COMPARING CONSTRUCTIONS OF THE CLASSIFYING SPACE FOR THE FIBRE OF THE DOUBLE SUSPENSION

PAUL SELICK AND STEPHEN THERIAULT

ABSTRACT. The different constructions of a classifying space for the fibre of the double suspension by Gray and the authors are shown to be essentially the same, up to a homotopy equivalence. We go on to compare a variety of maps $\Omega^2 S^{2np+1} \longrightarrow S^{2np-1}$ that are of degree p on the bottom cell.

1. Introduction

The double suspension $E^2\colon S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ is the double adjoint of the identity map on S^{2n+1}. Understanding the relation of E^2 to power maps on $\Omega^2 S^{2n+1}$ is important in determining the homotopy groups of spheres. To elaborate it will be assumed from now on that all spaces and maps are localized at a prime p.

In [CMN1, CMN2] for $p \geq 5$ and in [N] for $p = 3$, it was shown that there is a map $\pi\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ with the property that $E^2 \circ \pi$ is homotopic to the p^{th}-power map on $\Omega^2 S^{2n+1}$. The map π was constructed via a retraction of S^{2n-1} off the loops on the fibre of the pinch map $P^{2n+1}(p) \longrightarrow S^{2n+1}$, where $P^{2n+1}(p)$ is the mod-p Moore space of dimension $2n+1$. The formulation of π was later improved by Anick [An] for primes $p \geq 5$, and subsequently in a much simpler way by Gray and the second author [GT] for primes $p \geq 3$, by showing that it is the connecting map in an associated homotopy fibration. Phrased in the np-case that is relevant to this paper, there is a space T and a homotopy fibration sequence

$$\Omega^2 S^{2np+1} \xrightarrow{\pi} S^{2np-1} \longrightarrow T \longrightarrow \Omega S^{2np+1}. \tag{1}$$

The space T and this homotopy fibration sequence have been well studied and satisfy many favourable properties (see [AG, GT, G2].)

On the other hand, let W_n be the homotopy fibre of E^2. In [G1] it was shown that W_n has a classifying space BW_n and there are homotopy fibrations

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$$

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1}$$

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where $j\circ\nu$ is homotopic to ΩH, with $H\colon \Omega S^{2n+1}\longrightarrow\Omega S^{2np+1}$ being the p^{th}-James-Hopf invariant. Harper [H] showed that if p is odd then $\Omega E^2\circ\Omega\phi$ is homotopic to the p^{th}-power map on Ω^3S^{2np+1}, and this was later improved by Richter [R] who showed that if p is any prime then $E^2\circ\phi\simeq p$.

It would be ideal if the two constructions were linked. Pre-dating Anick's fibration, the map π in (1) was constructed by Cohen, Moore and Neisendorfer [CMN1]. In [CMN2, Introduction] it was conjectured that if p is odd there is a homotopy equivalence $W_n\simeq\Omega D$, where D is the homotopy fibre of π. In light of the existence of Anick's fibration, $D\simeq\Omega T$. Combined with Gray's classifying space for W_n the conjecture can be strengthened to the existence of a homotopy equivalence $BW_n\simeq\Omega T$. This would occur, for example, if the maps $\Omega^2S^{2np+1}\xrightarrow{\pi}S^{2np-1}$ and $\Omega^2S^{2np+1}\xrightarrow{\phi}S^{2np-1}$ were homotopic, up to a self-equivalence of Ω^2S^{2np+1}. In [G1] the space BW_n was shown to be an H-space if p is odd, so an even stronger version of the conjecture is that there is a homotopy equivalence of H-spaces $BW_n\simeq\Omega T$.

In [ST] the authors gave a different construction of a classifying space for W_n at odd primes, showing that there are homotopy fibrations

$$S^{2n-1}\xrightarrow{E^2}\Omega^2S^{2n+1}\xrightarrow{\nu'}B_n$$

$$B_n\xrightarrow{j'}\Omega^2S^{2np+1}\xrightarrow{\phi'}S^{2np-1}$$

where $j'\circ\nu'\simeq\Omega H$. They also used Gray's construction to produce a potentially different map $\Omega^2S^{2np+1}\xrightarrow{\overline{\phi}}S^{2np-1}$ with homotopy fibre BW_n but satisfying $E^2\circ\overline{\phi}\simeq p$ in a much simpler and more conceptual way than Richter's argument.

The current state of affairs, then, has two constructions of a classifying space for BW_n (a third by Moore and Neisendorfer [MN, Section 4] was shown in the same paper to be equivalent to Gray's in an appropriate manner) and four maps $\Omega^2S^{2np+1}\longrightarrow S^{2np-1}$. The purpose of this paper is to compare the various constructions. First, we show that BW_n and B_n are homotopy equivalent in a manner compatible with the maps ν,ν' and j,j'. Consequently, $\Omega\phi$ and $\Omega\phi'$ are shown to be homotopic up to a self-equivalence of ΩS^{2np-1}. Second, we show that $\Omega\phi$ and $\Omega\overline{\phi}$ are homotopic up to a self-equivalence of Ω^3S^{2np+1}. Third, we show that the conjectured H-space equivalence $BW_n\simeq\Omega T$ implies that $\Omega\phi$ and $\Omega\pi$ are homotopic up to a self-equivalence of Ω^3S^{2np+1}. This conjecture is known to hold in a small number of cases related to the existence of elements of mod-p Kervaire invariant one [Am]. Otherwise, the conjecture is very mysterious: we conclude the paper by giving homological evidence that it is true.

2. Comparing constructions for a classifying space of W_n

The comparison of BW_n and B_n is based on refining the construction of B_n in [ST]. The latter was based on linking Milnor's classifying space construction applied to Ω^2S^{2n+1} and the James construction on ΩS^{2n+1}.

In general, let $X^{\wedge k}$ be the smash product of k copies of X with itself and let X^{*k} be the join of k copies of X with itself. Observe that $X^{*k} \simeq \Sigma^{k-1} X^{\wedge k}$. Milnor's classifying space construction applied to $\Omega^2 S^{2n+1}$ gives, for each $k \geq 1$, a homotopy fibration diagram

$$
\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\partial_k} & (\Omega^2 S^{2n+1})^{*(k+1)} & \longrightarrow & \mathcal{P}_k(\Omega^2 S^{2n+1}) & \xrightarrow{ev_k} & \Omega S^{2n+1} \\
\Vert & & \downarrow & & \downarrow & & \Vert \\
\Omega^2 S^{2n+1} & \xrightarrow{\partial_{k+1}} & (\Omega^2 S^{2n+1})^{*(k+2)} & \longrightarrow & \mathcal{P}_{k+1}(\Omega^2 S^{2n+1}) & \xrightarrow{ev_{k+1}} & \Omega S^{2n+1}.
\end{array} \tag{2}
$$

The $k = 1$ case has $\mathcal{P}_k(\Omega^2 S^{2n+1}) = \Sigma \Omega^2 S^{2n+1}$ and ev_1 is the canonical evaluation map. Three properties will be relevant.

Lemma 2.1. *In (2), for $k \geq 1$ the following hold:*

(a) *∂_k is null homotopic;*
(b) *the map $(\Omega^2 S^{2n+1})^{*(k+1)} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)}$ is null homotopic;*
(c) *the space $(\Omega^2 S^{2n+1})^{*(k+1)}$ is $(2n(k+1)-2)$-connected.*

Proof. Part (a) follows from the fact that Ωev_1 has a right homotopy inverse, so the homotopy commutativity of the loops on the right square in (2) implies inductively that Ωev_k has a right homotopy inverse. Part (b) is from the fact that the upper direction around the middle square in (2) is a homotopy cofibration, so the map $(\Omega^2 S^{2n+1})^{*(k+1)} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)}$ lifts through ∂_{k+1}, which is null homotopic by part (a). Part (c) follows from the fact that $(\Omega^2 S^{2n+1})^{*(k+1)} \simeq \Sigma^k (\Omega^2 S^{2n+1})^{\wedge(k+1)}$. $\square$

The connectivity statement in Lemma 2.1 (c) immediately implies the following.

Corollary 2.2. *If X is a CW-complex of dimension $\leq 2n(k+1)-2$ then any map $X \longrightarrow \Omega S^{2n+1}$ has a unique lift (up to homotopy) through ev_k to a map $X \longrightarrow \mathcal{P}_k(\Omega^2 S^{2n+1})$.* $\square$

For a path-connected space X let $X^{\times k}$ be the product of k copies of X with itself. Let $J_k(X)$ be the quotient space obtained from $X^{\times k}$ given by identifying $(x_1, \ldots, x_i, *, x_{i+2}, \ldots, x_k)$ with $(x_1, \ldots, x_i, x_{i+2}, *, \ldots, x_k)$. There is an inclusion $J_k(X) \longrightarrow J_{k+1}(X)$ given by sending $(x_1, \ldots, x_k)$ to $(x_1, \ldots, x_k, *)$, and $J(X)$ is defined as the colimit of the spaces $J_k(X)$. James [J] showed that there is a homotopy equivalence $J(X) \simeq \Omega\Sigma X$. In particular, the space $J_k(S^{2n})$ has dimension $2nk$ and the map $J_k(S^{2n}) \longrightarrow J(S^{2n}) \simeq \Omega S^{2n+1}$ can be regarded as the inclusion of the $2nk$-skeleton.

Since $J_k(S^{2n})$ has dimension $2nk$, Corollary 2.2 implies the inclusion $J_k(S^{2n}) \longrightarrow \Omega S^{2n+1}$ lifts through ev_k to a map $J_k(S^{2n}) \longrightarrow \mathcal{P}_k(\Omega^2 S^{2n+1})$. From this lift we obtain a homotopy fibration diagram

$$
(3)\qquad
\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\delta_k} & Y_k & \longrightarrow & J_k(S^{2n}) & \longrightarrow & \Omega S^{2n+1} \\
\Vert & & \downarrow & & \downarrow & & \Vert \\
\Omega^2 S^{2n+1} & \xrightarrow{\partial_k} & (\Omega^2 S^{2n+1})^{*(k+1)} & \longrightarrow & \mathcal{P}_k(\Omega^2 S^{2n+1}) & \xrightarrow{ev_k} & \Omega S^{2n+1}
\end{array}
$$

that defines the space Y_k and the map δ_k. Suppose that $k \geq 1$ and consider the square

$$
(4)\qquad
\begin{array}{ccc}
J_k(S^{2n}) & \longrightarrow & J_{k+1}(S^{2n}) \\
\downarrow & & \downarrow \\
\mathcal{P}_k(\Omega^2 S^{2n+1}) & \longrightarrow & \mathcal{P}_{k+1}(\Omega^2 S^{2n+1}).
\end{array}
$$

Both directions around the diagram are lifts of the map $J_k(S^{2n}) \longrightarrow \Omega S^{2n+1}$ through ev_k, so as the dimension of $J_k(S^{2n})$ is $2nk$, the uniqueness property in Lemma 2.1 (b) implies that the two lifts are homotopic. That is, the square homotopy commutes. Mapping all four corners into ΩS^{2n+1} and taking homotopy fibres gives homotopy fibration diagrams

$$
(5)\qquad
\begin{array}{ccccc}
Y_k & \xrightarrow{y_k} & Y_{k+1} & \longrightarrow & (\Omega^2 S^{2n+1})^{*(k+2)} \\
\downarrow & & \downarrow & & \downarrow \\
J_k(S^{2n}) & \longrightarrow & J_{k+1}(S^{2n}) & \longrightarrow & P_{k+1}(\Omega^2 S^{2n+1}),
\end{array}
$$

where y_k is an induced map of fibres, and

$$
(6)\qquad
\begin{array}{ccccc}
Y_k & \longrightarrow & (\Omega^2 S^{2n+1})^{*(k+1)} & \longrightarrow & (\Omega^2 S^{2n+1})^{*(k+2)} \\
\downarrow & & \downarrow & & \downarrow \\
J_k(S^{2n}) & \longrightarrow & P_k(\Omega^2 S^{2n+1}) & \longrightarrow & P_{k+1}(\Omega^2 S^{2n+1}).
\end{array}
$$

Lemma 2.3. *In (5) the following hold:*

(a) *taking fibration connecting maps for the left square gives* $\delta_{k+1} \simeq y_k \circ \delta_k$*;*

(b) *the composite* $Y_k \xrightarrow{y_k} Y_{k+1} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)}$ *is null homotopic.*

Proof. Part (a) is immediate from the definitions of the maps. For part (b), it suffices to show that the composite

$$
(7)\qquad Y_k \xrightarrow{y_k} Y_{k+1} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)} \longrightarrow P_{k+1}(\Omega^2 S^{2n+1})
$$

is null homotopic. For if so then $Y_k \xrightarrow{y_k} Y_{k+1} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)}$ lifts through ∂_{k+1}, which by Lemma 2.1 (a) is null homotopic. By (5), the composite (7) is homotopic to

$$
Y_k \longrightarrow J_k(S^{2n}) \longrightarrow J_{k+1}(S^{2n}) \longrightarrow P_{k+1}(\Omega^2 S^{2n+1}),
$$

which by (4) is homotopic to

$$
Y_k \longrightarrow J_k(S^{2n}) \longrightarrow P_k(\Omega^2 S^{2n+1}) \longrightarrow P_{k+1}(\Omega^2 S^{2n+1}),
$$

which in turn by (6) is homotopic to

$$Y_k \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)} \longrightarrow P_{k+1}(\Omega^2 S^{2n+1}).$$

But by Lemma 2.1 (b), the map $(\Omega^2 S^{2n+1})^{*(k+1)} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)}$ is null homotopic, and therefore the composite (7) is null homotopic. □

If $k \geq 1$ the evaluation map $\Sigma^2\Omega^2 S^{2n+1} \longrightarrow S^{2n+1}$ can be used iteratively to obtain a map

$$g_k \colon (\Omega^2 S^{2n+1})^{*(k+1)} \longrightarrow S^{2n(k+1)-1}.$$

Define h_k by the composite

$$h_k \colon Y_k \longrightarrow (\Omega^2 S^{2n+1})^{*(k+1)} \xrightarrow{g_k} S^{2n(k+1)-1}$$

and let $B_{n,k}$ be the homotopy fibre of h_k.

Lemma 2.4. *For $k \geq 1$ there is a lift*

$$\begin{array}{ccc} & & B_{n,k+1} \\ & \nearrow & \downarrow \\ Y_k & \xrightarrow{y_k} & Y_{k+1}. \end{array}$$

Proof. By Lemma 2.3 (b), the composite $Y_k \xrightarrow{y_k} Y_{k+1} \longrightarrow (\Omega^2 S^{2n+1})^{*(k+2)}$ is null homotopic. By definition, h_{k+1} factors through the right map, so $h_{k+1} \circ y_k$ is null homotopic. Thus y_k lifts to the fibre $B_{n,k+1}$ of h_{k+1}. □

Observe that if $k = p-1$ then the homotopy fibration in the top row of (3) is

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \longrightarrow J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2n+1}$$

where H is the p^{th}-James-Hopf invariant. That is, $Y_{p-1} = \Omega^2 S^{2np+1}$ and $\delta_{p-1} = \Omega H$. In [ST] it was observed that $h_{p-1} \circ \Omega H$ is null homotopic, giving a lift

$$\begin{array}{ccc} & & B_{n,p-1} \\ & \overset{\widetilde{\nu}}{\nearrow} & \downarrow \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega H} & \Omega^2 S^{2np+1} \end{array}$$

for some map $\widetilde{\nu}$, and that for any choice of lift $\widetilde{\nu}$ there is a homotopy fibration

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\widetilde{\nu}} B_{n,p-1}.$$

Thus $B_{n,p-1}$ is a classifying space for the fibre of the double suspension.

In light of Lemma 2.4, the lift $\widetilde{\nu}$ can be chosen more deliberately. Define ν' by the composite

$$\nu' \colon \Omega^2 S^{2n+1} \xrightarrow{\partial_{p-2}} Y_{p-2} \longrightarrow B_{n,p-1}$$

where the right map is from Lemma 2.4.

Lemma 2.5. *If $p \geq 3$ then the map ν' is a lift of ΩH.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\partial_{p-1}} & Y_{p-2} & \longrightarrow & B_{n,p-1} \\
\| & & \big\downarrow{\scriptstyle y_{p-2}} & & \big\downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{\Omega H} & \Omega^2 S^{2np+1} & = & \Omega^2 S^{2np+1}.
\end{array}$$

Remembering that $\Omega^2 S^{2np+1} = Y_{p-1}$, $\Omega H = \delta_{p-1}$ and $B_n = B_{n,p-1}$, the left square homotopy commutes by Lemma 2.3 (a) and the right square homotopy commutes by Lemma 2.4. Note that having $p \geq 3$ ensures that the map y_{p-2} exists. The top row of the diagram is the definition of ν'. Its homotopy commutativity therefore implies that ν' is a lift of ΩH. □

To summarise, let $B_n = B_{n,p-1}$ and let $\phi' \colon \Omega^2 S^{2np+1} \longrightarrow S^{2np-1}$ be h_{p-1}. Then there are homotopy fibrations

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu'} B_n$$

$$B_n \xrightarrow{j'} \Omega^2 S^{2np+1} \xrightarrow{\phi'} S^{2np-1}$$

where $j' \circ \nu' \simeq \Omega H$ and ν' factors as $\Omega^2 S^{2n+1} \xrightarrow{\partial_{p-2}} Y_{p-2} \longrightarrow B_n$.

In comparison, Gray [G1] constructed homotopy fibrations

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$$

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1}$$

where $j \circ \nu \simeq \Omega H$ and showed that there is a factorization

(8)
$$\begin{array}{ccc}
 & & BW_n \\
 & \overset{\nu}{\nearrow} & \big\downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{\delta_1} & Y_1.
\end{array}$$

Our first main result is to show that there is a homotopy equivalence between B_n and BW_n that is compatible with the maps j', ν' and j, ν.

Theorem 2.6. *If $p \geq 3$ then there is a homotopy commutative diagram*

$$\begin{array}{ccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n & \xrightarrow{j} & \Omega^2 S^{2np+1} \\
\| & & \big\downarrow{\scriptstyle e} & & \| \\
\Omega^2 S^{2n+1} & \xrightarrow{\nu'} & B_n & \xrightarrow{j'} & \Omega^2 S^{2np+1}
\end{array}$$

where e is a homotopy equivalence.

Proof. Define e by the composite

$$e\colon BW_n \longrightarrow Y_1 \xrightarrow{y_1} Y_2 \longrightarrow \cdots \longrightarrow Y_{p-2} \longrightarrow B_{p-1}.$$

Consider the diagram

$$\begin{array}{ccccccccccc}
\Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} & = & \cdots & \longrightarrow & \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\
\big\downarrow{\scriptstyle \nu} & & \big\downarrow{\scriptstyle \delta_1} & & \big\downarrow{\scriptstyle \delta_2} & & & & \big\downarrow{\scriptstyle \delta_{p-2}} & & \big\downarrow{\scriptstyle \nu'} \\
BW_n & \longrightarrow & Y_1 & \xrightarrow{y_1} & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_{p-2} & \longrightarrow & B_n.
\end{array}$$

The left square homotopy commutes by (8), the middle squares homotopy commute by Lemma 2.3 (a), and the right square homotopy commutes by definition of ν'. The bottom row is the definition of e. Thus the homotopy commutativity of the diagram as a whole implies that $e \circ \nu \simeq \nu'$.

Since the homotopy fibre of both ν and ν' is S^{2n-1}, the homotopy $e \circ \nu \simeq \nu'$ implies that there is a homotopy fibration diagram

$$\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{t} & S^{2n-1} & \longrightarrow & X \\
\Big\| & & \big\downarrow{\scriptstyle E^2} & & \big\downarrow \\
S^{2n-1} & \xrightarrow{E^2} & \Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n \\
 & & \big\downarrow{\scriptstyle \nu'} & & \big\downarrow{\scriptstyle e} \\
 & & B_n & = & B_n
\end{array} \tag{9}$$

that defines the space X and the map t. Since E^2 induces an isomorphism on H_{2n-1}, the commutativity of the upper left square implies that t must induce an isomorphism on H_{2n-1}. Thus t is a homotopy equivalence, implying that X is contractible. Hence e is a homotopy equivalence.

Now consider the diagram

$$\begin{array}{ccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n & \xrightarrow{j} & \Omega^2 S^{2np+1} \\
\Big\| & & \big\downarrow{\scriptstyle e} & & \Big\| \\
\Omega^2 S^{2n+1} & \xrightarrow{\nu'} & B_n & \xrightarrow{j'} & \Omega^2 S^{2np+1}.
\end{array} \tag{10}$$

The left square homotopy commutes by (9). The composites $j \circ \nu$ and $j' \circ \nu'$ are both homotopic to ΩH, so the outer rectangle also homotopy commutes. We wish to show that the right square also homotopy commutes. It is equivalent to show that the difference $d = j - j' \circ e$ is null homotopic. The homotopy commutativity of the left square and outer rectangle in (10) implies that $d \circ \nu$ is null homotopic. Thus if $\tilde{d}$ is the double adjoint of d then the composite

$$\Sigma^2\Omega^2 S^{2n+1} \xrightarrow{\Sigma^2\nu} \Sigma^2 BW_n \xrightarrow{\tilde{d}} S^{2np+1}$$

is null homotopic. By [G1], $\Sigma^2\nu$ has a right homotopy inverse. Hence $\tilde{d}$ is null homotopic, and therefore so is d. □

Theorem 2.6 also lets us compare the maps ϕ and ϕ'.

Corollary 2.7. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^3 S^{2np+1} & \xrightarrow{\Omega\phi} & \Omega S^{2np-1} \\ \| & & \downarrow e' \\ \Omega^3 S^{2np+1} & \xrightarrow{\Omega\phi'} & \Omega S^{2np-1} \end{array}$$

where e' is a homotopy equivalence.

Proof. From the right square in the statement of Theorem 2.6 we obtain a homotopy fibration diagram

$$\begin{array}{ccccccc} \Omega^3 S^{2np+1} & \xrightarrow{\Omega\phi} & \Omega S^{2np-1} & \longrightarrow & BW_n & \xrightarrow{j} & \Omega^2 S^{2np+1} \\ \| & & \downarrow e' & & \downarrow e & & \| \\ \Omega^3 S^{2np+1} & \xrightarrow{\Omega\phi'} & \Omega S^{2np-1} & \longrightarrow & B_{p-1} & \xrightarrow{j'} & \Omega^2 S^{2np+1} \end{array}$$

that defines the map e'. Since e is a homotopy equivalence, the Five-Lemma implies that e' induces an isomorphism on homotopy groups and so is a homotopy equivalence by Whitehead's Theorem. □

3. Comparing ϕ and $\overline{\phi}$

In general, if X is an H-space with multiplication m then there is a homotopy fibration sequence

$$\Omega\Sigma X \xrightarrow{r} X \longrightarrow X * X \xrightarrow{m^*} \Sigma X$$

where m^* is the Hopf construction on m and the map r has a right homotopy inverse. If the multiplication m is homotopy associative then by [St] the map r can be chosen to be an H-map.

In our case, localize at an odd prime p. Then Gray [G1] shows that BW_n is a homotopy asssociative H-space and in the homotopy fibration sequence

$$\Omega S^{2np-1} \xrightarrow{k} BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1} \tag{11}$$

the maps j and k are H-maps. In [ST] it was shown that there is a homotopy pullback

$$\begin{array}{ccc} \Omega^2 S^{2np+1} & \xrightarrow{\overline{\phi}} & S^{2np-1} \\ \downarrow g & & \downarrow i \\ BW_n * BW_n & \xrightarrow{m^*} & \Sigma BW_n \end{array} \tag{12}$$

where i is the inclusion of the bottom cell and g and $\overline{\phi}$ are induced by the pullback. The maps ϕ and $\overline{\phi}$ need not be homotopic. The map ϕ is interesting because of its immediate association with BW_n; the map $\overline{\phi}$ is interesting because in [ST] a straightforward argument was given to show that $E^2 \circ \overline{\phi}$ is homotopic to the p^{th}-power map on $\Omega^2 S^{2np+1}$. We now compare ϕ and $\overline{\phi}$.

Proposition 3.1. *If $p \geq 3$ then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^3 S^{2np+1} & \xrightarrow{\Omega\phi} & \Omega S^{2np-1} \\ \big\downarrow{\scriptstyle \overline{e}} & & \big\| \\ \Omega^3 S^{2np+1} & \xrightarrow{\Omega\overline{\phi}} & \Omega S^{2np-1} \end{array}$$

where $\overline{e}$ is a homotopy equivalence.

Proof. From (12) we obtain a homotopy fibration diagram

$$\begin{array}{ccccccc} \Omega S^{2np-1} & \xrightarrow{h} & BW_n & \longrightarrow & \Omega^2 S^{2np+1} & \xrightarrow{\overline{\phi}} & S^{2np-1} \\ \big\downarrow{\scriptstyle \Omega i} & & \big\| & & \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \Omega\Sigma BW_n & \xrightarrow{r} & BW_n & \longrightarrow & BW_n * BW_n & \xrightarrow{m^*} & \Sigma BW_n \end{array} \tag{13}$$

that defines the maps $\overline{\phi}$ and h. The map r is an H-map since BW_n is homotopy associative, so the homotopy $h \circ r \circ \Omega i$ in the leftmost square implies that h is also an H-map.

In general, for a path-connected space A, let $E\colon A \longrightarrow \Omega\Sigma A$ be the suspension. By the James construction [J], if Y is a homotopy associative H-space then any map $f\colon A \longrightarrow Y$ extends to an H-map $\overline{f}\colon \Omega\Sigma A \longrightarrow Y$, and this is the unique H-map, up to homotopy, such that $\overline{f} \circ E \simeq f$.

In our case this implies that the H-maps $\Omega S^{2np-1} \xrightarrow{h} BW_n$ in (13) and $\Omega S^{2np-1} \xrightarrow{k} BW_n$ in (11) are determined by their restrictions to the bottom cell. In both cases the restrictions are the same – the inclusion of the bottom cell – so $h \simeq k$. This homotopy implies that there is a homotopy fibration diagram

$$\begin{array}{ccccc} \Omega^3 S^{2np+1} & \xrightarrow{\Omega\phi} & \Omega S^{2np-1} & \xrightarrow{k} & BW_n \\ \big\downarrow{\scriptstyle \overline{e}} & & \big\| & & \big\| \\ \Omega^3 S^{2np+1} & \xrightarrow{\Omega\overline{\phi}} & \Omega S^{2np-1} & \xrightarrow{h} & BW_n \end{array}$$

that defines the map $\overline{e}$. The Five-Lemma implies that $\overline{e}$ induces an isomorphism on homotopy groups and so is a homotopy equivalence by Whitehead's Theorem. □

4. Comparing ϕ and π

There is an analogue of the homotopy pullback (12) with respect to ΩT. In this case the homotopy fibration involving the Hopf construction extends to

$$\Omega T * \Omega T \xrightarrow{m^*} \Sigma\Omega T \xrightarrow{ev} T,$$

where ev is the evaluation map. Let $i'\colon S^{2np-1} \longrightarrow \Sigma\Omega T$ be the inclusion of the bottom cell. Since $ev \circ i'$ is the inclusion of the bottom cell into T, its homotopy fibre is $\Omega^2 S^{2np+1} \xrightarrow{\pi} S^{2np-1}$ and we obtain a homotopy pullback

$$\begin{array}{ccc} \Omega^2 S^{2np+1} & \xrightarrow{\pi} & S^{2np-1} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle i'} \\ \Omega T * \Omega T & \xrightarrow{m^*} & \Sigma\Omega T \end{array} \tag{14}$$

for some map g'.

We next show that if BW_n and ΩT are homotopy equivalent as H-spaces then the maps $\overline{\phi}$ and π are homotopic, up to a self-equivalence of $\Omega^2 S^{2np+1}$.

Proposition 4.1. *If there is an H-equivalence $h\colon BW_n \longrightarrow \Omega T$ then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2np+1} & \xrightarrow{\overline{\phi}} & S^{2np-1} \\ \downarrow{\scriptstyle \epsilon} & & \| \\ \Omega^2 S^{2np+1} & \xrightarrow{\pi} & S^{2np-1} \end{array}$$

where ϵ is a homotopy equivalence.

Proof. Consider the cube

$$\begin{array}{ccccccc} \Omega^2 S^{2np+1} & & \xrightarrow{\quad\overline{\phi}\quad} & & S^{2np-1} & & \\ & \searrow^{\epsilon} & & & \downarrow & \searrow^{=} & \\ \downarrow{\scriptstyle g} & & \Omega^2 S^{2np+1} & \xrightarrow{\quad\pi\quad} & & & S^{2np-1} \\ & & & & \downarrow{\scriptstyle i} & & \\ BW_n * BW_n & & \xrightarrow{\quad m^*\quad} & & \Sigma BW_n & & \downarrow{\scriptstyle i'} \\ & \searrow_{h*h} & \downarrow{\scriptstyle g'} & & & \searrow^{\Sigma h} & \\ & & \Omega T * \Omega T & \xrightarrow{\quad m^*\quad} & & & \Sigma\Omega T \end{array} \tag{15}$$

where the map ϵ is to be defined momentarily. The lower face homotopy commutes since h is an H-map. The right face homotopy commutes since both i and i' are the inclusion of the bottom cell. The rear and front faces are homotopy pullbacks by (12) and (14) respectively. Thus $i' \circ \overline{\phi}$ is homotopic to $m^* \circ (h * h) \circ g$, implying that there is a pullback map ϵ such that $g' \circ \epsilon \simeq (h * h) \circ g$ and $\pi \circ \epsilon \simeq \overline{\phi}$. Since π and $\overline{\phi}$ are both degree p on the bottom cell, the homotopy $\pi \circ \epsilon \simeq \overline{\phi}$ implies that ϵ is degree 1 on the bottom cell. Since $\Omega^2 S^{2np+1}$ is atomic [CM], ϵ is therefore a homotopy equivalence. □

Combining Propositions 3.1 and 4.1 lets us compare ϕ and π.

Corollary 4.2. *If there is an H-equivalence $h\colon BW_n \longrightarrow \Omega T$ then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^3 S^{2np+1} & \xrightarrow{\Omega\phi} & \Omega S^{2np-1} \\ \downarrow{\scriptstyle \varepsilon} & & \| \\ \Omega^3 S^{2np+1} & \xrightarrow{\Omega\pi} & \Omega S^{2np-1} \end{array}$$

where ε is a homotopy equivalence. □

In general it is not known whether BW_n and ΩT are homotopy equivalent, let alone homotopy equivalent as H-spaces. However, there are a small number of cases where a homotopy equivalence is known and in all such cases the equivalences is an H-equivalence. In [T2] it was shown that there is an H-equivalence $BW_n \simeq \Omega T$ if p is odd and $n \in \{1, p\}$, and in [Am] it was shown that there is also ab H-equivalence if $p = 3$ and $n \in \{9, 27\}$. Thus Corollary 4.2 immediately implies the following.

Corollary 4.3. *The maps $\Omega^3 S^{2np+1} \xrightarrow{\Omega\phi} \Omega S^{2np-1}$ and $\Omega^3 S^{2np+1} \xrightarrow{\Omega\pi} \Omega S^{2np-1}$ are homotopic, up to a self-equivalence of $\Omega^3 S^{2np+1}$, provided either:*

(a) *p is odd and $n \in \{1, p\}$;*

(b) *$p = 3$ and $n \in \{9, 27\}$.* □

5. Homological evidence for an H-equivalence $BW_n \simeq \Omega T$

Let p be an odd prime and let $S^{2n+1}\{p\}$ be the homotopy fibre of the p^{th}-power map on S^{2n+1}. In [S] it was shown that there is a lift

$$\begin{array}{ccccc} & & & & \Omega S^{2np+1}\{p\} \\ & & \overset{S'}{\nearrow} & & \downarrow \\ \mathcal{P}_{p-1}(\Omega^2 S^{2n+1}) & \xrightarrow{ev_{p-1}} & \Omega S^{2n+1} & \xrightarrow{H} & \Omega S^{2np+1} \end{array}$$

for some map S'. In [S] it was also shown that the composite

$$s\colon \Omega^2 S^{2n+1} \xrightarrow{E} \Omega\Sigma\Omega^2 S^{2n+1} = \Omega\mathcal{P}_1(\Omega^2 S^{2n+1}) \longrightarrow \Omega\mathcal{P}_{p-1}(\Omega^2 S^{2n+1})$$

is an H-map (in fact, the same argument shows it is an A_{p-1}-map, in the sense of Stasheff). Let S be the composite

$$S\colon \Omega^2 S^{2n+1} \xrightarrow{s} \Omega\mathcal{P}_{p-1}(\Omega^2 S^{2n+1}) \xrightarrow{\Omega S'} \Omega S^{2np+1}\{p\}.$$

Then S is an H-map (an A_{p-1}-map) since it is the composite of H-maps (A_{p-1}-maps) and as s is a right homotopy inverse for Ωev_{p-1}, the map S is a lift of ΩH.

There is a potential improvement. In [GT] it was shown that there is a homotopy fibration

$$T \longrightarrow \Omega S^{2np+1}\{p\} \longrightarrow BW_{np}.$$

Conjecture 5.1. If $p \geq 3$ then there is a lift

$$\begin{array}{ccc} & & T \\ & \overset{S''}{\nearrow} & \downarrow \\ \mathcal{P}_{p-1}(\Omega^2 S^{2n+1}) & \xrightarrow{S'} & \Omega S^{2np+1}\{p\} \end{array}$$

for some map S''.

Conjecture 5.1 is a strong form of the conjecture that $BW_n \simeq \Omega T$.

Proposition 5.2. *If Conjecture 5.1 holds then there is an H-equivalence $BW_n \simeq \Omega T$.*

Proof. Let $\mathcal{S}$ be the composite

$$\mathcal{S}\colon \Omega^2 S^{2n+1} \xrightarrow{s} \Omega\mathcal{P}_{p-1}(\Omega^2 S^{2n+1}) \xrightarrow{\Omega S''} \Omega T.$$

Arguing as in [T1, Lemma 2.2] and using the fact that T is an H-space [GT] implies that from $\mathcal{S}$ one obtain an H-map $BW_n \longrightarrow \Omega T$. (The statement of [T1, Lemma 2.2] is for $p \geq 5$ but the $p = 3$ case is also valid.) $\square$

We close the paper by giving homological evidence that Conjecture 5.1 is true. Let p be an odd prime and assume that homology is taken with mod-p coefficients. Theorem 5.4 shows that the image of S'_* lifts to $H_*(T)$.

5.1. **The Eilenberg-Moore Spectral Sequence.** For a topological group G, the Eilenberg-Moore spectral sequence for $G \to EG \to BG$ can be identified with the one associated to the filtration

$$\mathrm{pt} = \mathcal{P}_0(G) \subset \mathcal{P}_1(G) \subset \ldots \subset \mathcal{P}_\infty(G) = BG.$$

Let $\alpha_{j,k}(G) : \mathcal{P}_j(G) \to \mathcal{P}_k(G)$ denote the inclusion and write $\alpha_k(G)$ for $\alpha_{k,\infty}(G)$. As there is a homotopy cofibration $G^{*k} \longrightarrow \mathcal{P}_{k-1}(G) \longrightarrow \mathcal{P}_k(G)$, there is a commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}_{k-1}(G) & \xrightarrow{\alpha_{k-1,k}(G)} & \mathcal{P}_k(G) & \longrightarrow & \Sigma(G)^{*k} \\
& \searrow^{\alpha_{k-1}(G)} & \downarrow{\scriptstyle \alpha_k(G)} & & \\
& & \mathcal{P}_\infty(G) & &
\end{array}$$

where the row is a cofibration and $\Sigma(G)^{*k} \simeq \Sigma^k G^{\wedge k}$. We examine $\alpha_k(G)_*$.

For $v \in \operatorname{Im} \alpha_{k-1,k}(G)_*$ the map is determined by its restriction to $\mathcal{P}_{k-1}(G)$. Given $v \in H_*\big(\mathcal{P}_k(G)\big)$ write $v = \alpha_{k-1,k}(G)_*(v') + w'$ where w' has image w in $H_*\big(\Sigma(G)^{*k}\big)$. Working modulo the inductively known image of $\alpha_{k-1}(G)_*$ we have the following. By exactness $w \in \ker H_*\big(\Sigma(G)^{*k}\big) \to H_*\big(\Sigma\mathcal{P}_{k-1}(G)\big)$. Since $w \mapsto 0$ under $\mathcal{P}_k(G)/\mathcal{P}_{k-1}(G) = \Sigma(G)^{*k} \to \mathcal{P}_{k-1}(G)/\mathcal{P}_{k-2}(G)$ which is the d^1 differential of the spectral sequence, it represents an element $[w]$ in E^2. If $[w]$ is in the image of some differential d_r then $v \mapsto \alpha_{k-1,r}(v')$ under $\alpha_{k,r} : \mathcal{P}_k(G) \to \mathcal{P}_r(G)$, and in particular $\alpha_k(v) = \alpha_{k-1}(v')$. Otherwise $[w]$ survives to E^∞ and contributes to the filtration quotient for some element of $H_*(BG)$, which gives the equivalence class modulo lower filtration of the image of v under $\alpha^*_{k,\infty} : H_*\big(\mathcal{P}_k(G)\big) \to H_*\big(\mathcal{P}_\infty(BG)\big)$.

5.2. **Known homology.** We record the homology of several spaces. This is often phrased in terms of Dyer-Lashof operations Q_t and the calculations can be found in [CLM]. By Q_t^j we mean j copies of Q_t composed with itself. First, there are Hopf algebra isomorphisms

$$H_*(\Omega^2 S^{2n+1}) \cong H_*(S^{2n+1}) \otimes H_*(BW_n) \cong \Lambda[\{a_j\}_{j=0}^{\infty}] \otimes \mathbf{Z}/p[\{b_j\}_{j=1}^{\infty}]$$

where $a_j = Q^j_{p-1}(a_0)$ and $b_j = \beta(a_j)$ and $|Q_{s(p-1)}y| = p|y| + s(p-1)$. Thus $|a_j| = 2np^j - 1$ and $|b_j| = 2np^j - 2$. We will also alternatively write

$$H_*(\Omega^2 S^{2n+1}) \cong \Lambda[\{Q^j_{p-1}(\iota_{2n-1})\}_{j=0}^{\infty}] \otimes \mathbf{Z}/p[\{\beta Q^j_{p-1}(\iota_{2n-1})\}_{j=1}^{\infty}].$$

Second, there is a Hopf algebra isomorphism

$$H_*(\Omega^3 S^{2n+1}) \cong \mathbf{Z}/p[\{Q^j_{2(p-1)}\iota_{2n-2}\}_{j=0}^{\infty}] \otimes \Lambda[\{Q^i_{p-1}\beta Q^j_{2(p-1)}\iota_{2n-2}\}_{j=1}^{\infty}{}_{i=0}^{\infty}] \otimes \mathbf{Z}/p[\{\beta Q^i_{p-1}\beta Q^j_{2(p-1)}\iota_{2n-2}\}_{j=1}^{\infty}{}_{i=0}^{\infty}].$$

Third, there is a Hopf algebra isomorphism

$$H_*(\Omega^2 S^{2np+1}\{p\}) \cong H_*(\Omega^3 S^{2np+1}) \otimes H_*(\Omega^2 S^{2np+1}).$$

We will denote algebra generators of $H_*(\Omega^2 S^{2np+1}\{p\})$ which are images under $H_*(\Omega^3 S^{2np+1}) \to H_*(\Omega^2 S^{2np+1}\{p\})$ by their names in $H_*(\Omega^3 S^{2np+1})$ and abuse notation by writing our choice of preimages of generators in $H_*(\Omega^2 S^{2np+1})$ by their names in $H_*(\Omega^2 S^{2np+1})$, written in Dyer-Lashof notation. We have $\beta(\iota_{2np-1}) = \iota_{2np-2}$, and otherwise the Bockstein is given on the generators by their Bocksteins in $H_*(\Omega^3 S^{2np+1})$ and $H_*(\Omega^2 S^{2np+1})$ respectively.

Fourth, there is a Hopf algebra isomorphism

$$H_*(\Omega S^{2np+1}\{p\}) \cong H_*(\Omega^2 S^{2np+1}) \otimes H_*(\Omega S^{2np+1}).$$

The naming convention for generators follows as in $H_*(\Omega^2 S^{2np+1}\{p\})$, and the Bockstein is given by $\beta(\iota_{2np}) = \iota_{2np-1}$ and the Bocksteins in $H_*(\Omega^2 S^{2np+1})$. Alternatively, using $H_*(\Omega^2 S^{2np+1}) \cong H_*(S^{2np-1}) \otimes H_*(BW_{np})$ we also have

$$H_*(\Omega S^{2np+1}\{p\}) \cong H_*(T) \otimes H_*(BW_{np}).$$

5.3. **The images of S_* and S'_*.** The map $\Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{2np+1}\{p\}$ is an A_{p-1}-map in the sense of Stasheff. In particular, as S is an H-map, the description of $H_*(\Omega^2 S^{2n+1})$ implies that S_* is determined by its images of the odd degree generators and the Bockstein.

Lemma 5.3. $S_*(a_j) = Q^{j-1}_{p-1}\iota_{2np-1}$ *where by convention* $Q_{-1} = 0$.

Proof. Since S is a lift of ΩH and a_j is primitive, we have $S_*(a_j) = Q^{j-1}_{p-1}\iota_{2np-1} + X$ for some primitive $X \in H_*(\Omega^3 S^{2np+1})$. The odd degree primitives in $H_*(\Omega^3 S^{2np+1})$ are $Q^i_{p-1}\beta Q^j_{2(p-1)}\iota_{2np-2}$. Observe that

$$|Q^j_{2(p-1)}\iota_{2np-2}| = 2np^{j+1} - 2$$

so

$$|Q^i_{p-1}\beta Q^j_{2(p-1)}\iota_{2np-2}| = 2np^{j+1+i} - 2p^i - 1.$$

Given j, there is no pair (j',i) for which $2np^{j+1} - 1 = 2np^{j'+1+i} - 2p^i - 1$ since it simplifies to $np^{j+1-i} = np^{j'+1} - 1$ which has the wrong congruence modulo p except possibly when $i = j+1$ in which case the right is larger than the left. Therefore $X = 0$ giving $S_*(a_j) = Q^{j-1}_{p-1}\iota_{2np-1}$. □

The map $\Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{2np+1}\{p\}$ does not induce a map of Eilenberg-Moore spectral sequences with respect to the classifying space construction since it is not a loop map. However, as it is an A_{p-1} map, for $k < p$ we do have

$$\begin{array}{ccccc}
\mathcal{P}_{k-1}(\Omega^2 S^{2n+1}) & \longrightarrow & \mathcal{P}_k(\Omega^2 S^{2n+1}) & \longrightarrow & \Sigma(\Omega^2 S^{2n+1})^{*k} \\
\downarrow{\scriptstyle \mathcal{P}_{k-1}(S)} & & \downarrow{\scriptstyle \mathcal{P}_k(S)} & & \downarrow{\scriptstyle \Sigma S^{*k}} \\
\mathcal{P}_{k-1}(\Omega^2 S^{2np+1}\{p\}) & \longrightarrow & \mathcal{P}_k(\Omega^2 S^{2np+1}\{p\}) & \longrightarrow & \Sigma(\Omega^2 S^{2np+1}\{p\})^{*k} \\
 & & \downarrow{\scriptstyle \alpha_k(\Omega^2 S^{2np+1}\{p\})} & & \\
 & & \mathcal{P}_\infty(\Omega^2 S^{2np+1}\{p\}) = \Omega S^{2np+1}\{p\}. & &
\end{array}$$

We wish to compute $\big(\alpha_k(\Omega^2 S^{2np+1}\{p\}) \circ \mathcal{P}_k(S)\big)_*$. Note that $S' = \alpha_{p-1}(\Omega^2 S^{2np+1}\{p\}) \circ \mathcal{P}_{p-1}(S)$. Assume by induction that $\big(\alpha_{k-1}(\Omega^2 S^{2np+1}\{p\}) \circ \mathcal{P}_{k-1}(S)\big)_*$ is understood. Let $v \in H_*\big(\mathcal{P}_k(\Omega^2 S^{2n+1})\big)$ and write $v = v' + w'$ where $v' \mapsto 0$ under $H_*\big(\mathcal{P}_k(\Omega^2 S^{2n+1})\big) \to H_*\big(\Sigma(\Omega^2 S^{2n+1})^{*p}\big)$ and w' has image $w \in H_*\big(\Sigma(\Omega^2 S^{2n+1})^{*p}\big)$. Applying Lemma 5.3 and our knowledge of the Eilenberg-Moore spectral sequence for $\Omega^2 S^{2np+1}\{p\}$, we see that the only elements w for which $\Sigma(S^{*k})_*(w)$ survives the spectral sequence for $\Omega^2 S^{2np+1}\{p\}$ are $\underbrace{\sigma(a_j) \otimes \sigma(a_j) \cdots \otimes \sigma(a_j)}_{k \text{ times}}$ for some j, which become representatives for $\iota^{kp^j}_{2np} \in H_*(\Omega S^{2np+1}\{p\})$. The restriction to $\mathcal{P}_1(\Omega^2 S^{2n+1})$ is determined by $a_0 \mapsto 0$ and $\sigma(a_j) \mapsto \iota^{p^{j-1}}_{2np}$ together with the action of the Bockstein which is determined by $\beta(\iota_{2np)} = \iota_{2np-1}$. Thus $\mathrm{Im}\big(\alpha_k(\Omega^2 S^{2np+1}\{p\}) \circ \mathcal{P}_k(S)\big)_*$ equals

$$\mathrm{Im}\big(\alpha_{k-1}(\Omega^2 S^{2np+1}\{p\}) \circ \mathcal{P}_k(S)\big)_* + \langle \{\iota^{kp^j}_{2np}\}^\infty_{j=1}\rangle = \langle (\{\iota^{ip^j}_{2np}\}^\infty_{j=1})^k_{i=1}\rangle$$

together with their Bocksteins. Thus, inductively, for $k = p-1$ we obtain

$$\mathrm{Im}\, S'_* = \langle (\{\iota^{ip^j}_{2np}\}^\infty_{j=1})^{p-1}_{i=1} \cup (\{\beta(\iota^{ip^j}_{2np})\}^\infty_{j=1})^{p-1}_{i=1}\rangle.$$

The right side of this equation, via the quotient map $H_*(\Omega S^{2np+1}\{p\}) \longrightarrow QH_*(\Omega S^{2np+1}\{p\})$, identifies with the submodule of indecomposables in $H_*(\Omega S^{2np+1}\{p\})$ obtained from the image in homology of the map $T \longrightarrow \Omega S^{2np+1}\{p\}$. That is, $\mathrm{Im}\, S'_*$ identifies with $QH_*(T)$. Consequently, we obtain the following.

Theorem 5.4. *Conjecture 5.1 holds homologically.* □

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Department of Mathematics, University of Toronto, Toronto ON, M5S 2E4, Canada

E-mail address: `selick@math.toronto.edu`

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail address: `S.D.Theriault@soton.ac.uk`