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Homotopy types of gauge groups of principal bundles with certain non-simply connected structure groups

by

Simon Rea

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ABSTRACT

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES
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HOMOTOPY TYPES OF GAUGE GROUPS OF PRINCIPAL BUNDLES WITH CERTAIN NON-SIMPLY CONNECTED STRUCTURE GROUPS

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The gauge group of a principal G-bundle P over a space X is the group of G-equivariant homeomorphisms of P that cover the identity on X. To date, the study of the homotopy theory of gauge groups has been focused primarily on principal bundles whose structure groups are simply-connected, mainly due to the inherent complexity of the case of non-simply-connected structure groups.

In this thesis, we carry out a systematic study of the homotopy types of gauge groups of principal bundles with two families of non-simply connected structure groups: namely, the projective unitary groups $\mathrm{PU}(n)$, particularly with n prime, and the complex spin groups $\mathrm{Spin}^c(n)$. These are defined as quotients of $\mathrm{U}(n)$ by its centre, and of the product $\mathrm{Spin}(n) \times \mathrm{U}(1)$ by the diagonal action, respectively.

We examine the relation between the gauge groups of $\mathrm{SU}(n)$- and $\mathrm{PU}(n)$-bundles over the even dimensional sphere S^{2i}, with $2 \leq i \leq n$. As special cases, for $\mathrm{PU}(5)$-bundles over S^4, we show that there is a rational or p-local equivalence $\mathcal{G}_{2,k} \simeq_{(p)} \mathcal{G}_{2,l}$ for any prime p if, and only if, $(120, k) = (120, l)$, while for $\mathrm{PU}(3)$-bundles over S^6 there is an integral equivalence $\mathcal{G}_{3,k} \simeq \mathcal{G}_{3,l}$ if, and only if, $(120, k) = (120, l)$.

We also study the gauge groups of bundles over S^4 with $\mathrm{Spin}^c(n)$ as structure group and show that there is a decomposition $\mathcal{G}_k(\mathrm{Spin}^c(n)) \simeq S^1 \times \mathcal{G}_k(\mathrm{Spin}(n))$. This implies that the homotopy theory of $\mathrm{Spin}^c(n)$-gauge groups reduces to that of $\mathrm{Spin}(n)$-gauge groups over S^4. We then advance on what is known by providing a partial classification for $\mathrm{Spin}(7)$- and $\mathrm{Spin}(8)$-gauge groups over S^4.

Contents

UNIVERSITY OF Southampton

Academic thesis: Declaration of authorship

I, **Simon Rea**, declare that this thesis and the work presented in it are my own and have been generated by me as the result of my own original research.

Homotopy types of gauge groups of principal bundles with certain non-simply connected structure groups

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
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3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Either none of this work has been published before submission, or parts of this work have been published as:
    - S. Rea, *Homotopy types of gauge groups of* $\mathrm{PU}(p)$*-bundles over spheres*, J. Homotopy Relat. Str. **16** (2021), 61–74.

Signed:

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Ai miei genitori

To my parents

Chapter 1

Introduction

Let G be a topological group and X a space. The *gauge group* $\mathcal{G}(P)$ of a principal G-bundle P over X is defined as the group of G-equivariant bundle automorphisms of P which cover the identity on X. An introduction to the topology of gauge groups of bundles can be found in [Hus94, PS98]. The study of gauge groups is important for the classification of principal bundles, as well as understanding moduli spaces of connections on principal bundles [CM94, The11, The13].

Gauge groups also play a key role in theoretical physics, where they are used [BM94] to describe the parallel transport of point particles by means of connections on bundles. Famously, Donaldson [Don83, Don90] computed the rational cohomology of the classifying space of the gauge group of an SU(2)-bundle over a simply-connected 4-manifold and used it to define a new polynomial invariant of manifolds. This allowed Donaldson to deduce strong results about the differential topology of 4-manifolds. This work was cited as the primary contribution for his Fields medal in 1986 and, to this day, constitutes one of the strongest examples of successful interaction between pure mathematics and theoretical physics.

Key properties of gauge groups are invariant under continuous deformation and so studying their homotopy theory is important. Having fixed a topological group G and a space X, an interesting problem is that of classifying the possible homotopy types of the gauge groups $\mathcal{G}(P)$ of principal G-bundles P over X.

Crabb and Sutherland showed [CS00, Theorem 1.1] that if G is a compact, connected, Lie group and X is a connected, finite CW complex, then the number of distinct homotopy types of $\mathcal{G}(P)$, as $P \to X$ ranges over all principal G-bundles over X, is finite. In fact, since isomorphic G-bundles give rise to homeomorphic gauge groups, it will suffice to the let $P \to X$ range over the set of isomorphism classes of principal G-bundles over X.

Explicit classification results have been obtained, especially for the case of gauge groups of bundles with low rank, compact, Lie groups as structure groups and $X = S^4$ as base space. In particular, the first such result was obtained by Kono [Kon91] in 1991. Using the fact that isomorphism classes of principal SU(2)-bundles over S^4 are classified by $k \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(2))$ and denoting by $\mathcal{G}_k$ the gauge group of the principal SU(2)-bundle $P_k \to S^4$ corresponding to the integer k, Kono showed that there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ if, and only if, $(12, k) = (12, l)$, where (m, n) denotes the greatest common divisor of m and n. Since 12 has six divisors, it follows that there are precisely six homotopy types of SU(2)-gauge groups over S^4.

Results formally similar to that of Kono have been obtained for principal bundles over S^4 with different structure groups, among others, by: Hamanaka and Kono [HK06] for SU(3)-gauge groups; Theriault [The15, The17] for SU(n)-gauge groups, as well as [The10a] Sp(2)-gauge groups; Cutler [Cut18a, Cut18b] for Sp(3)-gauge groups and U(n)-gauge groups; Kishimoto and Kono [KK19] for Sp(n)-gauge groups; Kishimoto, Theriault and Tsutaya [KTT17] for G_2-gauge groups; Kamiyama, Kishimoto, Kono and Tsukuda [KKKT07] for SO(3)-gauge groups; Kishimoto, Membrillo-Solis and Theriault [KMST21] for SO(4)-gauge groups; Hasui, Kishimoto, Kono and Sato [HKKS16] for PU(3)- and PSp(2)-gauge groups; and Hasui, Kishimoto, So and Theriault [HKST19] for bundles with exceptional Lie groups as structure groups.

There are also several classification results for gauge groups of principal bundles with base spaces other than S^4 [CS09, HK07, KT96, KT13, The12, MST21, MS19, HKK08, TS19, MAG19a, MAG19b, HKKS16, Moh21, Wes17, Hua21b, Hua21a, So19b, So19a].

Most of early work on the homotopy types of gauge groups has focused on principal bundles whose structure groups are simply connected. In this work, we investigate the homotopy types of gauge groups of principal bundles whose structure groups belong to two families of non-simply connected Lie groups: the projective unitary groups PU(n) and the complex spin groups $\mathrm{Spin}^c(n)$.

In Chapter 5, we examine how the close relationship between the groups SU(n) and PU(n) is reflected in the homotopy properties of the gauge groups of the corresponding bundles, particularly when n is a prime. We do this by generalising certain results relating the classification of PU(n)-gauge groups to that of SU(n)-gauge groups from the paper [KKKT07] for the case $n = 2$, which we restate with proofs in Chapter 4, and from [HKKS16] for the case $n = 3$.

Our first main result compares certain Samelson products on SU(p) and PU(p), with $p \geq 3$ a prime. As will be discussed in Chapter 4, the finiteness of the orders of Samelson products (in the appropriate groups of homotopy classes of maps) plays a crucial role in the homotopy classification of gauge groups. In Chapter 5, we will show the following.

Theorem A. *Let p be an odd prime and let $2 \leq i \leq p$. Let ϵ_i and δ_i denote generators of $\pi_{2i-1}(\mathrm{PU}(p))$ and $\pi_{2i-1}(\mathrm{SU}(p))$, respectively. The orders of the Samelson products*

$\langle \epsilon_i, 1\rangle \colon S^{2i-1} \wedge \mathrm{PU}(p) \to \mathrm{PU}(p)$ *and* $\langle \delta_i, 1\rangle \colon S^{2i-1} \wedge \mathrm{SU}(p) \to \mathrm{SU}(p)$*, where* 1 *denotes the appropriate identity map, coincide.*

Theorem A will be the key ingredient for the necessary direction of our classification results. The converse direction, on the other hand, will require that suitable homotopy invariants of the gauge groups be identified. In Chapter 5, we introduce the notation $\mathcal{G}_{i,k}(\mathrm{PU}(n))$, with $k \in \mathbb{Z}$, for gauge groups of $\mathrm{PU}(n)$-bundles over S^{2i} in analogy with the notation used by Kono [Kon91] and others. In Section 5.3, we give a sufficient condition for certain homotopy invariants of $\mathrm{SU}(n)$- and $\mathrm{PU}(n)$-gauge groups to coincide.

Our methods then allow us to deduce classification results for $\mathrm{PU}(p)$-gauge groups from the corresponding classification results for $\mathrm{SU}(p)$-gauge groups. As examples of applications of our results, we obtain the following complete classifications.

Theorem B. *For* $\mathrm{PU}(5)$*-bundles over* S^4*, it is the case that*

(a) if $\mathcal{G}_{2,k}(\mathrm{PU}(5)) \simeq \mathcal{G}_{2,l}(\mathrm{PU}(5))$*, then* $(120, k) = (120, l)$*;*

(b) if $(120, k) = (120, l)$*, then* $\mathcal{G}_{2,k}(\mathrm{PU}(5)) \simeq \mathcal{G}_{2,l}(\mathrm{PU}(5))$ *when localised rationally or at any prime.*

Theorem C. *For* $\mathrm{PU}(3)$*-bundles over* S^6*, we have* $\mathcal{G}_{3,k}(\mathrm{PU}(3)) \simeq \mathcal{G}_{3,l}(\mathrm{PU}(3))$ *if, and only if,* $(120, k) = (120, l)$*.*

In Chapter 6 we examine $\mathrm{Spin}^c(n)$-gauge groups over S^4. We begin by recalling some basic properties of the complex spin group $\mathrm{Spin}^c(n)$ and showing that, provided $n \geq 3$, it can be expressed as a product of a circle and the real spin group $\mathrm{Spin}(n)$. For $n \geq 6$, we show that this decomposition is reflected in the corresponding gauge groups.

Theorem D. *For* $n \geq 6$ *and any* $k \in \mathbb{Z}$*, we have*

$$\mathcal{G}_k\big(\mathrm{Spin}^c(n)\big) \simeq S^1 \times \mathcal{G}_k\big(\mathrm{Spin}(n)\big).$$

The homotopy theory of $\mathrm{Spin}^c(n)$-gauge groups over S^4 therefore reduced to that of the corresponding $\mathrm{Spin}(n)$-gauge groups. We advance on what is known on $\mathrm{Spin}(n)$-gauge groups by providing a partial classification for the homotopy types of $\mathrm{Spin}(7)$- and $\mathrm{Spin}(8)$-gauge groups over S^4.

Theorem E. *(a) If* $(168, k) = (168, l)$*, there is a homotopy equivalence*

$$\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$$

after localising rationally or at any prime;

(b) If $\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$*, then* $(84, k) = (84, l)$*.*

We note that the discrepancy by a factor of 2 between parts (a) and (b) is due to the same discrepancy for G_2-gauge groups.

Theorem F. *(a) If* $(168, k) = (168, l)$*, there is a homotopy equivalence*

$$\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$$

after localising rationally or at any prime;

(b) If $\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$*, then* $(28, k) = (28, l)$.

For the Spin(8) case, in addition to the same 2-primary indeterminacy appearing in the Spin(7) case, there are also known [KK10, The10b] difficulties at the prime 3 due to the non-vanishing of $\pi_{10}(\mathrm{Spin}(8))_{(3)}$.

Chapter 2

Preliminary homotopy theory

We introduce the topological and homotopy theoretic background that will be necessary in the subsequent chapters. The material in this chapter is based mainly on [Ark11, DK01, Hat00, Sel97, Spa66, Str11, Whi78].

We will be working in the categories **Top** and **Top*** of topological spaces and continuous maps, and of pointed spaces and pointed maps, respectively.

2.1 Basic topological constructions

In this section we introduce several ways of constructing new topological spaces from given ones, and present certain results relating some of these constructions.

Definition 2.1. Let X and Y be topological spaces. The *(unpointed) mapping space* $\mathrm{Map}(X, Y)$ is the set of all continuous maps from X to Y equipped with the compact-open topology.

Recall that the collection

$$\mathcal{U} = \{W_{K,U} \mid K \subseteq X \text{ is compact}, U \subseteq Y \text{ is open}\},$$

where $W_{K,U} = \{f \in \mathrm{Map}(X, Y) \mid f(K) \subseteq U\}$ forms a sub-basis for the compact-open topology on $\mathrm{Map}(X, Y)$.

Given maps $f\colon A \to X$ and $g\colon Y \to B$, there is an induced, continuous map

$$\begin{aligned} g^f : \mathrm{Map}(X, Y) &\to \mathrm{Map}(A, B) \\ \lambda &\mapsto g \circ \lambda \circ f, \end{aligned}$$

which gives rise to two important endofunctors of **Top**:

(i) For any $Y \in \mathbf{Top}$, we have a contravariant functor $\mathrm{Map}(-, Y)$ acting on spaces by sending $X \mapsto \mathrm{Map}(X, Y)$ and on maps by sending $f\colon A \to X$ to

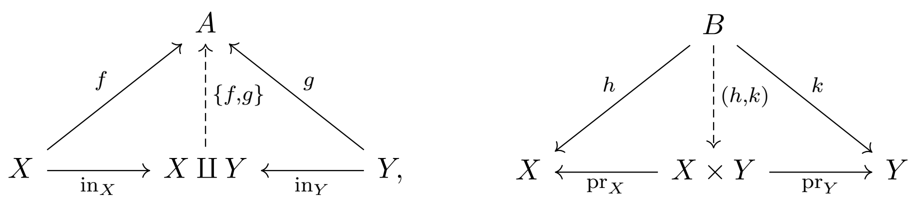
$$\begin{aligned} \mathrm{Map}(f, Y)\colon \mathrm{Map}(X, Y) &\to \mathrm{Map}(A, Y) \\ \lambda &\mapsto \lambda \circ f. \end{aligned}$$

(ii) For any $X \in \mathbf{Top}$, we have a covariant functor $\mathrm{Map}(X, -)$ acting on spaces by sending $Y \mapsto \mathrm{Map}(X, Y)$ and on maps by sending $g\colon Y \to B$ to

$$\begin{aligned} \mathrm{Map}(X, g)\colon \mathrm{Map}(X, Y) &\to \mathrm{Map}(X, B) \\ \lambda &\mapsto g \circ \lambda. \end{aligned}$$

The induced maps $\mathrm{Map}(f, Y)$ and $\mathrm{Map}(X, g)$ are more commonly denoted as f_* and g_*, respectively.

Let $X, Y \in \mathbf{Top}$. Then we denote by $X \amalg Y$ and $X \times Y$ the *disjoint union* and *product* of X and Y, respectively. These are the categorical sum and product of X and Y in the category **Top**, in that they satisfy the following universal properties. Given any spaces A and B and maps f, g, h, k, there exist unique maps $\{f, g\}$ and (h, k) making the diagrams



commute, where in_X, in_Y, pr_X, and pr_Y denote the canonical injections and projections. Taking $X = Y = A = B$ and $f = g = h = k = \mathrm{id}_X$ in the diagrams, we obtain two important canonical maps.

Definition 2.2. For any space X, we define the *folding map* $\nabla\colon X \amalg X \to X$ and the *diagonal map* $\Delta\colon X \to X \times X$ as $\nabla = \{\mathrm{id}_X, \mathrm{id}_X\}$ and $\Delta = (\mathrm{id}_X, \mathrm{id}_X)$, respectively.

Definition 2.3. Given maps $f\colon X \to A$, $g\colon Y \to B$, $h\colon C \to X$ and $k\colon D \to Y$, we define the maps

$$f \amalg g = \{\mathrm{in}_A \circ f, \mathrm{in}_B \circ g\}\colon X \amalg Y \to A \amalg B$$

and

$$h \times k = (h \circ \mathrm{pr}_C, k \circ \mathrm{pr}_D)\colon C \times D \to X \times Y.$$

A consequence of the universal properties for sums and products is that maps out of a disjoint union are completely determined by the restrictions to each of the summands, and that maps into a product are completely determined by their projections onto each of the factors.

Proposition 2.4. *Let X, Y, Z be spaces. If X and Y are Hausdorff, then there is a homeomorphism*

$$\begin{aligned} \mathrm{Map}(X \amalg Y, Z) &\xrightarrow{\cong} \mathrm{Map}(X, Z) \times \mathrm{Map}(Y, Z) \\ f &\longmapsto (f \circ \mathrm{in}_X, f \circ \mathrm{in}_Y) \end{aligned}$$

and, if X is Hausdorff, then there is a homeomorphism

$$\begin{aligned} \mathrm{Map}(X, Y \times Z) &\xrightarrow{\cong} \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \\ f &\longmapsto (\mathrm{pr}_X \circ f, \mathrm{pr}_Y \circ f). \end{aligned}$$

The following is a key result for mapping spaces.

Theorem 2.5 (Exponential law)**.** *Let $X, Y, Z \in$ **Top** and suppose that X and Y are locally compact and Hausdorff. Then there is a homeomorphism*

$$\begin{aligned} \mathrm{Map}(X \times Y, Z) &\xrightarrow{\cong} \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \\ f &\longmapsto \big(x \mapsto f(x, -)\big). \end{aligned}$$

Several important constructions in homotopy theory rely on the choice of a particular point in a space. This leads to the following notion.

Definition 2.6. A *pointed space* is a pair (X, x_0) where X is a space and $x_0 \in X$. The point x_0 is called the *basepoint* of X.

We will denote the pointed space (X, x_0) simply as X when x_0 is understood. Sometimes, basepoints are collectively denoted by the symbol $*$.

Definition 2.7. A map $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces is *pointed* if it preserves the basepoints, i.e. $f(x_0) = y_0$.

If X and Y are pointed, we denote by $\mathrm{Map}^*(X, Y)$ the space of all pointed maps $X \to Y$. Observe that $\mathrm{Map}^*(X, Y)$ is a subspace of $\mathrm{Map}(X, Y)$.

Any one-point space $* = \{*\}$ constitutes a *zero object* for $\mathbf{Top}^*$. This means that $*$ is both initial (i.e. for every other space $X \in \mathbf{Top}^*$ there is a unique map $* \to X$) and terminal (i.e. for every other space $X \in \mathbf{Top}^*$ there is a unique map $X \to *$).

If $X, Y \in \mathbf{Top}^*$, then their categorical sum is the pointed space $(X \amalg Y/(x_0 \sim y_0), [x_0])$, denoted $X \vee Y$ and called the *wedge sum* of X and Y. The categorical product is simply $X \times Y$ with basepoint (x_0, y_0).

Mutatis mutandis, the notations in_X, pr_X, $\{f, g\}$, (h, k), ∇, Δ, $f \vee g$ and $h \times k$ carry over (with $\vee$ replacing $\amalg$) from the unpointed case. Dually to Proposition 2.4, we have the following.

Proposition 2.8. *Let X, Y, Z be pointed spaces. If X and Y are Hausdorff, then there is a homeomorphism*

$$\begin{aligned} \mathrm{Map}^*(X \vee Y, Z) &\xrightarrow{\cong} \mathrm{Map}^*(X, Z) \times \mathrm{Map}^*(Y, Z) \\ f &\longmapsto (f \circ \mathrm{in}_X, f \circ \mathrm{in}_Y) \end{aligned}$$

and, if X is Hausdorff, then there is a homeomorphism

$$\begin{aligned} \mathrm{Map}^*(X, Y \times Z) &\xrightarrow{\cong} \mathrm{Map}^*(X, Y) \times \mathrm{Map}^*(X, Z) \\ f &\longmapsto (\mathrm{pr}_X \circ f, \mathrm{pr}_Y \circ f). \end{aligned}$$

Due to the existence of a zero object in $\mathbf{Top}^*$, there is a canonical map $X \vee Y \to X \times Y$, no analogue of which exists in **Top**. Let $*$ denote either of the composites $X \to * \to Y$ and $Y \to * \to X$. The universal property of products gives rise to the maps

$$(\mathrm{id}_X, *)\colon X \to X \times Y \qquad \text{and} \qquad (*, \mathrm{id}_Y)\colon Y \to X \times Y.$$

In turn, the universal property of sums defines the map

$$w_{X,Y} = \{(\mathrm{id}_X, *), (*, \mathrm{id}_Y)\}\colon X \vee Y \to X \times Y.$$

Note that w is self-dual in $\mathbf{Top}^*$, since $\{(\mathrm{id}_X, *), (*, \mathrm{id}_Y)\} = (\{\mathrm{id}_X, *\}, \{*, \mathrm{id}_Y\})$, and natural in that, given maps $f\colon X \to A$ and $g\colon Y \to B$, there is a commutative diagram

$$\begin{array}{ccc} X \vee Y & \xrightarrow{f \vee g} & A \vee B \\ \Big\downarrow{\scriptstyle w_{X,Y}} & & \Big\downarrow{\scriptstyle w_{A,B}} \\ X \times Y & \xrightarrow{f \times g} & A \times B. \end{array}$$

There is another useful notion of product for pointed spaces, distinct from the categorical product. It is denoted $X \wedge Y$, the *smash product* of X and Y, and defined as the quotient space $X \times Y/(X \vee Y)$ with basepoint $[(x_0, y_0)]$. The smash product replaces the usual topological product in the pointed version of the exponential law.

Theorem 2.9 (Pointed exponential law)**.** *Let $X, Y, Z \in \mathbf{Top}^*$ and suppose that X and Y are locally compact and Hausdorff. Then there is a homeomorphism*

$$\begin{aligned} \mathrm{Map}^*(X \wedge Y, Z) &\xrightarrow{\cong} \mathrm{Map}^*(X, \mathrm{Map}^*(Y, Z)) \\ f &\longmapsto \big(x \mapsto f(x, -)\big). \end{aligned}$$

We will usually denote the image of a map f under the pointed or unpointed exponential law, and in either direction, simply by $\overline{f}$ and call it the *adjoint* of f.

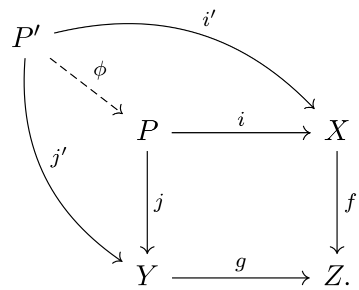
Remark 2.10. The homeomorphisms of the both the pointed and unpointed exponential laws are natural in X and Z. In particular, this means that, for every locally compact Hausdorff Y, the functors $- \times Y$ and $\mathrm{Map}(Y, -)$ form a pair of adjoint functors in (a suitable subcategory of) **Top**, and provided Y is also pointed, then the functors $- \wedge Y$ and $\mathrm{Map}^*(Y, -)$ form a pair of adjoint functors in (a suitable subcategory of) $\mathbf{Top}^*$.

Two other very useful constructions are pullbacks and pushouts. A diagram consisting of two maps with the same target space is called a pre-pullback diagram. Pre-pushout diagrams are defined dually.

Definition 2.11. The *pullback* of the pre-pullback diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ is a space P together with maps $i\colon P \to X$ and $j\colon P \to Y$ such that

(1) $f \circ i = g \circ j$;

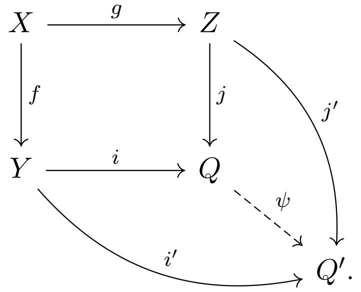
(2) for every other space P' together with maps $i'\colon P' \to X$ and $j'\colon P' \to Y$ such that $f \circ i' = g \circ j'$, there is a unique map $\phi\colon P' \to P$ such that $i \circ \phi = i'$ and $j \circ \phi = j'$. That is, there is a commutative diagram



Definition 2.12. The *pushout* of the pre-pushout diagram $Y \xleftarrow{f} X \xrightarrow{g} Z$ is a space Q together with maps $i\colon Y \to Q$ and $j\colon Z \to Q$ such that

(1) $i \circ f = j \circ g$;

(2) for every other space Q' together with maps $i'\colon Y \to Q'$ and $j'\colon Z \to Q'$ such that $i' \circ f = j' \circ g$, there is a unique map $\psi\colon Q \to Q'$ such that $\psi \circ i = i'$ and $\psi \circ j = j'$. That is, there is a commutative diagram



Pullbacks and pushouts make sense in both **Top** and **Top***, and are unique up to homeomorphism. They can be described explicitly as follows.

Given maps $X \xrightarrow{f} Z \xleftarrow{g} Y$, the space

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

together with the maps pr_X and pr_Y restricted to P is a pullback of $X \xrightarrow{f} Z \xleftarrow{g} Y$.

Dually, given maps $Y \xleftarrow{f} X \xrightarrow{g} Z$, the quotient space

$$Q = \frac{Y \amalg Z}{\{f(x) \sim g(x) \mid x \in X\}}$$

together with in_Y and in_Z followed by the quotient map is a pushout of $Y \xleftarrow{f} X \xrightarrow{g} Z$.

Cellular or CW complexes, first introduced by Whitehead in [Whi49a], are a class of spaces that is particularly amenable to the tools of homotopy theory, and can be defined in terms of pushouts.

Definition 2.13. A 0-dimensional *CW complex* X_0 is a space equipped with the discrete topology. Suppose, inductively, that X_{n-1} is an $(n-1)$-dimensional CW complex and let $f\colon \coprod S^{n-1} \to \coprod X_{n-1}$ be a map from a (possibly empty) disjoint union of $(n-1)$-dimensional spheres into X_{n-1} and let $\iota\colon S^{n-1} \to D^n$ denotes the inclusion of the boundary of the n-disk. The pushout X_n in the diagram

$$\begin{array}{ccc} \coprod S^{n-1} & \xrightarrow{\coprod \iota} & \coprod D^n \\ \downarrow{\scriptstyle f} & & \downarrow \\ X_{n-1} & \longrightarrow & X_n, \end{array}$$

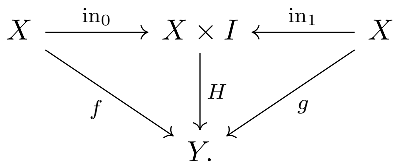
is then an *n-dimensional CW complex.*

If X is a CW complex, then, for $k \in \mathbb{N}$, the homeomorphic image of each D^k into X is denoted e_k and called a k-cell of X. A CW complex is said to be of *finite type* if it has finitely many cells in each dimension.

2.2 Homotopies and homotopy sets

Let I denote the unit interval $[0, 1]$ and, for any space X and any $t \in I$, let $\mathrm{in}_t\colon X \to X \times I$ denote the injection $x \mapsto (x, t)$.

Definition 2.14. A *(free) homotopy* between two maps $f, g\colon X \to Y$ is a continuous map $H\colon X \times I \to Y$ making the following diagram commute



Definition 2.15. A *pointed homotopy* between two pointed maps $f, g\colon X \to Y$ is a free homotopy $H\colon X \times I \to Y$ such that the composite

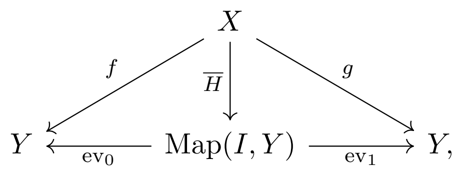
$$H \circ \mathrm{in}_t\colon X \to Y$$

is pointed for all $t \in I$.

By the exponential law, to each homotopy$H\colon X \times I \to Y$, there corresponds an adjoint map $\overline{H}\colon X \to \mathrm{Map}(I, Y)$ defined by

$$\overline{H}\colon x \mapsto \big(t \mapsto H(x,t)\big).$$

Proposition 2.16. *Two maps $f, g\colon X \to Y$ are homotopic in the sense of Definition 2.14 if, and only if, they are homotopic through $\overline{H}$, in the sense that there is a commutative diagram*



where $\mathrm{ev}_t\colon \mathrm{Map}(I, Y) \to Y$ *denotes evaluation at* $t \in I$*. And similarly in the case of a pointed homotopy.*

Remark 2.17. The space $X \times I$, together with the injection maps $\mathrm{in}_0, \mathrm{in}_1\colon X \to X \times I$, is an example of a categorical construction called a *cylinder object* on X, while the space $\mathrm{Map}(I, Y)$, together with the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Map}(I, Y) \to Y$ is an example of a *path object* on Y. Cylinder and path objects are dual to each other in the usual categorical sense.

For any $X, Y \in \mathbf{Top}$ (resp. $\mathbf{Top}^*$), the relation of homotopy (resp. pointed homotopy) is an equivalence relation on $\mathrm{Map}(X, Y)$ (resp. $\mathrm{Map}^*(X, Y)$).

Definition 2.18. For spaces (rep. pointed spaces) X and Y, we denote the set of homotopy classes of maps (resp. pointed homotopy classes of pointed maps) between X and Y as $[X, Y]_{\mathrm{free}}$ (resp. $[X, Y]$).

There is a category **hTop** whose objects are the same as in **Top** but whose morphisms are homotopy classes of maps. Isomorphisms in **hTop** are called homotopy equivalences. More explicitly, we have the following.

Definition 2.19. A map $f\colon X \to Y$ is called a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X \quad \text{and} \qquad f \circ g \simeq \mathrm{id}_Y.$$

If there is a homotopy equivalence between X and Y, we write $X \simeq Y$ and say that X and Y are homotopy equivalent, or that they have the same homotopy type. The pointed notions are defined analogously. If X is homotopy equivalent to a one-point space, then X is said to be *contractible*.

The most prominent example of homotopy sets is given by the fundamental group and the higher homotopy groups of a space.

Definition 2.20. Let X be a pointed space. The nth *homotopy groups* of X at $x_0 \in X$ is defined as

$$\pi_n(X, x_0) := [S^n, X].$$

For $n = 0$, $\pi_0(X, x_0)$ is the set of path components of X and does not possess a natural group structure. The group $\pi_n(X, x_0)$ is abelian for $n \geq 2$.

If $x_1, x_2 \in X$ belong to the same path component of X, then there are (non-canonical) isomorphisms $\pi_n(X, x_1) \cong \pi_n(X, x_2)$ for all $n \geq 1$. In particular, if X is path connected, then $\pi_n(X, x_0)$ does not depend on the choice of $x_0 \in X$, and we will simply write $\pi_n(X)$ in place of $\pi_n(X, x_0)$.

A pointed map $f\colon X \to Y$ induces a set map $f_*\colon \pi_0(X, x_0) \to \pi_0(Y, f(x_0))$ and group homomorphisms $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for $n \geq 1$ that are compatible with identities and composition of maps. In other words, we have a sequence of functors

$$\begin{aligned}
&\pi_0\colon \mathbf{Top}^* \to \mathbf{Set}, \\
&\pi_1\colon \mathbf{Top}^* \to \mathbf{Grp}, \\
&\pi_n\colon \mathbf{Top}^* \to \mathbf{AbGrp}, \quad n \geq 2.
\end{aligned}$$

Definition 2.21. Let X be a space and $A \subseteq X$ a subspace. If $i\colon A \hookrightarrow X$ denotes the inclusion map, then we have the following notions:

- a *retraction* of X onto A is a map $r\colon X \to A$ which is a left inverse of i in **Top**, i.e. such that $r \circ i = \mathrm{id}_A$;
- a *deformation retraction* of X onto A is a homotopy $H\colon i \circ r \simeq \mathrm{id}_X$, where r is a retraction of X onto A;

- a *strong deformation retraction* of X onto A is a deformation retraction H of X onto A for which $H(a,t) = a$ for all $a \in A$ and $t \in I$.

Remark 2.22. Note that an inclusion map $i\colon A \hookrightarrow X$ never admits a right inverse in **Top** except in the trivial case when $A = X$. The existence of a right inverse of i in **hTop** essentially coincides with the notion of deformation retraction.

Definition 2.23. A pointed space (X, x_0) is said to be *well pointed* if x_0 admits a neighbourhood which is a strong deformation retract of $\{x_0\}$ (or, equivalently, if the inclusion $\{x_0\} \hookrightarrow X$ is a closed cofibration, cf. Section 2.4).

For a well pointed space X and any pointed space Y, there is an action of the fundamental group of Y on the set $[X, Y]$. Indeed, one can show (see, e.g. [DK01, Lemma 6.56]) that given $\gamma \in \pi_1(Y, y_0)$ and $f \in [X, Y]$, there exists $f' \in [X, Y]$ and a homotopy $H\colon X \times I \to Y$ for which $H(x_0, -)$ is homotopic to the loop γ. The action is defined by setting $\gamma f = f'$. We then have the following.

Proposition 2.24. *If X is well pointed and Y is path-connected, then the natural map $[X, Y] \to [X, Y]_{\text{free}}$ induces a bijection*

$$[X, Y]/\pi_1(Y, y_0) \longrightarrow [X, Y]_{\text{free}}.$$

In particular, $[X, Y] = [X, Y]_{\text{free}}$ whenever Y is simply-connected.

Definition 2.25. A space X is said to be *n-connected* if $\pi_k(X, x_0) \cong 0$ for all $k \leq n$. Moreover, if X is n-connected for all n, then X is said to be *weakly contractible.*

The homotopy groups of a space are related to its singular homology groups with integer coefficients by the following.

Theorem 2.26. *Let X be a path-connected space. Then, for every n, there is a group homomorphism*

$$h_n\colon \pi_n(X) \longrightarrow H_n(X; \mathbb{Z}),$$

called the Hurewicz homomorphism. When $n \geq 2$, if X is $(n-1)$-connected, then h_k is an isomorphism for $k \leq n$ and an epimorphism for $k = n + 1$.

In particular, the first non-trivial homotopy and homology groups of X appear in the same dimension.

Remark 2.27. For $n = 1$, the Hurewicz homomorphism h_1 is just the abelianisation and it induces an isomorphism

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X; \mathbb{Z}).$$

Definition 2.28. A map $f\colon X \to Y$ is called an *n-equivalence* if it induced a bijection on π_0 and an isomorphism on π_k for all $k \leq n$. If the map f is an n-equivalence for all n, then f is said to be a *weak homotopy equivalence.*

Theorem 2.29 (Whitehead [Whi49a, Whi49b])**.** *A weak homotopy equivalence between CW complexes is a homotopy equivalence.*

In particular, it follows that a weakly contractible CW complex is contractible.

Remark 2.30. Another useful version of Whiteahead's theorem states that if a map $f\colon X \to Y$ between simply-connected CW complexes induces isomorphisms on all homology groups, then it is a homotopy equivalence.

Homotopy groups do not determine the homotopy type of a space in general. However, there is a class of spaces whose homotopy types are completely determined by their homotopy groups.

Definition 2.31. Let $n \in \mathbb{N}$ and let G be a group, abelian if $n \geq 2$. A space X is said to be an *Eilenberg-Maclane space* of type (G, n) if

$$\pi_k(X, x_0) \cong \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Eilenberg-Maclane spaces, which were first introduced by Eilenberg and Mac Lane in a 1945 paper [EM45], exist and are unique up to weak homotopy equivalence for each n and G. By a slight abuse of notation, any Eilenberg-Maclane space of type (G, n) is denoted as $K(G, n)$.

The Eilenberg-Maclane space $K(G, n)$ is a representing object for the nth singular cohomology group with coefficients in G, in the sense of Brown [Bro62].

Theorem 2.32. *Let X be a CW complex and G an abelian group. For every $n \in \mathbb{N}$ there is a natural bijection*

$$H^n(X; G) \cong [X, K(G, n)].$$

The dual notion to Eilenberg-Maclane space involves homology groups.

Definition 2.33. Let $n \in \mathbb{N}$ and let G be an abelian group. A space X is said to be a *Moore space* of type (G, n) if

$$H_k(X; \mathbb{Z}) \cong \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

2.3 H-spaces and co-H-spaces

If X is any space and G is a topological group, then G induces a group structure on $\mathrm{Map}^*(X, G)$ in the obvious way. As $[X, Y]$ is the quotient of $\mathrm{Map}^*(X, Y)$ by the pointed homotopy relation, in order for Y to induce a group structure on $[X, Y]$, it suffices that Y satisfy the group axioms only up to homotopy.

Definition 2.34. An *H-space* is a pointed topological space Y endowed with a *multiplication* map $\mu\colon Y \times Y \to Y$ such that the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{(*,\mathrm{id}_Y)} & Y \times Y & \xleftarrow{(\mathrm{id}_Y,*)} & Y \\ & \searrow\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\swarrow & \\ & & Y & & \end{array}$$

commutes up to homotopy. Equivalently, Y is an H-space with multiplication μ if μ is an extension of the folding map in the diagram

$$\begin{array}{ccc} Y \vee Y & \xrightarrow{w_{Y,Y}} & Y \times Y \\ & \underset{\nabla}{\searrow} & \downarrow{\scriptstyle \mu} \\ & & Y. \end{array}$$

Remark 2.35. It is noted in Hubbuck [Hub99] that the "H" in H-space was suggested by Serre in recognition of the influence exerted on the subject by Hopf.

Note that an H-space is the homotopy-theoretic generalisation of a unital magma (i.e. a set endowed with a binary operation which admits an identity). An H-space may, of course, also satisfy the remaining group axioms, associativity and existence of inverses, as well as commutativity, up to homotopy.

Definition 2.36. An H-space Y is said to be *homotopy associative* if the diagram

$$\begin{array}{ccc} Y \times Y \times Y & \xrightarrow{\mu \times \mathrm{id}_Y} & Y \times Y \\ \downarrow{\scriptstyle \mathrm{id}_Y \times \mu} & & \downarrow{\scriptstyle \mu} \\ Y \times Y & \xrightarrow{\mu} & Y \end{array}$$

commutes up to homotopy.

A *homotopy inverse* for Y is a map $\iota\colon Y \to Y$ such that the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{(\iota,\mathrm{id}_Y)} & Y \times Y & \xleftarrow{(\mathrm{id}_Y,\iota)} & Y \\ & \underset{*}{\searrow} & \downarrow{\scriptstyle \mu} & \underset{*}{\swarrow} & \\ & & Y & & \end{array}$$

commutes up to homotopy.

The H-space Y is said to be *homotopy commutative* if the diagram

$$\begin{tikzcd} Y \times Y \arrow[r, "T"] \arrow[d, "\mu"'] & Y \times Y \arrow[d, "\mu"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

commutes up to homotopy, where T is the map $(x, y) \mapsto (y, x)$.

Definition 2.37. An H-space is called an *H-group* if it is homotopy associative and admits a homotopy inverse.

Dually, we have the following definition.

Definition 2.38. A *co-H-space* is a pointed topological space X endowed with a *comultiplication* map $\sigma\colon X \to X \vee X$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[d, "\sigma"] \arrow[dr, equal] & \\ X & X \vee X \arrow[l, "\{*,\mathrm{id}_X\}"] \arrow[r, "\{\mathrm{id}_X,*\}"'] & X \end{tikzcd}$$

commutes up to homotopy. Equivalently, X is a co-H-space with comultiplication σ if σ is a lift of the diagonal map in the diagram

$$\begin{tikzcd} X \arrow[d, "\sigma"] \arrow[dr, "\Delta"] & \\ X \vee X \arrow[r, "w_{X,X}"] & X \times X. \end{tikzcd}$$

Definition 2.39. A co-H-space X is said to be *homotopy coassociative* if the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\sigma"] & X \vee X \arrow[d, "\mathrm{id}_X \vee \sigma"] \\ X \vee X \arrow[r, "\sigma \vee \mathrm{id}_X"] & X \vee X \vee X \end{tikzcd}$$

commutes up to homotopy.

A *homotopy coinverse* for X is a map $\nu\colon Y \to Y$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "*"'] \arrow[d, "\sigma"] \arrow[dr, "*"] & \\ X & X \vee X \arrow[l, "\{\nu,\mathrm{id}_X\}"] \arrow[r, "\{\mathrm{id}_X,\nu\}"'] & X \end{tikzcd}$$

commutes up to homotopy.

The co-H-space X is said to be *homotopy cocommutative* if the diagram

$$\begin{array}{ccc} X & = & X \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ X \vee X & \xrightarrow{T} & X \vee X \end{array}$$

commutes up to homotopy, where T is the map $(x, y) \mapsto (y, x)$.

Definition 2.40. A co-H-space is called a *co-H-group* if it is homotopy coassociative and admits a homotopy coinverse.

If Y is an H-space and X is any space, then the multiplication μ of Y induces a binary operation on $\mathrm{Map}^*(X, Y)$, usually written additively, defined as

$$f + g := \mu \circ (f \times g) \circ \Delta,$$

where $\Delta \colon X \to X \times X$ denotes the diagonal map, making $\mathrm{Map}^*(X, Y)$ into an H-space. This, in turn, descends to a well-defined operation on $[X, Y]$ by

$$[f] + [g] := [f + g].$$

Dropping the equivalence class notation and denoting by $* \in [X, Y]$ the nullhomotopic class, note that the commutative diagram in Definition 2.34 implies that

$$* + f = f = f + *$$

for all $f \in [X, Y]$, making $[X, Y]$ into a unital magma. If, in addition, Y is homotopy associative, then $[X, Y]$ is a monoid, while if Y is an H-group, then $\mathrm{Map}^*(X, Y)$ is also an H-group, while $[X, Y]$ is actually a group. In the latter case, the inverse of $f \in [X, Y]$ is given by $\iota \circ f$, where ι is the homotopy inverse of Y.

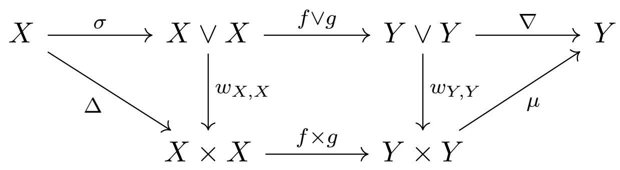
Dually, if X is a co-H-space and Y is any space, then the comultiplication σ of X induces a binary operation on $\mathrm{Map}^*(X, Y)$ defined as

$$f + g := \nabla \circ (f \vee g) \circ \sigma,$$

where $\nabla \colon X \to X \times X$ denotes the folding map, satisfying analogous properties to those of the operation induced by an H-space.

Proposition 2.41. *If X is a co-H-space and Y is an H-space, the operations on $[X, Y]$ induced by X and Y are one and the same.*

Proof. Let σ denote the comultiplication on X and μ denote the multiplication on Y, and let $f, g \in [X, Y]$. Consider the following diagram.



The middle square commutes by the naturality of $w\colon (- \vee -) \to (- \times -)$ (cf. p.4), while the left and right triangles homotopy commute by the definitions of co-H-space and H-space, respectively. Hence

$$\nabla \circ (f \vee g) \circ \sigma \simeq \mu \circ (f \times g) \circ \Delta$$

and the two induced operations are therefore one and the same. □

Remark 2.42. The Eckmann–Hilton argument [EH62] shows, additionally, that the resulting group structure on $[X, Y]$ is commutative.

Definition 2.43. Let Y be an H-space and let $f \in [X, Y]$. The *order* of f is the smallest $k \in \mathbb{N}$ such that

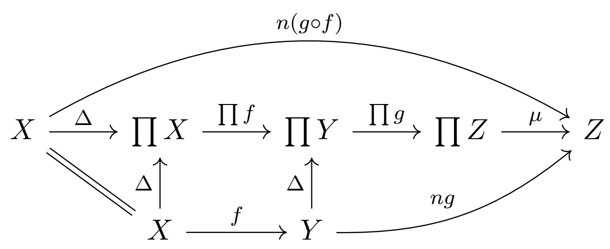
$$kf := \underbrace{f + f + \cdots + f}_{k \text{ times}} = *.$$

If no such k exists, we say that f has *infinite order.*

Proposition 2.44. *(a) Let $f \in [X, Y]$ and $g \in [Y, Z]$, where X and Y are spaces, and Z is an H-space. Then, for $n \in \mathbb{Z}$, we have $n(g \circ f) \simeq ng \circ f$.*

(b) Let $f \in [X, Y]$ and $g \in [Y, Z]$, where Y and Z are spaces, and X is a co-H-space. Then, for $n \in \mathbb{Z}$, we have $n(g \circ f) \simeq g \circ nf$.

Proof. We only show (a) as (b) is formally dual. The case $n = 0$ is trivial. So suppose $n > 0$. By considering the commutative diagram



we see that $n(g \circ f) \simeq ng \circ f$. For $n < 0$, note that $ng \simeq \iota \circ (-ng)$. □

Definition 2.45. Let Y be an H-space. For a non-negative integer $k \in \mathbb{Z}$, we define the kth *power map* $k\colon Y \to Y$ recursively by $0 := *$, $1 := \mathrm{id}_Y$, and

$$k := \mu \circ \big((k-1) \circ \mathrm{id}_Y\big) \circ \Delta.$$

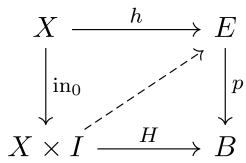
If, in addition, Y possesses a homotopy inverse $\iota\colon Y \to Y$, then, for a negative $k \in \mathbb{Z}$, we define $k := \iota \circ (-k)$.

Remark 2.46. Observe that for $f \in [X, Y]$ we have $kf = k \circ f$, where the left-hand side is the k-multiple of f in the group $[X, Y]$, while the right hand side is the composition of f with kth power map on Y.

2.4 Fibrations and cofibrations

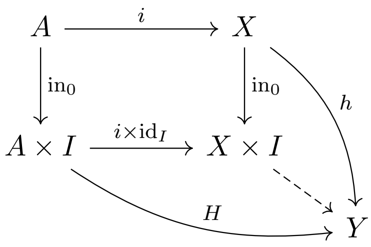
Fibrations and cofibrations are important classes of maps in homotopy theory. Intuitively speaking, fibrations and cofibrations are the homotopy theoretic generalisations of projection and inclusion maps, respectively. The precise definitions are as follows.

Definition 2.47. A map $p\colon E \to B$ is called a *fibration* if it possess the *homotopy lifting property*. Namely, if given any space X, any map $h\colon X \to E$, and any homotopy $H\colon X \times I \to B$ such that $H \circ \mathrm{in}_0 = p \circ h$, there exists a homotopy $X \times I \to E$ making the diagram



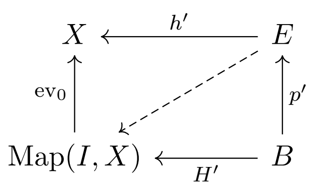
commute.

Definition 2.48. A map $i\colon A \to X$ is called a *cofibration* if it possess the *homotopy extension property*. Namely, if given any space Y, any map $h\colon X \to Y$, and any homotopy $H\colon A \times I \to Y$ such that $H \circ \mathrm{in}_0 = h \circ i$, there exists a homotopy $X \times I \to Y$ making the diagram

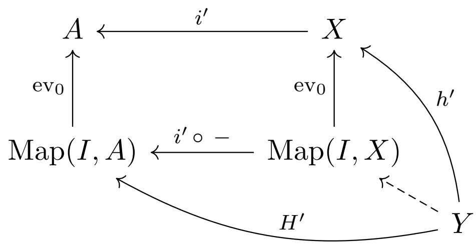


commute.

Dualising the commutative diagram in Definition 2.47 produces the diagram



which can easily be shown to be equivalent to the diagram in Definition 2.48, in that it expresses the homotopy extension property for the map $p' \colon B \to E$. Similarly, dualising the diagram in Definition 2.48, we obtain



which can be shown to be equivalent to the diagram in Definition 2.47, expressing the homotopy lifting property for the map $i' \colon X \to A$.

A cofibration $i \colon A \to X$ is called a *closed cofibration* if $i(A)$ is closed in X.

Definition 2.49. Let $p \colon E \to B$ be a fibration and $i \colon A \to X$ be a cofibration. The subspace $p^{-1}(\{b\}) \subset E$ is called the *fibre* of p over b, and the quotient $C_i := X/i(A)$ is called the *cofibre* of i. If the fibration $p \colon E \to B$ is pointed, then by its fibre we always mean the preimage $p^{-1}(\{*\})$ of the basepoint $* \in B$.

We also say that $F \to P \to E$ and $A \to X \to C_i$ are a fibre sequence and a cofibre sequence, respectively.

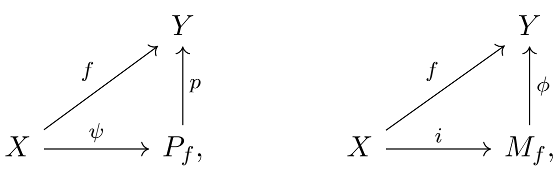
Proposition 2.50 (Selick [Sel97, Proposition 7.1.3])**.** *Let $p \colon E \to B$ be a fibration with B path-connected. Then $p^{-1}(\{b_1\}) \simeq p^{-1}(\{b_2\})$ for all $b_1, b_2 \in B$.*

Example 2.51. Let X and Y be pointed spaces. Denote by $\mathrm{ev} \colon \mathrm{Map}(X, Y) \to Y$ the evaluation at the basepoint $* \in X$, that is, $\mathrm{ev}(f) := f(*) \in Y$. Then, we have a fibre sequence

$$\mathrm{Map}^*(X, Y) \longrightarrow \mathrm{Map}(X, Y) \xrightarrow{\mathrm{ev}} Y.$$

A fact that lies at the heart of homotopy theory is that every map can be converted, up to homootpy, into both a fibration and a cofibration, in the sense of the following proposition.

Proposition 2.52. *Let $f \colon X \to Y$ be a map. Then there exist spaces P_f, M_f, and factorisations of f through P_f and M_f*



such that p is a fibration, i is a cofibration, and ϕ and ψ are homotopy equivalences.

The standard choices (see, for instance, Arkowitz [Ark11, Section 3.5]) for the space P_f and the corresponding maps ψ and p are

$$P_f := \big\{(x,\gamma) \in X \times \mathrm{Map}(I,Y) \mid f(x) = \gamma(0)\big\},$$

called the *mapping path space* of $f\colon X \to Y$, together with

$$\begin{aligned} \psi\colon X &\to P_f \\ x &\mapsto (x, c_x), \end{aligned} \qquad \begin{aligned} p\colon \quad P_f &\to Y \\ (x,\gamma) &\mapsto \gamma(0), \end{aligned}$$

where $c_x \in \mathrm{Map}(I,Y)$ denotes the constant path at $x \in X$, while those for the space M_f and the corresponding maps i and ϕ are

$$M_f := \frac{(X \times I) \amalg Y}{\{(x,0) \sim f(x)\} \cup \{(*,t) \sim *\}},$$

called the *mapping cylinder* associated to the map $f\colon X \to Y$, together with

$$\begin{aligned} i\colon X &\to M_f \\ x &\mapsto [(x,1)], \end{aligned} \qquad \begin{aligned} \phi\colon \quad M_f &\to Y \\ [(x,t)] &\mapsto f(x) \\ [y] &\mapsto y. \end{aligned}$$

In fact, the mapping path space P_f is obtained as the pullback

$$\begin{array}{ccc} P_f & \longrightarrow & \mathrm{Map}(I,Y) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow{f} & Y \end{array}$$

while the mapping cylinder M_f is obtained as the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{in}_0}\downarrow & & \downarrow \\ X \times I & \longrightarrow & M_f. \end{array}$$

Example 2.53. Let $f\colon * \to Y$ be the inclusion of the basepoint into Y. Then

$$P_f = PY := \big\{\gamma \in \mathrm{Map}(I,Y) \mid \gamma(0) = *\big\}$$

and $p\colon PY \to Y$ is just ev_1. The fibre $p^{-1}(\{*\})$ is the loop space

$$\Omega Y := \big\{\gamma \in \mathrm{Map}(I,Y) \mid \gamma(0) = * = \gamma(1)\big\}.$$

Dually, if $g\colon X \to *$ is the terminal map, then

$$M_g = \frac{(X \times I) \amalg *}{\{(x,0) \sim *\} \cup \{(*,t) \sim *\}} \simeq CX,$$

and the map $i\colon X \to M_g$ is just the inclusion of the base $x \mapsto [(x,1)]$, so the cofibre $M_g/i(X)$ is the reduced suspension

$$\Sigma X = \frac{X \times I}{\{(x,0) \sim (x,1) \sim (*,t)\}}.$$

Converting maps into fibrations and cofibrations up to homotopy allows us to consider fibres and cofibres of arbitrary maps, up to homotopy.

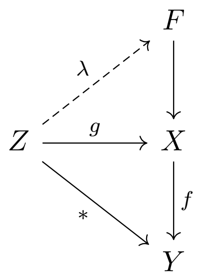
Definition 2.54. Let $f\colon X \to Y$ be an arbitrary map. The *homotopy fibre* of f is the fibre of the fibration $P_f \to Y$, and the *homotopy cofibre* of f is the cofibre of the cofibration $X \to M_f$.

Evidently, homotopic maps possess homotopy equivalent homotopy fibres and cofibres, respectively. Note that, if the map $f\colon X \to Y$ is already a fibration (resp. a cofibration), then its fibre (resp. cofibre) is homotopy equivalent to its homotopy fibre (resp. homotopy cofibre).

An important property of homotopy fibres and cofibres that often allows us to deduce the existence of certain maps is the following.

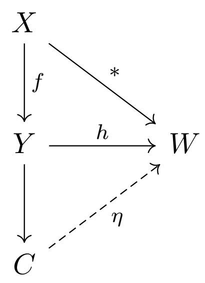
Given $f\colon X \to Y$, a map into X lifts to the homotopy fibre of f if, and only if, it composes trivially with f and, dually, a map out of Y extends to the homotopy cofibre of f if, and only if, it composes trivially with f. More precisely, we have the following result.

Proposition 2.55. *Let $f\colon X \to Y$ be a map and let F be its homotopy fibre. Then, for any map $g\colon Z \to X$, there exists a map $\lambda\colon Z \to F$ making the upper triangle in the diagram*



homotopy commute if, and only if, the lower triangle homotopy commutes.

Dually, if C denotes the homotopy cofibre of $f\colon X \to Y$, then, for any map $h\colon Y \to W$, there exists a map $\eta\colon C \to W$ making the lower triangle in the diagram



homotopy commute if, and only if, the upper triangle homotopy commutes.

Homotopy commutative squares give rise to induced maps of fibres and cofibres. More precisely, the following statements hold.

Proposition 2.56. *Suppose that there is a homotopy $H\colon hg \simeq kf$ making the square*

$$\begin{CD} X @>{f}>> Y \\ @V{h}VV @VV{k}V \\ Z @>{g}>> W \end{CD}$$

homotopy commute. Then H induces a map $\Phi_H\colon F_f \to F_g$ between the homotopy fibres of f and g such that extending the diagram to the left as

$$\begin{CD} \Omega Y @>>> F_f @>>> X @>{f}>> Y \\ @V{\Omega k}VV @V{\Phi_H}VV @V{h}VV @VV{k}V \\ \Omega W @>>> F_g @>>> Z @>{g}>> W, \end{CD}$$

the leftmost square is homotopy commutative and the middle square is strictly commutative.

Dually, the homotopy H also induces a map $\Psi\colon C_f \to C_g$ between the homotopy cofibres of f and g such that extending the diagram to the right as

$$\begin{CD} X @>{f}>> Y @>>> C_f @>>> \Sigma X \\ @V{h}VV @V{k}VV @V{\Psi_H}VV @VV{\Sigma h}V \\ Z @>{g}>> W @>>> C_g @>>> \Sigma Z, \end{CD}$$

the rightmost square is homotopy commutative and the middle square is strictly commutative.

One of the many reasons why fibrations and cofibrations are essential tools in homotopy theory is that they induce (long) exact sequences of sets of homotopy classes of maps.

Proposition 2.57. *Let $F \to X \to Y$ be a fibration sequence. Then, for any pointed space A, there is an exact sequence*

$$[A, F] \longrightarrow [A, X] \longrightarrow [A, Y].$$

Dually, if $X \to Y \to C$ is a cofibration sequence, then, for any pointed space A, there is an exact sequence

$$[C, A] \longrightarrow [Y, A] \longrightarrow [X, A].$$

Let $F \to X \to Y$ be a fibration sequence and let F' be the homotopy fibre of $F \to X$. Then there is a homotopy equivalence $F' \simeq \Omega Y$. Iterating the process of replacing a map with a fibration, we obtain a fibration sequence

$$\cdots \longrightarrow \Omega^2 Y \longrightarrow \Omega F \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow F \longrightarrow X \longrightarrow Y.$$

Dually, if $X \to Y \to C$ is a cofibration sequence and and C' denotes the homotopy cofibre of $Y \to C$, then there is a homotopy equivalence $C' \simeq \Sigma X$ and we obtain a cofibration sequence

$$X \longrightarrow Y \longrightarrow C \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C \longrightarrow \Sigma^2 X \longrightarrow \cdots.$$

Proposition 2.58. *Let $F \to X \to Y$ be a fibration sequence. Then, for any pointed space A, there is an exact sequence*

$$\cdots \longrightarrow [A, \Omega X] \longrightarrow [A, \Omega Y] \longrightarrow [A, F] \longrightarrow [A, X] \longrightarrow [A, Y].$$

Dually, if $X \to Y \to C$ is a cofibration sequence, then, for any pointed space A, there is an exact sequence

$$\cdots \longrightarrow [\Sigma Y, A] \longrightarrow [\Sigma X, A] \longrightarrow [C, A] \longrightarrow [Y, A] \longrightarrow [X, A].$$

Another important application of fibrations and cofibrations is to find decompositions of spaces as a product or a wedge of other spaces, up to homotopy.

Definition 2.59. A fibration sequence $F \to X \to Y$ is said to *split* if the inclusion of the fibre $F \to X$ admits a homotopy inverse.

Dually, a cofibration sequence $X \to Y \to C$ is said to *split* if the quotient map to the cofibre $Y \to C$ admits a homotopy inverse.

Proposition 2.60. *(a) Suppose that the fibration sequence $F \to X \to Y$ splits. Then there is a homotopy equivalence*

$$X \simeq F \times Y.$$

(b) If the cofibration sequence $X \to Y \to C$ splits and X, Y and C are simply-connected, then there is a homotopy equivalence

$$Y \simeq X \vee C.$$

The existence of a homotopy inverse for $X \to Y$ in either the fibration or cofibration sequence in Proposition 2.60 does not guarantee a splitting. However, we do have the following.

Proposition 2.61. *(a) Let $F \to X \xrightarrow{f} Y$ be a homotopy fibration sequence in which X is an H-space and suppose f admits a homotopy inverse. Then there is a homotopy equivalence*

$$X \simeq F \times Y.$$

(b) Let $X \xrightarrow{f} Y \to C$ be a homotopy cofibration sequences in which X, Y and C are simply connected and B is a co-H-space. If f admits a homotopy inverse, then there is a homotopy equivalence

$$Y \simeq X \vee C.$$

Note that if $F \to X \xrightarrow{f} Y$ is a homotopy fibration sequence in which F is an H-space and $f \simeq *$, then the homotopy fibration $\Omega Y \to F \to X$ satisfies the hypotheses of 2.61 and hence we have

$$F \simeq \Omega Y \times X.$$

Similarly, if $X \xrightarrow{f} Y \to C$ is a homotopy cofibration sequences in which X, Y and C are simply-connected, C is a co-H-space and $f \simeq *$, then

$$C \simeq \Sigma X \vee Y.$$

We conclude this section with the notions of homotopy actions and coactions.

Definition 2.62. If Y is an H-space and X is a space, then a *homotopy action* of Y on X is a map $\phi\colon Y \times X \to X$ such that there are homotopy commutative diagrams

$$\begin{array}{ccc} X & & \\ {\scriptstyle j_2}\downarrow & \searrow^{=} & \\ Y \times X & \xrightarrow[\phi]{} & X, \end{array} \qquad \begin{array}{ccc} Y \times Y \times X & \xrightarrow{\mathrm{id}_Y \times \phi} & Y \times X \\ \downarrow{\scriptstyle \mu \times \mathrm{id}_X} & & \downarrow{\scriptstyle \phi} \\ Y \times X & \xrightarrow[\phi]{} & X, \end{array}$$

where j_2 denotes the inclusion into the second factor.

Dually, If X is a co-H-space and Y is a space, then a *homotopy coaction* of X on Y is a map $\psi\colon Y \to Y \vee X$ such that there are homotopy commutative diagrams

$$
\begin{array}{ccc}
Y & & \\
{\scriptstyle\psi}\downarrow & \searrow\!\!\!\!\!= & \\
Y \vee X & \xrightarrow[p_1]{} & Y,
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y \vee X \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y \vee \sigma} \\
Y \vee X & \xrightarrow{\psi \vee \mathrm{id}_X} & Y \vee X \vee X,
\end{array}
$$

where p_1 denotes the projection onto the first factor.

Example 2.63. In a principal fibration sequence $\Omega Y \to F \to X$ associated to a map $f\colon X \to Y$, there is an action ϕ_0 of the loop space ΩY on F given by

$$
\begin{aligned}
\phi_0\colon \quad \Omega Y \times F &\longrightarrow F \\
\big(\gamma, (x, \omega)\big) &\longmapsto (x, \gamma\omega),
\end{aligned}
$$

where $\gamma\omega \in \mathrm{Map}^*(I, X)$ denotes the concatenation of paths, i.e.

$$
\gamma\omega(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \omega(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}
$$

Dually, if $Y \to C \to \Sigma X$ is the principal cofibration associated to a map $f\colon X \to Y$, there is a coaction ψ_0 of the reduced suspension ΣX on C

$$
\psi_0\colon C \longrightarrow C \vee \Sigma X
$$

given by $\psi_0([y]) = ([y], *)$ for $[y] \in C$ and by

$$
\psi_0([x,t]) = \begin{cases} ([x, 2t], *), & \text{for } 0 \le t \le \frac{1}{2}, \\ (*, [x, 2t-1]), & \text{for } \frac{1}{2} \le t \le 1, \end{cases}
$$

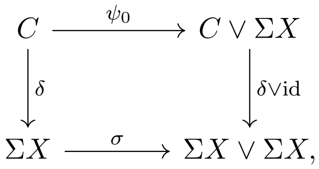
for $[x,t] \in C$.

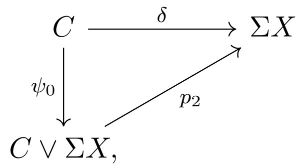
Proposition 2.64. *Given a homotopy fibration $F \to X \to Y$, let $\partial\colon \Omega Y \to F$ denote the fibration connecting map and let $\phi_0\colon \Omega Y \times F \to F$ be the homotopy action defined in Example 2.63. Then, of the following diagrams,*

$$
\begin{array}{ccc}
\Omega Y \times \Omega Y & \xrightarrow{\mu} & \Omega Y \\
{\scriptstyle \mathrm{id}\times\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\
\Omega Y \times F & \xrightarrow[\phi_0]{} & F,
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Omega Y & \xrightarrow{\partial} & F \\
{\scriptstyle j_1}\downarrow & \nearrow{\scriptstyle\phi_0} & \\
\Omega Y \times F, & &
\end{array}
$$

the square on the left is commutative and the triangle on the right is homotopy commutative.

Dually, if $X \to Y \to C$ is a homootpy cofibration, $\delta\colon C \to \Sigma X$ is the cofibration connecting map and $\psi_0\colon C \to C \vee \Sigma X$ denotes the homotopy coaction defined in Example 2.63, then, of the following diagrams,





the square on the left is commutative and the triangle on the right is homotopy commutative.

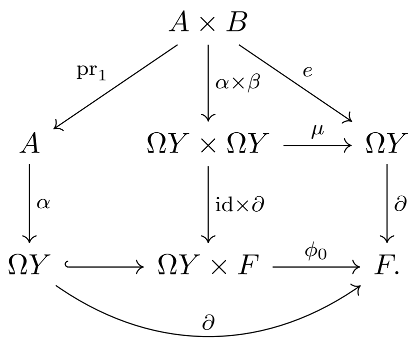
The next lemma will be needed in Chapter 6.

Lemma 2.65. *Let $F \to X \to Y$ be a homotopy fibration, where F is an H-space, and let $\partial\colon \Omega Y \to F$ be the homotopy fibration connecting map. Let $\alpha\colon A \to \Omega Y$ and $\beta\colon B \to \Omega Y$ be maps such that*

1. *$\mu \circ (\alpha \times \beta)\colon A \times B \to \Omega Y$ is a homotopy equivalence, where μ is the loop multiplication on ΩY;*
2. *$\partial \circ \beta\colon B \to F$ is nullhomotopic.*

Then the orders of ∂ and $\partial \circ \alpha$ coincide.

Proof. Let $\phi_0\colon \Omega Y \times F \to F$ denote the canonical homotopy action of the loop space ΩY on the homotopy fibre F as defined in Example 2.63, and let $e = \mu \circ (\alpha \times \beta)$. Consider the diagram



The left portion of the diagram commutes by the assumption that $\partial \circ \beta \simeq *$, while the right and bottom portions commute by properties of the canonical action θ. Therefore

$$\partial \simeq \partial \circ \alpha \circ \mathrm{pr}_1 \circ e^{-1},$$

and hence the orders of ∂ and $\partial \circ \alpha$ coincide. □

2.5 Localisation of spaces

Let us first recall the basics of localisation of the integers and of abelian groups.

Definition 2.66. Denote by Π the set of all prime numbers and let $P \subseteq \Pi$. We define the ring of *integers localised at* P to be the set

$$\mathbb{Z}_P := \left\{ \tfrac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\; b \neq 0, \text{ and } (b,p) = 1 \text{ for all } p \in P \right\} \subseteq \mathbb{Q},$$

where (b, p) denotes the greatest common divisor of b and p, with ring operations being the restriction of those for $\mathbb{Q}$.

From the definition, it follows that $\mathbb{Z}_\varnothing = \mathbb{Q}$ and $\mathbb{Z}_\Pi = \mathbb{Z}$. , Moreover, if $P \subset Q \subseteq \Pi$, then we have a homomorphism of rings $\mathbb{Z}_Q \to \mathbb{Z}_P$. In fact, there is a contravariant functor

$$\mathbb{Z}_- : \mathcal{P}(\Pi) \longrightarrow \mathbf{Rng},$$

where $\mathcal{P}(\Pi)$ is the poset category with objects the subsets of Π and morphisms given by inclusions of subsets, and **Rng** is the category of rings and ring homomorphisms.

Example 2.67. (a) If $P = \{p\}$ is a singleton, $\mathbb{Z}_P$ consists of all those fractions whose denominators are not divisible by p. In this case, we denote $\mathbb{Z}_P$ by $\mathbb{Z}_{(p)}$ and refer to it as the *integers localised at* p.

(b) If $P = \Pi \setminus \{p\}$, then $\mathbb{Z}_P$ consists of all those fractions whose denominators are powers of p. In this case, we denote $\mathbb{Z}_P$ by $\mathbb{Z}_p$ and refer to it as the *integers localised away from* p. To avoid confusion, we will always use the quotient notation for the integers mod p, that is, we will always write $\mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{Z}_p$.

Definition 2.68. Given a set of primes P, the *localisation* of an abelian group G at P is the tensor product $G \otimes \mathbb{Z}_P$ as $\mathbb{Z}$-modules. An abelian group G is said to be P*-local* if the map $g \mapsto g \otimes 1$ is a group isomorphism of G with $G \otimes \mathbb{Z}_P$.

The localisation of a group allows us to filter out certain primary torsion summands from the group. Indeed, if p is a prime and $r > 0$, we have

$$(\mathbb{Z}/p^r\mathbb{Z}) \otimes \mathbb{Z}_P = \begin{cases} \mathbb{Z}/p^r\mathbb{Z} & \text{if } p \in P, \\ 0 & \text{if } p \notin P. \end{cases}$$

The idea of localisation in homotopy theory, first introduced by Sullivan [SRe05] in the early 1970s, is to construct, from a given space X and a set of primes P, a space X_P whose homotopy groups are the P-localisations of the homotopy groups of X. Of course, for this to make sense, X should at least have an abelian fundamental group. In fact, localisations can be constructed for all nilpotent spaces (see Kane [Kan88, Chapter 9]), which include all connected H-spaces and all simply connected CW complexes.

Theorem 2.69 (Sullivan [SRe05])**.** *For any nilpotent space X, there exists a nilpotent space X_P, unique up to homotopy equivalence, together with a canonical map $L_P\colon X \to X_P$ which induces isomorphisms*

$$\pi_n(X_P) \cong \pi_n(X) \otimes \mathbb{Z}_P$$

for all $n \in \mathbb{N}$. □

The construction is functorial, in the sense that if $f\colon X \to Y$ is a map, there exists a map $f_P\colon X_P \to Y_P$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle L_P} & & \big\downarrow{\scriptstyle L_P} \\ X_P & \xrightarrow{f_P} & Y_P \end{array}$$

commute. Moreover, given that there is a canonical map $X_P \to X_Q$ if $P \subseteq Q$ [Kan88], we can assemble all the localisations functors into a bifunctor

$$\mathbf{NTop} \times \mathcal{P}(\Pi) \longrightarrow \mathbf{NTop},$$

covariant in the first variable and contravariant in the second, where **NTop** is the category of nilpotent spaces and continuous maps.

In line with our previous notation, we denote by $X_{(p)}$ the localisation of X at the prime p. We also write $X_{(0)}$ in place of $X_\varnothing$, the rationalisation of X.

Definition 2.70. Two spaces X and Y are said to be *mod-P homotopy equivalent* if their P-localisations are homotopy equivalent, that is, $X_P \simeq Y_P$. Similarly, we say that X and Y are *rationally equivalent* if $X_{(0)} \simeq Y_{(0)}$.

The most subtle point of localisation theory is reassembling local information into global information. Indeed, the existence of homotopy equivalences $X_{(p)} \simeq Y_{(p)}$ for all primes $p \in \Pi$ do not necessarily imply that $X \not\simeq Y$ integrally. For instance, it is the case that $\mathrm{Sp}(2) \simeq_{(p)} E_5$ for all primes p, but $\mathrm{Sp}(2) \not\simeq E_5$ (see [Kan88, Section 10.3]). So the local information does not, by itself, determine the global picture. However, if X is a nilpotent CW complex of finite type, then X can be recovered from knowledge of its rational and p-local homotopy types.

Proposition 2.71 (Hilton, Mislin, and Roitberg [HMR75])**.** *Let X be a nilpotent CW complex of finite type. Then, the following diagram is a homotopy pullback square*

$$\begin{array}{ccc} X & \xrightarrow{\left(\prod_{p\in\Pi} L_{(p)}\right)\circ\Delta} & \prod_{p\in\Pi} X_{(p)} \\ \downarrow & & \downarrow \\ X_{(0)} & \longrightarrow & \left(\prod_{p\in\Pi} X_{(p)}\right)_{(0)}. \end{array}$$

This means that X can be recovered as a subspace of $X_{(0)} \times \prod_{p\in\Pi} X_{(p)}$. With additional assumptions on X, we have the following.

Corollary 2.72. *If X is an H-space whose underlying space is a nilpotent CW complex of finite type with $\pi_j(X)$ finite for all $j \in \mathbb{N}$, then there is a homotopy equivalence*

$$X \longrightarrow \prod_{p\in\Pi} X_{(p)}.$$

Proof. As $\pi_j(X)$ is finite for all $j \in \mathbb{N}$, we have $\pi_j(X_{(0)}) = \pi_j(X)_{(0)} = 0$. Hence, both $X_{(0)}$ and $\left(\prod_{p\in\Pi} X_{(p)}\right)_{(0)}$ are contractible, whence it follows that the pullback square in Proposition 2.71 reduces to the desired equivalence. □

Recall that, for an H-group Y, we denote by $k\colon Y \to Y$ the kth power map. The localised $Y_{(p)}$ is also an H-group and the map $k_{(p)}\colon Y_{(p)} \to Y_{(p)}$ is homotopic to the kth power map $k\colon Y_{(p)} \to Y_{(p)}$.

If $(k, p) = 1$, then $k\colon Y_{(p)} \to Y_{(p)}$ is a homotopy equivalence. Let a/b be a reduced fraction in $\mathbb{Z}_{(p)}$. Then $(a, b) = 1$ and $(b, p) = 1$, so $b\colon Y_{(p)} \to Y_{(p)}$ is a homotopy equivalence, and we can define $a/b := a \circ b^{-1}\colon Y_{(p)} \to Y_{(p)}$. If, in addition, $(a, p) = 1$, then a/b is also a homotopy equivalence.

Lemma 2.73 (Hamanaka, Kono [HK06])**.** *Let X be a space and let Y be an H-group such that $\pi_j(Y)$ is finite for all j. Let $f \in [X, Y]$ be of finite order n, and let $k, l \in \mathbb{Z}$ satisfy $(n, k) = (n, l)$. Then, there exists a homotopy equivalence $\psi\colon Y \to Y$ making the following diagram commute*

$$\begin{array}{ccc} X & \xrightarrow{k\circ f} & Y \\ \| & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{l\circ f} & Y. \end{array}$$

Proof. Let n be the order of $f\colon X \to Y$, and let $k, l \in \mathbb{Z}$ be integers such that $(n, k) = (n, l)$. For an integer m, denote by $\nu_p(m)$ the exponent with which the prime p appears in the prime factorisation of m.

Define a map $h_p\colon Y_{(p)} \to Y_{(p)}$ by

$$h_p := \begin{cases} l/k & \text{if } \nu_p(n) > \nu_p(k), \\ \mathrm{id}_{Y_{(p)}} & \text{if } \nu_p(n) \le \nu_p(k) \end{cases}$$

and let $h := \prod_{p\in\Pi} h_p$. As $\nu_p\big((n,k)\big) = \min\big(\nu_p(n), \nu_p(k)\big)$, if $\nu_p(n) > \nu_p(k)$, then we have $\nu_p(n) > \nu_p(l)$ and $\nu_p(k) = \nu_p(l)$. Therefore, the order of $L_{(p)} \circ k \circ f$ is a power of p. If, on the other hand, $\nu_p(n) \le \nu_p(k)$, then $\nu_p(n) \le \nu_p(l)$ also, and hence the compositions $L_{(p)} \circ k \circ f$ and $L_{(p)} \circ l \circ f$ are both nullhomotopic.

Let $L\colon Y \to \prod_{p\in\Pi} Y_{(p)}$ denote the homotopy equivalence of Corollary 2.72 and define $\psi\colon Y \to Y$ by

$$\psi := L^{-1} \circ h \circ L,$$

which is clearly a homotopy equivalence. It remains to show that it makes the diagram in Lemma 2.73 commute. If $\nu_p(n) > \nu_p(k)$, then

$$\begin{aligned} h_p \circ L_{(p)} \circ k \circ f &\simeq h_p \circ k \circ L_{(p)} \circ f \\ &\simeq l \circ k^{-1} \circ k \circ L_{(p)} \circ f \\ &\simeq l \circ L_{(p)} \circ f \\ &\simeq L_{(p)} \circ l \circ f. \end{aligned}$$

If, on the other hand, $\nu_p(n) \le \nu_p(k)$, then $h_p = \mathrm{id}_{Y_{(p)}}$, and thus

$$h_p \circ L_{(p)} \circ k \circ f \simeq * \simeq h_p \circ L_{(p)} \circ l \circ f.$$

Hence, $h \circ L \circ k \circ f \simeq L \circ l \circ f$, and therefore $\psi \circ k \circ f = l \circ f$. □

The following is a weaker version of Lemma 2.73 which is useful when not enough information is known about the homotopy groups of Y, or when the condition of finiteness of the homotopy groups fails.

Lemma 2.74 (Theriault [The10a, Lemma 3.1])**.** *Let X be a space and let Y be an H-space with a homotopy inverse. Suppose that $f \in [X, Y]$ has finite order n. Then, if $k, l \in \mathbb{Z}$ are such that $(n,k) = (n,l)$, the homotopy fibres of $k \circ f$ and $k \circ l$ are homotopy equivalent when localised rationally or at any prime.*

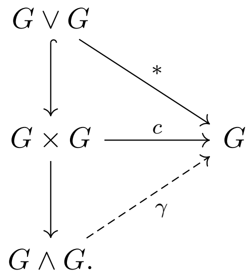
2.6 The Samelson product

The Samelson product, first introduced by Samelson [Sam53] in 1953, and its order in particular, plays a crucial role in the determination of the number of homotopy types of certain gauge groups, as we will discuss in the later sections.

Let G be a topological group. We denote by c the commutator map

$$\begin{aligned} c\colon G \times G &\to G \\ (g,h) &\mapsto ghg^{-1}h^{-1}. \end{aligned}$$

Note that the restriction $c|_{G\vee G}$ of c to $G \vee G := (* \times G) \cup (G \times *)$ is nullhomotopic, and hence c factors through the smash product $G \wedge G$ via a map γ as in the diagram



Let $f\colon X \to G$ and $g\colon Y \to G$ be pointed maps into a topological group G and define

$$\langle f, g\rangle = \gamma \circ (f \wedge g).$$

Proposition 2.75 (Whitehead [Whi78])**.** *The homotopy class of the map $\langle f, g\rangle$ depends only on the homotopy classes of f and g, thereby giving a well-defined operation*

$$\langle -, -\rangle\colon [X, G] \times [Y, G] \to [X \wedge Y, G].$$

Definition 2.76. For $f \in [X, G]$ and $g \in [Y, G]$, the map $\langle f, g\rangle \in [X \wedge Y, G]$ is called the *Samelson product* of the maps f and g.

It is clear that γ is null-homotopic if, and only if, the group G is homotopy commutative. Simply connected compact Lie groups of rank 1 and above are known [AJT60] to not be homotopy commutative. However, some Lie groups do become homotopy commutative after localisation at a prime. The question of precisely how and when this happens was answered by McGibbon in [McG84].

Proposition 2.77 (Whitehead [Whi78, Theorem 5.1])**.** *The Samelson product is bilinear with respect to the operation $+$ induced on $[X \wedge Y, G]$ by G. Namely,*

$$\begin{aligned} \langle f + g, h\rangle &= \langle f, h\rangle + \langle g, h\rangle, \\ \langle f, g + h\rangle &= \langle f, g\rangle + \langle f, h\rangle, \end{aligned}$$

for all f, g, h belonging to the appropriate homotopy sets.

In particular, if $k\colon G \to G$ denotes the kth power map on G, we have

$$\langle k \circ f, g\rangle = k \circ \langle f, g\rangle.$$

An interesting example of a family of non-trivial Samelson products associated to the unitary groups was found by Bott [Bot60].

Lemma 2.78. *Let r, s, t be positive integers such that $t = r + s + 1$. Let*

$$\alpha \in \pi_{2r+1}(\mathrm{U}(t)) \cong \mathbb{Z}, \quad \beta \in \pi_{2s+1}(\mathrm{U}(t)) \cong \mathbb{Z}, \quad \gamma \in \pi_{2t}(\mathrm{U}(t)) \cong \mathbb{Z}/t!\mathbb{Z}$$

denote suitable generators. Then $\langle \alpha, \beta \rangle = r!s!\gamma$. Furthermore, $\langle \alpha, \beta \rangle$ does not vanish unless $\gamma = 0$, that is, unless $r = s = 1$.

Chapter 3

Principal bundles and their gauge groups

3.1 Principal bundles

We begin with the most general definition of bundle from [Hus94], with the lead-up to the definition of vector and principal bundles being inspired primarily by [MM92].

Definition 3.1. A topological *bundle* is a triple (E, p, B) where E and B are topological spaces and $p\colon E \to B$ is a map.

In addition to the triple (E, p, B), the map $p\colon E \to B$, as well as the space E when B and p are understood, are common alternative notations for a bundle; we also say that E is a bundle over B when p is understood. The spaces E and B are called the *total* and *base space* of the bundle, respectively; the map p is called the bundle *projection*; the preimage $p^{-1}(\{b\}) \subseteq E$, with $b \in B$, is called the *fibre* over b and is denoted by E_b. We note that the fibres form a partition of the total space, that is, they are pairwise disjoint and $E = \bigcup_{b\in B} E_b$.

Example 3.2. The *product* (or *trivial*) *bundle* $(E \times F, \mathrm{pr}_1, E)$, where pr_1 denotes the projection onto the first factor, is a bundle whose fibre over e is $\{e\} \times F$ for every $e \in E$.

Whilst the notion of bundle as presented in Definition 3.1 can be used to state and prove some basic results, it is exceedingly broad and applies to some very pathological constructions. In order for it to be of any practical use, it will need to be enriched appropriately by imposing certain conditions, such as fibre uniformity and local triviality.

Definition 3.3. A *fibre bundle* (E, p, B, F) with *typical fibre* F is a bundle (E, p, B) such that $E_b \cong F$ for all $b \in B$.

Before introducing local triviality, let us first define some basic necessary concepts related to bundles.

Definition 3.4. If U is an open subset of B, then the *restriction* of the bundle (E, p, B) to U is the bundle $(E_U, p|_{E_U}, U)$, where $E_U := p^{-1}(U)$ in analogy with the notation for fibres.

Definition 3.5. A *bundle map* from (E, p, B) to (E', p', B') consists of a pair of maps $f\colon E \to E'$ and $g\colon B \to B'$ such that $gp = p'f$, that is, such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & E' \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{\;g\;} & B'. \end{array}$$

The commutativity of this diagram is equivalent to the requirement that, for each $b \in B$, the map f sends the fibre $E_b \subseteq E$ into the fibre $E'_{g(b)} \subseteq E'$. We note that, if p is surjective, the map f uniquely determines g by sending $b \in B$ to the unique element in the singleton $p'(f(E_b))$.

If we confine our attention to bundles having a fixed base space B, we obtain a stronger notion of map between bundles *as bundles over* B.

Definition 3.6. A *bundle map over* B (or B*-map*) from (E, p, B) to (E', p', B) is a map $f\colon E \to E'$ such that $p = p'f$, that is, such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B. & \end{array}$$

Clearly, there is a bijective correspondence between B-maps and bundle maps (f, g) with $g = \mathrm{id}_B$. For this reason, one also says that a B-map is a map $E \to E'$ that *covers the identity*. Unless otherwise stated, maps between bundles over the same base space are always assumed to be of this type.

In both cases, composition of bundle maps is defined in the obvious way.

Definition 3.7. The *composition* of bundle maps $(f, g)\colon (E, p, B) \to (E', p', B')$ and $(h, k)\colon (E', p', B') \to (E'', p'', B'')$ is $(hf, kg)\colon (E, p, B) \to (E'', p'', B'')$

$$\begin{array}{ccccc} E & \xrightarrow{\;f\;} & E' & \xrightarrow{\;h\;} & E'' \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p''} \\ B & \xrightarrow{\;g\;} & B' & \xrightarrow{\;k\;} & B''. \end{array}$$

Consequently, the composition of two B-maps f and g is just gf. We can thus form the corresponding category **Bun** of bundles and bundle maps and, for each space B, the category **Bun**$/B$ of bundles over B and maps of bundles over B. We observe that **Bun**$/B$ is not a full subcategory of **Bun** since any bundle map (f, g) between two bundles over B with $g \neq \mathrm{id}_B$ is a morphism in **Bun** but not in **Bun**$/B$.

Definition 3.8. A bundle map $(f, g)\colon (E, p, B) \to (E', p', B')$ is a *bundle isomorphism* if there exists a bundle map $(h, k)\colon (E', p', B') \to (E, p, B)$ such that

$$(h, k) \circ (f, g) = \mathrm{id}_{(E,p,B)} \qquad \text{and} \qquad (f, g) \circ (h, k) = \mathrm{id}_{(E',p',B')}.$$

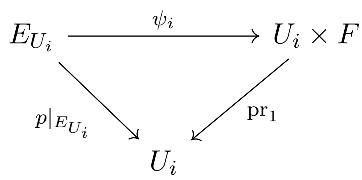
Given bundles (E, p, B) and (E', p', B) in **Bun**$/B$, an *isomorphism as bundles over* B is simply a homeomorphism of the total spaces $f\colon E \to E'$ such that $p'f = p$.

We write $(E, p, B) \cong (E', p', B')$ to denote isomorphisms in **Bun**, or even $E \cong E'$ whenever p, B, p', and p are understood, and similarly for **Bun**$/B$.

We can now state the local triviality condition for bundles.

Definition 3.9. A bundle (E, p, B) is said to be *locally trivial* if there is an open cover $\{U_i\}_{i\in I}$ of B such that, for each $i \in I$, the restricted bundle $(E_{U_i}, p|_{E_{U_i}}, U_i)$ is isomorphic to the trivial bundle $(U_i \times F, \mathrm{pr}_1, U_i)$ for some fixed space F, as bundles over U_i.

More explicitly, this means that there exists a collection $\{\psi_i\}_{i\in I}$ of homeomorphisms such that, for each $i \in I$, the following diagram commutes.



Definition 3.10. The collection $\{\psi_i\}_{i\in I}$ above is called a *local trivialisation*.

Example 3.11. The product bundle $(E\times F, \mathrm{pr}_1, E)$ is trivial is the sense of being *globally trivial*, in that it admits a local trivialisation as defined above where I can be taken to be a singleton.

From this point onward, we will only consider locally trivial bundles. Observe that locally trivial bundles over a numerable base space are a special case of fibrations. Also note that the imposition of a local trivialisation as above on (E, p, B) immediately implies that (E, p, B) is a fibre bundle with typical fibre F. Indeed, for each $b \in B$, letting U_i be such that $b \in U_i$, we have

$$E_b = (p|_{E_{U_i}})^{-1}(\{b\}) = \psi_i^{-1} \circ \mathrm{pr}_1^{-1}(\{b\}) = \psi_i^{-1}(\{b\} \times F)$$

and since $\{b\} \times F \cong F$ and each ψ_i is a homeomorphism, we conclude that there are homeomorphisms $E_b \cong F$ for all $b \in B$.

Furthermore, note that the restriction of $\psi_i^{-1} : U_i \times F \to E_{U_i}$ to $\{b\} \times F$ is a homeomorphism onto its image E_b. Hence, supposing $b \in U_i \cap U_j$ and letting $\psi_{i,b}^{-1} : F \to E_b$ denote the homeomorphism $f \mapsto \psi_i^{-1}(b, f)$, we have

$$\psi_{j,b} \circ \psi_{i,b}^{-1} \in \mathrm{Homeo}(F),$$

where $\psi_{j,b} : E_b \to F$ is the inverse of $\psi_{j,b}^{-1}$. Whenever $U_i \cap U_j \neq \varnothing$ we can thus define a (not necessarily continuous) map

$$\begin{aligned} \psi_{ij} : U_i \cap U_j &\to \mathrm{Homeo}(F) \\ b &\mapsto \psi_{j,b} \circ \psi_{i,b}^{-1}. \end{aligned}$$

Definition 3.12. The collection of maps $\{\psi_{ij}\}_{i,j \in I}$ are called the *transition functions* for the given local trivialisation.

Transition functions play a key role in the introduction of the concept of structure group of a bundle.

Definition 3.13. Let (E, p, B, F) be a locally trivial fibre bundle and let G be a topological subgroup of $\mathrm{Homeo}(F)$. We say that (E, p, B, F) is a fibre bundle with *structure group* G if, for each transition function ψ_{ij}, we have $\psi_{ij}(b) \in G$ for all $b \in U_i \cap U_j$ and ψ_{ij} is a continuous map of $U_i \cap U_j$ into G.

The notion of fibre bundle with structure group underlies both of the two most important cases of bundles, namely vector bundles and principal bundles. In fact, their definitions consist merely in the imposition of appropriate conditions on the typical fibre and the structure group of a fibre bundle. To illustrate this, let us first define vector bundles.

Definition 3.14. A *vector bundle* of (finite) *rank* n is a fibre bundle (E, p, B, F) with F an n-dimensional vector space and structure group $\mathrm{GL}(F)$.

Our main focus will be on principal bundles.

Definition 3.15. A *principal bundle* is a fibre bundle (E, p, B, F) where F is a topological group and whose structure group G is the subgroup of $\mathrm{Homeo}(F)$ consisting of homeomorphisms h such that $h(f_1 f_2) = h(f_1) f_2$ for all $f_1, f_2 \in F$.

The term *principal G-bundle* is also used to emphasize the structure group. Note that the map $h \mapsto h(1_F)$, where 1_F denotes the group identity of F, is a homeomorphism of G with F, so the structure group of a principal bundle is also its typical fibre. This observation is the basis for the following equivalent definition of a principal bundle.

Definition 3.16. A *principal G-bundle* over B is a locally trivial fibre bundle (E, p, B, G) together with a right action $\rho\colon E \times G \to E$ of G on E such that

(a) the action ρ is free, continuous, and fibre-preserving, that is

$$\forall e \in E, \forall g \in G, \qquad p(eg) = p(e);$$

(b) for each $\psi_i\colon E_{U_i} \to U_i \times G$ in a local trivialisation, we have

$$\forall e \in E_{U_i}, \forall g \in G, \qquad \psi_i(eg) = \psi_i(e)g,$$

where the action on the right-hand side is given by $(b, g') \mapsto (b, g'g)$.

That these two definitions are equivalent is shown as follows. Given the first definition, for any $(e, g) \in E \times G$ let i be such that $e \in E_b \subset E_{U_i}$ and set

$$\rho(e, g) := \psi_{i,b}^{-1}\big(\psi_{i,b}(e)g\big),$$

the action on the right-hand side being the same as in Definition 3.16 (b) above. These assignments give a well-defined right action $\rho\colon E \times G \to E$ of G on E which satisfies conditions (a) and (b) of Definition 3.16.

Conversely, condition (b) of Definition 3.16 implies that the transition functions associated to the local trivialisation in (b) satisfy

$$\forall b \in U_i \cap U_j, \forall g_1, g_2 \in G, \qquad \psi_{ij}(b)(g_1, g_2) = \psi_{ij}(b)(g_1)g_2,$$

which is the defining condition in Definition 3.15.

While the first definition makes the role of the structure group more readily apparent, the advantages of emphasising the action ρ, as in the second definition, can be seen in the next proposition.

Proposition 3.17 (Husemöller[Hus94])**.** *Let (E, p, B, G) be a principal G-bundle and let ρ denote the action of G on E. Then the following hold true.*

(a) The orbits of ρ coincide with the fibres of the bundle;

(b) The restriction of ρ to each of the fibres is transitive;

(c) The orbit space E/G is homeomorphic to the base space B;

(d) For each $e \in E$, the map $G \to E_{p(e)}$ given by $g \mapsto eg$ is a homeomorphism.

3.2 Homotopy classification of principal bundles

The following is a construction that plays a key role in the classification of bundles.

Definition 3.18. Let $\pi\colon P \to X$ be a principal G-bundle and let $f\colon Y \to X$ be a map. The *pullback* of P along f is the space

$$f^*P := \big\{(y,p) \in Y \times P \mid f(y) = \pi(p)\big\}.$$

We can now use f^*P as the total space of a principal G-bundle over Y, also called the *pullback* of $\pi\colon P \to X$ along f.

Proposition 3.19 (Husemöller [Hus94, Proposition 4.4.1])**.** *With the above notation,* $\mathrm{pr}_1\colon f^*P \to Y$ *is a principal G-bundle and there is a commutative diagram*

$$\begin{array}{ccc} f^*P & \xrightarrow{\mathrm{pr}_2} & P \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X. \end{array}$$

We also say that $\mathrm{pr}_1\colon f^*P \to Y$ is *induced* from $\pi\colon P \to X$ by f. We remark that the pullback operation allows us to construct, for any topological group G, a contravariant functor

$$\mathrm{Prin}_G\colon \mathbf{Top} \to \mathbf{Set}$$

associating to each topological space X the set of isomorphism classes of principal G-bundles over X, and to each map $f\colon X \to Y$ the map induced by the pullback construction

$$\begin{aligned} f^*\colon \mathrm{Prin}_G(Y) &\to \mathrm{Prin}_G(X) \\ P &\mapsto f^*P. \end{aligned}$$

The pullblack construction also behaves well with respect to homotopy.

Proposition 3.20. *If $\pi\colon P \to X$ is a principal G-bundle and $f, g\colon Y \to X$ are two maps such that $f \simeq g$, then $f^*P \cong g^*P$ as principal G-bundles over Y.*

It follows that, for any space Y and for any fixed principal G-bundle over X, there is a well-defined map of sets

$$\begin{aligned} [Y, X]_{\text{free}} &\to \mathrm{Prin}_G(Y) \\ f &\mapsto f^*P \end{aligned}$$

which is, however, not a bijection in general.

Definition 3.21. A principal G-bundle $P \to X$ is called a *universal G-bundle* if, for any space Y, the above function is a bijection.

It is a result of Milnor [Mil56] that, for any topological group G, there exists a universal principal G-bundle $\pi_G \colon \mathrm{E}G \to \mathrm{B}G$.

For a fixed G, both $\mathrm{E}G$ and $\mathrm{B}G$ are unique up to homotopy equivalence. Hence, by a slight abuse of language, we call $\mathrm{B}G$ the *classifying space* of G, and we refer to a map $f \colon X \to \mathrm{B}G$ as the *classifying map* of the bundle $f^*(\mathrm{E}G) \to X$ induced by it. Furthermore, we have the following.

Theorem 3.22 (Steenrod [Ste51, Theorem 19.4])**.** *A principal G-bundle $P \to X$ is a universal G-bundle if, and only if, its total space P is contractible.*

It follows that if $\mathrm{E}G \to \mathrm{B}G$ is a universal G-bundle, there is a homotopy equivalence

$$G \simeq \Omega \mathrm{B}G.$$

3.3 The gauge group of a principal bundle

We are now in a position to define the main object of study in this thesis.

Definition 3.23. Let $\pi \colon P \to X$ be a principal G-bundle. The *gauge group* $\mathcal{G}(P)$ of P is the topological group of G-equivariant homeomorphisms of P that cover the identity. The group operation is composition of maps and the topology is that inherited as a subspace of $\mathrm{Map}(P, P)$. In symbols, we have

$$\mathcal{G}(P) := \big\{ f \in \mathrm{Homeo}(P) \mid \pi f = \pi \text{ and } \forall p \in P, \forall g \in G,\ f(pg) = f(p)g \big\}.$$

Elements of $\mathcal{G}(P)$ are sometimes called *gauge transformations* of P. We remark that a gauge transformation $f \colon P \to P$ preserves the following structures on P: its topological structure, as f is a homeomorphism; its right G-space structure, as f is G-equivariant; and its fibre structure, as the requirement $\pi f = \pi$ implies that $f(P_x) = P_x$ for all $x \in X$.

We also note a slight redundancy in Definition 3.23 as, given $f \in \mathrm{Map}(P, P)$, the condition $\pi f = \pi$ together with the G-equivariance of f automatically imply (see Husemöller [Hus94]) that f^{-1} exists and is continuous.

If X is pointed, we have the corresponding notion of pointed gauge group.

Definition 3.24. Let $\pi \colon P \to X$ be a principal G-bundle over a pointed space X. The *pointed gauge group* $\mathcal{G}^*(P)$ of P is the topological subgroup of $\mathcal{G}(P)$ consisting of gauge transformations which restrict to the identity on the fibre over the basepoint. In symbols, we thus have

$$\mathcal{G}^*(P) := \big\{ f \in \mathcal{G}(P) \mid f|_{P_*} = \mathrm{id}_{P_*} \big\},$$

where, of course, $P_* = \pi^{-1}(\{*\})$.

The gauge group $\mathcal{G}(P)$, pointed gauge group $\mathcal{G}^*(P)$, and structure group G of a principal G-bundle satisfy the following relation.

Proposition 3.25. *Let $\pi\colon P \to X$ be a principal G-bundle. Then, there is a short exact sequence of groups*

$$1 \longrightarrow \mathcal{G}^*(P) \longrightarrow \mathcal{G}(P) \longrightarrow G \longrightarrow 1.$$

Proof. We show, equivalently, that $\mathcal{G}^*(P)$ is a normal subgroup of $\mathcal{G}(P)$, and that the quotient group $\mathcal{G}(P)/\mathcal{G}^*(P)$ is isomorphic to G.

Firstly, let $f \in \mathcal{G}^*(P)$ and $h \in \mathcal{G}(P)$. Then, $f(p) = p$ for all $p \in P_*$, the fibre over the basepoint, and $h(p) \in P_*$ for all $p \in P_*$, as $h(P_*) = P_*$. Hence, for every $p \in P_*$, we have $f\big(h(p)\big) = h(p)$, that is, $(h^{-1}fh)(p) = p$, so $h^{-1}fh \in \mathcal{G}^*(P)$.

For the second claim, fix $p \in P_*$ and let $f \in \mathcal{G}(P)$. As the action of G on P is free and transitive when restricted to the fibres, there exists a unique $g_f \in G$ such that $f(p) = pg_f$. We can therefore define a map

$$\begin{aligned} \phi\colon \mathcal{G}(P) &\to G \\ f &\mapsto g_f, \end{aligned}$$

which is easily seen to be a surjective homomorphism with kernel $\mathcal{G}^*(P)$. By the first isomorphism theorem, ϕ induces an isomorphism $\mathcal{G}(P)/\mathcal{G}^*(P) \cong G$. □

There is an alternative characterisation of the gauge group of a principal bundle which is often useful in arguments. Let $\mathrm{Ad}(G)$ denote the right G-space G with the right action given by conjugation in G, and denote by $\mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big)$ the subspace of $\mathrm{Map}(P, G)$ consisting of G-equivariant maps $P \to \mathrm{Ad}(G)$. So

$$\mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big) := \big\{ \phi \in \mathrm{Map}(P, G) \mid \forall p \in P, \forall g \in G,\ \phi(pg) = g^{-1}\phi(p)g \big\}.$$

Proposition 3.26. *Let $\pi\colon P \to X$ be a principal G-bundle. There is a bijection between the gauge group $\mathcal{G}(P)$ and the space $\mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big)$.*

Proof. Let $f \in \mathcal{G}(P)$ and let $p \in P$. As the elements p and $f(p)$ belong to the same fibre, there exists (Proposition 3.17) a unique element $g_p \in G$ such that $f(p) = pg_p$. Define a map $\phi_f\colon P \to G$ by $p \mapsto g_p$ and let $h \in G$ be arbitrary. Then, using this notation, we have

$$(ph)g_{ph} = f(ph) = f(p)h = (pg_p)h,$$

which, as the action of G on P is free, implies that $hg_{ph} = g_p h$. Therefore, we have that $g_{ph} = h^{-1}g_p h$, that is, $\phi_f(ph) = h^{-1}\phi_f(p)h$. Hence, $\phi_f \in \mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big)$.

Conversely, let $\phi \in \mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big)$ and define $f_\phi\colon P \to P$ by $p \mapsto p\phi(p)$. First, observe that, since the action of G on P is free, f_ϕ is a continuous bijection with continuous inverse given by $p \mapsto p\phi(p)^{-1}$. Moreover, as G permutes the elements within each fibre, we have

$$\pi\big(f_\phi(p)\big) = \pi\big(p\phi(p)\big) = \pi(p)$$

for all $p \in P$, so $\pi f_\phi = \pi$, and for any $g \in G$,

$$f_\phi(pg) = (pg)\phi(pg) = (pg)g^{-1}\phi(p)g = p\phi(p)g = f_\phi(p)g,$$

so f_ϕ is G-equivariant, and thus $f_\phi \in \mathcal{G}(P)$. Finally, as $\phi_{f_\phi} = \phi$ and $f_{\phi_f} = f$, we have the desired bijection. □

We remark that, as G induces a group structure on $\mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big)$ in the obvious way, the bijection in Proposition 3.26 can, in fact, be promoted to a continuous group isomorphism (Husemöller [Hus94, Remark 7.1.4]). The advantages of this identification can be seen in the following proposition.

Proposition 3.27. *Let $\pi\colon P \to X$ be a principal G-bundle. If either $\pi\colon P \to X$ is the trivial bundle, or G is abelian, then*

$$\mathcal{G}(P) \cong \mathrm{Map}(X, G).$$

Proof. Suppose first that $\pi\colon P \to X$ is trivial, that is, $P = X \times G$, $\pi = \mathrm{pr}_1$, and the action of G on $X \times G$ is right multiplication on G. By Proposition 3.26, we have

$$\mathcal{G}(X \times G) \cong \mathrm{Map}_G\big(X \times G, \mathrm{Ad}(G)\big).$$

Let $f \in \mathrm{Map}_G\big(X \times G, \mathrm{Ad}(G)\big)$. Then, for all $(x, g) \in X \times G$, we find

$$f(x, g) = f\big((x, 1_G)g\big) = g^{-1}f(x, 1_G)g.$$

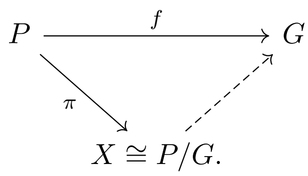
That is, f is completely determined by $f \circ \mathrm{in}_1 \in \mathrm{Map}(X, G)$, where in_1 denotes the injection $x \mapsto (x, 1_G)$, and thus

$$\mathcal{G}(X \times G) \cong \mathrm{Map}(X, G).$$

If, on the other hand, G is abelian, then

$$\mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big) = \big\{f \in \mathrm{Map}(P, G) \mid \forall p \in P, \forall g \in G,\ f(pg) = f(p)\big\}.$$

Hence, by Proposition 3.17, every $f \in \mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big)$ factors through X as



Conversely, for every $f \in \mathrm{Map}(X, G)$, we clearly have $f \circ \pi \in \mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big)$. These correspondences are continuous and inverses to each other. □

By the characterisation of global sections of bundles as maps from the total space into the typical fibre, given in Husemöller [Hus94, Theorem 4.8.1], Proposition 3.26 implies that $\mathcal{G}(P)$ can also be identified with the group of global sections $\Gamma\big(\mathrm{Ad}(P)\big)$ of the adjoint bundle of P. Indeed, this is precisely how Atiyah and Bott chose to define $\mathcal{G}(P)$ in [AB83, p. 43]. This approach was also used by Kono and Tsukuda in [KT10] to obtain a localised version of Proposition 3.27. However, we shall not make use of this characterisation of $\mathcal{G}(P)$ in this thesis.

The next result is fundamental for the work that will be presented in the next chapters, and can also be found in Gottlieb [Got72].

Theorem 3.28 (Atiyah and Bott [AB83, Proposition 2.4])**.** *Let $f\colon X \to \mathrm{B}G$ be a map into the classifying space of G and let $\pi\colon P \to X$ be the principal G-bundle induced by the map f. Then, there is a homotopy equivalence*

$$\mathrm{B}\mathcal{G}(P) \simeq \mathrm{Map}_f(X, \mathrm{B}G),$$

where $\mathrm{Map}_f(X, \mathrm{B}G) := \{h \in \mathrm{Map}(X, \mathrm{B}G) \mid h^*(\mathrm{E}G) \cong P\}$ *is the connected component of the space* $\mathrm{Map}(X, \mathrm{B}G)$ *containing the classifying map* f*.*

Proof. Let $\pi_G\colon \mathrm{E}G \to \mathrm{B}G$ denote the universal G-bundle and consider the space of G-equivariant maps $\mathrm{Map}_G(P, \mathrm{E}G)$. There is a right action of $\mathcal{G}(P)$ on the space $\mathrm{Map}_G(P, \mathrm{E}G)$ defined as follows. Let $\phi \in \mathrm{Map}_G\big(P, \mathrm{Ad}(G)\big) \cong \mathcal{G}(P)$ and let $h \in \mathrm{Map}_G(P, \mathrm{E}G)$. Define $h\phi \in \mathrm{Map}_G(P, \mathrm{E}G)$ by

$$(h\phi)(p) := h(p)\phi(p),$$

where the action on right-hand side is that of G on $\mathrm{E}G$, implying that this right $\mathcal{G}(P)$-action is also free.

The quotient map to the orbit space of this action can be identified with the map

$$q\colon \mathrm{Map}_G(P, \mathrm{E}G) \to \mathrm{Map}_f(X, \mathrm{B}G)$$

sending $h\colon P \to \mathrm{E}G$ to the unique $q(h)\colon X \to \mathrm{B}G$ making the diagram

$$\begin{array}{ccc} P & \xrightarrow{h} & \mathrm{E}G \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_G} \\ X & \xrightarrow{q(h)} & \mathrm{B}G \end{array}$$

commute (see the remark following Definition 3.5). Note that the range of q is $\mathrm{Map}_f(X, \mathrm{B}G)$ as, clearly, $q(h)$ is also a classifying map for $\pi\colon P \to X$. The map

$$q\colon \mathrm{Map}_G(P, \mathrm{E}G) \to \mathrm{Map}_f(X, \mathrm{B}G), \tag{3.3.1}$$

together with the right $\mathcal{G}(P)$-action defined above, can be shown to be a principal $\mathcal{G}(P)$-bundle (see [Hus94, Remark 7.2.2]).

Moreover, as $\mathrm{E}G$ is contractible by Theorem 3.22, so is $\mathrm{Map}_G(P, \mathrm{E}G)$. Thus, again by Theorem 3.22, (3.3.1) is a universal $\mathcal{G}(P)$-bundle, and therefore

$$\mathrm{B}\mathcal{G}(P) \simeq \mathrm{Map}_f(X, \mathrm{B}G),$$

as was sought. □

Chapter 4

Homotopy classification of gauge groups

The first key step in the classification of the homotopy types of gauge groups is to exhibit the gauge groups as the homotopy fibres of certain maps.

4.1 Gauge groups as homotopy fibres

For a fixed space X and a topological group G, we are interested in the problem of classifying the homotopy types of the gauge groups $\mathcal{G}(P)$ as P ranges over all principal G-bundles over X.

If there exists a G-equivariant bundle isomorphism $P \cong P'$, then conjugation by such an isomorphism yields a homeomorphism $\mathcal{G}(P) \cong \mathcal{G}(P')$. It therefore suffices to let P range over a set of representatives of all the isomorphism classes of G-bundles over X.

Suppose $X = S^n$. Since S^n is paracompact, the set of isomorphism classes of principal G-bundles over S^n is in bijection with the set $[S^n, \mathrm{B}G]_{\text{free}}$ of free homotopy classes of maps from S^n to $\mathrm{B}G$, the classifying space of G. If G is a simply-connected, simple Lie group, then as $\mathrm{B}G$ is also simply-connected there is a bijection $[S^n, \mathrm{B}G]_{\text{free}} \cong [S^n, \mathrm{B}G]$ and, furthermore, we have bijections

$$[S^n, \mathrm{B}G] \cong \pi_n(\mathrm{B}G) \cong \pi_{n-1}(G).$$

It is possible to introduce a labelling for the gauge group of a principal G-bundle over S^n, in terms of which the classification results can be conveniently stated, which requires one index for each generator in a presentation of $\pi_{n-1}(G)$.

In this thesis, we will only consider cases where $\pi_{n-1}(G)$ is infinite cyclic on one generator[1]. This happens, for instance, whenever $n = 4$ and G is any of the classical matrix Lie groups other than SO(4). When $n > 4$, examples still exist (e.g., $n = 5$ and $G = \mathrm{SO}(6)$ or Spin(6)), but are less frequent. A table of homotopy groups of Lie groups, together with some relevant general results, can be found in [Mim95].

We proceed as follows. Let $\epsilon\colon S^{n-1} \to G$ denote a generator of $\pi_{n-1}(G)$. Each isomorphism class of G-bundles is represented by the bundle $P_k \to S^n$ induced by pulling back the universal G-bundle along the classifying map $k\bar{\epsilon}\colon S^n \to \mathrm{B}G$, where $\bar{\epsilon}$ denotes the adjoint of ϵ and generates $\pi_n(\mathrm{B}G)$.

$$\begin{array}{ccc} P_k & \longrightarrow & \mathrm{E}G \\ \downarrow & & \downarrow \\ S^n & \xrightarrow{k\bar{\epsilon}} & \mathrm{B}G \end{array}$$

We let $\mathcal{G}_k(P)$ (or simply $\mathcal{G}_k$ when the context is clear) denote the gauge group of $P_k \to S^n$.

By Theorem 3.28, there is a homotopy equivalence $\mathrm{B}\mathcal{G}_k \simeq \mathrm{Map}_k(S^n, \mathrm{B}G)$, the latter space being the k-th component of $\mathrm{Map}(S^n, \mathrm{B}G)$, meaning the connected component containing the classifying map $k\bar{\epsilon}$.

There is an evaluation fibration

$$\mathrm{Map}^*_k(S^n, \mathrm{B}G) \longrightarrow \mathrm{Map}_k(S^n, \mathrm{B}G) \xrightarrow{\mathrm{ev}} \mathrm{B}G,$$

where ev evaluates a map at the basepoint of S^n and the fibre is the k-th component of the pointed mapping space $\mathrm{Map}^*(S^n, \mathrm{B}G)$. This fibration extends to a homotopy fibration sequence

$$\mathcal{G}_k \longrightarrow G \longrightarrow \mathrm{Map}^*_k(S^n, \mathrm{B}G) \longrightarrow \mathrm{B}\mathcal{G}_k \longrightarrow \mathrm{B}G,$$

where we used the equivalences $\mathrm{B}\mathcal{G}_k \simeq \mathrm{Map}_k(S^n, \mathrm{B}G)$, $\Omega\mathrm{B}G \simeq G$, and $\Omega\mathrm{B}\mathcal{G}_k \simeq \mathcal{G}_k$.

Moreover, by adapting an argument of [Sut92] and [KT96], we have the following.

Lemma 4.1. *For each $k \in \mathbb{Z}$, there is a homotopy equivalence*

$$\mathrm{Map}^*_k(S^n, \mathrm{B}G) \simeq \mathrm{Map}^*_0(S^n, \mathrm{B}G)$$

between the connected component of $\mathrm{Map}^*(S^n, \mathrm{B}G)$ *containing the map* $k\bar{\epsilon}\colon S^n \to \mathrm{B}G$ *and that containing the constant map* $*\colon S^n \to \mathrm{B}G$.

[1]See, e.g., [MS19] for a case in which two indices are needed.

Proof. Let $\sigma\colon S^n \to S^n \vee S^n$ be the comultiplication giving the standard co-H-group structure on S^n. Recall from Section 2.3 that there is an induced H-group structure on $\mathrm{Map}^*(S^n, \mathrm{B}G)$ with multiplication μ and homotopy inverse ν.

For $f \in \mathrm{Map}^*(S^n, \mathrm{B}G)$, let $\alpha(f)$ and $\beta(f)$ be the composites

$$\alpha(f)\colon S^n \xrightarrow{\sigma} S^n \vee S^n \xrightarrow{\nu(k\bar{\epsilon}) \vee f} \mathrm{B}G \vee \mathrm{B}G \xrightarrow{\nabla} \mathrm{B}G$$

and

$$\beta(f)\colon S^n \xrightarrow{\sigma} S^n \vee S^n \xrightarrow{k\bar{\epsilon} \vee f} \mathrm{B}G \vee \mathrm{B}G \xrightarrow{\nabla} \mathrm{B}G.$$

In other words, we set $\alpha = \mu(\nu(k\bar{\epsilon}), -)$ and $\beta = \mu(k\bar{\epsilon}, -)$.

It is clear that if $f \simeq k\bar{\epsilon}$, then $\alpha(f) \simeq *$, and that if $f \simeq *$, then $\beta(f) \simeq k\bar{\epsilon}$, so that we can consider α and β as maps

$$\alpha\colon \mathrm{Map}^*_k(S^n, \mathrm{B}G) \longrightarrow \mathrm{Map}^*_0(S^n, \mathrm{B}G)$$

and

$$\beta\colon \mathrm{Map}^*_0(S^n, \mathrm{B}G) \longrightarrow \mathrm{Map}^*_k(S^n, \mathrm{B}G),$$

respectively. Furthermore, observe that, for $f \in \mathrm{Map}^*_k(S^n, \mathrm{B}G)$, we have

$$\beta(\alpha(f)) = \mu\big(k\bar{\epsilon}, \mu(\nu(k\bar{\epsilon}), f)\big) \simeq \mu\big(\mu(k\bar{\epsilon}, \nu(k\bar{\epsilon})), f\big) \simeq \mu(*, f) \simeq f.$$

So $\beta\alpha \simeq \mathrm{id}$ and, similarly, $\alpha\beta \simeq \mathrm{id}$. □

By the pointed exponential law, the space $\mathrm{Map}^*_0(S^n, \mathrm{B}G)$ is homotopy equivalent to the space $\mathrm{Map}^*_0(S^{n-1}, G)$, more commonly denoted as $\Omega^{n-1}_0 G$.

We therefore have the following homotopy fibration sequence

$$\mathcal{G}_k \longrightarrow G \xrightarrow{\partial_k} \Omega^{n-1}_0 G \longrightarrow \mathrm{B}\mathcal{G}_k \longrightarrow \mathrm{B}G,$$

which exhibits the gauge group $\mathcal{G}_k$ as the homotopy fibre of the map ∂_k. This is a key observation, as it suggests that the homotopy theory of the gauge groups $\mathcal{G}_k$ depends on the maps ∂_k. In fact, more is true.

Lemma 4.2 (Lang [Lan73, Theorem 2.6])**.** *The adjoint of $\partial_k\colon G \to \Omega^{n-1}_0 G$ is homotopic to the Samelson product $\langle k\epsilon, 1\rangle\colon S^{n-1} \wedge G \to G$, where $\epsilon \in \pi_{n-1}(G)$ is a generator and 1 denotes the identity map on G.*

By the bilinearity of the Samelson product, we have

$$\langle k\epsilon, 1\rangle \simeq k\langle \epsilon, 1\rangle,$$

and hence, taking adjoints once more, $\partial_k \simeq k\partial_1$.

Thus, each of the gauge groups $\mathcal{G}_k$ is the homotopy fibre of the map ∂_1 composed with the k-th power map on $\Omega_0^{n-1}G$.

Hence, if there is a finite m for which $m\partial_1 \simeq *$ in the group $[G, \Omega_0^{n-1}G]$, then applying Theriault's Lemma 2.74 we obtain the forward direction of the classification results for $\mathcal{G}_k$. Namely, if $k, l \in \mathbb{Z}$ and $(m, k) = (m, l)$, then there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ after localising rationally or at any prime.

Furthermore, if G is such that $\pi_s(\Omega_0^{n-1}G)$ is finite for all $s \geq 1$, then Hamanaka and Kono's Lemma 2.73 improves the local equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ to an integral homotopy equivalence.

The order of ∂_1 is often determined by studying the Samelson product $\langle \epsilon, 1 \rangle$, whose order coincides with that of ∂_1 by Lang's Lemma 4.2.

For the converse direction of the classification results, one looks for a suitable homotopy invariant of $\mathcal{G}_k$ which is not independent of k. For example, in Kono's 1991 paper [Kon91] on the classification of SU(2)-gauge groups over S^4, a suitable invariant was found to be the second homotopy group. Indeed, from Kono's calculation that

$$\pi_2(\mathcal{G}_k) \cong \mathbb{Z}/(12, k)\mathbb{Z},$$

it follows at once that a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ of SU(2)-gauge groups over S^4 implies that $(12, k) = (12, l)$.

4.2 Homotopy types of SO(3)-gauge groups over S^4

In preparation for the more general case of PU(p)-gauge groups over S^{2i}, with $p \geq 3$ and $2 \leq i \leq p$, examined in the next chapter, we restate here the classification result for the homotopy types of SO(3)-gauge groups over S^4 (note that SO(3) $\cong$ PU(2)) with proofs, in order to emphasise the analogy with more general results presented in Chapter 5. The reference for this section is the paper [KKKT07] by Kamiyama, Kishimoto, Kono, and Tsukuda where the following result is obtained.

Theorem 4.3. *Let $\mathcal{G}_k$ denote the gauge group of the principal* SO(3)*-bundle over S^4 classified by $k \in \mathbb{Z}$. Then there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ if, and only if, we have $(12, k) = (12, l)$.*

Remark 4.4. Additionally, [KKKT07] also determines the number of path-components and the fundamental group of $\mathcal{G}_k$.

Following the general strategy laid out in Section 4.1, we first seek to determine the order of the boundary map $\partial_1 \colon \mathrm{SO}(3) \to \Omega_0^3 \mathrm{SO}(3)$ by studying its triple adjoint, the Samelson product

$$\langle \epsilon_3, 1 \rangle \colon S^3 \wedge \mathrm{SO}(3) \to \mathrm{SO}(3),$$

where ϵ_3 is a generator of $\pi_3(\mathrm{SO}(3)) \cong \mathbb{Z}$ and 1 denotes the identity map on $\mathrm{SO}(3)$.

Lemma 4.5. *There is a homotopy equivalence*

$$S^2 \wedge \mathrm{SO}(3) \simeq (S^2 \wedge \mathbb{R}P^2) \vee S^5.$$

Proof. By a result of Atiyah [Ati61], the parallelisability of $\mathrm{SO}(3)$ implies that there is a stable homotopy equivalence $\mathrm{SO}(3) \simeq \mathbb{R}P^2 \vee S^3$. Hence, by the Freudenthal suspension theorem, the cofibration

$$S^2 \wedge \mathbb{R}P^2 \xrightarrow{1 \wedge \iota} S^2 \wedge \mathbb{R}P^3 \longrightarrow S^5,$$

where $\iota \colon \mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$ is the inclusion of the 2-skeleton, splits as

$$S^2 \wedge \mathbb{R}P^3 \simeq (S^2 \wedge \mathbb{R}P^2) \vee S^5.$$

The result now follows since $\mathbb{R}P^3 \cong \mathrm{SO}(3)$. □

Suspending once more, we obtain

$$S^3 \wedge \mathrm{SO}(3) \simeq (S^3 \wedge \mathbb{R}P^2) \vee S^6. \tag{4.2.1}$$

The homotopy equivalence (4.2.1) induces a group isomorphism

$$[S^3 \wedge \mathrm{SO}(3), \mathrm{SO}(3)] \cong [S^3 \wedge \mathbb{R}P^2, \mathrm{SO}(3)] \oplus \pi_6(\mathrm{SO}(3)).$$

In particular, the Samelson product $\langle \epsilon_3, 1 \rangle \colon S^3 \wedge \mathrm{SO}(3) \to \mathrm{SO}(3)$ factors as

$$\begin{array}{ccc} S^3 \wedge \mathrm{SO}(3) & \xrightarrow{\langle \epsilon_3, 1 \rangle} & \mathrm{SO}(3) \\ \downarrow \simeq & & \| \\ (S^3 \wedge \mathbb{R}P^2) \vee S^6 & \xrightarrow{\langle \epsilon_3, \iota \rangle \vee \alpha} & \mathrm{SO}(3) \end{array}$$

for some map $\alpha \colon S^6 \to \mathrm{SO}(3)$.

We investigate the orders of the two summands $\langle \epsilon_3, \iota \rangle$ and α separately.

Lemma 4.6. *We have* $[S^3 \wedge \mathbb{R}P^2, \mathrm{SO}(3)] \cong \mathbb{Z}/4\mathbb{Z}$.

Proof. Applying the functor $[S^3 \wedge \mathbb{R}P^2, -]$ to the fibration sequence

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(3) \longrightarrow \mathrm{B}(\mathbb{Z}/2\mathbb{Z})$$

yields an isomorphism

$$[S^3 \wedge \mathbb{R}P^2, \mathrm{SO}(3)] \cong [S^3 \wedge \mathbb{R}P^2, \mathrm{Sp}(1)],$$

since $S^3 \wedge \mathbb{R}P^2$ is 3-connected.

In turn, applying the functor $[S^3 \wedge \mathbb{R}P^2, -]$ to the fibration sequence

$$\Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(1)) \longrightarrow \mathrm{Sp}(1) \longrightarrow \mathrm{Sp}(\infty) \longrightarrow \mathrm{Sp}(\infty)/\mathrm{Sp}(1)$$

yields a further isomorphism

$$[S^3 \wedge \mathbb{R}P^2, \mathrm{Sp}(1)] \cong [S^3 \wedge \mathbb{R}P^2, \mathrm{Sp}(\infty)],$$

since $S^3 \wedge \mathbb{R}P^2$ is 5-dimensional and $\mathrm{Sp}(\infty)/\mathrm{Sp}(1)$ is 6-connected.

By the exponential law, we therefore have

$$[S^3 \wedge \mathbb{R}P^2, \mathrm{Sp}(1)] \cong [S^4 \wedge \mathbb{R}P^2, \mathrm{BSp}(\infty)] \cong \mathbb{Z}/4\mathbb{Z},$$

where the latter isomorphism is due to Adams [Ada62]. □

Let ϵ_1 denote a generator of $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$ and let $q\colon \mathrm{U}(2) \to \mathrm{SO}(3)$ be the natural quotient map. Let $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_3$ represent generators of $\pi_1(\mathrm{U}(2))$ and $\pi_3(\mathrm{U}(2))$ respectively, such that $q_*(\tilde{\epsilon}_1) = \epsilon_1$ and $q_*(\tilde{\epsilon}_3) = \epsilon_3$. Then, as q_* is an isomorphism in π_n for $n \geq 2$, the element $\langle \epsilon_3, \epsilon_1 \rangle = q_*(\langle \tilde{\epsilon}_3, \tilde{\epsilon}_1 \rangle)$ is a generator of $\pi_4(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.7. *The Samelson product* $\langle \epsilon_3, \iota \rangle\colon S^3 \wedge \mathbb{R}P^2 \to \mathrm{SO}(3)$ *has order* 4.

Proof. Let $j\colon S^1 \to \mathbb{R}P^2$ be the inclusion of the 1-skeleton. Then $\iota \circ j \simeq \epsilon_1$ and we therefore have a homotopy commutative diagram

$$\begin{array}{ccc} S^3 \wedge S^1 & \xrightarrow{\langle \epsilon_3, \epsilon_1 \rangle} & \mathrm{SO}(3) \\ \big\downarrow{\scriptstyle 1 \wedge j} & & \big\| \\ S^3 \wedge \mathbb{R}P^2 & \xrightarrow{\langle \epsilon_3, \iota \rangle} & \mathrm{SO}(3). \end{array}$$

Since $\langle \epsilon_3, \epsilon_1 \rangle$ generates $\pi_4(\mathrm{SO}(3))$ and hence has order 2, the element $\langle \epsilon_3, \iota \rangle$ is non-trivial and not divisible by 2 in the group $[S^3 \wedge \mathbb{R}P^2, \mathrm{SO}(3)] \cong \mathbb{Z}/4\mathbb{Z}$. It must therefore be a generator and thus of order 4. □

Lemma 4.8. *The order* $\alpha\colon S^6 \to \mathrm{SO}(3)$ *is a divisor of* 12 *and is divisible by* 3.

Proof. After localisation at any odd prime p we have $\alpha \simeq_{(p)} q_*(\langle \tilde{\epsilon}_3, \tilde{\epsilon}_3 \rangle)$ and, by [Sam54], the latter is a generator of $\pi_6(\mathrm{U}(2)) \cong \mathbb{Z}/12\mathbb{Z}$. □

Corollary 4.9. *The order of* $\langle \epsilon_3, 1\rangle \colon S^3 \wedge \mathrm{SO}(3) \to \mathrm{SO}(3)$ *is* 12.

This essentially completes the "if" direction of Theorem 4.3. For the "only if", we have the following.

Lemma 4.10. *For each* $k \in \mathbb{Z}$*, we have* $\pi_2(\mathcal{G}_k) \cong \mathbb{Z}/(12, k)\mathbb{Z}$.

Proof. The quotient map $\mathrm{SU}(2) \to \mathrm{PU}(2) \cong \mathrm{SO}(3)$ induces an isomorphism

$$\pi_2(\mathcal{G}_k(\mathrm{SU}(2))) \cong \pi_2(\mathcal{G}_k(\mathrm{SO}(3)))$$

and we know the left-hand side to be isomorphic to $\mathbb{Z}/(12, k)\mathbb{Z}$ by Kono's calculation for SU(2)-gauge groups in [Kon91]. □

Therefore, we have the following.

Proof of Theorem 4.3. By Corollary 4.9, the order of the boundary map ∂_1 is 12. Since the homotopy groups $\pi_j(\Omega^3_0 \mathrm{SO}(3))$ are finite for all j, it follows from Lemma 2.73 that if $(12, k) = (12, l)$, then $\mathcal{G}_k \simeq \mathcal{G}_l$.

The converse direction follows at once from Lemma 4.10. □

Chapter 5

Homotopy types of $\mathrm{PU}(p)$-gauge groups over S^{2i}

The results of this chapter have been published as [Rea21]. In this chapter, we examine how the close relationship between the groups $\mathrm{SU}(n)$ and $\mathrm{PU}(n)$ is reflected in the homotopy properties of the gauge groups of the corresponding bundles, particularly when n is a prime. We do this by generalising certain results relating the classification of $\mathrm{PU}(n)$-gauge groups to that of $\mathrm{SU}(n)$-gauge groups from Chapter 4 for the case $n = 2$, and from [HKKS16] for the case $n = 3$.

First, observe that for $2 \leq m \leq n$, we have

$$[S^m, \mathrm{BPU}(n)] \cong \pi_{m-1}(\mathrm{PU}(n)) \cong \begin{cases} \mathbb{Z} & \text{for } m \text{ even}, \\ 0 & \text{for } m \text{ odd}, \end{cases}$$

since, for $k \geq 2$, we have $\pi_k(\mathrm{PU}(n)) \cong \pi_k(\mathrm{SU}(n)) \cong \pi_k(\mathrm{U}(n))$, the latter homotopy groups being determined for $k < n$ in, for instance, [Bot57].

Hence, for m odd, there is only one isomorphism class of $\mathrm{PU}(n)$-bundles over S^m, and hence only possible homotopy type for the corresponding gauge groups. We thus consider the case $m = 2i$, and use that notation $\mathcal{G}_{i,k}$ for the gauge group of the principal $\mathrm{PU}(n)$-bundle over S^{2i} classified by $k \in \pi_{2i-1}(\mathrm{PU}(n))$.

As in Section 4.1, there is a homotopy fibration

$$\mathcal{G}_{i,k} \longrightarrow \mathrm{PU}(n) \xrightarrow{\partial_{i,k}} \Omega_0^{2i-1}\mathrm{PU}(n) \longrightarrow \mathrm{B}\mathcal{G}_{i,k} \longrightarrow \mathrm{B}PU(n)$$

where $\partial_{i,k} \simeq k\partial_{i,1}$. We seek to determine the order of $\partial_{i,1}$ by studying the Samelson product $\langle \epsilon_i, 1 \rangle$, where ϵ_i denotes a generator of $\pi_{2i-1}(\mathrm{PU}(n))$.

5.1 Samelson products on PU(p)

Our first main result compares certain Samelson products on SU(p) and PU(p), with $p \geq 3$ a prime.

Having fixed $n \geq 3$ and $2 \leq i \leq n$, let $\delta_i \colon S^{2i-1} \to \mathrm{SU}(n)$ denote the generator of

$$\pi_{2i-1}(\mathrm{SU}(n)) \cong \mathbb{Z}$$

corresponding to the generator ϵ_i of $\pi_{2i-1}(\mathrm{PU}(n))$. That is, such that $q_*(\delta_i) = \epsilon_i$, where q denotes the quotient map $q \colon \mathrm{SU}(n) \to \mathrm{PU}(n)$.

In this section, we wish to compare the orders of the Samelson products $\langle \delta_i, 1 \rangle$ and $\langle \epsilon_i, 1 \rangle$ on SU(n) and PU(n), respectively. First, observe that there is a commutative diagram

$$\begin{array}{ccc} S^{2i-1} \wedge \mathrm{SU}(n) & \xrightarrow{\langle \delta_i, 1 \rangle} & \mathrm{SU}(n) \\ \big\downarrow{\scriptstyle 1\wedge q} & & \big\downarrow{\scriptstyle q} \\ S^{2i-1} \wedge \mathrm{PU}(n) & \xrightarrow{\langle \epsilon_i, 1 \rangle} & \mathrm{PU}(n) \end{array} \tag{5.1.1}$$

and recall the following property of the quotient map q.

Lemma 5.1. *The quotient map $q \colon \mathrm{SU}(n) \to \mathrm{PU}(n)$ induces a p-local homotopy equivalence $\mathrm{SU}(n) \simeq_{(p)} \mathrm{PU}(n)$ for any prime p which does not divide n.*

Proof. The long exact sequence of homotopy groups induced by the fibration

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}$$

implies that SU(n) simply-connected, so $\pi_1(\mathrm{SU}(n)) \cong 0$, and that we have

$$\pi_k(\mathrm{SU}(n)) \cong \begin{cases} 0 & \text{if } k \text{ is even} \\ \mathbb{Z} & \text{if } k \text{ is odd} \end{cases}$$

for $2 \leq k < 2n$. The quotient map $q \colon \mathrm{SU}(n) \to \mathrm{PU}(n)$ induces a homotopy fibration sequence

$$\cdots \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{SU}(n) \longrightarrow \mathrm{PU}(n) \longrightarrow \mathrm{B}(\mathbb{Z}/n\mathbb{Z})$$

where $\mathrm{B}(\mathbb{Z}/n\mathbb{Z})$ is an infinite-dimensional lens space $L_n^\infty = S^\infty/(\mathbb{Z}/n\mathbb{Z})$.

As $\pi_k(\mathbb{Z}/n\mathbb{Z}) \cong 0$ for all $k \geq 1$, applying the functor π_0 to the above sequence, we find that the quotient map $q \colon \mathrm{SU}(n) \to \mathrm{PU}(n)$ induces isomorphisms of homotopy groups $\pi_k(\mathrm{SU}(n)) \cong \pi_k(\mathrm{PU}(n))$ for all $k \geq 2$, as well as an isomorphism of fundamental groups $\pi_1(\mathrm{PU}(n)) \cong \pi_1(\mathrm{B}\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Furthermore, as SU($n$) and PU($n$) are both connected, q also induces a bijection $\pi_0(\mathrm{SU}(n)) \cong \pi_0(\mathrm{PU}(n))$.

Hence, after localisation at any prime p which does not divide n, we have

$$\pi_1(\mathrm{PU}(n)_{(p)}) \cong 0,$$

and thus the quotient map q induces isomorphisms on the p-localised homotopy groups, and hence it is a p-local homotopy equivalence by [HMR75, Theorem 3B(ii)]. □

Lemma 5.2. *If the prime p does not divide n, then the p-primary components of the orders of the Samelson products $\langle \delta_i, 1\rangle$ and $\langle \epsilon_i, 1\rangle$ coincide.*

Proof. Let p be a prime which does not divide n. Then q is a p-local homotopy equivalence by Lemma 5.1, and hence the commutativity of (5.1.1) yields

$$\langle \delta_i, 1\rangle_{(p)} = q_{(p)}^{-1} \circ \langle \epsilon_i, 1\rangle_{(p)} \circ (1 \wedge q_{(p)}),$$

so the p-primary components of the orders of $\langle \delta_i, 1\rangle$ and $\langle \epsilon_i, 1\rangle$ coincide. □

Hence, when n is prime, the orders of $\langle \delta_i, 1\rangle$ and $\langle \epsilon_i, 1\rangle$ coincide, except possibly in the number of factors of n appearing in their respective prime decompositions.

Lemma 5.3. *For any n, the quotient map $q\colon \mathrm{SU}(n) \to \mathrm{PU}(n)$ induces an isomorphism*

$$q_*\colon [S^{2i-1} \wedge \mathrm{SU}(n), \mathrm{SU}(n)] \to [S^{2i-1} \wedge \mathrm{SU}(n), \mathrm{PU}(n)].$$

Proof. Recall that $q\colon \mathrm{SU}(n) \to \mathrm{PU}(n)$ fits into a homotopy fibration sequence

$$\cdots \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{SU}(n) \xrightarrow{q} \mathrm{PU}(n) \longrightarrow \mathrm{B}(\mathbb{Z}/n\mathbb{Z}).$$

Since $\mathbb{Z}/n\mathbb{Z}$ is discrete, applying the functor $[S^{2i-1} \wedge \mathrm{SU}(n), -]$ yields

$$\cdots \longrightarrow 0 \longrightarrow [S^{2i-1} \wedge \mathrm{SU}(n), \mathrm{SU}(n)] \xrightarrow{q_*} [S^{2i-1} \wedge \mathrm{SU}(n), \mathrm{PU}(n)] \longrightarrow 0,$$

whence the statement. □

Lemma 5.4. *The order of $\langle \delta_i, 1\rangle$ divides the order of $\langle \epsilon_i, 1\rangle$.*

Proof. Let p be a prime. If p^k divides the order of $\langle \delta_i, 1\rangle$ for some $k \geq 1$, then p^k also divides the order of the composite $q \circ \langle \delta_i, 1\rangle_{(p)}$ by Lemma 5.3. It then follows, by the commutativity of (5.1.1), that the order of $\langle \epsilon_i, 1\rangle_{(p)}$ is at least p^k. □

5.2 A mod-p decomposition of PU(p) and of its double suspension

For the remainder of this section, we shall restrict to considering $\mathrm{PU}(n)$ when n is an odd prime p.

Since $\mathrm{SU}(p)$ is the universal cover of $\mathrm{PU}(p)$ and $H_*(\mathrm{SU}(p);\mathbb{Z})$ is torsion-free, by [KK08, Theorem 1.1] we have the following decomposition of $\mathrm{PU}(p)$.

Lemma 5.5. *For an odd prime p, there is a p-local homotopy equivalence*

$$\mathrm{PU}(p) \simeq_{(p)} L \times \prod_{j=2}^{p-1} S^{2j-1}$$

where L is an H-space with $\pi_1(L) \cong \mathbb{Z}/p\mathbb{Z}$. □

Remark 5.6. Note that for $n = 2$ we have the result that $\mathrm{PU}(2)$ is diffeomorphic to $\mathbb{R}P^3$, the latter space being equivalently described as the lens space $S^3/(\mathbb{Z}/2\mathbb{Z})$.

Let $\alpha\colon L_{(p)} \to \mathrm{PU}(p)_{(p)}$ be the inclusion. Then we can write the equivalence of Lemma 5.5 as

$$L_{(p)} \times \prod_{j=2}^{p-1} S^{2j-1}_{(p)} \xrightarrow{\alpha \times \prod_j \epsilon_{j\,(p)}} \left(\mathrm{PU}(p)_{(p)}\right)^{p-1} \xrightarrow{\mu} \mathrm{PU}(p)_{(p)},$$

where μ is the group multiplication in $\mathrm{PU}(p)_{(p)}$. We note that this composite is equal to the product

$$(\alpha \circ \mathrm{pr}_1) \cdot \prod_{j=2}^{p-1} (\epsilon_{j\,(p)} \circ \mathrm{pr}_j)$$

in the group $[L_{(p)} \times \prod_{j=2}^{p-1} S^{2j-1}_{(p)}, \mathrm{PU}(p)_{(p)}]$, where pr_j denotes the projection onto the jth factor.

Lemma 5.7. *With the above notation, the localised Samelson product*

$$\langle \epsilon_i, 1\rangle_{(p)}\colon S^{2i-1}_{(p)} \wedge \mathrm{PU}(p)_{(p)} \to \mathrm{PU}(p)_{(p)}$$

is trivial if, and only if, each of $\langle \epsilon_{i\,(p)}, \alpha\rangle$ and $\langle \epsilon_i, \epsilon_j\rangle_{(p)}$, for $2 \le j \le p-1$, are trivial.

Proof. By [HKKS16, Lemmas 3.3 and 3.4], $\langle \epsilon_i, 1\rangle_{(p)}$ is trivial if, and only if, both $\langle \epsilon_{i\,(p)}, \alpha\rangle$ and $\langle \epsilon_i, \prod_j \epsilon_j\rangle_{(p)}$ are trivial. Applying the same lemmas to the second factor a further $p-3$ times gives the statement. □

We therefore calculate the groups $[S^{2i-1} \wedge L, \mathrm{PU}(p)]_{(p)}$ and, for $2 \le j \le p-1$, the homotopy groups $\pi_{2i+2j-2}(\mathrm{PU}(p))_{(p)}$ in order to get an upper bound on the order of the Samelson product $\langle \epsilon_i, 1\rangle_{(p)}$.

Remark 5.8. Note that, as the product on the right-hand side in Lemma 5.5 reduces to a single factor when $p = 2$, Lemma 5.7 does not say anything useful in the $p = 2$ case, since $\mathrm{PU}(2) \cong L$.

Lemma 5.9. *For* $2 \leq i \leq p$ *and* $2 \leq j \leq p-1$*, the group* $\pi_{2i+2j-2}\big(\mathrm{PU}(p)\big)_{(p)}$ *has exponent at most* p*.*

Proof. Decompose $\mathrm{PU}(p)$ as in Lemma 5.5. Observe that, by [KK08, Proposition 2.2], we have $\pi_n(L) \cong \pi_n(S^{2p-1})$ for $n \geq 2$, and hence

$$\pi_{2i+2j-2}\big(\mathrm{PU}(p)\big)_{(p)} \cong \pi_{2i+2j-2}\left(L \times \prod_{k=2}^{p-1} S^{2k-1}\right)_{(p)} \cong \bigoplus_{k=2}^{p} \pi_{2i+2j-2}(S^{2k-1})_{(p)}.$$

By Toda [Tod66, Theorem 7.1], if $k \geq 2$ and $r < 2p(p-1)-2$, the p-primary component of $\pi_{(2k-1)+r}(S^{2k-1})$ is either 0 or $\mathbb{Z}/p\mathbb{Z}$. Since $2i+2j-2 \leq 4p-4$ and

$$4p-4 < 2p(p-1) - 2 + (2k-1)$$

for all $k \geq 2$, the statement follows. □

For the next part of our calculation, we will need a certain mod-p decomposition of $\Sigma^2 L$ which will, in turn, require some cohomological information. The mod-p cohomology algebra of $\mathrm{PU}(n)$, with p any prime and n arbitrary, was determined by Baum and Browder in [BB65, Corollary 4.2]. In particular, we have:

Lemma 5.10. *For* p *an odd prime, there is an algebra isomorphism*

$$H^*(\mathrm{PU}(p);\mathbb{Z}/p\mathbb{Z}) \cong \Lambda(x_1, x_3, \ldots, x_{2p-3}) \otimes \frac{\mathbb{Z}/p\mathbb{Z}[y]}{(y^p)},$$

with $|x_d| = d$*,* $|y| = 2$*, and* $\beta(x_1) = y$*. where* β *is the Bockstein operator.* □

For $m \geq 2$, denote by $P^m(p)$ the mod-p Moore space defined as the homotopy cofibre of the degree p map

$$S^{m-1} \xrightarrow{p} S^{m-1} \longrightarrow P^m(p)$$

on the sphere S^{m-1}. In other words, $P^m(p) = S^{m-1} \cup_p e^m$. Note that, by extending the cofibre sequence to the right, we see that $\Sigma P^m(p) \simeq P^{m+1}(p)$.

Lemma 5.11. *For* p *an odd prime, there is a* p*-local homotopy equivalence*

$$\Sigma L \simeq_{(p)} A \vee \bigvee_{k=2}^{p-1} P^{2k+1}(p),$$

with $H_*(A;\mathbb{Z}/p\mathbb{Z})$ *generated by* $\{u, v, w\}$*, with* $|u| = 2$*,* $|v| = 3$*,* $|w| = 2p$*, and subject to the relation* $\beta(v) = u$*.*

Proof. By decomposing PU(p) as in Lemma 5.5, taking mod-p cohomology and comparing with Lemma 5.10, we obtain

$$H^*(L;\mathbb{Z}/p\mathbb{Z}) \cong \Lambda(x_1) \otimes \frac{\mathbb{Z}/p\mathbb{Z}[y]}{(y^p)}.$$

Since $H^*(L;\mathbb{Z}/p\mathbb{Z})$ is of finite type and self-dual, we have an isomorphism

$$H^*(L;\mathbb{Z}/p\mathbb{Z}) \cong H_*(L;\mathbb{Z}/p\mathbb{Z})$$

of Hopf algebras. Then, as $H_*(L;\mathbb{Z}/p\mathbb{Z})$ is primitively generated and L is a connected H-space (being a retract of PU(p)), by [Coh76, Theorem 4.1] there is a decomposition

$$\Sigma L \simeq_{(p)} A_1 \vee A_2 \vee \cdots \vee A_{p-1},$$

with each summand A_j having homology $H_*(A_j;\mathbb{Z}/p\mathbb{Z})$ generated by the suspensions of monomials in $H_*(L;\mathbb{Z}/p\mathbb{Z})$ of length j (modulo $p-1$), where by length of a monomial one means the number of (not necessarily distinct) factors in that monomial.

Let $\overline{x}_1$ and $\overline{y}$ denote the duals of x_1 and y, and let σ denote the suspension isomorphism for homology. Then, $H_*(A_1;\mathbb{Z}/p\mathbb{Z})$ is generated by $\sigma(\overline{x}_1)$, $\sigma(\overline{y})$, and $\sigma(\overline{x}_1\overline{y}^{p-1})$, in degrees 2, 3, and $2p$, respectively. Furthermore, by the stability of the Bockstein operator β, we also have $\beta(\sigma(\overline{y})) = \sigma(\overline{x}_1)$.

On the other hand, for $j \neq 1$, the homology $H_*(A_j;\mathbb{Z}/p\mathbb{Z})$ is generated by the elements $\sigma(\overline{x}_1\overline{y}^{j-1})$ and $\sigma(\overline{y}^j)$, in degrees $2j$ and $2j+1$, respectively, subject to the relation $\beta(\sigma(\overline{y}^j)) = \sigma(\overline{x}_1\overline{y}^{j-1})$. As the mod-$p$ homotopy type of Moore spaces is uniquely characterised by their mod-p homology, we must have $A_j \simeq P^{2j+1}(p)$ for $j \neq 1$, yielding the decomposition in the statement. □

With A as in Lemma 5.11, we have as follows.

Lemma 5.12. *There is a p-local homotopy equivalence*

$$\Sigma A \simeq_{(p)} P^4(p) \vee S^{2p+1}.$$

Proof. Localise at p throughout. By looking at the degrees of the generators of $H_*(A;\mathbb{Z}/p\mathbb{Z})$ in Lemma 5.11, we see that the 3-skeleton of A is $P^3(p)$.

Let $f\colon S^{2p-1} \to P^3(p)$ be the attaching map of the top cell of A, and let F be the homotopy fibre of $\rho\colon P^3(p) \to S^3$, the pinch map to the top cell of $P^3(p)$. As $\pi_{2p-1}(S^3) \cong 0$, the map f lifts through F via some map $\lambda\colon S^{2p-1} \to F$, as in the diagram

$$\begin{array}{ccc} & & F \\ & \overset{\lambda}{\nearrow} & \downarrow \\ S^{2p-1} & \xrightarrow{f} & P^3(p) \\ & \underset{*}{\searrow} & \downarrow{\scriptstyle \rho} \\ & & S^3. \end{array}$$

Let $j\colon S^3 \to P^4(p)$ be the inclusion of the bottom cell and let $S^3\{p\}$ be the homotopy fibre of the degree p map on S^3. As j has order p, there is a homotopy fibration diagram

$$\begin{array}{ccccccc} \Omega S^3 & \longrightarrow & S^3\{p\} & \longrightarrow & S^3 & \xrightarrow{p} & S^3 \\ \downarrow{\scriptstyle \Omega j} & & \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle j} \\ \Omega P^4(p) & = & \Omega P^4(p) & \longrightarrow & * & \longrightarrow & P^4(p) \end{array}$$

which defines a map $s\colon S^3\{p\} \to \Omega P^4(p)$. For connectivity reasons, the suspension map $P^3(p) \xrightarrow{E} \Omega P^4(p)$ factors as the composite $P^3(p) \xrightarrow{\iota} S^3\{p\} \xrightarrow{s} \Omega P^4(p)$, where ι is the inclusion of the bottom Moore space. Furthermore, there is a homotopy fibration diagram

$$\begin{array}{ccccc} F & \longrightarrow & P^3(p) & \xrightarrow{\rho} & S^3 \\ \downarrow & & \downarrow{\scriptstyle \iota} & & \| \\ \Omega S^3 & \longrightarrow & S^3\{p\} & \longrightarrow & S^3. \end{array}$$

Putting this together gives a commutative diagram

$$\begin{array}{ccccccc} & & F & \longrightarrow & \Omega S^3 & = & \Omega S^3 \\ & \overset{\lambda}{\nearrow} & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega j} \\ S^{2p-1} & \xrightarrow{f} & P^3(3) & \xrightarrow{\iota} & S^3\{p\} & \xrightarrow{s} & \Omega P^4(p). \end{array}$$

Thus $E \circ f$ factors through Ωj, implying that Σf factors as the composite

$$S^{2p} \xrightarrow{\hat{f}} S^3 \xrightarrow{j} P^4(p)$$

for some map $\hat{f}$.

As $\pi_{2p}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$, we must have $\hat{f} = t\alpha$, where α is a generator of $\pi_{2p}(S^3)$. If Σf were essential, then $t \neq 0$. However, the element α would then be detected by the Steenrod operation $\mathcal{P}^1$ in the cohomology of ΣA. This would, in turn, imply that $\mathcal{P}^1$ were non-trivial in $H^*(A;\mathbb{Z}/p\mathbb{Z})$, and hence in $H^*(\mathrm{PU}(p);\mathbb{Z}/p\mathbb{Z})$. However, $\mathcal{P}^1(H^*(\mathrm{PU}(p);\mathbb{Z}/p\mathbb{Z})) = 0$, and thus we must have had $\Sigma f \simeq *$. □

Remark 5.13. The decomposition of the iterated suspension of L resulting from combining Lemmas 5.11 and 5.12 corresponds to (4.2.1). It is interesting to note that, in passing from the $p = 2$ case to the general p case, of the two real projective spaces in the equation

$$S^3 \wedge \mathbb{R}P^3 \simeq (S^3 \wedge \mathbb{R}P^2) \vee S^6,$$

appearing in Chapter 4, we now see the former to be a special 3-dimensional case of a lens space, while the latter is a special 2-dimensional Moore space.

Lemma 5.14. *The exponent of the group* $[S^{2i-1} \wedge L, \mathrm{PU}(p)]_{(p)}$ *is at most* p.

Proof. By the decompositions in Lemmas 5.11 and 5.12, we have

$$\begin{aligned}[S^{2i-1} \wedge L, \mathrm{PU}(p)]_{(p)} &\cong \big[S^{2i-2} \wedge \big(A \vee \textstyle\bigvee_{k=2}^{p-1} P^{2k+1}(p)\big), \mathrm{PU}(p)\big]_{(p)} \\ &\cong \big[S^{2i-3} \wedge \big(S^{2p+1} \vee \textstyle\bigvee_{k=1}^{p-1} P^{2k+2}(p)\big), \mathrm{PU}(p)\big]_{(p)} \\ &\cong \pi_{2i+2p-2}(\mathrm{PU}(p))_{(p)} \oplus \bigoplus_{k=1}^{p-1} [P^{2k+2i-1}(p), \mathrm{PU}(p)]_{(p)}.\end{aligned}$$

Since $2i+2p-2 \leq 4p-2 < 2p(p-1)+1$ for $p \geq 3$, the group $\pi_{2i+2p-2}(\mathrm{PU}(p))_{(p)}$ consists of elements of order at most p by the same argument as in Lemma 5.9.

On the other hand, by [Nei80, Theorem 7.1], the groups $[P^{2k+2i-1}(p), \mathrm{PU}(p)]$ have exponent at most p (since, for $m \geq 3$, the identity on $P^m(p)$ has order p), whence the statement. □

Combining Lemmas 5.7, 5.9, and 5.14 we obtain the following statement.

Lemma 5.15. *The order of the Samelson product*

$$\langle \epsilon_i, 1 \rangle_{(p)} \colon S^{2i-1}_{(p)} \wedge \mathrm{PU}(p)_{(p)} \to \mathrm{PU}(p)_{(p)}$$

is at most p. □

We now have all the ingredients necessary to prove the main result of this chapter.

Theorem 5.16. *Let* p *be an odd prime and let* $2 \leq i \leq p$. *Let* ϵ_i *and* δ_i *denote generators of* $\pi_{2i-1}(\mathrm{PU}(p))$ *and* $\pi_{2i-1}(\mathrm{SU}(p))$, *respectively. Then the orders of the Samelson products*

$$\langle \epsilon_i, 1 \rangle \colon S^{2i-1} \wedge \mathrm{PU}(p) \to \mathrm{PU}(p)$$

and

$$\langle \delta_i, 1 \rangle \colon S^{2i-1} \wedge \mathrm{SU}(p) \to \mathrm{SU}(p),$$

where 1 *denotes the appropriate identity map, coincide.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} S^{2i-1} \wedge S^{2(p-i)+1} & \xrightarrow{\langle \eta_i, \eta_{p-i-1} \rangle} & \mathrm{U}(p) \\ \big\downarrow{\scriptstyle 1 \wedge \delta_{p-i-1}} & & \big\uparrow{\scriptstyle \iota} \\ S^{2i-1} \wedge \mathrm{SU}(p) & \xrightarrow{\langle \delta_i, 1 \rangle} & \mathrm{SU}(p) \end{array}$$

where $\iota\colon \mathrm{SU}(p) \to \mathrm{U}(p)$ is the inclusion and $\eta_i := \iota_*(\delta_i)$.

By the unnumbered corollary in Bott [Bot60, p. 250], the map $\langle \eta_i, \eta_{p-i-1} \rangle$ is non-trivial and p divides its order. Hence, the order of $\langle \delta_i, 1 \rangle_{(p)}$ is at least p. The result now follows from Lemmas 5.2, 5.4 and 5.15. □

5.3 Homotopy invariants of $\mathrm{PU}(p)$-gauge groups

The content of Lemma 5.17 is a straightforward observation about how certain homotopy invariants of $\mathrm{SU}(n)$-gauge groups relate to the corresponding invariants of $\mathrm{PU}(n)$-gauge groups.

Lemma 5.17. *Let n be arbitrary and X be a simply-connected space. Suppose further that we have $[X, \mathrm{SU}(n)] \cong 0$. Then, the quotient map $q\colon \mathrm{SU}(n) \to \mathrm{PU}(n)$ induces an isomorphism of groups*

$$[X, \mathcal{G}_{i,k}(\mathrm{SU}(n))] \cong [X, \mathcal{G}_{i,k}(\mathrm{PU}(n))]$$

for any $2 \le i \le n$ and any $k \in \mathbb{Z}$.

Proof. Since $[X, \mathrm{SU}(n)] \cong 0$ and X is simply-connected, applying the functor $[X, -]$ to the homotopy fibration sequence

$$\cdots \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{SU}(n) \xrightarrow{q} \mathrm{PU}(n) \longrightarrow \mathrm{B}(\mathbb{Z}/n\mathbb{Z})$$

shows that $[X, \mathrm{PU}(n)] \cong 0$ also.

Applying now the functor $[\Sigma X, -]$ to the homotopy fibration sequence

$$\mathrm{PU}(n) \xrightarrow{\partial_{i,k}} \Omega_0^{2i-1}\mathrm{PU}(n) \longrightarrow \mathrm{B}\mathcal{G}_{i,k}(\mathrm{PU}(n)) \longrightarrow \mathrm{BPU}(n),$$

as well as to its $\mathrm{SU}(n)$ analogue, yields the following commutative diagram

$$\begin{array}{ccccccc} [\Sigma X, \mathrm{SU}(n)] & \xrightarrow{(\partial_{i,k})_*} & [\Sigma^{2i} X, \mathrm{SU}(n)] & \longrightarrow & [X, \mathcal{G}_{i,k}(\mathrm{SU}(n))] & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle q_*} & & \big\downarrow{\scriptstyle q_*} & & \big\downarrow & & \big\downarrow \\ [\Sigma X, \mathrm{PU}(n)] & \xrightarrow{(\partial_{i,k})_*} & [\Sigma^{2i} X, \mathrm{PU}(n)] & \longrightarrow & [X, \mathcal{G}_{i,k}(\mathrm{PU}(n))] & \longrightarrow & 0 \end{array}$$

where the rows are exact and the two leftmost vertical maps are isomorphisms by the same argument as in the proof of Lemma 5.3. The statement now follows from the five lemma. □

Remark 5.18. Clearly, Lemma 5.17 generalises Lemma 4.10 appearing in Section 4.2.

Hamanaka and Kono showed in [HK06, Theorem 1.2] that, for principal SU(n)-bundles over S^4, the homotopy equivalence $\mathcal{G}_{2,k}(\mathrm{SU}(n)) \simeq \mathcal{G}_{2,l}(\mathrm{SU}(n))$ implies that

$$(n(n^2-1), k) = (n(n^2-1), l).$$

As an application of Lemma 5.17, let us show that the analogue of this result holds for PU(n)-gauge groups.

Corollary 5.19. *Let* $n > 3$. *For principal* PU(n)*-bundles over* S^4*,if*

$$\mathcal{G}_{2,k}(\mathrm{PU}(n)) \simeq \mathcal{G}_{2,l}(\mathrm{PU}(n)),$$

then $(n(n^2-1), k) = (n(n^2-1), l)$.

Proof. First, suppose that n is even. Note that we have

$$\pi_{2n-4}(\mathrm{SU}(n)) \cong \pi_{2n-2}(\mathrm{SU}(n)) \cong 0.$$

Hence, applying Lemma 5.17 with $X = S^{2n-4}$ and $X = S^{2n-2}$, we find

$$\pi_{2n-4}\big(\mathcal{G}_{2,k}(\mathrm{PU}(n))\big) \cong \pi_{2n-4}\big(\mathcal{G}_{2,k}(\mathrm{SU}(n))\big)$$

and

$$\pi_{2n-2}\big(\mathcal{G}_{2,k}(\mathrm{PU}(n))\big) \cong \pi_{2n-2}\big(\mathcal{G}_{2,k}(\mathrm{SU}(n))\big).$$

So the result follows for n even by [Sut92, Proposition 4.2].

When n is odd, we have from [HK06] that $[\Sigma^{2n-6}\mathbb{C}P^2, \mathrm{SU}(n)] \cong 0$. Hence, applying Lemma 5.17 with $X = \Sigma^{2n-6}\mathbb{C}P^2$, we find

$$[\Sigma^{2n-6}\mathbb{C}P^2, \mathcal{G}_{2,k}(\mathrm{PU}(n))] \cong [\Sigma^{2n-6}\mathbb{C}P^2, \mathcal{G}_{2,k}(\mathrm{SU}(n))].$$

So the result follows for n odd by [HK06, Corollary 2.6]. □

Following the work of [HK07], Mohammadi and Asadi-Golmankhaneh [MAG19b] recently showed that, for SU(n)-bundles over S^6, an equivalence $\mathcal{G}_{3,k}(\mathrm{SU}(n)) \simeq \mathcal{G}_{3,l}(\mathrm{SU}(n))$ implies that

$$\big((n-1)n(n+1)(n+2), k\big) = \big((n-1)n(n+1)(n+2), l\big).$$

Hence, we also have:

Corollary 5.20. *Let* $n \geq 3$*. For principal* PU(n)*-bundles over* S^6*, if there is a homotopy equivalence* $\mathcal{G}_{3,k}(\mathrm{PU}(n)) \simeq \mathcal{G}_{3,l}(\mathrm{PU}(n))$*, then*

$$((n-1)n(n+1)(n+2), k) = ((n-1)n(n+1)(n+2), l).$$

Proof. Apply Lemma 5.17 with $X = \Sigma^{2n-6}\mathbb{C}P^2$ and the result of [MAG19b]. □

5.4 Notable special cases

5.4.1 PU(p)-bundles over S^4

Theriault showed in [The17] that, after localisation at an odd prime p and provided $n < (p-1)^2 + 1$, the order of the Samelson product $\langle \delta_2, 1\rangle \colon S^3 \wedge \mathrm{SU}(n) \to \mathrm{SU}(n)$ is the p-primary component of the integer $n(n^2-1)$. It then follows immediately from Theorem 5.16 that

Corollary 5.21. *After localisation at an odd prime, the order of the Samelson product* $\langle \epsilon_2, 1\rangle \colon S^3 \wedge \mathrm{PU}(p) \to \mathrm{PU}(p)$ *is* $p(p^2-1)$*.* □

5.4.2 PU(5)-bundles over S^4

In [The15], Theriault showed that the order of $\langle \delta_2, 1\rangle \colon S^3 \wedge \mathrm{SU}(5) \to \mathrm{SU}(5)$ is 120. Hence, by Theorem 5.16, the order of $\langle \epsilon_2, 1\rangle \colon S^3 \wedge \mathrm{PU}(5) \to \mathrm{PU}(5)$ is also 120.

Theorem 5.22. *For* PU(5)*-bundles over* S^4*, it is the case that*

(a) if $\mathcal{G}_{2,k}(\mathrm{PU}(5)) \simeq \mathcal{G}_{2,l}(\mathrm{PU}(5))$*, then* $(120, k) = (120, l)$*;*

(b) if $(120, k) = (120, l)$*, then* $\mathcal{G}_{2,k}(\mathrm{PU}(5)) \simeq \mathcal{G}_{2,l}(\mathrm{PU}(5))$ *when localised rationally or at any prime.*

Proof. Part (i) follows from Corollary 5.19, while part (ii) follows from Lemma 2.74. □

5.4.3 PU(3)-bundles over S^6

Hamanaka and Kono showed in [HK07] that the order of $\langle \delta_3, 1\rangle \colon S^5 \wedge \mathrm{SU}(3) \to \mathrm{SU}(3)$ is 120. It follows from Theorem 5.16 that the order of $\langle \epsilon_3, 1\rangle \colon S^5 \wedge \mathrm{PU}(3) \to \mathrm{PU}(3)$ is also 120.

Theorem 5.23. *For* PU(3)*-bundles over* S^6*, we have* $\mathcal{G}_{3,k}(\mathrm{PU}(3)) \simeq \mathcal{G}_{3,l}(\mathrm{PU}(3))$ *if, and only if,* $(120, k) = (120, l)$*.*

Proof. As the homotopy groups $\pi_n(\Omega_0^5\mathrm{PU}(3)) \cong \pi_{n+5}(\mathrm{PU}(3))$ are all finite, the "if" direction follows from Lemma 2.73, while the "only if" direction follows from Corollary 5.20. □

We should note that in [HKKS16], the PU(3)-gauge group $\mathcal{G}_{2,k}$ is shown to be homotopy equivalent to $\widehat{\mathcal{G}}_{2,k} \times S^1$, where $\widehat{\mathcal{G}}_{2,k}$ is a space whose homotopy groups are all finite. This allows the authors of [HKKS16] to apply Lemma 2.73 to obtain a classification result for $\mathcal{G}_{2,k}$ that holds integrally. We expect the same result to apply more generally to gauge groups of PU(n)-bundles over S^{2n-2}. However, there are currently no other cases, besides that of [HKKS16], in which such a result would be applicable.

Finally, it is worth noting that, should any further classifications of gauge groups of SU(p)-bundles over even-dimensional spheres be obtained, our results would imply the corresponding classifications for PU(p)-gauge groups as immediate corollaries, provided the SU(p) results were arrived at as consequences of Lemmas 2.73 or 2.74 by calculating the orders of the relevant Samelson products.

Chapter 6

Homotopy types of $\mathrm{Spin}^c(n)$-gauge groups over S^4

The complex spin group $\mathrm{Spin}^c(n)$ was first introduced in 1964 in a paper of Atiyah, Bott and Shapiro [ABS64]. There has been an increasing interest in the $\mathrm{Spin}^c(n)$ groups ever since the publication of the Seiberg-Witten equations for 4-manifolds [Wit94], whose formulation requires the existence of $\mathrm{Spin}^c(n)$-structures, and more recently for the role they play in string theory [BS99, FW99, Sat12].

In this chapter we examine $\mathrm{Spin}^c(n)$-gauge groups over S^4. We begin by recalling some basic properties of the complex spin group $\mathrm{Spin}^c(n)$ and showing that, provided $n \geq 3$, its underlying topological space can be expressed as a product of a circle and the real spin group $\mathrm{Spin}(n)$.

For $n \geq 6$, we show that this decomposition is reflected in the corresponding gauge groups. The homotopy theory of $\mathrm{Spin}^c(n)$-gauge groups over S^4 therefore reduced to that of the corresponding $\mathrm{Spin}(n)$-gauge groups. We advance on what is known on $\mathrm{Spin}(n)$-gauge groups by providing a partial classification for Spin(7)- and Spin(8)-gauge groups over S^4.

6.1 $\mathrm{Spin}^c(n)$ groups

For $n \geq 1$, the complex spin group $\mathrm{Spin}^c(n)$ is defined as the quotient

$$\frac{\mathrm{Spin}(n) \times \mathrm{U}(1)}{\mathbb{Z}/2\mathbb{Z}}$$

where

$$\mathbb{Z}/2\mathbb{Z} \cong \{(1,1), (-1,-1)\} \subseteq \mathrm{Spin}(n) \times \mathrm{U}(1)$$

denotes the central subgroup of order 2. The group $\mathrm{Spin}^c(n)$ is special case of the more general notion of $\mathrm{Spin}^k(n)$ group introduced in [AM21].

The first low rank $\mathrm{Spin}^c(n)$ groups can be identified as follows:

- $\mathrm{Spin}^c(1) \cong \mathrm{U}(1) \simeq S^1$;
- $\mathrm{Spin}^c(2) \cong \mathrm{U}(1) \times \mathrm{U}(1) \simeq S^1 \times S^1$;
- $\mathrm{Spin}^c(3) \cong \mathrm{U}(2) \simeq S^1 \times S^3$;
- $\mathrm{Spin}^c(4) \cong \{(A, B) \in \mathrm{U}(2) \times \mathrm{U}(2) \mid \det A = \det B\}$.

The group $\mathrm{Spin}^c(n)$ fits into a commutative diagram

$$\begin{array}{ccccc}
\{\pm 1\} & \xleftarrow{\mathrm{pr}_1} & \{(1,1),(-1,-1)\} & \xrightarrow{\mathrm{pr}_2} & \{\pm 1\} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spin}(n) & \xleftarrow{\mathrm{pr}_1} & \mathrm{Spin}(n) \times S^1 & \xrightarrow{\mathrm{pr}_2} & S^1 \\
\downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 2} \\
\mathrm{SO}(n) & \longleftarrow & \mathrm{Spin}^c(n) & \longrightarrow & S^1,
\end{array}$$

where q is the quotient map, $\lambda\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ denotes the double covering map of the group $\mathrm{SO}(n)$ by $\mathrm{Spin}(n)$ and $2\colon S^1 \to S^1$ denotes the degree 2 map. Furthermore, we observe that the map

$$\lambda \times 2\colon \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times S^1$$

is a double covering of $\mathrm{SO}(n) \times S^1$ by $\mathrm{Spin}^c(n)$.

We begin with a decomposition of $\mathrm{Spin}^c(n)$ as a product of spaces which will be reflected in an analogous decomposition of $\mathrm{Spin}^c(n)$-gauge groups.

Lemma 6.1. *For* $n \geq 1$*, the space* $\mathrm{Spin}^c(n)$ *is homeomorphic to* $S^1 \times \widetilde{\mathrm{Spin}}^c(n)$*, where* $\widetilde{\mathrm{Spin}}^c(n)$ *denotes the universal cover of* $\mathrm{Spin}^c(n)$*.*

Proof. We have $\pi_1(\mathrm{Spin}^c(n)) \cong \mathbb{Z}$ for $n \geq 3$ (see, e.g. [Jür08]). By the Hurewicz and the universal coefficient theorems, we have isomorphisms

$$\mathbb{Z} \cong \pi_1(\mathrm{Spin}^c(n)) \cong H_1(\mathrm{Spin}^c(n); \mathbb{Z}) \cong H^1(\mathrm{Spin}^c(n); \mathbb{Z}).$$

Therefore, we have maps $S^1 \to \mathrm{Spin}^c(n)$ and $\mathrm{Spin}^c(n) \to K(\mathbb{Z}, 1) \simeq S^1$ representing generators of $\pi_1(\mathrm{Spin}^c(n))$ and of $H^1(\mathrm{Spin}^c(n); \mathbb{Z})$, respectively, such that the composite induces an isomorphism in π_1. Therefore, the homotopy fibration

$$\widetilde{\mathrm{Spin}}^c(n) \longrightarrow \mathrm{Spin}^c(n) \longrightarrow K(\mathbb{Z}, 1) \simeq S^1$$

defining the universal cover of $\mathrm{Spin}^c(n)$ admits a right homotopy splitting and hence, as $\mathrm{Spin}^c(n)$ is a group, we have

$$\mathrm{Spin}^c(n) \simeq S^1 \times \widetilde{\mathrm{Spin}}^c(n). \qquad \square$$

Note that as $\mathrm{Spin}^c(n)$ is a Lie group, we can equip $\widetilde{\mathrm{Spin}}^c(n)$ with a group structure for which there is a covering map

$$\varrho\colon \widetilde{\mathrm{Spin}}^c(n) \to \mathrm{Spin}^c(n)$$

which is a group homomorphism.

Lemma 6.2. *For* $n \geq 3$*, we have* $\widetilde{\mathrm{Spin}}^c(n) \simeq \mathrm{Spin}(n)$*.*

Proof. Since $\mathrm{Spin}(n) \times S^1$ is a double cover of $\mathrm{Spin}^c(n)$ and the covering space of a product of spaces is the product of their respective covering spaces, there are coverings

$$\mathrm{Spin}(n) \times \mathbb{R} \longrightarrow \mathrm{Spin}(n) \times S^1 \longrightarrow \mathrm{Spin}^c(n),$$

and hence $\widetilde{\mathrm{Spin}}^c(n) \simeq \mathbb{R} \times \mathrm{Spin}(n) \simeq \mathrm{Spin}(n)$. $\square$

6.2 A decomposition of $\mathrm{Spin}^c(n)$-gauge groups

Isomorphism classes of principal $\mathrm{Spin}^c(n)$-bundles over S^4 are classified by the free homotopy classes of maps $S^4 \to \mathrm{BSpin}^c(n)$. Since $\mathrm{Spin}^c(n)$ is connected, $\mathrm{BSpin}^c(n)$ is simply-connected and hence there are isomorphisms

$$[S^4, \mathrm{BSpin}^c(n)]_{\mathrm{free}} \cong \pi_3(\mathrm{Spin}^c(n)) \cong \pi_3(\mathrm{SO}(n)) \cong \begin{cases} 0 & n = 1, 2 \\ \mathbb{Z}^2 & n = 4 \\ \mathbb{Z} & n = 3, n \geq 5. \end{cases}$$

Remark 6.3. Note that for $n = 3$ we have $\mathrm{Spin}^c(3) \cong \mathrm{U}(2)$, and the homotopy types of $\mathrm{U}(2)$-gauge groups over S^4 have been studied by Cutler in [Cut18b].

For $n \geq 5$, let $\mathcal{G}_k$ denote the gauge group of the $\mathrm{Spin}^c(n)$-bundle $P_k \to S^4$ classified by $k \in \mathbb{Z}$. Arguing as in Section 4.1 , we obtain the following homotopy fibration sequence

$$\mathcal{G}_k \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\partial_k} \Omega^3_0\mathrm{Spin}^c(n) \longrightarrow B\mathcal{G}_k \longrightarrow \mathrm{BSpin}^c(n),$$

which exhibits the gauge group $\mathcal{G}_k$ as the homotopy fibre of the map ∂_k.

We will now show that the decomposition of $\mathrm{Spin}^c(n)$ induces a corresponding decomposition of $\mathrm{Spin}^c(n)$-gauge groups.

Theorem 6.4. *For $n \geq 6$ and any $k \in \mathbb{Z}$, we have*

$$\mathcal{G}_k\big(\mathrm{Spin}^c(n)\big) \simeq S^1 \times \mathcal{G}_k\big(\mathrm{Spin}(n)\big).$$

Proof. Identifying the universal cover of $\mathrm{Spin}^c(n)$ as $\mathrm{Spin}(n)$ as in Lemma 6.2, there is a covering fibration

$$\mathrm{Spin}(n) \xrightarrow{\varrho} \mathrm{Spin}^c(n) \xrightarrow{g} S^1$$

where ϱ is a group homomorphism. Let $s\colon S^1 \to \mathrm{Spin}^c(n)$ be a right homotopy inverse of g, which exists by Lemma 6.1.

As $\pi_4(\mathrm{Spin}^c(n)) \cong 0$ for $n \geq 6$, there is a lift in the diagram

$$\begin{array}{ccccc} & & S^1 & & \\ & \overset{a}{\swarrow} & \downarrow s & \overset{*}{\searrow} & \\ \mathcal{G}_k(\mathrm{Spin}^c(n)) & \longrightarrow & \mathrm{Spin}^c(n) & \xrightarrow{\partial_k} & \Omega_0^3\mathrm{Spin}^c(n). \end{array}$$

Define the map b to be the composite

$$b\colon \mathcal{G}_k(\mathrm{Spin}^c(n)) \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{g} S^1.$$

Since s is a right homotopy inverse for g, the map a is a right homotopy inverse for b. Therefore we have $\mathcal{G}_k(\mathrm{Spin}^c(n)) \simeq S^1 \times F_b$, where F_b denotes the homotopy fibre of b.

As the covering map $\varrho\colon \mathrm{Spin}(n) \to \mathrm{Spin}^c(n)$ is a group homomorphism, it classifies to a map

$$\mathrm{B}\varrho\colon \mathrm{BSpin}(n) \to \mathrm{BSpin}^c(n).$$

Since ϱ induces an isomorphism in π_3, it respects path-components in $\mathrm{Map}_k(S^4, -)$ and $\mathrm{Map}_k^*(S^4, -)$ for any $k \in \mathbb{Z}$. We therefore have a diagram of fibration sequences

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Map}_k^*(S^4, \mathrm{BSpin}(n)) & \longrightarrow & \mathrm{Map}_k(S^4, \mathrm{BSpin}(n)) & \longrightarrow & \mathrm{BSpin}(n) \\ & \downarrow (\mathrm{B}\varrho)_* & & \downarrow (\mathrm{B}\varrho)_* & & \downarrow \mathrm{B}\varrho \\ \cdots \longrightarrow & \mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)) & \longrightarrow & \mathrm{Map}_k(S^4, \mathrm{BSpin}^c(n)) & \longrightarrow & \mathrm{BSpin}^c(n). \end{array} \tag{6.2.1}$$

Furthermore, observe that for all $k \in \mathbb{Z}$ we have

$$\pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}(n))) \cong \pi_m(\Omega_0^3\mathrm{Spin}(n)) \cong \pi_{m+3}(\mathrm{Spin}(n))$$

and, similarly, $\pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n))) \cong \pi_{m+3}(\mathrm{Spin}^c(n))$. Since ϱ induces isomorphisms on π_m for $m \geq 2$, it follows that $(\mathrm{B}\varrho)_*$ induces isomorphisms

$$\pi_m((\mathrm{B}\varrho)_*)\colon \pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}(n))) \xrightarrow{\cong} \pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)))$$

for all m and is therefore a homotopy equivalence by Whitehead's theorem.

We can extend the fibration diagram (6.2.1) to the left as

$$\begin{array}{ccccccc} \mathcal{G}_k(\mathrm{Spin}(n)) & \longrightarrow & \mathrm{Spin}(n) & \xrightarrow{\partial'_k} & \mathrm{Map}^*_k(S^4, \mathrm{BSpin}(n)) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \mathcal{G}_k(\varrho)} & & \downarrow{\scriptstyle \varrho} & & \simeq \downarrow{\scriptstyle (\mathrm{B}\varrho)_*} & & \\ \mathcal{G}_k(\mathrm{Spin}^c(n)) & \longrightarrow & \mathrm{Spin}^c(n) & \xrightarrow{\partial_k} & \mathrm{Map}^*_k(S^4, \mathrm{BSpin}^c(n)) & \longrightarrow & \cdots \end{array}$$

where ∂'_k denotes the boundary map associated to $\mathrm{Spin}(n)$-gauge groups over S^4.

Since $(\mathrm{B}\varrho)_*$ is a homotopy equivalence, the leftmost square is a homotopy pull-back. Since we know that there is a fibration

$$\mathrm{Spin}(n) \xrightarrow{\varrho} \mathrm{Spin}^c(n) \xrightarrow{g} S^1,$$

it follows that we also have a fibration

$$\mathcal{G}_k(\mathrm{Spin}(n)) \xrightarrow{\mathcal{G}_k(\varrho)} \mathcal{G}_k(\mathrm{Spin}^c(n)) \xrightarrow{b} S^1.$$

In particular, the space $\mathcal{G}_k(\mathrm{Spin}(n))$ is seen to be the homotopy fibre F_b of the map $b\colon \mathcal{G}_k(\mathrm{Spin}^c(n)) \to S^1$ and hence we have

$$\mathcal{G}_k(\mathrm{Spin}^c(n)) \simeq S^1 \times \mathcal{G}_k(\mathrm{Spin}(n)).$$ □

In light of Theorem 6.4, the homotopy theory of $\mathrm{Spin}^c(n)$-gauge groups over S^4 for $n \geq 6$ is completely determined by that of $\mathrm{Spin}(n)$-gauge groups over S^4.

Remark 6.5. By a result of Cutler [Cut18b], there is a decomposition

$$\mathcal{G}_k(\mathrm{U}(2)) \simeq S^1 \times \mathcal{G}_k(\mathrm{SU}(2))$$

of $\mathrm{U}(2)$-gauge groups over S^4 whenever k is even. Given that $\mathrm{Spin}^c(3) \cong \mathrm{U}(2)$ and $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, the statement of Theorem 6.4 still holds true when $n = 2$ provided that k is even. Cutler also shows that $\mathcal{G}_k(\mathrm{U}(2)) \simeq S^1 \times \mathcal{G}_k(\mathrm{PU}(2))$ for odd k, so Theorem 6.4 does not hold for $n = 2$.

6.3 Spin(n)-gauge groups

We now shift our focus to principal $\mathrm{Spin}(n)$-bundles over S^4 and the classification of their gauge groups. In the interest of completeness, we recall that, for $n \leq 6$, the following exceptional isomorphisms hold.

n	$\mathrm{Spin}(n)$
1	$\mathrm{O}(1)$
2	$\mathrm{U}(1)$
3	$\mathrm{SU}(2)$
4	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
5	$\mathrm{Sp}(2)$
6	$\mathrm{SU}(4)$

Table 6.1: The exceptional isomorphisms.

The cases $n = 1, 2$ are trivial. Indeed, as $\pi_3(\mathrm{O}(1)) \cong \pi_3(\mathrm{U}(1)) \cong 0$, there is only one isomorphism class of $\mathrm{O}(1)$- and $\mathrm{U}(1)$-bundles over S^4, namely, that of the trivial bundle, and hence there is only one possible homotopy type for the corresponding gauge groups. The case $n = 3$ was studied by Kono in [Kon91]. The case $n = 4$ can be reduced to the $n = 3$ case by [BHMP81, Theorem 5]. The case $n = 5$ was studied by Theriault in [The10a]. Finally, the case $n = 6$ was studied by Cutler and Theriault in [CT19].

We shall now explore the $n = 7$ case. Recall that we have a fibration sequence

$$\mathcal{G}_k(\mathrm{Spin}(7)) \longrightarrow \mathrm{Spin}(7) \xrightarrow{k\partial_1} \Omega^3_0\mathrm{Spin}(7).$$

Lemma 6.6. *Localised away from the prime 2, the boundary map*

$$\mathrm{Spin}(7) \xrightarrow{\partial_1} \Omega^3_0\mathrm{Spin}(7)$$

has order 21*.*

Proof. Harris [Har61] showed that $\mathrm{Spin}(2m+1) \simeq_{(p)} \mathrm{Sp}(m)$ for odd primes p. This result was later improved by Friedlander [Fri75] to a p-local homotopy equivalence of the corresponding classifying spaces. Then, in particular, localising at an odd prime p, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Spin}(7) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(7) & \longrightarrow & \mathrm{Map}_1(S^4, B\mathrm{Spin}(7)) & \longrightarrow & B\mathrm{Spin}(7) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{Sp}(3) & \xrightarrow{\partial_1'} & \Omega^3_0\mathrm{Sp}(3) & \longrightarrow & \mathrm{Map}_1(S^4, B\mathrm{Sp}(3)) & \longrightarrow & B\mathrm{Sp}(3)
\end{array}$$

where $\partial_1' \colon \mathrm{Sp}(3) \to \Omega^3_0\mathrm{Sp}(3)$ denotes the boundary map associated to $\mathrm{Sp}(3)$-gauge groups over S^4 studied in [Cut18a]. Hence the result follows from the calculation in [Cut18a, Theorem 1.2] where it is shown that ∂_1' has order 21 after localising away from 2. □

Lemma 6.7. *Localised at the prime 2, the order of the boundary map*

$$\mathrm{Spin}(7) \xrightarrow{\partial_1} \Omega^3_0\mathrm{Spin}(7)$$

is at most 8.

Proof. The strategy here will be to show that ∂_8 is nullhomotopic. This will suffice as we have $\partial_8 \simeq 8\partial_1$ by Lemma 4.2.

By a result of Mimura [Mim67, Proposition 9.1], the fibration

$$G_2 \xrightarrow{\alpha} \mathrm{Spin}(7) \longrightarrow S^7$$

splits at the prime 2. Let $\beta\colon S^7 \to \mathrm{Spin}(7)$ denote a right homotopy inverse for $\mathrm{Spin}(7) \to S^7$. Then the composite

$$G_2 \times S^7 \xrightarrow{\alpha\times\beta} \mathrm{Spin}(7) \times \mathrm{Spin}(7) \xrightarrow{\mu} \mathrm{Spin}(7)$$

is a 2-local homotopy equivalence.

Observe that we have $\partial_8 \circ \beta \simeq *$ since $\pi_{10}(\mathrm{Spin}(7)) \cong \mathbb{Z}/8\mathbb{Z}$ and $\partial_8 \circ \beta \simeq 8\partial_1 \circ \beta$. Therefore, by Lemma 2.65, the order of ∂_8 equals the order of $\partial_8 \circ \alpha$. As α is a group homomorphism, there is a diagram of evaluation fibrations

$$\begin{array}{ccc} G_2 & \xrightarrow{\partial_8'} & \Omega^3_0 G_2 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\Omega^3\alpha} \\ \mathrm{Spin}(7) & \xrightarrow{\partial_8} & \Omega^3_0\mathrm{Spin}(7). \end{array}$$

Since $\partial_8' \simeq 8\partial_1' \simeq *$ by [KTT17, Theorem 1.1], we must have $\partial_8 \simeq *$. □

We now move on to consider Spin(8)-gauge groups.

Lemma 6.8. *Localised at the prime 2 (resp. 3), the order of the boundary map*

$$\mathrm{Spin}(8) \xrightarrow{\partial_1} \Omega^3_0\mathrm{Spin}(8)$$

is at most 8 *(resp. 3).*

Proof. There is a fibration

$$\mathrm{Spin}(7) \longrightarrow \mathrm{Spin}(8) \longrightarrow S^7$$

which splits after localisation at any prime. Therefore, we have a local homotopy equivalence $\mathrm{Spin}(8) \simeq \mathrm{Spin}(7) \times S^7$ realised by maps

$$\alpha\colon \mathrm{Spin}(7) \to \mathrm{Spin}(8), \qquad \beta\colon S^7 \to \mathrm{Spin}(8),$$

where α is a group homomorphism and β a homotopy inverse for the map $\mathrm{Spin}(8) \to S^7$. Integrally, we have

$$\pi_{10}(\mathrm{Spin}(8)) \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z},$$

(see, e.g. the table in [Mim95]). Hence the same argument presented in the proof of Lemma 6.7 shows that $8\partial_1 \simeq *$ and $3\partial_1 \simeq *$ after localising at $p = 2$ and $p = 3$, respectively. □

Lemma 6.9. *Let $p \neq 3$ be an odd prime. Then the p-primary orders of the boundary maps $\partial_1 \colon \mathrm{Spin}(7) \to \Omega_0^3\mathrm{Spin}(7)$ and $\partial_1 \colon \mathrm{Spin}(8) \to \Omega_0^3\mathrm{Spin}(8)$ coincide.*

Proof. As $\pi_{10}(\mathrm{Spin}(8)) \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, any map $S^7 \to \mathrm{Spin}(8)$ is nullhomotopic after localisation at an odd prime p different from 3. Thus, decomposing $\mathrm{Spin}(8)$ as $\mathrm{Spin}(7) \times S^7$ and arguing as in the proof of Lemma 6.7 yields the statement. □

6.4 Homotopy invariants of $\mathrm{Spin}(n)$-gauge groups

In this section we investigate certain homotopy invariants of Spin(7)- and Spin(8)-gauge groups.

Lemma 6.10. *If $\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$, then $(21, k) = (21, l)$.*

Proof. As in the proof of Lemma 6.6, localising at an odd prime, we have an equivalence $\mathrm{BSpin}(7) \simeq_{(p)} \mathrm{BSp}(3)$. We therefore have a diagram of homotopy fibrations

$$\begin{array}{ccccccc}
\mathrm{Spin}(7) & \xrightarrow{\partial_k} & \Omega_0^3\mathrm{Spin}(7) & \longrightarrow & \mathrm{B}\mathcal{G}_k(\mathrm{Spin}(7)) & \longrightarrow & \mathrm{BSpin}(7) \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow & & \downarrow{\simeq} \\
\mathrm{Sp}(3) & \xrightarrow{\partial'_k} & \Omega_0^3\mathrm{Sp}(3) & \longrightarrow & \mathrm{B}\mathcal{G}_k(\mathrm{Sp}(3)) & \longrightarrow & \mathrm{BSp}(3)
\end{array}$$

where $\partial'_k \colon \mathrm{Sp}(3) \to \Omega_0^3\mathrm{Sp}(3)$ denotes the boundary map studied in [Cut18a]. Thus, by the five lemma, we have

$$\pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Spin}(7))) \cong \pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Sp}(3))).$$

Hence the result now follows from the calculations in [Cut18a, Theorem 1.1] where it is shown that, integrally,

$$\pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Sp}(3))) \cong \mathbb{Z}/120(84, k)\mathbb{Z}.$$ □

In their study of the homotopy types of G_2-gauge groups over S^4 in [KTT17], Kishimoto, Theriault and Tsutaya constructed a space C_k for which

$$H^*(C_k) \cong H^*(\mathcal{G}_k(G_2))$$

in mod 2 cohomology in dimensions 1 through 6. The cohomology of C_k is then shown to be as follows.

Lemma 6.11 ([KTT17, Lemma 8.3])**.** *We have*

- *if* $(4,k) = 1$ *then* $C_k \simeq S^3$*, so* $H^*(C_k) \cong H^*(S^3)$*;*
- *if* $(4,k) = 2$ *or* $(4,k) = 4$ *then* $H^*(C_k) \cong H^*(S^3) \oplus H^*(P^5(2)) \oplus H^*(P^6(2))$*, where* $P^n(p)$ *denotes the nth dimensional mod p Moore space;*
- *if* $(4,k) = 2$ *then* Sq^2 *is non-trivial on the degree 4 generator in* $H^*(C_k)$*;*
- *if* $(4,k) = 4$ *then* Sq^2 *is trivial on the degree 4 generator in* $H^*(C_k)$*.* □

We make use of the same spaces C_k as follows.

Lemma 6.12. *If* $\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$*, then we have* $(4,k) = (4,l)$*.*

Proof. As in the proof of Lemma 6.7, recall that we have a 2-local homotopy equivalence

$$G_2 \times S^7 \xrightarrow{\alpha\times\beta} \mathrm{Spin}(7)\times\mathrm{Spin}(7) \xrightarrow{\mu} \mathrm{Spin}(7).$$

Since the map $\alpha\colon G_2 \to \mathrm{Spin}(7)$ is a homomorphism, we have a commutative diagram

$$\begin{array}{ccc} G_2 & \xrightarrow{\partial_1'} & \Omega^3_0 G_2 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\Omega^3\alpha} \\ \mathrm{Spin}(7) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(7). \end{array}$$

Furthermore, as $\pi_7(\Omega^3_0 G_2) \cong \pi_{10}(G_2) \cong 0$, we have

$$\pi_7(\Omega^3_0\mathrm{Spin}(7)) \cong \pi_7(\Omega^3_0 G_2) \oplus \pi_7(\Omega^3 S^7) \cong \pi_7(\Omega^3 S^7),$$

and thus there is a commutative diagram

$$\begin{array}{ccc} S^7 & \xrightarrow{\gamma} & \Omega^3 S^7 \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\Omega^3\beta} \\ \mathrm{Spin}(7) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(7) \end{array}$$

for some γ representing a class in $\pi_7(\Omega^3 S^7) \cong \pi_{10}(S^7) \cong \mathbb{Z}/8\mathbb{Z}$.

We therefore have a commutative diagram

$$\begin{array}{ccc} G_2 \vee S^7 & \xrightarrow{k\partial_1' \vee k\gamma} & \Omega_0^3 G_2 \times \Omega^3 S^7 \\ \downarrow{\scriptstyle \alpha\vee\beta} & & \simeq \downarrow{\scriptstyle \Omega^3\alpha \times \Omega^3\beta} \\ \mathrm{Spin}(7) & \xrightarrow{k\partial_1} & \Omega_0^3 \mathrm{Spin}(7) \end{array}$$

which induces a map of fibres $\phi\colon M \to \mathcal{G}_k(\mathrm{Spin}(7))$, where M denotes the homotopy fibre of the map $k\partial_1' \vee k\gamma$.

Since the lowest dimensional cell in $G_2 \times S^7/(G_2 \vee S^7)$ appears in dimension 10, the canonical map $G_2 \vee S^7 \to G_2 \times S^7$ is a homotopy equivalence in dimensions less than 9. It thus follows that M is homotopy equivalent to the homotopy fibre of $k\partial_1' \times k\gamma$ in dimensions up to 8. Since the homotopy fibre of $k\partial_1' \times k\gamma$ is just the product $\mathcal{G}_k(G_2) \times F_k$, the composite

$$C_k \times F_k \longrightarrow \mathcal{G}_k(G_2) \times F_k \longrightarrow M \xrightarrow{\phi} \mathcal{G}_k(\mathrm{Spin}(7))$$

induces an isomorphism in mod-2 cohomology in dimensions 1 through 6, and therefore we have

$$H^*(\mathcal{G}_k(\mathrm{Spin}(7))) \cong H^*(C_k) \otimes H^*(F_k), \qquad * \leq 6.$$

From the fibration sequence

$$\Omega^4 S^7 \longrightarrow F_k \longrightarrow S^7$$

we see that $H^*(F_k) \cong H^*(\Omega^4 S^7)$ in dimensions 1 through 6 for dimensional reasons, and hence we have

$$H^*(F_k) \cong \mathbb{Z}/2\mathbb{Z}[y_3, y_6], \qquad * \leq 6,$$

where $|y_i| = i$, which, in turn, yields

$$H^*(\mathcal{G}_k(\mathrm{Spin}(7))) \cong H^*(C_k) \otimes \mathbb{Z}/2\mathbb{Z}[y_3, y_6], \qquad * \leq 6.$$

Since $H^*(F_k)$ does not contribute any generators in degree 4 to $H^*(\mathcal{G}_k(\mathrm{Spin}(7)))$, the result now follows from Lemma 6.11. Indeed, the presence of a degree 4 generator allows us to distinguish between the $(4,k) = 1$ case and the $2|k$ cases, whereas the vanishing of the Steenrod square Sq^2 on the degree 4 generator in $H^*(\mathcal{G}_k(\mathrm{Spin}(7)))$ coming from $H^*(C_k)$ can be used to distinguish between the $(4,k) = 2$ and $(4,k) = 4$ cases. □

This completes our partial classification result for Spin(7)-gauge groups over S^4.

Theorem 6.13. *(a) If* $(168,k) = (168,l)$*, there is a homotopy equivalence*

$$\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$$

after localising rationally or at any prime;

(b) If $\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$ *then* $(84,k) = (84,l)$*.*

Proof. For part (a), Lemmas 6.6 and 6.7 imply that $168\partial_1 \simeq *$, so the result follows from Lemma 2.74.

For part (b), combine Lemmas 6.10 and 6.12. □

Lemma 6.14. *If* $\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$*, then* $(4,k) = (4,l)$.

Proof. As in the proof of Lemma 6.7, the splitting of $G_2 \to \mathrm{Spin}(7) \to S^7$ at the prime 2 implies that there is a 2-local homotopy equivalence

$$\mu \circ (\alpha \times \beta)\colon G_2 \times S^7 \longrightarrow \mathrm{Spin}(7).$$

Since the fibration $\mathrm{Spin}(7) \to \mathrm{Spin}(8) \to S^7$ also splits after localising at any prime, there is a decomposition

$$\mu \circ \big((\iota \circ \alpha) \times (\iota \circ \beta) \times \gamma\big)\colon G_2 \times S^7 \times S^7 \longrightarrow \mathrm{Spin}(8),$$

where $\iota\colon \mathrm{Spin}(7) \to \mathrm{Spin}(8)$ is the inclusion homomorphism and γ is a homotopy inverse for the map $\mathrm{Spin}(8) \to S^7$.

Since the map $\iota \circ \alpha$ is a homomorphism, we have a commutative diagram

$$\begin{array}{ccc} G_2 & \xrightarrow{\partial_1'} & \Omega^3_0 G_2 \\ \downarrow{\scriptstyle \iota\circ\alpha} & & \downarrow{\scriptstyle \Omega^3(\iota\circ\alpha)} \\ \mathrm{Spin}(8) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(8). \end{array}$$

Furthermore, as $\pi_7(\Omega^3_0 G_2) \cong \pi_{10}(G_2) \cong 0$, we have

$$\pi_7(\Omega^3_0\mathrm{Spin}(8)) \cong \pi_7(\Omega^3_0 G_2) \oplus \pi_7(\Omega^3 S^7 \times \Omega^3 S^7) \cong \pi_7(\Omega^3 S^7 \times \Omega^3 S^7),$$

and thus there are commutative diagrams

$$\begin{array}{ccc} S^7 & \xrightarrow{\delta} & \Omega^3 S^7 \times \Omega^3 S^7 \\ \downarrow{\scriptstyle \iota\circ\beta} & & \downarrow{\scriptstyle \Omega^3(\iota\circ\beta)\times\Omega^3\gamma} \\ \mathrm{Spin}(8) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(8) \end{array} \qquad \begin{array}{ccc} S^7 & \xrightarrow{\delta'} & \Omega^3 S^7 \times \Omega^3 S^7 \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Omega^3(\iota\circ\beta)\times\Omega^3\gamma} \\ \mathrm{Spin}(8) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(8) \end{array}$$

for some δ, δ' representing classes in $\pi_7(\Omega^3 S^7 \times \Omega^3 S^7) \cong (\mathbb{Z}/8\mathbb{Z})^2$. We therefore have a commutative diagram

$$\begin{array}{ccc} G_2 \vee (S^7 \vee S^7) & \xrightarrow{k\partial_1' \vee k(\delta\vee\delta')} & \Omega^3_0 G_2 \times (\Omega^3 S^7 \times \Omega^3 S^7) \\ \downarrow{\scriptstyle \iota\alpha\vee(\iota\beta\vee\gamma)} & & \simeq\downarrow{\scriptstyle \Omega^3\iota\alpha\times(\Omega^3\iota\beta\times\Omega^3\gamma)} \\ \mathrm{Spin}(8) & \xrightarrow{k\partial_1} & \Omega^3_0\mathrm{Spin}(8). \end{array}$$

Arguing as in the proof of Lemma 6.12, we conclude that

$$H^*(\mathcal{G}_k(\mathrm{Spin}(7))) \cong H^*(\mathcal{G}_k(\mathrm{Spin}(8))), \qquad * \leq 6,$$

hence the statement follows from Lemma 6.11. □

Lemma 6.15. *If* $\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$*, then* $(7, k) = (7, l)$.

Proof. Localising at $p = 7$, we have

$$\mathrm{Spin}(8) \simeq \mathrm{Spin}(7) \times S^7 \simeq G_2 \times S^7 \times S^7.$$

Applying the functor π_{11} and noting that

$$\pi_{10}(S^7) \cong \pi_{11}(S^7) \cong \pi_{14}(S^7) \cong 0,$$

(see, e.g. [Tod62]) we find that the evaluation fibration

$$\mathrm{Spin}(8) \xrightarrow{\partial_k} \Omega_0^3\mathrm{Spin}(8) \longrightarrow \mathrm{B}\mathcal{G}_k(\mathrm{Spin}(8)) \longrightarrow \mathrm{BSpin}(8)$$

reduces to the exact sequence

$$\pi_{11}(G_2) \longrightarrow \pi_{11}(\Omega_0^3 G_2) \longrightarrow \pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Spin}(8))) \longrightarrow 0.$$

Hence the result follows from [KTT17]. □

This completes our partial classification result for Spin(8)-gauge groups over S^4.

Theorem 6.16. *(a) If* $(168, k) = (168, l)$*, there is a homotopy equivalence*

$$\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$$

after localising rationally or at any prime;

(b) If $\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$ *then* $(28, k) = (28, l)$.

Proof. For part (a), Lemmas 6.8 and 6.9 imply that $168\partial_1 \simeq *$, so the result follows from Lemma 2.74.

For part (b), combine Lemmas 6.14 and 6.15. □

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