POLYHEDRAL PRODUCTS FOR WHEEL GRAPHS AND THEIR GENERALIZATIONS

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Abstract. A general homotopy decomposition is established for the based loops on certain polyhedral products. This is then specialized to obtain an explicit homotopy decomposition for the loops on the moment-angle complex $\mathcal{Z}_K$, where K is a wheel graph or a generalization thereof.

1. Introduction

Polyhedral products have emerged as an important class of topological spaces and a key problem is identifying their homotopy type. While there has been a certain amount of success in doing this in special cases it is a very difficult problem in general. In this paper we make the case that it is sometimes, perhaps paradoxically, easier to determine the homotopy type of the loop space. This may be a new way forward in the analysis of the homotopy theory of polyhedral products.

Let K be an abstract simplicial complex on the vertex set $[m]=\{1,2,\ldots,m\}$. In other words, K is a collection of subsets $\sigma\subseteq[m]$ such that for any $\sigma\in K$ all subsets of σ also belong to K. We refer to K as a simplicial complex rather than an abstract simplicial complex. A subset $\sigma\in K$ is a *simplex* or *face* of K. The emptyset $\emptyset$ is assumed to belong to K.

Given a simplicial complex K on the vertex set $[m]$, for $1\leq i\leq m$ let (X_i,A_i) be a pair of pointed CW-complexes, where A_i is a pointed subspace of X_i. Let $(\underline{X},\underline{A})=\{(X_i,A_i)\}_{i=1}^m$ be the sequence of CW-pairs. For each simplex (face) $\sigma\in K$, let $(\underline{X},\underline{A})^\sigma$ be the subspace of $\prod_{i=1}^m X_i$ defined by

$$(\underline{X},\underline{A})^\sigma=\prod_{i=1}^m Y_i\qquad\text{where}\qquad Y_i=\begin{cases}X_i & \text{if } i\in\sigma\\ A_i & \text{if } i\notin\sigma.\end{cases}$$

The *polyhedral product* determined by $(\underline{X},\underline{A})$ and K is

$$(\underline{X},\underline{A})^K=\bigcup_{\sigma\in K}(\underline{X},\underline{A})^\sigma\subseteq\prod_{i=1}^m X_i.$$

A fundamental case is the *moment-angle complex* $\mathcal{Z}_K$ that is central to toric topology, which occurs when each pair of spaces (X_i,A_i) is (D^2,S^1).

We first give a general decomposition for the loops on certain polyhedral products that generalizes work of Félix and Tanré [FT]. Let K, L and M be simplicial complexes with L a sub-complex of K, and let $\overline{K}$ be the pushout of the simplicial maps $L\longrightarrow K$ and $L\longrightarrow L*M$, where $L*M$ is the join

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of L and M. In Theorem 2.6 a homotopy equivalence for $\Omega(\underline{X}, \underline{A})^{\overline{K}}$ is given in terms of $\Omega(\underline{X}, \underline{A})^{K}$, $\Omega(\underline{X}, \underline{A})^{L}$ and two related spaces. This is then specialized considerably in order to get concrete, explicit homotopy equivalences for a family of moment-angle complexes. Taking $K = P_m$ as the boundary of the m-gon, $L = V_m$ as its vertex set, and M a single vertex, the simplicial complex $\overline{K} = W_m$ is known as a *wheel graph*. With the same K and L but taking M to be any simplicial complex we obtain what will be called a *wheel complex* $W_m(M)$ (as it need no longer be a graph). In Theorem 5.9 explicit homotopy equivalences for $\Omega\mathcal{Z}_{P_m}$ and the homotopy fibre of $\mathcal{Z}_{V_m} \longrightarrow \mathcal{Z}_{P_m}$ are used to give an explicit homotopy equivalence for $\Omega\mathcal{Z}_{W_m(M)}$ in terms of spheres, loops on spheres, and $\Omega\mathcal{Z}_M$. In particular, $\Omega\mathcal{Z}_{W_m}$ is homotopy equivalent to a product of spheres and loops on spheres.

This suggests there may be a wide class of simplicial complexes K with the property that $\Omega\mathcal{Z}_K$ is homotopy equivalent to a product of spheres and loops on spheres. It would be interesting to investigate this problem further.

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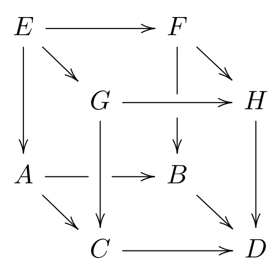
2. A general decomposition for the loops on certain polyhedral products

This section generalizes work of Félix and Tanré [FT] on the homotopy theory of certain polyhedral products. The main result is Theorem 2.6; it requires two tools, presented in Lemmas 2.1 and 2.3. The first is Mather's Cube Lemma [Mat].

Lemma 2.1. *Suppose that there is a homotopy commutative diagram of spaces and maps*



where the bottom face is a homotopy pushout and the four sides are homotopy pullbacks. Then the top face is a homotopy pushout. □

A typical construction of such a cube is to start with a homotopy pushout A-B-C-D and a map $f \colon D \longrightarrow Z$. Define the space H as the homotopy fibre of f and define F, G and E by pulling back with the map $H \longrightarrow D$. This gives a homotopy commutative cube with the bottom face a homotopy pushout and all four sides being homotopy pullbacks, so Lemma 2.1 implies that the top face is also a homotopy pushout.

The second tool requires some setup. In general, if L and M are simplicial complexes the *join* of L and M is the simplicial complex

$$L * M = \{\sigma \cup \tau \mid \sigma \in L \text{ and } \tau \in M\}.$$

By the definitions of the join and the polyhedral product, there is a homeomorphism

$$(\underline{X},\underline{A})^{L*M} = (\underline{X},\underline{A})^{L} \times (\underline{X},\underline{A})^{M}.$$

Let K be a simplicial complex on the vertex set $\{1,\ldots,m\}$ and let M be a simplicial complex on the vertex set $\{m+1,\ldots,n\}$. Let L be a subcomplex of K and define the simplicial complex $\overline{K}$ by the pushout

$$\begin{array}{ccc} L & \longrightarrow & L*M \\ \big\downarrow & & \big\downarrow \\ K & \longrightarrow & \overline{K}. \end{array} \tag{1}$$

Note that $\overline{K}$ has vertex set $\{1,\ldots,n\}$.

Example 2.2. If $M=\{v\}$ has only a single vertex then (1) attaches a cone to the subcomplex L of K. In terms of the standard star-link-restriction pushout with respect to the vertex v we have $K=\overline{K}\backslash v$, $L*M=\mathrm{star}_{\overline{K}}(v)$ and $L=\mathrm{link}_{\overline{K}}(v)$.

The pushout (1) induces a commutative diagram of polyhedral products but this can be strengthened to a pushout of polyhedral products if correctly interpreted. Regard K, L and M as simplicial complexes on the vertex set $\{1,\ldots,n\}$. In particular, K has *ghost vertices* $m+1,\ldots,n$, and the simplicial map $K\longrightarrow\overline{K}$ induces a map of polyhedral products $(\underline{X},\underline{A})^{K}\times\prod_{i=1}^{m+1}A_i\longrightarrow(\underline{X},\underline{A})^{\overline{K}}$. While L, as a subcomplex of K, may have fewer vertices it will be convenient to distinguish the ghost vertices $m+1,\ldots,n$ and regard the simplicial map $L\longrightarrow K$ as inducing a map of polyhedral products $(\underline{X},\underline{A})^{L}\times\prod_{i=m+1}^{n}A_i\stackrel{g\times 1}{\longrightarrow}(\underline{X},\underline{A})^{K}\times\prod_{i=m+1}^{n}A_i$ where g is the map of polyhedral products induced when restricted to the vertex set $\{1,\ldots,m\}$ and 1 is the identity map on $\prod_{i=m+1}^{n}A_i$. The simplicial map $L*M\longrightarrow\overline{K}$ induces the map $(\underline{X},\underline{A})^{L}\times(\underline{X},\underline{A})^{M}\longrightarrow(\underline{X},\underline{A})^{\overline{K}}$ and the simplicial inclusion $L\longrightarrow L*M$ induces the map $(\underline{X},\underline{A})^{L}\times\prod_{i=m+1}^{n}A_i\stackrel{1\times h}{\longrightarrow}(\underline{X},\underline{A})^{L}\times(\underline{X},\underline{A})^{M}$ where 1 is the identity map on $(\underline{X},\underline{A})^{L}$ and h is induced by the simplicial map $\overline{\emptyset}\longrightarrow M$ (where $\overline{\emptyset}$ is the simplicial complex on ghost vertices $\{m+1,\ldots,n\}$). By [GT2, Proposition 3.1] all this combines to give the following.

Lemma 2.3. *There is a (point-set) pushout*

$$\begin{array}{ccc} (\underline{X},\underline{A})^{L}\times\prod_{i=m+1}^{n}A_i & \stackrel{1\times h}{\longrightarrow} & (\underline{X},\underline{A})^{L}\times(\underline{X},\underline{A})^{M} \\ \big\downarrow{\scriptstyle g\times 1} & & \big\downarrow \\ (\underline{X},\underline{A})^{K}\times\prod_{i=m+1}^{n}A_i & \longrightarrow & (\underline{X},\underline{A})^{\overline{K}}. \end{array}$$

□

The pushout in Lemma 2.3 will serve as the starting point for a cube that lets us apply Lemma 2.1. To produce the four sides of the cube we will construct a map from $(\underline{X},\underline{A})^{\overline{K}}$ to an appropriate polyhedral product and take fibres.

Including K and $L * M$ into $K * M$, by (1) there is a pushout map

$$\overline{K} \longrightarrow K * M.$$

Since the composite $K \longrightarrow \overline{K} \longrightarrow K * M$ is the inclusion of the first factor the composite and $L * M \longrightarrow \overline{K} \longrightarrow K * M$ is the join of the inclusion of L into K and the identity map on M, there are induced maps of polyhedral products

$$(\underline{X}, \underline{A})^K \times \prod_{i=m+1}^{n} A_i \xrightarrow{1\times h} (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^M$$

$$(\underline{X}, \underline{A})^L \times (\underline{X}, \underline{A})^M \xrightarrow{g\times 1} (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^M.$$

Therefore, if the four corners of the diagram in Lemma 2.3 are composed with the map $(\underline{X}, \underline{A})^{\overline{K}} \longrightarrow (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^M$ we obtain homotopy fibrations

$$F \longrightarrow (\underline{X}, \underline{A})^{\overline{K}} \longrightarrow (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^M$$

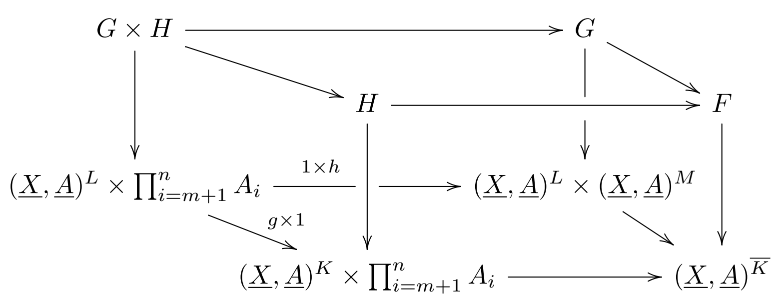
$$H \longrightarrow (\underline{X}, \underline{A})^K \times \prod_{i=m+1}^{n} A_i \xrightarrow{1\times h} (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^M$$

$$G \longrightarrow (\underline{X}, \underline{A})^L \times (\underline{X}, \underline{A})^M \xrightarrow{g\times 1} (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^M$$

$$G \times H \longrightarrow (\underline{X}, \underline{A})^L \times \prod_{i=m+1}^{n} A_i \xrightarrow{g\times h} (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^M$$

where the first fibration defines F, and G and H are defined as the homotopy fibres of the maps g and h respectively. Since all homotopy fibres are given by composing into a common base, we obtain a homotopy commutative cube

(2)



where the bottom face is a pushout and the four sides are homotopy pullbacks. Therefore, by Lemma 2.1 the top face is also a homotopy pushout.

The (reduced) *join* of two spaces A and B is the quotient space $A * B = (A \times [0,1] \times B)/\sim$ where $(a, 0, b) \sim (a', 0, b)$, $(a, 1, b) \sim (a, 1, b')$ and $(*, t, *) \sim (*, 0, *)$ for all $a, a' \in A$, $b, b' \in B$ and $t \in [0, 1]$. It is well known that there is a homotopy equivalence $A * B \simeq \Sigma A \wedge B$.

Lemma 2.4. *The maps $G \times H \longrightarrow G$ and $G \times H \longrightarrow H$ in (2) can be chosen to be the projections. Consequently, there is a homotopy equivalence $F \simeq G * H$.*

Proof. Consider the homotopy fibration diagram

$$\begin{array}{ccccc} G\times H & \longrightarrow & (\underline{X},\underline{A})^{L}\times\prod_{i=m+1}^{n}A_i & \xrightarrow{g\times h} & (\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M} \\ \downarrow & & \downarrow{\scriptstyle g\times 1} & & \| \\ H & \longrightarrow & (\underline{X},\underline{A})^{K}\times\prod_{i=m+1}^{n}A_i & \xrightarrow{1\times h} & (\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M}. \end{array}$$

Regarding H as $*\times H$, this fibration diagram is the product of the fibration diagrams for the lefthand and righthand factors. Thus one choice of the map between fibres is $G\times H\longrightarrow *\times H$ is the product $*\times 1$. That is, this is the projection $G\times H\longrightarrow H$. The argument that $G\times H\longrightarrow G$ can be chosen to be a projection is similar.

In general, it is well known that the homotopy pushout of projections $S\times T\longrightarrow S$ and $S\times T\longrightarrow T$ is the joint $S*T$. So in our case, we obtain $F\simeq G*H$. □

Remark 2.5. Félix and Tanré [FT] considered the special case of (2) when M is a single vertex.

We now identify a homotopy decomposition for $\Omega(\underline{X},\underline{A})^{\overline{K}}$.

Theorem 2.6. *Let $\overline{K}$ be a pushout as in (1). Then there is a homotopy fibration*

$$G*H\longrightarrow(\underline{X},\underline{A})^{\overline{K}}\longrightarrow(\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M}$$

where G is the homotopy fibre of $(\underline{X},\underline{A})^{L}\longrightarrow(\underline{X},\underline{A})^{K}$ and H is the homotopy fibre of $\prod_{i=m+1}^{n}A_i\longrightarrow(\underline{X},\underline{A})^{M}$. Further, this fibration splits after looping, giving a homotopy equivalence

$$\Omega(\underline{X},\underline{A})^{\overline{K}}\simeq\Omega(\underline{X},\underline{A})^{K}\times\Omega(\underline{X},\underline{A})^{M}\times\Omega(G*H).$$

Proof. Consider the homotopy fibration $F\longrightarrow(\underline{X},\underline{A})^{\overline{K}}\longrightarrow(\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M}$ that defines F. By Lemma 2.4, $F\simeq G*H$, proving the first statement.

For the splitting, we have seen that the composite $(\underline{X},\underline{A})^{K}\longrightarrow(\underline{X},\underline{A})^{\overline{K}}\longrightarrow(\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M}$ is the inclusion of the left factor while $(\underline{X},\underline{A})^{L}\times(\underline{X},\underline{A})^{M}\longrightarrow(\underline{X},\underline{A})^{\overline{K}}\longrightarrow(\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M}$ is $g\times 1$. Restricting the latter case to $(\underline{X},\underline{A})^{M}$ is the inclusion of the right factor. Taking the wedge sum therefore gives a composite

$$(\underline{X},\underline{A})^{K}\vee(\underline{X},\underline{A})^{M}\longrightarrow(\underline{X},\underline{A})^{\overline{K}}\longrightarrow(\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M}$$

which is the inclusion of the wedge into the product. It is well known that the inclusion of the wedge into a product has a right homotopy inverse after looping. Hence the fibration $G*H\longrightarrow(\underline{X},\underline{A})^{\overline{K}}\longrightarrow(\underline{X},\underline{A})^{K}\times(\underline{X},\underline{A})^{M}$ splits after looping, and the asserted homotopy equivalence for $\Omega(\underline{X},\underline{A})^{\overline{K}}$ follows. □

3. An initial analysis of Theorem 2.6

The decomposition for $\Omega(\underline{X},\underline{A})^{\overline{K}}$ in Theorem 2.6 has four ingredients: (i) $\Omega(\underline{X},\underline{A})^{K}$, (ii) $\Omega(\underline{X},\underline{A})^{M}$, (iii) the homotopy fibre of $(\underline{X},\underline{A})^{L} \longrightarrow (\underline{X},\underline{A})^{K}$, and (iv) the homotopy fibre of $\prod_{i=m+1}^{n} A_i \longrightarrow (\underline{X},\underline{A})^{M}$. To go further, we would like to identify some or all of these components.

It will be helpful to reduce to analyzing a special case of polyhedral products. In general, for $1 \leq i \leq m$, let Y_i be the homotopy fibre of the inclusion $A_i \longrightarrow X_i$. In [HST] the following was proved when each pair (X_i, A_i) has both X_i and A_i path-connected, but the same argument works in the more general case when only X_i is path-connected.

Theorem 3.1. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X},\underline{A})$ be any sequence of pointed pairs (X_i, A_i) where each X_i is path-connected. Then there is a homotopy fibration*

$$(\underline{CY},\underline{Y})^{K} \longrightarrow (\underline{X},\underline{A})^{K} \longrightarrow \prod_{i=1}^{m} X_i.$$

Further, this fibration splits after looping to give a homotopy equivalence

$$\Omega(\underline{X},\underline{A})^{K} \simeq \left(\prod_{i=1}^{m} \Omega X_i\right) \times \Omega(\underline{CY},\underline{Y})^{K}.$$

□

In our case, we obtain $\Omega(\underline{X},\underline{A})^{\overline{K}} \simeq (\prod_{i=1}^{n} \Omega X_i) \times \Omega(\underline{CY},\underline{Y})^{\overline{K}}$, so to determine the homotopy type of $\Omega(\underline{X},\underline{A})^{\overline{K}}$ it is equivalent to determine the homotopy type of $\Omega(\underline{CY},\underline{Y})^{\overline{K}}$. Put another way, in the context of Theorem 2.6 it suffices to consider the case when $(\underline{X},\underline{A})$ is of the form $(\underline{CA},\underline{A})$.

Now, from the point of view of $(\underline{CA},\underline{A})$, the decomposition for $\Omega(\underline{CA},\underline{A})^{\overline{K}}$ in Theorem 2.6 has four ingredients: (i)′ $\Omega(\underline{CA},\underline{A})^{K}$, (ii)′ $\Omega(\underline{CA},\underline{A})^{M}$, (iii)′ the homotopy fibre of $(\underline{CA},\underline{A})^{L} \longrightarrow (\underline{CA},\underline{A})^{K}$, and (iv)′ the homotopy fibre of $\prod_{i=m+1}^{n} A_i \longrightarrow (\underline{CA},\underline{A})^{M}$. Component (iv)′ can be handled generically. In general, in [GT2, Corollary 3.4] the following was proved.

Proposition 3.2. *If K is a simplicial complex on the vertex set $[m]$ then the inclusion $\prod_{i=1}^{m} A_i \longrightarrow (\underline{CA},\underline{A})^{K}$ is null homotopic.* □

In our case, we immediately obtain the following.

Corollary 3.3. *The homotopy fibre of the map $\prod_{i=m+1}^{n} A_i \longrightarrow (\underline{CA},\underline{A})^{M}$ is homotopy equivalent to $\left(\prod_{i=m+1}^{n} A_i\right) \times \Omega(\underline{CA},\underline{A})^{M}$.* □

Consequently, (iv)′ has been rewritten in terms of (ii)′. Components (i)′ to (iii)′ cannot be handled generically, so special cases need to be identified. In particular, (iii)′ is particularly contentious.

In what follows we will specialize considerably. The polyhedral products will be taken to be moment-angle complexes. The simplicial complex K will be the boundary of an m-gon and L will be the vertex set of the m-gon. The point in specializing so much is that then the homotopy types of $\mathcal{Z}_K$ and $\mathcal{Z}_L$ are known, and with a nontrivial amount of work we will be able to identify the homotopy types of $\Omega\mathcal{Z}_K$ and the homotopy fibre of the map $\mathcal{Z}_L \longrightarrow \mathcal{Z}_K$.

4. Spaces having the homotopy type of a wedge of spheres

This section establishes some preliminary properties for spaces having the homotopy type of a wedge of spheres.

Lemma 4.1. *If X is homotopy equivalent to a finite type product of path-connected spheres and loops on simply-connected spheres then ΣX is homotopy equivalent to a wedge of simply-connected spheres.*

Proof. By hypothesis,

$$X \simeq \left(\prod_{\alpha\in\mathcal{I}} S^{n_\alpha}\right) \times \left(\prod_{\beta\in\mathcal{J}} \Omega S^{n_\beta}\right)$$

for some index sets $\mathcal{I}$ and $\mathcal{J}$, with each $n_\alpha \geq 1$ and each $n_\beta \geq 2$. By the James construction [J], $\Sigma\Omega S^n$ is homotopy equivalent to a wedge of spheres if $n \geq 2$ and it is well known that $\Sigma(S \times T) \simeq \Sigma S \vee \Sigma T \vee (\Sigma S \wedge T)$. Iteratively using these two properties implies that ΣX is homotopy equivalent to a wedge of simply-connected spheres. $\square$

The *right half-smash* of pointed spaces A and B is the quotient space $A \rtimes B = (A \times B)/\sim$ where $(a, *) \sim (*, *)$ for all $a \in A$. It is well known that if A is a co-H-space then there is a homotopy equivalence $A \rtimes B \simeq A \vee (A \wedge B)$. A modest variation on Lemma 4.1 is the following.

Lemma 4.2. *If X is homotopy equivalent to a finite type product of path-connected spheres and loops on simpy-connected spheres, and Y is homotopy equivalent to a wedge of simply-connected spheres, then $Y \rtimes X$ is homotopy equivalent to a wedge of simply-connected spheres.*

Proof. Since Y is homotopy equivalent to a wedge of simply-connected spheres we have $Y \simeq \Sigma Y'$ for some wedge of path-connected spheres Y'. Therefore Y is a co-H-space so there is a homotopy equivalence $Y \rtimes X \simeq Y \vee (Y \wedge X)$. Further, $Y \wedge X \simeq Y' \wedge (\Sigma X)$ and by Lemma 4.1 ΣX is homotopy equivalent to a wedge of simply-connected spheres. Hence as Y' is a wedge of spheres so is $Y' \wedge (\Sigma X)$, and the spheres are all simply-connected because ΣX is. Thus $Y \rtimes X$ is homotopy equivalent to a wedge of simply-connected spheres. $\square$

Lemma 4.3. *Suppose that R and S are wedges of simply-connected spheres and $R \xrightarrow{f} S$ induces an epimorphism in homology. Then f has a right homotopy inverse.*

Proof. Take homology with integral coefficients. Since f_* is an epimorphism, for each generator $x_\alpha \in H_*(S)$ there is an element $y_\alpha \in H_*(R)$ such that $f_*(y_\alpha) = x_\alpha$. Since R is a wedge of spheres, the basis for $H_*(R)$ induced by including each sphere into the wedge implies that each basis generator is in the image of the Hurewicz homomorphism. As the Hurewicz homomorphism is a homomorphism, any linear combination of basis elements in $H_*(R)$ is also in the image of the

Hurewicz homomorphism. In particular, y_α is in the image of the Hurewicz homomorphism and so there is a map $s_\alpha\colon S^{n_\alpha}\longrightarrow R$ whose Hurewicz image is y_α. Let

$$s\colon \bigvee_\alpha S^{n_\alpha}\longrightarrow R$$

be the wedge sum of the maps s_α as α runs over a basis for $H_*(S)$. Then the composite $\bigvee_\alpha S^{n_\alpha}\stackrel{s}{\longrightarrow} R\stackrel{f}{\longrightarrow} S$ induces an isomorphism in homology. As all spaces are simply-connected, this isomorphism in homology implies that $f\circ s$ is a homotopy equivalence by Whitehead's Theorem. Thus f has a right homotopy inverse. $\square$

Improving on Lemma 4.3, the next lemma shows that the map $R\stackrel{f}{\longrightarrow} S$ is obtained by taking the cofibre of some map $T\stackrel{u}{\longrightarrow} R$.

Lemma 4.4. *Suppose that R and S are wedges of simply-connected spheres and $R\stackrel{f}{\longrightarrow} S$ induces an epimorphism in homology. Then there is a wedge of simply-connected spheres T and a map $u\colon T\longrightarrow R$ such that there is a homotopy cofibration $T\stackrel{u}{\longrightarrow} R\stackrel{f}{\longrightarrow} S$.*

Proof. Let $s\colon S\longrightarrow R$ be the right homotopy inverse of f in Lemma 4.3. Define the space T and the map t by the homotopy cofibration

$$S\stackrel{s}{\longrightarrow} R\stackrel{t}{\longrightarrow} T. \tag{3}$$

As R is a wedge of spheres it is a co-H-space so it has a comultiplication σ. The right homotopy inverse for f implies that the composite

$$e\colon R\stackrel{\sigma}{\longrightarrow} R\vee R\stackrel{f\vee t}{\longrightarrow} S\vee T$$

is a homotopy equivalence. Note that as T retracts off a simply-connected space it is simply-connected, and as it retracts off a wedge of spheres it is homotopy equivalent to a wedge of spheres.

Define the map u by the composite

$$u\colon T\stackrel{i_2}{\longrightarrow} S\vee T\stackrel{e^{-1}}{\longrightarrow} R$$

where i_2 is the inclusion of the second wedge summand. Let C be the homotopy cofibre of u. By definition of u, the composite $e\circ u\simeq i_2$. Therefore there is a homotopy pushout diagram

$$\begin{array}{ccccc} T & \stackrel{u}{\longrightarrow} & R & \longrightarrow & C \\ \Vert & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e'} \\ T & \stackrel{i_2}{\longrightarrow} & S\vee T & \stackrel{p_1}{\longrightarrow} & S \end{array}$$

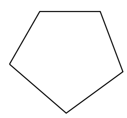
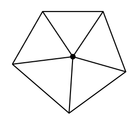
where p_1 is the pinch map to the first wedge summand and e' is an induced map of cofibres. Since e is a homotopy equivalence, the Five-Lemma implies that e' induces an isomorphism in homology so as all spaces are simply-connected, e' is a homotopy equivalence by Whitehead's Theorem. Thus there is a homotopy cofibration $T \xrightarrow{u} R \xrightarrow{p_1 \circ e} S$. It remains to show that $p_1 \circ e \simeq f$. But this follows from the definition of e and the naturality of the pinch map p_1. □

5. Wheel graphs

Let P_m be the boundary of an m-gon and let V_m be its vertex set, so V_m consists of m disjoint points. Define the simplicial complex W_m by the pushout

$$\begin{array}{ccc} V_m & \longrightarrow & V_m * \{v\} \\ \downarrow & & \downarrow \\ P_m & \longrightarrow & W_m \end{array}$$

where $\{v\}$ is a vertex disjoint from those in V_m. The simplicial complex W_m is called a *wheel graph*, where v is regarded as a hub with spokes (edges) connecting it to each vertex in the n-gon. Pictorially, representations of P_5 and W_5 are as follows:

P_5:  W_5: 

The homotopy type of $\mathcal{Z}_{P_m}$ is known. In fact, a much stronger identification was proved by MacGavran [Mac] in work that predated moment-angle complexes. Since P_m is a triangulation of a sphere, it is known [BP] that the corresponding moment-angle complex $\mathcal{Z}_{P_m}$ is a manifold. Reformulating MacGavran's result in terms of moment-angle complexes, he showed that for $m \geq 4$ there is a diffeomorphism

$$\mathcal{Z}_{P_m} \cong \#_{k=3}^{m-1} (S^k \times S^{m+2-k})^{\#(k-2)\binom{m-2}{k-1}} \tag{4}$$

where the right side is an iterated connected sum of products of two spheres.

It would be ideal to identify the homotopy type of $\mathcal{Z}_{W_m}$ as well. However, this seems to be difficult, but it is possible to determine the homotopy type of $\Omega\mathcal{Z}_{W_m}$. In fact, we will do more. Let M be any simplicial complex. Define the simplicial complex $W_m(M)$ by the pushout

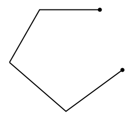
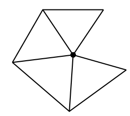
$$\begin{array}{ccc} V_m & \longrightarrow & V_m * M \\ \downarrow & & \downarrow \\ P_m & \longrightarrow & W_m(M). \end{array} \tag{5}$$

The wheel graph W_m is the special case when M is a single point. We will determine the homotopy type of $\mathcal{Z}_{W_m(M)}$, provided the homotopy type of $\Omega\mathcal{Z}_M$ is known.

It is worth mentioning two facts that give some context to the homotopy type of $W_m(M)$.

- Observe that P_m is the full subcomplex of $W_m(M)$ on the vertex set V_m. Therefore, by [DS], $\mathcal{Z}_{P_m}$ is a retract of $\mathcal{Z}_{W_m(M)}$. That is, $\mathcal{Z}_{W_m(M)}$ has the connected sum of products of two spheres in (4) retracting off it.

- Removing an edge from P_m and also the corresponding edge from W_m gives simplicial complexes that pictorially look like:

P_5':  W_5': 

Observe that P_m' can be formed by iteratively gluing an edge to the previous one along a common endpoint, and W_m' can be formed by iteratively gluing a 2-simplex to the previous one along a common edge. By [T], both $\mathcal{Z}_{P_m'}$ and $\mathcal{Z}_{W_m'}$ are homotopy equivalent to wedges of spheres. So inserting the final edge to form P_m from P_m' and W_m from W_m' dramatically changes the homotopy type, and also significantly changes cohomology by introducing nontrivial cup products.

To get started, suppose that M is on the vertex set $\{m+1, \ldots, n\}$. By Theorem 2.6 applied to (5) there is a homotopy equivalence

$$\Omega(\underline{X}, \underline{A})^{W_m(M)} \simeq \Omega(\underline{X}, \underline{A})^{P_m} \times \Omega(\underline{X}, \underline{A})^{M} \times \Omega(G * H) \tag{6}$$

where G is the homotopy fibre of the map $(\underline{X}, \underline{A})^{V_m} \longrightarrow (\underline{X}, \underline{A})^{P_m}$ and H is the homotopy fibre of the map $\prod_{i=m+1}^{n} A_i \longrightarrow (\underline{X}, \underline{A})^{M}$. Specialize to the case when each pair (X_i, A_i) is (D^2, S^1). Then the polyhedral products in (6) are moment-angle complexes, Corollary 3.3 applies to identify H, and we obtain the following.

Lemma 5.1. *Let V_m, P_m and M be as in (5) and suppose that M is on the vertex set $\{m+1, \ldots, n\}$. Then there is a homotopy equivalence*

$$\Omega\mathcal{Z}_{W_m(M)} \simeq \Omega\mathcal{Z}_{P_m} \times \Omega\mathcal{Z}_M \times \Omega(G * H)$$

where G is the homotopy fibre of the map $\mathcal{Z}_{V_m} \longrightarrow \mathcal{Z}_{P_m}$ and $H \simeq \left(\prod_{i=m+1}^{n} S^1\right) \times \Omega\mathcal{Z}_M$. □

Lemma 5.1 implies that to understand the homotopy type of $\mathcal{Z}_{W_m(M)}$ we need to understand the homotopy type of (i) $\Omega\mathcal{Z}_{P_m}$, (ii) $\Omega\mathcal{Z}_M$ and (iii) G. Both (i) and (iii) depend only on V_m and P_m, so the next few lemmas will focus solely on these cases.

While $\mathcal{Z}_{P_m}$ is a connected sum of products of two spheres, the homotopy type of the loops on a connected sum is not easy to explicitly identify. However, in this case, by [BT1, Example 3.1] we have the following.

Lemma 5.2. *For $m \geq 4$ there is a homotopy equivalence*

$$\Omega\mathcal{Z}_{P_m} \simeq \Omega S^3 \times \Omega S^{m-1} \times \Omega S(P_m)$$

where $S(P_m)$ is a wedge of simply-connected spheres. □

The construction in [BT1] describes the wedge $S(P_m)$ explicitly. In general, let $A^{\vee t}$ be the wedge sum of t copies of A. Let

$$R_m = \Big(\bigvee_{k=3}^{m-1} (S^k \vee S^{m+2-k})^{\vee (k-2)\binom{m-2}{k-1}}\Big).$$

Observe that R_m is the $(m+1)$-skeleton of $\#_{k=3}^{m-1}(S^k \times S^{m+2-k})^{\#(k-2)\binom{m-2}{k-1}}$. Equivalently, R_m is homotopy equivalent to the connected sum with a puncture. Define R'_m by the cofibration

$$S^3 \vee S^{m-1} \longrightarrow R_m \longrightarrow R'_m$$

where the left map is the inclusion of one copy of $S^3 \vee S^{m-1}$ into R_m. Then R'_m is a wedge of simply-connected spheres and

$$S(P_m) = R'_m \rtimes (\Omega S^3 \times \Omega S^{m-1}).$$

By Lemma 4.2, $S(P_m)$ is homotopy equivalent to a wedge of simply-connected spheres.

Next, we aim towards Lemma 5.7, which identifies the space G in Lemma 5.1. Since V_m is m disjoint points, by [GT1, P] the following holds.

Lemma 5.3. *For $m \geq 4$ there is a homotopy equivalence*

$$\mathcal{Z}_{V_m} \simeq \bigvee_{k=3}^{m+1} (S^k)^{\vee (k-2)\binom{m}{k-1}}.$$

□

Observe that the dimensions of $\mathcal{Z}_{V_m}$ and $\mathcal{Z}_{P_m}$ are $m+1$ and $m+2$ respectively, so the map $\mathcal{Z}_{V_m} \longrightarrow \mathcal{Z}_{P_m}$ factors through the $(m+1)$-skeleton R_m of $\mathcal{Z}_{P_m}$, giving a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_{V_m} & \longrightarrow & \mathcal{Z}_{P_m} \\ \downarrow \theta & & \| \\ R_m & \longrightarrow & \mathcal{Z}_{P_m} \end{array} \qquad (7)$$

for some map θ.

Lemma 5.4. *The map θ induces a surjection in homology.*

Proof. In general, by [BBCG, Corollary 2.23] there is a homotopy equivalence

$$\Sigma\mathcal{Z}_K \simeq \bigvee_{I \notin K} \Sigma^{2+|I|}|K_I|$$

where $I = \{i_1, \ldots, i_k\}$ is a subsequence of $[m]$ with $1 \leq i_1 < \cdots < i_k \leq m$, K_I is the full subcomplex of K on the vertex set I, $|K_I|$ is the geometric realization of K_I, and $|I|$ is the number of vertices in I. This homotopy equivalence is natural for simplicial maps $L \longrightarrow K$ and induces a $\mathbb{Z}$-module decomposition in integral homology,

$$H_*(\mathcal{Z}_K) \cong \bigoplus_{I \notin K} H_*(\Sigma^{1+|I|}|K_I|). \tag{8}$$

In our case, if $I = [m]$ then $|(P_m)_I| = |P_m| \simeq S^1$ and this case accounts for the generator of $H_{m+2}(\mathcal{Z}_{P_m})$. Therefore, as R_m is the $(m+1)$-skeleton of $\mathcal{Z}_{P_m}$, there is an isomorphism

$$H_*(R_m) \cong \bigoplus_{\substack{I \notin P_m \\ I \neq [m]}} H_*(\Sigma^{1+|I|}|(P_m)_I|).$$

In general, the inclusion of the vertex set V into a simplicial complex K induces an epimorphism $H_0(|V|) \longrightarrow H_0(|K|)$ since H_0 counts the number of connected components, $|K|$ has at most m components where m is the number of vertices in V, and each connected component of $|K|$ contains at least one of the vertices of V. Consequently, if $|K|$ is homotopy equivalent to some number of disjoint points then the inclusion $V \longrightarrow K$ induces an epimorphism $H_*(|V|) \longrightarrow H_*(|K|)$.

In our case, consider (8) applied to the simplicial map $V_m \longrightarrow K_m$. Assume that $I \notin P_m$ and $I \neq [m]$. Observe that $I \notin V_m$ as well. Since I is a proper subset of $[m]$ we have $|(P_m)_I|$ homotopy equivalent to some number of disjoint points. Therefore, as the vertex set of $(P_m)_I$ is $(V_m)_I$, the simplicial map $(V_m)_I \longrightarrow (P_m)_I$ induces an epimorphism $H_*(|(V_m)_I|) \longrightarrow H_*(|(P_m)_I|)$. Hence there is an epimorphism

$$\bigoplus_{\substack{I \notin V_m \\ I \neq [m]}} H_*(\Sigma^{1+|I|}|(V_m)_I|) \longrightarrow \bigoplus_{\substack{I \notin P_m \\ I \neq [m]}} H_*(\Sigma^{1+|I|}|(P_m)_I|) \cong H_*(R_m).$$

Observe that the left side is a submodule of $H_*(\mathcal{Z}_{V_m})$ by (8), and therefore the homotopy commutativity of (7) implies that θ_* is an epimorphism. □

Observe that $\mathcal{Z}_{V_m}$ is a wedge of simply-connected spheres by Lemma 5.3, R_m is a wedge of simply-connected spheres by definition, and $\mathcal{Z}_{V_m} \xrightarrow{\theta} R_m$ induces an epimorphism in homology by Lemma 5.4. Therefore Lemmas 4.3 and 4.4 imply that θ has a right homotopy inverse and there is a homotopy cofibration

$$T_m \xrightarrow{u} \mathcal{Z}_{V_m} \xrightarrow{\theta} R_m$$

where T_m is a wedge of simply-connected spheres.

By definition, R_m is the $(m+1)$-skeleton of $\#_{k=3}^{m-1}(S^k \times S^{m+2-k})^{\#(k-2)\binom{m-2}{k-1}}$. Writing the connected sum as $\mathcal{Z}_{P_m}$, there is a homotopy cofibration

$$S^{m+1} \xrightarrow{g} R_m \longrightarrow \mathcal{Z}_{P_m} \tag{9}$$

where g attaches the top cell. Since θ has a right homotopy inverse, g lifts to a map $g' \colon S^{m+1} \longrightarrow \mathcal{Z}_{V_m}$.

Lemma 5.5. *There is a homotopy cofibration $S^{m+1} \vee T_m \xrightarrow{g' \vee u} \mathcal{Z}_{V_n} \longrightarrow \mathcal{Z}_{P_m}$.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
T_m & = & T_m & & \\
\downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle u} & & \\
S^{m+1} \vee T & \xrightarrow{g' \vee u} & \mathcal{Z}_{V_m} & \longrightarrow & \mathcal{Z}_{P_m} \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \theta} & & \| \\
S^{m+1} & \xrightarrow{g} & R_m & \longrightarrow & \mathcal{Z}_{P_m}.
\end{array}$$

The upper left square clearly commutes. Taking cofibres vertically gives the lower left square. The lower left square is therefore a homotopy pushout, so taking cofibres horizontally we obtain the lower right square (which matches (7)). □

We will use the following result proved in [BT2].

Proposition 5.6. *Suppose that $A \longrightarrow X \xrightarrow{h} Z$ is a homotopy cofibration and Ωh has a right homotopy inverse. Then there is a homotopy fibration*

$$A \rtimes \Omega Z \longrightarrow X \xrightarrow{h} Z.$$ □

In our case, consider the homotopy cofibration $S^{m+1} \vee T_m \longrightarrow \mathcal{Z}_{V_m} \xrightarrow{h} \mathcal{Z}_{P_m}$ from Lemma 5.5, where h is simply a label for the right map. Since P_m is a flag simplicial complex and V_m is its vertex set, by [PT] the map Ωh has a right homotopy inverse. Therefore the hypotheses of Proposition 5.6 are satisfied, implying that the homotopy fibre G of h can be identified.

Lemma 5.7. *There is a homotopy equivalence $G \simeq (S^{m+1} \vee T_m) \rtimes \Omega \mathcal{Z}_{P_m}$.* □

Remark 5.8. By definition, T_m is a wedge of simply-connected spheres, and by Lemma 5.2, $\Omega \mathcal{Z}_{P_m}$ is homotopy equivalent to a product of loops on simply-connected spheres. Therefore, Lemma 4.2 implies that $G \simeq (S^{m+1} \vee T_m) \rtimes \Omega \mathcal{Z}_{P_m}$ is homotopy equivalent to a wedge of simply-connected spheres.

The homotopy equivalence for $\Omega \mathcal{Z}_{W_m(M)}$ in Lemma 5.1 can now be refined by substituting in the homotopy equivalences for $\Omega \mathcal{Z}_{P_m}$ and G in Lemmas 5.2 and 5.7 respectively.

Theorem 5.9. *For $m \geq 4$ there is a homotopy equivalence*

$$\Omega \mathcal{Z}_{W_m(M)} \simeq \Omega S^3 \times \Omega S^{m+1} \times \Omega S(P_m) \times \Omega \mathcal{Z}_M \times \Omega(G * H)$$

where $G \simeq (S^{m+1} \vee T_m) \rtimes \Omega \mathcal{Z}_{P_m}$ and $H = (\prod_{i=m+1}^{n} S^1) \times \Omega \mathcal{Z}_M$. □

Corollary 5.10. *If $\Omega \mathcal{Z}_M$ is homotopy equivalent to a product of path-connected spheres and loops on simply-connected spheres then so is $\Omega \mathcal{Z}_{W_M}$.*

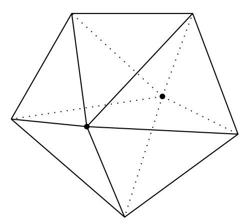
Proof. Since $\Omega\mathcal{Z}_M$ is homotopy equivalent to a product of path-connected spheres and loops on simply-connected spheres, so is $H = (\prod_{i=m+1}^{n} S^1) \times \Omega\mathcal{Z}_M$. Therefore, by Lemma 4.1, ΣH homotopy equivalent to a wedge of simply-connected spheres. Since G is also homotopy equivalent to a wedge of simply-connected spheres by Remark 5.8, so is $G * H$. The Hilton-Milnor Theorem then implies that $\Omega(G * H)$ is homotopy equivalent to a product of loops on simply-connected spheres. Thus in the homotopy equivalence for $\Omega\mathcal{Z}_{W_m(M)}$ in Theorem 5.9, each of the factors is homotopy equivalent to a product of path-connected spheres and loops on simply-connected spheres and hence so is $\Omega\mathcal{Z}_{W_m(M)}$. □

Example 5.11. Return to the wheel graph W_m itself. This is $W_m(M)$ with $M = \{v\}$ being a single vertex. By definition of the polyhedral product, $\mathcal{Z}_{\{v\}} = D^2$, which is contractible, so Theorem 5.9 implies that there is a homotopy equivalence

$$\Omega\mathcal{Z}_{W_m} \simeq \Omega S^3 \times \Omega S^{m+1} \times \Omega S(P_m) \times \Omega(G * H)$$

where $G \simeq (S^{m+1} \vee T_m) \rtimes \Omega\mathcal{Z}_{P_m}$ and $H = S^1$. In particular, $G * H \simeq \Sigma^2 G$.

Example 5.12. Take $M = \{u, v\}$ be two disjoint points. A pictorial representation of $W_5(\{u, v\})$ is:

$W_5(\{u, v\})$: 

By Lemma 5.3, $\mathcal{Z}_M \simeq S^3$. Theorem 5.9 therefore implies that there is a homotopy equivalence

$$\Omega\mathcal{Z}_{W_m(\{u,v\})} \simeq \Omega S^3 \times \Omega S^{m+1} \times \Omega S(P_m) \times \Omega S^3 \times \Omega(G * H)$$

where $G \simeq (S^{m+1} \vee T_m) \rtimes \Omega\mathcal{Z}_{P_m}$ and $H = S^1 \times S^1 \times \Omega S^3$.

More generally, there is a wide class of simplicial complexes M with the property that $\mathcal{Z}_M$ is homotopy equivalent to a wedge of simply-connected spheres, implying by the Hilton-Milnor Theorem that $\Omega\mathcal{Z}_M$ is homotopy equivalent to a product of loops on simply-connected spheres, and hence Corollary 5.10 can be applied to decompose $\Omega\mathcal{Z}_{W_m(M)}$. This class of simplicial complexes includes shifted complexes [GT2, IK1], or more generally extractible simplicial complexes [IK2], and flag simplicial complexes whose 1-skeleton is a chordal graph [GPTW].

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