POLYHEDRAL PRODUCTS FOR CONNECTED SUMS OF SIMPLICIAL COMPLEXES

STEPHEN THERIAULT

ABSTRACT. We investigate how the homotopy type of a polyhedral product changes under the operation of taking the connected sum of two simplicial complexes. This is obtained as a consequence of a more general result that considers how the homotopy type of a polyhedral products changes under the operation of gluing two simplicial complexes together along a common full subcomplex.

1. Introduction

Polyhedral products are topological spaces formed by gluing together products of ingredient spaces in a manner governed by a simplicial complex. They unify several constructions appearing in diverse area of mathematics, including moment-angle complexes in toric topology, complements of complex coordinate subspaces in combinatorics, graph products in geometric group theory, and intersection of quadrics in complex geometry. Understanding their topology has implications in all these areas.

A fundamental problem is to determine how the topology of a polyhedral product changes given a change to the underlying simplicial complex. For example, it well known that the polyhedral product of the join $K * L$ of two simplicial complexes is the Cartesian product of the polyhedral products for K and L. This essentially follows from the definition of a polyhedral product, stated below. Other operations on simplicial complexes are not so straightforward to analyze. The purpose of this paper is to understand how the homotopy theory of polyhedral products changes after taking the connected sum of simplicial complexes.

We begin by defining terms. Let K be an abstract simplicial complex on the vertex set $[m] = \{1, 2, \ldots, m\}$. That is, K is a collection of subsets $\sigma \subseteq [m]$ such that for any $\sigma \in K$ all subsets of σ also belong to K. We refer to K as a simplicial complex rather than an abstract simplicial complex. A subset $\sigma \in K$ is a *simplex* or *face* of K. The emptyset $\emptyset$ is assumed to belong to K. The *boundary* of a face $\sigma \in K$, written $\partial\sigma$, is the simplicial complex consisting of all the proper subsets of σ.

Let K be a simplicial complex on the vertex set $[m]$. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed subspace of X_i. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{m}$ be the sequence of CW-pairs. For each simplex (face) $\sigma \in K$, let $(\underline{X}, \underline{A})^{\sigma}$ be the subspace of $\prod_{i=1}^{m} X_i$

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defined by

$$(\underline{X},\underline{A})^{\sigma}=\prod_{i=1}^{m}Y_i \qquad \text{where} \qquad Y_i=\begin{cases} X_i & \text{if } i\in\sigma \\ A_i & \text{if } i\notin\sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X},\underline{A})$ and K is

$$(\underline{X},\underline{A})^{K}=\bigcup_{\sigma\in K}(\underline{X},\underline{A})^{\sigma}\subseteq\prod_{i=1}^{m}X_i.$$

For example, suppose each A_i is a point. If K is a disjoint union of m points then $(\underline{X},\underline{*})^K$ is the wedge $X_1\vee\cdots\vee X_m$, and if K is the standard $(m-1)$-simplex then $(\underline{X},\underline{*})^K$ is the product $X_1\times\cdots\times X_m$. Comprehensive accounts of polyhedral products can be found in [BP, BBC]. Of particular importance is the case when $(\underline{X},\underline{A})$ is of the form $(\underline{CA},\underline{A})$, where CA_i is the cone on A_i. Our main result is framed in terms of $(\underline{CA},\underline{A})$.

The result on the connected sum of simplicial complexes is a consequence of a much more general result on gluing two simplicial complexes together over a common full subcomplex. If K is a simplicial complex on the vertex set $[m]$ and M is a subcomplex on the vertex set $[\ell]$ (relabelling the vertices if necessary) then M is a *full subcomplex* of K if every face in K on the vertex set $[\ell]$ is also a face of M. The *join* of two pointed topological spaces A and B is denoted by $A*B$, it is homotopy equivalent to $\Sigma A\wedge B$; and the *left half-smash* $A\ltimes B$ and *right half-smash* $A\rtimes B$ are the quotient spaces obtained from $A\times B$ respectively by identifying A or B to a point.

Theorem 1.1. *Let K be a simplicial complex on the vertex set $\{1,\ldots,m\}$, L a simplicial complex on the vertex set $\{\ell+1,\ldots,n\}$ and M a full subcomplex of both K and L on the vertex set $\{\ell+1,\ldots,m\}$, where $\ell<m<n$. Then there is a homotopy fibration*

$$(\mathcal{A}*\mathcal{A}')\vee(G\rtimes\mathcal{A}')\vee(\mathcal{A}\ltimes H)\longrightarrow(\underline{CA},\underline{A})^{K\cup_M L}\stackrel{r}{\longrightarrow}(\underline{CA},\underline{A})^{M}$$

where: (i) $\mathcal{A}=\prod_{i=1}^{\ell}A_i$, (ii) $\mathcal{A}'=\prod_{j=m+1}^{n}A_i$, and (iii) G and H are the homotopy fibres of the composites $(\underline{CA},\underline{A})^K\longrightarrow(\underline{CA},\underline{A})^{K\cup_M L}\stackrel{r}{\longrightarrow}(\underline{CA},\underline{A})^M$ and $(\underline{CA},\underline{A})^L\longrightarrow(\underline{CA},\underline{A})^{K\cup_M L}\stackrel{r}{\longrightarrow}(\underline{CA},\underline{A})^M$ respectively. Further, this fibration splits after looping to give a homotopy equivalence

$$\Omega(\underline{CA},\underline{A})^{K\cup_M L}\simeq\Omega(\underline{CA},\underline{A})^{M}\times\Omega((\mathcal{A}*\mathcal{A}')\vee(G\rtimes\mathcal{A}')\vee(\mathcal{A}\ltimes H)).$$

Let K and L be simplicial complexes and let σ be a facet of both K and L. A *facet* is a maximal face, in the sense that there is no other face of K or L that has σ as a proper subface. The *connected sum* $K\#_\sigma L$ of K and L over σ is defined by deleting σ from both K and L and then gluing the two deleted complexes together along $\partial\sigma$. More precisely, let K' be the subcomplex of K consisting of all the simplices of K except σ, and similarly let L' be the subcomplex of L consisting of all the simplices of L except σ. Note that K' and L' are well-defined since σ is a facet. Then $K\#_\sigma L=K'\cup_{\partial\sigma}L'$. The deletion of σ in K and L implies that $\partial\sigma$ is a full subcomplex of both K' and L', so Theorem 1.1

applies to give a homotopy fibration

$$(\mathcal{A} * \mathcal{A}') \vee (G \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes H) \longrightarrow (\underline{CA}, \underline{A})^{K\#_\sigma L} \xrightarrow{r} (\underline{CA}, \underline{A})^{\partial\sigma}$$

where G and H are the homotopy fibres of the composites $(\underline{CA}, \underline{A})^{K'} \longrightarrow (\underline{CA}, \underline{A})^{K\#_\sigma L} \xrightarrow{r} (\underline{CA}, \underline{A})^{\partial\sigma}$ and $(\underline{CA}, \underline{A})^{L'} \longrightarrow (\underline{CA}, \underline{A})^{K\#_\sigma L} \xrightarrow{r} (\underline{CA}, \underline{A})^{\partial\sigma}$ respectively, and this fibration splits after looping.

As examples, in Section 4 we give explicit loop space decompositions of the moment-angle manifolds $\mathcal{Z}_{P_n}$ where P_n is the boundary of an n-gon for $n \geq 5$. These manifolds are known to be diffeomorphic to connected sums of products of two spheres [Mc], so one byproduct of our work is a description of the homotopy type of the loops on these manifolds. Moreover, the factors that appear can be read off from the combinatorics of P_n.

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2. Three tools

2.1. **Polyhedral products and full subcomplexes.** Let K be a simplicial complex on the vertex set $[m]$. Let M be a proper subcomplex of K and, relabelling the vertices if necessary, assume that M has vertex set $[\ell]$ for $\ell < m$. The simplicial inclusion $M \longrightarrow K$ induces a map of polyhedral products $(\underline{X}, \underline{A})^M \longrightarrow (\underline{X}, \underline{A})^K$. Refining a bit, we may regard M as also being a simplicial complex on $[m]$ by thinking of $\ell + 1, \ldots, m$ as *ghost vertices*. From this point of view, the induced map of polyhedral products is

$$(\underline{X}, \underline{A})^M \times \prod_{j=\ell+1}^{m} A_j \longrightarrow (\underline{X}, \underline{A})^K.$$

If M is a full subcomplex of K then, while there is no simplicial map that projects K to M, there is a version of such a map for polyhedral products (stated in [DS, Lemma 2.2.3] without proof; a proof can be found in [T2, Proposition 3.6] or [PV, Proposition 2.2]).

Lemma 2.1. *Let K be a simplicial complex on the vertex set $[m]$ and let M be a full subcomplex of K on the vertex set $[\ell]$. Then the induced map of polyhedral products $(\underline{X}, \underline{A})^M \longrightarrow (\underline{X}, \underline{A})^K$ has a left inverse*

$$r\colon (\underline{X}, \underline{A})^K \longrightarrow (\underline{X}, \underline{A})^M.$$

□

Lemma 2.1 is proved by projecting $\prod_{i=1}^{m} X_i$ onto $\prod_{j=k+1}^{m} X_j$. This implies that more is true.

Lemma 2.2. *Let M, K and r be as in Lemma 2.1. Suppose in addition that there is a composite of simplicial inclusions $M \longrightarrow K' \longrightarrow K$ where K' is on the vertex set $[\ell']$ for $\ell \leq \ell' \leq m$. Then* *there is a homotopy commutative diagram*

$$\begin{tikzcd} (\underline{X},\underline{A})^{K'}\times\prod_{i=\ell'+1}^{m}A_i \arrow[r] \arrow[d, "\pi_1"'] & (\underline{X},\underline{A})^{K} \arrow[d, "r"] \\ (\underline{X},\underline{A})^{K'} \arrow[r, "r'"] & (\underline{X},\underline{A})^{M} \end{tikzcd}$$

where π_1 is the projection onto the first factor and r' is the map obtained from Lemma 2.1 applied to the full subcomplex M of K'. $\square$

2.2. **The Cube Lemma.** Next, we state Mather's Cube Lemma [Ma].

Lemma 2.3. *Suppose that there is a homotopy commutative diagram of spaces and maps*

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] \arrow[dd] & & F \arrow[dr] \arrow[dd] & \\ & G \arrow[rr] \arrow[dd] & & H \arrow[dd] \\ A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\ & C \arrow[rr] & & D \end{tikzcd}$$

where the bottom face is a homotopy pushout and the four sides are homotopy pullbacks. Then the top face is a homotopy pushout. $\square$

A typical construction of such a cube is the following.

Example 2.4. Given a homotopy pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

and a map $f\colon D \longrightarrow Z$, define H as the homotopy fibre of f. Define F, G and E by pulling back with the map $H \longrightarrow D$. Equivalently, F, G and E are defined by the homotopy fibrations

$$F \longrightarrow B \longrightarrow Z \qquad G \longrightarrow C \longrightarrow Z \qquad E \longrightarrow A \longrightarrow Z$$

where each of the right maps is obtained by composing the given maps from B, C and A to D with f. Then Lemma 2.3 implies that there is a homotopy pushout

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[d] \\ G \arrow[r] & H. \end{tikzcd}$$

2.3. **The homotopy type of a certain pushout.** Let $I = [0,1]$ be the unit interval with base-point 0. Given pointed spaces A and B the (reduced) *join* of A and B is the quotient space

$$A * B = (A \times I \times B)/\sim$$

where $(a,0,b) \sim (a,0,b')$, $(a,1,b) \sim (a',1,b)$ and $(*,t,*) \sim (*,0,*)$ for all $a,a' \in A$, $b,b' \in B$ and $t \in I$. It is well known that there is a homotopy equivalence $A * B \simeq \Sigma A \wedge B$. The *left half-smash* and *right half-smash* of A and B are the quotient spaces

$$A \ltimes B = (A \times B)/\sim_1 \qquad A \rtimes B = (A \times B)/\sim_2$$

respectively, where $(a,*) \sim_1 *$ and $(*,b) \sim_2 *$. The following was proved in [GT1, Lemma 3.3].

Lemma 2.5. *Let A, B, C and D be spaces. Define Q as the homotopy pushout*

$$\begin{tikzcd}
A \times B \arrow[r, "1\times *"] \arrow[d, "*\times 1"] & A \times C \arrow[d] \\
D \times B \arrow[r] & Q.
\end{tikzcd}$$

Then $Q \simeq (A * B) \vee (D \rtimes B) \vee (A \ltimes C)$.

3. The proof of Theorem 1.1

Let K and L be simplicial complexes that have a common subcomplex M, where M is a full subcomplex of both K and L. Relabelling the vertices if necessary, suppose that K has vertex set $\{1,\ldots,m\}$, the full subcomplex M has vertex set $\{\ell+1,\ldots,m\}$ where $1 \leq \ell$, and L has vertex set $\{\ell+1,\ldots,n\}$ for $n > m$. To eliminate trivial cases, we deliberately exclude the case when $\ell = 0$, which would imply that $K = M$ so K is a full subcomplex of L, and the case when $m = n$, which would imply that $L = M$ so L is a full subcomplex of K. There is a pushout of simplicial complexes

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & L \arrow[d] \\
K \arrow[r] & K \cup_M L.
\end{tikzcd} \tag{1}$$

Include ghost vertices as in Subsection 2.1 so that M, L, K and $K \cup_M L$ are all regarded as simplicial complexes on the vertex set $\{1,\ldots,n\}$. To compress notation, let

$$\mathcal{A} = \prod_{i=1}^{\ell} A_i \qquad \mathcal{A}' = \prod_{j=m+1}^{n} A_j.$$

The simplicial inclusions $M \longrightarrow K$ and $M \longrightarrow L$ induce maps of polyhedral products

$$p\colon \mathcal{A} \times (\underline{X},\underline{A})^M \longrightarrow (\underline{X},\underline{A})^K$$

$$q\colon (\underline{X},\underline{A})^M \times \mathcal{A}' \longrightarrow (\underline{X},\underline{A})^L$$

respectively. By [GT2, Proposition 3.1] the pushout (1) of simplicial complexes induces a (point-set) pushout of polyhedral products

$$\begin{tikzcd}
\mathcal{A}\times(\underline{X},\underline{A})^{M}\times\mathcal{A}' \arrow[r, "1\times q"] \arrow[d, "p\times 1"] & \mathcal{A}\times(\underline{X},\underline{A})^{L} \arrow[d] \\
(\underline{X},\underline{A})^{K}\times\mathcal{A}' \arrow[r] & (\underline{X},\underline{A})^{K\cup_M L}.
\end{tikzcd} \tag{2}$$

Since M is a full subcomplex of both K and L on the vertex set $\{\ell+1,\ldots,m\}$, it is also the full subcomplex of $K\cup_M L$ on the same vertex set. Therefore by Lemma 2.1, there is a map

$$r\colon(\underline{X},\underline{A})^{K\cup_M L}\longrightarrow(\underline{X},\underline{A})^{M}. \tag{3}$$

By Lemma 2.2 the map r satisfies homotopy commutative diagrams

$$\begin{tikzcd}
(\underline{X},\underline{A})^{K}\times\mathcal{A}' \arrow[r] \arrow[d, "\pi_1"'] & (\underline{X},\underline{A})^{K\cup_M L} \arrow[d, "r"] \\
(\underline{X},\underline{A})^{K} \arrow[r, "r'"] & (\underline{X},\underline{A})^{M}
\end{tikzcd} \qquad \begin{tikzcd}
\mathcal{A}\times(\underline{X},\underline{A})^{L} \arrow[r] \arrow[d, "\pi_2"'] & (\underline{X},\underline{A})^{K\cup_M L} \arrow[d, "r"] \\
(\underline{X},\underline{A})^{L} \arrow[r, "r''"] & (\underline{X},\underline{A})^{M}
\end{tikzcd} \tag{4}$$

where π_1 and π_2 are projections and r' and r'' are the maps obtained from Lemma 2.1 applied to M regarded as a full subcomplex of K and L respectively. Further, Lemma 2.2 also implies that there is a homotopy commutative diagram

$$\begin{tikzcd}
\mathcal{A}\times(\underline{X},\underline{A})^{M}\times\mathcal{A}' \arrow[r] \arrow[dr, "\pi"'] & (\underline{X},\underline{A})^{K\cup_M L} \arrow[d, "r"'] \\
 & (\underline{X},\underline{A})^{M}
\end{tikzcd} \tag{5}$$

where π is the projection. Note that we also have $\pi\simeq(r'\circ\pi_1)\circ(p\times 1)$ and $\pi\simeq(r''\circ\pi_2)\circ(1\times q)$.

Define F, G and H as the homotopy fibres of r, r' and r'' respectively. Then (4) and (5) imply that there are homotopy fibrations

$$F\longrightarrow(\underline{X},\underline{A})^{K\cup_M L}\xrightarrow{r}(\underline{X},\underline{A})^{M}$$

$$G\times\mathcal{A}'\longrightarrow(\underline{X},\underline{A})^{K}\times\mathcal{A}'\longrightarrow(\underline{X},\underline{A})^{M}$$

$$\mathcal{A}\times H\longrightarrow\mathcal{A}\times(\underline{X},\underline{A})^{L}\longrightarrow(\underline{X},\underline{A})^{M}$$

$$\mathcal{A}\times\mathcal{A}'\longrightarrow\mathcal{A}\times(\underline{X},\underline{A})^{M}\times\mathcal{A}'\longrightarrow(\underline{X},\underline{A})^{M}.$$

Thus, composing the four corners of the pushout (2) with r and taking homotopy fibres gives, as in Example 2.4, a homotopy commutative cube in which the bottom face is a homotopy pushout and the four sides are homotopy pullbacks. Lemma 2.3 therefore implies that the top face

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{A}' \arrow[r, "a"] \arrow[d, "b"] & \mathcal{A}\times H \arrow[d] \\
G\times\mathcal{A}' \arrow[r] & F.
\end{tikzcd} \tag{6}$$

is a homotopy pushout, for some maps a and b.

Lemma 3.1. *In (6) the maps a and b may be chosen to be $\mathcal{A}\times\mathcal{A}'\xrightarrow{1\times a'}\mathcal{A}\times H$ and $\mathcal{A}\times\mathcal{A}'\xrightarrow{b'\times 1}G\times\mathcal{A}'$ respectively, for some maps a' and b'.*

Proof. Consider the homotopy fibration diagram

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{A}' \arrow[r] \arrow[d] & \mathcal{A}\times(\underline{X},\underline{A})^M\times\mathcal{A}' \arrow[r, "\pi"] \arrow[d, "p\times 1"] & (\underline{X},\underline{A})^M \arrow[d, equal] \\
G\times\mathcal{A}' \arrow[r] & (\underline{X},\underline{A})^K\times\mathcal{A}' \arrow[r, "r'\circ\pi_1"] & (\underline{X},\underline{A})^M.
\end{tikzcd}$$

The left square is the left face of the cube that leads to (6). As both horizontal maps in the right square project the factor $\mathcal{A}'$ to a point, the homotopy pullback in the left square can be chosen to be $b'\times 1$ for some map $b'\colon\mathcal{A}\longrightarrow G$. The argument for $a=1\times a'$ is similar. □

To go further we need to identify a' and b'. This may be difficult in general but it is possible in the special case when $(\underline{X},\underline{A})$ is of the form $(\underline{CA},\underline{A})$.

Lemma 3.2. *When $(\underline{X},\underline{A})$ is of the form $(\underline{CA},\underline{A})$, the maps a' and b' in Lemma 3.1 are null homotopic.*

Proof. In general, consider the homotopy fibration $H\longrightarrow(\underline{X},\underline{A})^L\xrightarrow{r''}(\underline{X},\underline{A})^M$. Since r'' has a right homotopy inverse, the fibration connecting map is null homotopic. Therefore the induced map of homotopy classes $[\mathcal{A}',H]\longrightarrow[\mathcal{A}',(\underline{X},\underline{A})^L]$ is an injection. Thus it suffices to show that the composite $\mathcal{A}'\longrightarrow H\longrightarrow(\underline{X},\underline{A})^L$ is null homotopic. Observe that this composite is the same as $\mathcal{A}'=\prod_{j=m+1}^n A_i\hookrightarrow\prod_{k=\ell+1}^n A_k\xrightarrow{c}(\underline{X},\underline{A})^L$ where the c is induced by the simplicial inclusion $\emptyset\longrightarrow L$. When $(\underline{X},\underline{A})$ is of the form $(\underline{CA},\underline{A})$, the map c is null homotopic by [GT2, Corollary 3.3], and hence $\mathcal{A}'\longrightarrow H\longrightarrow(\underline{X},\underline{A})^L$ is null homotopic. □

Proof of Theorem 1.1. Consider the homotopy fibration $F\xrightarrow{f}(\underline{CA},\underline{A})^{K\cup_M L}\xrightarrow{r}(\underline{CA},\underline{A})^M$. By Lemmas 3.1 and 3.2, the homotopy pushout (6) has the form

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{A}' \arrow[r, "1\times *"] \arrow[d, "*\times 1"] & \mathcal{A}\times H \arrow[d] \\
G\times\mathcal{A}' \arrow[r] & F.
\end{tikzcd}$$

Therefore Lemma 2.5 implies that there is a homotopy equivalence

$$F\simeq(\mathcal{A}*\mathcal{A}')\vee(G\rtimes\mathcal{A}')\vee(\mathcal{A}\ltimes H).$$

In general, if $F\xrightarrow{f}E\xrightarrow{p}B$ is a homotopy fibration and p has a right homotopy inverse $s\colon B\longrightarrow E$ then, looping to multiply, the composite $\Omega B\times\Omega F\xrightarrow{\Omega s\times\Omega f}\Omega E\times\Omega E\xrightarrow{\mu}\Omega E$ is a homotopy equivalence. In our case, since r has a right inverse by Lemma 2.1, we obtain the asserted homotopy equivalence for $\Omega(\underline{CA},\underline{A})^{K\cup_M L}$. □

Another instance when the maps a' and b' can be identified in when $(\underline{X}, \underline{A})$ is of the form $(\underline{X}, \underline{*})$. Then both $\mathcal{A} = \prod_{i=1}^{\ell} A_i$ and $\mathcal{A}' = \prod_{j=m+1}^{n} A_j$ are points, so the pushout (6) takes the form

$$\begin{tikzcd} * \arrow[r] \arrow[d] & H \arrow[d] \\ G \arrow[r] & F \end{tikzcd}$$

implying that $F \simeq G \vee H$. Now arguing as for Theorem 1.1 gives the following.

Proposition 3.3. *With K, L and M as in Theorem 1.1, there is a homotopy fibration*

$$G \vee H \longrightarrow (\underline{X}, \underline{*})^{K \cup_M L} \xrightarrow{r} (\underline{X}, \underline{*})^M$$

that splits after looping to give a homotopy equivalence

$$\Omega(\underline{X}, \underline{*})^{K \cup_M L} \simeq \Omega(\underline{X}, \underline{*})^M \times \Omega(G \vee H). \qquad \square$$

4. Examples and Applications

Consider the special case when M is a simplex. Then the definition of the polyhedral product implies that $M = \prod_{k=\ell+1}^{m} X_i$. Specializing further by taking each pair (X_i, A_i) to be of the form (CA_i, A_i), we obtain $(\underline{CA}, \underline{A})^M \simeq \prod_{k=\ell+1}^{m} CA_i$, which is contractible. Therefore $G \simeq (\underline{CA}, \underline{A})^K$, $H \simeq (\underline{CA}, \underline{A})^L$ and $F \simeq (\underline{CA}, \underline{A})^{K \cup_M L}$. Applying Theorem 1.1 recovers [GT2, Theorem 7.2].

Corollary 4.1. *If K is a simplicial complex on the vertex set $\{1, \ldots, m\}$, L is a simplicial complex on the vertex set $\{\ell+1, \ldots, n\}$, and K and L are glued together along a common face $\sigma = (\ell+1, \ldots, m)$, then there is a homotopy equivalence*

$$(\underline{CA}, \underline{A})^{K \cup_\sigma L} \simeq (\mathcal{A} * \mathcal{A}') \vee ((\underline{CA}, \underline{A})^K \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes (\underline{CA}, \underline{A})^L). \qquad \square$$

Next, consider the special case when $M = \partial\sigma$ is the boundary of the simplex σ. Take each pair (X_i, A_i) to be of the form (CA_i, A_i). By [GT2, Proposition 2.2] there is a homotopy equivalence $(\underline{CA}, \underline{A})^M = A_{\ell+1} * \cdots * A_m$. In this case the fact that $M = \partial\sigma$ is the full subcomplex of K and L on the vertex set $\{\ell+1, \ldots, m\}$ implies that σ is not a face of K or L. We say σ is a *missing face* of K and L. Theorem 1.1 implies the following.

Corollary 4.2. *With notation as in Theorem 1.1, if K and L are simplicial complexes that are glued together along the boundary of a common missing face $\sigma = (\ell+1, \ldots, m)$, then there is a homotopy fibration*

$$(\mathcal{A} * \mathcal{A}') \vee (G \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes H) \longrightarrow (\underline{CA}, \underline{A})^{K \cup_{\partial\sigma} L} \xrightarrow{r} A_{\ell+1} * \cdots * A_m$$

that splits after looping to give a homotopy equivalence

$$\Omega(\underline{CA}, \underline{A})^{K \cup_{\partial\sigma} L} \simeq \Omega(A_{\ell+1} * \cdots * A_m) \times \Omega((\mathcal{A} * \mathcal{A}') \vee (G \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes H)).$$

$\square$

Connected sums of simplicial complexes. These will be dealt with by applying Corollary 4.2. As in the Introduction, let K and L be simplicial complexes and let σ be a maximal face of both K and L. Let K' be the subcomplex of K consisting of all the simplices of K except σ, and similarly let L' be the subcomplex of L consisting of all the simplices of L except σ. Then $K\#_\sigma L = K' \cup_{\partial\sigma} L'$. Since $\partial\sigma$ is a full subcomplex of both K' and L', Corollary 4.2 implies that there is a homotopy fibration

$$(\mathcal{A} * \mathcal{A}') \vee (G \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes H) \longrightarrow (\underline{CA}, \underline{A})^{K\#_\sigma L} \xrightarrow{r} A_{\ell+1} * \cdots * A_m$$

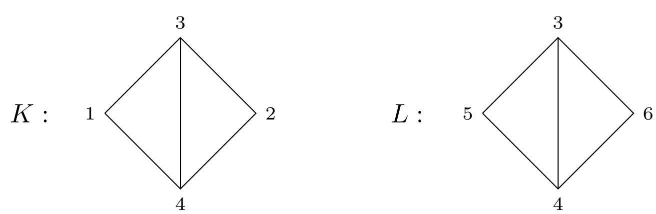
that splits after looping.

For example, the *moment-angle complex* $\mathcal{Z}_K$ is the polyhedral product determined by taking each pair $(CA_i, A_i) = (D^2, S^1)$. Then $A_{\ell+1} * \cdots * A_m \simeq S^{2m-2\ell-1}$. Observe that $\mathcal{A} = \prod_{i=1}^{\ell} A_i$ and $\mathcal{A}' = \prod_{j=m+1}^{n} A_j$ are the tori T^ℓ and T^{n-m} formed by taking the product of ℓ and $n-m$ circles, respectively. We therefore obtain a homotopy fibration

$$(T^\ell * T^{n-m}) \vee (G \rtimes T^{n-m}) \vee (T^\ell \ltimes H) \longrightarrow \mathcal{Z}_{K\#_\sigma L} \xrightarrow{r} S^{2m-2\ell-1}$$

that splits after looping.

As an explicit example, let



and take the connected sum along $\sigma = (3,4)$. Observe that $\ell = 2$, $m = 4$, and $n = 6$ in this case, so $S^{2m-2\ell-1} = S^3$. Observe that K' and L' are both boundaries of squares, implying that $\mathcal{Z}_{K'} \simeq S^3 \times S^3 \simeq \mathcal{Z}_{L'}$. The retraction of S^3 off both of these induced by r implies that $G \simeq H \simeq S^3$. Thus there is a homotopy fibration

$$(T^2 * T^2) \vee (S^3 \rtimes T^2) \vee (T^2 \ltimes S^3) \longrightarrow \mathcal{Z}_{K\#_\sigma L} \xrightarrow{r} S^3$$

that splits after looping. In this case the fibre of r is homotopy equivalent to a wedge of spheres. Write $t\,S^k$ for a wedge of t copies of S^k. As $T^2 = S^1 \times S^1$ we obtain $T^2 * T^2 \simeq \Sigma(S^1 \times S^1) \wedge (S^1 \times S^1) \simeq 4S^3 \vee 4S^4 \vee S^5$. In general, $\Sigma A \rtimes B \simeq \Sigma A \vee (\Sigma A \wedge B)$ and $A \ltimes \Sigma B \simeq \Sigma B \vee (\Sigma A \wedge B)$, so $S^3 \rtimes T^2 \simeq S^3 \vee 2S^4 \vee S^5 \simeq T^2 \ltimes S^3$. Hence the fibre of r is homotopy equivalent to $6\,S^3 \vee 8\,S^4 \vee 3\,S^5$.

The stacking operation. A special case of the connected sum operation on simplicial complexes is *stacking*. Suppose that K is a simplicial complex and let σ be a maximal face of K. If σ is a k-dimensional simplex, let $C\sigma$ be the cone on σ. That is, $C\sigma = \sigma * \{v\}$ where v is a vertex disjoint from σ and $*$ is the join operation on simplicial complexes. The stacking operation is the connected sum $K\#_\sigma C\sigma$. We will show that the homotopy fibration in Corollary 4.2 simplifies in the case of stacking.

As before, regard $K\#_\sigma C\sigma$ as $K'\cup_{\partial\sigma}(C\sigma)'$ where K' and $(C\sigma)'$ are K and $C\sigma$ with the face σ deleted. Observe that $(C\sigma)'=C(\partial\sigma)$. Observe also that if the vertex set of K is $\{1,\ldots,m\}$ and that for σ is $\{\ell+1,\ldots,m\}$ then the vertex set for $K\#_\sigma C\sigma$ is $\{1,\ldots,m,v\}$. In the context of Corollary 4.2 this implies that the space $\mathcal{A}'$ equals A_v. Next, consider the homotopy fibration $H\longrightarrow(\underline{CA},\underline{A})^{C(\partial\sigma)}\longrightarrow(\underline{CA},\underline{A})^{\partial\sigma}$ from Corollary 4.2. In general, the definition of the polyhedral product implies that there is a homeomorphism $(\underline{X},\underline{A})^{P*Q}\cong(\underline{X},\underline{A})^{P}\times(\underline{X},\underline{A})^{Q}$. Further, Q is the full subcomplex of $P*Q$ on the vertex set of Q, and the induced map of polyhedral products $(\underline{X},\underline{A})^{P*Q}\longrightarrow(\underline{X},\underline{A})^{Q}$ is the projection $(\underline{X},\underline{A})^{P}\times(\underline{X},\underline{A})^{Q}\longrightarrow(\underline{X},\underline{A})^{Q}$. In our case, as $(\underline{CA},\underline{A})^{\{v\}}=CA_v$, there is a homeomorphism $(\underline{CA},\underline{A})^{C(\partial\sigma)}=(\underline{CA},\underline{A})^{\partial\sigma*\{v\}}\cong(\underline{CA},\underline{A})^{\partial\sigma}\times CA_v$, and the map $(\underline{CA},\underline{A})^{C(\partial\sigma)}\longrightarrow(\underline{CA},\underline{A})^{\partial\sigma}$ is a projection. Since CA_v is contractible, the map $(\underline{CA},\underline{A})^{C(\partial\sigma)}\longrightarrow(\underline{CA},\underline{A})^{\partial\sigma}$ is therefore a homotopy equivalence. Hence its homotopy fibre H is contractible. Corollary 4.2 therefore implies that there is a homotopy fibration

$$(\mathcal{A}*A_v)\vee(G\rtimes A_v)\longrightarrow(\underline{CA},\underline{A})^{K\#_\sigma C\sigma}\xrightarrow{r}A_{\ell+1}*\cdots*A_m \tag{7}$$

that splits after looping to give a homotopy equivalence

$$\Omega(\underline{CA},\underline{A})^{K\#_\sigma C\sigma}\simeq\Omega(A_{\ell+1}*\cdots*A_m)\times\Omega((\mathcal{A}*A_v)\vee(G\rtimes A_v)). \tag{8}$$

This will be used in the next subsection to study the homotopy theory of certain manifolds.

The based loops on certain manifolds. For $n\geq 3$ let P_n be the boundary of an n-gon. By [Mc] there is a diffeomorphism

$$\mathcal{Z}_{P_n}\cong\#_{k=3}^{n-1}(S^k\times S^{n+2-k})^{\#(k-2)\binom{n-2}{k-1}}$$

where $(M\times N)^{\#t}$ is the connected sum of t copies of $M\times N$. We will apply our results to give a homotopy decomposition of the based loops on the manifold $\#_{k=3}^{n-1}(S^k\times S^{n+2-k})^{\#(k-2)\binom{n-2}{k-1}}$, viewed through the lens of it being $\Omega\mathcal{Z}_{P_n}$.

Observe that $P_n=P_{n-1}\#_\sigma C\sigma$ where σ is an edge. Applying (7) in the moment-angle complex case when the pairs (CA_i,A_i) are all (D^2,S^1), we obtain a homotopy fibration

$$(T^\ell*S^1)\vee(G\rtimes S^1)\longrightarrow\mathcal{Z}_{P_n}=\mathcal{Z}_{P_{n-1}\#_\sigma C\sigma}\longrightarrow S^3.$$

The space G is the homotopy fibre of the composite $g\colon\mathcal{Z}_{P'_n}\longrightarrow\mathcal{Z}_{P_{n-1}\#_\sigma C\sigma}\longrightarrow S^3$ where P'_{n-1} is P_{n-1} without the edge σ. On the one hand, the S^3 in the range of g corresponds to the boundary of the edge in σ that has been removed from both $C\sigma$ and P_{n-1} in order to perform the connected sum. In particular, the boundary of this edge is a full subcomplex of P'_{n-1}, so g has a right inverse.

On the other hand, by [T1] there is a homotopy equivalence $\mathcal{Z}_{P'_{n-1}} \simeq \mathcal{Z}_{K_{n-2}}$ where K_{n-2} is the simplicial complex consisting of $n-2$ disjoint points, and by [GT1] there is a homotopy equivalence

$$\mathcal{Z}_{K_{n-2}} \simeq \bigvee_{k=3}^{n-1} (S^k)^{\wedge (k-2)\binom{n-2}{k-1}}$$

where $M^{\wedge t}$ is the smash product of t copies of M. Thus we may write $\mathcal{Z}_{P'_{n-1}}$ as $S^3 \vee W$ where W is a wedge of simply-connected spheres and the restriction of g to S^3 is the identity map. This implies that the homotopy fibre of g has the same homotopy type as the homotopy fibre of the pinch map $S^3 \vee W \longrightarrow S^3$. That is, $G \simeq \Omega S^3 \ltimes W$. Thus from (8) there is a homotopy equivalence

$$\Omega Z_{P_n} = \Omega \mathcal{Z}_{P_{n-1} \#_\sigma C\sigma} \simeq \Omega S^3 \times \Omega\Big((T^\ell * S^1) \vee ((\Omega S^3 \ltimes W) \rtimes S^1) \Big). \tag{9}$$

Since $T^\ell = \prod_{i=1}^{\ell} S^1$, the space $T^\ell * S^1$ is homotopy equivalent to a wedge of spheres. Since W is a wedge of simply-connected spheres, it is a suspension, so $\Omega S^3 \ltimes W \simeq (\Omega S^3 \wedge W) \vee W$, implying that $\Omega S^3 \ltimes W$ is homotopy equivalent to a wedge of spheres. This then implies that $(\Omega S^3 \ltimes W) \rtimes S^1$ is homotopy equivalent to a wedge of spheres. Thus $\Omega \mathcal{Z}_{P_n} \simeq \Omega S^3 \times \Omega V$ where V is a wedge of spheres. That is, there is a homotopy equivalence

$$\Omega\Big(\#_{k=3}^{n-1} (S^k \times S^{n+2-k})^{\#(k-2)\binom{n-2}{k-1}} \Big) \simeq \Omega S^3 \times \Omega V.$$

Since V is a wedge of spheres, the Hilton-Milnor Theorem may be applied to decompose ΩV as a finite type product of looped spheres. Consequently, the homotopy groups of the connected sum $\#_{k=3}^{n-1} (S^k \times S^{n+2-k})^{\#(k-2)\binom{n-2}{k-1}}$ can be calculated to the same extent as can the homotopy groups of spheres.

Using different methods, a homotopy decomposition for $\Omega\big(\#_{k=3}^{n-1} (S^k \times S^{n+2-k})^{\#(k-2)\binom{n-2}{k-1}}\big)$, and hence for $\Omega \mathcal{Z}_{P_n}$ was given in [BT]. This had the advantage of working for all connected sums of products of two simply-connected spheres, but being derived from purely homotopy theoretic methods, it obscured the combinatorial contribution to the decomposition coming from the connected sum of simplicial complexes. The decomposition (9) has the advantage of retaining this combinatorial information. Precisely, the S^3 arises from the missing edge in $\partial\sigma$, the torus T^ℓ arises from the vertices of P_{n-1} that are not in σ, the circle S^1 arises from the vertex of $C\sigma$ that is not in σ, and the space $\Omega S^3 \ltimes W$ arises from the homotopy fibre of the map $\mathcal{Z}_{P'_{n-1}} \longrightarrow \mathcal{Z}_{\partial\sigma}$.

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Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail address: S.D.Theriault@soton.ac.uk