UNIVERSITY OF SOUTHAMPTON

Faculty of Social Sciences
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Inverse Semigroups Acting on Graphs

by

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A thesis for the degree of
Doctor of Philosophy

August 2021

University of Southampton

Abstract

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In group theory we are able to derive many properties about a group from how it acts on a graph. Knowing this, we aimed to find similar results for inverse semigroups acting on graphs. We were able to find a consistent method of defining an action for a free product of inverse semigroups provided we already have actions for the semigroups that make up this product. Furthermore, this action will deliver back to us a fundamental inverse semigroup that is isomorphic to the free product. Following this, we looked at how our method works with polycyclic, Bruck-Reilly and Brandt semigroups. After finding that it does not work in the general case, we looked at what additional properties we will need for our semigroup in order to make it work. In particular, we found that a zero element in an inverse semigroup causes a lot of problems for our current method.

Acknowledgements

I would like to thank those in the School of Mathematics at the University of Southampton who have helped me over the course of my research. In particular, I would like to thank my supervisors, Dr J.H. Renshaw and Prof P.H. Kropholler for the help they gave me in developing my research.

Contents

1 Introduction

In group theory, we are able to determine many properties of a group based on how it acts on a graph. It is therefore a logical next step to investigate if we can replicate such results with inverse semigroups. A method of defining an action of an inverse semigroup on a non-trivial graph has been around for some time but it is yet to be proven either way if this tells us anything about the inverse semigroup. This project aimed to take inverse semigroups with certain properties and see if said properties can be proven with the inverse semigroup action.

We begin by covering the method currently in place for defining an action of an inverse semigroup on a graph and how we get back what we call the *fundamental inverse semigroup* from this method. As we demonstrate this, we will be comparing it to how the same process is done in group theory so that we may better understand the logic behind the process.

Following this, we look at how we can create a graph that a free product of inverse semigroups can act on when we already have graphs that these semigroups act on independently of each other. Furthermore, we demonstrate how the fundamental inverse semigroup from this sytem is isomorphic to the free product of inverse semigroups. Next we investigate polycyclic semigroups and demonstrate how this method does not always work for such semigroups. The final sections show how our method can be used for Bruck-Reilly semigroups and Brandt semigroups and more importantly, when this method will give the desired results for such semigroups.

2 Preliminary Ideas

The motivation for this project comes from the Bass-Serre theory of groups. As such it will be worthwhile giving a brief overview of this theory and how we try to reflect this method with inverse semigroups. The notation and terminlogy used in Bass-Serre theory will be the same used by Dicks and Dunwoody [6]. Similarly the ideas behind how we can adapt these ideas to work with inverse semigroups comes from James Renshaw's paper [22].

We will also be defining graphs by the sets V and E where V is the set of vertices of the graph and E is the set of edges that connect them.

2.1 Semigroup Actions

We will begin with the definiton of a group action outlined by Dicks and Dunwoody [6].

Definition 2.1.1. Let G by a group and X a set of elements. We say G acts on X if we can define a function $f : G \times X \to X$ such that:

- $\forall g_1, g_2 \in G, x \in X,\ f(g_1 g_2, x) = f(g_1, f(g_2, x))$

- $\forall x \in X,\ f(1, x) = x$

If such a function exists we call X a G-set and say that a (left) G-act exists on X.

Note that this is equivalent to saying there exists a group homomorphism from G to $Sym(X)$ (the set of permutations of X). Another important property that we can derive from this definition is that $\forall g \in G$ and $x, y \in X$, $f(g, x) = y \Rightarrow x = f(g^{-1}, y)$.

Similarly we can expand this definition to apply to inverse semigroups.

Definition 2.1.2. Let S by an inverse semigroup and X a set of elements. We say S acts on X if we can define a partial function $f : S \times X \to X$ such that:

- $\forall s_1, s_2 \in S, x \in X,\ f(s_1 s_2, x)$ exists $\Leftrightarrow$ $f(s_1, f(s_2, x))$ exists. Then $f(s_1 s_2, x) = f(s_1, f(s_2, x))$.
- $\forall x, y \in X, s \in S,\ f(s, x) = f(s, y) \Rightarrow x = y$

If such a partial function exists it defines X as an S-act and we say that a (left) S-act exists on X.

This definition when applied to groups is in fact the same definition as a group action. In other words, all group actions can be considered to be semigroup actions. In fact it is easy to prove that inverse semigroup actions share some properties with group actions.

Lemma 2.1.1. *[22, Lemma 3.1.] Let X be an S-act of an inverse semigroup S:*

- $\forall s \in S, x, y \in X,\ f(s, x) = y \Rightarrow x = f(s^{-1}, y)$
- $\forall e \in E(S), x \in X,\ f(e, x)$ *exists implies that* $f(e, x) = x$.

It is important to mention one difference in these two actions that will complicate our objective. By definition, in a group action $f(g, x)$ will exists for all $g \in G$ and $x \in X$, however the same cannot be said for inverse semigroup actions. Going forward, we will say $f(s, x) = sx$ when the context is clear.

Right group and inverse semigroup actions are also defined dually, but unless otherwise specified we will be considering all our actions to be left actions. It will also be helpful to define some other terminology that we will be using alongside their roots in group theory when applicable. For the following definitions, we say G, S and X are a group, inverse semigroup and set respectively. Furthermore, we are assuming that X is a G-act and S-act.

Definition 2.1.3. Let X and Y also be G-sets. If a function $f : X \to Y$ exists such that $\forall x \in X,\ g \in G,\ f(gx) = gf(x)$ then we define f to be a G-map.

Definition 2.1.4. Let X and Y be S-sets. If a function $f : X \to Y$ exists such that $\forall x \in X$, $s \in S$, $f(sx) = sf(x)$ then we define f to be a S-map.

Definition 2.1.5. Let X be a graph defined by a set of vertices, V that is a G-set and a set of directed edges, E. Say the graph is such that E is also a G-set where any $g \in G$ acts on $e \in E$ if it acts on both the edge's initial and terminal vertices in V which we shall label v_1 and v_2 respectively. The action will then map the edge to another edge in E whose inital vertex is gv_1 and whose terminal vertex is gv_2. We define two G-maps $\iota : E \to V$ and $\tau : E \to V$ which map the graph's edges to their initial and terminal vertices respectively. Collectively, (X, V, E, ι, τ) is defined to be a G-graph which will just be denoted by X when the context is clear.

Definition 2.1.6. Let X be a graph defined by a set of vertices, V that is a S-set and a set of directed edges, E. Say the graph is such that E is also a S-set where any $s \in S$ acts on $e \in E$ if it acts on both the edge's initial and terminal vertices in V which we shall label v_1 and v_2 respectively. The action will then map the edge to another edge in E whose inital vertex is sv_1 and whose terminal vertex is sv_2. We define two S-maps $\iota : E \to V$ and $\tau : E \to V$ which map the graph's edges to their initial and terminal vertices respectively. Collectively, (X, V, E, ι, τ) is defined to be a S-graph which will just be denoted by X when the context is clear.

Definition 2.1.7. For every $s \in S$ we define the domain of s to be given by the set;

$$D_s^X = \{x \in X | sx \in X\} \tag{1}$$

In other words, the domain of s is the set of elements in X that s can act on. Likewise, the domain of any $x \in X$ is the set;

$$D_S^x = \{s \in S | sx \in X\} \tag{2}$$

Remark. In a group action, the domain of every element would be X.

Knowing this,we can prove certain properties of the domains of an S-act based on the properties of S.

Lemma 2.1.2. *Let $s \in S$ such that $D_s \neq \emptyset$:*

- *If s is the zero element in S, then $\forall t \in S$, $D_s \subseteq D_t$.*
- *If s is the identity element in S, then $\forall t \in S$, $D_t \subseteq D_s$.*

Proof. Let $s \in S$ and $x \in X$ be defined so $x \in D_s$. Say s is the zero element in S. Then, $\forall t \in S$, $st = s$. So, $x \in D_s \Rightarrow sx = s(tx)$ exists. Therefore, $x \in D_t$ which tells us that $D_s \subseteq D_t$. If instead, s is the identity element in S, then $ts = t$. Let $y \in D_t$, then $ty = t(sy)$ exists. Consequently, $y \in D_s$ and thus $D_t \subseteq D_s$. $\square$

Corollary 2.1.3. *If S is an inverse semigroup with zero and $x \in X$ is such that $x \in D_0$, then $\forall s \in S$, $sx = x$.*

Proof. From Lemma 2.1.2, we know that $D_0 \subseteq D_s$ and hence sx will exist. Therefore, since $0 = 0s$, we can say that $0x = 0sx$ which then implies that $x = sx$. $\square$

Lemma 2.1.4. *$\forall s, t \in S$, $D_{st} \subseteq D_t$.*

Proof. Let $s, t \in S$. If $D_{st} = \emptyset$, then $D_{st} \subseteq D_t$ by definition. Assume $D_{st} \neq \emptyset$. Then, $\forall x \in D_{st}$, $(st)x$ exists $\Rightarrow tx$ exists. Hence, $x \in D_t$ and $D_{st} \subseteq D_t$. $\square$

A consequence of Lemma 2.1.4 is that if $s, t \in S$ are such that $ss^{-1} = tt^{-1}$, then $D_{s^{-1}} = D_{t^{-1}}$.

Lemma 2.1.5. *Let $s \in S$. If $|D_s| \neq |D_{s^{-1}}|$ then $|X| = \infty$.*

Proof. Say X is a finite set. This then implies that $|D_s|$ and $|D_{s^{-1}}|$ are also finite. So, say $D_s = \{x_i | i \in \{1, 2, \dots, n\}\}$ for some $n \in \mathbb{N}^+$. Then, for each $x_i \in D_s$, $\exists y_i \in X$ such that $sx_i = y_i$. Furthermore, this implies that $x_i = s^{-1}y_i$ and hence $y_i \in D_{s^{-1}}$. Given that the action of s is a bijection $D_s \to D_{s^{-1}}$ we can say that $|D_s| = |D_{s^{-1}}|$. $\square$

Lemma 2.1.6. *If $\exists x \in X$ such that D^x is an inverse semigroup, then $S = D^x$ or S is a union of two disjoint inverse semigroups.*

Proof. Say x is an element in an S-act such that D^x is an inverse semigroup. Assume $S \neq D^x$, so $\exists s, t \in S$ such that $s, t \notin D^x$. If $st \in D^x$, then $(st)x = s(tx)$ exists. However, this then implies that tx exists and hence $t \in D^x$ which contradicts our definition of t. Hence, $s, t \notin D^x \Rightarrow st \notin D^x$. Finally, since D^x is an inverse semigroup, we know that for any $s \notin D^x \Rightarrow s^{-1} \notin D^x$. This then makes $S \backslash D^x$ an inverse semigroup. S can therefore be thought of as a union of the inverse semigroups D^x and $S \backslash D^x$ which, by definition, must be disjoint. $\square$

Lemma 2.1.7. *If D^x is a proper inverse subsemigroup of S for some $x \in X$, then $S \backslash D^x$ is an ideal of S.*

Proof. Let s be an element in S such that $s \notin D^x$. Say $\exists t \in S$ such that $ts \in D^x$. This would then imply that $(ts)x = t(sx)$ exists and hence $s \in D^x$ which contradicts our definition of s. Therefore, $S \backslash D^x$ is a left ideal of S.

Now instead assume that t is such that $st \in D^x$. Since D^x is an inverse semigroup, it must be the case that $(st)^{-1} = t^{-1}s^{-1} \in D^x$. This further implies that $s^{-1}x$ exists meaning that $s^{-1} \in D^x$. However, since D^x is an inverse semigroup, $s^{-1} \in D^x \Rightarrow (s^{-1})^{-1} = s \in D^x$ which again contradicts our definition of s. This means that $st \in S \backslash D^x$ making $S \backslash D^x$ a right ideal of S. $\square$

Remark. To prove that $S \backslash D^x$ is a left ideal of S, we never needed to use the fact that D^x is an inverse semigroup and so this must be true for the domain of any element in an S-act.

Definition 2.1.8. $\forall x \in X$ the S-orbit of x is given by;

$$Sx = \{sx | s \in D_S^x\} \tag{3}$$

These orbits are used to define the quotient set of S-acts. For example, $S \backslash X = \{Sx | x \in X\}$.

Definition 2.1.9. The G-stabilizer of any $x \in X$ is defined to be the set of elements in G that fix x. This set is given by;

$$G_x = \{g \in G | gx = x\}. \tag{4}$$

Similarly we define the S-stabilizer of x to be the set;

$$S_x = \{s \in S | sx = x\}. \tag{5}$$

Remark. It is easy to prove that any G-stabilizer is itself a group. Likewise any S-stabilizer is an inverse semigroup.

Any G-stabilizer or S-stabilizer can be considered to be a subgroup and inverse subsemigroup of G and S respectively. Furthermore, we define the set of G-orbits of X to be the quotient set of X given by $G \backslash X$. Similarly, if we define X to be an S-act then the quotient set, $S \backslash X$, is defined to be the set of S-orbits of X. It is easy to see that we can map a G-set (or S-set) to its quotient set by mapping the elements to their orbits.

Definition 2.1.10. Consider X to be a G-graph. The graph $(G \backslash X, G \backslash V, G \backslash E, \bar{\iota}, \bar{\tau})$ is called the quotient graph of a G-graph where $\bar{\iota}$ and $\bar{\tau}$ are functions from $G \backslash E$ to $G \backslash V$. They are defined by $\bar{\iota}(Ge) = G\iota e$ and $\bar{\tau}(Ge) = G\tau e$. The quotient graph of an S-graph is defined dually.

Definition 2.1.11. A G-transversal of X is a subset of X that meets each of its G-orbits exactly once. We also define an S-transversal dually.

Now consider our action to be a G-graph (resp. S-graph) and let Y be a G-transversal (resp. S-transversal) of X. If $\exists Y_0 \subseteq Y \subseteq X$ such that Y_0 is a subtree of X, $V \cap Y \subseteq Y_0$ and $\iota(e) \in V \cap Y$ $(\forall e \in E \cap Y)$ then we call Y a fundamental (G-)transversal in X. The definition of a fundamental (S-)transversal of an S-graph is given dually.

Remark. Note that for any G-transversal, Y the composite $Y \subseteq X \to G \backslash X$ given by mapping elements in Y to their orbits is a bijection.

It has been proven that if the quotient graph $G \backslash X$ is connected then such subsets of X can be shown to exist [6, Proposition I.2.6.].

For the following definitions, we will be assuming our set X gives us a fundamental transversal when we take it to be a G-graph or an S-graph.

Definition 2.1.12. A graph of groups associated to X is a graph derived from a fundamental G-transversal of X and its quotient graph whose vertices and edges are all groups. If we take a fundamental G-transversal of the quotient graph then there exists a bijection between it and a fundamental G-transversal of X, say Y. For every vertex v in Y we assign the group G_v. Similarly for every edge e in Y we wish to assign the group G_e, however $\tau(e)$ may not exist in Y. If such an edge exists, the graph formed from these stabilizers will not be a graph. To account for this, we also need to assign to every edge stabilizer a group monomorphism from G_e to $G_{\bar{\tau}(Ge)}$ (which is the stabilizer of the unique vertex in Y whose orbit contains $\tau(e)$).

Given our definition of Y, we know that it contains a unique vertex that exists in the same orbit as $\tau(e)$. If we call this vertex v we can say that $G_{\tau(e)} = G_v$. Therefore, $\exists t_e \in G$ such that $t_e v = \tau(e)$. Note that $t_e G_v t_e^{-1} = G_{\tau(e)}$. Hence we can say that there exists an embedding $f_e : G_e \to G_v$ given by $g \mapsto t_e^{-1} g t_e$ since G_e is a subset of $G_{\tau(e)}$. So in our graph of groups the terminal vertex of G_e will be given by G_v. Using this we are able to create a connected graph whose vertices and edges are stabilizers of the vertices and edges in Y.

Definition 2.1.13. We define a graph of inverse semigroups of an S-graph from a fundamental S-transversal and its quotient graph in a similar way to how we would define a graph of groups of a G-graph. The only difference comes from how we find embeddings of edges in our graph. In the definition of the graph of groups, the embedding that we presented is always guarenteed to exist. In the inverse semigroup case, however, this is not always true. As such it may not be possible to find a suitable embedding and hence the graph of inverse semigroups cannot be defined.

We do however have a method of proving the existence of certain embeddings. Say we have an edge $e \in Y$ such that $\tau(e) \notin Y$, and therefore need to find an embedding from S_e to S_v where v is the unique vertex in $Y \cap S\tau(e)$. Since $v \in S\tau(e)$ we know that $\exists t_e \in S$ such that $t_e v = \tau(e)$. Unlike with the group scenario, it is not always possible for us to say that $t_e S_v t_e^{-1} = S_{\tau(e)}$. We can however say that $t_e^{-1} S_{\tau(e)} t_e \subseteq S_v$. Knowing this, it can be shown that an embedding exists from $S_{\tau(e)}$ to a subset of S_v if $S_{\tau(e)}$ is a monoid whose identity is $t_e t_e^{-1}$ [22, Theorem 3.8.]. Again we label this embedding with f_e and define it as mapping every $s \in S_{\tau(e)}$ to $f_e(s) = t_e^{-1} s t_e \in S_v$.

If these graphs exist we are able to use them to create fundamental groups and inverse semigroups.

Definition 2.1.14. Let Y be a fundamental G-transversal of X such that we can define a graph of groups from it and V_Y to be the vertices in Y. Also let E_τ be the set of edges in Y whose terminal vertices do not exist in Y and $\forall e \in E_\tau$ let t_e be an element in S such that the map $f_e : G_e \to G_v$ given by $g \mapsto t_e^{-1} g t_e$ is an embedding. Note that we know such a t_e must exist due to the existence of the graph of groups.

From this we define a group interms of its generators and relations. The generators of the group are given by the set $\{t_e | e \in E_\tau\} \cup \bigcup_{v \in V_Y} G_v$. In other words, the set is a union of the vertices in the graph of groups and any other elements in G that are required for the existence of the relavent embeddings in said graph. The relations of our group are the relations of the groups G_v $(\forall v \in V_Y)$ and that $f_e(g) = t_e^{-1} g t_e$ $(\forall e \in E_\tau,\ g \in G_e)$. We call this group the fundamental group of G and label it G'.

The fundamental (inverse) semigroup is defined dually from a graph of inverse semigroups where the genrators and relations we obtain are instead used to define an inverse semigroup instead of a group. That is to say, if Y is instead a fundamental S-transversal of X for every edge $e \in E_\tau$ we require an element $t_e \in S$such that $f_e : S_e \to S_v$ given by $s \mapsto t_e^{-1} s t_e$ is an embedding. Then the generators of the fundamental inverse semigroup is given by the set $\{t_e | e \in E_\tau\} \cup \bigcup_{v \in V_Y} S_v$ and the relations are the relations of the inverse semigroups S_v $(\forall v \in V_Y)$ and that $f_e(s) = t_e^{-1} s t_e$ $(\forall e \in E_\tau,$ $s \in S_e)$. Similarly we label the fundamental inverse semigroup S'

It is known that if we have a group that acts on a tree then the fundamental group we obtain will be isomorphic to the original group. In particular, for any group we can define a graph that it acts on.

Definition 2.1.15. For any group G take a subset $T \subseteq G$. Define a graph whose vertices are the elements of G and whose edges are given by $E = G \times T$ where $\forall (g, t) \in E$, $\iota((g, t)) = g$ and $\tau((g, t)) = gt$. Such a graph is called the Cayley graph of G with respect to T.

If we take the Cayley graph of G with respect to a generating set of G, then the quotient graph is connected. As such, we are able to find a fundamental group from this system that is isomorphic to the original group.

There is not, however, an equivalent theory for inverse semigroups acting on graphs. Like with group actions, we are able to make a graph from a generating set of any inverse semigroup that it can subsequently act on.

Definition 2.1.16. [22] Let X be a partial S-biact of an inverse semigroup S and $T \subseteq S$. We define an S-graph, (X, V, E, ι, τ) to be given by $X = V$, $E = \{(x, t) \in X \times E : xt \text{ exists and } xt \neq x\}$, $\iota(x, t) = x$ and $\tau(x, t) = xt$ for any $(x, t) \in E$. Such a graph is called the *Schützenberger graph* of X with respect to T and is denoted by $\Gamma = \Gamma(X, T)$.

The particular Schützenberger graph we are interested in is when $V = S$ (where S is an inverse semigroup) and T is a generating set of S. In such a case, the Schützenberger graph will be a set of trees where the set of vertices of any such tree is equivalent to an $\mathcal{R}$-class of S.

Definition 2.1.17. Let S be an inverse semigroup. We define $*_S$ to be a function from S to itself where, $\forall s, t \in S$, $s *_S t$ is defined if and only if

$t = s^{-1}st$. If $s *_S t$ does exist, then $s *_S t = st$. Such a function is then called the *Preston-Wagner representation* with respect to S

Remark. The Preston-Wagner representation is usually defined in a slightly different way. For any $s \in S$, we define ρ_s to be a partial bijective map given by $\rho_s(t) = ts$ where $t \in dom(p_s)$ if and only if $t \in Ss^{-1}$. The map $s \mapsto \rho_s$ is then called the Preston-Wagner representation of S. It is easy to see that this definition is equivalent to the one that is used in this paper.

Using this, we can define an action of S on itself by setting $f(s,t) = s *_S t$ ($\forall s, t \in S$). By doing this, we find that the orbits of elements to be the $\mathcal{L}$-classes of S. The problem with this method is that, unlike the Cayley graph of groups, the quotient graph of this system may not be connected. Therefore, we not be able to define a graph of inverse semigroups. Even if we are able to do this, we would still need to show that the fundamental inverse semigroup would be isomorphic to the original. As such,we do not always use this method when finding an action of our inverse semigroup on a graph.

Example 2.1.8. For clarity, it will be helpful to look at an example of when we have an action of an inverse semigroup on a graph which returns to us a fundamental inverse semigroup that is isomorphic to the original. Let S be a monogenic inverse semigroup given by $S = Inv\langle \alpha \mid \rangle$ and define the set $V = \{a_1, a_0, a_{-1}, \ldots\}$ to be the set of vertices of a graph.

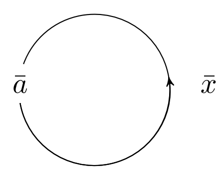
We then define a partial map $\phi : S \times V \to V$ to be such that $\phi(s,v) = \rho_s(v)$ where ρ_s is given by;

$$\rho_\alpha = \begin{pmatrix} a_1 & a_0 & a_{-1} & \cdots \\ a_0 & a_{-1} & a_{-2} & \cdots \end{pmatrix}. \tag{6}$$

Say our graph with vertices V is;

$$\cdots \xrightarrow{\alpha^3 x} a_{-2} \xrightarrow{\alpha^2 x} a_{-1} \xrightarrow{\alpha x} a_0 \xrightarrow{x} a_1$$

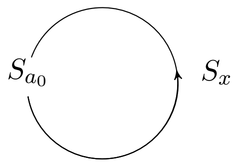
Now we need to find our vertex and edge orbits to get our quotient graph. From our definition of ρ_α we can see that all our vertices share the same orbit and all the edges share the same orbit. Let $\bar{a}$ represent the orbit of the vertices in the graph and $\bar{x}$ represent the orbit of the edges. Then we have the following quotient graph;



By picking the vertex a_0 to represent $\bar{a}$ in an S-transversal, we must then pick x to represent $\bar{x}$ as this is the only edge in the graph that has a_0 as its start point. This then means that our S-transversal is;

$$a_0 \xrightarrow{x}$$

Therefore, we wish to construct a graph of inverse semigroups from the stabilizers of a_0 and x which looks like;



However, for such a graph to exist, we require a map from $S_{a_1} \to S_{a_0}$ such that it gives a monomorphism from $S_x \subseteq S_{a_1}$ to S_{a_0}. We know that such a monomorphism exists if $\exists t \in S$ such that tt^{-1} is the identity in S_{a_1} and $ta_0 = a_1$. Said monomorphism is then defined by $s \mapsto t^{-1}st \; \forall s \in S_{a_1}$.

Before we can identify t, we need to find the values of S_{a_0}, S_{a_1} and S_x. From examination of ρ_α we can see that a_0 can be acted on by α^{-1} and all positive powers of α. Hence, $S_{a_0} = Inv\langle \alpha^{-n}\alpha^n, \alpha\alpha^{-1} | n \in \mathbb{N}\rangle$. Similarly we can find that $S_{a_1} = \{\alpha^{-n}\alpha^n | n \in \mathbb{N}\}$ and $S_x = \{\alpha^{-n}\alpha^n | n \in \mathbb{N}\}$. Therefore, we can now say that we are looking for a $t \in S$ such that $tt^{-1} = \alpha^{-1}\alpha$. An obvious choice is α^{-1} and since $\alpha^{-1}a_0 = a_1$ we can say that $t = \alpha^{-1}$.

We now construct a semigroup presentation, say S', from the information we have from our graph of inverse semigroups. First we will say that $\gamma = \alpha\alpha^{-1}$ and $\beta_n = \alpha^{-n}\alpha^n \; \forall n \in \mathbb{N}$. These along with t then give the generators of S'.

The first relations we set for S' are those of S_{a_0}. These are ($\forall n, m \in \mathbb{N}$) $\beta_n^2 = \beta_n$, $\gamma^2 = \gamma$ and $\beta_n\beta_m = \beta_{max\{n,m\}}$. We also need the following relations, $tt^{-1} = \beta_1$ and $t^{-1}\beta_n t = \gamma\beta_{n-1}$ if $n > 1$ and γ otherwise.

We now show that S' is equivalent to S. We prove this by simplifying the presentation of S'. First, $tt^{-1} = \beta_1$ and $t^{-1}\beta_1 t = \gamma$ imply that $\gamma = t^{-1}t$. So we can now remove γ from our generators providing we account for the relations in S' that contain γ. Both $\gamma^2 = \gamma$ and $t^{-1}\beta_1 t = \gamma$ are given by $\gamma = t^{-1}t$ and can therefore also be removed. However, we can not do the same for $t^{-1}\beta_n t = \gamma\beta_{n-1}$ if $n > 1$. We therefore replace this relation with$t^{-1}\beta_n t = t^{-1}t\beta_{n-1}$ if $n > 1$, but this is equivalent to another equation.

$$\begin{aligned} t^{-1}\beta_n t = t^{-1}t\beta_{n-1} \text{ if } n > 1 &\Leftrightarrow t^{-1}\beta_{n+1}t = t^{-1}t\beta_n \\ &\Leftrightarrow tt^{-1}\beta_{n+1}tt^{-1} = t\beta_n t^{-1} \\ &\Leftrightarrow \beta_1\beta_{n+1}\beta_1 = t\beta_n t^{-1} \\ &\Leftrightarrow \beta_{n+1} = t\beta_n t^{-1} \end{aligned} \tag{7}$$

So we can replace our relation with $\beta_{n+1} = t\beta_n t^{-1}\ \forall n \in \mathbb{N}$.

Furthermore, we can prove by induction that $\beta_{n+1} = t\beta_n t^{-1}\ \forall n \in \mathbb{N} \Rightarrow \beta_n = t^n t^{-n}$. We know this holds when $n = 1$ as $\beta_1 = tt^{-1}$ is given in the relations. Say $\beta_k = t^k t^{-k}$ for some $k \in \mathbb{N}$. Then we can say;

$$\beta_{k+1} = t\beta_k t^{-1} = tt^k t^{-k} t^{-1} = t^{k+1} t^{-(k+1)}. \tag{8}$$

Also note that $\beta_n = t^n t^{-n} \Rightarrow \beta_{n+1} = t\beta_n t^{-1}$, so again we can replace our relation with a new equivalent equation.

$\beta_n = t^n t^{-n}$ then allows us to remove the generators β_n from our presentation of S'. It is immediately obvious that $\beta_{n+1} = t\beta_n t^{-1}$, $\beta_n^2 = \beta_n$ and $\beta_1 = tt^{-1}$ are all given by $\beta_n = t^n t^{-n}$. So we now need only check if it also implies that $\beta_n \beta_m = \beta_{max\{n,m\}}\ \forall n, m \in \mathbb{N}$. Assume that $n < m$, then $\beta_n \beta_m = t^n t^{-n} t^m t^{-m} = (t^n t^{-n} t^n) t^{m-n} t^{-m} = t^n t^{m-n} t^{-m} = t^m t^{-m} = \beta_m$. Similarly, if $n > m$ then $\beta_n \beta_m = \beta_n$. Note that we don't need to check for when $n = m$ as this is covered by $\beta_n^2 = \beta_n$.

So, $S' = Inv\langle t \mid \rangle$ which is equivalent to S.

We could equally have gotten a working example if we had defined ϕ by $\phi(s, v) = \rho_s^{-1}(v)$ and our graph given by;

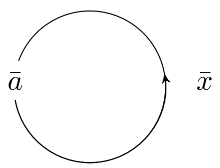
$$\cdots \xrightarrow{\alpha^{-3}x} a_{-2} \xrightarrow{\alpha^{-2}x} a_{-1} \xrightarrow{\alpha^{-1}x} a_0 \xrightarrow{x} a_1$$

Though this example does work, we cannot say that we can always follow a similar method whenever we have such a semigroup acting on a graph.

Example 2.1.9. Consider ϕ to be the same partial mapping as in Example 2.1.8 but this time we have an S-act on the graph;

$$a_1 \xrightarrow{x} a_0 \xrightarrow{\alpha x} a_{-1} \xrightarrow{\alpha^2 x} a_{-2} \xrightarrow{\alpha^3 x} \cdots$$

Again as in Example 2.1.8 we see that the vertices all share the same orbit which we shall denote $\bar{a}$. Similarly, the edges of our graph also share the same orbit which we denote by $\bar{x}$. Therefore, we have the same quotient graph as in the last example.



However, this example differentiates from Example 2.1.8 when we find the S-transversal of this system is. First say we take a_1 to represent $\bar{a}$. Since a_1 is the initial vertex of only one edge in our graph, we must take the edge representing $\bar{x}$ to be said edge, which is x. This then gives us the following S-transversal;

$$a_1 \xrightarrow{x}$$

So, we wish to find a monomorphism from S_x to S_{a_1} given by some element $t \in S$ such that $ta_1 = a_0$ and tt^{-1} is the identity element in S_x. As $S_x = \{\alpha^{-n}\alpha^n | n \in \mathbb{N}\}$, we can see that it is a monoid with identity $\alpha^{-1}\alpha$. We also note that a_1 is mapped to a_0 by α, elements of the form $\alpha^{-n}\alpha^{n+1}$ and those of the form $\alpha\alpha^{-n}\alpha^n$ (where $n \in \mathbb{N}$) and hence are all the possible values of t. However, none of these elements will satisfy $tt^{-1} = \alpha^{-1}\alpha$. Therefore we can go no further with this S-transversal and must choose a different representative of $\bar{a}$ when constructing an S-transversal.

Note though, that S_x is the only stabilizer of an edge in this system that contains an identity element. However, we have already examined the only S-transversal that contains x. So, there is no S-tranversal in this system that will allow us to create a graph of inverse semigroups from our method.

3 Free Products of Inverse Semigroups

The topic of the free product of inverse semigroups was first investigated by Preston [20] when he examined the free products of semigroups in general. Though he did not go in depth with his analysis, Preston did suggest the idea of a theorem that tells us the strucre of this product similar to what Scheiblich [23] had recently done with free inverse semigroups. Later work was done on the inverse free product of groups [12] [16] and E-unitary semigroups [10], but it was work by Jones [11] that gave us a general structure of the free product of inverse semigroups that we will be using in this section.

3.1 Previous Results of the Free Product

We will begin by covering work presented in *Free Products of Inverse Semigroups* [11] that gives us some of the terminology and foundational knowledge that I will use in this section. Note that in this paper, Jones presents results for both the free product of inverse semigroups in the categories of semigroups and of inverse semigroups. It should be mentioned that we will be taking the free product to be in the category of inverse semigroups. We do this in the following way.

Definition 3.1.1. Let S and T be inverse semigroups. The inverse free product is defined to be the set of words $a = a_1 a_2 \cdots a_m$ over $S \cup T$ such that no two adjacent letters belong to the same factor of S or T along with the relation that $a^{-1} = a_m^{-1} a_{m-1}^{-1} \cdots a_1^{-1}$ and that letters in $E(S) \cup E(T)$ may commute with each other in a word. We then label such a product by $S * T$.

It can then be proven that this product is an inverse semigroup [7]. We then need some terminology used by Jones. In particular, we will be using the notation that any word $a \in S * T$ can be given by $a_1 a_2 \cdots a_m$ (as in the previous definition) and the integer m is defined to be the length of a.

Definition 3.1.2. Let $a \in S * T$.

- a is called *reduced* if $a_i \notin E(S) \cup E(T)$ for any letter a_i in a.
- If a_m is idempotent then a is called *right idempotent.*
- a is defined to be *left reduced* if a_m is the only idempotent letter in a.
- The set of prefixes of a is the set, $pre(a) = \{a_1 a_1^{-1}, a_1 a_2 a_2^{-1}, \ldots, a_1 \cdots a_m a_m^{-1}\}$.
- If a is right idempotent, then we define $\hat{a}_r$ for any positive integer r in the following way;

$$\hat{a}_r = \begin{cases} a_1 \cdots (a_{r-1} a_{r+1}) \cdots a_m & r < m-1 \\ a_1 \cdots a_{m-3}(a_{m-2} a_m a_{m-2}^{-1}) & r = m-1 \\ a & r \geqslant m \end{cases} \tag{9}$$

  We define $\hat{a}_r$ differently when $r = m-1$ so that $\hat{a}_r$ is always right-idempotent.

If $A \subseteq S * T$ is such that $\forall a \in A$, $pre(a) \subseteq A$ then A is called prefix closed. Furthermore, A is called precanonical if it is a finite, nonempty, prefix closed set of right idempotent words. Note that $pre(a)$ is a precanonical set.

Definition 3.1.3. Let $A \subseteq S * T$. A is said to have unique last letters if $a_1 a_2 \cdots a_{m-1} a_m, a_1 a_2 \cdots a_{m-1} b_m \in A \Rightarrow a_m = b_m$.

Definition 3.1.4. Let A be a precanonical subset of $S * T$. $\forall a \in A$, $\epsilon(a) = aa^{-1}$. Furthermore, $\epsilon(A) = \prod_{a \in A} \epsilon(a)$.

Using this notation, Jones showed that $\epsilon(pre(a)) = \epsilon(a) = aa^{-1}$ and hence we can write any word $a \in S * T$ by $\epsilon(pre(a))a$. Using certain opertaions, we are then able to change this into a form that is unique.

Definition 3.1.5. Let A be a precanonical subset of $S * T$. We can then define the following functions:

- $L(A) = A$ when A is left reduced. Otherwise, let i be the least positive integer such that $\exists a \in A$ whose ith letter is an internal idempotent (that being an idempotent that is not the last letter in a). Then, $L(A) = \langle \hat{a}_r | a \in A \rangle$.
- $R(A) = A$ if A has unique last letters. Otherwise, $\exists k \in \mathbb{N}^+$ such that for some $x, y \in A$, $x_j = y_j, \forall j < k$ and $x_k \neq y_k$. $R(A)$ is then obtained by replacing each word $a \in A$ of length $m \geqslant k$ by $a_1 \cdots (e_k(a) a_k) \cdots a_m$ where $e_k(a) = \prod \{f | a_1 \cdots a_{k-1} f \in A\}$. Notice that since A contains right idempotent words, $e_k(a)$ is a product of idempotents. Therefore, if a is a word of length k, $e_k(a) a_k = e_k(a)$.

Note that the L and R operations give a left reduced set and a set with unique last letters respectively. Jones shows us that if we start with a precanonical set A, the sequence $L(A)$, $R(L(A))$, $L(R(L(A)))$, $\ldots$ eventually terminates [11, Corollary 3.2.] with a set we label $cl(A)$. Furthermore, $cl(A)$ is called a canonical set which is defined in the following way;

Definition 3.1.6. Let A be a precanonical set. If A is both left reduced and has unique last letters then A is called canonical.

As previously stated, any element $a \in S * T$ can be written as $\epsilon(pre(a))a$. Further work by Jones also shows that we can also use this to further say that $a = \epsilon(cl(pre(a)))x$ for some element $x \in S * T$ is an associate of $cl(pre(A))$ [11, Theorem 3.3.]. An associative element of a set is defined in the succeeding definition.

Definition 3.1.7. Let A be a precanonical set and $a \in S * T$. If $a = 1$ or $a_m \notin E(S) \cup E(T)$ and $aa_m^{-1} \in A$ then we define a to be an associate of A.

So all that remains is to explain how x is found. To do so, we need to define how the L and R functions act on words in $S * T$.

Definition 3.1.8. Let a be an associate of a precanonical set A and $E = E(S) \cup E(T)$.

- If $a = 1$ or $L(A) = A$, then $L_A(a) = a$. Otherwise, let i be as defined for L(A). If a_i is nonidempotent or $m \leqslant i$, then $L_A(a) = a$. If a_i does not satisfy either condition then:

$$L_A(a) = \begin{cases} \hat{a}_i & 1 \leqslant i < m-1 \\ \hat{a}_i & i = m-1 \text{ and } a_{m-2}a_m \notin E \\ a_1 a_2 \cdots a_{m-3} & i = m-1,\ m \neq 3 \text{ and } a_{m-2}a_m \in E \\ 1 & i = m-1,\ m = 3 \text{ and } a_{m-2}a_m \in E \end{cases} \tag{10}$$

- If $a = 1$ or $R(A) = A$, then $R_A(a) = a$. Otherwise, let k and e_k be as defined for $R(A)$.

$$R_A(a) = \begin{cases} a & m < k \\ a_1 \cdots (e_k a_k) \cdots a_m & m \geqslant k \end{cases} \tag{11}$$

Finally, given that $cl(A)$ is defined to be the terminating set of the sequence $L(A), R(L(A)), \ldots$, we define $cl_A(a)$ to be the corresponding word in the sequence $L_A(a), R_{L(A)}(L_A(a)), \ldots$.

Jones uses this to say that any word in $S * T$ can be given by a canonical set and an associate of that set [11, Theorem 3.3.]. This is shown by the following equation;

$$a = aa^{-1}a = \epsilon(pre(a))a = \epsilon(cl(pre(a)))cl_{pre(a)}(a) \tag{12}$$

In fact, this form is unique to the word [11, Theorem 5.4.] and we define it to be the canonical form of a. Note that a is idempotent if and only if $cl_{pre(a)}(a) = 1$.

3.2 Stabilizer of S-action

Using this information, we wish to investigate if there is a relation between the graphs that inverse semigroups act on and those that their free inverse product will act on (under inverse semigroup actions). Depending on which action I started with, I defined the corresponding actions in the following way;

Definition 3.2.1. Let S and T be inverse semigroups.

- If we have a known action of $S * T$ on a set X, we also have actions of S and T on X since $S, T \subseteq S * T$.
- If S and T both have known actions on a set X, we can define an inverse semigroup action of $S * T$ on X by having words in $S * T$ act on elements from our set with respect to how the letters that compose them act on elements. For example, let $a \in S * T$, $x \in X$, $a_m \in T$ and $\cdot_S$ and $\cdot_T$ be the S-actions of S and T on X respectively. Then we define a acting on x by;

$$\begin{aligned} a \cdot_{S*T} x &= (a_1 \cdots a_{m-1}) \cdot_{S*T} (a_m \cdot_T x) \\ &= (a_1 \cdots a_{m-2}) \cdot_{S*T} (a_{m-1} \cdot_S (a_m \cdot_T x)) \\ &= \dots \end{aligned} \tag{13}$$

  Note that since idempotents preserve elements they act on, this still holds as an $(S*T)$-act given that idempotents commute under the free inverse product.

It is easy to show that the second case would give us an inverse semigroup action. First let $a, b \in S*T$. So, $a = a_1 a_2 \cdots a_p$ and $b = b_1 b_2 \cdots b_q$ (for some $p, q \in \mathbb{N}$) where $a_1, \dots a_p, b_1, \dots b_q \in S \cup T$ such that no two adjacent letters belong to the same factor of S or T. Assume that $(ab) \cdot_{S*T} x$ is defined for some $x \in X$. By definition of $\cdot_{S*T}$, it is the case that:

$$\begin{aligned} (ab) \cdot_{S*T} x &= (a_1 \cdots a_p b_1 \cdots b_q) \cdot_{S*T} x \\ &= a_1 \cdot_{S*T} (a_2 \cdot_{S*T} \dots (a_p \cdot_{S*T} (b_1 \cdot_{S*T} \dots (b_q \cdot_{S*T} x) \dots). \end{aligned} \tag{14}$$

Since $b_1, \dots, b_q \in S \cup T$, we know that when any of these values act on an element in X under the function defined by $\cdot_{S*T}$ then $\cdot_{S*T}$ is equivalent to an inverse semigroup action (either $\cdot_S$ or $\cdot_T$). Knowing this, we can say that $b_1 \cdot_{S*T} (b_2 \cdot_{S*T} \dots (b_q \cdot_{S*T} x) \dots) = b \cdot_{S*T} x$. Therefore, $(ab) \cdot_{S*T} x = a_1 \cdot_{S*T} (a_2 \cdot_{S*T} \dots (a_p \cdot_S (b \cdot_{S*T} x) \dots)$.

Let $y := b \cdot_{S*T} x$. So, $(ab) \cdot_{S*T} x = a_1 \cdot_{S*T} (a_2 \cdot_{S*T} \dots (a_p \cdot_S y) \dots)$. Using the same method that let us say $b_1 \cdot_{S*T} (b_2 \cdot_{S*T} \dots (b_q \cdot_{S*T} x) \dots) = b \cdot_{S*T} x$ we can say that $a_1 \cdot_{S*T} (a_2 \cdot_{S*T} \dots (a_p \cdot_{S*T} y) \dots) = a \cdot_{S*T} x$.

$$\begin{aligned}(ab) \cdot_{S*T} x &= a_1 \cdot_{S*T} (a_2 \cdot_{S*T} \dots (a_p \cdot_S (b \cdot_{S*T} x) \dots) \\ &= a \cdot_{S*T} (b \cdot_{S*T} x).\end{aligned} \tag{15}$$

Now assume $a \cdot_{S*T} x_1 = a \cdot_{S*T} x_2$ for some $x_1, x_2 \in X$.

When working on this topic it became helpful to have a method of defining the stabilizer of a point with respect to $S * T$ using the stabilizers with respect to S and T. One idea as to how the stabilizer might be defined was $(S_x^S * S_x^T)\omega = S_x^{S*T}$ (where x is any element in a set that $S * T$ acts on and for any inverse semigroup A, $A\omega$ is the closure of A) since it is easy to see that one is a subset of the other;

Lemma 3.2.1. *Let S and T be inverse semigroups and X a set they both act on. For all $x \in X$;*

$$(S_x^S * S_x^T)\omega \subseteq S_x^{S*T} \tag{16}$$

Proof. Say $w \in (S_x^S * S_x^T)\omega$. By definition, $\exists a \in S_x^S * S_x^T$ and $e \in E(S_x^S * S_x^T)$ such that $a = we$. Furthermore, we can say that $a \cdot_{S*T} x = e \cdot_{S*T} x = x$. Then;

$$w \cdot_{S*T} x = w \cdot_{S*T} (e \cdot_{S*T} x) = (we) \cdot_{S*T} x = a \cdot_{S*T} x = x. \tag{17}$$

Therefore, $w \in S_x^{S*T}$ □

The obvious next step was to take an $s \in S_x^{S*T}$ and see if it exists in $(S_x^S * S_x^T)\omega$. It is not known if the converse is true, however, given that s has the canonical form $\epsilon(cl(pre(s)))cl_{pre(s)}(s)$ we can determine some more properties of the stabilizer.

Let $c = cl_{pre(s)}(s)$. Clearly $c \in S_x^{S*T}$ (since $s \in S_x^{S*T} \Rightarrow \epsilon(cl(pre(s))) \in S_x^{S*T}$) and $s \le c$. It is also a simple matter to find the canonical form of c.

Lemma 3.2.2. *$c = \epsilon(pre(c))c$ is the unique canonical form of c.*

Proof. Given Jones' method of finding the canonical form of a word, we begin with $c = \epsilon(pre(c))c$. The next step will be to find $cl(pre(c))$. Given that c is defined to be an associate of a canonical set, we can say that it is left reduced. Therefore, $pre(c)$ is left reduced and $L(pre(c)) = pre(c)$. Furthermore, by the definition of $pre(c)$, $R(pre(c)) = pre(c)$. So, we can say $cl(pre(c)) = pre(c)$ and hence $cl_{pre(c)}(c) = c$. □

Using this we can then show the following;

Corollary 3.2.3. *c is a maximal element in S_x^{S*T}*

Proof. Say $\exists d \in S_x^{S*T}$ such that $c \leq d$. By definition, $\exists e, f \in E(S_x^{S*T})$ such that $c = ed = df$.

$$\begin{aligned} c &= ed = edd^{-1}d = e^2dd^{-1}d = edd^{-1}ed \\ &= (ed)(d^{-1}e)d = (df)(df)^{-1}d = \epsilon(pre(df))d. \end{aligned} \tag{18}$$

Note that d is an associate of $pre(df)$. Therefore, $\epsilon(cl(pre(df)))cl_{pre(df)}(d)$ is a canonical form of c. Given the uniqueness of this form and Lemma 3.2.2, we can say that $pre(c) = cl(pre(df))$ and $c = cl_{pre(df)}(d)$.

However, $c = df \Rightarrow c = cl_{pre(df)}(d) = cl_{pre(c)}(d)$. Also from the proof of Lemma 3.2.2 we know that $L(pre(c)) = pre(c)$ and $R(pre(c)) = pre(c)$ and so we can say that $cl_{pre(c)}(d) = d$. Therefore $c = d$. □

3.3 Action of a Free Product of Bicyclic Semigroups

Say we have two copies of the Bicyclic semigroup;

$$\begin{aligned} A &= Inv\langle a | aa^{-1} = 1_A \rangle \\ B &= Inv\langle b | bb^{-1} = 1_B \rangle \end{aligned} \tag{19}$$

where 1_A and 1_B are the identities in A and B respectively.

The set of elements $X = \{x, \alpha_0, \alpha_{w_a}, \beta_0, \beta_{w_b} | w_a \in a^+, w_b \in b^+\}$ can be considered to be an A-act under the semigroup action;

$$a^{-1}y = \begin{cases} \alpha_a & \text{for } y = x \\ x & \text{for } y = \alpha_0 \\ \alpha_{aw} & \text{for } y = \alpha_w \end{cases} \tag{20}$$

Similarly X is also a B-act with the action;

$$b^{-1}y = \begin{cases} \beta_b & \text{for } y = x \\ x & \text{for } y = \beta_0 \\ \beta_{bw} & \text{for } y = \beta_w \end{cases} \tag{21}$$

We include x in X so that there exists an element that both A and B can act on. Note that though a^{-1} does not act on β_0 and b^{-1} does not act on α_0, this action still satisfies to the properties of the Bicyclic semigroup in A and B. Even though the domain of any element in a monoid must be a subset of the domain of the identity, aa^{-1} is defined to be an identity element only on A. Therefore, our action need only satisfy $D_{w_a}^X \subseteq D_{a^{-1}}^X$ $(\forall w_a \in A)$ for our action to hold and this is clearly true given that $D_{a^{-1}}^X = \{x, \alpha_0, \alpha_{w_a} | w_a \in a^+\}$ and elements in A cannot act on β_0 or β_{w_b} $(\forall w_b \in b^+)$. The justification for why b^{-1} not acting on α_0 does not contradict $bb^{-1} = 1_B$ is given dually. Hence, we can say that we have n inverse semigroup action of $A * B$ on the set X.

Lemma 3.3.1. *Let $w \in a^+ \cup b^+$. Then;*

$$w^{-1}x = \begin{cases} \alpha_w & \textit{for } w \in a^+ \\ \beta_w & \textit{for } w \in b^+ \end{cases} \tag{22}$$

Proof. Say $w \in a^+$. This would mean that $w = a^n$ for some $n \in \mathbb{N}$.

$$\begin{aligned} w^{-1}x &= a^{-n}x \\ &= a^{-(n-1)}(a^{-1}x) = a^{-(n-1)}\alpha_a \\ &\dots \\ &= a^{-1}\alpha_{a^{n-1}} = \alpha_{a^n} = \alpha_w. \end{aligned} \tag{23}$$

The same method also shows us that $w \in b^+ \Rightarrow w^{-1}x = \beta_w$. □

Corollary 3.3.2. *Under our action, there is only one vertex orbit.*

Proof. By Lemma 3.3.1, all elements in $\{\alpha_w, \beta_w | w \in a^+ \cup b^+\}$ can be mapped to x. Furthermore, the definition of our action tells us that $a^{-1}\alpha_0 = x$ and $b^{-1}\beta_0 = x$. So, $\forall y \in X$, $\exists s \in (A * B)^1$ such that $sy = x$. □

At this point it would be helpful to demostrate how we can get back the Bicyclic semigroup as the fundamental semigroup using this action on a graph.

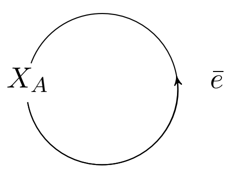
Example 3.3.3. Let $X_A = \{x, \alpha_0, \alpha_{a^n} | n \in \mathbb{N}\}$ and define a directed edge e given by $\iota(e) = x$ and $\tau(e) = \alpha_0$. We take this edge to be the single base edge of a graph where the other edges are given by the semigroup action of A on e. Call this graph G_A. As shown below, this graph is a chain.

$$\alpha_0 \xleftarrow[e]{\qquad} x \xleftarrow[a^{-1}e]{\qquad} \alpha_a \xleftarrow[a^{-2}e]{\qquad} \alpha_{a^2} \xleftarrow[a^{-3}e]{\qquad} \cdots$$

Lemma 3.3.4. $V(G_A) = X_A$.

Proof. We can immediately say that $x, \alpha_0 \in V(G_A)$ since they are the vertices that define e. From Lemma 3.3.1 we can also say that $(\forall n \in \mathbb{N})$ $\iota(\alpha^{-n}e) = \alpha_{a^n}$ and so $\alpha_{a^n} \in V(G_A)$. □

Given that our graph can be defined from a single base edge, there is only one edge orbit, say $\bar{e}$. We also know from Corollary 3.3.2 that there is a single vertex orbit. Hence, the quotient graph of this system will be given by:



One $(A * B)$-transversal we can get from this is;

$$x \xrightarrow{\quad e \quad} (\alpha_0)$$

Lemma 3.3.5. *Under our action of A on X_A;*

$$\begin{aligned} D_A^x =& \{1_A, a^{-n}, a^{-n}a | n \in \mathbb{N}^0\}, \\ D_A^{\alpha_0} =& \{1_A, a^{-n} | n \in \mathbb{N}\}. \end{aligned} \tag{24}$$

Proof. Let $w \in D_A^x$. We can write w in the form of a normal element in the bicyclic semigroup, in other words, $w = a^{-p}a^q$ for some $p, q \in \mathbb{N}^0$ where $p = 0$ and $q = 0 \Rightarrow w = 1_A$. Then, $w \in D_A^x \Rightarrow a^{-p}a^q x$ exists $\Rightarrow a^q x$ exists. By definition of our action, $a^q x$ exists $\Rightarrow q = 0$ or 1. If $q = 0$, then $w = a^{-p}$ which acts on x $\forall p \in \mathbb{N}^0$. If $q = 1$, then $w = a^{-p}a$. According to our action, $a^{-p}ax = a^{-p}\alpha_0$ and hence;

$$wx = \begin{cases} \alpha_0 & \text{when } p = 0 \\ x & \text{when } p = 1 \\ \alpha_{a^{p-1}} & \text{when } p > 1. \end{cases} \tag{25}$$

Therefore, $D_A^x = \{1_A, a^{-n}, a^{-n}a | n \in \mathbb{N}^0\}$.

Similarly, if $w \in D_A^{\alpha_0}$ we again say $w = a^{-p}a^q$ for some $p, q \in \mathbb{N}^0$ where $p = 0$ and $q = 0 \Rightarrow w = 1_A$. Then $w\alpha_0$ exists $\Rightarrow a^{-p}a^q\alpha_0$ exists $\Rightarrow a^q\alpha_0$ exists. However, $a^q\alpha_0$ is undefined when $q \neq 0$ so it must be the case that $q = 0$ and $w = a^{-p}$. By our action, $a^{-p}\alpha_0$ exists $\forall p \in \mathbb{N}^0$ and so $D_A^{\alpha_0} = \{1_A, a^{-n} | n \in \mathbb{N}\}$.

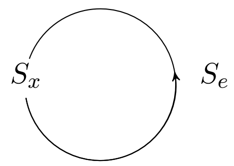
□

Corollary 3.3.6.

$$\begin{aligned} S_x &= \{1_A, a^{-1}a\} \subseteq A, \\ S_{\alpha_0} &= \{1_A | 1_A^2 = 1_A\}. \end{aligned} \tag{26}$$

Proof. Lemma 3.3.5 tells us that $S_x \subseteq \{1_A, a^{-(n+1)}, a^{-n}a | n \in \mathbb{N}^0\}$. So we need only check which of the elements in this set will fix x. By definition, $1_A x = x$ since 1_A is idempotent. Also, given Lemma 3.3.1 it is not possible for $a^{-n}x = x$ for any $n \in \mathbb{N}$. Therefore, it only remains to check $a^{-n}a$ fixes x for any $n \in \mathbb{N}$. Say $a^{-n}ax = x$. This implies that $ax = a^n x$, but $a^n x$ is only defined when $n = 1$. So, $a^{-1}a$ is the only possible stabilizer of x of the form $a^{-n}a$. Furthermore, since $a^{-1}a$ is idempotent, we can say that $a^{-1}ax = x$. Hence, $S_x = \{1_A, a^{-1}a | 1_A^2 = 1_A,\ 1_A(a^{-1}a) = (a^{-1}a)1_A = a^{-1}a\}$. Similarly, $S_{\alpha_0} = \{1_A | 1_A^2 = 1_A\}$. □

Now that we know S_x and S_{α_0} we can find a $t \in A$ such that $tx = \alpha_0$ and $tt^{-1} = Id(S_{\alpha_0})$. Such a t is given by $t = a$. Therefore this $(A*B)$-transversal will give us the following graph of inverse semigroups.



Lemma 3.3.7. *The fundamental inverse semigroup of this system is equal to A.*

Proof. Let S be the fundamental inverse semigroup of this system. By definition the generating set of S will be given by $\{s_1, s_2, t\}$ where $s_1 = 1_A$, $s_2 = a^{-1}a$ and $t = a$. The relations of S are then given by $s_1^2 = s_1$, $s_2^2 = s_2$, $s_1 s_2 = s_2 s_1 = s_2$, $tt^{-1} = s_1$ and $t^{-1}s_1 t = s_2$.

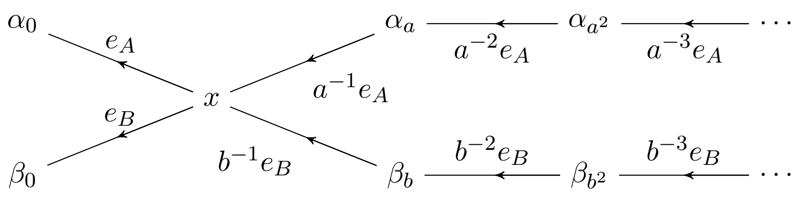
Note that $tt^{-1} = s_1$ and $t^{-1}s_1 t = s_2$ imply that $s_2 = t^{-1}tt^{-1}t = t^{-1}t$. Since s_2 can be expressed this way, we can remove it from our generators and replace it in our relations with $t^{-1}t$. Similarly, since $tt^{-1} = s_1$ we can also remove s_1 from the generators and replace it in the relations. This will then give us $S = Inv\langle t | tt^{-2}t = t^{-1}t^2 t^{-1} = t^{-1}t \rangle$. Note that we have already removed the relations $(tt^{-1})^2 = tt^{-1}$ and $(t^{-1}t)^2 = t^{-1}t$ since these are given by definition of an inverse semigroup.

Given the relation we have in S we can say that;

$$t(tt^{-1}) = (tt^{-1}t)(tt^{-1}) = t(tt^{-1})(t^{-1}t) = t(tt^{-2}t) = tt^{-1}t = t. \tag{27}$$

Since we also know that $(tt^{-1})t = t$, we can say that tt^{-1} acts as an identity on t in S. This would further imply that it also fixes t^{-1} and since t is the only generator of S we can say that tt^{-1} is the identity in S. Knowing this we can rewrite our original relation as $(tt^{-1})(t^{-1}t) = (t^{-1}t)(tt^{-1}) = t^{-1}t$ and see it is given by $tt^{-1} = 1_S$. Therefore, we can say that $S = Inv\langle t | tt^{-1} = 1_S \rangle$ which is the bicyclic semigroup. □

Example 3.3.8. Now consider an action of $A * B$ on X as defined by Definition 3.2.1. Create a graph with edges e_A and e_B as our base edges where $x = \iota(e_A) = \iota(e_B)$, $\alpha_0 = \tau(e_A)$ and $\beta_0 = \tau(e_B)$. This then gives us the following graph that we shall label G.



Lemma 3.3.9. $V(G) = X$

Proof. Given our definition, we know that $x, \alpha_0, \beta_0 \in V(G)$. Also, using Lemma 3.3.1 we can say that $\forall w \in \langle a \rangle$, $\tau(w^{-1}e_A) = \alpha_w$ and $\forall w \in \langle b \rangle$, $\tau(w^{-1}e_B) = \beta_w$. Therefore, $\alpha_{w_A}, \beta_{w_B} \in V(G)$ $(\forall w_A \in \langle a \rangle, w_B \in \langle b \rangle)$. □

Lemma 3.3.10. *$\forall w \in A * B$;*

$$\begin{aligned} we_A \text{ exists } &\Rightarrow w \in A \\ we_B \text{ exists } &\Rightarrow w \in B \end{aligned} \tag{28}$$

Proof. Say we_A exists. This implies that $w\iota(e_A) = wx$ and $w\tau(e_A) = w\alpha_0$ are both defined. Let n be the word length of w in $A * B$ and assume $n > 1$. Then, $w = w_1 w_2 \cdots w_n$ for some $w_1, w_2, \ldots, w_n \in A \cup B$ where $(\forall i \in [1, n])$ $w_i \in A$ (resp. B) implies that $w_{i-1}, w_{i+1} \in B$ (resp. A) if they exist. From our action, we know that $w\alpha_0$ can only exist if $w_n \in A$. Therefore, we_A exists $\Rightarrow w_n \in A$. We also know that if w acts on e_A, then w_n acts on x. Since $w_n \in A$ either $w_n x = x$ or α_z for some $z \in \langle a | \rangle$. If $w_n x = x$, then Corollary 3.3.6 tells us that $w_n = 1_A$, since w_n must also act on α_0. In which case $\iota(w_n e_A) = x$ and $\tau(w_n e_A) = \alpha_0$. However, this would mean that w_{n-1} can not exist, since it must exist in B and act on α_0 which is not possible. Alternitively, if $w_n x = \alpha_z$ for some $z \in \langle a | \rangle$ then agaain w_{n-1} cannot be defined since it must act on α_z and exist in B. So in both cases w can only be a word of length 1 and hence it exists in A. The existence of we_B implying $w \in B$ is defined dually. ☐

Corollary 3.3.11.

$$D^{e_A}_{A*B} = \{1_A, a^{-n} | n \in \mathbb{N}\} \tag{29}$$

$$D^{e_B}_{A*B} = \{1_B, b^{-n} | n \in \mathbb{N}\} \tag{30}$$

Proof. By definition, $D^{e_A}_{A*B} = D^{\alpha_0}_{A*B} \cap D^{x}_{A*B}$. However, Lemma 3.3.10 tells us that $D^{e_A}_{A*B} \subseteq A$. Therefore, $D^{e_A}_{A*B} = D^{\alpha_0}_{A} \cap D^{x}_{A}$. Using Lemma 3.3.5 we get that $D^{e_A}_{A*B} = \{1_A, a^{-n} | n \in \mathbb{N}\}$. Similarly, we can use a dual proof of Lemma 3.3.5 in B to define D^{x}_{B} and $D^{\beta_0}_{B}$ and Lemma 3.3.10 to say that $D^{e_B}_{A*B} = \{1_B, b^{-n} | n \in \mathbb{N}\}$. ☐

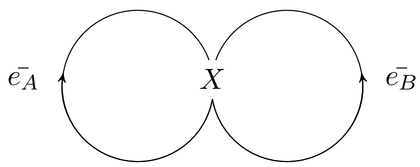
From the construction of G, it is safe to assume it is a tree. However, it will be worth while to prove this properly to be certain of this fact.

Corollary 3.3.12. *G is a tree*

Proof. We know that edges e_A and e_B are connected since they both share an initial vertex. From Lemma 3.3.10 we have that every other edge in G can be written as we_A for some $w \in A$ or we_B for some $w \in B$. Say we have an edge given by we_A. Corollary 3.3.11 tells us that $w \in \{1_A, a^{-n} | n \in \mathbb{N}\}$. If $w = 1_A$, then $we_A = e_A$ and so is connected to our base edges by definition. Instead assume $w = a^{-n}$. If $n = 1$, then $\tau(we_A) = \iota(e_A) = x$ and so we_A is connected to a base edge. Now assume $n \geq 2$. We can then say that we_A is connected to the edge $a^{-(n-1)}e_A$, since $\tau(we_A) = \iota(a^{-(n-1)}e_A) = \alpha_{a^{n-1}}$. Then either $a^{-(n-1)}e_A$ is connected to e_A or $a^{-(n-2)}e_A$ depending on the value of n. If we continue this method, we find that we can show that there is a chain of edges creating a path of edges that connect we_A to e_A for any $n \in \mathbb{N}$. Similarly, we can show that any edge of the form we_B is conencted to e_B and hence our graph must be connected.

Now assume $\exists n \in \mathbb{N}$ such that $n > 1$ and α_{a^n} exists on an edge where the other endpoint is not $\alpha_{a^{n-1}}$ or $\alpha_{a^{n+1}}$. Then $\exists s \in A * B$ such that $\iota(se_A)$, $\iota(se_B)$, $\tau(se_A)$ or $\tau(se_B)$ equals α_{a^n} and the correponding endpoint does not equal $\alpha_{a^{n-1}}$ or $\alpha_{a^{n+1}}$. Using Corollary 3.3.11 we can immediately say that $\iota(se_B)$ and $\tau(se_B)$ can't possibly equal α_{a^n} due to the fact that there is no element in $D^{e_B}_{A*B}$ that satisfies this property. Similarly, given the value of $D^{e_A}_{A*B}$ we know that $\iota(se_A) = \alpha_{a^n} \Rightarrow s = a^{-n}$ and $\tau(se_A) = \alpha_{a^n} \Rightarrow s = a^{-(n+1)}$. However, in these circumstances the other endpoints are $\alpha_{a^{n-1}}$ and $\alpha_{a^{n+1}}$ respectively. Hence the only other vertices that α_{a^n} is directly connected to are $\alpha_{a^{n-1}}$ and $\alpha_{a^{n+1}}$. It can also be proven dually that the only vertices β_{b^n} is directly connected to are $\beta_{b^{n-1}}$ and $\beta_{b^{n+1}}$. Knowing this we can say that no loops can exist in G since it would require the existence of a unique set of edges that create a path from a vertex to itself. $\square$

From Corollary 3.3.2 and Lemma 3.3.10 we know there is one vertex orbit and two edge orbits in this system. Hence the quotient graph of this system is given by;



From this we then have the following as an $(A * B)$-transversal.

$$(\alpha_0) \xleftarrow{\quad e_A \quad} x \xrightarrow{\quad e_B \quad} (\beta_0)$$

The next step is then to find the stabilizers.

Lemma 3.3.13.

$$S_x^{A*B} = S_x^A * S_x^B = \langle 1_A, 1_B, a^{-1}a, b^{-1}b \rangle \tag{31}$$

Proof. It is clear to see from our action that $S_x^A * S_x^B \subseteq S_x^{A*B}$, so let $w \in A*B$ be such that $wx = x$. Take w to be a word of length n in $A * B$ we can say that, $w = w_1 w_2 \cdots w_n$. Assume $n = 1$. If $w_n \in A$, then we can say that $w_n \in D_A^x$. Then $w = w_n$ and so $w_n x = x \Rightarrow w_n \in S_x^A$. Similarly, $w_n \in B \Rightarrow w_n \in S_x^B$. So in this case, $w \in S_x^A * S_x^B$.

Now assume our lemma holds when $n = m$ for some $m \in \mathbb{N}$. Now let $n = m + 1$. If $w_{m+1} \in A$ then we require $w_{m+1}x$ to exist Furthermore, $w_m w_{m+1} x$ must also exist and so $w_{m+1}x$ is a value that an element in B can act on. This is only possible if $w_{m+1}x = x$ since this is the only value in Ax that an element in B can act on. Therefore, $x = wx = w_1 \cdots w_m(w_{m+1}x) = w_1 \cdots w_m x$. From our assumption, $w_1 \cdots w_m x = x \Rightarrow w_1 \cdots w_m \in S_x^A * S_x^B$. Since $w_{m+1} \in S_x^A$ we can say that $w \in S_x^A * S_x^B$. It can be shown dually that $w_{m+1} \Rightarrow w_{m+1} \in S_x^B \Rightarrow w \in S_x^A * S_x^B$. Hence, $S_x^{A*B} \subseteq S_x^A * S_x^B \Rightarrow S_x^{A*B} = S_x^A * S_x^B$.

Finally, from Corollary 3.3.6 we know that $S_x^A = \{1_A, a^{-1}a\}$. The proof of this corollary can be used dually with B instead of A to show that $S_x^B = \{1_B, b^{-1}b\}$. Therefore, $S_x^A * S_x^B = \langle 1_A, 1_B, a^{-1}a, b^{-1}b\rangle$. □

Lemma 3.3.14.

$$\begin{aligned} S_{\alpha_0}^{A*B} =& \{1_A, awa^{-1} | w \in S_x^{A*B}\}, \\ S_{\beta_0}^{A*B} =& \{1_B, bwb^{-1} | w \in S_x^{A*B}\}. \end{aligned} \tag{32}$$

Proof. Let $w \in S_{\alpha_0}$. From the definition of our action, we know that if $w\alpha_0$ exists then $w = w'a^{-1}$ for some $w' \in (A*B)^1$ since a^{-1} is the only generator of $A * B$ that will act on α_0. By the same logic, since $w\alpha_0 = \alpha_0 \Rightarrow \alpha_0 = w^{-1}\alpha_0$ we can say that $w^{-1} = w''a^{-1}$ for some $w'' \in (A * B)^1$. Therefore, we can say that $w = a\omega a^{-1}$ for some $\omega \in (A * B)^1$.

If $\omega = 1$, then $w = aa^{-1} = 1_A$. Alternatively, assume $\omega \in A * B$. Then, $w\alpha_0 = \alpha_0 \Rightarrow a\omega a^{-1}\alpha_0 = \alpha_0 \Rightarrow \omega(a^{-1}\alpha_0) = a^{-1}\alpha_0 \Rightarrow \omega x = x$. Therefore, $\omega \in S_x^{A*B}$ and so $w \in aS_x^{A*B}a^{-1}$. From this we can say that $S_{\alpha_0}^{A*B} \subseteq \{1_A, awa^{-1} | w \in S_x^{A*B}\}$.

Conversely, the definition of our action tells us that $1_A\alpha_0 = \alpha_0$. Furthermore, if $w \in S_x^{A*B}$, then $awa^{-1}\alpha_0 = awx = ax = \alpha_0$. From this we can say that, $\{1_A, awa^{-1} | w \in S_x^{A*B}\} \subseteq S_{\alpha_0}$ Hence it must be the case that $S_{\alpha_0}^{A*B} = \{1_A, awa^{-1} | w \in S_x^{A*B}\}$. It can also be proven dually that $S_{\beta_0}^{A*B} = \{1_B, bwb^{-1} | w \in S_x^{A*B}\}$. □

Corollary 3.3.15. *$S_{\alpha_0}^{A*B}$ and $S_{\beta_0}^{A*B}$ are monoids with identities 1_A and 1_B respectively.*

Proof. Let $s \in S_{\alpha_0}^{A*B}$. Lemma 3.3.14 tells us that $s = 1_A$ or is of the form awa^{-1} for some $w \in S_x^{A*B}$. If $s = 1_A$, then by definition 1_A acts as an identity on s. Alternatively, if $s = awa^{-1}$ for some $w \in S_x^{A*B}$, then $1_As = 1_A(awa^{-1}) = (1_Aa)wa^{-1} = awa^{-1} = s$ and $s1_A = (awa^{-1})1_A = aw(a^{-1}1_A) = awa^{-1} = s$. Hence, $1_A = Id(S_{\alpha_0}^{A*B})$. Similarly, we can show that $1_B = Id(S_{\beta_0}^{A*B})$. □

From Lemmas 3.3.13 and 3.3.14 we can begin to construct the graph of inverse semigroups from the aforementioned $(A * B)$-transversal. As previously established, we would need to find an embedding from $S_{\alpha_0}^{A*B}$ into $S_{e_A}^{A*B}$. To do this, we need to find a $t_A, t_B \in A * B$ such that $t_Ax = \alpha_0$, $t_At_A^{-1} = Id(S_{\alpha_0}^{A*B})$, $t_Bx = \beta_0$ and $t_Bt_B^{-1} = Id(S_{\beta_0}^{A*B})$. Using the identities found in Cororllary 3.3.15, we know that $t_A = a$ and $t_B = b$ satisfy these conditions. Note that t_A is equal to the t we defined in Example 3.3.3.

Theorem 3.3.16. *If S is the fundamental inverse semigroup of this system, then $S = A * B$.*

Proof. The generators of S are the generators of S_x^{A*B} as well as t_A and t_B. From Lemma 3.3.13, S_x^{A*B} is generated by 1_A, 1_B, $a^{-1}a$ and $b^{-1}b$. We label these values g_1, g_2, g_3 and g_4 respectively. So the generating set of S is $\{g_1, g_2, g_3, g_4, t_A, t_B\}$. Let R be the set of relations in S that come from the relations in S_x^{A*B}. Then;

$$R = \{g_1^2 = g_1, g_1g_3 = g_3g_1 = g_3 = g_3^2, g_2^2 = g_2, g_2g_4 = g_4g_2 = g_4 = g_4^2\}. \quad (33)$$

We also have relations of S given by the embeddings. These are $t_At_A^{-1} = g_1$, $t_A^{-1}g_1t_A = g_3$, $t_Bt_B^{-1} = g_2$ and $t_B^{-1}g_2t_B = g_4$.

Given that we have $t_At_A^{-1} = g_1$ and $t_Bt_B^{-1} = g_2$ we can remove g_1 and g_2 from the set of generators of S. Since $t_At_A^{-1}$ and $t_Bt_B^{-1}$ are idempotent, we can remove the relations $g_1^2 = g_1$ and $g_2^2 = g_2$ as well. Furthermore, if we substiture this new value for g_1 into the relation $t_A^{-1}g_1t_A = g_3$ we find that $g_3 = t_A^{-1}(t_At_A^{-1})t_A = t_A^{-1}t_A$. Simlarly it can be shown that $g_4 = t_B^{-1}t_B$. Therefore, as with g_1 and g_2, we can remove g_3 and g_4 from our generators. Also since we are values for g_3 and g_4 are idempotent by definition we can remove he relations $g_3 = g_3^2$ and $g_4 = g_4^2$. This then leaves us with S in the following form;

$$\begin{aligned} S = Inv\langle t_A, t_B | (t_At_A^{-1})(t_A^{-1}t_A) = (t_A^{-1}t_A)(t_At_A^{-1}) = t_A^{-1}t_A, \\ (t_Bt_B^{-1})(t_B^{-1}t_B) = (t_B^{-1}t_B)(t_Bt_B^{-1}) = t_B^{-1}t_B\rangle. \end{aligned} \quad (34)$$

Using what we did in the proof of Lemma 3.3.7 we can simplify this to get $S = \langle t_A, t_B | t_At_A^{-1} = 1_{T_A}, t_Bt_B^{-1} = 1_{T_B}\rangle$. It is clear from this that S is isomomorphic to $A * B$. □

3.4 Action of the Free Product

Say we have two inverse semigroups A and B that act on the graphs G_A and G_B respectively. Furthermore, say that the actions of A on G_A and B on G_B in such a way that we can get fundamental inverse semigroups $S_A \cong A$ and $S_B \cong B$ respectively. Knowing this it is possible to construct a graph G that $A * B$ can act on such that the fundamental inverse semigroup is isomorphic to $A * B$.

In order to define S_A it must be possible to define an A-transversal from our action of A acting on G_A that we in turn can obtain the fundamental inverse semigroup, S_A from. Similarly, we know that S_B was defined from a B-transversal. Let v_A and v_B be any fixed vertices from the A-transversal and B-transversal respectively. We then have an action of $A * B$ on the set $V = (V(G_A) \cup V(G_B))/(v_A = v_B)$ given by Definition 3.2.1. Let $v = v_A = v_B$. G is then defined to be the graph where $V(G) = V$ and $E(V) = E(G_A) \cup E(G_B)$. Note that by the definiton of our action, v is the only value in V that can be acted on by elements in both A and B.

Lemma 3.4.1. *For all $x \in V(G)$ if $x \notin (A * B)^1 v$:*

$$D^x_{A*B} = \begin{cases} D^x_A & \text{when } x \in V(G_A) \\ D^x_B & \text{when } x \in V(G_B) \end{cases} \tag{35}$$

Proof. Let $w \in D^x_{A*B}$ and assume $x \in V(G_A)$. We can take w to be a word of length n with respect to $A * B$ and so can write it as $w = w_1 w_2 \cdots w_n$ where each letter alternates between being an element of A or an element of B. So, wx exists $\Rightarrow w_n x$ exists. Since $w_n \in A \cup B$ and $x \neq v$ it must be the case that $w_n \in A$. If $n = 1$, $w \in A$. Otherwise, $w_{n-1}(w_n x)$ exists. Since $x \notin (A * B)^1 v$ implies that $w_n x \neq v$ we require $w_{n-1} \in A$. However, this contradicts the definition of w_{n-1}. We therefore conclude that w can only be a word of length 1 and it exists in A. Hence, $D^x_{A*B} = D^x_A$. Dually it can be shown that $D^x_{A*B} = D^x_B$ if $x \in V(G_B)$. □

Lemma 3.4.2. *For all $x \in (A * B)^1 v$ such that $x \neq v$ define $X_A = \{w \in A | wx = v\}$ and $X_B = \{w \in B | wx = v\}$. If $x \in V(G_A)$ then;*

$$D^x_{A*B} = D^x_A \cup \{\omega_1\omega_2\omega_3, \omega_1\omega_3 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v, \omega_3 \in X_A\}. \tag{36}$$

Similarly, if $x \in V(G_A)$;

$$D^x_{A*B} = D^x_B \cup \{\omega_1\omega_2\omega_3, \omega_1\omega_3 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v, \omega_3 \in X_B\}. \tag{37}$$

Proof. Without loss of generality, assume $x \in V(G_A)$. Let $w \in D^x_{A*B}$ be a word of length n. Assume $n = 1$. Given that $x \neq v$, the only words of length 1 that can act on x exist in A. Thereofore, $w \in D^x_A$.

Now say $n = 2$. So $w = w_2 w_1$ for some $w_1, w_2 \in A \cup B$. Given that $w_1 x$ must exist we can say that $w_1 \in A$ since $x \neq v \Rightarrow w_1 \notin B$. Additionally, w_2 must be an element in B that can act on $w_1 x$. This is only possible if $w_1 x = v$ and so $w_1 \in X_A$. It must then also be the case that $w_2 \in D^v_B$.

We have now shown that $w \in D^x_A \cup \{\omega_1\omega_2\omega_3, \omega_1\omega_3 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v, \omega_3 \in X_A\}$ when $n = 1$ or 2. Now assume this statement is true $\forall n \leq m$ (where $m \geq 2$) and say $n = m+1$. In this case $w = w_{m+1} w_m \cdots w_1$ where every letter w_i alternates between existing in A or B. In particular, this means $w_m \cdots w_1$ is a word of length m that exists in D^x_{A*B}. It therefore also exists in $D^x_A \cup \{\omega_1\omega_2\omega_3, \omega_1\omega_3 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v, \omega_3 \in X_A\}$. Since $m \geq 2$, $w_m \cdots w_1 \notin D^x_A$. So, $w = w_{m+1}\omega_1\omega_2\omega_3$ or $w_{m+1}\omega_1\omega_3$ for some $\omega_1 \in D^v_{A\cup B}$, $\omega_2 \in S^A_v * S^B_v$, and $\omega_3 \in X_A$.

Assume $w_m \in A$. This implies that $w_{m+1} \in B$. Given the definition of ω_1, ω_2 and ω_3, $w = w_{m+1}\omega_1\omega_2\omega_3$ or $w_{m+1}\omega_1\omega_3 \Rightarrow \omega_1 = w_m$ and $\omega_3 = w_1$.

If we assume $w = w_{m+1}\omega_1\omega_3$ then we can say that $\omega_1 \in B \Rightarrow \omega_1 \in D^v_B$. Therefore, $w_{m+1} \in A$ and we require $\omega_1\omega_3 x = v$ so that wx is defined Hence, $w_{m+1} \in D^v_A$. Furthermore, $\omega_3 x = v$ and $\omega_1\omega_3 x = v \Rightarrow \omega_1 v = v$ and so $\omega_1 \in S^B_v$.

Now instead take $w = w_{m+1}\omega_1\omega_2\omega_3$ and assume m is a value such that $\omega_1 \in A$. By definition, $\omega_2\omega_3 x = v$ and $w_{m+1} \in B$. Therefore, $\omega_1 \in S^A_v$. since the only possible value that ω_1 can map v to so that an element of B can act on it is v itself. So, $w_{m+1} \in D^v_B$. A dual proof tells us that $\omega_1 \in B \Rightarrow w_{m+1} \in D^v_A$ and $\omega_1 \in S^B_v$. So by induction we can say that $D^x_{A*B} \subseteq D^x_A \cup \{\omega_1\omega_2\omega_3, \omega_1\omega_3 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v, \omega_3 \in X_A\}$.

Conversely, say $w \in \{\omega_1\omega_2\omega_3, \omega_1\omega_3 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v, \omega_3 \in X_A\}$. If $w = \omega_1\omega_2\omega_3$, then $wx = \omega_1\omega_2 v = \omega_1 v$ and $\omega_1 v$ exists by definition of ω_1. Simlarly if $w = \omega_1\omega_3$ we again have $wx = \omega_1 v$. So in both cases, $w \in D^x_{A*B}$. Since $D^x_A \subseteq D^x_{A*B}$ we can conclude that $D^x_{A*B} = D^x_A \cup \{\omega_1\omega_2\omega_3, \omega_1\omega_3 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v, \omega_3 \in X_A\}$. A dual proof for when $x \in V(G_B)$ gives us the value for D^x_A stated in the lemma. □

Lemma 3.4.3.

$$D^v_{A*B} = D^v_{A\cup B} \cup \{\omega_1\omega_2 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v\} \tag{38}$$

Proof. Let $w \in D^v_{A*B}$ be a word of length n in $A * B$. It can therefore be written as $w = w_n \cdots w_2 w_1$ where w_i are the letters that compose w. Then $w \in D^v_{A*B} \Rightarrow w_1 \in D^v_{A*B}$. Given the definition of w_1, $w_1 \in A \cup B$. Therefore if $n = 1$, $w = w_1 \in D^v_{A\cup B}$.

Assume our lemma holds for $n = m$. If $n = m+1$, then $w = w_{m+1}w_m \cdots w_1$. Given that wv exists know that $w_m \cdots w_1$ is a word of length m that acts on v. Therefore it exists in $D^v_{A\cup B} \cup \{\omega_1\omega_2 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v\}$. Let $w' = w_m \cdots w_1$. If $w' \in D^v_{A\cup B}$ then w' has to have a length of 1 since $D^v_{A\cup B} \subseteq A \cup B$. If $w' \in A$ then $w_{m+1} \in B$. Furthermore, for $w_{m+1}w'v$ to be defined we require $w' \in S^A_v$ since this is the only value in Av that w_{m+1} can possibly act on. We could then say that $w_{m+1} \in D^v_B$. Similarly, $w' \in B \Rightarrow w' \in S^B_v$ and $w_{m+1} \in D^v_A$.

If $w' \notin D^v_{A\cup B}$ then $w' = \omega_1\omega_2$ for some $\omega_1 \in D^v_{A\cup B}$ and $\omega_2 \in S^A_v * S^B_v$. By definition of ω_1, $w_m = \omega_1$. This further implies that $m > 1$ and $w_{m-1} \cdots w_1 = \omega_2$. So, $wv = w_{m+1}\omega_1\omega_2 v = w_{m+1}\omega_1 v$ by definition of ω_2. It is then the case that $\omega_1 v$ must be a value that w_{m+1} can act on. Since $\omega_1 \in A$ (resp. B) $\Rightarrow w_{m+1} \in B$ (resp. A) we can again say that this would further imply that $\omega_1 \in S^A_v$ (resp. S^B_v) and $w_{m+1} \in D^v_B$ (resp. D^v_A).

We have therefore shown that $D^v_{A*B} \subseteq D^v_{A\cup B} \cup \{\omega_1\omega_2 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v\}$. It is also clear that $D^v_{A\cup B} \cup \{\omega_1\omega_2 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v\} \subseteq D^v_{A*B}$. We can therefore conclude that $D^v_{A*B} = D^v_{A\cup B} \cup \{\omega_1\omega_2 | \omega_1 \in D^v_{A\cup B}, \omega_2 \in S^A_v * S^B_v\}$. □

Lemma 3.4.4. *Let Q be the quotient graph we get when $A*B$ acts on G. Q is the graph obtained by equating the vertices Av_A and Bv_B in the quotient graphs derived from A acting on G_A and B acting on G_B respectively.*

Proof. Let Q_A and Q_B be the quotient graphs of A acting on G_A and B acting on G_B respectively. Furthermore, let α_1 and α_2 be vertices in G_A such that $A^1\alpha_1$ and $A^1\alpha_2$ are seperate vetices in Q_A. Assume α_1 and α_2 exist in the same orbit under the action of $A * B$ on $V(G)$. Since $A^1\alpha_1 \neq A^1\alpha_2$ it must be the case that $\exists b \in B, a_1, a_2 \in A$ such that $(a_2ba_1)\alpha_1 = \alpha_2$.

Under our action, the only vertex in $V(G_A)$ that can be acted on by an element in B is v. Therefore, it must be the case that $a_1\alpha_1 = v$. Consequently, $A^1\alpha_1 = A^1v$. So, two elements that exist in seprate orbits under the action of A on $V(G_A)$ can only possibly exist in the same orbit when acted on by $A*B$ if one of those vertices is v. Also, $(a_2ba_1)\alpha_1 = \alpha_2 \Rightarrow \alpha_1 = (a_1^{-1}b^{-1}a_2^{-1})\alpha_2$ and so we can also say that $A^1\alpha_2 = A^1v$. Therefore, if two vertices have unique orbits under an action of A on G_A then they are also give unique orbits under the action of $A * B$ on G. A dual proof tells us the same is true for vertex orbits obtained from an action of B on G_B.

If instead we start with the assumption that α_1 and α_2 exist in the same vertex orbit when G_A is acted on by A then they must also exist in the same vertex orbit when $A * B$ acts on G. This is because $A^1\alpha \subseteq (A * B)^1\alpha$ ($\forall\alpha \in V(G_A)$). A dual statement can be made for the vertex orbits obtained from our B-act. We can therefore conclude that $\forall\alpha \in V(G_A)$ (resp. $\beta \in V(G_B)$), $\alpha \notin A^1v$ (resp. $\beta \notin B^1v$) $\Rightarrow (A * B)^1\alpha = A^1\alpha$ (resp. $(A * B)^1\beta = B^1\beta$).

Under our action of $A * B$ on $V(G)$, v can be mapped to any element in A^1v or B^1v. Therefore, $A^1v \cup B^1v \subseteq (A*B)^1v$. Conversely, say $x \in (A*B)^1v$ and $x \neq v$. By definition, x exists in $V(G_A)$ or $V(G_B)$ but not both since $V(G_A) \cap V(G_B) = \{v\}$. Without loss of generality, let $x \in V(G_A)$. Since $x \in (A*B)^1v$, we know that $\exists w \in A*B$ such that $wx = v$. If we assume w is a word of length n in $A*B$ then it can be written in the form $w = w_1w_2 \cdots w_n$ where each w_i is a letter with respect to $A * B$. Given our action, $Bx = \emptyset$. Hence $w_n \in A$. So if $n = 1$, $w = w_n \Rightarrow w \in A \Rightarrow x \in Av$. If instead $n > 1$ then it must be the case that $w_{n-1} \in B$. For wx to be defined we require $w_{n-1}w_nx$ to exist. Since $w_n \in A \Rightarrow w_nx \in V(G_A)$ we conclude that $w_nx = v$ since v is the only vertex in $V(G_A)$ such that $B^1v \neq \emptyset$. Then, $w_nx = v \Rightarrow x \in A^1v$. Simlarly, $x \in V(G_B) \Rightarrow x \in B^1v$. Therefore, $(A * B)^1v \subseteq A^1v \cup B^1v \Rightarrow (A * B)^1v \subseteq A^1v \cup B^1v$.

It must therefore be the case that the vertices in Q are the same as in Q_A and Q_B except for the vertices represented A^1v and B^1v which have been replaced by a single vertex, $(A * B)^1x = A^1v \cup B^1v$. Given how an inverse semigroup acts on an edge is determined by the edge's endpoints, we can say that the edge orbits remain the same under the action of $A * B$. Therefore, Q is as described in the lemma. $\square$

Corollary 3.4.5. *$\forall x \in V(G)$:*

$$(A*B)^1 x = \begin{cases} A^1 v \cup B^1 v & \text{when } x \in A^1 v \cup B^1 v \\ A^1 x & \text{when } x \in V(G_A), x \notin A^1 v \\ B^1 x & \text{when } x \in V(G_B), x \notin B^1 v \end{cases} \tag{39}$$

Proof. Given in the proof of Lemma 3.4.4. □

Now consider the A-transversal and B-transversal used in our original actions. By definition, they include the vertices v_A and v_B respectively. Given what we know the quotient graph of $A * B$ acting on G will look like, we can say that an $(A * B)$-transversal is given by the A-transversal and B-transversal when we equate the vertices v_A and v_B. We label this transversal T. Now that we have a transversal, we need to establish some properties of the stabilizers if we wish to say that it can create a graph of inverse semigroups.

Lemma 3.4.6. *Let $x \in V(G)$ such that $x \notin (A*B)^1 v$.*

$$S_x^{A*B} = \begin{cases} S_x^A & \text{when } x \in V(G_A) \\ S_x^B & \text{when } x \in V(G_B) \end{cases} \tag{40}$$

Proof. By definition, $S_x^{A*B} \subseteq D_{A*B}^x$. Say $x \in V(G_A)$. From Lemma 3.4.1 we know this implies that $D_{A*B}^x = D_A^x$. Therefore, $S_x^{A*B} \subseteq D_A^x$. In other words, $wx = x \Rightarrow w \in A$ and so $S_x^{A*B} \subseteq S_x^A$. Conversely, $w \in S_x^A \Rightarrow w \in S_x^{A*B}$ by definition of our action. It must then be te case that $S_x^{A*B} = S_x^A$. It can be shown dually that $x \in V(G_B) \Rightarrow S_x^{A*B} = S_x^B$. □

Lemma 3.4.7. *$\forall x \in (A*B)^1 v$ such that $x \neq v$ define $X_A = \{w \in A | wx = v\}$ and $X_B = \{w \in B | wx = v\}$. If $x \in V(G_A)$ then;*

$$S_x^{A*B} = S_x^A \cup \{\omega_1^{-1}\omega_2\omega_3 | \omega_1, \omega_3 \in X_A, \omega_2 \in S_v^A * S_v^B\}. \tag{41}$$

Similarly, if $x \in V(G_A)$ then;

$$S_x^{A*B} = S_x^B \cup \{\omega_1^{-1}\omega_2\omega_3 | \omega_1, \omega_3 \in X_B, \omega_2 \in S_v^A * S_v^B\}. \tag{42}$$

Proof. Assume $x \in V(G_A)$. Let $w \in S_x^{A*B}$. From Lemma 3.4.2 we know that either $w \in D_A^x$ or $\exists \omega_1 \in D_{A \cup B}^v$, $\omega_2 \in S_v^A * S_v^B$, $\omega_3 \in X_A$ such that $w = \omega_1\omega_2\omega_3$ or $\omega_1\omega_3$. If $w \in D_A^x$,then $w \in A$. Furthermore, $w \in S_x^{A*B}$ and $w \in A \Rightarrow w \in S_x^A$.

If instead we assume that $w = \omega_1\omega_2\omega_3$ or $\omega_1\omega_3$, then $wx = \omega_1 v$ (by definition of ω_2 and ω_3). Then $wx = x \Rightarrow \omega_1 v = x \Rightarrow v = \omega_1^{-1} x$. So, $\omega_1^{-1} \in X_A$. Therefore, $w \in \{\omega_1^{-1}\omega_2\omega_3, \omega_1^{-1}\omega_3 | \omega_1, \omega_3 \in X_A, \omega_2 \in S_v^A * S_v^B\}$.

It is then the case that $S_x^{A*B} \subseteq S_x^A \cup \{\omega_1^{-1}\omega_2\omega_3, \omega_1^{-1}\omega_3 | \omega_1, \omega_3 \in X_A, \omega_2 \in S_v^A * S_v^B\}$. We can also easily see that $\omega_1^{-1}\omega_2\omega_3, \omega_1^{-1}\omega_3 \in S_x^{A*B}$ $(\forall \omega_1, \omega_3 \in$

$X_A, \omega_2 \in S_x^A * S_x^B$). However given that $\omega_1, \omega_3 \in A$, $\omega_1\omega_3 \in S_x^A$. Therefore, we can say that S_x^{A*B} is as defined in the lemma. A dual proof can be used to define S_x^{A*B} when $x \in V(G_B)$. □

Lemma 3.4.8.

$$S_v^{A*B} = S_v^A * S_v^B \tag{43}$$

Proof. Let $w \in S_v^{A*B}$. Lemma 3.4.3 tells us $w \in D_{A\cup B}^v$ or $\exists \omega_1 \in D_{A*B}^v, \omega_2 \in S_v^A * S_v^B$ such that $w = \omega_1\omega_2$. If $w \in D_{A\cup B}$, then $w \in A \cup B \Rightarrow w \in S_v^A \cup S_v^B$. Alternitively, if $w = \omega_1\omega_2$ then $wv = \omega_1\omega_2 v = \omega_1 v$ (by definition of ω_2). We have defined w to be a stabilizer of v and so $v = wv = \omega_1 v \Rightarrow \omega_1 \in S_v^{A*B}$. Furthermore, $\omega_1 \in D_{A\cup B}^v \Rightarrow \omega_1 \in S_v^A \cup S_v^B$. Hence, $w = \omega_1\omega_2 \in S_v^A * S_v^B$. □

Corollary 3.4.9. *$\forall e \in E(T)$ such that $\tau(e) \notin V(T)$, $\exists t_e \in A * B$ such that $t_e\iota(e) = \tau(e)$ and $t_e t_e^{-1} = Id(S_{\tau(e)}^{A*B})$.*

Proof. Let $e \in E(T)$ and $\tau(e) \notin V(T)$. By definition of the edges in G, $e \in E(G_A)$ or $e \in E(G_B)$. Assume $e \in E(G_A)$. Our defintion of T means that e must have also been an edge in the A-transversal taken when A acted on G_A. Since this system was defined to give us a fundamental inverse semigroup we know that we could create a graph of inverse semigroups from this transversal. In other words, $\exists t_e \in A$ such that $t_e\iota(e) = \tau(e)$ and $t_e t_e^{-1} = Id(S_{\tau(e)}^A)$.

If $\tau(e) \notin (A * B)^1 v$, then Lemma 3.4.6 states that $S_{\tau(e)}^{A*B} = S_{\tau(e)}^A$. Therefore, $t_e t_e^{-1} = Id(S_{\tau(e)}^{A*B})$ and t_e satisfies the required properties. If instead $\tau(e) \in (A * B)^1 v$ then we know that $\tau(e) \neq v$ since T is defined so that $v \in V(T)$. Lemma 3.4.7 tells us that $\forall w \in S_{\tau(e)}^{A*B}$, $w \in S_{\tau(e)}^A$ or $w = \omega_1^{-1}\omega_2\omega_3$ for some $\omega_2 \in S_{\tau(e)}^A * S_{\tau(e)}^B$ and $\omega_1, \omega_3 \in X_A = \{a \in A | a\tau(e) = v\}$.

If $w \in S_{\tau(e)}^A$, then $t_e t_e^{-1} w = w t_e t_e^{-1} = w$ by definition of t_e. Alternitively, say $w = \omega_1^{-1}\omega_2\omega_3$. Note that $\omega_1^{-1}\omega_1, \omega_3^{-1}\omega_3 \in S_{\tau(e)}^A$ and so $t_e t_e^{-1}\omega_1^{-1}\omega_1 = \omega_1^{-1}\omega_1$ and $\omega_3^{-1}\omega_3 t_e t_e^{-1} = \omega_3^{-1}\omega_3$. Knowing this we can say that;

$$\begin{aligned} w = \omega_1^{-1}\omega_2\omega_3 &= (\omega_1^{-1}\omega_1)\omega_1^{-1}\omega_2\omega_3 = (t_e t_e^{-1}\omega_1^{-1}\omega_1)\omega_1^{-1}\omega_2\omega_3 \\ &= (t_e t_e^{-1})(\omega_1^{-1}\omega_1\omega_1^{-1})\omega_2\omega_3 = (t_e t_e^{-1})\omega_1^{-1}\omega_2\omega_3 = t_e t_e^{-1} w. \end{aligned} \tag{44}$$

Similarly it can be shown that $\omega_3^{-1}\omega_3 t_e t_e^{-1} = \omega_3^{-1}\omega_3 \Rightarrow w = w t_e t_e^{-1}$. So in both cases $t_e t_e^{-1}$ acts as an identity on w and hence $t_e t_e^{-1} = Id(S_{\tau(e)}^{A*B})$. □

This Corollary means that we can create a graph of inverse semigroups from T which we can then use to create a fundamental inverse semigroup which we shall call S.

Lemma 3.4.10. *Let $G(S)$, $G(S_A)$ and $G(S_B)$ be the generating sets of S, S_A and S_B respectively that we intially get from their transversals. Then $G(S_A) \cup G(S_B) = G(S)$.*

Proof. Let $g \in G(S_A) \cup G(S_B)$. Assume $g \in G(S_A)$ then by the defintion of $G(S_A)$ we know the either g is a generator of the stabilizer of a vertex in the A-transversal or g is an element that defines an embedding of an edge in the transversal to it's initial point. If g is the latter then Corollary 3.4.9 tells us that $g \in G(S)$ since it again defines an embedding this time in the $(A * B)$-transversal.

Alternatively, g is a generator of S_v^A or S_x^A for some $x \notin A^1 v$ where x is a vertex in the A-transversal. By Lemmas 3.4.6 and 3.4.8, $S_x^{A*B} = S_x^A$ and $S_v^{A*B} = S_v^A * S_v^B$. Therefore, since all vertices in the A-transversal are also vertices in the $(A * B)$-transversal we can say that g is a generator of S_v^A or S_x^A implies that g is a generator of the stabilizers of a vertex in the $(A * B)$-transversal. By definition, this makes g a generator of S. The same can be shown if $g \in G(S_B)$.

Conversely, say $g \in G(S)$. By definition, this meanst that either g comes from an embedding of an edge in the $(A*B)$-transversal into it's intial vertex or it is a generator of the stabilizer of a vertex in the $(A * B)$-transversal. If it's the former, then we know from the proof of Corollary 3.4.9 that $g \in G(S_A) \cup G(S_B)$.

Otherwise g is a generator of S_x^{A*B} for some vertex x in the $(A * B)$-transversal. Given the defintion of the $(A * B)$-transversal, either $x = v$ or $x \notin (A * B)^1 v$. If $x \notin (A * B)^1 v$ then g is a generator of S_x^A or S_x^B depending on if $x \in V(G_A)$ or $V(G_B)$ (Lemma 3.4.6). In either case this means $g \in G(S_A) \cup G(S_B)$. If instead $x = v$ then g is a generator of $S_v^{A*B} = S_v^A * S_v^B$ (Lemma 3.4.8). This makes g a generator of either S_v^A or S_v^B which in turn means $g \in G(S_A) \cup G(S_B)$ (since v exists in both the A and B-transversals). □

Lemma 3.4.11. *Let $R(S)$, $R(S_A)$ and $R(S_B)$ be the sets of relations of S, S_A and S_B respectively that we initially get from their transversals. Then $R(S_A) \cup R(S_B) \subseteq R(S)$.*

Proof. Let $r \in R(S_A)$. By definiton, r can originate from two places. The first is when r is a relation in S_x^A for some vertex x in the A-transversal. The second is if r is of the form $t^{-1} w_1 t = w_2$ for some $w_1 \in A$ that is a generator of a stabilizer of an edge in the A-transversal, $w_2 \in S_x^A$ for some vertex x in the A-transversal and $t \in A$ such that t is the required element to define an embedding of a stabilizer of an edge in the A-transversal (whose terminal vertex does not exist in the A-transversal) into the stabilizer of the intial vertex of the edge.

If it's the former and $x \notin A^1 v$ then Lemma 3.4.6 tells us r must be a relation in S_x^{A*B}. Therefore, $r \in R(S)$. If instead $x \in A^1 v$ then $x = v$ (since the A-transversal can only contain one element from each vertex orbit). Therefore, r is a relation in S_v^A. Given the value of S_v^{A*B} (from Lemma 3.4.8) we can the also say that r is a relation in S_v^{A*B}.

Alternatively, say r is of the form $t^{-1}w_1t = w_2$. From Corollary 3.4.9 we know that the embedding t defines in the graph of inverse semigroups for the A-act is also used in the graph of inverse semigroups for the $(A*B)$-act. Given this, we know that $r \in R(S)$. The same logic can be used to say that $r \in R(S_B) \Rightarrow r \in R(S)$. □

Lemma 3.4.12. *Let $R(S)$, $R(S_A)$ and $R(S_B)$ be the sets of relations of S, S_A and S_B respectively that we initially get from their transversals. Then $R(S) \subseteq R(S_A) \cup R(S_B)$.*

Proof. By definition the elements in $R(S)$ either came from relations in the stabilizers of vertices of the $(A * B)$-transversal or they come from embeddings of stabilizers of edges in the transversal without terminal vertices into the stabilizer of their inital vertices. If it's the former, then Lemmas 3.4.6 and 3.4.8 tells us that such relations would also exist in $R(S_A) \cup R(S_B)$. If it's the latter then there exists an edge e in the $(A*B)$-transversal such that $\tau(e)$ does not exist in the transversal. This then gives us relations of the form $t_e^{-1}w_1t_e = w_2$ where $t_e \in A*B$ is the element with respect to e defined by Corollary 3.4.9, $w_1 \in A * B$ is a generator of S_e^{A*B} and $w_2 \in S_{\iota(e)}^{A*B}$.

Say e is an edge in $G_A \Rightarrow \tau(e) \in G_A$. Then since $S_e^{A*B} = S_{\iota(e)}^{A*B} \cap S_{\tau(e)}^{A*B}$ it must be the case that $\iota(e)$ or $\tau(e) \notin (A*B)^1v \Rightarrow S_e^{A*B} = S_e^A$ (by definiton of the stabilizers of $\iota(e)$ and $\tau(e)$ from Lemmas 3.4.6, 3.4.7 and 3.4.8). In which case, any relation we obtain from the embedding of e would be equivalent to a relation obtained from embedding e in the A-transversal. In other words, if r is a relation in $R(S)$ that we obtained from such an edge, then $r \in R(S_A)$.

Now instead say $\iota(e), \tau(e) \in (A * B)^1v$. By definition of our $(A * B)$-transversal this means that $\iota(e) = v$. So using Lemmas 3.4.7 and 3.4.8 we say $S_{\iota(e)}^{A*B} = S_v^A * S_v^B$ and $S_{\tau(e)}^{A*B} = S_{\tau(e)}^A \cup \{\omega_1^{-1}\omega_2\omega_3 | \omega_1, \omega_3 \in X_A, \omega_2 \in S_v^A * S_v^B\}$ where we define $X_A = \{a \in A | a\tau(e) = v\}$. So, to define S_e^{A*B} we need only define the union of these sets. Since $S_{\tau(e)}^A \subseteq A$, we can say that $S_{\tau(e)}^A \cap (S_v^A * S_v^B) = S_{\tau(e)}^A \cap S_v^A = S_e^A$. It now only remains to find out which elements (if any) exist in both $\{\omega_1^{-1}\omega_2\omega_3 | \omega_1, \omega_3 \in X_A, \omega_2 \in S_v^A * S_v^B\}$ and $S_v^A * S_v^B$. Let w be such an element, so $w = \omega_1^{-1}\omega_2\omega_3$ for some $\omega_1, \omega_3 \in X_A$, $\omega_2 \in S_v^A * S_v^B$. If $w \in S_v^A * S_v^B$, then ω_3 must preserve v. However, this means $\omega_3 v = v = \omega_3\tau(e)$ which contradicts our action. Therefore, the union of these sets is empty, meaning that $S_e^{A*B} = S_e^A$. Again this means any relation we have from the embedding of e in the $(A * B)$-transversal we would also get from the A-transversal. The same can also be said for when e is an edge in G_B. □

Corollary 3.4.13.

$$R(S_A) \cup R(S_B) = R(S) \tag{45}$$

Proof. Follows from Lemmas 3.4.11 and 3.4.12. □

Theorem 3.4.14.

$$S \simeq A * B \tag{46}$$

Proof. From Lemma 3.4.10 and Corollary 3.4.13 we have the intial set of generators and relations that define S and can therefore say;

$$S = Inv\langle G(S_A), G(S_B)|R(S_A), R(S_B)\rangle. \tag{47}$$

Given that there are no relations in $R(S_A)$ (resp. $R(S_B)$) that involve any of the generators in $G(S_B)$ (resp. $G(S_A)$) our value of S implies that;

$$S = Inv\langle G(S_A)|R(S_A)\rangle * Inv\langle G(S_B)|R(S_B)\rangle. \tag{48}$$

By defintion, $S_A = Inv\langle G(S_A)|R(S_A)\rangle$ and $S_B = Inv\langle G(S_B)|R(S_B)\rangle$. Hence, $S = S_A * S_B$. Since S_A and S_B are defined to be isomorphic to A and B respectively we can say that $S \simeq A * B$. □

Given that if C is also an inverse semigroup $A * (B * C) = (A * B) * C$ we can use the same logic used in this section to define actions for the free product of any number of inverse semigroups provided we have an action of them on a graph for each semigroup seperately. One area where this is useful is for inverse semigroups that can be expressed as a free product of groups. Given that we have a method of defining a group action on a graph that will give us back a fundamental group isomorphic to the original group, we now in turn have a method of defining an action of their free product. Note though that the definition of the free product used here is different to the defintion used in group theory which equates the idenity of the two groups so that the identity is itself a group. Under our defintion the free product of two groups would be an inverse semigroup, but not a group.

4 Actions of Polycyclic Monoids

In this section, we will examine a way of defining an action of a polycyclic monoid on a set of infinitely many elements. Polycyclic monoids were first defined by Nivat and Perrot [19]. We will begin by establishing some of the properties of a polycyclic monoid that we will then use to define a semigroup action.

4.1 Polycyclic Monoid Action

Definition 4.1.1. A polycyclic monoid with n generators is an inverse semigroup with zero given by the following presentation;

$$P_n = Inv\langle p_1, p_2, \ldots, p_n | p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ when } i \neq j\rangle. \tag{49}$$

We can define a set and an action on this set by P_n. To do this we require an action so that p_i^{-1} acts on every element in the set (so that the identoty element acts on every element), but also p_i cannot act on every element (since otherwise the zero element will act on every element in the set). To define our P_n-act, first let V be a set of elements given by $V = \{v_1, v_2, \dots\}$ and define a partial map $\phi_n : P_n \times V \to V$ to be the partial map such that $\phi_n(s, v) = \rho_{(n,s)}(v)$ where $\rho_{(n,s)}$ is given by $\rho_{(n,p_i^{-1})}v_j = v_{n(j-1)+i}$ (note that the inverses of generators of an inverse semigroup will also generate the same semigroup and hence our definition of $\rho_{(n,p_i^{-1})}$ will define $\rho_{(n,s)}$, $\forall s \in P_n$). One immediate property that can be observed is that $\forall i \in \{1, 2, \dots, n\}$, if $p_i^{-1}v_j = v_k$, then $j \leq k$.

Lemma 4.1.1. *If P_n is a polycyclic monoid with generators $p_1, p_2, \dots, p_n$, then $\rho_{(n,p_i)}v_j$ exists if and only if $j = n(j'-1)+i$ for some $j' \in \{1, 2, \dots, n\}$.*

Proof. $\rho_{(n,p_i)}$ is defined to be the inverse map of $\rho_{(n,p_i^{-1})}$. Therefore, the domain of $\rho_{(n,p_i)}$ is equivalent to the image of $\rho_{(n,p_i^{-1})}$. Given the definition of $\rho_{(n,p_i^{-1})}$ and the fact that it acts on every element in V, we know that the image of $\rho_{(n,p_i^{-1})}$ consists of all elements of V of the form v_j where $j = n(j'-1) + i$ for some $j' \in \{1, 2, \dots, n\}$. □

Remark. Note that we can further prove that $\rho_{(n,p_i)}v_j = v_{j'}$. It can also be inferred that $\rho_{(n,p_i)}v_j$ exists if and only if $j - i$ is divisible by n.

In fact, the images of v_1 under $p_1^{-1}, p_2^{-1}, \dots, p_n^{-1}$ are $v_1, v_2, \dots v_n$ respectively. Furthermore, our action will map any v_i to a complete set of representatives $mod(n)$. We will now prove that our action is in fact a well-defined P_n-act.

Lemma 4.1.2. *For any polycyclic monoid P_n, V is a P_n-act with respect to the semigroup action on V given by $\rho_{(n,s)}$.*

Proof. As $\rho_{(n,s)}$ is defined from the generators of s, we know $\forall s, t \in P_n$ and any $v_i \in V$, $(st)v_i = s(tv_i)$. Now say, $\exists s \in P_n$ and $v_j, v_k \in V$ such that $sv_j = sv_k$. As $s \in P_N$ we know it can be written as a product of the generators of P_n and their inverses. Therefore, $s = s_{i_1}s_{i_2} \cdots s_{i_m}$ where $s_{i_l} = v_{i_l}$ or $v_{i_l}^{-1}$, $\forall l \in \{1, 2, \dots, m\}$. Hence, $sv_j = sv_k$ implies that $s_{i_1}s_{i_2} \cdots s_{i_m}v_j = s_{i_1}s_{i_2} \cdots s_{i_m}v_k$. So, given that we have already proven the associativity property of our potential S-act, we can say that $s_{i_1}(s_{i_2} \cdots s_{i_m}v_j) = s_{i_1}(s_{i_2} \cdots s_{i_m}v_k)$. Define $v_{j'} = s_{i_2} \cdots s_{i_m}v_j$ and $v_{k'} = s_{i_2} \cdots s_{i_m}v_k$ and suppose $s_{i_1} = p_{i_1}^{-1}$. Then we can say the following;

$$\begin{aligned} p_{i_1}^{-1}v_{j'} = p_{i_1}^{-1}v_{k'} &\Rightarrow v_{n(j'-1)+i} = v_{n(k'-1)+i} \\ &\Rightarrow n(j'-1) + i = n(k'-1) + i \Rightarrow j' = k'. \end{aligned} \tag{50}$$

Therefore, $v_{j'} = v_{k'}$. Alternatively, if $s_{i_1} = p_{i_1}$, then for $p_{i_1}v_{j'} = p_{i_1}v_{k'}$ to exist, we require $j' = n(j'' - 1) + i_1$ and $k' = n(k'' - 1) + i_1$ for some $j'', k'' \in \{1, 2, \dots, n\}$ (by Lemma 4.1.1). This further tells us that;

$$\begin{aligned} p_{i_1}v_{j'} = p_{i_1}v_{k'} &\Rightarrow v_{j''} = v_{k''} \Rightarrow j'' = k'' \\ &\Rightarrow \frac{j' - i_1}{n} + 1 = \frac{k' - 1}{n} + 1 \\ &\Rightarrow j' = k' \Rightarrow v_{j'} = v_{k'}. \end{aligned} \tag{51}$$

Hence, $s_{i_1}(s_{i_2} \cdots s_{i_m} v_j) = s_{i_1}(s_{i_2} \cdots s_{i_m} v_k) \Rightarrow s_{i_2} \cdots s_{i_m} v_j = s_{i_2} \cdots s_{i_m} v_k$. Using the method we just used to prove this, we can further imply that $s_{i_3} \cdots s_{i_m} v_j = s_{i_3} \cdots s_{i_m} v_k$ and hence we can keep repeating this process until we find that $v_j = v_k$. □

Now that we have shown that this is a semigroup action, we can say the following.

Lemma 4.1.3. *In P_n, $\forall m, i \in \mathbb{N}^+$:*

- $p_1^{-m} v_i = v_{n^m i - n^m + 1}$
- $p_n^{-m} v_i = v_{in^m}$

Proof. Say $m = 1$, then $p_1^{-1} v_i = v_{n(i-1)+1} = v_{ni-n+1}$ and so our first statement holds when $m = 1$. If we assume it holds when $m = k$, then we can say that $p_1^{-(k+1)} v_i = p_1^{-1} v_{n^k i - n^k + 1} = v_{n(n^k i - n^k + 1 - 1) + 1} = v_{n^{k+1} i - n^{k+1} + 1}$ and so our first statement holds by induction.

Similarly, when $m = 1$ we can say that $p_n^{-1} v_i = v_{n(i-1)+n} = v_{in}$ and if we assume the second statement holds when $m = k$, then $p_n^{k+1} v_i = p_n^{-1} v_{in^k} = v_{n(in^k - 1) + n} = v_{in^{k+1}}$. □

At this point it is worth mentioning that elements in P_n have a normal form. That being any non-zero element has the form $a^{-1}b$ where $a$and b are positive words generated by the generators of P_n or the identity element in P_n. We can also notice the following property of v_1 under this action.

Lemma 4.1.4. *For any polycyclic monoid P_n;*

$$S_{v_1} = Inv\langle p_1 \rangle = \{1, p_1^m, p_1^{-m}, p_1^{-r} p_1^m | r, m \in \mathbb{N}^+\}. \tag{52}$$

Proof. Let $w \in S_{v_1}$, so $w = a^{-1}b$ for some positive words a and b generated by the generators of P_n or the identity elemnet in P_n. From our action, we know that $a^{-1}bv_1$ exists $\Rightarrow bv_1$ exists $\Rightarrow b \in \langle p_1 \rangle$ or $b = 1_{P_n}$. So, $b = 1_{P_n}$ or p_1^m for some $m \in \mathbb{N}^+$. In either case, $a^{-1}bv_1 = a^{-1}v_1$. Therefore, $w \in S_{v_1} \Rightarrow a^{-1}v_1 = v_1$. By the rules of inverse semigroup actions,$a^{-1}v_1 = v_1 \Rightarrow v_1 = av_1$. Again, we can use this to say that $a \in \langle p_1 \rangle$ or $a = 1_{P_n}$. Therefore, $w \in Inv\langle p_1 \rangle = \{1, p_1^m, p_1^{-m}, p_1^{-r} p_1^m | r, m \in \mathbb{N}^+\}$ and hence S_{v_1} must be a subset of this set. Finally, it is clear that every word in this set will fix v_1 and so we conclude that $S_{v_1} = Inv\langle p_1 \rangle = \{1, p_1^m, p_1^{-m}, p_1^{-r} p_1^m | r, m \in \mathbb{N}^+\}$. □

It will also be important for us to establish how elements of P_n will act on v_1 in particular.

Lemma 4.1.5. *Think of V as a P_n-act for some fixed n and take any $v_i \in V$ such that $i = j_0 + j_1 n + j_2 n^2 + \cdots + j_m n^m$ for some $m \in \mathbb{N}^0$, $j_0, j_1, \ldots, j_m \in \{0, 1, \ldots, n-1\}$ and $j_0 \neq 0$. Then, $\exists s \in P_n$ such that $sv_1 = v_i$. Such an s is given by $p_{j_0}^{-1} p_{j_1+1}^{-1} p_{j_2+1}^{-1} \cdots p_{j_m+1}^{-1}$ when $m > 0$ and $p_{j_0}^{-1}$ when $m = 0$.*

Proof. First say $m = 0$, then;

$$p_{j_0}^{-1} v_1 = v_{n(1-1)+j_0} = v_{j_0} = v_i \tag{53}$$

and so our lemma holds in this case.

If $m = 1$, then $i = j_0 + j_1 n$ and;

$$\begin{aligned} p_{j_0}^{-1} p_{j_1+1}^{-1} v_1 = p_{j_0}^{-1} v_{n(1-1)+j_1+1} &= p_{j_0}^{-1} v_{j_1+1} \\ = v_{n(j_1+1-1)+j_0} &= v_{j_0+j_1 n} = v_i. \end{aligned} \tag{54}$$

Now assume our lemma holds for $m = k$ and say i is equivalent to $j_0 + j_1 n + j_2 n^2 + \cdots + j_{k+1} n^{k+1}$. As our Lemma holds for $m = k$, we know that $p_{j_1+1}^{-1} p_{j_2+1}^{-1} \cdots p_{j_{k+1}+1}^{-1} v_1 = v_{(j_1+1)+j_2 n + \cdots + j_{k+1} n^k}$. From this, we can then say that;

$$\begin{aligned} p_{j_0}^{-1} p_{j_1+1}^{-1} p_{j_2+1}^{-1} \cdots p_{j_{k+1}+1}^{-1} v_1 &= p_{j_0}^{-1} v_{(j_1+1)+j_2 n + \cdots + j_{k+1} n^k} \\ &= v_{n((j_1+1)+j_2 n + \cdots + j_{k+1} n^k - 1)+j_0} \\ &= v_{j_0 + j_1 n + \cdots + j_{k+1} n^{k+1}} = v_i \end{aligned} \tag{55}$$

$\square$

Remark. The proof of this Lemma does not apply to all $v_i \in V$ as if $j_0 = 0$, then $p_{j_0}^{-1}$ is not defined. It is also worth noting that we know that p_{j_l+1} exists for any $l \in \{1, 2, \ldots, m\}$ as $j_l \in \{0, 1, \ldots, n-1\}$.

Corollary 4.1.6. *Consider V to be a P_n-act for some n. If $v_i \in V$ is such that $i = j_k n^k + j_{k+1} n^{k+1} + \cdots + j_m n^m$ for some $m \in \mathbb{N}^0$, $j_k, j_{k+1}, \ldots, j_m \in \{0, 1, \ldots, n-1\}$ and $j_k \neq 0$, then $p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_m+1}^{-1} v_1 = v_i$.*

Proof. From Lemma 4.1.5, we know that;

$$p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_m+1}^{-1} v_1 = v_{j_k + j_{k+1} n + \cdots + j_m n^{m-k}}. \tag{56}$$

Note that since $j_k \neq 0$, we know that $p_{j_k}^{-1}$ exists. This then implies that;

$$\begin{aligned}
p_n^{-k}p_{j_k}^{-1}p_{j_{k+1}+1}^{-1}\cdots p_{j_m+1}^{-1}v_1 &= p_n^{-(k-1)}p_n^{-1}v_{j_k+j_{k+1}n+\cdots+j_mn^{m-k}} \\
&= p_n^{-(k-1)}v_{n(j_k+j_{k+1}n+\cdots+j_mn^{m-k}-1)+n} \\
&= p_n^{-(k-1)}v_{j_kn+j_{k+1}n^2+\cdots+j_mn^{m-k+1}-n+n} \\
&= p_n^{-(k-1)}v_{j_kn+j_{k+1}n^2+\cdots+j_mn^{m-k+1}} \\
&= \dots \\
&= p_n^{-1}v_{n(j_kn^{k-1}+j_{k+1}n^k+\cdots+j_mn^{m-1}} \\
&= v_{j_kn^k+j_{k+1}n^{k+1}+\cdots+j_mn^m} = v_i
\end{aligned} \tag{57}$$

□

These lemmas then give us a relation for the elements in V.

Proposition 4.1.7. *All elements of V exist in the same P_n-orbit for a fixed n with respect to the action given by $\phi_n : P_n \times V \to V$. In particular, $\forall v_i \in V$, $\exists s \in P_n$ such that $sv_1 = v_i$ where $s = p_n^{-k}p_{j_k}^{-1}p_{j_{k+1}+1}^{-1}\cdots p_{j_m+1}^{-1}$ for some $j_k, j_{k+1}, \dots, j_m \in \{0, 1, \dots, n-1\}$ such that $j_k, j_m \neq 0$ and $k, m \in \mathbb{N}^0$, such that $k \leq m$.*

Proof. By Lemma 4.1.5 and Corollary 4.1.6, $\forall v \in V$, $\exists s \in P_n$ such that $sv_1 = v$. Therefore, all elements in V are in the same orbit as V_1 and hence they must all exist in the same orbit. Furtheremore, if $j_m = 0$, then $p_{j_m+1}^{-1}v_1 = v_1$ and so then $s' = p_n^{-k}p_{j_k}^{-1}p_{j_{k+1}+1}^{-1}\cdots p_{j_{m-1}+1}^{-1}$ satisfies $s'v_1 = v_i$. Therefore, we can always assume that $j_m \neq 0$. □

Remark. Note that $\forall i \in \mathbb{N}^+$, i is uniquely defined as a polynomial of n. Hence, the value of s defined in Proposition 4.1.7 is unique for every i. Therefore it will be helpful to define $s_i \in P_n$ to be given by $s_iv_1 = v_i$ where $s_i = p_n^{-k}p_{j_k}^{-1}p_{j_{k+1}+1}^{-1}\cdots p_{j_m+1}^{-1}$ when $i = j_kn^k + j_{k+1}n^{k+1} + \cdots + j_mn^m$ (so s_i is uniquely defined for each $v_i \in V$).

4.2 Polycyclic Monoids Acting on a Graph

If we take V to be the set of vertices of a graph, we wish to find a set of edges of such a graph such that it is connected and if an element of P_n acting on an edge exists, it is equivalent to an edge that exists in the graph. One such example of a set of edges is defined by the following;

Definition 4.2.1. Let E_n be a set of edges connecting elements in V given by base edges $e_i = (v_1, v_i)$ which are defined $\forall i \in \{2, 3, \dots, n\}$. In other words, $E_n = \{se_i = (v_1, v_i) | i \in \{2, 3, \dots, n\}, s \in P_n\}$. Furthermore, we define G_n to be the graph whose vertices are given by V and edges are given by E_n. Part of this graph is shown in the diagram below (where $m \in \mathbb{N}$).

$$\begin{tikzcd}
v_3 & v_{2n+2} & \\
v_1 \arrow[u, "e_3"] \arrow[r, "e_2"] \arrow[d, "p_1^{-m}e_2"] & v_2 \arrow[u, "p_2^{-1}e_3"] \arrow[r, "p_2^{-1}e_2"] \arrow[d, "p_2^{-1}p_1^{-m}e_2"] & v_{n+2} \\
v_{n^m+1} & v_{n^{m+1}+2} &
\end{tikzcd}$$

From the injectivity of our action, it is clear that G_n will contain no loops.

Lemma 4.2.1. *If $i \neq 1$, then v_i is the terminal vertex of an edge in G_n.*

Proof. Say $i \in \{2, 3, \dots, n\}$. Then we know that v_i is the terminal vertex of the edge e_i. Now say $i > n$. Let $i = j_k n^k + j_{k+1} n^{k+1} + \cdots + j_m n^m$ for some $j_k, j_{k+1}, \dots, j_m \in \{0, 1, \dots, n-1\}$ such that $j_k, j_m \neq 0$ and $k, m \in \mathbb{N}^0$, such that $k \leq m$. By Proposition 4.1.7, $s_i = p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_m+1}^{-1}$ is such that $s_i v_1 = v_i$. Now assume $k < m$ and define $s_i' = p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_{m-1}+1}^{-1}$. So, $s_i = s_i' p_{j_m+1}^{-1}$. The endpoint of the edge $s_i' e_{j_m+1}$ is then given by $s_i' v_{j_m+1} = s_i' p_{j_m+1}^{-1} v_1 = s_i v_1 = v_i$ (we know that the edge e_{j_m+1} exists as $j_m \in \{1, 2, \dots n-1\}$). Similarly, if $k = m$ and $i = j_m n^m$ for some $j_m \in \{2, 3, \dots n-1\}$, then we define $s_i' = p_n^{-k}$ so $s_i = s_i' p_{j_m+1}^{-1}$ and find v_i is the endpoint of the edge $s_i' e_{j_m+1}$. Finally, if $i = n^m$ then we define $s_i' = p_n^{-(m-1)}$ and can show that v_i is the endpoint of the edge $s_i' e_n$. ☐

Remark. Our definition of s_i' with respect to some $v_i \in V$ will be used throughout the rest of this section. Furthermore, we also define $i' \in \mathbb{N}^+$ to be the element such that $s_i' v_1 = v_{i'}$ is satisfied. In other words $s_i' = s_{i'}$. From our proof of this Lemma, we can then see that since v_i is the terminal vertex of the edge given by $s_i' e_j$ for some $j \in \{2, 3, \dots, n\}$ the other endpoint will be given by $v_{i'}$.

Theorem 4.2.2. *Let $v_i \in V$ be such that $i \neq 1$. Then v_i is the terminal vertex of exactly one edge in G_n.*

Proof. From Lemma 4.2.1, we know that every such v_i is an endpoint of an edge, so it only remains to show that it is unique. By definition, we know that $\forall v_i \in V$, $\exists j_k, j_{k+1}, \dots, j_m \in \{0, 1, \dots, n-1\}, k, m \in \mathbb{N}^0$ such that $i = j_k n^k + j_{k+1} n^{k+1} + \cdots + j_m n^m$ where $j_k, j_m \neq 0$ and $k \leq m$. Proposition 4.1.7 tells us that $s_i = p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_m+1}^{-1}$ is such that $s_i v_1 = v_i$. Now assume $k < m$ and so $i' = i - j_m n^m$ and $s_{i'} = p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_{m-1}+1}^{-1}$. Then, from our proof of Lemma 4.2.1, we can say that $\iota(s_{i'} e_{j_m+1}) = v_{i'}$ and

$\tau(s_{i'}e_{j_m+1}) = v_{i'}$. Finally assume $\exists t \in P_n$ and $l \in \{2, 3, \ldots, n\}$ such that $\tau(te_l) = v_i$ and $\iota(te_l) \neq v_{i'}$, so te_l is an edge with v_i as an endpoint that is not equal to $s_{i'}e_{j_m+1}$.

As $t \in P_n$, we can think of it as a product of generators of P_n and their inverses. In other words, let $t = q_1q_2 \cdots q_x$ for some $x \in \mathbb{N}^+$ where $\forall y \in \{1, 2, \ldots x\}$, $q_y = p_{i_y}$ or $p_{i_y}^{-1}$ for some $i_y \in \{1, 2, \ldots n\}$ such that t can not be reduced to a word of a shorter length. Let's first assume that $q_1 = p_{i_1}$, then for t to be in its reduced form, we require $q_2 = p_{i_2}$ (as otherwise $q_1q_2 = 1$ or 0). By repeating this logic, we eventually find that $t = p_{i_1}p_{i_2} \cdots p_{i_x}$. Therefore, te_l exists if and only if $p_{i_x}v_1$ and $p_{i_x}v_l$ both exist. In other words, $i_x \in \{1, 2, \ldots n\}$ must be such that $1 - i_x$ and $l - i_x$ must be divisible by n (by Lemma 4.1.1). The only i_x that will satisfy $1 - i_x$ being divisible by n is 1, but then there is no $l \in \{2, 3, \ldots, n\}$ such that $l - 1$ is divisible by n. So no such t exists and hence our assumption that $q_1 = p_{i_1}$ is not true, meaning that $q_1 = p_{i_1}^{-1}$. Furthermore, this then implies that every q_y is of the form $p_{i_y}^{-1}$ as otherwise our relations of P_n tell us that t would be reducible.

By definition, $tv_l = q_1q_2 \cdots q_xv_l = v_i$ so it must be the case that $q_1^{-1}v_i$ exists and hence $p_{i_1}v_i$ to exists. Then, by Lemma 4.1.1, $i - i_1$ must be divisible by n. Given that $i = j_kn^k + j_{k+1}n^{k+1} + \cdots + j_mn^m$ and our definition of i_1, we know this is only possible if $i_1 = n$ if $k \neq 0$ or j_k if $k = 0$. Therefore, $p_{i_1}v_i = v_{i'_1}$ where $i'_1 = j_kn^{k-1} + j_{k+1}n^k + \cdots + j_mn^{m-1}$ if $k > 0$ and $1 + j_1 + j_2n + \cdots + j_mn^{m-1}$.

So, we have found that $p_{i_2}^{-1}p_{i_3}^{-1} \cdots p_{i_x}^{-1} = v_{i'_1}$ and hence we require that $p_{i_2}v_{i'_1}$ to exist. Using the same method used to find the possible values of i_1, we can show that $i_2 = n$ if $k > 1$, j_k if $k = 1$ or $j_1 + 1$ if $k = 0$. If we keep repeating this method, we find that $q_y = p_{j_{y-1}}^{-1}, \forall y \in \{1, 2, \ldots min\{x, m+1\}\}$ where when $k \neq 0$, we will define $p_{j_w}^{-1} = p_n^{-1}, \forall w \in \{1, 2, \ldots k-1\}$.

So, we now have 3 possible values of t depending on if $x < m + 1$, $x > m + 1$ or $x = m + 1$. If $x < m + 1$, then $t = p_n^{-k}p_{j_k}^{-1}p_{j_{k+1}+1}^{-1} \cdots p_{j_{x-1}+1}^{-1}$ if $k < x$ or p_n^{-x} otherwise. Assume $t = p_n^{-k}p_{j_k}^{-1}p_{j_{k+1}+1}^{-1} \cdots p_{j_{x-1}+1}^{-1}$, then we can say that;

$$\begin{aligned} tv_l = v_i &\Rightarrow p_n^{-k}p_{j_k}^{-1}p_{j_{k+1}+1}^{-1} \cdots p_{j_{x-1}+1}^{-1}p_l^{-1}v_1 = s_iv_1 \\ &\Rightarrow p_l^{-1}v_1 = p_{j_x+1}^{-1} \cdots p_{j_m+1}^{-1}v_1 = v_{(j_x+1)+j_{x+1}n+\cdots+j_mn^{m-x}} \\ &\Rightarrow v_1 = p_lp_{j_x+1}^{-1} \cdots p_{j_m+1}^{-1}v_1 = p_lv_{(j_x+1)+j_{x+1}n+\cdots+j_mn^{m-x}} \end{aligned} \tag{58}$$

For $p_lp_{j_x+1}^{-1} \cdots p_{j_m+1}^{-1}v_1$ to be defined, we require $p_lp_{j_x+1}^{-1} \cdots p_{j_m+1}^{-1} \neq 0$ and so the relations of P_n tell us that $p_l = p_{j_x+1}$. In which case, $p_lp_{j_x+1}^{-1} \cdots p_{j_m+1}^{-1}v_1 = p_{j_{x+1}+1}^{-1} \cdots p_{j_m+1}^{-1}v_1 = v_1$. Hence, $p_{j_{x+1}+1}^{-1} \cdots p_{j_m+1}^{-1} \in S_{v_1}$. However, as p_j^{-1} will map any $v_i \in V$ to some $v_{i'} \in V$ such that $i \leq i'$ it must be the case that element in this product is equal to p_1^{-1} for it to exist in S_{v_1}. In particular, this means that $p_{j_m+1}^{-1} = p_1^{-1}$, but this would further imply that $j_m = 0$

which contradicts our definition of j_m. If instead we assumed that $k < x$ we use the same method to show that $l = n$ and again finding this implies that $j_m = 0$. So, it is not possible that $x < m + 1$.

Now assume $x > m+1$. We can therefore define $q_y, \forall y \in \{1, 2, \ldots, m+1\}$. So we can say that;

$$\begin{aligned} t &= p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_m+1}^{-1} q_{m+2} q_{m+3} \cdots q_x \\ &= p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_m+1}^{-1} p_{i_{m+2}}^{-1} p_{i_{m+3}}^{-1} \cdots p_{i_x}^{-1} \end{aligned} \tag{59}$$

We can now take $tv_l = v_i$ to imply the following.

$$\begin{aligned} tv_l = v_i &\Rightarrow p_n^{-k} p_{j_k}^{-1} p_{j_{k+1}+1}^{-1} \cdots p_{j_m+1}^{-1} p_{i_{m+2}}^{-1} p_{i_{m+3}}^{-1} \cdots p_{i_x}^{-1} p_l v_1 = v_1 \\ &\Rightarrow p_{i_{m+2}}^{-1} p_{i_{m+3}}^{-1} \cdots p_{i_x}^{-1} p_l v_1 = v_1 \end{aligned} \tag{60}$$

We therefore require $p_l v_1$ to exist for $p_{i_{m+2}}^{-1} p_{i_{m+3}}^{-1} \cdots p_{i_x}^{-1} p_l$ to exist. However, $p_l v_1$ exists if and only if $1 - l$ is divisible by n. Given that l is defined such that the edge e_l exists, we know that $1 - l$ can never be divisible by n (as $l \in \{2, 3, \ldots, n\}$. Therefore, such a $p_l v_1$ can not exist and our assumption that $x > m + 1$ is incorrect.

This leaves us with only one option, $x = m + 1$. However, since $q_y = p_{j_{y-1}}^{-1}$, this then implies that $t = s_{i'}$. As $\iota(e_i) = v_1$, $\forall i \in \{2, 3, \ldots, n\}$, this tells us that $\iota(te_l) = v_{i'}$ which contradicts our initial definition of t. This same method could have also been applied if we had instead assumed that $k = m$. □

Remark. In the proof of this Theorem, we also showed that the value of the other endpoint on the edge with a terminal vertex of v_i will be given by $v_{i'}$. We could then find the edge that $v_{i'}$ is the terminal vertex of and the other endpoint of this edge. It is possible for us to repeat this method until we reach a base edge of G_n. As these base edges are all connected, we can say that G_n will be connected.

Corollary 4.2.3. *G_n contains no cycles.*

Proof. Assume that $v_{i_1}, v_{i_2}, \ldots, v_{i_x} \in V$ and their adjacent edges make a cycle in G_n. In other words, $\exists t_1, t_2, \ldots t_x \in P_n$ and $l_1, l_2, \ldots, l_x \in \{2, 3, \ldots, n\}$ such that;

$$v_{i_1} \xrightarrow{t_1 e_{l_1}} v_{i_2} \xrightarrow{t_2 e_{l_2}} \cdots \xrightarrow{t_{x-1} e_{l_{x-1}}} v_{i_x} \xrightarrow{t_x e_{l_x}} v_{i_1}$$

We know that for any $v_{i_y} \in V$, $\exists j_{k_y}, j_{k_y+1}, \ldots, j_{m_y} \in \{0, 1, \ldots n-1\}$ and $k_y, m_y \in \mathbb{N}^0$ such that $i_y = j_{k_y} n^{k_y} + j_{k_y+1} n^{k_y+1} + \cdots + j_{m_y} n^{m_y}$ where $j_{k_y}, j_{m_y} \neq 0$ and $k_y \leq m_y$. Furthermore, from the proof of Theorem 4.2.2, we know that v_{i_y} can only be the terminal vertex of the edge $s_{i'_y} e_{j_{m_y}+1}$ where $i'_y = i_y - j_{m_y} n^{m_y}$ if $m_y \neq k_y$, n^{m_y} if $k_y = m_y$ and $j_{m_y} \neq 1$ and $n^{m_y - 1}$ if

$k_y = m_y$ and $j_{m_y} = 1$. The other endpoint of the edge will then be given by $v_{i'_y}$. Furthermore, Theorem 4.2.2 tells us that for our cycle to exist, we require that $i'_y = i_{y-1}$, $\forall y \in \{2, 3, \ldots, x\}$ and $i'_1 = i_x$.

Note that if there is some v_{i_y} in our cycle such that $i_y = j_{k_y} n^{k_y} + j_{k_y+1} n^{k_y+1} + \cdots + j_{m_y} n^{m_y}$ where $j_{k_y}, j_{m_y} \neq 0$ and $k_y < m_y$ then the highest order of n in i_y will be greater than the highest order of n in i'_y. The same can also be said if $i_y = n^{m_y}$ for some $m_y \in \mathbb{N}^+$. Finally, if $i_y = j_{m_y} n^{m_y}$ for some $j_{m_y} \in \{2, 3, \ldots n-1\}$, $m_y \in \mathbb{N}^+$ then the highest order of n in i_y is equal to the highest order of n in i'_y. So, if we define $h(i_y)$ to be the highest order of n in i_y, then we can say that $h(i_1) \leq h(i_2) \leq \cdots \leq h(i_x) \leq h(i_1)$. Therefore, for this property to hold, it must be the case that every i_y in our cycle is of the form $j_{m_y} n^{m_y}$ for some $j_{m_y} \in \{2, 3, \ldots n-1\}$ and $m_y \in \mathbb{N}^+$. However, if i_y is in such a form, then $i_{y-1} = i'_y = n^{m_y}$ which contradicts the required form for i_{y-1}. Therefore, no such cycle can exist. □

Theorem 4.2.4. *Let $V = \{v_1, v_2, \ldots\}$ be a P_n-act with respect to $\phi_n(s, v) = \rho_{(n,s)}(v)$ where $\rho_{(n,s)}$ is given by $\rho_{(n,p_i^{-1})} v_j = v_{n(j-1)+i}$. Let E be a set of edges connecting elements in V given by base edges $e_i = (v_1, v_i)$ which are defined $\forall i \in \{2, 3, \ldots, n\}$. By base edges, we mean the set of edges such that $E = \bigcup_{i \in \{2,3,\ldots,n\}} Se_i$. Then the graph with vertices V and edges E will give us a connected tree.*

Proof. By definition, $\forall i \in \{2, 3, \ldots, n\}$, there exists an edge from v_1 to v_i. So the elements $v_1, v_2, \ldots v_n$ are connected in this graph. Theorem 4.2.2 then tells us that every other v_i is the terminal vertex of a multiple of one of these edges. Therefore, our graph is connected. Finally, Corollary 4.2.3 tells us our graph will have no cycles and therefore, it must be a connected tree. □

4.3 The Graph of Inverse Semigroups from the Polycyclic Monoid

To begin with, we need to find the vertex and edge orbits of our P_n-action. We know from Proposition 4.1.7 that we will have a single vertext orbit which we will define by $\bar{v}$. It only remains to find the edge orbits of our graph.

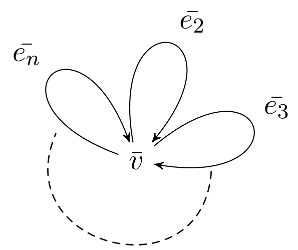
Lemma 4.3.1. *$\forall i, j \in \{2, 3, \ldots n\}$, $P_n e_i \cap P_n e_j = \emptyset$.*

Proof. Take edges e_i and e_j such that $i \neq j$ and assume they exist in the same edge orbit. In other words, $\exists s \in P_n$ such that $se_j = e_i$. Given our definition of e_i and e_j, it must be the case that $sv_1 = v_1$ and $sv_j = v_i$. From Lemma 4.1.4, we know that $S_{v_1} = \{1, p_1^m, p_1^{-m}, p_1^{-r} p_1^m | r, m \in \mathbb{N}^+\}$ and hence s must be equivalent to one of the values in this set. By definition of the action of idempotents, we know that $s \neq 1$ or $p_1^{-m} p_1^m$ as $sv_j = v_i$ would then imply that $i = j$.

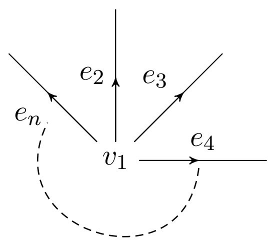
Now say $s = p_1^{-m}$ for some $m \in \{1, 2, \dots, n\}$. Then, $sv_j = v_i$ implies that $v_j = p_1^m v_i$ and hence $p_1 v_i$ must be defined. This is only possible if $i = n(i' - 1) + 1$ for some $i' \in \{1, 2, \dots\}$. It is simple to show this then means that $i' = \frac{i-1}{n} + 1$. For this to hold, we require $\frac{i-1}{n} \in \mathbb{N}^0$. However, given that $i \in \{2, 3, \dots, n\}$ this is not possible.

Similarly, if $s = p_1^m$, then $p_1 v_j$ must exist, but there is no $j \in \{2, 3, \dots n\}$ that satisfy this. Finally if $s = p_1^{-r} p_1^m$ then $p_1 v_i$ and $p_1 v_j$ must both exist which again contradicts our definition of i and j. Therefore, no such s can exist. $\square$

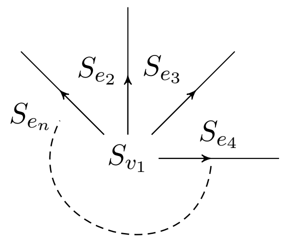
It must therefore be the case that P_n will give $n - 1$ edge orbits when acting on the graph G_n since each orbit is defined by one of our base edges, e_l for some $l \in \{2, 3, \dots n\}$. Hence, we shall define $\bar{e_l}$ to be the edge orbit containing the base edge e_l. This then gives us the following quotient graph:



When picking a vertex to represent $\bar{v}$ for our S-transversal, we will need a vertex that is the initial point of an edge in each of our edge orbits. The only such vertex in V is v_1. Therefore we find our S-transversal will be given by;



So we wish to construct a graph of inverse semigroups given by;



To make use of this graph, for each edge S_{e_l} in our graph we need to find $t_l \in P_n$ such that $t_l v_1 = v_l$ and $t_l t_l^{-1} = Id(S_{v_l}) = 1$. However, no such t_l exists given the following lemma.

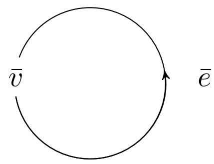
Lemma 4.3.2. *$\forall s \in P_n$, $ss^{-1} = 1_{P_n} \Rightarrow s \in \langle p_i | i \in [1, n] \rangle$.*

Proof. Say $s \in P_n$ is such that $ss^{-1} = 1$. Given the aforementioned normal form of elements in P_n we can say that $s = a^{-1}b$ for some positive words a and b generated by the generators of P_n or are the identity element in P_n. So, $1_{P_n} = ss^{-1} \Rightarrow 1_{P_n} = a^{-1}bb^{-1}a = a^{-1}a$. Therefore, $a = 1_{P_n}$ and $s = b$. $\square$

Knowing this, we can say that $tt^{-1} = 1 \Rightarrow tv_1 \neq v_l$ since $p_i v_j = v_k \Rightarrow k < j$. Therefore, we cannot continue our process any further with this graph.

If our base edges where the other way round, then we are able to find such a t_l, however this would require a vertex in V to be the initial point of an edge in each orbit. Proposition 4.1.7 will then tell us that each vertex is the inital point of a single edge in our graph (except v_1 which is not a initial point of any edge) and so we will be unable to find a vertex to represent $\bar{v}$ unless $n = 2$. It is worth mentioning that when $n = 2$, we can say some additional properties for P_n. Birget found a connection between such a monoid and certain Thompson groups [1], though relations between P_n and Thompson groups where later found $\forall n \in \mathbb{N}$ [15]. However, we will still be unable to complete our method as shown in the following example.

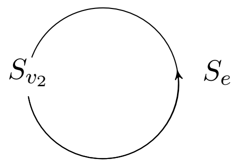
Example 4.3.3. We define P_2 to act on the graph G whose vertices are given by V and whose edges are generated from the base edge $e = (v_2, v_1)$. We know that all the elements in V exists in the same orbit which we shall call $\bar{v}$. By definition, there is only a single edge orbit which we label $\bar{e}$. This then gives us the following quotient graph;



If we then pick v_2 to represent $\bar{v}$, we find our S-transversal will be given by;

$$v_2 \xrightarrow{\quad e \quad} $$

So, the graph of inverse semigroups that we will construct is;



Therefore, we wish to find a $t \in P_2$ such that $tv_2 = v_1$ and $tt^{-1} = Id(S_{v_1})$. These conditions will be satisfied by $t = p_2$, however before we continue, we need to find the value of S_{v_2}.

Lemma 4.3.4. *In P_2, the words of length 2 in D_{v_2} are given by the set $\{p_2^{-m_1}p_1^{-m_0}, p_1^{-m_1}p_2^{-m_0}, p_1^{-m_1}p_2, p_2^{-m_1}p_2, p_1^{m_1}p_2 | m_0, m_1 \in \mathbb{N}^+\}$.*

Proof. We know that the words of length 1 that act on v_2 are given equal to 1 or p_2 our are of the form $p_1^{-m_0}$ or $p_2^{-m_0}$ for some $m_0 \in \mathbb{N}^+$. Therefore to find the words of length 2 that act on v_2 we need only multiply these values on the left by $p_1^{m_1}, p_1^{-m_1}, p_2^{m_1}, p_2^{-m_1}$ where m_1 is any element in $\mathbb{N}^+$. By definition of the identity, we know that we need not check if multiplying 1 on the left by any of these terms gives a word of length 2 as we know this will never happen. The solutions to our remaining equations are then given in the following table;

	$p_1^{-m_0}$	$p_2^{-m_0}$	p_2
$p_1^{m_1}$	$p_1^{m_1}p_1^{-m_0}$	$p_1^{m_1}p_2^{-m_0}$	$p_1^{m_1}p_2$
$p_1^{-m_1}$	$p_1^{-(m_1+m_0)}$	$p_1^{-m_1}p_2^{-m_0}$	$p_1^{-m_0}p_2$
$p_2^{m_1}$	$p_2^{m_1}p_1^{-m_0}$	$p_2^{m_1}p_2^{-m_0}$	$p_2^{m_1+1}$
$p_2^{-m_1}$	$p_2^{-m_1}p_1^{-m_0}$	$p_2^{-(m_1+m_0)}$	$p_2^{-m_1}p_2$

So we can immediately see that $p_1^{-m_1}p_1^{-m_0}$, $p_2^{-m_1}p_2^{-m_0}$ and $p_2^{m_1}p_2$ are words of length one. Furthermore, our relations of P_2 tell us that $p_i^{m_1}p_j^{-m_0}$ will equal a word of length 1 if $i = j$ and will equal 0 if $i \neq j$ (in which case it won't act on v_2). The remaining values then form the set defined in the lemma. $\square$

Lemma 4.3.5. *The words of length 3 in D_{v_2} are of the form $p_1^{-m_2}p_2^{-m_1}p_1^{-m_0}$, $p_2^{-m_2}p_1^{-m_1}p_2^{-m_0}$, $p_2^{-m_2}p_1^{-m_1}p_2$, $p_1^{-m_2}p_2^{-m_2}p_2$, $p_1^{-m_2}p_1^{m_1}p_2$ or $p_2^{-m_2}p_1^{m_1}p_2$ for some $m_0, m_1, m_2 \in \mathbb{N}^+$.*

Proof. Similiar to how we proved Lemma 4.3.4, we obtain words of length 3 by multiplying words of length 2 on the left by $p_1^{m_2}, p_1^{-m_2}, p_2^{m_2}, p_2^{-m_2}$ where m_2 is any element in $\mathbb{N}^+$ and checking that our new word is of length 3 and still exists in D_{v_2}. Our equations are given in the following table;

	$p_2^{-m_1}p_1^{-m_0}$	$p_1^{-m_1}p_2^{-m_0}$	$p_1^{-m_1}p_2$	$p_2^{-m_1}p_2$	$p_1^{m_1}p_2$
$p_1^{m_2}$	$p_1^{m_2}p_2^{-m_1}p_1^{-m_0}$	$p_1^{m_2}p_1^{-m_1}p_2^{-m_0}$	$p_1^{m_2}p_1^{-m_1}p_2$	$p_1^{m_2}p_2^{-m_1}p_2$	$p_1^{m_2+m_1}p_2$
$p_1^{-m_2}$	$p_1^{-m_2}p_2^{-m_1}p_1^{-m_0}$	$p_1^{-(m_2+m_1)}p_2^{-m_0}$	$p_1^{-(m_2+m_1)}p_2$	$p_1^{-m_2}p_2^{-m_1}p_2$	$p_1^{-m_1}p_1^{m_1}p_2$
$p_2^{m_2}$	$p_2^{m_2}p_2^{-m_1}p_1^{-m_0}$	$p_2^{m_2}p_1^{-m_1}p_2^{-m_0}$	$p_2^{m_2}p_1^{-m_1}p_2$	$p_2^{m_2}p_2^{-m_1}p_2$	$p_2^{m_2}p_1^{m_1}p_2$
$p_2^{-m_2}$	$p_2^{-(m_2+m_1)}p_1^{-m_0}$	$p_2^{-m_2}p_1^{-m_1}p_2^{-m_0}$	$p_2^{-m_2}p_1^{-m_1}p_2$	$p_2^{-(m_2+m_1)}p_2$	$p_2^{-m_2}p_1^{m_1}p_2$

In the same way we ruled out scenarios in Lemma 4.3.4 using the relations of P_2, we can also rule out the values in our table when our word equals 0 or is not of length 3. This then leaves us with the required forms and $p_2^{m_2}p_1^{m_1}p_2$. However, for $p_2^{m_2}p_1^{m_1}p_2$ to act on v_2 we need p_2v_1 to exist as $p_1^{m_1}p_2v_2 = v_1$. Lemma 4.1.1 rells us this value doesn't exist and so $p_2^{m_2}p_1^{m_1}p_2 \in D_{v_2}$. We can also use Lemmas 4.1.1 and 4.1.3 to say that all the other words of length 3 we have defined will act on v_2. □

It will now be helpful to define the following form of words in P_2.

Definition 4.3.1. Let s be a word of length n in P_2. If s is of the form $w_{(n,1)}$ then $\exists m_1, m_2, \dots, m_n \in \mathbb{N}^0$ it is equal to $p_1^{-m_n}p_2^{-m_{n-1}} \cdots p_2^{-m_2}p_1^{-m_1}$ if n is odd or $p_2^{-m_n}p_1^{-m_{n-1}} \cdots p_2^{-m_2}p_1^{-m_1}$ if n is even.

Similarly, we say s is of the form $w_{(n,2)}$ if $\exists m_1, m_2, \dots, m_n \in \mathbb{N}^0$ such that s is equal to $p_1^{-m_n}p_2^{-m_{n-1}} \cdots p_1^{-m_2}p_2^{-m_1}$ if n is even or $p_2^{-m_n}p_1^{-m_{n-1}} \cdots p_1^{-m_2}p_2^{-m_1}$ if n is odd.

Corollary 4.3.6. *Words of length $n \geq 3$ in D_{v_2} are given by words of the form $w_{(n,i)}$, $w_{(n-1,i)}p_2$ or $w_{(n-2,i)}p_1^{m_0}p_2$ ($\forall m_0 \in \mathbb{N}^+$, $i \in \{1,2\}$).*

Proof. Lemma 4.3.5 tells us our Corollary holds when $n = 3$, so we now assume it holds when $n = k$. Then, the words of length k in D_{v_2} are of the form $w_{(n,i)}$, $w_{(n-1,i)}p_2$ or $w_{(n-2,i)}p_1^{m_0}p_2$ ($\forall m_0 \in \mathbb{N}^+$, $i \in \{1,2\}$). The words of length $k+1$ will be given by multiplying words of length k on the left by $p_1^{m_{k+1}}, p_1^{-m_{k+1}}, p_2^{m_{k+1}}$ or $p_2^{-m_{k+1}}$ for any $m_{k+1} \in \mathbb{N}^+$.

If we multiply a word of the form $w_{(n,1)}$ on the left by $p_1^{m_{k+1}}$ or $p_2^{m_{k+1}}$ then our relations tell us we will either get the zero element or a word whose length is less then or equal to n (depending on if the first term of our word is p_1^{-1} or p_2^{-1}). The same can also be said for words of the form $w_{(n,2)}$.

If instead we multiplied a word of the form $w_{(n,1)}$ on the left by $p_1^{-m_{k+1}}$, then we get a word of length n if n is odd (in which case the first term in our word is p_1^{-1}) or a word of length $n+1$ if n is even which would then give us a word of the form $w_{(n+1,1)}$. Similarly, if we multiplied our word on the left by $p_2^{-m_{k+1}}$ instead we will get a word of length n if n is even or a word of the form $w_{(n+1),1)}$ if n is odd.

The same logic will also tell us that if we multiply a word of the form $w_{(n,2)}$ on the left by $p_1^{-m_{k+1}}$, then we get a word of length n if n is even and a word of the form $w_{(n+1,2)}$ if n is odd. If we instead multiply our word on the left by ${p_2}^{-m_{k+1}}$ we will get a word of length n if n is odd and a word of the form $w_{(n,2)}$ if n is even.

Therefore, if we multiply our words of length k on the left by $p_1^{m_{k+1}}, {p_1}^{-m_{k+1}}$, $p_2^{m_{k+1}}$ or ${p_2}^{-m_{k+1}}$ we will see that the words we get will be in the forms required to say our Corollary holds for $n = k + 1$. □

Now that we know the form elements in D_{v_2} take, we can identify which of them will also exist in S_{v_2}.

Lemma 4.3.7. *The stabilizer of v_2 with respect to P_2 is given by;*

$$S_{v_2} = \{1, p_2^{-1}p_2, p_2^{-1}p_1^{-m_1}p_2, p_2^{-1}p_1^{m_0}p_2, p_2^{-1}p_1^{-m_1}p_1^{m_0}p_2 | m_0, m_1 \in \mathbb{N}^+\} \quad (61)$$

Proof. We begin by finding the words with a length greater than 3 that are stabilizers of v_2. From Corollary 4.3.6, we know the forms that these elements in D_{v_2} will take. First we shall check if any elements of the form $w_{(n,i)}$ (for some $i \in \{1, 2\}$) will exist in S_{v_2}. By definition, p_2^{-1} will map any $v_j \in V$ to some $v_{j'} \in V$ such that $j < j'$. Simliarly, we can say that p_1^{-1} will map v_j to some $v_{j''} \in V$ such that $j \leq j''$, but $j = j''$ only when $j = 1$. Therefore, we can say that no word of the form $w_{(n,i)}$ will fix v_2.

Now assume $s \in S_{v_2}$ is a word of length $n \geq 3$ that's of the form $w_{(n-1,i)}p_2$. We can say that $p_2v_2 = v_1$ and so we wish to see which words of the form $w_{(n-1,i)}$ will map v_1 to v_2. Given that $p_1^{-1}v_1 = v_1$ and $p_2^{-1}v_1 = v_2$, we can say that the only words of the form $w_{(2,i)}$ and $w_{(1,i)}$ that'll satisfy this are $p_2^{-1}p_1^{-m_1}$ and p_2^{-1} respectively (for any $m_1 \in \mathbb{N}^+$). Given that p_2^{-1} will always map v_j to a vertex with a greater index and p_1^{-1} will do the same unless $j = 1$ (in which case $p_1^{-1}v_j = v_1$) we can say that these will be the only possible words of the form $w_{(n-1,i)}$ that map v_1 to v_2. Therefore, the only possible value of s is $p_2^{-1}p_1^{-m_1}p_2$ for some $m_1 \in \mathbb{N}^+$ (as if s is given by $p_2^{-1}p_2$, then it is a word of lenght 2).

Finally let's say $s \in S_{v_2}$ is a word of length $n \geq 3$ that's of the form $w_{(n-2,i)}p_1^{m_0}p_2$ for some $m_0 \in \mathbb{N}^+$. We can use Lemmas 4.1.1 and 4.1.3 to say that $p_1^{m_0}p_2v_2 = v_1$. So, we are looking for elements of the form $w_{(n-2,i)}$ that map v_1 to v_2. We have previously found that the only such elements are p_2^{-1} and $p_2^{-1}p_1^{-m_1}$ for any $m_1 \in \mathbb{N}^+$. Therefore, we can say that s equals $p_2^{-1}p_1^{m_0}p_2$ or $p_2^{-1}p_1^{-m_1}p_1^{m_0}p_2$ for some $m_0, m_1 \in \mathbb{N}^+$.

Finally, from observation of our result of Lemma 4.3.4 we can see that the words of length 1 or 2 that are stabilizers of v_2 are $p_2^{-1}p_2$ and 1. □

Corollary 4.3.8. *The stabilizer of v_2 with respect to P_2 is given by;*

$$S_{v_2} = Inv\langle 1, p_2^{-1}p_2, p_2^{-1}p_1p_2\rangle \quad (62)$$

Proof. From Lemma 4.3.7 we know that the stabilizer of v_2 is given by $S_{v_2} = \{1, p_2^{-1}p_2, p_2^{-1}p_1^{-m_1}p_2, p_2^{-1}p_1^{m_0}p_2, p_2^{-1}p_1^{-m_1}p_1^{m_0}p_2 | m_0, m_1 \in \mathbb{N}^+\}$. However, note that;

$$p_2^{-1}p_1^{-m_1}p_1^{m_0}p_2 = (p_2^{-1}p_1^{-m_1}p_2)(p_2^{-1}p_1^{m_0}p_2) \tag{63}$$

since $p_2p_2^{-1} = 1$. So all elements of the form $p_2^{-1}p_1^{-m_1}p_1^{m_0}p_2$ can be written as a product of elements of the forms $p_2^{-1}p_1^{-m_1}p_2$ and $p_2^{-1}p_1^{m_0}p_2$.

Also note that we are also able to use the relation that $p_2p_2^{-1} = 1$ to say that;

$$p_2^{-1}p_1^{m_0}p_2 = (p_2^{-1}p_1p_2)(p_2^{-1}p_1^{m_0-1}p_2) = \cdots = (p_2^{-1}p_1p_2)^{m_0} \tag{64}$$

Similarly we can say that $p_2^{-1}p_1^{-m_1}p_2 = (p_2^{-1}p_1^{-1}p_2)^{m_1}$. Finally, note that $(p_2^{-1}p_1p_2)^{-1} = p_2^{-1}p_1^{-1}p_2$. □

We may now continue with our example. From Lemma 4.1.4 we know that $S_{v_1} = Inv\langle p_1\rangle = \{1, p_1^m, p_1^{-m}, p_1^{-r}p_1^m | r, m \in \mathbb{N}^+\}$ and hence we can say that $S_e = S_{v_1} \cap S_{v_2} = \{1\}$.

Therefore, the generators of our fundamental inverse semigroup will be given by, $\beta_1 = p_2^{-1}p_2$, $\beta_2 = p_2^{-1}p_1p_2$, 1 and t. Furthermore, the relations will be given by:

- The relations in the stabilizers which are $\beta_1^2 = \beta_1$, $\beta_2\beta_2^{-1} = \beta_1$, $\beta_1\beta_2 = \beta_2\beta_1 = \beta_2$ and $\beta_1\beta_2^{-1} = \beta_2^{-1}\beta_1 = \beta_2^{-1}$
- $tt^{-1} = 1$

First we note that since $\beta_2\beta_2^{-1} = \beta_1$ we can remove β_1 from our generators. So our fundamental inverse semigroup will be given by;

$$S' = Inv\langle \beta_2, t | \beta_2^{-1} = \beta_2\beta_2^{-2}tt^{-1} = 1\rangle \tag{65}$$

which is obviously not equivalent to P_2 since it does not include a zero element.

The main problem here seems to be that we lose our information about the zero element in our semigroup when it acts on the graph since there is no vertex or edge that the zero element acts on. If we had a vertex that 0 acted on, then every other element in our semigroup would not only also have to act on it, but will also exist in the stabilizer of that vertex (as seen in Corollary 2.1.3). Therefore, this vertex would exist in its own orbit and will have to be a vertex in our S-transversal. Since the stabilizer of this vertex is the whole semigroup, if we are able to create a fundamental semigroup from this system it will have to be equivalent to the initial semigroup, but not in a way that'll interest us.

It is obvious now that if we want an action on a graph to return P_n as the fundamental inverse semigroup, we require that 0 is not a stabilizer of any element in a P_n-transversal that we derive from this system. However, there is more to this property then we might originally expect.

Lemma 4.3.9. *Let T be any inverse semigroup and G a graph such that there exists a T-act on G. Let A be a T-transversal obtained from this system.*

$$\forall v \in V(A), 0 \notin S_v^T \Leftrightarrow \forall v \in V(G), 0 \notin S_v^T. \qquad (66)$$

Proof. The fact that $\forall v \in V(G), 0 \notin S_v^T \Rightarrow \forall v \in V(A), 0 \notin S_v^T$ is obvious. Therefore, assume our action and T-transversal are such that $\forall v \in V(A), 0 \notin S_v^T$ and assume $0 \in S_u^T$ for some $u \in V(G)$. By definition, this means that $u \notin V(A)$, however, given the definition of A, $\exists t \in T$, $v \in V(A)$ such that $tu = v$. However, since $u = 0u$, we can say that $tu = t(0u) \Rightarrow v = t(0u) = (t0)u = 0u = u$ which means $0 \in S_v^T$ which is a contradiction. So our intial assumption that 0 acts on some vertex in G is incorrect. □

Therefore any action must be such that 0 does not act on any element in the S-act since otherwise we would get 0 as an element in the stabilizer of a vertex in any S-transversal we may obtain.

So if we have an action of P_n on a graph in such a way that the fundamental inverse semigroup is isomorphic to P_n and $0 \in P_n$ does not act on any element in the graph, we would need to be able to derive the 0-element property from the relations we obtain from the embeddings of edges into their intial points that are sometimes required to create the graph of inverse semigroups. In other words, we need to be able to define a zero element existing from equations of the form $w^{-1}w_1w = w_2$ for some $w_1, w_2 \in P_n$ and $w \in \langle p_i | p_i \in P_n \rangle$ (since $ww^{-1} = 1$ implies w must exist in this subset of P_n (by Lemma 4.3.2)). Furthermore, we would also require $w_1, w_2 \neq 0$ since this would then make 0 an element in a stabilizer of a vertex. Lawson tells us that the only possible values for a non-zero element in P_n are given by the set $\{s_1^{-1}s_2 | s_1, s_2 \in P^1\}$ where $P = \langle p_1, \ldots, p_n | \rangle$ [14]. Knowing this, I have been unable to find any embeddings from relations that (along with relations that can exist in a set of stabilizers) will define a zero element in the fundamental inverse semigroup.

It is also worth mentioning that polycyclic monoids can be thought of as an amalgamated free product of inverse moniods. In the previous section we discussed how we can create actions for free products, however the difference here is that the identity (resp. zero) element in each monoid that makes the polycyclic monoid must also act as an identity (resp. zero) element in every other monoid in the product. As such, we cannot apply many of the ideas discussed there to the polycyclic monoid.

5 Bruck-Reilly Semigroup Actions

For this section, we will define $S = \mathbb{N}^0 \times T \times \mathbb{N}^0$ to be a Bruck-Reilly semigroup where T is an inverse monoid with identity 1. By doing so, we are also defining S to be an inverse semigroup [8, Proposition 5.6.6.(4)]. We will also define θ to be a morphism from T to $U(T)$ where $U(T)$ is the $\mathcal{H}$-class of 1 in T, otherwise known as the group of units of T. In such a case, we write $S = BR(T, \theta)$. The semigroup operation of S is defined to be given by $(\forall (m, a, n), (p, b, q) \in S)$;

$$(m, a, n)(p, b, q) = (m - n + t, \theta^{t-n}(a)\theta^{t-p}(b), q - p + t) \tag{67}$$

where $t = max\{n, p\}$ and θ^0 is defined to be the identity map on T. Note that S is also a monoid with an identity given by $(0, 1, 0)$ [8, Proposition 5.6.6.(1)].

Such semigroups were first described by Bruck in his book *a Survey of Binary Systems* [3]. However, he only examined the properties when θ is the morphism that maps any element in T to the identity $1_T \in T$. Reilly [21] also proposed something similar, but he only considered the case when T was a group. It was Munn [18] who put these two ideas together to give us what we are working with today.

5.1 Decomposition of Elements

We begin by looking at what different elements in S can be written as with the goal of finding a presnetation of S. The presentation that we find is similar to the one proposed by Yamamura [25], however, we will derive the result in a different way. Similar work was also done by Lavers [13], but the results are presented differently there. A presentation for the Bruck-Reilly semigroups is also given by Howie and Ruškuc [9].

A lot of the material presented in this section is already well known for Bruck-Reilly semigroups, however, it is helpful to prove them within the context of the presentation we will be using throughout our work on Bruck-Reilly semigroups.

Lemma 5.1.1. *Let $(m, t, n) \in S$ where $t = t_1 t_2 \dots t_k$ for generators $t_1, t_2, \dots, t_k$ of T and $n > m$. If $n = m + k$, then;*

$$(m, t, n) = (m, t_1, m + 1)(m + 1, t_2, m + 2) \dots (m + k - 1, t_k, m + k). \tag{68}$$

If $n > m + k$, then;

$$\begin{aligned}(m, t, n) =&(m, t_1, m + 1)(m + 1, t_2, m + 2) \dots \\ &\dots (m + k - 1, t_k, m + k)(m + k, 1, m + k + 1) \dots \\ &\dots (n - 1, 1, n).\end{aligned} \tag{69}$$

Finally, if $n < m + k$ then;

$$\begin{aligned}(m, t, n) =&(m, t_1, m + 1)(m + 1, t_2, m + 2) \dots \\ &\dots (n - 1, t_{n-m}, n)(n, t_{n-m+1}, n) \dots (n, t_k, n).\end{aligned} \tag{70}$$

Proof. By the definition of the semigroup action of a Bruck-Reilly semigroup, we know that for any $p, q, r \in \mathbb{N}^0$ and $u, v \in T$ we can say;

$$(p, u, q)(q, v, r) = (p, uv, r). \tag{71}$$

Therefore, if we set (m, t, n) to be the same as in our Lemma, we can immediately say that our statement holds when $n = m + k$. For the other two cases, note the following for any $(p, u, q) \in S$ and any generator t_i of T;

$$(p, u, q)(q, 1, q + 1) = (p, u, q + 1) \tag{72}$$

$$(p, u, q)(q, t_i, q) = (p, ut_i, q). \tag{73}$$

It then follows that our Lemma must also hold when $n > m + k$ or $n < m + k$. □

A similar proof is used to give the following Lemma for when $n < m$.

Lemma 5.1.2. *Let $(m, t, n) \in S$ where $t = t_1t_2 \dots t_k$ for generators $t_1, t_2, \dots, t_k$ of T and $n < m$. If $m = n + k$, then;*

$$(m, t, n) = (m, t_1, m - 1)(m - 1, t_2, m - 2) \dots (m - k + 1, t_k, m - k). \tag{74}$$

If $m > n + k$, then;

$$\begin{aligned}(m, t, n) =&(m, t_1, m - 1)(m - 1, t_2, m - 2) \dots \\ &\dots (m - k + 1, t_k, m - k)(m - k, 1, m - k - 1) \dots \\ &\dots (n + 1, 1, n).\end{aligned} \tag{75}$$

Finally, if $m < n + k$, then;

$$\begin{aligned}(m, t, n) =&(m, t_1, m - 1)(m - 1, t_2, m - 2) \dots \\ &\dots (n + 1, t_{m-n}, n)(n, t_{m-n+1}, n) \dots (n, t_k, n).\end{aligned} \tag{76}$$

Lemmas 5.1.1 and 5.1.2 can be used to identify what elements are needed to generate any element in S, as shown in the following Corollary.

Corollary 5.1.3. *Let A be a set of elements in S. If any element of the form $(m, t_i, m+1), (m, t_i, m-1), (m, 1, m+1), (m, 1, m-1)$ or (m, t_i, m) in S (where t_i is any generator of T and $m \in \mathbb{N}^0$) can be generated by A, then A is a set of generators of S.*

Proof. By Lemmas 5.1.1 and 5.1.2, any element in S can be written as a combination of words of the from $(m, t_i, m + 1), (m, t_i, m - 1), (m, 1, m + 1), (m, 1, m - 1)$ or (m, t_i, m). Hence, the Corollary holds. □

This then leads us to make a test for determining if a set generates a Bruck-Reilly semigroup.

Proposition 5.1.4. *Let A be a set of elements in S. Say all elements of the form $(m,1,m+1)$ or $(m,1,m-1)$ in S (where $m \in \mathbb{N}^0$) are generated by the set A. If A also generates all elements of one of the following forms, then A generates all of S:*

- $(m, t_i, m+1)$
- $(m, t_i, m-1)$
- (m, t_i, m)

where t_i is any generator of T.

Proof. From Corollary 5.1.3 we know that to prove a set generates the whole of a Bruck-Reilly extension, we need only show that it generates all elements in S of the form $(m,t_i,m+1),(m,t_i,m-1),(m,1,m+1),(m,1,m-1)$ or (m,t_i,m) (where t_i is any generator of T and $m \in \mathbb{N}^0$). However, note the following:

$$\begin{aligned}(m,t_i,m+1) &= (m,1,m+1)(m+1,t_i,m)(m,1,m+1)\\ &= (m,t_i,m)(m,1,m+1).\end{aligned} \tag{77}$$

Therefore, elements of the form $(m,t_i,m+1)$ can be written as a product of elements of the form $(m,1,m+1),(m,t_i,m-1)$ and an element of the form $(m,t_i,m-1)$ or (m,t_i,m). We can also give corresponding statements for elements of the form $(m,t_i,m-1)$ or (m,t_i,m) using the following equations:

$$\begin{aligned}(m,t_i,m-1) &= (m,t_i,m+1)(m+1,1,m)(m,1,m-1)\\ &= (m,t_i,m)(m,1,m-1)\end{aligned} \tag{78}$$

$$\begin{aligned}(m,t_i,m) &= (m,t_i,m+1)(m+1,1,m)\\ &= (m,1,m+1)(m+1,t_i,m).\end{aligned} \tag{79}$$

Hence, our Proposition holds. □

Remark. Note that the process we used to create this Proposition can be done for any Bruck-Reilly extension, not just those that are inverse semigroups.

In fact, this Proposition can then be used to find a subset of a Bruck-Reilly extension that will always generate the semigroup.

Theorem 5.1.5. *Let $S = BR(T,\theta)$ be the Bruck-Reilly extension of T determined by θ. If $\{t_i | i \in I\}$ is the generating set of T for some index set I, then $\{(1,1,0),(0,1,1)\} \cup \{(0,t_i,0)| i \in I\}$ is a generating set of S.*

Proof. First, note that by the definition of θ, any such morphism must map the identity element in T to itself. Knowing this, we can say that;

$$(1,1,0)(1,1,0) = (2,1,0) \tag{80}$$

$$(1,1,0)(n,1,0) = (n+1,1,0) \ (\forall n \in \mathbb{N}). \tag{81}$$

Hence, $\forall n \in \mathbb{N}$, $(n,1,0)$ can be generated by $\{(1,1,0),(0,1,1)\}$ in S. Similarly, we can show that $(0,1,n)$ can also be generated by $\{(1,1,0),(0,1,1)\}$ in S for any $n \in \mathbb{N}$.

We can then say that $(m,1,n) \in S$ is generated by $\{(1,1,0),(0,1,1)\}$ $\forall m,n \in \mathbb{N}$ by the equation;

$$(m,1,0)(0,1,n) = (m,1,n). \tag{82}$$

In particular, this means that $(m,1,m+1),(m+1,1,m) \in S$ are generated by $\{(1,1,0),(0,1,1)\}$ $\forall m \in \mathbb{N}^0$.

Now we note the following where t_i is any generator of T and $m \in \mathcal{N}$;

$$(m,1,0)(0,t_i,0)(0,1,m) = (m,t_i,m). \tag{83}$$

So, (m,t_i,m) must be generated by the set $\{(1,1,0),(0,1,1),(0,t_i,0)\}$ when t_i is a generator of T and $m \in \mathbb{N}^0$ since $(m,1,0)$ and $(0,1,m)$ can be generated from the set $\{(1,1,0),(0,1,1)\}$. Hence, for any generator t_i of T and $m \in \mathbb{N}^0$, (m,t_i,m) is generated by $\{(1,1,0),(0,1,1)\} \cup \{(0,t_i,0)|i \in I\}$. Then by Proposition 5.1.4, our Theorem holds. □

We now wish to find the relations of our Bruck-Reilly semigroup. These are given to us by the relations in the semigroup that generates the Bruck-Reilly semigroup. To see how, we first note the following;

Lemma 5.1.6. *Let T be an inverse monoid such that there exists a morphism $\theta : T \to U(T)$. Then, $\{0\} \times T \times \{0\} \subset BR(T,\theta)$ is isomorphic to T.*

Proof. Define a map $\phi : T \to \{0\} \times T \times \{0\}$ to be given by $\phi(t) = (0,t,0)$, $\forall t \in T$. It is immediately obvious that ϕ is a bijection, so we need only check that it is a homomorphism. Take any $t_1, t_2 \in T$, then;

$$\begin{aligned} \phi(t_1)\phi(t_2) &= (0,t_1,0)(0,t_2,0) \\ &= (0,(t_1\theta^0)(t_2\theta^0),0) \\ &= (0,t_1t_2,0) = \phi(t_1t_2). \end{aligned} \tag{84}$$

□

The existence of such an isomorphism then gives us the relations in $BR(T,\theta)$.

Corollary 5.1.7. *Let $R(t_1, t_2, \ldots, t_n)_T$ describe a relation between elements $t_1, t_2, \ldots, t_n \in T$ with respect to the semigroup operation of T. Then the relations of $BR(T, \theta)$ contain $R((0, t_1, 0), (0, t_2, 0), \ldots, (0, t_n, 0))_{BR(T,\theta)}$.*

Remark. What this Corollary is saying is that any relation in T gives rise to a relation in $\{0\} \times T \times \{0\}$ and consequently a relation in $BR(T, \theta)$. For example, if $t_1, t_2, t_3 \in T$ are such that $t_1^2 t_2^{-1} = t_3^3$, then $(0, t_1, 0)^2(0, t_2, 0)^{-1} = (0, t_3, 0)^3$ in $BR(T, \theta)$.

5.2 Bruck-Reilly Action

If we know that there exists an action of the monoid T on a set V, then we can define an action for $BR(T, \theta)$.

Lemma 5.2.1. *Say the set V is a T-act with an action given by $tv = f(t, v)$ ($\forall t \in T$, $v \in V$). The set $\mathbb{N}^0 \times V$ can be considered to be an S-act with an action such that $\forall (m, a, n) \in S$, $(\alpha, v) \in \mathbb{N}^0 \times V$, $(m, a, n)(\alpha, v)$ is defined if and only if $n \leq \alpha$ and $\theta^{\alpha-n}(a)v$ is defined in the T-act. This action is given by;*

$$(m, a, n)(\alpha, v) = (m - n + \alpha, \theta^{\alpha-n}(a)v) \tag{85}$$

Proof. In this proof, we will be defining $t = max\{n, p\}$. Say $\exists (m, a, n)$, $(p, b, q) \in S$ such that $(m, a, n)(p, b, q) = (m - n + t, \theta^{t-n}(a)\theta^{t-p}(b), q - p + t)$ acts on $(\alpha, v) \in \mathbb{N}^0 \times V$. By definition, this means that $q - p + t \leq \alpha$ and $\theta^{\alpha-q+p-t}(\theta^{t-n}(a)\theta^{t-p}(b))v = (\theta^{\alpha-q+p-n}(a)\theta^{\alpha-q}(b))v$ exists. Since $p \leq t$ we can say that $0 \leq -p + t \Rightarrow q \leq q - p + t \leq \alpha$. Also, by the definition of an inverse semigroup action, $(\theta^{\alpha-q+p-n}(a)\theta^{\alpha-q}(b))v$ exists $\Rightarrow \theta^{\alpha-q}(b)v$ exists. Therefore, $(p, b, q)(\alpha, v)$ is defined and equals $(p - q + \alpha, \theta^{\alpha-q}(b)v)$. Furthermore, $q - p + t \leq \alpha \Rightarrow n \leq t \leq p - q + \alpha$ and $(\theta^{\alpha-q+p-n}(a)\theta^{\alpha-q}(b))v$ exists $\Rightarrow \theta^{\alpha-q+p-n}(a)(\theta^{\alpha-q}(b)v)$ exists. So, $(m, a, n)((p, b, q)(\alpha, v))$ is defined and equals $(m - n + p - q + \alpha, \theta^{\alpha-q+p-n}(a)(\theta^{\alpha-q}(b)v))$.

Conversely, say (m, a, n) and (p, b, q) are such that $(m, a, n)((p, b, q)(\alpha, v))$ is defined. So, since $(p, b, q)(\alpha, v) = (p - q + \alpha, \theta^{\alpha-q}(b)v)$, we can say that $q \leq \alpha$, $n \leq p - q + \alpha$, $\theta^{\alpha-q}(b)v$ exists and $\theta^{p-q+\alpha-n}(a)(\theta^{\alpha-q}(b)v)$ exists. If $t = p$, then $q - p + t = q \leq \alpha$. Alternitively, if $t = n$, then $q - p + t = q - p + n$. Since $n \leq p - q + \alpha$, $q - p + t \leq q - p + p - q + \alpha = \alpha$. Therefore, $q - p + t \leq \alpha$ is true regardless of the value of t.

By defintion of an inverse semigroup action $(\theta^{p-q+\alpha-n}(a)\theta^{\alpha-q}(b))v$ exists and is equal to $\theta^{p-q+\alpha-n}(a)(\theta^{\alpha-q}(b)v)$. The properties of a morphism also tell us that $\theta^{p-q+\alpha-n}(a)\theta^{\alpha-q}(b) = \theta^{\alpha-q+p-t}(\theta^{t-n}(a)\theta^{t-p}(b))$ and so $\theta^{\alpha-q+p-t}(\theta^{t-n}(a)\theta^{t-p}(b))v = \theta^{p-q+\alpha-n}(a)(\theta^{\alpha-q}(b)v)$. Note that $q \leq \alpha$ and $n \leq p - q + \alpha$ tell us that these morphisms are always defined (which is to say their index is always a value in $\mathbb{N}^0$). From this, we can say that $((m, a, n)(p, b, q))(\alpha, v)$ exists.

Finally, note that;

$$\begin{aligned}(m,a,n)((p,b,q)(\alpha,v)) &= (m-n+p-q+\alpha, \theta^{\alpha-q+p-n}(a)(\theta^{\alpha-q}(b)v)) \\ &= (m-n+p-q+\alpha, \theta^{\alpha-q+p-t}(\theta^{t-n}(a)\theta^{t-p}(b))v) \\ &= (m-n+t, \theta^{t-n}(a)\theta^{t-p}(b), q-p+t)(\alpha,v) \\ &= ((m,a,n)(p,b,q))(\alpha,v).\end{aligned} \tag{86}$$

□

Now that we have defined our action, we can find some properties of it. As stated when we defiend the action, $(m,a,n) \in S$ acts on $(\alpha,v) \in \mathbb{N}^0 \times V$ if and only if $n \leq \alpha$ and $\theta^{\alpha-n}(a)v$ exists. Therefore, $\forall(\alpha,v) \in \mathbb{N}^0 \times V$;

$$D_S^{(\alpha,v)} = \{(m,a,n) \in S | n \leq \alpha, \theta^{\alpha-n}(a) \in D_T^v\}. \tag{87}$$

Lemma 5.2.2. *$\forall(\alpha,v) \in \mathbb{N}^0 \times V$;*

$$S^S_{(\alpha,v)} = \{(n,a,n) \in S | n \leq \alpha, \theta^{\alpha-n}(a) \in S_v^T\}. \tag{88}$$

Proof. Let $(m,a,n) \in S^S_{(\alpha,v)}$. Since $S^S_{(\alpha,v)} \subseteq D_S^{(\alpha,v)}$ we can say that $n \leq \alpha$ and $\theta^{\alpha-n}(a) \in D_T^v$. By definition, $(\alpha,v) = (m,a,n)(\alpha,v) = (m-n+\alpha, \theta^{\alpha-n}(a)v)$. Then, $\alpha = m-n+\alpha \Rightarrow m = n$ and $v = \theta^{\alpha-n}(a)v \Rightarrow \theta^{\alpha-n}(a) \in S_v^T$. Therefore, $S^S_{(\alpha,v)} \subseteq \{(n,a,n) \in S | n \leq \alpha, \theta^{\alpha-n}(a) \in S_v^T\}$.

If $(n,a,n) \in S$ is such that $n \leq \alpha$ and $\theta^{\alpha-n}(a) \in S_v^T \subseteq D_T^v$ then $(n,a,n)(\alpha,v)$ is defined. Furthermore, $(n,a,n)(\alpha,v) = (n-n+\alpha, \theta^{\alpha-n}(a)v) = (\alpha,v)$. Hence, $S^S_{(\alpha,v)} = \{(n,a,n) \in S | n \leq \alpha, \theta^{\alpha-n}(a) \in S_v^T\}$. □

Lemma 5.2.3. *$\forall(\alpha,v) \in \mathbb{N}^0 \times V$;*

$$S^1(\alpha,v) = \{(\beta,u) \in \mathbb{N}^0 \times V | u \in T^1 v\} \tag{89}$$

where $T^1 v$ is the stabilizer of v under the T-act.

Proof. Let $(\beta,u) \in S^1(\alpha,v)$. So, $\exists(m,a,n) \in S$ such that $(\alpha,v) = (m,a,n)(\beta,u)$ (note that since S is a monoid, $S^1 = S$). Therefore, $(\alpha,v) = (m-n+\beta, \theta^{\beta-n}(a)u)$ Since $v = \theta^{\beta-n}(a)u \Rightarrow (\theta^{\beta-n}(a))^{-1}v = u$ we can say that $u \in T^1 v$ and hence $S^1(\alpha,v) \subseteq \{(\beta,u) \in \mathbb{N}^0 \times V | u \in T^1 v\}$.

Conversely, say $(\beta,u) \in \mathbb{N}^0 \times V$ is such that $u \in T^1 v$. Then, $\exists x \in T$ such that $xv = u$ (again since T is a monoid, $T^1 = T$). Then;

$$\begin{aligned}(\beta,x,\alpha)(\alpha,v) &= (\beta-\alpha+\alpha, \theta^{\alpha-\alpha}(x)v) \\ &= (\beta,xv) = (\beta,u)\end{aligned} \tag{90}$$

and so, by definition, $(\beta,u) \in S^1(\alpha,v)$ since $(\beta,x,\alpha) \in S$. □

Corollary 5.2.4. *$\forall\alpha,\beta \in \mathbb{N}^0$, $v_1,v_2 \in V$;*

$$S^1(\alpha,v_1) = S^1(\beta,v_2) \Leftrightarrow T^1 v_1 = T^1 v_2. \tag{91}$$

Proof. Follows from Lemma 5.2.3. □

5.3 Action on a Graph

Say that we had an action of T on a tree G_T such that we not only get back a fundamental inverse semigroup but said semigroup is isomorphic to T. In other words, in this system we get a quotient graph, Q_T, which then gives us a T-transversal, A_T, which then gave us a fundamental inverse semigroup T' such that $T' \simeq T$. Furthermore, let E_T be the set of base edges that generate $E(G_T)$ under the T-act. Knowing this, it is possible to create a graph that $S = BR(T, \theta)$ will act on such that the fundamental inverse semigroup is isomorphic to S provided T and θ have additional properties. What these properties are and why we require them will be discussed later.

First we define a graph G_S in the following way. Let $V(G_S) = \mathbb{N}^0 \times V(G_T)$. For all $u \in V(A_T)$, define an edge $f_u \in E(G_S)$ by $\iota(f_u) = (1, u)$ and $\tau(f_u) = (0, u)$. We define the set of base edges of G_S (with respect to the S-act) to be given by $\{(0, e)|e \in E_T\} \cup \{f_u\}$ where the edge $(0, e)$ is defined by $\iota((0, e)) = (0, \iota(e))$ and $\tau((0, e)) = (0, \tau(e))$. It is then the case that every edge in G_S that is not in the orbit of an edge of the form f_u is given by $(\alpha, e) \in \mathbb{N}^0 \times E(G_T)$ where $\iota((\alpha, e)) = (\alpha, \iota(e))$ and $\tau((\alpha, e)) = (\alpha, \tau(e))$. This structure means that G_S can be thought of as an infinite number of graphs that are isomorphic to G_T with neighboring layers connected edges that exist in the same orbit as edges of the form f_u.

Lemma 5.3.1. *G_S is a connected graph.*

Proof. Let $(\alpha, v) \in \mathbb{N}^0 \times V$ and $u \in V(A_T)$. By definition of G_T, there exists a set, $P \subseteq E(G_T)$, that defines a path connecting v and u. Therefore, in G_S, there exists a path connecting $(0, u)$ to $(0, v)$ given by the set $P_0 = \{(0, e) \in E(G_S)|e \in P\}$. Similarly, there is a path between (α, u) and (α, v) given by $P_\alpha = (\alpha, 1, 0)P_0$.

Note that $\iota((1, 1, 0)^n f_u) = (1, 1, 0)^n \iota(f_u) = (n+1, u)$ and $\tau((1, 1, 0)^n f_u) = (1, 1, 0)^n \tau(f_u) = (n, u)$ and so $\iota((1, 1, 0)^n f_u) = \tau((1, 1, 0)^{n+1} f_u)$ $(\forall n \in \mathbb{N}^0)$. Therefore, there is a path in G_S connecting $(0, u)$ and (α, u) meaning that (α, v) is connected to $(0, u)$. Since this is true for any element in $\mathbb{N}^0 \times V$ we can say that G_S is connected. $\square$

It is possible that G_S will contain a loop. However, this does not stop us from using our method since the quotient graph of this system will be a tree no matter the properties of S as we will show later. We can still identify the circumstances under which G_S contains no loops.

Lemma 5.3.2. *G_S contains no loops $\Leftrightarrow T^1 u = \{u\}$ and $V(A_T) = \{u\}$ (for some $u \in V$).*

Proof. Say $T^1 u \neq \{u\}$ and hence $\exists v \in T^1 u$ such that $v \neq u$. So, $\exists t \in T$ such that $v = tu$ (where $t \neq 1$ since this will contradict $v \neq u$). Given that G_T is a tree there must exist a set of edges $P \subseteq E(G_T)$ that define the unique path going from u to v. Therefore, the sets $P_0 = \{(0,e) \in E(G_S) | e \in P\}$ and the set $P_1 = \{(1,e) \in E(G_S) | e \in P\}$ define paths from $(0,u)$ and $(1,u)$ to $(0,v)$ and $(1,v)$ respectively. Also, we know that the edge $(0,t,0)f_u$ is in $E(G_S)$ (by definition of a base edge) where $\iota((0,t,0)f_u) = (1,v)$ and $\tau((0,t,0)f_u) = (0,v)$. This gives us two paths in G_S from $(0,u)$ to $(1,v)$: going against the direction of f_u and following the path given by P_1 or following the path P_0 and going against the direction of $(0,t,0)f_u$. These paths share no edges and hence there is a loop in G_S.

Conversely, say $T^1u = \{u\}$ and $V(A_T) = \{u\}$. Then, $S^1(0,u) = S^1(1,u) = \{(\alpha,u) | \alpha \in \mathbb{N}^0\}$ (by Lemma 5.2.3) and so $S^1 f_u = \{f \in E(G_S) | \iota(f) = (n+1,u), \tau(f) = (n,u) (\forall n \in \mathbb{N}^0)\}$. Since G_T contains no loops, any loop in G_S must contain an edge in the orbit of f_u. Given the elements in $S^1 f_u$, there is only one path between the different subgraphs of G_S that are isomorphic to G_T. Therefore, we are unable to create a loop in G_S. $\square$

Note that this condition on G_T does not mean that it has a single vertex, only that the G_T-transversal we obtain from the T-act will have a single vertex. We have seen previously in this paper examples of graphs with multiple vertices whose transversal has just a siungle vertex like in Example 2.1.8.

Now we can examine the quotient graph we will obtain when S acts on $\mathbb{N}^0 \times V$. Let Q_S the quotient graph obtained when S acts on G_S.

Lemma 5.3.3. *$V(Q_S) = \{S^1(0,v) | v \in V(Q_T)\}$ and $E(Q_S) = \{S^1(0,e) | e \in E'\} \cup \{S^1 f_u | u \in V(A_T)\}$ where $\iota(S^1(0,e)) = S^1\iota((0,e))$ and $\tau(S^1(0,e)) = S^1\tau((0,e))$.*

Proof. By Lemma 5.2.3 we know that $\forall v \in V(G_T), \alpha \in \mathbb{N}^0$, $(\alpha,v) \in S^1(0,v)$. Therefore, the orbits of elements in the S-act is determined by the orbits of elements in the T-act. In other words, $\forall \alpha,\beta \in \mathbb{N}^0$, $v_1,v_2 \in V$, $S^1(\alpha,v_1) = S^1(\beta,v_2) \Leftrightarrow T^1v_1 = T^1v_2$. So, $V(Q_S) = \{S^1(0,v) | v \in V(Q_T)\}$. Note that a consequence of this is that the number of vertices in Q_S is equal to the number of vertices in Q_T. Furthermore, by definition of the base edges of G_S we know that $E(Q_S)$ is as described in the Lemma. $\square$

Remark. Notice that if we did not include the edges $S^1 f_u$ in Q_S then Corollary 5.2.4 tells us $Q_S \simeq Q_T$.

From Q_S we can define an S-transversal. We define a graph A_S by saying $V(A_S) = \{(1,v) | v \in V(A_T)\}$ and $E(A_S) = \{(1,e) | e \in E(A_T)\} \cup \{f_u | u \in V(A_T)\}$.

Lemma 5.3.4. *A_S is an S-transversal.*

Proof. Since every vertex and edge that isn't of the form f_u corresponds to a vertex or edge in A_T it must be the case that A_S is connected since A_T

is connected. Similarly, since A_T contains no loops, a loop in A_S is only possible if said loop includes an edge f_u for some $u \in V(A_T)$. However, $(\forall u \in V(A_T))\ \tau(f_u) \notin V(A_S)$ and so no such loop can exist.

Now assume $\exists (1, v_1), (1, v_2) \in V(A_S)$ such that these vertices exist in the same orbit. In other words, $S^1(1, v_1) = S^1(1, v_2)$. This implies that $T^1 v_1 = T^1 v_2$ (Corollary 5.2.4). However, $v_1, v_2 \in V(A_T)$ and therefore can not exist in the same orbit with respect to the T-act. Therefore, all vertices in A_S exist in seperate orbits. Consequently every edge in A_S exist in their own orbit as well. Also note that $\forall (\alpha, v) \in \mathbb{N}^0 \times V$, $\exists v_A \in V(A_T)$ such that $v_A \in T^1 v$. Therefore, $(1, v_A) \in S^1(\alpha, v)$ is an element in $V(A_S)$. Similarly, we can show that every edge orbit in our S-act has an element in $E(A_S)$. S0, A_S contains exactly one element in every vertex and edge orbit.

Finally, let $(1, e) \in E(A_S)$. By definition, $e \in E(A_T)$ and hence $\iota(e) \in V(A_T)$ since A_T is a T-transversal. So, $(1, \iota(e)) = \iota((1, e)) \in V(A_S)$. Similarly, $\iota(f_u) = (1, u) \in V(A_S)$ $(\forall u \in V(A_T))$. Therefore, the intial vertex of every edge in $E(A_S)$ exists in $V(A_S)$. $\square$

Now that we have an S-transversal, we need to examine it to see if it will give us a graph of inverse semigroups. This is where the restriction on T and θ that was mentioned at the start of this section becomes important. We add the restriction that θ preserves the T-act. In other words, $tv = u$ for some $t \in T$, $v, u \in V \Rightarrow \theta(t)v = u$.

Lemma 5.3.5. *$\forall \epsilon \in E(A_S)$ such that $\tau(\epsilon) \notin V(A_S)$, $\exists s_\epsilon \in S$ such that $s_\epsilon s_\epsilon^{-1} = Id(S_\epsilon)$ and $s_\epsilon \iota(\epsilon) = \tau(\epsilon)$.*

Proof. Recall that since $(0, 1, 0) = Id(S)$, $(0, 1, 0)$ exists in every domain and hence is the identity in every stabilizer. Let $\epsilon \in E(A_S)$ be such that $\tau(\epsilon) \notin V(A_S)$. If $\epsilon = f_u$ for some $u \in V(A_T)$, then $\iota(f_u) = (1, u)$ and $\tau(f_u) = (0, u)$. Then, $(0, 1, 1)(1, u) = (0, u)$ and $(0, 1, 1)(1, 1, 0) = (0, 1, 0) = Id(S_{f_u})$. Hence $s_\epsilon = (0, 1, 1)$ satisfies our conditions $\forall u \in V(A_T)$.

If ϵ is not of this form, then $\epsilon = (1, e)$ for some $e \in E(A_T)$. Furthermore, $\tau(\epsilon) \notin V(A_S) \Rightarrow \tau(e) \notin V(A_T)$. Since we know that we can create a graph of inverse semigroups from the T-transversal A_T it must be the case that $\exists t_e \in T$ such that $t_e t_e^{-1} = Id(S_e)$ and $t_e \iota(e) = \tau(e)$. Since $Id(S_e) = 1$, we can say that $(0, t_e, 0)(0, t_e, 0)^{-1} = (0, t_e, 0)(0, t_e^{-1}, 0) = (0, 1, 0)$. Also, $(0, t_e, 0)\iota(\epsilon) = (0, t_e, 0)(1, \iota(e)) = (1, \theta(t_e)\iota(e))$. Due to the aforementioned restritions on T and θ, $\theta(t_e)\iota(e) = t_e \iota(e) = \tau(e)$. So, $s_\epsilon = (0, t_e, 0)$ satisfies our required conditions for s_ϵ. $\square$

Remark. Without the additional properties of T and θ we would have been unable to say that $(0, t_e, 0)(1, \iota(e)) = (1, \tau(e))$. If, however, it can be shown that $\exists t \in T$ such that $tt^{-1} = 1$ and $\theta(t)\iota(e) = \tau(e)$ then such a restriction is not required to create a graph of groups.

A consequence of this restriction on the T-act comes from the fact that θ maps elements in T to H_1^T. Therefore, because θ now preserves our actions, $\theta(tt^{-1}) = \theta(t^{-1}t) = 1 \Rightarrow tt^{-1}$ and $t^{-1}t$ act on and preserve any element in V $(\forall t \in T)$.

Other restrictions are possible that would make $(0, t_e, 0)$ satisfy our conditions. For example, saying $\forall t \in H_1^T, \theta(t) = t$ and $R_1^T \subseteq H_1^T$ was considered, but this would not give us the desired set of generators for the fundamental inverse semigroup.

5.4 Fundamental Inverse Semigroup of the Bruck-Reilly Action

Now that we have a graph of inverse semigroups, we can find the fundamental inverse semigroups of this system, S'. As with the previous section, we will find the need to add another property to T in order to get back the desired result. Again, we will introduce this property when it is needed. First, we begin by looking at the generators we obtain from the verticies of the graph of inverse semigroups.

Lemma 5.4.1. *Let $T_{\theta,v} = \{t \in T | \theta(t) \in S_v^T\}$. For all $(1, v) \in V(A_S)$, the set of elements in the generating set of S' that come from $S_{(1,v)}^S$ is given by $\Gamma_{(1,v)} = \{\gamma_{0,t_0}, \gamma_{1,t_1} | t_0 \in T_{\theta,v}, t_1 \in S_v^T\}$ with the relations $\gamma_{0,t}\gamma_{0,t'} = \gamma_{0,tt'}$, $\gamma_{1,t}\gamma_{0,t'} = \gamma_{0,t}\gamma_{1,t'} = \gamma_{1,tt'}$ and $\gamma_{1,t}\gamma_{1,t'} = \gamma_{1,tt'}$.*

Proof. From Lemma 5.2.2 we know that;

$$S_{(1,v)}^S = \{(1, t, 1) | t \in S_v^T\} \cup \{(0, t, 0) | \theta(t) \in S_v^T\}. \tag{92}$$

If we set $\gamma_{1,t} := (1, t, 1)$ and $\gamma_{0,t} := (0, t, 0)$ then this satisfies $\Gamma_{(1,v)}$ being the set of elements in the generating set of S' that come from $S_{(1,v)}^S$. The relations defined in the Lemma then follow. $\square$

Remark. Given these relations, we are able to say that $\gamma_{0,t}^{-1} = \gamma_{0,t^{-1}}$ and $\gamma_{1,t}^{-1} = \gamma_{1,t^{-1}}$.

Corollary 5.4.2. *The set of elements in the intial generating set of S' is given by the set $\Gamma := \bigcup_{v \in V(A_T)} \Gamma_{(1,v)}$ with the relations $\gamma_{0,t}\gamma_{0,t'} = \gamma_{0,tt'}$, $\gamma_{1,t}\gamma_{0,t'} = \gamma_{0,t}\gamma_{1,t'} = \gamma_{1,tt'}$ and $\gamma_{1,t}\gamma_{1,t'} = \gamma_{1,tt'}$ $(\forall \gamma_{0,t}, \gamma_{0,t'}, \gamma_{1,t}, \gamma_{1,t'} \in \Gamma)$.*

Proof. Follows from Lemma 5.4.1. $\square$

Remark. As stated previously, since 1 is the identity in T then $(0, 1, 0)$ is the identity in S and consequently exists in the set of stabilizers of every vertex and edge in G_S. Hence, Γ contains an identity element given by $\gamma_{0,1}$.

Lemma 5.4.3. *$\gamma_{1,t}$ exists $\Rightarrow \gamma_{0,t}$ exists.*

Proof. $\gamma_{1,t}$ exists $\Rightarrow t \in S^T_v$ for some $v \in V(A_T)$. Hence, $\theta(t) \in S^T_v \Rightarrow \gamma_{0,t}$ exists. □

All other generators are the elements $s_\epsilon \in S$ that are defined in Lemma 5.3.5. Let E'_X be the set of edges $\epsilon \in E(A_X)$ such that $\tau(\epsilon) \notin V(A_X)$ where $X = S$ or T. Then the remaining elements in the initial generating set of S' are given by $\mathcal{E} := \{s_\epsilon \in S | \epsilon \in E'_S, s_\epsilon s_\epsilon^{-1} = (0,1,0), s_\epsilon \iota(\epsilon) = \tau(\epsilon)\}$. However, it will be beneficial to break down this set into two disjoint sets. In the following Lemma we will be using the term t_e as it was defined in the proof of Lemma 5.3.5.

Lemma 5.4.4. $\mathcal{E} = \mathcal{E}_0 \cup \{f\}$ *where* $\mathcal{E}_0 := \{\delta_{t_e} | e \in E'_T\}$. *Furthermore, these give us the following relations in* S'*:*

- $\forall \delta_{t_e} \in \mathcal{E},\ \delta_{t_e}\delta_{t_e}^{-1} = ff^{-1} = \gamma_{0,1}$,
- $\forall \gamma_{1,t} \in \Gamma,\ f^{-1}\gamma_{0,t}f = \gamma_{1,t}$,
- $\forall \delta_{t_e} \in \mathcal{E},\ \gamma_{0,t} \in \Gamma_{(1,\iota(e))},\ \delta_{t_e}^{-1}\gamma_{0,t}\delta_{t_e} = \gamma_{0,t_e^{-1}tt_e}$,
- $\forall \delta_{t_e} \in \mathcal{E},\ \gamma_{0,t} \in \Gamma_{(1,\iota(e))},\ \delta_{t_e}^{-1}\gamma_{1,t}\delta_{t_e} = \gamma_{1,\theta(t_e^{-1})t\theta(t_e)}$.

Proof. From the proof of Lemma 5.3.5 we know what the remaining elements in the initial generating set of S' are since they all are the elements we use to define the required embeddings. First, $\forall (1,e) \in E'_S$ we have an element $(0, t_e, 0) \in S$ to give us an embedding from $S_{(1,e)}$ into $S_{\iota((1,e))}$. We label such elements δ_{t_e}. If we take any δ_{t_e} then the first relation that we would add to S' is that $\delta_{t_e}\delta_{t_e}^{-1}$ is the identiy of S. In other words, $\delta_{t_e}\delta_{t_e}^{-1} = \gamma_{0,1}$.

The other relations that we get from δ_{t_e} are found from finding what $\delta_{t_e}^{-1}\gamma_{0,t}\delta_{t_e}$ and $\delta_{t_e}^{-1}\gamma_{1,t}\delta_{t_e}$ equal in S ($\forall \gamma_{0,t}, \gamma_{1,t} \in S^S_{(1,e)}$).

$$\delta_{t_e}^{-1}\gamma_{1,t}\delta_{t_e} = (0,t_e,0)(1,t,1)(0,t_e,0) = (1, \theta(t_e^{-1})t\theta(t_e), 1) \tag{93}$$

Furthermore, our definition of t_e means that $t_e^{-1}tt_e \in S^T_{\iota(e)} \Rightarrow \theta(t_e^{-1})t\theta(t_e) \in S^T_{\iota(e)}$ and hence $\gamma_{1,\theta(t_e^{-1})t\theta(t_e)}$ is defined. Therefore, we can say that $\delta_{t_e}^{-1}\gamma_{1,t}\delta_{t_e} = \gamma_{1,\theta(t_e^{-1})t\theta(t_e)}$. Similarly, it can be show that $\delta_{t_e}^{-1}\gamma_{0,t}\delta_{t_e} = \gamma_{0,t_e^{-1}tt_e}$ using this method.

By Lemma 5.3.5, the only other element required for our embeddings is $(0,1,1)$. If we label this element f, then we will immdiately have the relation $ff^{-1} = \gamma_{0,1}$ and so it only remains to define what values $f^{-1}\gamma f$ equal $\forall \gamma \in S^S_{f_u}, u \in V(A_T)$. Given that $S^S_{f_u} = S^S_{\iota(f_u)} \cup S^S_{\tau(f_u)}$ and Lemma 5.2.2 we can say that we need only add the relation for what $f^{-1}\gamma_{0,t}f$ equals ($\forall \gamma_{0,t} \in \Gamma$).

$$f^{-1}\gamma_{0,t}f = (1,1,0)(0,t,0)(0,1,1) = (1,t,1) = \gamma_{1,t}, \tag{94}$$

provided $\gamma_{1,t}$ is defined. By Lemma 5.4.3, $\gamma_{1,t}$ being defined implies $\gamma_{0,t}$ is also defined and so we can say this relation exists $\forall \gamma_{1,t} \in \Gamma$. □

Corollary 5.4.5. *$\forall \gamma_{0,t} \in S'$, $\gamma_{0,t}f^{-1} = f^{-1}\gamma_{0,\theta(t)}$ and $f\gamma_{0,t} = \gamma_{0,\theta(t)}f$.*

Proof. Let $\gamma_{0,t} \in \Gamma_0$. Knowing this and the fact that $\gamma_{1,1}$ is also in the intial generating set of S' we can say that $\gamma_{0,t}\gamma_{1,1} = \gamma_{1,\theta(t)}$. Since $\gamma_{1,\theta(t)} = f^{-1}\gamma_{0,\theta(t)}f$, $\gamma_{0,t}\gamma_{1,1} = \gamma_{1,\theta(t)} \Rightarrow \gamma_{0,t}f^{-1}\gamma_{0,1}f = f^{-1}\gamma_{0,\theta(t)}f$. Given that $\gamma_{0,1}$ is the identity in S' this equation can be simplified to get $\gamma_{0,t}f^{-1}f = f^{-1}\gamma_{0,\theta(t)}f \Rightarrow \gamma_{0,t}f^{-1} = f^{-1}\gamma_{0,\theta(t)}$ (since $ff^{-1} = \gamma_{0,1}$). Similarly, it can be shown that $\gamma_{1,1}\gamma_{0,t} = \gamma_{1,\theta(t)} \Rightarrow f\gamma_{0,t} = \gamma_{0,\theta(t)}f$. □

Now that we have an intial set of generators and relation of S', we can begin to simplify it. First, $\forall v \in V(A_T)$, we can define a set $\Gamma_{(0,v)} := \{\gamma_{0,t} | t \in S_v^T\}$. From this definition, it is clear that $\Gamma_{(0,v)} \subseteq \Gamma_{(1,v)}$. Therefore, if we define another set $\Gamma_0 := \bigcup_{v \in V(A_T)} \Gamma_{(0,v)}$ it is clear that $\Gamma_0 \subseteq \Gamma$.

Lemma 5.4.6. *$T \simeq Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$.*

Proof. Consider the intial set of generators and relations that we obtain from A_T to define T'. Label this generating set G_T and let R_T be the set of relations. By definition, every $\gamma_{0,t} \in \Gamma_0$ corresponds to some $(0, t, 0) \in S$ such that $t \in S_v^T$ for some $v \in V(A_T)$. Therefore, when finding the initial set of genrators of T' we would have equated t to some element $g_t \in G_T$. Similarly, $\forall \delta_t \in \mathcal{E}_0$, δ_t corresponds to some $(0, t, 0) \in S$ that defines an embedding of an edge $\epsilon \in E(A_S)$ such that $\tau(\epsilon) \notin V(A_S)$ and $\epsilon \neq f_u$ (for some $u \in V(A_T)$). By Lemma 5.3.5, we know that ϵ is defined by an edge $e \in E(A_T)$ such that $\tau(e) \notin V(A_T)$ and $t \in T$ satisfies the properties we have set to define the necessary embedding. Therefore, t would be equated to some $\bar{g}_t \in G_T$.

Knowing this, we can create a morphism $\rho : Inv\langle \Gamma_0, \mathcal{E}_0 \rangle \to T'$ given by $\rho(\gamma_{0,t}) = g_t$ and $\rho(\delta_t) = \bar{g}_t$ ($\forall \gamma_{0,t} \in \Gamma_0$, $\delta_t \in \mathcal{E}_0$). Given Corollary 5.1.7 we know that any relation between elements in $Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$ will be preserved by ρ. Also, the definition of how we define the set of relations of the fundmental inverse semigroup tells us that any relation that exists in T' must also exist between the same elements when mapped to $Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$ by ρ^{-1}. Therefore, we can say that ρ is an isomorphism and hence $Inv\langle \Gamma_0, \mathcal{E}_0 \rangle \simeq T'$. By definition, $T' \simeq T$ and hence $T \simeq Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$. □

Corollary 5.4.7. *$\forall \gamma_{0,t} \in S'$ such that $\gamma_{0,t} \notin \Gamma_0$, $\gamma_{0,t}$ can be expressed as a product of elements in $Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$.*

Proof. Let $\gamma_{0,t} \in S'$ be such that $\gamma_{0,t} \notin \Gamma_0$. By definition, $t \in T$ and is therefore isomorphic to an element in T', say t'. However, $\gamma_{0,t} \notin \Gamma_0$ implies that t' is not one of the initial generators of T'. Therefore, the relations in T' allow us to write t' as a product of elements in the generating set of T'. Since $T' \simeq Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$ (by Lemma 5.4.6) these same relations exist in $Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$. Hence, $\gamma_{0,t}$ can be expressed as a product of elements in $Inv\langle \Gamma_0, \mathcal{E}_0 \rangle$. □

Using this Corollary we can simplify the generating set of S' by removing elements of the form $\gamma_{0,t}$ that do not exist in Γ_0. Furthermore, given Lemma 5.4.6 and the original values of our elements in S', we wish to show that $\forall w \in T$ we can define a $\gamma_{0,w} \in S'$ such that all our relations relating to elements of this form still hold. Say $\gamma_{0,w} := \rho^{-1}(w')$ where ρ is the same as in the proof of Lemma 5.4.6 and w' is the unique element in T' that is equated to w under the isomorphism between T and T'. Before we continue, $\gamma_{0,w}$ is already defined for some $w \in T$. Therefore, we neeed to check this definition does not contradict how we have defined these elements previously.

Lemma 5.4.8. *$\forall \gamma_{0,w} \in S'$, $\gamma_{0,w} := \rho^{-1}(w')$.*

Proof. Say $\gamma_{0,w} \in S'$. Then $w \in T$ can be written as $w = t_1^{n_1} t_2^{n_2} \cdots t_m^{n_m}$ where $t_1, t_2, \ldots, t_m$ are generators of T and $n_1, n_2, \ldots, n_m \in \mathbb{Z}$. Given $T \simeq T'$, T' has a generating set isomorphic to the generating set of T. In other words, the generating set includes elements $t'_1, t'_2 \ldots, t'_m$ such that $t_1 \simeq t'_1, t_2 \simeq t'_2, \ldots, t_m \simeq t'_m$. Therefore, w is isomorphic to the element $w' = (t'_1)^{n_1}(t'_2)^{n_2} \cdots (t'_m)^{n_m}$ in T'. Hence, $\rho^{-1}(w') = \gamma_{0,t_1}^{n_1} \gamma_{0,t_2}^{n_2} \cdots \gamma_{0,t_m}^{n_m}$. Given that $\gamma_{0,t}\gamma_{0,\bar{t}} = \gamma_{0,t\bar{t}}$ ($\forall \gamma_{0,t}, \gamma_{0,\bar{t}} \in S'$) we can say that $\rho^{-1}(w') = \gamma_{0,t_1^{n_1} t_2^{n_2} \cdots t_m^{n_m}} = \gamma_{0,w}$. □

It now remains to show that our relations hold when we set $\gamma_{0,w} := \rho^{-1}(w')$ ($\forall w \in T$). To do this we need only show that those elements given by $\delta_{t_e} := \gamma_{0,t_e}$ ($\forall \delta_{t_e} \in \mathcal{E}_0$) satisfy the relations since (together with Γ_0) these will generate all other new values of $\gamma_{0,t}$.

Lemma 5.4.9. *$\forall \gamma_{0,t} \in \Gamma_0, \delta_{t_e} \in \mathcal{E}_0$, $\gamma_{0,t}\delta_{t_e} = \gamma_{0,tt_e}$ and $\delta_{t_e}\gamma_{0,t} = \gamma_{0,t_e t}$.*

Proof. $\forall \gamma_{0,t} \in \Gamma_0, \delta_{t_e} \in \mathcal{E}_0$, $\gamma_{0,t}\delta_{t_e} = \rho^{-1}(t')\rho^{-1}(t'_e)$ for some t' and t'_e that exist in the initial generating set of T'. So, $\gamma_{0,t}\delta_{t_e} = \rho^{-1}(t't'_e) = \gamma_{0,tt_e}$ (given that $T \simeq T'$ we know that $t't'_e = (tt_e)')$. Similarly, it can be shown that $\delta_{t_e}\gamma_{0,t} = \gamma_{0,t_e t}$. □

Before we continue proving these relations we wish to also say that $\forall w \in T$, $\gamma_{1,t} := f^{-1}\gamma_{0,t}f$. Given Lemma 5.4.4 we know that this defintion corresponds with what we know about elements of the form $\gamma_{1,t}$ that we have defined previously. Again, to prove that the relations relating to elements of this form still hold, we need only check that they hold for $\gamma_{1,t_e} = f^{-1}\delta_{t_e}f$ ($\forall \delta_{t_e} \in \mathcal{E}_0$).

Lemma 5.4.10. *$\forall \gamma_{1,t} \in S', \delta_{t_e}, \delta_{t_{e'}} \in \mathcal{E}_0$, $\gamma_{1,t}\gamma_{1,t_e} = \gamma_{1,tt_e}$, $\gamma_{1,t_e}\gamma_{1,t} = \gamma_{1,t_e t}$ and $\gamma_{1,t_e}\gamma_{1,t_{e'}} = \gamma_{1,t_e t_{e'}}$.*

Proof. $\forall \gamma_{1,t} \in \Gamma_0, \delta_{t_e} \in \mathcal{E}_0$;

$$\begin{aligned} \gamma_{1,t}\gamma_{1,t_e} &= (f^{-1}\gamma_{0,t}f)(f^{-1}\gamma_{0,t_e}f) = (f^{-1}\gamma_{0,t})(ff^{-1})(\gamma_{0,t_e}f) \\ &= (f^{-1}\gamma_{0,t})\gamma_{0,1}(\gamma_{0,t_e}f) = f^{-1}\gamma_{0,tt_e}f = \gamma_{1,tt_e}. \end{aligned} \tag{95}$$

Similarly, it can be shown that $\gamma_{1,t_e}\gamma_{1,t} = \gamma_{1,t_e t}$ and $(\forall \delta_{t_{e'}} \in \mathcal{E}_0)$ $\gamma_{1,t_e}\gamma_{1,t_{e'}} = \gamma_{1,t_e t_{e'}}$. □

Lemma 5.4.11. *$\forall \gamma_{0,t} \in \Gamma_0, \delta_{t_e} \in \mathcal{E}_0$, $\gamma_{0,t}\gamma_{1,t_e} = \gamma_{1,\theta(t)t_e}$ and $\gamma_{1,t_e}\gamma_{0,t} = \gamma_{1,t_e\theta(t)}$.*

Proof. Let $\gamma_{0,t} \in \Gamma_0$ and $\delta_{t_e} \in \mathcal{E}_0$. By definition, $\gamma_{0,t}\gamma_{1,t_e} = \gamma_{0,t}f^{-1}\gamma_{0,t}f$. Given that $\gamma_{0,t} \in \Gamma_0$ we can use Corollary 5.4.5 to say that $\gamma_{0,t}f^{-1} = f^{-1}\gamma_{0,\theta(t)}$. Hence, $\gamma_{0,t}\gamma_{1,t_e} = f^{-1}\gamma_{0,\theta(t)}\gamma_{0,t}f = \gamma_{1,\theta(t)t_e}$. A similar method shows that $\gamma_{1,t_e}\gamma_{0,t} = \gamma_{1,t_e\theta(t)}$. □

At this point, we would wish to show that $\forall \gamma_{1,t} \in S', \delta_{t_e} \in \mathcal{E}_0$, $\gamma_{1,t}\gamma_{0,t_e} = \gamma_{1,t\theta(t_e)}$ and $\gamma_{0,t_e}\gamma_{1,t} = \gamma_{1,\theta(t_e)t}$. Though we have been unable to prove this, there are some properties of S' that suggest this may be possible. For example, $\forall \delta_{t_e} \in \mathcal{E}$, $t_e^{-1}t_e$, $\theta(t_e)t_e^{-1}$, $t_e^{-1}\theta(t_e) \in S^T_{\iota(e)}$ (since θ preserves the action). Therefore, $\gamma_{1,t_e^{-1}t_e}$, $\gamma_{1,\theta(t_e)t_e^{-1}}$ and $\gamma_{1,t_e^{-1}\theta(t_e)}$ are all elements in the initial generating set of S' and hence satisfy the initial set of relations. Furthermore, the embedding that δ_{t_e} is required for would give us the relations $\delta_{t_e}^{-1}\gamma_{1,t_e^{-1}t_e}\delta_{t_e} = \gamma_{1,\theta(t_e^{-1})t_e^{-1}t_e\theta(t_e)}$, $\delta_{t_e}^{-1}\gamma_{1,\theta(t_e)t_e^{-1}}\delta_{t_e} = \gamma_{1,t_e^{-1}\theta(t_e)}$ and $\delta_{t_e}^{-1}\gamma_{1,t_e^{-1}\theta(t_e)}\delta_{t_e} = \gamma_{1,\theta(t_e^{-1})t_e^{-1}\theta(t_e^2)}$.

Given that we have been unable to prove that the remaining relations hold, we need to add another property to T that will allow us to continue. In this case, we will add the property that any initial generator of T' that comes from an embedding of an edge in A_T can be generated by the initial generators of T' that come from generators of stabilizers. Consequently, it can then be shown that $\mathcal{E}_0 \subseteq \Gamma_0$ and our defintions of γ_{0,t_e} and γ_{1,t_e} $(\forall \delta_e \in \mathcal{E}_0)$ satisfy our relations.

Therefore, we can now say that $S' = Inv\langle \gamma_{0,t}, \gamma_{1,t}, f | t \in T, R'\rangle$, where R' is the set of relations $R' = \{f^{-1}\gamma_{0,t_1}f = \gamma_{1,t_1}, \gamma_{0,t_1}\gamma_{0,t_2} = \gamma_{0,t_1t_2}, \gamma_{1,t_1}\gamma_{1,t_2} = \gamma_{1,t_1t_2}, \gamma_{0,t_1}\gamma_{1,t_2} = \gamma_{1,\theta(t_1)t_2}, \gamma_{1,t_1}\gamma_{0,t_2} = \gamma_{1,t_1\theta(t_2)}, ff^{-1} = \gamma_{0,1} | t_1, t_2 \in T\}$. Note that we are not missing the relation that $\gamma_{0,1}$ is the identity of S' since this can be implied by the relations in R' which we will demostrate in the next corollary. It is clear to see that this form can be simplified to remove the set $\{\gamma_{1,t} | t \in T\}$ from our generating set by using the relation $f^{-1}\gamma_{0,t_1}f = \gamma_{1,t_1}$. This would also allow us to remove the relation $\gamma_{1,t_1}\gamma_{1,t_2} = \gamma_{1,t_1t_2}$ since this would already be given by $\gamma_{0,t_1}\gamma_{0,t_2} = \gamma_{0,t_1t_2}$.

Lemma 5.4.12. *$(\forall t \in T)$ $\gamma_{0,t}f^{-1} = f^{-1}\gamma_{0,\theta(t)} \Leftrightarrow (\forall t_1, t_2 \in T)$ $\gamma_{0,t_1}\gamma_{1,t_2} = \gamma_{1,\theta(t_1)t_2}$. Similarly, $(\forall t \in T)$ $f\gamma_{0,t} = \gamma_{0,\theta(t)}f \Leftrightarrow (\forall t_1, t_2 \in T)$ $\gamma_{1,t_1}\gamma_{0,t_2} = \gamma_{1,t_1\theta(t_2)}$.*

Proof. Say $(\forall t \in T)$ $\gamma_{0,t}f^{-1} = f^{-1}\gamma_{0,\theta(t)}$. Then, $(\forall t_1, t_2 \in T)$;

$$\begin{aligned}\gamma_{0,t_1}\gamma_{1,t_2} &= \gamma_{0,t_1}f^{-1}\gamma_{0,t_2}f = f^{-1}\gamma_{0,\theta(t_1)}\gamma_{0,t_2}f \\ &= f^{-1}\gamma_{0,\theta(t_1)t_2}f = \gamma_{1,\theta(t_1)t_2}.\end{aligned} \tag{96}$$

Conversely, if $(\forall t_1, t_2 \in T)\ \gamma_{0,t_1}\gamma_{1,t_2} = \gamma_{1,\theta(t_1)t_2}$ then we can set $t_2 = 1 \in T$ to get $(\forall t_1 \in T)\ \gamma_{0,t_1}\gamma_{1,1} = \gamma_{1,\theta(t_1)} \Rightarrow \gamma_{0,t_1}f^{-1}\gamma_{0,1}f = f^{-1}\gamma_{0,\theta(t_1)}f$. Furthermore, this implies that $\gamma_{0,t_1}f^{-1}\gamma_{0,1}ff^{-1} = f^{-1}\gamma_{0,\theta(t_1)}ff^{-1} \Rightarrow \gamma_{0,t_1}f^{-1}\gamma_{0,1} = f^{-1}\gamma_{0,\theta(t_1)} \Rightarrow \gamma_{0,t_1}f^{-1}ff^{-1} = f^{-1}\gamma_{0,\theta(t_1)}$ (since $ff^{-1} = \gamma_{0,1}$). Hence, $(\forall t \in T)\ \gamma_{0,t_1}f^{-1} = f^{-1}\gamma_{0,\theta(t_1)}$. It can similarly be shown that $(\forall t \in T)$ $f\gamma_{0,t} = \gamma_{0,\theta(t)}f \Leftrightarrow (\forall t_1, t_2 \in T)\ \gamma_{1,t_1}\gamma_{0,t_2} = \gamma_{1,t_1\theta(t_2)}$. □

Note that $(\forall t \in T)\ \gamma_{0,t}f^{-1} = f^{-1}\gamma_{0,\theta(t)}$ is equivalent to saying $(\forall t \in T)$ $f\gamma_{0,t} = \gamma_{0,\theta(t)}$ since one equation is the inverse of the other. Therefore, we can simplify our set of relations by removing $(\forall t_1, t_2 \in T)$, $\gamma_{0,t_1}\gamma_{1,t_2} = \gamma_{1,\theta(t_1)t_2}$ and $\gamma_{1,t_1}\gamma_{0,t_2} = \gamma_{1,t_1\theta(t_2)}$ and replacing them with $(\forall t \in T)\ f\gamma_{0,t} = \gamma_{0,\theta(t)}$.

Corollary 5.4.13. *$\gamma_{0,1}$ is the identity in S'.*

Proof. Since $\forall t_1, t_2 \in T$, $\gamma_{0,t_1}\gamma_{0,t_2} = \gamma_{0,t_1t_2}$ we can say that $\forall t \in T$, $\gamma_{0,1}\gamma_{0,t} = \gamma_{0,t} = \gamma_{0,t}\gamma_{0,1}$ and so $\gamma_{0,1}$ acts as an identity on elements of the form $\gamma_{0,t} \in S'$. Therefore, we need only show it acts as an identity on f to say it is the identity in S'. Since $ff^{-1} = \gamma_{0,1}$, we know that $\gamma_{0,1}f = ff^{-1}f = f$. Similarly, $f\gamma_{0,1} = \gamma_{0,\theta(1)}f = \gamma_{0,1}f = f$ (since θ being a morphism implies that $\theta(1) = 1$). □

We have now simplified our presentation of S' to get the following;

$$S' = Inv\langle \gamma_{0,t}, f | \gamma_{0,t_1}\gamma_{0,t_2} = \gamma_{0,t_1t_2}, f\gamma_{0,t} = \gamma_{0,\theta(t)}f, ff^{-1} = \gamma_{0,1}, t, t_1, t_2 \in T\rangle. \tag{97}$$

Now that we have simplified S' we can begin to look at the how we can describe the elements in it.

Lemma 5.4.14. *Every word of length 1 in $S' = Inv\langle \gamma_{0,t}, f | \gamma_{0,t_1}\gamma_{0,t_2} = \gamma_{0,t_1t_2}, f\gamma_{0,t} = \gamma_{0,\theta(t)}f, ff^{-1} = \gamma_{0,1}, t, t_1, t_2 \in T\rangle$ can be written in the form $f^{-n_1}\gamma_{0,t}f^{n_2}$ for some $t \in T$ and $n_1, n_2 \in \mathbb{N}^0$ where $f^0 = \gamma_{0,t}$.*

Proof. The words in S' of length 1 are given by f^n, f^{-n}, $\gamma_{0,t}^n$ $(\forall n \in \mathbb{N}^0, t \in T)$. First, $f^n = \gamma_{0,1}\gamma_{0,1}f^n = f^0\gamma_{0,t}f^n$. Similarly, $f^{-n} = f^{-n}\gamma_{0,1}f^0$. Finally, $\gamma_{0,t}^n = \gamma_{0,t^n} = \gamma_{0,1}\gamma_{0,t^n}\gamma_{0,1} = f^0\gamma_{0,t^n}f^0$. □

Theorem 5.4.15. *Every word in $S' = Inv\langle \gamma_{0,t}, f | \gamma_{0,t_1}\gamma_{0,t_2} = \gamma_{0,t_1t_2}, f\gamma_{0,t} = \gamma_{0,\theta(t)}f, ff^{-1} = \gamma_{0,1}, t, t_1, t_2 \in T\rangle$ can be written in the form $f^{-n_1}\gamma_{0,t}f^{n_2}$ for some $t \in T$ and $n_1, n_2 \in \mathbb{N}^0$ where $f^0 = \gamma_{0,t}$.*

Proof. Say that this theorem is true for words up to length $m \in \mathbb{N}$. Any word of length $m+1$ in S' is therefore given by the product of $f^{-n_1}\gamma_{0,t}f^{n_2}$ for some $t \in T$ and $n_1, n_2 \in \mathbb{N}^0$ and an element in the set $\{\gamma_{0,\bar{t}}, f^n, f^{-n} | \bar{t} \in T, n \in \mathbb{N}\}$. That is two say the words of length $m+1$ in S' can be written as $f^{-n_1}\gamma_{0,t}f^{n_2}\gamma_{0,\bar{t}}$, $f^{-n_1}\gamma_{0,t}f^{n_2}f^n$ or $f^{-n_1}\gamma_{0,t}f^{n_2}f^{-n}$ for some $t, \bar{t} \in T$ and $n_1, n_2, n \in \mathbb{N}^0$.

First look at $f^{-n_1}\gamma_{0,t}f^{n_2}\gamma_{0,\bar{t}}$. This is equal to $f^{-n_1}\gamma_{0,t}\gamma_{0,\theta^{n_2}(\bar{t})}f^{-n_2}$ (by the relation $f\gamma_{0,t} = \gamma_{0,\theta(t)}f$ in S'). It must therefore be the case that $f^{-n_1}\gamma_{0,t}f^{n_2}\gamma_{0,\bar{t}} = f^{-n_1}\gamma_{0,t\theta^{n_2}(\bar{t})}f^{-n_2}$. The next value, $f^{-n_1}\gamma_{0,t}f^{n_2}f^{n}$, can easily be put in the desired form since $f^{-n_1}\gamma_{0,t}f^{n_2}f^{n} = f^{-n_1}\gamma_{0,t}f^{n_2+n}$. Our final value, $f^{-n_1}\gamma_{0,t}f^{n_2}f^{-n}$ is slightly trickier. Since $ff^{-1} = \gamma_{0,1}$ which is the identity in S' we have two possible values this can equal.

$$f^{-n_1}\gamma_{0,t}f^{n_2}f^{-n} = \begin{cases} f^{-n_1}\gamma_{0,t}f^{n_2-n} & \text{when } n \leq n_2 \\ f^{-n_1}\gamma_{0,t}f^{-(n-n_2)} & \text{when } n > n_2. \end{cases} \tag{98}$$

Given the realtions in S', we can say that $\forall t \in T$, $\gamma_{0,t}f^{-1} = f^{-1}\gamma_{0,\theta(t)}$. Therefore, $f^{-n_1}\gamma_{0,t}f^{-(n-n_2)} = f^{-(n_1+n-n_2)}\gamma_{0,\theta^{n-n_2}(t)}f^{0}$. □

Now that we have a standard form for the elementsin S' we can create a map from S to S'. From Theorem 5.1.5 we know that a generating set of S is given by $\{(1,1,0),(0,1,1)\} \cup \{(0,t_i,0)|i \in I\}$ where $\{t_i|i \in I\}$ is the generating set of T for some index set I. In particular, $\forall (m,a,n) \in S$, $(m,a,n) = (0,1,1)^{-m}(0,a,0)(0,1,1)^{n}$ where $(0,1,1)^0 = (0,1,0)$. Knowing this, we can create a semigroup morphism $\mu : S' \to S$ given by $\mu(f^{-m}\gamma_{0,t}f^{n}) = (m,t,n)$.

Theorem 5.4.16. *The morphism μ is an isomorphism and hence $S \simeq S'$.*

Proof. Say $\mu(s_1) = \mu(s_2)$ for some $s_1, s_2 \in S'$. By Theorem 5.4.15, $\exists m_1$, m_2, n_1, $n_2 \in \mathbb{N}^0$ and $t_1, t_2 \in T$ such that $s_1 = f^{-m_1}\gamma_{0,t_1}f^{n_1}$ and $s_2 = f^{-m_2}\gamma_{0,t_2}f^{n_2}$. So, $\mu(f^{-m_1}\gamma_{0,t_1}f^{n_1}) = \mu(f^{-m_2}\gamma_{0,t_2}f^{n_2})$. This implies that $(m_1,t_1,n_1) = (m_2,t_2,n_2) \Rightarrow m_1 = m_2$, $t_1 = t_2$ and $n_1 = n_2$. Therefore, $s_1 = f^{-m_1}\gamma_{0,t_1}f^{n_1} = f^{-m_2}\gamma_{0,t_2}f^{n_2} = s_2$ and hence μ is injective. Also $\forall (m,a,n) \in S$, $a \in T$. Therefore, $\gamma_{0,a} \in S' \Rightarrow f^{-m}\gamma_{0,a}f^{n} \in S'$. Hence, $\mu(f^{-m}\gamma_{0,a}f^{n}) = (m,a,n)$ and μ is surjective. □

6 Action of B_n

In this section, we define S to be an inverse semigroup given by;

$$S = Inv\langle x_1, x_2, \ldots x_{n-1} | x_i^{-1}x_i = x_{i+1}x_{i+1}^{-1}, x_1^3 = x_jx_k\rangle \tag{99}$$

where all the generators of S are non-zero and i, j and k are integer values such that $i, j, k \in \{1, 2, \ldots n-1\}$, $i < n-1$ and $k \neq j+1$. We wish to prove that S is isomorphic to B_n when $n \geq 3$ (so throughout this proof we will be assuming $n \geq 3$) where B_n is defined to be a Brandt semigroup that is generated by the trivial group and a finite index set of n elements as defined by Howie [8]. Using this presentation of B_n, we will then define a B_n-act and see how it acts on a graph.

The origins of the Brandt semigroup come from work by Brandt [2] on the Brandt groupoid. Clifford [4] found that adjoining a zero to this groupoid gives us a semigroup that we call the Brandt semigroup. Further properties of this semigroup were found by Munn [17] as well as further work done by Clifford with Preston [5]. In particular, Vagner [24] found that the Brandt semigroup is an inverse semigroup.

6.1 Properties of S

In this section, we will prove that S is isomorphic to B_n and we will determine some properties of S.

Lemma 6.1.1. *Any word in S that contains a subword $x_i x_j$ or $x_j^{-1}x_i^{-1}$ where $j \neq i+1$ is the zero element in S.*

Proof. For all, $x_i, x_j \in S$, $x_i^2 = x_j^2$ since both these values equal x_1^3. In particular, $x_1^3 = x_1^2$ and $x_i x_j = x_1^3$ when $j \neq i+1$. Therefore, $\forall x_k \in S$, $x_k^2 = x_1^2$.

If $x_k \in S$, then;

$$x_1^2 x_k = x_k^2 x_k = x_k^3 = x_k x_1^2 = (x_k x_1)x_1 = x_1^2 x_1 = x_1^3 = x_1^2 \tag{100}$$

Similarly, we can show that $x_k x_1^2 = x_1^2$. So x_1^2 acts as a zero element on all $x_k \in S$ making it the zero element of S.

Therefore, $x_1^2 = 0$, but since $x_1^3 = x_1^2$ and $x_i x_j = x_1^3$ when $j \neq i+1$ we conclude that any $x_i x_j$ or $x_j^{-1}x_i^{-1}$ where $j \neq i+1$ is the zero element in S. □

Lemma 6.1.2. *Any word in S that contains a subword $x_i x_j^{-1}$ or $x_i^{-1}x_j$ where $i \neq j$ is the zero element in S*

Proof. Let $x_i, x_j \in S$ be such that $i \neq j$. If $i < n-1$, then;

$$x_i x_j^{-1} = x_i x_i^{-1} x_i x_j^{-1} = x_i x_{i+1}(x_{i+1}^{-1} x_j^{-1}) = 0 \tag{101}$$

If $i = n-1$, then;

$$x_{n-1} x_j^{-1} = x_{n-1} x_j^{-1} x_j x_j^{-1} = (x_{n-1} x_{j+1}) x_{j+1}^{-1} x_j^{-1} = 0 \tag{102}$$

So, $x_i x_j^{-1} = 0$ if $i \neq j$.

Similarly, say $i > 1$, then;

$$x_i^{-1} x_j = x_i^{-1} x_i x_i^{-1} x_j = x_i^{-1} x_{i-1}^{-1}(x_{i-1} x_j) = 0 \tag{103}$$

If $i = 1$, then;

$$x_1^{-1} x_j = x_1^{-1} x_j x_j^{-1} x_j = (x_1^{-1} x_{j-1}^{-1}) x_{j-1} x_j = 0 \tag{104}$$

So, $x_i^{-1} x_j = 0$ if $i \neq j$. □

It is then clear that these Lemmas give us the structure of all non-zero words in S.

Corollary 6.1.3. *Let $i \in \{1, 2, \dots n-1\}$. All non-zero elements in S can be written in one of the following forms:*

(i). $x_i x_{i+1} \dots x_{i+p}$ *where* $p \in \{0, 1, \dots n-i-1\}$

(ii). $x_i^{-1} x_{i-1}^{-1} \dots x_{i-p}^{-1}$ *where* $p \in \{0, 1, \dots i-1\}$

(iii). $x_i x_i^{-1}$

(iv). $x_{n-1}^{-1} x_{n-1}$

Later on, it will be helpful to have a more concise way describing these non-zero words. We will do this with the following notation and properties.

Definition 6.1.1. Define $w_{a,b} = x_a x_{a+1} \dots x_{a+b-1}$, so that b is the length of $w_{a,b}$.

Remark. Notice that if $w_{a,b}$ is well defined if and only if a and b are positive integers such that $2 \leq a + b \leq n$.

By Corollary 6.1.3, any element in S that is not an idempotent can be written as $w_{a,b}$ or $w_{a,b}^{-1}$ for some a or b. Also note the following properties of this form.

Lemma 6.1.4. *The following are true for any $w_{a,b}$ and $w_{c,d}$ in S:*

(i). $$w_{a,b} w_{c,d} = \begin{cases} 0 & \text{if } c \neq a+b \\ w_{a,b+d} & \text{if } c = a+b \end{cases}$$

(ii). $$w_{a,b}^{-1} w_{c,d}^{-1} = \begin{cases} 0 & \text{if } a \neq c+d \\ w_{c,b+d}^{-1} & \text{if } a = c+d \end{cases}$$

(iii). $$w_{a,b} w_{c,d}^{-1} = \begin{cases} 0 & \text{if } a+b \neq c+d \\ w_{a,b-d} & \text{if } a+b = c+d \text{ and } b > d \\ w_{c,d-b}^{-1} & \text{if } a+b = c+d \text{ and } b < d \\ x_a x_a^{-1} & \text{if } a = c \text{ and } b = d \end{cases}$$

(iv). $$w_{a,b}^{-1} w_{c,d} = \begin{cases} 0 & \text{if } a \neq c \\ w_{a+d,b-d}^{-1} & \text{if } a = c \text{ and } b > d \\ w_{a+b,d-b} & \text{if } a = c \text{ and } b < d \\ x_{a+b-1}^{-1} x_{a+b-1} & \text{if } a = c \text{ and } b = d \end{cases}$$

Proof. First note that $w_{a,b}w_{c,d} = x_a x_{a+1} \dots x_{a+b-1} x_c x_{c+1} \dots x_{c+d-1}$. By Lemma 6.1.1 this word equals zero unless $x_c = x_{a+b}$, which is to say $c = a+b$. In this case, $w_{a,b}w_{c,d} = x_a x_{a+1} \dots x_{a+b+d-1} = w_{a,b+d}$ so (i) holds. A similar proof is used to show that (ii) holds as well.

Now look at $w_{a,b}w_{c,d}^{-1}$. This equals $x_a x_{a+1} \dots x_{a+b-1} x_{c+d-1}^{-1} x_{c+d-2}^{-1} \dots x_c^{-1}$, but this word equals zero unless x_{c+d-1}^{-1} is the inverse of x_{a+b-1}^{-1}. In other words, this element is equal to zero unless $a+b = c+d$. Now we can then say that;

$$w_{a,b}w_{c,d}^{-1} = x_a x_{a+1} \dots x_{a+b-2} x_{a+b-1} x_{a+b-1}^{-1} x_{a+b-2}^{-1} \dots x_{a+b-d}^{-1}. \tag{105}$$

Recall from our relations of S that $x_i x_i^{-1} = x_{i-1}^{-1} x_{i-1}$. Therefore;

$$\begin{aligned} x_{a+b-2} x_{a+b-1} x_{a+b-1}^{-1} x_{a+b-2}^{-1} &= x_{a+b-2} x_{a+b-2}^{-1} x_{a+b-2} x_{a+b-2}^{-1} \\ &= x_{a+b-2} x_{a+b-2}^{-1}. \end{aligned} \tag{106}$$

Therefore, we can reduce the length of $w_{a,b}w_{c,d}^{-1}$. We can keep using this method to reduce our element unti it is of one of our known forms of words in S (as given by Corollary 6.1.3). If $b > d$, then $a < a+b-d$. Hence, if we keep reducing our word, we find that;

$$\begin{aligned} w_{a,b}w_{c,d} &= x_a x_{a+1} \dots x_{a+b-1} x_{a+b-1}^{-1} x_{a+b-2}^{-1} \dots x_{a+b-d}^{-1} \\ &= x_a x_{a+1} \dots x_{a+b-2} x_{a+b-2}^{-1} x_{a+b-3}^{-1} \dots x_{a+b-d}^{-1} \\ &= \dots \\ &= x_a x_{a+1} \dots x_{a+b-d-1} x_{a+b-d} x_{a+b-d}^{-1} \\ &= x_a x_{a+1} \dots x_{a+b-d-1} = w_{a,b-d}. \end{aligned} \tag{107}$$

Similarly, if $b < d$, then $a > a+b-d$. So;

$$\begin{aligned} w_{a,b}w_{c,d} &= x_{a-1}^{-1} x_{a-2}^{-1} \dots x_{a+b-d}^{-1} \\ &= w_{a+b-d,d-b}^{-1} = w_{c,d-b}^{-1}. \end{aligned} \tag{108}$$

Finally, if $b = d$, then $a+b = c+d \Rightarrow a = c$ and we find that $w_{a,b}w_{c,d}^{-1} = x_a x_a^{-1}$. Hence (iii) holds. A similar method is used to sow that (iv) holds. □

Now that we know the non-zero elements in S, we can find the cardinality of S.

Lemma 6.1.5. $|S| = |B_n|$

Proof. From our presentation of S, we can see there is a surjective homomorphism $S \to B_n$ given by $x_i \mapsto (i, i+1)$. Therefore, $|S| \geq n^2 + 1$ since $|B_n| = n^2 + 1$.

From the found forms of non-zero elements in S (given by Corollary 6.1.3) we can also say that $|S| \leq n^2 + 1$. Therefore, $|S| = n^2 + 1 = |B_n|$ □

Given that $|S| = |B_n|$ we can conclude that S and B_n are isomorphic.

Going forward, we will continue to use S as our presentation for B_n rather then any more well known presentations since it allows us to better demonstrate how our action works and its relation to the reltions and generators given by S.

6.2 B_n acting on a Set

Now we have a presentation for B_n, we can look at how it could act on a set, V. Define a map $\phi : B_n \times V \to V$ to be the map such that $\forall s \in B_n, v \in V$, $\phi(s, v) = \rho_s(v)$ where ρ_s is an injective function from V to itself. To define ϕ, we need to define $\rho_{x_i}, \forall i \in \{1, 2, \ldots n-1\}$ such that the relations of B_n still hold. In particular, we start by looking at the zero element in B_n

Lemma 6.2.1. *For any generator x_i of B_n, $D_{x_1^3} \subset D_{x_i}$.*

Proof. Since x_1^3 is the zero element in B_n this is given by Lemma 2.1.2. □

Lemma 6.2.2. *$\forall i \in \{1, 2, \ldots n-1\}$, ρ_{x_i} maps $D_{x_1^3}$ to $D_{x_1^3}$. Furthermore, $\forall i, j \in \{1, 2, \ldots n-1\}, v \in D_{x_1^3}$, $x_i v = x_j v$.*

Proof. Let $v \in D_{x_1^3}$, then by Lemma 6.2.1 we can say that $x_i v = u$ for some $u \in V$. We also know from our relations that $x_i^2 = x_1^3$ and so $x_i x_i v = v$ which implies that $x_i v = x_i^{-1} v = u$. Then $x_i^{-1} v = u \Rightarrow v = x_i u$. As $v \in D_{x_i}$, $x_i v = x_i x_i u = x_1^3 u$ exists and hence $u \in D_{x_1^3}$.

Since $v \in D_{x_1^3}$ we know that for any $j \in \{1, 2, \ldots n-1\}$, $x_j v$ exists (again by Lemma 6.2.1). Then, $x_j v = x_j x_i^{-1} u$. If $x_j x_i^{-1} \neq x_1^3$, it must be the case that $j = i$. In which case $x_j v = x_i x_i^{-1} u = u = x_i v$ (as $x_i x_i^{-1}$ is idempotent). Alternatively, if $x_j x_i^{-1} \neq x_1^3$, then $x_j v = u$. □

Lemma 6.2.3. *If v is an element in $D_{x_1^3}$, then $x_i v = v$ or $x_i^2 v = v$ for any generator x_i of B_n.*

Proof. First, we know from Lemma 6.2.1 that for any $v \in D_{x_1^3}$ and generator x_i, $x_i v$ exists as $D_{x_1^3} \subset D_{x_i}$. Also, our relations tell us that we require that $\rho_{x_i^2} = \rho_{x_1^3}$. Therefore, $x_i^2 v$ and must be equivalent to v (as x_1^3 is idempotent). This means that $x_i v = v$ or $\exists v' \in D_{x_1^3}$ such that $x_i v = v'$ and $x_i v' = v$. □

Remark. This Lemma as well as Lemma 6.2.2 tells us that the orbit of any element in v contains at most 2 elements.

We must now consider what happens to elements in D_{x_i} that are not in $D_{x_1^3}$.

Lemma 6.2.4. *If $v_i \in D_{x_i}$ is such that $v_i \notin D_{x_1^3}$, then $\forall j \in \{1, 2, \ldots n-1\}$, D_{x_j} contains at least one element in $B_n v_i$.*

Proof. If v_i is such an element and $i \neq n-1$, then we know that since $x_i^{-1}x_i = x_{i+1}x_{i+1}^{-1}$, $v_i \in D_{x_{i+1}^{-1}}$. Therefore, $\exists v_{i+1} \in D_{x_{i+1}}$ such that $x_{i+1}v_{i+1} = v_i$. Furthermore, it must be the case that $v_{i+1} \notin D_{x_1^3}$ as if $x_1^3 v_{i+1} = v_{i+1}$ exists, then $x_1^3 x_{i+1}^{-1} v_i = x_1^3 v_i$ must also exist. This means that if $i+1 \neq n-1$ we can repeat our process and find a $v_{i+2} \in D_{x_{i+2}}$ such that $x_{i+2}v_{i+2} = v_{i+1}$. In other words, we can keep repeating this process all the way up to finding a $v_{n-1} \in D_{x_{n-1}}$ such that $x_{i+1}x_{i+2}\cdots x_{n-1}v_{n-1} = v_i$.

Similarly if $i \neq 1$ then we can say that $x_{i-1}^{-1}x_{i-1} = x_i x_i^{-1}$. As $v_i \in D_{x_i}$, we know that $\exists v_{i-1} \in V$ such that $x_i v_i = v_{i-1}$. This then implies that $v_i = x_i^{-1}v_{i-1}$ meaning $v_{i-1} \in D_{x_i^{-1}}$. Since $x_{i-1}^{-1}x_{i-1} = x_i x_i^{-1}$, this means that $v_{i-1} \in D_{x_{i-1}}$. Furthermore, we can then prove that $v_{i-1} \notin D_{x_1^3}$ in the same way we showed that $v_{i+1} \notin D_{x_1^3}$. This allows us to say $\exists v_{i-2} \in V$ such that $x_{i-1}v_{i-1} = v_{i-2}$. Again we keep repeating this process until we find a $v_1 \in D_1$ such that $x_1 x_2 \cdots x_i v_i = v_1$. □

Remark. Note that in our proof, we also showed that the elements we found in the orbit of v_i will also not exist in $D_{x_1^3}$.

Lemma 6.2.5. $\forall i \in \{1, 2, \dots n-1\}$*, any two unique elements in* D_{x_i} *that don't exist in* $D_{x_1^3}$ *exist in separate orbits.*

Proof. Define v_i and u_i to be two distinct elements in D_{x_i} for some $i \in \{1, 2, \cdots n-1\}$ that do not exist in $D_{x_1^3}$. If we assume that v_i and u_i are in the same orbit, then $\exists s \in B_n$ such that $u_i = sv_i$. By Corollary 6.1.3, we know the four possible forms s can take. We can immediately rule out s being of the form (iii) and (iv) as these are idempotents and will hence map v_i to itself if they act at all on it.

Since $u_i \in D_{x_i}$ we know that $x_i u_i = x_i s v_i$ exists. We require that $x_i s \neq x_1^3$ as otherwise v_i must exist in $D_{x_1^3}$. So by Corollary 6.1.3 and the relation $x_1^3 = x_j x_k$ when $k \neq j+1$, meaning that $s = x_{i+1}x_{i+2}\cdots x_{i+p_1}$ for some $p_1 \in \{0, 1, \dots n-i-1\}$ or $x_i^{-1}x_{i-1}^{-1}\cdots x_{i-q_1}^{-1}$ for some $q_1 \in \{0, 1, \dots i-1\}$.

Say $i < n-1$, then from the relation $x_i^{-1}x_i = x_{i+1}x_{i+1}^{-1}$ we can say that $\exists v_{i+1} \in V$ such that $x_{i+1}v_{i+1} = v_i$ and $v_{i+1} \notin D_{x_1^3}$ (as shown in the proof of Lemma 6.2.4). We know from, $u_i = sv_i$ that $v_i = s^{-1}u_i$ and hence $x_{i+1}v_{i+1} = s^{-1}u_i$. This then implies that $u_i = sx_{i+1}v_{i+1}$. Again, for $v_{i+1} \notin D_{x_1^3}$, we require $sx_{i+1} \neq x_1^3$ and hence our Corollary 6.1.3 and our relations we can say that $s = x_{i-p_2}x_{i-p_2+1}\cdots x_i$ for some $p_2 \in \{0, 1, \dots i-1\}$ or $x_{i+q_2}^{-1}x_{i+q_2-1}^{-1}\cdots x_{i+1}^{-1}$ for some $q_2 \in \{0, 1, \dots n-i-1\}$. However, neither of these potential values of s correspond to the two previously established values of s and so we can say that no such s exists when $i < n-1$.

If $i = n-1$, then (as shown in the proof of Lemma 6.2.4) $\exists v_{n-2} \in D_{x_{n-1}^{-1}}$ such that $x_{n-1}v_{n-1} = v_{n-2}$ and $v_{n-2} \notin D_{x_1^3}$. We can then say that since $v_{n-1} = x_{n-1}^{-1}v_{n-2}$, $u_{n-1} = sx_{n-1}^{-1}v_{n-2}$. We know that v_{n-2} does not exist in the domain of x_1^3 and so $sx_{n-1}^{-1} \neq x_1^3$. Therefore, by Corollary 6.1.3 and the relation $x_1^3 = x_j x_k$ when $k \neq j+1$, we know that s can only be of the form $x_{p_3} x_{p_3+1} \cdots x_{n-1}$ for some $p_3 \in \{1, 2, \ldots n-1\}$. However, this contradicts $s = x_{i+1} x_{i+2} \cdots x_{i+p_1}$ or $x_i^{-1} x_{i-1}^{-1} \cdots x_{i-q_1}^{-1}$. So, we must conclude that v_i and u_i cannot exist in the same orbit. $\square$

Lemmas 6.2.4 and 6.2.5 then prove the following.

Corollary 6.2.6. *If $v_i \in D_{x_i}$ is such that $v_i \notin D_{x_1^3}$, then $\forall j \in \{1, 2, \ldots n-1\}$, D_{x_j} contains at exactly one element in $B_n v_i$.*

It is also helpful at this point to explicitly state how ρ_{x_i} maps elements.

Proposition 6.2.7. *For any $i \in \{2, 3, \ldots n-1\}$, ρ_{x_i} maps D_{x_i} to $D_{x_{i-1}}$. Also, ρ_{x_1} maps elements in D_{x_1} that don't exist in $D_{x_1^3}$ to elements that are not acted on by any generator of B_n.*

Proof. Let $v \in D_{x_i}$ for some $i \in \{1, 2, \ldots n-1\}$. If $v \in D_{x_1^3}$, then $v \in D_{x_{i-1}}$ as $D_{x_1^3}$ is a subset of the domain of any generator of B_n. Alternatively, if $v \notin D_{x_1^3}$, then $\exists u \in V$ such that $x_i v = u$. This implies that $v = x_i^{-1} u$ and hence $u \in D_{x_i^{-1}}$. Since x_{i-1} exists we can say that $x_{i-1}^{-1} x_{i-1} = x_i x_i^{-1}$. Therefore, $D_{x_i^{-1}} = D_{x_{i-1}}$ meaning that $u \in D_{x_{i-1}}$.

Now define $v \in D_{x_1}$ such that $v \notin D_{x_1^3}$. Then $x_1 v = u$ for some $u \in V$. If $x_i u$ exists for some $i \in \{2, 3, \ldots n-1\}$, then $x_i x_1 v = x_i u$ also exists. Since $v \notin D_{x_1^3}$, we require $x_i x_1 \neq x_1^3$, but our relations tell us this is only possible if $1 = i+1$. However, this would then imply that $i = 0$ which contradicts our definition of i. Therefore, $x_i u$ cannot exist. $\square$

6.3 B_n acting on a Graph

Using what we now know about B_n-acts, we can now see what graphs B_n will act on. However, before we start we note the following;

Lemma 6.3.1. *Let e be an edge of a graph with endpoints v and u such that $v \notin D_{x_1^3}$. Then e is acted on by an element of B_n if and only if $u \notin B_n v$.*

Proof. Say e, v and u are defined as they are in the lemma. Let $s \in B_n$ be the element that acts on e. By the definition of a generator, we know that $s = t x_i$ or $t x_i^{-1}$ for some $t \in B_n^1$ and generator x_i of B_n. First assume that $s = t x_i$, in which case x_i acts on e. This then implies that $x_i v$ and $x_i u$ both exist and hence $v, u \in D_{x_i}$. Corollary 6.2.6 then tells us that v and u must exist in different orbits.

Alternatively, if $s = t x_i^{-1}$ then $x_i^{-1} v$ and $x_i^{-1} u$ exist. Therefore, $\exists v', u' \in V$ such that $x_i^{-1} v = v'$ and $x_i^{-1} u = u'$. This then implies that $v = x_i v'$ and $u = x_i u'$ meaning that $v', u' \in D_{x_i}$. By Corollary 6.2.6, $B_n v' \neq B_n u'$. Also, by definition, $B_n v = B_n v'$ and $B_n u = B_n u'$ which allows us to say that $B_n v \neq B_n u$. Therefore, v and u exist in different orbits. $\square$

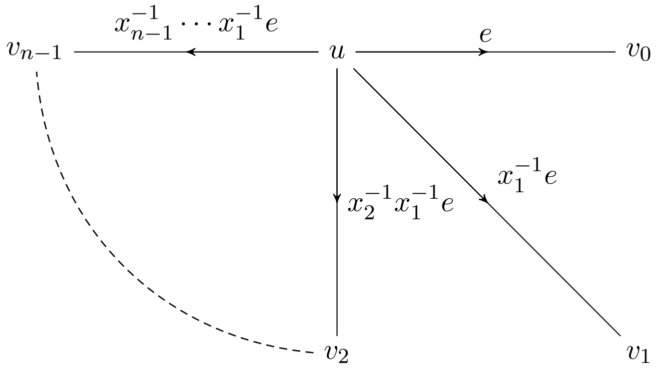
Example 6.3.2. By Corollary 6.2.6, we know that the number of orbits of elements that x_1^3 does not act on is equal to the number of elements in D_{x_i} that x_1^3 does not act on for any generator x_i of B_n. We also know from Lemmas 6.2.2 and 6.2.3 how x_i will act on elements in $D_{x_1^3}$. So for simplicity, we can say that V is such that there is only one orbit of vertices that x_1^3 does not act on and that there is only one element in $D_{x_1^3}$ that is fixed by every generator x_i.

This then leads us to say that $V = \{v_0, v_1, \dots v_{n-1}, u\}$ and $\forall s \in B_n$, ρ_s is defined by;

$$\rho_{x_i} = \begin{pmatrix} v_i & u \\ v_{i-1} & u \end{pmatrix} \tag{109}$$

Note that this form along with Lemma 6.3.1 mean that only an edge between u and v_i (for any $i \in \{1, 2, \dots n-1\}$) can be acted on by an element of B_n.

If say we had a directed edge e such that $\iota(e) = u$ and $\tau(e) = v_1$ we get the following graph;



However, if we now look at the orbits of the elements in V, we see that there are are only two, $\bar{v} = \{v_0, v_1, \dots v_{n-1}\}$ and $\bar{u} = \{u\}$. Also, $B_n e = \{e, x_1^{-1} e, x_2^{-1} x_1^{-1} e, \dots x_{n-1}^{-1} \cdots x_1^{-1} e\}$ which we label $\bar{e}$. This means our quotient graph will be given by;

$$\bar{u} \xrightarrow{\quad \bar{e} \quad} \bar{v}$$

Therefore, if we now look at the S-transversal we get if we choose v_0 to represent $\bar{v}$ then we get a graph that is similar to our quotient graph.

$$u \xrightarrow{\quad e \quad} v_0$$

Note that no matter what element we pick to represent $\bar{v}$, we will always get a graph that is the same form as the quotient graph. The graph of inverse semigroups would then be given by;

$$S_u \xrightarrow{S_e} S_{v_0}$$

However, since $S_u = B_n$, this example would not give us any information we did not already know about the semigroup. Alternatively, we could've defined $D_{x_1^3} = \emptyset$ but we would then lose information about the zero element in B_n. This is the same problem we cam across with the polycyclic semigroup action (Example 4.3.3). As with that case, we need to find an action such that we get a quotient graph where the zero element does not exist in the stabilizer of any of it vertices. Consequently our action must be such that the zero element does not act on any element in our graph (as shown by Lemma 4.3.9).

6.4 Action without 0

Since we have now established that we wish for our action to be such that 0 does not act on any element in the S-act we again examine what actions will work. With this new restriction, there are some properties of our desired action that we can derive. From here on, say that we have an action of $S \simeq B_n$ on a graph G such that the zero element of S does not act on any element in G.

Lemma 6.4.1. *Only non-zero idempotents fix elements under our action.*

Proof. Let $s \in S_v$ for some $v \in G$. Consider the forms of non-zero words in B_n that we gave in Corollary 6.1.3. If $s = w_{a,b}$ then $s^2 v = v \Rightarrow w_{a,b}^2 v = v$. However, $w_{a,b}^2 = 0 \Rightarrow 0 \in S_v$ (by Lemma 6.1.4). Hence, $s \neq w_{a,b}$. Similarly we can show that $s \neq w_{a,b}^{-1}$. This only leaves non-zero idempotent values that s can equal. $\square$

Corollary 6.4.2. *Every element in G can only be acted on by at most one generator of S. Similarly, every element in G can only be acted on by at most one inverse of a generator of S.*

Proof. Let x_i and x_j exist in the generating set of S and suppose $\exists v \in G$ such that $x_i v$ and $x_j v$ both exist. This implies that $x_i^{-1} x_i v = v = x_j^{-1} x_j v \Rightarrow x_j x_i^{-1} x_i v = x_j v$. If $x_j x_i^{-1} = 0$, then we find that $x_j v = 0 x_i v = 0v = v$ which means $x_j \in S_v$. This contradicts Lemma 6.4.1, so it must be the case that $x_j x_i^{-1} \neq 0$. This implies that $x_i^{-1} = x_j^{-1} \Rightarrow x_i = x_j$. The proof that $x_i^{-1} v$, $x_j^{-1} v$ exist $\Rightarrow x_i^{-1} = x_j^{-1}$ is given dually. $\square$

Note that if no generators of S or their inverses act on an element in G then said element will have an empty domain. Therefore, for this action we will assume that every element in G must be acted on by a single element or inverse of an element in the generating set of S. We will also assume that every generator of S or every inverse of said generators acts on at least one element of G since otherwise we would definintely lose said element from any fundamental inverse semigroup we might obtain.

Knowing this, we can begin constructing a set of elements that S will act on. Without loss of generality, we will consider our action to be such that all elements in the genrating set of S act on at least one value in G and begin by finding a set that S will act on to give us a set of vertices of S. First we define a set $V = \{v_i | i \in \{1, 2, \ldots, n-1\}\}$ and say that $x_i v_i$ is defined $\forall i \in \{1, 2, \ldots, n-1\}$. We will add more elements to our set later to equal these values, but for now consider the following.

Lemma 6.4.3. *Let F be an S-act. Then, $\forall i \in \{2, 3, \ldots, n-1\}$ and $v \in F$, $x_i^{-1}v$ exists $\Rightarrow x_{i-1}v$ exists. Similarly, $\forall j \in \{1, 2, \ldots, n-2\}$ and $v \in F$, $x_j v$ exists $\Rightarrow x_{j+1}^{-1}v$ exists.*

Proof. Say $x_i^{-1}v$ exists for some $i \in \{2, 3, \ldots, n-1\}$ and $v \in F$. This implies that $x_i x_i^{-1} v$ exists $\Rightarrow x_{i-1}^{-1} x_{i-1} v$ exists. Note that we know x_{i-1} is defined since $i \in \{2, 3, \ldots, n\}$. Since $x_{i-1}^{-1} x_{i-1} v = x_{i-1}^{-1}(x_{i-1}v)$ it must be the case that $x_{i-1}v$ exists. The rest of the Lemma can be proven in a similar way. $\square$

So, according to this Lemma, it must be the case that $x_{i+1}^{-1} v_i$ is defined ($\forall i \in [1, n-2]$). We must now define more elements for S to act on. First, $\forall i \in [1, n-1]$ we define a value $y_{1,i}$ and say $x_i v_i = y_{1,i}$. This then implies that $x_i^{-1} y_{1,i}$ exists. Furthermore, Lemma 6.4.3 then tells us that $x_{i-1} y_{1,i}$ exists $\forall i \in [2, n-1]$ and so we need values in our S-act that these can be equal to. In this case, we define the values $y_{2,i}$ ($\forall i \in \{2, 3, \ldots, n\}$) to be given by $y_{2,i} = x_{i-1} y_{1,i}$. However, as happened earlier, Lemma 6.4.3 tells us that more elements need to be defined. Eventually we will find that we have added the set Y to our intial set where $Y := \{y_{j,i} | i, j \in \mathbb{N}, j \leq i \leq n-1\}$. We also define how S acts on elements in this set by saying $\forall k \in [1, n-1]$, $x_k y_{j,i}$ exists $\Leftrightarrow k = i - j$ (and hence $j \neq i$) in which case $x_k y_{j,i} = y_{j+1,i}$.

Similarly, since $x_{i+1}^{-1} v_i$ is defined $\forall i \in [1, n-2]$ we also add the set $Z := \{z_{j,i} | i, j \in \mathbb{N}, i + j < n\}$ to our proposed S-act where ($\forall i \in [1, n-2]$) $x_{i+1}^{-1} v_i = z_{1,i}$ and ($\forall k \in [1, n-1]$, $z_{j,i} \in Z$) $x_k^{-1} z_{j,i} = z_{j+1,i}$ when $k - 1 = i + j$ but is undefined otherwise. Knowing this, we can define how certain elements will act on values in this set.

Lemma 6.4.4. *$\forall w_{a,b} \in S$, $w_{a,b} v_i$ exists $\Leftrightarrow a + b = i + 1$. Then, $w_{a,b} v_i = y_{b,i}$. Similarly, $\forall w_{a,b}^{-1} \in S$, $w_{a,b}^{-1} v_i$ exists $\Leftrightarrow a = i + 1$ in which case $w_{a,b}^{-1} v_i = z_{b,i}$.*

Proof. Given that $w_{a,b} = x_a x_{a+1} \cdots x_{a+b-1}$ and the conditions under which $x_k y_{j,i}$ exists we can say that $w_{a,b} v_i$ exists $\Leftrightarrow a + b = i + 1$. Knowing this, we can say that;

$$\begin{aligned} w_{a,b} v_i &= x_a x_{a+1} \cdots x_{a+b-1} v_i = x_{i-b+1} x_{i-b} \cdots x_i v_i \\ &= x_{i-b+1} \cdots x_{i-1}(x_i v_i) = x_{i-b} \cdots x_{i-1} y_{1,i} \\ &= \cdots = x_{i-b+1} y_{b-1,i} = y_{b,i}. \end{aligned} \tag{110}$$

The existence and value of $w_{a,b}^{-1}v_i$ is given dually. $\square$

Corollary 6.4.5. *$\forall y_{j,i} \in Y$, $\exists w_{a,b} \in S$ such that $w_{a,b}v_i = y_{j,i}$, $a = i - j +$ and $b = j$. Also, $\forall z_{j,i} \in Z$, $\exists w_{a,b}^{-1} \in S$ such that $w_{a,b}^{-1}v_i = z_{j,i}$, $a = i + 1$ and $b = j$.*

Proof. Take $y_{j,i} \in Y$. From this we know that $i, j \in \mathbb{N}$ and $j \leq i \leq n - 1$. If we set $a = i - j + 1$ and $b = j$ note that $a \geq j - j + 1 = 1$ and $a \leq n - 1 - j + 1 \leq n - j \leq n - 1$, so $a \in [1, n-1]$ and $x_a \in S$. Also, $a + b - 1 = i - j + 1 + j - 1 = i \in [1, n-1]$ which tells us that $w_{a,b} \in S$. Since $a + b = i + 1$, we know that $w_{a,b}v_i = y_{b,i}$ (by Lemma 6.4.4).

Now take $z_{j,i} \in Z$ and set $a = i + 1$ and $b = j$. Given the definition of Z, $i, j \in \mathbb{N}$ and $i + j < n$. Since $i, j \in \mathbb{N}$ we can say that $1 \leq i, j$ and therefore, $2 \leq i + j \leq n - 1$. Also, $i + j < n \Rightarrow i < n - j \Rightarrow i < n - 1$ (since $1 \leq j$). Therefore, $a = i + 1 \in [2, n-1]$. We also know that $a + b - 1 = i + j \in [2, n-1]$. Hence, $w_{a,b} \in S$ and by Lemma 6.4.4, $w_{a,b}^{-1}v_i = z_{j,i}$. $\square$

The different actions we can get from S acting on the set $V \cup Y \cup Z$ will come from equating different values in the set. However there are some restrictions on what can be equated.

Lemma 6.4.6. *$\forall i, j \in [1, n-1]$, $i \neq j \Rightarrow v_i \neq v_j$.*

Proof. Say $i \neq j$ and $v_i = v_j$. Then both x_i and x_j act on $v_i = v_j$ which contradicts Corollary 6.4.2. $\square$

Lemma 6.4.7. *$\forall y_{j,i}, y_{q,i}, y_{j,p} \in Y$ and $z_{j,i}, z_{q,i}, z_{j,p} \in Z$:*

- $y_{j,i} = y_{q,i} \Rightarrow j = q$,
- $y_{j,i} = y_{j,p} \Rightarrow i = p$,
- $z_{j,i} = z_{q,i} \Rightarrow j = q$,
- $z_{j,i} = z_{j,p} \Rightarrow i = p$.

Proof. Say $y_{j,i} = y_{q,i}$. Corollary 6.4.5 tells us this is equivalent to saying $w_{i-j+1,j}v_i = w_{i-q+1,q}v_i \Rightarrow v_i = w_{i-j+1,j}^{-1}w_{i-q+1,q}v_i$. Since are action is defined such that zero element does not act on anything in our set, we can use Lemma 6.1.4 to say that $i - j + 1 = i - q + 1 \Rightarrow j = q$. The same method of using Corollary 6.4.5 and Lemma 6.1.4 tells us that $y_{j,i} = y_{j,p} \Rightarrow i - j + 1 = p - j + 1 \Rightarrow i = p$.

A dual proof can then be used to show that $z_{j,i} = z_{q,i} \Rightarrow j = q$ and $z_{j,i} = z_{j,p} \Rightarrow i = p$. $\square$

7 Conclusion

A working method of gaining information about inverse semigroups from their actions is a very promising idea. It would allow us to be able to learn more about a semigroup without the need of fully understanding theoir structure. Though I was not able to achieve such a high goal in my studies, I believe that the work I have shown here demonstrates the current problems we have.

Our current method for having inverse semigroups act on graphs shows promising results. In particular, given what we now know about the actions of free products of inverse semigroups, we can now create working examples of very complex inverse semigroups if we know we can express them as a free product of simple semigroups with a known working action. This includes free inverse semigroups with an infinite number of generators. The ideas explored can now be applied to any inverse semigroup that can be expressed as a free product of inverse semigroups.

Our investigation into polycyclic, Bruck-Reilly and Brandt semigroups suggests we might need to change our approach, however. As demonstrated in their respective sections, our current way of defining an inverse semigroup action can not give us back a fundamental inverse semigroup that is isomorphic to the original semigroup in some cases. This is due to such semigroups containing a zero element. I think I have been able to demonstrate the issues that come with the zero element and our current method of finding an action. It is clear that any working method would probably involve diverging more from the Bass-serre theory of groups then our method currently does. If such a method can be found that will allow the existence of zero elements in the fundamental inverse semigroup then it we can re-examine our current working models to see if they will still work under this new method.

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