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Separable subsets of GFERF negatively curved groups ☆

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Abstract

A word hyperbolic group G is called GFERF if every quasiconvex subgroup coincides with the intersection of finite index subgroups containing it. We show that in any such group, the product of finitely many quasiconvex subgroups is closed in the profinite topology on G.

Keywords: Word hyperbolic groups; Profinite topology; GFERF

1. Introduction

Let G be a finitely generated group. The *profinite topology* $\mathcal{PT}(G)$ on G is defined by proclaiming all finite index normal subgroups to be the basis of open neighborhoods of the identity element. It is easy to see that G equipped with this topology becomes a topological group. This topology is Hausdorff if and only if G is residually finite.

A subset $P \subseteq G$ will be called *separable* if it is closed in the profinite topology on G. Thus, a subgroup $H \leqslant G$ is separable whenever it is an intersection of finite index subgroups. The group G is said to be *locally extended residually finite* (LERF) if every finitely generated subgroup $H \leqslant G$ is separable.

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A famous theorem of M. Hall states that free groups are LERF. Among other well-known examples of LERF groups are surface groups and fundamental groups of compact Seifert fibred 3-manifolds [26]. In [25] P. Schupp provided certain sufficient conditions for a Coxeter group to be LERF. More recently, R. Gitik [7] constructed an infinite family of LERF hyperbolic groups that are fundamental groups of hyperbolic 3-manifolds.

In 1991 Pin and Reutenauer [23] conjectured that a product of finitely many finitely generated subgroups in a free group is separable and listed some possible applications to groups and semigroups. In 1993 Ribes and Zalesskiĭ [24] showed that the statement of this conjecture is true. Later a similar question was studied in other LERF groups by Coulbois [5], Gitik [8], Niblo [21], Steinberg [28] and others.

In particular, Gitik in [8, Theorem 1] proved that in a LERF hyperbolic group, a product of two quasiconvex subgroups, one of which is malnormal, is separable.

However, many word hyperbolic groups are not LERF. For example, an ascending HNN-extension of a finite rank free group is never LERF but very often hyperbolic (see [13]). So, it makes sense to use the weaker notion below.

We will say that a (word) hyperbolic group G is GFERF if every quasiconvex subgroup $H \leqslant G$ is separable. The definition of a GFERF Kleinian group Γ was given by Long and Reid in [17]: Γ is called *geometrically finite extended residually finite* (GFERF) if each geometrically finite subgroup $H \leqslant \Gamma$ is separable. Our definition is in the same spirit because in any word hyperbolic group (more generally, in any automatic group) a subgroup is geometrically finite if and only if it is quasiconvex (see [29]).

Long, Reid and Agol gave several examples of GFERF groups [2,17,18]. Hsu and Wise [12] proved that certain right-angled Artin groups are GFERF. Some negatively curved (i.e., word hyperbolic) groups with this property were studied by Gitik in [7]. In the paper [31] Wise provided another large family of GFERF hyperbolic groups; he also showed that Figure 8 knot group is GFERF. The fact that this group is LERF follows from the recent proofs by Agol [1] and Galegari and Gabai [4] of Marden's "tameness" conjecture. This conjecture provides a new way for obtaining LERF and GFERF groups as fundamental groups of 3-manifolds.

The main goal of this paper is to prove the following

Theorem 1.1. *Assume G is a GFERF word hyperbolic group, $G_1, G_2, \dots, G_s$ are quasiconvex subgroups, $s \in \mathbb{N}$. Then the product $G_1 G_2 \cdots G_s$ is separable in G.*

Since a finitely generated subgroup of a finite rank free group is quasiconvex, the above theorem generalizes the result of Ribes and Zalesskiĭ [24] and provides an alternative proof of the conjecture [23]. An application of Theorem 1.1 to the case when $s = 2$ and G_2 is malnormal gives the statement of Gitik's theorem [8, Theorem 1].

Our proof of Theorem 1.1 uses geometry of quasigeodesics in negatively curved spaces and basic properties of quasiconvex subgroups.

A subgroup H of a group G will be called *almost malnormal* if for every $x \in G \setminus H$ the intersection $H \cap xHx^{-1}$ is finite. H is said to be *elementary* if it is virtually cyclic. It is well known that in a hyperbolic group G any element of infinite order belongs to a unique *maximal elementary subgroup*. Thus, any maximal elementary subgroup of G is almost malnormal.

A famous open problem in Geometric Group Theory addresses the existence of a (word) hyperbolic group that is not residually finite. The author would like to emphasize the importance of studying GFERF hyperbolic groups through the proposition below.

Proposition. *The following are equivalent.*

(1) *There exists a non-residually finite hyperbolic group.*
(2) *There is a hyperbolic group G having an almost malnormal quasiconvex subgroup H which is not separable.*

Proof. Assume the first condition holds. In this case Kapovich and Wise [14, Theorem 1.2], and, independently, Ol'shanskii [22, Theorem 2], proved that there exists a non-trivial hyperbolic group G which has no proper subgroups of finite index at all. Choose an arbitrary maximal elementary subgroup H of G. Obviously H satisfies the condition (2).

Now, suppose (2) holds. Then, according to a theorem of Kharlampovich and Myasnikov [15, Theorem 2], the double $D = G *_H G$ is a hyperbolic group. If the group D were residually finite then we could apply the theorem of Long and Niblo [16, §2, Lemma] (see also [21]) stating that H is separable in G. The latter contradicts our assumptions. Hence, D is not residually finite. □

Presently, the author does not know of any examples of hyperbolic groups that are not GFERF. So, it seems reasonable to ask

Question. Does there exist a non-GFERF word hyperbolic group?

As one can see from the proposition, this question may be quite difficult.

Finally, we note that in the case when a hyperbolic group G is GFERF, Theorem 1.1 provides a positive solution for Problem 3.11 posed by D. Wise in [30]. This problem asks whether the double coset HK is separable if G is residually finite and $H, K \leqslant G$ are separable quasiconvex subgroups.

2. Auxiliary information

Suppose G is a group with a fixed finite symmetrized generating set $\mathcal{A}$. If $g \in G$, $|g|_G$ will denote the length of a shortest word over $\mathcal{A}$ representing g. Now we can define the standard left-invariant word metric $d(\cdot,\cdot)$ on G by setting $d(x, y) \stackrel{\text{def}}{=} |x^{-1}y|_G$ for arbitrary $x, y \in G$. This metric extends to a metric on the Cayley graph $\Gamma(G, \mathcal{A})$ of the group G after endowing every edge with the metric of the segment $[0, 1] \subset \mathbb{R}$.

A subset Q of G is said to be ε-quasiconvex (where $\varepsilon \geqslant 0$) if for any pair of elements $u, v \in Q$ and any geodesic segment p connecting u and v, p belongs to a closed ε-neighborhood of Q in $\Gamma(G, \mathcal{A})$. A subset $Q \subset G$ is quasiconvex if it is ε-quasiconvex for some $\varepsilon \geqslant 0$.

For any two points $x, y \in \Gamma(G, \mathcal{A})$ we fix a geodesic path between them and denote it by $[x, y]$. If $x, y, w \in \Gamma(G, \mathcal{A})$, then the number

$$(x|y)_w \stackrel{\text{def}}{=} \frac{1}{2}\big(d(x, w) + d(y, w) - d(x, y)\big)$$

is called the *Gromov product* of x and y with respect to w.

Remark 2.1. Since the metric is left-invariant, for arbitrary $x, y, w \in G$ we have $(x|y)_w = (w^{-1}x|w^{-1}y)_{1_G}$.

Let abc be a geodesic triangle in $\Gamma(G,\mathcal{A})$. There exist "special" points $O_a \in [b,c]$, $O_b \in [a,c]$, $O_c \in [a,b]$ with the properties: $d(a,O_b) = d(a,O_c) = \alpha$, $d(b,O_a) = d(b,O_c) = \beta$, $d(c,O_a) = d(c,O_b) = \gamma$. It is easy to see that $\alpha = (b|c)_a$, $\beta = (a|c)_b$, $\gamma = (a|b)_c$. Two points $O \in [a,b]$ and $O' \in [a,c]$ are called *a-equidistant* if $d(a,O) = d(a,O') \leqslant \alpha$. The triangle abc is said to be *δ-thin* if for any two points O, O' lying on its sides and equidistant from one of its vertices, $d(O,O') \leqslant \delta$ holds.

The group G is said to be (word) hyperbolic (or negatively curved) if there is $\delta \geqslant 0$ such that every geodesic triangle in $\Gamma(G,\mathcal{A})$ is δ-thin (for more theory the reader is referred to [3,6]).

For a hyperbolic group G, the property of a subset to be quasiconvex does not depend on the choice of a generating set $\mathcal{A}$ (see [10]). A quasiconvex subgroup of a finitely generated group is finitely generated itself [3,27]. A conjugate of a quasiconvex subgroup is quasiconvex as well [19, Remark 2.2].

Fix an arbitrary GFERF hyperbolic group G. Then for $n \in \mathbb{N}$, $f_0, f_1, \dots, f_n \in G$ and any quasiconvex subgroups $G_1, \dots, G_n \leqslant G$, the subset

$$P = f_0 G_1 f_1 G_2 \cdots f_{n-1} G_n f_n \tag{1}$$

is called a *quasiconvex product* (here we use the terminology from [19]). Such a subset is always quasiconvex [11, Proposition 3.14], [19, Corollary 2.1].

Remark 2.2. Assume that $n \in \mathbb{N}$ and for any n quasiconvex subgroups of the group G, their product is closed in $\mathcal{PT}(G)$. Then any quasiconvex product P defined by (1) is also closed in $\mathcal{PT}(G)$.

Indeed, observe that $P = f\hat{G}_1 \cdots \hat{G}_n$ where $f = f_0 f_1 \cdots f_n \in G$ and $\hat{G}_i = (f_i f_{i+1} \cdots f_n)^{-1} G_i (f_i f_{i+1} \cdots f_n)$—quasiconvex subgroups of G. By the assumptions, $\hat{G}_1 \cdots \hat{G}_n$ is separable, and since G (endowed with $\mathcal{PT}(G)$) is a topological group, left translation by the element $f^{-1} \in G$ is a continuous operation, hence P is also separable.

Lemma 2.3. *Assume that G is a δ-hyperbolic group with respect to a finite generating set $\mathcal{A}$ and A, B are ε-quasiconvex subgroups. There exists a constant $C_0 = C_0(\delta,\varepsilon,G,\mathcal{A}) \geqslant 0$ such that for any $a \in A$, $b \in B$ the inequality $(a^{-1}|b)_{1_G} \leqslant C_0$ holds whenever a is a shortest representative of the coset $a(A \cap B)$.*

Proof. Define a finite subset of the group G by $\Theta = \{g \in AB \mid |g|_G \leqslant 2\varepsilon + \delta\}$. For every $g \in \Theta$ choose a pair $(x,y) \in A \times B$ satisfying $g = x^{-1}y$; let $\Omega \subset A \times B$ denote the (finite) set of these pairs. Consider

$$\Omega_1 = \bigl\{x \in A \bigm| (x,y) \in \Omega \text{ for some } y \in B\bigr\}.$$

Then one can define the number $C_0 = \max\{|x|_G \mid x \in \Omega_1\} + \varepsilon < \infty$.

Now, assume that $(a^{-1}|b)_{1_G} > C_0$, for some $a \in A$, $b \in B$ where a is a shortest representative of the coset $a(A \cap B)$. Let α and β denote the "special" points of the triangle $1_G a^{-1} b$ (in $\Gamma(G,\mathcal{A})$) on the sides $[1_G, a^{-1}]$ and $[1_G, b]$, respectively. Since A and B are ε-quasiconvex there are elements $a_1 \in A$ and $b_1 \in B$ that are ε-close to α and β correspondingly. Using the triangle inequality and δ-hyperbolicity of the space $\Gamma(G,\mathcal{A})$ we obtain

$$\begin{aligned}|aa_1|_G &= d\big(a^{-1}, a_1\big) \leqslant d\big(a^{-1}, \alpha\big) + \varepsilon = d\big(a^{-1}, 1_G\big) - d(\alpha, 1_G) + \varepsilon \\ &= d\big(a^{-1}, 1_G\big) - \big(a^{-1}\big|b\big)_{1_G} + \varepsilon < |a|_G - C_0 + \varepsilon, \\ \big|a_1^{-1}b_1\big|_G &= d(a_1, b_1) \leqslant d(a_1, \alpha) + d(\alpha, \beta) + d(\beta, b_1) \leqslant 2\varepsilon + \delta.\end{aligned}$$

By definition, there exists a pair of elements $(x, y) \in \Omega$ with $a_1^{-1}b_1 = x^{-1}y$, thus $a_1x^{-1} = b_1y^{-1} \in A \cap B$. Now, $a(a_1x^{-1}) \in a(A \cap B)$ and this element is shorter than a because

$$\big|aa_1x^{-1}\big|_G \leqslant |aa_1|_G + |x|_G < |a|_G - \big(C_0 - \varepsilon - |x|_G\big) \leqslant |a|_G.$$

Thus we achieve a contradiction with our assumptions. □

Let p be a path in the Cayley graph of G. Then p_-, p_+ will denote the initial and the final points of p, $\|p\|$—its length. We will use *elem*(p) to denote the element of the group G represented by the word written on p. A path q is called (λ, c)-*quasigeodesic* if there exist $0 < \lambda \leqslant 1$, $c \geqslant 0$, such that for any subpath p of q the inequality $\lambda\|p\| - c \leqslant d(p_-, p_+)$ holds.

The statement below is an analog of the fact that in a negatively curved space k-local geodesics are quasigeodesics for any sufficiently large k.

Lemma 2.4. [20, Lemma 4.2] *Let $\bar\lambda > 0$, $\bar c \geqslant 0$, $C_0 \geqslant 14\delta$, $C_1 = 12(C_0 + \delta) + \bar c + 1$ be given. Then for $\lambda = \bar\lambda/4 > 0$ there exist $c = c(\bar\lambda, \bar c, C_0) \geqslant 0$ satisfying the statement below.*

> *Assume $N \in \mathbb{N}$, $x_i \in \Gamma(G, \mathcal{A})$, $i = 0, \dots, N$, and q_i are $(\bar\lambda, \bar c)$-quasigeodesic paths between x_{i-1} and x_i in $\Gamma(G, \mathcal{A})$, $i = 1, \dots, N$. If $\|q_i\| \geqslant (C_1 + \bar c)/\bar\lambda$, $i = 1, \dots, N$, and $(x_{i-1}|x_{i+1})_{x_i} \leqslant C_0$ for all $i = 1, \dots, N-1$, then the path q obtained as a consecutive concatenation of $q_1, q_2, \dots, q_N$ is (λ, c)-quasigeodesic.*

For any element $x \in G$ and $N \geqslant 0$ the closed ball centered at x of radius N will be denoted by $\mathcal{O}_N(x) = \{y \in G \mid d(x, y) \leqslant N\}$.

Lemma 2.5. *Assume G is a δ-hyperbolic group, A and B are ε-quasiconvex subgroups. Then for any $N \geqslant 0$ there exists $N_1 = N_1(N, \delta, \varepsilon, G, \mathcal{A}) \geqslant 0$ such that the following holds. Suppose the subgroups $A' \leqslant A$ and $B' \leqslant B$ satisfy $A \cap B = A' \cap B'$, $\mathcal{O}_{N_1}(1_G) \cap (A' \cup B') \subset A \cap B$. Then for the subgroup $H = \langle A', B'\rangle \leqslant G$ one has*

$$\mathcal{O}_N(1_G) \cap AHB \subset AB.$$

Proof. First, let $\hat C_0 = \hat C_0(\delta, \varepsilon, G, \mathcal{A})$ be the constant given by Lemma 2.3. Define $C_0 = \max\{\hat C_0, 14\delta\}$, $\bar\lambda = 1$, $\bar c = 0$ and $C_1 = 12(C_0 + \delta) + \bar c + 1$. Now apply Lemma 2.4 to find $\lambda = \bar\lambda/4 = 1/4 > 0$ and $c = c(\bar\lambda, \bar c, C_0) \geqslant 0$ from its claim.

Set $N_1 = (N + c + 2C_1)/\lambda$ and let $A' \leqslant A$ and $B' \leqslant B$ satisfy the conditions of the lemma. Thus,

$$A' \cap \mathcal{O}_{N_1}(1_G) \subset A \cap B, \qquad B' \cap \mathcal{O}_{N_1}(1_G) \subset A \cap B. \tag{2}$$

Define the subgroup $H = \langle A', B'\rangle \leqslant G$ and consider an arbitrary element $g \in AHB \setminus (AB)$.

Then

$$g = x_0 y_1 x_1 y_2 \cdots x_l y_{l+1}, \tag{3}$$

where $l \in \mathbb{N} \cup \{0\}$, $x_0 \in A$, $x_i \in A' \setminus \{1_G\}$, $y_i \in B' \setminus \{1_G\}$, $i = 1, \ldots, l$, $y_{l+1} \in B$. Moreover, we can assume that $x_0, x_1, \ldots, x_l, y_1, \ldots, y_l$ are shortest representatives of their left cosets modulo $A \cap B$ (indeed, if there is $\tilde{x}_0 = x_0 z$ with $z \in A \cap B$ and $|\tilde{x}_0|_G < |x_0|_G$, then $\tilde{x}_0 \in A$, $g = \tilde{x}_0(zy_1)x_1 y_2 \cdots x_l y_{l+1}$ where $zy_1 \in B'$ because of the construction of B'; and then a similar procedure can be performed for zy_1, and so on) and l is the smallest such integer. Therefore

$$x_i \in A' \setminus (A \cap B), \qquad y_i \in B' \setminus (A \cap B), \quad i = 1, \ldots, l. \tag{4}$$

Observe that since $g \notin AB$, $l \geqslant 1$ and $y_1 \in B' \setminus (A \cap B)$. Choose geodesic paths q_1, q_2, $\ldots$, q_{2l+2} in $\Gamma(G, \mathcal{A})$ as follows: $(q_1)_- = 1_G$, $elem(q_1) = x_0$, $(q_2)_- = (q_1)_+$, $elem(q_2) = y_1$, $\ldots$, $(q_{2l+2})_- = (q_{2l+1})_+$, $elem(q_{2l+2}) = y_{l+1}$. Thus, $(q_{2l+2})_+ = g$. Using (4) and (2) we obtain $\|q_i\| > N_1 \geqslant C_1 = (C_1 + \bar{c})/\bar{\lambda}$, $i = 2, 3, \ldots, 2l+1$, and $((q_i)_- | (q_{i+1})_+)_{(q_i)_+} \leqslant C_0$ (by Remark 2.1 and Lemma 2.3) for $i = 1, \ldots, 2l+1$.

Now, there can occur four different situations depending on how long the paths q_1 and q_{2l+2} are:

(a) $\|q_1\| < C_1$ and $\|q_{2l+2}\| < C_1$;
(b) $\|q_1\| \geqslant C_1$ and $\|q_{2l+2}\| < C_1$;
(c) $\|q_1\| < C_1$ and $\|q_{2l+2}\| \geqslant C_1$;
(d) $\|q_1\| \geqslant C_1$ and $\|q_{2l+2}\| \geqslant C_1$.

Let us consider the situation (b) (the others can be resolved in a completely analogous fashion). Then the path $q = q_1 q_2 \cdots q_{2l+1}$ satisfies all the conditions of Lemma 2.4, hence it is (λ, c)-quasigeodesic (for the numbers λ, c defined in the beginning of the proof). Recalling (2) we get

$$\begin{aligned} |g|_G &\geqslant d(q_-, q_+) - d(q_+, g) \geqslant \lambda\|q\| - c - \|q_{2l+2}\| \geqslant \lambda\|q_2\| - c - C_1 \\ &= \lambda|y_1|_G - c - C_1 > \lambda\big((N + c + 2C_1)/\lambda\big) - c - C_1 \geqslant N. \end{aligned}$$

Similarly, one can show that $|g|_G > N$ in the other three situations.

Thus, we have $AHB \cap \mathcal{O}_N(1_G) \subset AB$ and the lemma is proved. □

Note that during the proof of Lemma 2.5 for each $g \in H = \langle A', B' \rangle$ we constructed a presentation (3) and a corresponding quasigeodesic path $q = q_1 \cdots q_{2l+2}$ connecting 1_G and g in $\Gamma(G, \mathcal{A})$. Since geodesics and quasigeodesics with same ends are mutually close [3, 3.3], the geodesic $[1_G, g]$ will lie in some neighborhood of q. If, in addition, the subgroups A' and B' are ε'-quasiconvex, q will belong to a closed ε'-neighborhood of H in $\Gamma(G, \mathcal{A})$. Hence H becomes quasiconvex itself. Thus, one obtains the statement below, first proved by R. Gitik:

Lemma 2.6. [9, Theorem 1] *Let A and B be ε-quasiconvex subgroups of a δ-hyperbolic group G. There exists a constant C_2, which depends only on G, δ and ε, with the following property. For any quasiconvex subgroups $A' \leqslant A$ and $B' \leqslant B$ with $A' \cap B' = A \cap B$, if all elements in A' and B' shorter than C_2 belong to $A \cap B$, then the subgroup $\langle A', B' \rangle$ is also quasiconvex in G.*

Corollary 2.7. *If G is a GFERF hyperbolic group and A, B are its quasiconvex subgroups then the double coset AB is separable in G.*

Proof. It is enough to show that for arbitrary $g \in G \setminus (AB)$ there exists a closed (in the profinite topology) subset K of G such that $AB \subseteq K$ and $g \notin K$. Let C_2 be the constant given by Lemma 2.6. Set $N = |g|_G$ and find the corresponding $N_1 \geqslant 0$ from the claim of Lemma 2.5. Denote $N_2 = \max\{N_1, C_2\}$. Since the subgroups A and B are closed in $\mathcal{PT}(G)$, then so is $A \cap B$; therefore there exist subgroups $A' \leqslant_f A$ and $B' \leqslant_f B$ (having finite indices in A and B correspondingly) such that $A \cap B \subset A'$, $A \cap B \subset B'$ and $\mathcal{O}_{N_2}(1_G) \cap (A' \cup B') \subset A \cap B$. Applying Lemma 2.5 to $H = \langle A', B' \rangle \leqslant G$ we achieve $g \notin AHB$.

Now, $A = \bigsqcup_{i=1}^m a_i A'$, $B = \bigsqcup_{j=1}^n B' b_j$ for some $m, n \in \mathbb{N}$, $a_i \in A$, $b_j \in B$ for all i, j. Since a finite index subgroup of a quasiconvex subgroup is itself quasiconvex, H is quasiconvex by Lemma 2.6, hence it is closed in $\mathcal{PT}(G)$ as G is GFERF. Therefore the sets $a_i H b_j$ are closed for any i, j, and, consequently, their finite union

$$K \stackrel{\text{def}}{=} \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} a_i H b_j$$

is closed too. It remains to observe that $AB \subset K = AHB$, thus $g \notin K$. $\square$

3. Proof of Theorem 1.1

We will use induction on s. If $s = 1$, the statement follows from the definition of a GFERF group. The case $s = 2$ is given by Corollary 2.7. So, we can now assume that $s > 2$ and the statement is already proved for a product of any $(s-1)$ quasiconvex subgroups.

For our convenience, denote $k = s - 2$, $A = G_{s-1}$, $B = G_s$. Let $\{A_i \mid i \in \mathbb{N}\}$, $\{B_i \mid i \in \mathbb{N}\}$ be enumerations of all finite index subgroups containing $A \cap B$ in A and B correspondingly. Define the sequences

$$A^{(i)} = \bigcap_{j=1}^{i} A_j, \qquad B^{(i)} = \bigcap_{j=1}^{i} B_j.$$

Now, due to the construction, $A \cap B \subset A^{(i)} \leqslant_f A$ and $A \cap B \subset B^{(i)} \leqslant_f B$ for all i. And (as we saw in the proof of Corollary 2.7) for every $i \in \mathbb{N}$ there are $m = m(i)$, $n = n(i) \in \mathbb{N}$ and elements $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_n \in B$ such that

$$A\langle A^{(i)}, B^{(i)} \rangle B = \bigcup_{p=1}^{m} \bigcup_{r=1}^{n} a_p \langle A^{(i)}, B^{(i)} \rangle b_r. \tag{5}$$

Remark 3.1. For any finite index subgroup H of G satisfying $A \cap B \leqslant H$ there exists $I \in \mathbb{N}$ such that $A^{(i)}, B^{(i)} \leqslant H$ for all $i \geqslant I$.

Since A and B are separable in G, their intersection $A \cap B$ is separable as well, and we have

$$A \cap B = \bigcap_{i=1}^{\infty} A^{(i)} = \bigcap_{i=1}^{\infty} B^{(i)}.$$

Without loss of generality, we can assume that the subgroups $G_1, \dots, G_k, A, B$ are ε-quasiconvex for a fixed $\varepsilon \geqslant 0$. Let $\hat{C}_0 = \hat{C}_0(\delta, \varepsilon, G, \mathcal{A})$ be the constant given by Lemma 2.3. Define $C_0 = \max\{\hat{C}_0, 14\delta\}$, $\bar{\lambda} = 1$, $\bar{c} = 0$ and $C_1 = 12(C_0 + \delta) + \bar{c} + 1$. Now apply Lemma 2.4 to find $\lambda = \bar{\lambda}/4 > 0$ and $c = c(\bar{\lambda}, \bar{c}, C_0) \geqslant 0$ from its claim.

Let $C_2 = C_2(\delta, \varepsilon, G)$ be the constant from the claim of Lemma 2.6. Since the group B is GFERF, there exist $A' \leqslant_f A$ and $B' \leqslant_f B$ such that $A' \cap B' = A \cap B$ and all the elements in A' and B' shorter than C_2 belong to $A \cap B$. Therefore, we can find an index $I_1 \in \mathbb{N}$ such that $A^{(i)} \leqslant A'$, $B^{(i)} \leqslant B'$ for all $i \geqslant I_1$, hence, according to Lemma 2.6, the subgroup $\langle A^{(i)}, B^{(i)} \rangle \leqslant G$ is quasiconvex.

Arguing by contradiction, suppose there exists $g \in G \setminus (G_1 G_2 \cdots G_k AB)$ which belongs to the closure of $G_1 G_2 \cdots G_k AB$ in $\mathcal{PT}(G)$. Keeping in mind formula (5) and Remark 2.2, for any $i \geqslant I_1$ we can apply the induction hypothesis to the product

$$P_i \stackrel{\text{def}}{=} G_1 G_2 \cdots G_k A \big\langle A^{(i)}, B^{(i)} \big\rangle B$$

to show that it is closed in $\mathcal{PT}(G)$.

Obviously, $G_1 G_2 \cdots G_k AB \subseteq P_i$, hence $g \in P_i$ for every $i \geqslant I_1$. Thus, for each $i \geqslant I_1$ one can find $l = l(i) \in \mathbb{N} \cup \{0\}$ and elements $z_1^{(i)} \in G_1, \dots, z_k^{(i)} \in G_k$, $x_0^{(i)} \in A$, $x_j^{(i)} \in A^{(i)} \setminus (A \cap B)$, $y_j^{(i)} \in B^{(i)} \setminus (A \cap B)$, $i = 1, \dots, l$, $y_{l+1}^{(i)} \in B$ satisfying

$$g = z_1^{(i)} \cdots z_k^{(i)} x_0^{(i)} y_1^{(i)} x_1^{(i)} \cdots x_l^{(i)} y_{l+1}^{(i)}. \tag{6}$$

Moreover, as in the proof of Lemma 2.5, we can assume that z_t is a shortest representative of its left coset modulo $G_t \cap G_{t+1}$ for $t = 1, \dots, k-1$, z_k is a shortest representative of its left coset modulo $G_k \cap A$, and x_0, x_j, y_j are shortest representatives of their left cosets modulo $A \cap B$ for $j = 1, \dots, l$.

Now we have to consider several possibilities.

Case 1. For some $t \in \{1, \dots, k\}$ we have $\liminf_{i \to \infty} |z_t^{(i)}|_G < \infty$.

Then, by passing to a subsequence, we can assume that $z_t^{(i)} = z_t \in G_t$ for all i. Using (6) and our assumptions on g we obtain

$$\begin{gathered} g \in G_1 \cdots G_{t-1} z_t G_{t+1} \cdots G_k A \big\langle A^{(i)}, B^{(i)} \big\rangle B \quad \text{and} \\ g \notin G_1 \cdots G_{t-1} z_t G_{t+1} \cdots G_k AB \quad \text{for all } i. \end{gathered} \tag{7}$$

By Remark 2.2 and the induction hypothesis, the subset

$$G_1 \cdots G_{t-1} z_t G_{t+1} \cdots G_k AB$$

is closed in $\mathcal{PT}(G)$, consequently, there exists a normal subgroup K of finite index in G such that

$$gK \cap G_1 \cdots G_{t-1} z_t G_{t+1} \cdots G_k AB = \emptyset.$$

Since $BK = KB = KBB$,

$$g \notin G_1 \cdots G_{t-1} z_t G_{t+1} \cdots G_k AKB = G_1 \cdots G_{t-1} z_t G_{t+1} \cdots G_k AHB, \tag{8}$$

where $H = KB$ is a finite index subgroup of G containing $A \cap B$. Applying Remark 3.1 we achieve that $\langle A^{(i)}, B^{(i)} \rangle \leqslant H$ for every sufficiently large i, thus

$$G_{t-1} z_t G_{t+1} \cdots G_k A \langle A^{(i)}, B^{(i)} \rangle B \subseteq G_{t-1} z_t G_{t+1} \cdots G_k AHB. \tag{9}$$

Combining (7), (8) and (9) together we obtain a contradiction.

Case 2. Suppose $\liminf_{i\to\infty} |x_0^{(i)}|_G < \infty$.

Again, by passing to a subsequence, we are able to assume that $x_0^{(i)} = x_0 \in A$ for all i. Thus,

$$g \in G_1 \cdots G_k x_0 \langle A^{(i)}, B^{(i)} \rangle B \quad \text{for all } i. \tag{10}$$

Now, since the subset $G_1 \cdots G_k x_0 B$ is closed in $\mathcal{PT}(G)$, we can find a normal subgroup K having finite index in G and satisfying

$$g \notin G_1 \cdots G_k x_0 KB = G_1 \cdots G_k x_0 HB, \tag{11}$$

where $H = KB \leqslant_f G$ and $A \cap B \leqslant H$. Similarly to Case 1, formula (11) leads to a contradiction with formula (10).

Case 3. Suppose $\liminf_{i\to\infty} |y_{l+1}^{(i)}|_G < \infty$ (though l may depend on i, it does not matter for us).

This case can be resolved in the same way as Case 2.

And, finally, the last

Case 4. For every $t \in \{1, \ldots, k\}$ we have $\lim_{i\to\infty} |z_t^{(i)}|_G = \infty$ and, in addition, $\lim_{i\to\infty} |x_0^{(i)}|_G = \lim_{i\to\infty} |y_{l+1}^{(i)}|_G = \infty$.

Then for some $i > I_1$, we will have $|z_t^{(i)}|_G > C_3$ for $t = 1, \ldots, k$, $|x_0^{(i)}|_G > C_3$, $|x_j^{(i)}|_G > C_3$ (since $x_j^{(i)} \in A^{(i)} \setminus (A \cap B)$ and $A \cap B = \bigcap_{i=1}^{\infty} A^{(i)}$), $|y_j^{(i)}|_G > C_3$ for $j = 1, \ldots, l$, $|y_{l+1}^{(i)}|_G > C_3$, where

$$C_3 \stackrel{\text{def}}{=} \max\left\{ \frac{C_1 + \bar{c}}{\bar{\lambda}}, \frac{|g|_G + c}{\lambda} \right\}.$$

Choose the geodesic paths $q_1, \ldots, q_{k+2l+2}$ in $\Gamma(G, \mathcal{A})$ as follows: $(q_1)_- = 1_G$, $elem(q_1) = z_1^{(i)}$, $\ldots$, $(q_k)_- = (q_{k-1})_+$, $elem(q_k) = z_k^{(i)}$, $(q_{k+1})_- = (q_k)_+$, $elem(q_{k+1}) = x_0^{(i)}$, $(q_{k+2})_- = (q_{k+1})_+$, $elem(q_{k+2}) = y_1^{(i)}$, $\ldots$, $(q_{k+2l+2})_- = (q_{k+2l+1})_+$, $elem(q_{k+2l+2}) = y_{l+1}^{(i)}$. Recalling (6) we see that $(q_{k+2l+2})_+ = g$.

Now, by the construction of presentation (6), we can first apply Lemma 2.3 and then Lemma 2.4 to the broken line $q = q_1 \cdots q_{k+2l+2}$. Thus, q is (λ, c)-quasigeodesic. Since $q_- = 1_G$, $q_+ = g$, we get

$$|g|_G = d(q_-, q_+) \geqslant \lambda \|q\| - c \geqslant \lambda \|q_1\| - c > \lambda C_3 - c \geqslant |g|_G.$$

The contradiction achieved finishes the proof. □

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