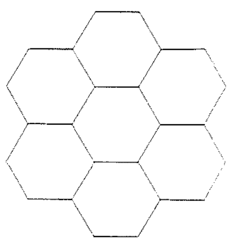


Fig. 5.8 Part of the Moussong complex of $\widetilde{A}_2$

Cubing: $\mathbb{E}^3$ with the integer lattice cubing. In M there are clearly three families of parallel hyperplanes each of which intersects every hyperplane from a distinct family. The cubing for each family is $\mathbb{R}$ and hence by

Lemma 5.11 the cubing is $\mathbb{E}^3$. Different orbits of 3-cubes are realised in M as different sized triangles formed by hyperplanes showing that the action is non-cocompact.

2. The three-dimensional Euclidean Coxeter groups.

There are precisely three Euclidean Coxeter groups whose Moussong complexes are isometric to $\mathbb{E}^3$. These are $\widetilde{A}_3$, $\widetilde{B}_3$, $\widetilde{C}_3$. (See graphs on page 73.) Below are pictures of part of their Moussong complexes. Each Moussong complex has a fundamental region consisting of four cells incident with a single common point (i.e. four special cells with the identity vertex in common). The pictures show three out of four of these cells. The missing cell is the same as the others in $\widetilde{A}_3$ case and is the same as the largest cell in the other two cases.

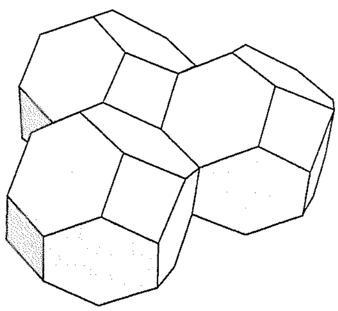


Fig. 5.9 Part of $M_{\widetilde{A}_3}$

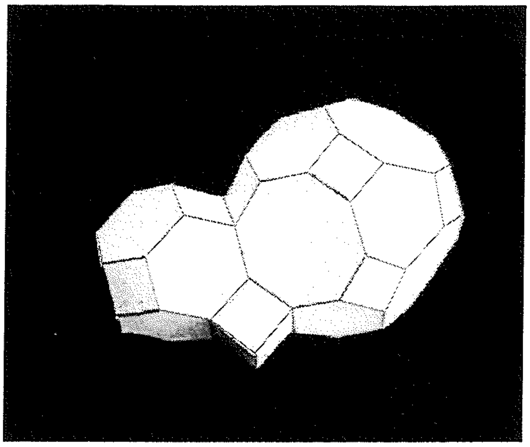


Fig. 5.10 Part of $M_{\widetilde{B}_3}$

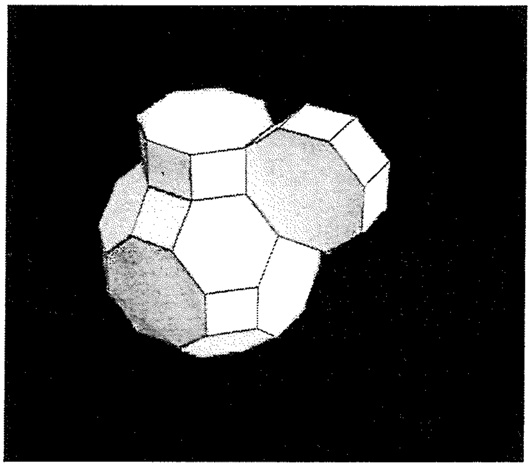
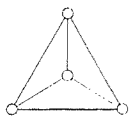


Fig. 5.11 Part of $M_{\widetilde{C}_3}$

The Coxeter cubings of these groups are $\mathbb{E}^6$, $\mathbb{E}^9$ and $\mathbb{E}^9$ respectively. The fact that they are cubings of Euclidean space is proved by the same reasoning as for $\widetilde{A}_2$ above. As above, the dimension is determined by the number of families of parallel hyperplanes in M.

3. The Coxeter group T with graph  contains four special subgroups isomorphic to $\widetilde{A}_2$ and hence by Corollary 5.14 acts non-cocompactly

on its Coxeter cubing.

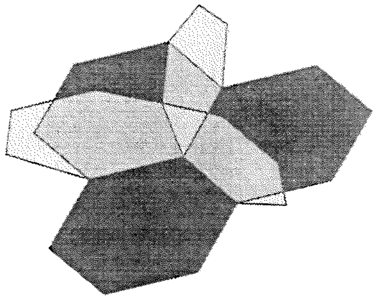


Fig. 5.12 The special cells of the Moussong complex for T

5.6 Triangle subgroups

Theorem 5.16. *Let G be a Coxeter group and X its Coxeter cubing. Then G acts non-cocompactly on X if and only if G contains infinitely many non-conjugate isomorphic infinite triangle groups.*

Proof: ($\Leftarrow$) Let the triangle subgroups be $\langle a_i, b_i, c_i \rangle$, with $a_i b_i$ of order p, $b_i c_i$ of order q, and $c_i a_i$ of order r for each i. Then the 2-generator subgroups $\langle a_i, b_i \rangle$ are all finite dihedral groups and each one is conjugate to a special subgroup $\langle A_i, B_i \rangle$ of rank 2, where the conjugacy takes a_i to A_i and b_i to B_i. Since G is finitely generated there are only finitely many such special subgroups, and so by the pigeonhole principle, infinitely many a_i and b_i are conjugate to each other and to some fixed special subgroup $\langle A, B \rangle$. Now discard the others and consider this infinite set. Let g_i be the element conjugating a_i to A and b_i to B, and let C_i be c_i conjugated by g_i. Then $\langle A, B, C_i \rangle$ is a special subgroup conjugate to (and therefore isomorphic to) $\langle a_i, b_i, c_i \rangle$. Since the subgroups $\langle a_i, b_i, c_i \rangle$ are not conjugate neither are the subgroups $\langle A, B, C_i \rangle$. Let h_A, h_B and h_{C_i} be the hyperplanes in which A, B and C_i respectively are reflections. We will show that the hyperplanes $\{h_A, h_B, h_{C_1}, h_{C_2}, \ldots\}$ contribute to an infinite ladder in M. The two uprights are h_A and h_B and the rungs will include the h_{C_i}. By the definition of infinite ladder on page 55 there are five things to check.

Properties (i) and (ii) are left to the end of the proof.

Property (iii) states that h_{C_i} must intersect h_A and h_B for all i, which is clearly satisfied.

For (iv) it is required to show that infinitely many of the h_{C_i} are pairwise non-intersecting in M, then these will be rungs for our ladder. Roughly speaking there are only finitely many ways that the h_{C_i} can intersect h_A and h_B and so by the pigeonhole principle infinitely many will be parallel. To prove this let $\alpha : [0, \infty) \rightarrow M$ be a geodesic ray with $\alpha(0) \in h_A \cap h_B$ and crossing infinitely many of the h_{C_i}. Let α_n be the geodesic that is the restriction of α to the interval $[0, n]$. For any n we can assume that α_n avoids $M^{(1)}$ by slightly moving the endpoints if necessary. Since M is a Euclidean complex α_n has a Euclidean neighbourhood, N say. Within N each h_{C_i} cuts h_A at the same angle and there are only finitely many ways it can do so. Since the metric on M is convex (a result of the CAT(0) property) planes that are parallel in N cannot meet in M. Hence there is no bound on the size of sets of pairwise non-intersecting h_{C_i}s. Now we discard the other hyperplanes and relabel this set of pairwise nested hyperplanes $h_{C_1}, h_{C_2}, h_{C_3}, \ldots$ with h_{C_i} closer to $h_A \cap h_B$ than h_{C_j} for all $i < j$.

For property (v) we note that if there are any such 'sandwiched hyperplanes' we may include them as rungs in the ladder.

Finally it is required to find a hyperplane h in M that satisfies properties (i) and (ii). Such a hyperplane always exists in the hyperplanes of any triangle group. Take h to be such a hyperplane in the triangle group $\langle a_i, b_i, c_1 \rangle$.

$(\Rightarrow)$ If there is an infinite ladder then we get an infinite family of triangle groups $\langle A, B, C_i \rangle$ but the orders of AC_i and BC_i may vary with C_i, so the triangle groups defined by the ladder are not all isomorphic. However there are only finitely many possibilities for the orders, so we may pass to an infinite subsequence where they are all the same. Hence we have an infinite family of isomorphic non-conjugate triangle subgroups. $\square$

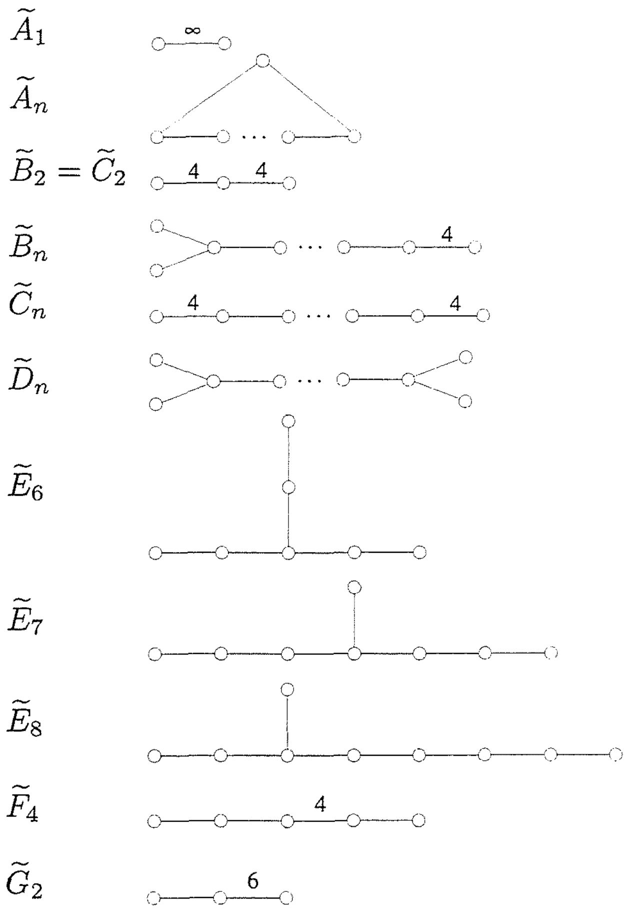


Fig. 5.13 The Euclidean Coxeter groups

References

A J.M.Alonso, T.Brady, D.Cooper, V.Ferlini, M.Lustig, M.Mihalik, M.Shapiro, H.Short, 'Notes on word-hyperbolic groups', in *Group theory from a geometric viewpoint*, World Scientific, Trieste, Italy 1990.

B Bridson, M.R., 'Geodesics and curvature in metric simplicial complexes', in *Group theory from a geometric viewpoint*, World Scientific, Trieste, Italy 1990.

BHa Martin Bridson, André Haefliger, *'Metric spaces of non-positive curvature'*, to appear in Grundlehren der mathematischen Wissenschaften series, Springer Verlag.

BHo Brigitte Brink and Robert B. Howlett, 'A finiteness property and an automatic structure for Coxeter groups', *Math. Ann.* **296** (1993), 179-190.

Br Brown, Kenneth, S., *'Buildings'*, Springer-Verlag, 1989.

C1 Coxeter, H.S.M., 'Discrete groups generated by reflections', *Annals of Mathematics*, **35** (1934), 588-621.

C2 Coxeter, H.S.M., 'The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$', *Journal of the London Mathematical Society*, **10** (1935), 21-25.

C3 Coxeter, H.S.M., *'Regular polytopes'*, Second Edition, Macmillan Mathematical Paperbacks, 1963.

DGS Eds. Chandler Davis, Branko Grünbaum, F. A. Sherk, *'The geometric vein: The Coxeter Festschrift'*, Springer-Verlag, 1981.

D Davis, M.W., 'Nonpositive curvature and reflection groups', Ohio State Mathematical Research Institute, Preprints 94-19.

dlH Pierre de la Harpe, 'Invitation to Coxeter groups',in *Group theory from a geometric viewpoint*, World Scientific, Trieste, Italy 1990.

Gr Graham, Ronald, L., *'Rudiments of Ramsey theory'*, Conference board of the mathematical sciences regional conference series, No. 45, American Mathematical Society, 1981.

H Humphries, James, E., *'Reflection groups and Coxeter groups'*, Cambridge studies in advanced mathematics 29, Cambridge University Press, 1990.

Mos Lee Mosher, *'Geometry of cubulated 3-manifolds'*, Topology **34** No. 4, (1995) 789-814.

Mou G.Moussong, 'Hyperbolic Coxeter groups', Ph.D. thesis, The Ohio State University, 1988.

NR1 Niblo, G.A., Reeves,L., 'The geometry of cube complexes and the complexity of their fundamental groups', *Topology* **37**, No. 3, (1998) 621-633.

NR2 Niblo, G.A., Reeves,L., 'Cubings for Coxeter groups', in preparation.

R Martin Roller,'Poc sets, median algebras and group actions: An extended study of Dunwoody's construction and Sageev's theorem', Preprint, University of Southampton, available on Department of Mathematics preprint server at http://www.maths.soton.ac.uk/pure/preprints/poc_mar.ps.gz, (1998).

S1 Michah Sageev, 'Ends of group pairs and non-positively curved cube-complexes', *Proc. London Math. Soc.*, (3) **71** (1995) 585-617.

S2 Michah Sageev, 'Codimension-1 subgroups and splittings of groups', *Journal of Algebra*, **189** (1997), 377-389.

Se Jean-Pierre Serre, *'Trees'*, Springer-Verlag, 1980.

T1 Tits, J., 'Groupes et géométries de Coxeter', unpublished manuscript, 1961.

T2 Tits, J., 'Le problème des mots dans les groupes de Coxeter', *Symposia Mathematica*, **1** (1969), Academic Press, 175-185.

Part II

Engulfing and subgroup separability for word-hyperbolic groups

Chapter 1

Introduction

A group is *subgroup separable* if all of its finitely generated subgroups are closed in a certain topology (the *profinite topology*) on the group and is *residually finite* if the trivial subgroup is closed in this topology. Subgroup separability is a very strong condition known only for a small class of groups. It is very useful in geometric topology where it can be used to pass from immersions to embeddings in some finite cover. Scott's paper [**Sc**] gives a good outline of this connection between subgroup separability and geometric topology.

A group has the *engulfing property* if every finitely generated proper subgroup is contained in a proper subgroup of finite index. The engulfing property is, in general, weaker than subgroup separability. However, in [**L**] Long proves that for any closed hyperbolic 3-manifold M, the engulfing property for $\pi_1(M)$ implies residual finiteness, and if additionally a subgroup

$H < G$ is geometrically finite then H is finite index in a separable subgroup of G. This then leads to the following result.

Theorem. *(Long) Let Γ be the fundamental group of a closed hyperbolic 3-manifold. Suppose that Γ has the engulfing property for those finitely generated subgroups H with $\Lambda(H) < S^2_\infty$. If Γ contains a surface group then $\mathbb{H}^3/\Gamma$ is virtually Haken.*

(Recall that 3-manifold is *Haken* if it is compact, orientable , irreducible and contains a two sided incompressible surface.)

The following result is proved about the engulfing property.

Theorem. *(Long) Let Γ_1, Γ_2 be fundamental groups of closed hyperbolic 3-manifolds such that Γ_2 is a finite index subgroup of Γ_1. Then Γ_1 has the property that it engulfs all its finitely generated subgroups H such that $\Lambda(H) < S^2$ if and only if Γ_2 also has this property.*

In fact Long conjectures the following.

Conjecture. *(Long) Let Γ be the fundamental group of a closed hyperbolic 3-manifold. Then Γ has the engulfing property for all subgroups H with $\Lambda(H) < S^2$.*

In his important and influential article 'Hyperbolic groups' [**G**] Gromov introduces the notion of a *δ-hyperbolic* or *negatively curved* space, a metric space with a simple condition requiring geodesic triangles to be 'thin', giving a generalisation of classical hyperbolic space. A *word-hyperbolic group* is a finitely generated group whose Cayley graph is δ-hyperbolic. Hyperbolic group theory is a very rich and interesting area of mathematics with still many unsolved problems. Among these, the question of subgroup separability and residual finiteness for word-hyperbolic groups is still open.

In this part we investigate these residual properties of word-hyperbolic groups adapting tools introduced by Long in [**L**].

The two main theorems are the following.

Theorem 3.5. *Let G be a non-elementary word-hyperbolic group and suppose that G engulfs all of its finitely generated free subgroups. Then G is almost residually finite, i.e. $\overline{\{e\}}$ (the closure of $\{e\}$ in the profinite topology) is finite. If G is also torsion-free, then G is residually finite.*

Theorem 3.11. *Let G be a non-elementary torsion-free word-hyperbolic group with the engulfing property and H a quasiconvex subgroup. Then H is finite index in a separable subgroup $(\overline{H})$ of G.*

Note that if G is elementary then it is virtually cyclic by [**G**] and 'almost'-residual finiteness and -separability of H are easy to prove.

This work can be seen as part of the tradition of taking results in the theory of Kleinian groups and adapting the proofs to fit the more general notions of word-hyperbolic groups. It also could be seen as providing more evidence for Thurston's hyperbolisation conjecture [**T**] that 3-manifolds with word-hyperbolic fundamental groups are in fact hyperbolic as well as giving another class of groups that satisfy Thurston's virtually Haken conjecture that says that if M is a closed orientable irreducible 3-manifold with infinite fundamental group, then M has a finite sheeted cover which is Haken. (See [**AR**].)

In [**Gi**] Rita Gitik constructs a large family of hyperbolic 3-manifolds that are subgroup separable (and hence also have the engulfing property).

In more recent work, Ilya Kapovich and Dani Wise [**KW**] prove that the following are equivalent:

(i) Every word-hyperbolic group is residually finite.

(ii) Every word-hyperbolic group has at least one finite quotient.

(iii) Every word-hyperbolic group is virtually torsion-free.

Chapter 2

Preliminaries

In this chapter we give a brief introduction to word-hyperbolic groups, introduce the profinite topology on a group and define the notion of separability.

2.1 Word-hyperbolic groups

This section is a brief introduction to word-hyperbolic groups. There are many books and articles giving a full treatment of this area including [**A**], [**CDP**], [**CP**], [**G**] and [**GH**].

Let G be a finitely generated group and let S be a finite generating set. Recall that the *Cayley graph* $\mathcal{G}(S)$ for G (with respect to the generating set S) is the connected graph with vertex set $\mathcal{VG}(S) = \{g | g \in G\}$ and edge set $\mathcal{EG}(S) = \{(g, h) | g, h \in \mathcal{VG}, g = hs, s \in S^{\pm 1}\}$. We metrise $\mathcal{G}$ by giving each edge length 1.

Let X be a geodesic metric space. A geodesic triangle in X is *δ-thin* if each edge is contained in the δ-neighbourhood of the other two edges. X is *δ-hyperbolic* if there exists $\delta \geq 0$ so that every geodesic triangle in X is δ-thin.

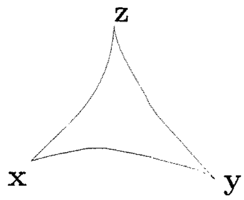


Fig. 2.1 A δ-thin triangle

Definition. G is *word-hyperbolic* if $\mathcal{G}(S)$ as metrised above is a δ-hyperbolic space for some $\delta \geq 0$ and for some finite generating set S.

It can be shown that Cayley graphs arising from different finite generating sets are quasi-isometric, i.e. geometrically 'very similar' and that δ-hyperbolicity is a quasi-isometry invariant. (See for example [**GH**] Chapter 5 §2.) Hence the definition above is independent of the generating set. From now on we fix a generating set S and write $\mathcal{G}(S) = \mathcal{G}$.

Definition. *Gromov's inner product* on $\mathcal{G}$ is defined as

$$(x.y) = \frac{1}{2}(d(x,1) + d(y,1) - d(x,y))$$

where d is the metric on $\mathcal{G}$. We say that a sequence $\{x_i\}$ of vertices of $\mathcal{G}$ *converges to infinity* if $lim_{i,j\to\infty}(x_i.x_j) = \infty$. We define an equivalence of sequences as follows. Two sequences $\{x_i\}$ and $\{y_i\}$ are defined to be equivalent if $lim_{i\to\infty}(x_i.y_i) = \infty$. The *boundary at infinity* $\partial\mathcal{G}$ of $\mathcal{G}$ is defined as the space whose points are equivalence classes of sequences converging to infinity.

The Gromov inner product extends naturally to the boundary and defines a metric on the boundary. A topology on $\mathcal{G} \cup \partial\mathcal{G}$ is defined by the basis consisting of the sets $N_k(x) = \{y \in \mathcal{G} \cup \partial\mathcal{G} | (x.y) > k\}$ for $k \geq 0$ for $x \in \partial G$ (known as *horoballs*) and the usual open balls $B_k(x) = \{y \in G | d(x,y) < k\}$ for $x \in G$. With this topology $\mathcal{G} \cup \partial\mathcal{G}$ is compact and $\partial\mathcal{G}$ is closed in $G \cup \partial\mathcal{G}$.

Given a subgroup H of G, the *limit set of H* which is denoted $\Lambda(H)$ is defined as the subset of $\partial\mathcal{G}$ attainable by sequences of elements of H. H acts properly discontinuously on $\partial\mathcal{G} - \Lambda(H)$.

The following describes the action of infinite order elements on the boundary. If g is an infinite order element of G it acts on the Cayley graph $\mathcal{G}$ by translation along a quasi-geodesic line, α say, (obtained by joining g^i to g^{i+i} for all $i \in \mathbb{Z}$ by a geodesic in $\mathcal{G}$). Denote by $\partial g = \{\partial g^+, \partial g^-\} = \{lim_{i \to \infty} g^i, lim_{i \to \infty} g^{-i}\}$ the endpoints of α in $\partial\mathcal{G}$ (which are fixed by g). There exist disjoint neighbourhoods U_+ and U_- of ∂g^+ and ∂g^- respectively such that for all $x \in \partial\mathcal{G} - (U_+ \cup U_-)$ we have $gx \in U_+$ and $g^{-1}x \in U_-$. We say that the pair (U_+, U_-) is *absorbing* for g. In fact any pair of disjoint neighbourhoods of ∂g^+ and ∂g^- is absorbing for g^k for sufficiently large k. (See [**GH**] Chapter 8.)

A word-hyperbolic group is called *elementary* if it is finite or contains a finite index infinite cyclic subgroup and is *non-elementary* otherwise. Elementary word-hyperbolic groups have either no boundary at infinity (if and only if the group is finite) or a boundary consisting of two points. Non-elementary word-hyperbolic groups have infinite boundaries.

A subgroup H of a group G with generating set S is *quasi-convex* if there exists $C \geq 0$ such that every geodesic in the Cayley graph $\mathcal{G}(S)$ lies within a C neighbourhood of H.

Examples.

1. Finite groups

Let G be a finite group. Any Cayley graph $\mathcal{G}$ of G is finite. Let δ be the diameter of $\mathcal{G}$. Then every geodesic triangle is δ-thin. G clearly has no boundary at infinity.

2. The Cayley graph for $\mathbb{Z}$ is isomorphic to $\mathbb{R}$ and has boundary $\partial\mathbb{Z} = \{-\infty, +\infty\}$.

Fig. 2.2 The Cayley graph for $\mathbb{Z}$

3. Finitely generated free groups

The Cayley graph of a rank n free group G is a $2n$-valent tree. Any geodesic triangle in a tree is 0-thin. If $n \geq 2$ the boundary of G is a Cantor set (e.g. isomorphic to the set $\{0,1\}^{\mathbb{N}}$).

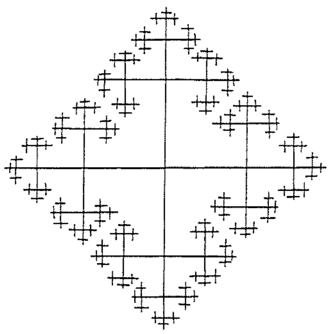


Fig. 2.3 The Cayley graph for the free group on two generators

4. Discrete groups of isometries of $\mathbb{H}^n$ with proper discontinuous cocompact actions. In fact if G is a group acting properly discontinuously and cocompactly on a geodesic space X then G is word-hyperbolic if and only if X is δ-hyperbolic.

5. Hyperbolic Coxeter groups

A Coxeter group as described in Part I is word-hyperbolic if and only if it contains no subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ [**Mou**].

6. Free products of hyperbolic groups

One can think of the Cayley graph of a free product of hyperbolic groups as a tree of hyperbolic graphs. Let $G = G_1 * G_2$ and $\mathcal{G}_1$ and $\mathcal{G}_2$ be Cayley graphs of G_1 and G_2 respectively such that $\mathcal{G}_1$ is δ_1-hyperbolic and $\mathcal{G}_2$ is δ_2-hyperbolic. To see what the Cayley graph of G looks like take $\mathcal{G}_1$ and attach a $\mathcal{G}_2$ at every vertex. Then attach a $\mathcal{G}_1$ to each vertex of the attached graphs and so on. Any geodesic triangle can be decomposed into triangles contained in a $\mathcal{G}_1$ or a $\mathcal{G}_2$. Hence G is δ-hyperbolic where $\delta = max(\delta_1, \delta_2)$.

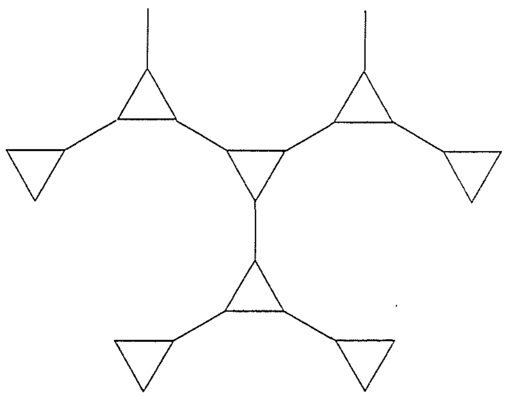


Fig. 2.4 The Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$

2.2 Separability

Given any group G we define a topology on G (different to that described in 2.1) called the *profinite topology.* A basis of closed sets is defined as the cosets of finite index normal subgroups of G. Hence G is a topological group as the actions of group elements are continuous.

Note that, in this topology, finite index subgroups are closed. To see this, let K be a finite index subgroup of G. Consider the group $K' = \cap\{gKg^{-1} | g \in G\}$. K has finitely many distinct conjugates in G and hence K' is a finite index normal subgroup. K' has finite index in K hence K is a finite union of closed sets (the cosets of K' in K) and hence is closed.

Definition. *Given a group G, a finitely generated subgroup H is separable in G if it is closed in the profinite topology on G. A group G is residually finite if $\{e\}$ is closed and G is subgroup separable or LERF (locally extended residually finite) if every finitely generated subgroup H is separable in G.*

We compare these definitions with the more standard definitions of separability, residual finiteness and subgroup separability: H is separable in G if for any $g \in G - H$ there exists a finite index subgroup $H \leq K < G$ so that g is not in K; as above G is residually finite if $\{e\}$ is separable in G and is subgroup separable if every finitely generated subgroup H is separable in G.

For the purpose of the following lemma we call these two definitions of separability *separability*$_1$ (profinite topology definition) and *separability*$_2$ (standard definition).

Lemma 2.1. *Let H be a finitely generated subgroup of a group G. H is separable$_1$ in G if and only it is separable$_2$ in G.*

Proof: Let $\overline{H}$ denote the closure of H in the profinite topology on G and let H^* denote the intersection of all proper finite index subgroups containing H. It is clear that $H = \overline{H}$ if and only if H is separable$_1$ and that $H^* = H$ if and only if H is separable$_2$. We will show that $H^* = \overline{H}$. Clearly $H^* \subseteq \overline{H}$. Suppose that $H^* \subset \overline{H}$. Then there exists $\overline{h} \in \overline{H}$ not in H^*. A general closed set is of the form $X_1 \cup g_2X_2 \cup \ldots \cup g_nX_n$ where each X_i is an intersection of finite index normal subgroups. H^* is closed and hence we set $H^* = X_1 \cup g_2X_2 \cup \ldots \cup g_nX_n$. $\overline{h} \notin H^*$ so we can assume that each X_i is a finite index normal subgroup. Now let $X = \cap X_i$ and consider the group HX. It is a finite index subgroup containing H and hence contains $\overline{h}$. $\overline{h} = hx$ for some $h \in H$ and $x \in X$. But $h = g_ix_i$ for some $g \in G$ and $x_i \in X_i$. Hence $\overline{h} \in H^*$, a contradiction. □

A useful corollary of residual finiteness is that given any finite list of nontrivial elements $x_1, x_2, \ldots, x_n$ of G there is a finite index normal subgroup containing none of them. To see this, take a finite index subgroup K_i not containing x_i. This is possible because G is residually finite. Let K_i' be the intersection of all conjugates of K_i which is a finite index normal subgroup not containing x_i. The intersection of the K_i's is the required group.

Definition. *(The engulfing property) A group G is said to engulf a subgroup H if H is contained in a proper finite index subgroup of G. G is said to have the engulfing property if G engulfs all of its finitely generated subgroups.*

Note that if a group is subgroup separable then it certainly has the engulfing property.

Examples.

1. Free groups are subgroup separable. This was first proved by Hall in [**H**].

2. Surface groups are subgroup separable. This was proved by Scott in [**Sc**] using 'the geometry of the hyperbolic plane and simple facts about groups generated by reflections'.

3. Finite extensions of subgroup separable groups are subgroup separable.

Proof: Let G be subgroup separable and a finite index normal subgroup of G'. First note that any closed (in the induced topology from G') subgroup of G is also closed in G' since G is closed in G'. Now let H be a subgroup of G'. $H \cap G$ is closed and is finite index in H and hence H is closed in G'. □

4. Fuchsian groups are subgroup separable.

This is a consequence of Examples 2. and 3. above since any Fuchsian group is a finite extension of a surface group.

Chapter 3

Results

Let H be a subgroup of a group G and $\overline{H}$ denote the closure of H in the profinite topology.

Lemma 3.1. *For any $H \leq G$, $\overline{\overline{H}} = \overline{H}$.*

Proof: Clearly a closure is closed. □

Corollary 3.2. *$\overline{H}$ is separable for any $H < G$.* □

The following well known fact can be viewed as an alternative definition of the limit set of a subgroup. A proof taken from [**GH**] is included for the convenience of the reader.

Lemma 3.3. *Let H be a non-elementary subgroup of a word-hyperbolic group G. Then $\Lambda(H)$ is the smallest non-empty closed H-invariant subset of $\partial\mathcal{G}$.*

Note that 'closed' in the statement of the Lemma refers to the standard topology on $\mathcal{G} \cup \partial\mathcal{G}$ as mentioned in Section 2.1.

Proof: We prove that if $A \subseteq \partial\mathcal{G}$ is closed and H-invariant then $\Lambda(H) \subseteq A$. Firstly, let $B \subset \partial\mathcal{G}$. Denote by $I(B)$ the set of points of $\mathcal{G}$ lying on geodesics between points of B. Suppose that $B \neq \emptyset$ and $|B| \neq 1$. Then $I(B) \neq \emptyset$. Let $\{x_i\} \subseteq I(B)$ be a sequence such that $x_i \to x \in \partial\mathcal{G}$. We claim that $x \in \overline{B}$. To see this, for each i choose a geodesic $l_i = [b_i', b_i'']$ with $b_i', b_i'' \in B$. Passing to a subsequence if necessary we get $b_i' \to b' \in \overline{B}$, $b_i'' \to b'' \in \overline{B}$, $l_i \to l$. $x_i \to x \in l \cup \{b', b''\}$ and hence $x \in \{b', b''\}$.

Now let $A \subseteq \partial\mathcal{G}$ be closed and H-invariant. Let $I(A)$ be as above. Then $I(A)$ is H-invariant. First suppose that $1 \in I(A)$. Then $H \subseteq I(A)$. Let $x \in \Lambda(H)$ and $\{x_i\} \subseteq H$ so that $x_i \to x$. By the first paragraph of the proof $x \in \overline{A} = A$ and hence $\Lambda(H) \subseteq A$. Now suppose that $1 \notin I(A)$. Then $I(A) \cap H = \emptyset$ and $I(A)$ is a union of right cosets of H. Suppose that $Hg \subseteq I(A)$. Let $x \in \Lambda(H)$ and $\{x_i\} \subseteq H$ with $x_i \to x$. Then since $x_i g$ and x_i are a distance exactly $|g|$ apart for all i we have $x_i g \to x \in \Lambda(H)$ and hence by the previous paragraph $x \in \overline{A} = A$ and $\Lambda(H) \subseteq A$ as required.

It is clear that $\Lambda(H)$ is H-invariant so it remains to prove that $\Lambda(H)$ is closed. We show that $\partial\mathcal{G} - \Lambda(H)$ is open. Let $y \in \partial\mathcal{G} - \Lambda(H)$ and let $\{y_i\}$ be a sequence converging to y. Let α_i be geodesics realising the distances $d(y_i, H)$. There is no bound on the lengths of the α_i. Let z_i lie on α_i so that there is no bound on the distances $d(y_i, z_i)$ and $d(z_i, H)$. Let $\{z_i\}$ converge to $z \in \partial\mathcal{G}$ then the horoball $N_{(y.z)}(y)$ is an open set containing y and disjoint from $\Lambda(H)$ as required. □

Corollary 3.4. *Let H be a non-elementary subgroup of a word-hyperbolic group G. Then $\Lambda(H)$ is the closure of the set*

$$\{\{h^i\} | h \in H,\ h \text{ has infinite order}\} \subset \partial\mathcal{G}.$$

Proof: By Lemma 3.3 $\Lambda(H)$ is the minimum non-empty closed H-invariant subset of $\partial\mathcal{G}$. Clearly $\{\{h^i\} | h \in H,\ h \text{ has infinite order}\} \subseteq \Lambda(H)$ and is H-invariant. Hence its closure must be $\Lambda(H)$. □

For the duration of Part II let $N = \overline{\{e\}}$, the closure of the identity element in the profinite topology.

First, we show that the engulfing property implies 'almost' residual finiteness. Recall that G is residually finite if and only if $\{e\}$ is closed in the profinite topology, i.e. $N = \{e\}$.

Theorem 3.5. *Let G be a non-elementary word-hyperbolic group and suppose that G engulfs all of its finitely generated free subgroups. Then G is almost residually finite, i.e. N is finite. If G is also torsion-free, then G is residually finite.*

Proof: Suppose that G is not residually finite. Then we have $N \neq \{e\}$. We claim that $N = \cap\{K | K \text{ is a proper finite index normal subgroup of } G\}$ is a non-trivial normal subgroup of G with $\Lambda(N) = \partial\mathcal{G}$ or $\emptyset$. N is clearly normal. Let $\Lambda(N) = X$. Then $NX = X$ and since N is normal $g^{-1}NgX = NX = X$ and hence $NgX = gX$ for all $g \in G$ so gX is N-invariant. By Lemma 3.3 X is either empty or is the minimal non-empty closed N-invariant subset of $\partial\mathcal{G}$ so we have $X \subseteq gX$ for all $g \in G$ and hence $g^{-1}X \subseteq X$ for all $g \in G$. Hence $gX = X$ for all $g \in G$ which implies that either $X = \emptyset$ or $X = \partial\mathcal{G}$.

Suppose that $X = \partial\mathcal{G}$. Let $\{g_1, g_2, \ldots, g_n\}$ be a generating set for G. Since G is non-elementary and hence $\partial\mathcal{G}$ is infinite we can choose infinite order elements $x_1, x_2, \ldots, x_n \in N$ so that the ∂x_i are distinct from each other and the ∂g_i. Since $\partial\mathcal{G}$ is metrisable (with metric d, say) we can choose $2n$ mutually disjoint neighbourhoods U^i_+, U^i_- of the ∂x_i (e.g. open balls of radius $r = 1/2min\{d(\partial x_i, \partial x_j)\}$ centred on ∂x_i). By taking sufficiently high powers of the x_i and relabelling we can ensure that these neighbourhoods satisfy the following:

(i) (U^i_+, U^i_-) is absorbing for x_i;

(ii) $(U^i_+, U^i_-) \cap g_i((U^i_+, U^i_-)) = \emptyset$ for all i.

Let $s_i = x_i g_i x_i$ and consider the group $S = \langle s_1, s_2, \ldots, s_n \rangle$. We prove that S is free by the usual 'ping-pong' argument. First note that (U^i_+, U^i_-) is absorbing for s_i. Let p be a point of $\partial\mathcal{G} - \cup_i(U^i_+ \cup U^i_-)$ and $s = s^{\epsilon_1}_{i_1} s^{\epsilon_2}_{i_2} \ldots s^{\epsilon_r}_{i_r}$ a reduced word in the generators of S and their inverses. Since $p \in \partial\mathcal{G} - \cup_i(U^i_+ \cup U^i_-)$ we have $s(p) = s^{\epsilon_1}_{i_1} s^{\epsilon_2}_{i_2} \ldots s^{\epsilon_{r-1}}_{i_{r-1}}(p_1)$ where p_1 is a point in $U^{i_r}_+ \cup U^{i_r}_-$ and so on until we see that $s(p) \in U^{i_1}_+ \cup U^{i_1}_-$ and hence p is not fixed by s so $s \neq e$. Hence S is free.

Next we prove that $\Lambda(S)$ is contained in the closure of the absorbing pairs (U^i_+, U^i_-). Let $h = s_{i_1} s_{i_2} \ldots s_{i_n}$ be an infinite order element of S. By Corollary 3.4 the closure of the set of fixed points of such elements is the limit set of S hence it is enough to show that ∂h is contained in the union of the $U^i_\pm$. Let $\partial h = \{x, y\}$ and suppose that x is not in any $U^i_\pm$ and consider $h(x)$. By a similar argument to above we see that $h(x) \in U^{i_1}_+ \cup U^{i_1}_-$ but h

fixes x, a contradiction. Exactly the same argument applies to y.

By the above $\Lambda(S) \subseteq \cup_i(U^i_+ \cup U^i_-) \subset \partial\mathcal{G}$, so S is a proper subgroup (of infinite index) in G. Since G engulfs its free subgroups, there exists a proper finite index subgroup K with $S < K < G$. All of the generators $s_i = x_i g_i x_i$ are in K and $N \subseteq K$ by definition of N and hence $x_1, x_2 \ldots, x_n \in K$, therefore $g_i \in K$ contradicting the fact that K is a proper subgroup of G.

Hence $\Lambda(N) = X = \emptyset$ and so N is finite. In particular if G is torsion free N is trivial and G is residually finite as required. □

Now we turn our attention to subgroup separability. First we deal with finite subgroups.

Lemma 3.6. *Let G be a non-elementary word-hyperbolic group with the engulfing property and let H be a finite subgroup of G. Then H is finite index in a separable subgroup of G, namely $\overline{H}$.*

Proof: Let $H = \{h_1, h_2, \ldots, h_n\}$. By Theorem 3.5 N is finite. $H = \cup_{i=1}^n h_i\{e\}$ and hence $\overline{H} = \overline{\cup_{i=1}^n h_i\{e\}} = \cup_{i=1}^n h_i N$, a finite union of finite sets, hence $\overline{H}$ is finite as required. □

Theorem 3.7. *Let G be a non-elementary torsion-free word-hyperbolic group. Suppose that G has the engulfing property. Then for all finitely generated infinite quasi-convex subgroups $H \leq G$ we have $\Lambda(H) = \Lambda(\overline{H})$.*

Proof: We have $H \leq \overline{H}$ and therefore $\Lambda(H) \subseteq \Lambda(\overline{H})$. If $\Lambda(H) = \partial\mathcal{G}$ then the result is clear so suppose that $\Lambda(H) < \partial\mathcal{G}$. Let $\{g_1, g_2, \ldots, g_n\}$ be a finite generating set for G. Assume, for a contradiction, that $\Lambda(H) \subset \Lambda(\overline{H})$. In particular $\overline{H}$ is non-elementary.

Claim. $\Lambda(\overline{H}) - \Lambda(H)$ *is infinite.*

Proof of Claim: Suppose, for a contradiction, that $\Lambda(\overline{H}) - \Lambda(H) = \{x_1, x_2, \ldots, x_n\} \subset \partial\mathcal{G}$. Each $\{x_i\}$ is a closed set and since by Lemma 3.4 $\Lambda(\overline{H})$ is the closure of the set $\{\partial h | h \in \overline{H}$, infinite order$\}$ each x_i must be a fixed point of an infinite order element $h_i \in \overline{H}$. Now let (U^j_{i+}, U^j_{i-}) be an absorbing pair for h^j_i and let $y \in \Lambda(\overline{H}) - (U^j_{i+}, U^j_{i-})$. By taking sufficiently large j we can ensure that $h^j_i(y)$ (which is clearly in $\Lambda(H)$) lies in the horoball $N_k(x_i) = \{x \in \partial\mathcal{G} | (x.x_i) > k\}$ for any $k > 0$ but x_i is a non-zero distance from $\Lambda(H)$ and hence there exists a k for which $N_k(x_i) \cap \Lambda(H) = \emptyset$, a contradiction. □ *(Claim)*

Now choose $y_1, y_2, \ldots, y_n \in \overline{H}$ so that $\partial y_1, \partial y_2, \ldots, \partial y_n \in \Lambda(\overline{H}) - \Lambda(H)$ and $\partial y_i \neq \partial g_j$ for any i, j. Let $C \subset \partial\mathcal{G} - \Lambda(H)$ be a compact set containing the y_i in its interior. Since $\mathcal{G} \cup \partial\mathcal{G}$ is compact and $\partial\mathcal{G}$ is closed in $\mathcal{G} \cup \partial\mathcal{G}$, $C = \cup_{i=1}^n \overline{N}_{k_i}(\partial y_i) \cap \partial\mathcal{G}$ will suffice for suitably large k_i. H acts properly discontinuously on $\partial\mathcal{G} - \Lambda(H)$ so there are finitely many non-trivial elements of H, $h_1, h_2, \ldots h_m$ say, taking C to intersect itself. By hypothesis G has the engulfing property and hence Theorem 3.6 proves that G and hence H is residually finite. Thus there exists a finite index normal subgroup $A \lhd H$ containing none of the h_i.

We now need the following technical Lemma.

Lemma 3.8. *Let G and H be as above and suppose that $A \lhd H$ is a normal subgroup of finite index in H. Then there is an integer t so that if $h \in \overline{H}$ then $h^t \in \overline{A}$.*

Proof: We will show that the result holds for $t = |H : A|$. Let K be a finite index subgroup of G containing A. Then we need to prove that for any $h \in H$ we have $h^t \in K$.

Let $K' = \cap\{hKh^{-1} | h \in H\}$. Since K is finite index in G, K' is the intersection of finitely many finite index subgroups and is hence of finite index in G. Also $hK'h^{-1} = K'$ for all $h \in H$, i.e. K' is normalised by H. Moreover since A was normal in H we have $A \leq K' \leq K$.

Since K' is normalised by H, the subgroup generated by K' and H is the group HK'. Now by the standard isomorphism theorems we have

$$|HK' : K'| = |H : H \cap K'| = \frac{|H : A|}{|H \cap K' : A|}.$$

Thus the order of the group HK'/K' divides $|H : A|$, and any element of the quotient group has order dividing $|H : A|$.

If now $h \in \overline{H}$, then $h \in HK'$ since HK' is a subgroup of finite index which contains H. Hence $h^t \in K' \leq K$ as required. □ (Lemma 3.8)

By Lemma 3.8 we can take powers and relabel to ensure that $y_i \in \overline{A}$. Since $\partial\mathcal{G}$ is metrisable we can choose n mutually disjoint pairs of neighbourhoods (U^i_+, U^i_-) for the ∂y_i so that the closure of each is contained in the interior of C. Ensure that (U^i_+, U^i_-) is absorbing for y_i by taking sufficiently large powers and relabelling.

Now let $s_i = y_i g_i y_i$ for each i and consider the group $S = \langle s_1, s_2, \ldots, s_n \rangle$. S is free by the same argument to that given in Theorem 3.7.

Now consider the group B generated by S and A. We claim that its limit set is contained in the closure of $\cup_i (U^i_+, U^i_-) \cup \partial\mathcal{G} - C$. To prove this consider an element $b \in B$ of infinite order. Write b as $s_{i_1}^{\epsilon_1} a_1 s_{i_2}^{\epsilon_2} a_2 \ldots s_{i_k}^{\epsilon_k} a_k$

where $a_i \in A$. By Corollary 3.4 the closure of the set of fixed points of such elements is the limit set of B hence it is enough to show that ∂b is contained in $\cup_i(U_+^i, U_-^i) \cup \partial\mathcal{G} - C$. Let $\partial b = \{x, y\}$ and suppose that x is not in any $\cup_i(U_+^i, U_-^i) \cup \partial\mathcal{G} - C$ and consider $b(x)$. Each a_i moves points out of C and hence out of the $U_\pm^i$ and each generator moves points from without into these absorbing sets. Hence $b(x) \in \cup_i(U_+^i, U_-^i) \cup \partial\mathcal{G} - C$ but b fixes x, a contradiction. The same argument applies to y. The claim now follows from Lemma 3.4.

Now A is finitely generated because it is a finite index subgroup of H and hence B is finitely generated. The engulfing property then implies there exists a proper finite index subgroup K containing B. K is finite index and contains A so $\overline{A} \leq K$ and hence K contains the elements $y_1, y_2, \ldots, y_n$. But K also contains the elements $s_i = y_i g_1 y_i$ and hence contains all of the generators of G and so $K = G$ contradicting the fact that K is a proper subgroup. □ *(Theorem 3.7)*

Corollary 3.9. *Let G be a non-elementary word-hyperbolic group with the engulfing property and H an infinite quasiconvex subgroup. Then H is finite index in a separable subgroup of G (namely $\overline{H}$).*

Proof: This follows from Theorem 3.7 and the following Lemma which was proved by Kapovich and Short in [**KS**] and Swenson in [**Sw**]. We follow the proof given in [**GH**].

Lemma 3.10. *Let H be a quasiconvex subgroup of a word-hyperbolic group G. If $H < L < G$ with $\Lambda(H) = \Lambda(L)$ then $|H : L| < \infty$.*

Proof: Let C denote the union of all geodesics in $\mathcal{G}$ joining points of $\Lambda(H)$ $(= \Lambda(L))$. Since H is quasiconvex $diam(C/H) = M < \infty$. Since $\Lambda(H) = \Lambda(L)$, L acts on C. Let $l \in L$. For any point $p \in C$ we have $l(p) \in C$. There exists a point $h(p) \in C$ with $h \in H$ such that $d(l(p), h(p)) \leq 2M$ and hence $d(h, l) \leq 2M$. We have shown that for any $l \in L$ there is a $g_l (= l^{-1}h) \in G$ such that $d(1, g_l) \leq 2M$ and $lg_l \in H$. There are clearly finitely many such elements g_i so that for all $l \in L$, $l \in Hg_i$ for some i. □

For completeness it is necessary to deal with the case when G is elementary. We have already dealt with the finite case so it remains to deal with G and H both containing an infinite cyclic subgroup of finite index, i.e. $\partial\mathcal{G} = \Lambda(H)$ and $|\partial\mathcal{G}| = |\Lambda(H)| = 2$. It is well known that G is residually finite since it contains a subgroup of finite index which is residually finite. Clearly $\Lambda(H) = \Lambda(\overline{H})$ and hence by Lemma 3.10 $|\overline{H} : H| < \infty$.

In conclusion we state our main theorems.

Theorem 3.5. *Let G be a non-elementary word-hyperbolic group and suppose that G engulfs all of its finitely generated free subgroups. Then G is almost residually finite, i.e. $\overline{\{e\}}$ (the closure of $\{e\}$ in the profinite topology) is finite. If G is also torsion-free, then G is residually finite.*

Theorem 3.11. *Let G be a non-elementary torsion-free word-hyperbolic group with the engulfing property and H a quasiconvex subgroup. Then H is finite index in a separable subgroup $(\overline{H})$ of G.*

It would be very interesting to know more about the engulfing property for word-hyperbolic groups as at present very little seems to be known. The following is a brief survey of work that has been done in this area.

Graham Niblo and Dani Wise [**NW**] have found examples of (non-hyperbolic) 3-manifold groups that do not satisfy the engulfing property.

Rita Gitik has found a large family of hyperbolic 3-manifold groups that are subgroup separable.

Dani Wise [**W**] has a construction which yields word-hyperbolic groups which are subgroup separable with respect to every quasiconvex subgroup but are not subgroup separable with respect to every finitely generated subgroup. The construction is as follows: For every finitely presented group Q, he gives an exact sequence:

$$1 \to N \to G \to Q \to 1$$

such that

(i) G is word-hyperbolic, and every quasiconvex subgroup of G is separable in G

(ii) N is finitely generated.

In particular, if we choose Q to be infinite with no finite quotients, then the only finite index subgroup containing N is G itself.

He has also proved the following and conjectures that the same proof will work for every prime alternating link group.

Theorem. *(Wise) The figure 8 knot group is subgroup separable with respect to its geometrically finite subgroups.* □

Theorem. *(Wise) Negatively curved n-gons (with $n > 3$) of finite groups are subgroup separable with respect to their quasiconvex subgroups.* □

The following theorem is from joint work of Dani Wise with Ilya Kapovich

[**KW**].

Theorem. *(Kapovich, Wise) The following are equivalent:*

(i) Every word-hyperbolic group is residually finite.

(ii) Every word-hyperbolic group has at least one finite quotient.

(iii) Every word-hyperbolic group is virtually torsion-free.

References

AR I.R.Aitchison, J.H.Rubinstein,''Incompressible surfaces and the topology of 3-dimensional manifolds', *Journal of the Australian Mathematical Society: Series A - Pure Mathematics and Statistics* **55** (1993), Pt 1, 1-22.

A J.M.Alonso, T.Brady, D.Cooper, V.Ferlini, M.Lustig, M.Mihalik, M.Shapiro, H.Short, 'Notes on word-hyperbolic groups', in *Group theory from a geometric viewpoint*, World Scientific, Trieste, Italy 1990.

CDP M.Coornaert, T.Delzant, A. Papadopoulos, *'Geometrie et theorie des groupes'*, Lecture notes in Mathematics, **1441**, Springer Verlag, 1990.

CP Michel Coornaert, Athanese Papadopoulos, 'Symbolic Dynamics and Hyperbolic groups', *Lecture Notes in Mathematics*, **1539**, Springer-Verlag 1993.

Gi Rita Gitik,'Doubles of groups and hyperbolic LERF 3-manifolds', MSRI Preprint No. 1996-090.

GH E. Ghys, P. de la Harpe editors, 'Sur les groupes hyperboliques d'après Mikhael Gromov', *Progress in Mathematics*, **83**, Birkhäuser, 1990.

G M. Gromov, 'Hyperbolic groups', in *Essays in Group Theory*, edited by S.M.Gersten, MSRI, Publ.8, Springer, New York, 1987, 75-264.

H M. Hall, 'Coset representatives in free groups', *Transactions of the American Mathematical Society*, **67** (1949), 421-432.

KS I. Kapovich, H.Short,'Some remarks on Quasiconvexity', Preprint.

KW I.Kapovich, D.Wise,'The equivalence of some residual properties of word hyperbolic groups', Preprint.

L D.D.Long,'The engulfing property and subgroup separability for hyperbolic groups', *Transactions of the American Mathematical Society*, **308** No. 2 (1988), 849-859.

Mou G.Moussong, 'Hyperbolic Coxeter groups', Ph.D. thesis, The Ohio State University, 1988.

NW G.A.Niblo, D.T.Wise,'The engulfing property for 3-manifolds',Preprint, University of Southampton (1997), to appear in the Proceedings of EpFest, Geometry and Topology.

Sc Peter Scott,'Subgroups of surface groups are almost geometric', *Journal of the London Mathematical Society* (2) **17** (1978), 555-565.

Sw E. Swenson,'Limit sets in the boundary of negatively curved groups', Preprint.

T W.P.Thurston,'Three-dimensional manifolds, Kleinian groups and hyperbolic geometry', *Bulletin of the American Mathematical Society* **6** (1982), 357-381.

W D.T.Wise, private communication.