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Residual and virtual properties of generalised Bestvina-Brady groups

by

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Abstract

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This work is about classical, group-theoretic finiteness properties of a generalisation of Bestvina-Brady groups due to Ian Leary [Lea18]. Bestvina-Brady groups originally appeared sporting exotic homological finiteness properties [BB97], yet enjoy a plethora of classical properties due to naturally being subgroups of right-angled Artin groups. Closely related is the notion of special cube complexes of Haglund-Wise [HW08], which can be used to establish residual finiteness and more. Generalised Bestvina-Brady groups arise from a branching construction and are almost never special, thus the need for studying finite-index subgroups.

It turns out that the property of being virtually torsion-free is key to resolving which generalised Bestvina-Brady groups are virtually special and residually finite, as it is the torsion from branched covers which causes pathological behaviour. We prove virtual specialness for different kinds of infinite families of generalised Bestvina-Brady groups [Van]. The ideas used also apply in a more general context of branching of cube complexes, and we include an application to hyperbolic groups joint with Robert Kropholler [KV].

We resolve the question of virtual torsion-freeness when the branching is governed by cyclic covers. This is done by generalising finite quotients having the structure of extensions of extraspecial groups, which were initially found by computer search. This is then further extended to any finite covers, provided the complexes involved can retract to graphs.

Contents

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
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7. Parts of this work have been published as: Vladimir Vankov. Virtually special non-finitely presented groups via linear characters. arXiv:2001.11868
   Robert Kropholler and Vladimir Vankov. Finitely generated groups acting uniformly properly on hyperbolic space. arXiv:2007.13880

Signed:... Date:..................

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To Natalia.

Definitions and Abbreviations

$o(g)$	Refers to the order of a group element.
$p_{\pm}^{1+2n}$	Extraspecial p-group of type $\pm$ and size $2n+1$.
C_n	Refers to the cyclic group of order n.
$\langle g_1, \ldots, g_n \rangle$	The subgroup generated by $g_1, \ldots, g_n$.
$C_G(g)$	The centraliser of g in G. When understood, the G may be dropped.
e	The identity element of a group.
WLOG	Without loss of generality.

Chapter 1

Introduction

The subsequent topics running throughout are interlinked as follows: Bestvina-Brady groups are subgroups of right-angled Artin groups, generalised Bestvina-Brady groups are groups of deck transformations of complexes having a Bestvina-Brady fundamental group, and special cube complexes have fundamental groups embedding into right-angled Artin groups.

1.1 Right-angled Artin groups

Given a simplicial graph Γ, the associated *right-angled Artin group* (RAAG) is the group A_Γ with generators the vertices of Γ, and relations being commutators between vertices joined by an edge. Complete graphs give free abelian groups and empty graphs give free groups. Subgroups of RAAGs have many rich properties, such as being residually finite.

Let $I = [0, 1]$ be the unit interval. An *n-cube* for $n > 0$ is a copy of I^n and a 0-cube is just a vertex. A *cube complex* is a cell complex where every cell is some n-cube, with the attaching maps being combinatorial on cubes [BH99]. This means that the attaching maps must send n-cubes isometrically to n-cubes when we consider the boundaries of cubes as being the union of lower-dimensional cubes.

The *link* of a vertex v is a simplicial complex whose vertices correspond to ends of 1-cubes attached to v, and these are joined by an n-simplex for each corner of each $(n + 1)$-cube being located at v.

A *flag complex* is a simplicial complex where any finite collection of vertices that are pairwise joined by a 1-simplex span a simplex. We say that a cube complex is *nonpositively curved* if the link of every vertex is a flag complex. We refer to simply connected nonpositively curved cube complexes as CAT(0) cube complexes.

If a space has fundamental group G and has contractible universal cover, we say that space is a *classifying space* for G.

Definition 1.1 (Salvetti complex)**.** Let A_Γ be a RAAG defined by a graph Γ. Then A_Γ has a natural classifying space (see e.g. [CD95]), called the Salvetti complex, formed from one vertex, one loop for each Artin generator, and an n-torus for each n-clique in Γ, glued appropriately.

The link of the vertex in the Salvetti complex corresponding to A_Γ is a "spherical double" of the flag complex with 1-skeleton Γ. This is formed by taking each vertex v and replacing it with two vertices v^+, v^-, such that a set of vertices form a simplex if the corresponding vertices formed a simplex when forgetting about the superscripts (without repeating v). Note that this remains a flag complex if the original complex was flag.

1.2 Bestvina-Brady groups

Bestvina-Brady groups (introduced in [BB97]) are by definition normal subgroups of Right-Angled Artin Groups (RAAGs), hence they are linear over $\mathbb{Z}$ and enjoy properties such as being residually finite.

Definition 1.2 (Bestvina-Brady Group)**.** Let L be a connected finite flag complex. Define BB_L to be the kernel of the homomorphism from the RAAG A_L (associated with the 1-skeleton of L) to $\mathbb{Z}$, which sends every Artin generator to 1 in $\mathbb{Z}$.

The finiteness properties of BB_L are controlled by the choice of complex L. Note that there is a natural correspondence between flag complexes L and their 1-skeletons Γ, hence we can refer to a group A_L. One can think of the naturally associated classifying space $\mathbb{B}_L$ either as the quotient of the universal cover of the Salvetti complex of A_L by BB_L, or as a $\mathbb{Z}$-cover of this Salvetti complex (which gives us a natural height function f). Notice that the ascending and descending links (which are the induced subcomplexes of the link on edges pointing either up or down, respectively) of vertices in $\mathbb{B}_L$ are all isomorphic to L. Since there are only countably many finite connected flag complexes, there are at most countably many groups BB_L.

Presentations for these groups are given in [DL99]. A particularly important example for much of what follows is for L being the minimal flag triangulation of S^1, see Figure 1.1.

The corresponding right-angled Artin group will have presentation:

$$A_L := \langle u, v, w, x \mid [u, v], [v, w], [w, x], [x, u] \rangle .$$

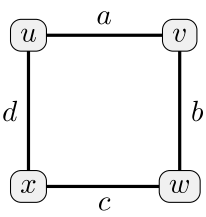


FIGURE 1.1: 4-vertex 1-dimensional square L.

Using the presentation from [DL99], which states that generators come from directed edges, the corresponding Bestvina-Brady group will have presentation:

$$BB_L = \left\langle a, b, c, d \;\middle|\; a^i b^i c^i d^i \ \forall i \in \mathbb{Z} \right\rangle,$$

where we can think of the relationship to the Artin generators as being:

$$uv^{-1} = a,\ vw^{-1} = b,\ wx^{-1} = c,\ xu^{-1} = d.$$

In this case, the RAAG splits as:

$$A_L = F_2 \times F_2 \cong F_{u,w} \times F_{v,x}.$$

We can use the fact that the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free group of rank 2 to obtain a faithful matrix representation for BB_L:

$$a = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}, \ d = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

This is only of dimension 4. We can compare this to a corresponding generalised Bestvina-Brady group, also on 4 generators, for the same L, at the end of Subsection 2.1.4, where the matrices we obtain are of size 1024.

1.3 Generalised Bestvina-Brady groups

The above construction was generalised by Leary in [Lea18], to give uncountably many more such groups for each finite connected flag complex.

Definition 1.3 (Generalised Bestvina-Brady Group)**.** Let $M \to L$ be a regular cover. Following the start of section 21 in [Lea18], define $G_L^M(S)$ to be the group of deck transformations of the branched cover $X_L^M(S)$ of the classifying space $\mathbb{B}_L$ of BB_L. The branching occurs at vertices with heights not in S, when the BB_L-orbits of vertices in the universal cover of $\mathbb{B}_L$ are naturally labelled with the integers (consider the height function f, mentioned above). The branching is such that the ascending and descending links of branching vertices are isomorphic to M.

The existence of such a branched cover is Theorem 9.1 in [Lea18]. In Subsection 2.1.1, we will give specific names to vertices, edges and squares in $X_L^M(S)$ as well as state more of its properties, using the results established in [Lea18].

When the group does not coincide with a Bestvina-Brady group (i.e. when S is not all of $\mathbb{Z}$), it does not act specially on its associated CAT (0) complex (using language from [HW10], see Definition 3.4 there), because of the action of the point stabilisers at the branching vertices. Indeed, those where this complex is locally finite (call these groups "of finite ramification") contain torsion, hence cannot be subgroups of RAAGs.

Definition 1.4 (Finite Ramification)**.** When the group of deck transformations of $M \to L$ is finite, we say that $G_L^M(S)$ is of finite ramification (i.e. $[\pi_1(L) : \pi_1(M)] < \infty$).

Note that when M is the universal cover of L and $\pi_1(L)$ is finite, we get finite ramification of the group in the main statement of [Lea18] without any modification. The first main results that we prove are:

Theorem A.1 There are infinitely many virtually special generalised Bestvina-Brady groups, coming from L being a flag subdivision of the circle, by varying the cover $M \to L$ and the branching set S.

Theorem A.2 For large enough subdivisions L of the circle and any regular finite covering $M \to L$, the group $G_L^M(S)$ is virtually special if and only if S is periodic.

Similar to Bestvina-Brady groups, we may write down a presentation for $G_L^M(S)$ in terms of the structure of $M \to L$, where as generators we take the directed edges x_i of L. The resulting relations, coming from Definition 1.1 in [Lea18], can be split into two kinds: *triangle relations* of the form $x_i x_j x_k$ coming from directed edge-loops of length 3 in L, and *long cycle relations* of the form $x_1^i \cdots x_n^i$ coming from edge-loops that (normally) generate the fundamental group of L. Note that while the assumption that $0 \in S$ is necessary for this presentation, in practice we can translate the height function without loss of generality to satisfy this, as long as S is non-empty. When S is empty, the structure of the group is then simply a semidirect product extension of the underlying Bestvina-Brady group by the group of deck transformations $M \to L$, so we do not study that case. Indeed, the interesting properties arise when the set S is neither empty nor all of the integers (when it just coincides with BB_L).

Theorem B.1 Generalised Bestvina-Brady groups of finite ramification are virtually torsion-free only if the branching set is periodic.

Theorem B.2 If Γ is a simplicial graph with sufficient independent edges and $M \to \Gamma$ is any regular finite cover, then the group $G_\Gamma^M(S)$ is virtually torsion-free if and only if S is periodic.

When $M \to L$ is the universal cover, then we may simply write $G_L(S)$ (as used in [Lea18]) to mean $G_L^M(S)$.

Theorem C.1 For L a flag triangulation of $\mathbb{RP}^2$, the group $G_L(S)$ is virtually torsion-free for $S = 2\mathbb{Z}$ and for $S = 3\mathbb{Z}$.

Theorem C.2 If L graph-retracts onto a graph with sufficient independent edges and $N \to L$ is any finite regular covering, then the group $G_L^N(S)$ is virtually torsion-free if and only if S is periodic.

Finally, we can identify the torsion elements in the group. The action on the edges of the branched cover is free, but vertices may have non-trivial stabilisers. In particular, each point of branching will have a stabiliser isomorphic to a copy of $\pi_1(L, M)$, the group of deck transformations $M \to L$. Since $G_L^M(S)$ acts as the group of deck transformations of the branched cover of the classifying space of the Bestvina-Brady group, it follows that each point stabiliser of some fixed height will be conjugate to any other (of the same height). Thus if h is a torsion element of $G_L^M(S)$, since it must fix a vertex of height n, it will have to be of the form gxg^{-1} for some $g \in G_L^M(S)$ and x being an element of a copy of $\pi_1(L, M)$ that is in the conjugacy class of height n.

1.4 Special cube complexes

Let X be a nonpositively curved cube complex. 'Square' will refer to a 2-cube in X, and 'edge' will refer to a 1-cube. For edges u, v which are opposite each other in some square of X, we write $u \sim v$ and say that u and v are elementary parallel. This induces an equivalence relation on the edges. By abuse of notation, we write $u \sim w$ if edges u, w lie in the same equivalence class and say they are parallel. We denote the equivalence class of u by $[u]$, and we call this a hyperplane. We also write $[u] \sim [v]$ if u, v lie in the same hyperplane. If we induce an orientation on edges and insist that elementary parallelism keeps track of orientation, we say that some hyperplane $[v]$ is not two-sided if there exists u such that $u \sim v$ and $u \sim -u$; we say the hyperplane is two-sided otherwise.

Definition 1.5 (Hyperplane interactions, [HW08]). If two edges u, v are adjacent in a square (intersect at a corner of the square), we write $u \perp v$. We write $[u] \perp [v]$ and say that hyperplanes $[u], [v]$ cross if there exist edges u', v' such that $u' \perp v'$ and $u' \sim u$, $v' \sim v$. If two edges w, x share a vertex (intersect at a 0-cell), but there does not exist a square that contains both of them where they are adjacent (i.e. $w \not\perp x$), we write $w \circlearrowright x$. We write $[w] \circlearrowright [x]$ and say that hyperplanes $[w], [x]$ osculate if there exist edges w', x' such that $w' \circlearrowright x'$ and $x' \sim x$, $w' \sim w$. If the edges are oriented, then we say the osculation is *direct* if the two edges in question both either point away or towards the vertex at which they osculate, and *indirect* otherwise.

One can define a special cube complex in terms of avoiding certain configurations of hyperplane interactions. Note that while $\circlearrowright$ and $\perp$ are relations, only $\sim$ is an equivalence relation.

Definition 1.6 (Special Cube Complex, [HW08]). If X is such that for all edges u, $[u]$ is two-sided, and for any pair of (not necessarily distinct) edges u, v, we have at most one of the relations $\sim$, $\perp$, $\circlearrowright$ holding between $[u]$ and $[v]$, then we say that X is a special cube complex (except we may allow indirect self-osculation).

If $\sim$ and $\perp$ hold, then a hyperplane crosses itself, so is not an embedded hyperplane. If $\sim$ and $\circlearrowright$ hold, then a hyperplane self-osculates, but note that we allow the case where this is an indirect self-osculation. If both $\perp$ and $\circlearrowright$ hold between a pair of hyperplanes, they inter-osculate. There are essentially 4 components to Definition 1.6, we will refer to them in the following chapter.

Theorem 1.7 (Haglund and Wise, [HW08]). *If X is a special cube complex, then $\pi_1(X)$ embeds into a Right-Angled Artin Group. In particular, if X contains finitely many hyperplanes, then $\pi_1(X)$ embeds into a finitely generated RAAG, and is linear over $\mathbb{Z}$.*

Having finitely many hyperplanes is important, as a finitely-generated Right Angled Artin Group in particular embeds into $SL_n(\mathbb{Z})$ for some finite n. Linear groups are residually finite [Mal40].

The embedding into a finitely generated RAAG and establishing finitely many hyperplanes usually involves some sort of compactness - the group in question, for example, may be acting cocompactly on a cube complex. The complexes we study will almost always not be compact, nor will the action of the groups we study be cocompact. Nonetheless, we will have finitely many orbits of hyperplanes and only finitely many hyperplanes in quotient complexes (see the notion of *cofinite* in the introduction of [HW10]), which will be good enough for embedding into a finitely generated RAAG.

Chapter 2

Virtually special groups

The group $G_L^M(S)$ of finite ramification for $S \neq \mathbb{Z}$ can never be the fundamental group of a special cube complex, but we can look for a finite-index subgroup. We have a natural action on a CAT(0) cube complex, so the subgroups we aim to find will arise as fundamental groups of quotients of this cube complex and kernels of homomorphisms to finite groups.

2.1 A family of virtually special groups via different L

We give here the first example of an infinite family of virtually special generalised Bestvina-Brady groups. This is achieved by varying the cover $M \to L$ and choosing an appropriate branching set S to go with it, and considering powers of elements in a quotient that has the structure of a direct product of cyclic groups, i.e. by utilising linear characters.

Theorem 2.1. *For each integer $m \geqslant 4$ and prime number $k \geqslant 2$, the group*

$$G_m^k := \left\langle x_1, x_2, \ldots, x_m \;\middle|\; \begin{array}{ll} x_1^i x_2^i \cdots x_m^i & \text{for } i \in k\mathbb{Z} \\ \left(x_1^i x_2^i \cdots x_m^i\right)^k & \text{for } i \in \mathbb{Z} \backslash k\mathbb{Z} \end{array} \right\rangle$$

is (non-cocompact, but cofinitely) virtually special. In particular, it is residually finite.

The group G_m^k corresponds to a generalised Bestvina-Brady group $G_L^M(S)$ by taking $M \to L$ to be the k-regular covering of the regular m-gon by the regular km-gon (thought of as flag triangulations of the circle), with branching set $S = k\mathbb{Z}$. This is of finite ramification, because the group of deck transformations corresponding to the finite regular cover $M \to L$ is C_k. As k varies, the covering space of L varies, and as m increases, L becomes a finer simplicial subdivision of the circle.

Note that the abelianisation of G_m^k is:

$$G_m^{k\ ab} = \mathbb{Z}^m / \langle (k, k, \ldots, k) \rangle .$$

Using Smith normal form, we get

$$\underbrace{(k, k, \ldots, k)}_{m} \mapsto (k, \underbrace{0, \ldots, 0}_{m-1}) \implies G_m^{k\ ab} \cong C_k \times \mathbb{Z}^{m-1}.$$

Thus G_m^k are distinct up to isomorphism for different integer pairs (m, k), and we indeed have an infinite family of pairwise non-isomorphic groups.

2.1.1 Fundamental domain

We apply Definition 1.3 to L being an m-vertex triangulation of the circle and M being an mk-triangulation of the circle. We refer to the resulting group as G_m^k, and to the branched cover as X_m^k. Using the presentation in [Lea18], we get generators $x_1, \ldots, x_m$ and relations as stated in Theorem 2.1.

The group G_m^k acts (on the left) on the square complex X_m^k which admits a height function $f : X_m^k \to \mathbb{R}$ such that vertices of the complex lie at integer heights (this is the complex from Theorem 9.1 in [Lea18]). There is one orbit of vertices at each height i, with a distinguished base vertex X^i, which has stabiliser $\left\langle x_1^i x_2^i \cdots x_m^i \right\rangle$ (see Section 14 in [Lea18]). Other vertices will have stabilisers being the appropriate conjugate of this.

Every edge joins two vertices of heights differing by 1. The edges are labelled by the heights of the top vertex. There are m free orbits of edges at each height, with distinguished orbit representatives $\mathbf{u}_j^i$ at height i for $1 \leqslant j \leqslant m$, such that $\mathbf{u}_1^i$ joins X^{i-1} to X^i. Every square has a top vertex of height $i+1$, a bottom vertex of height $i-1$ and two vertices of height i, with such a square being labelled with height i. Denote by $\operatorname{Orb}(\mathbf{u})$ the orbit of an edge. There are m free orbits of squares at each height, with distinguished orbit representatives $\underline{\mathbf{s}}_j^i$ at height i for $1 \leqslant j \leqslant m$. They are chosen such that $\underline{\mathbf{s}}_j^i$ contains edges in $\operatorname{Orb}\left(\mathbf{u}_j^{i+1}\right)$, $\operatorname{Orb}\left(\mathbf{u}_{j+1}^{i+1}\right)$, $\operatorname{Orb}\left(\mathbf{u}_j^{i}\right)$, $\operatorname{Orb}\left(\mathbf{u}_{j+1}^{i}\right)$, in that order, having picked an appropriate direction to read around the square (using cyclic indexing, which will also be used later on, which means that $j+1$ denotes 1 for $j = m$, for example). This order will be important as it will allow us to glue correctly the top and bottom halves of squares in the upcoming proof. In order to complete the convention for labelling the edges and squares, we use Figure 2.1 to order the edges and squares, and refer to this as Fact (1.). Note that the labels on the edges in the link refer to which squares in the complex X_m^k contribute towards this.

Note further that in Figure 2.1, when $i \in S$, the element $x_1^i \cdots x_m^i$ is the identity, so the last vertex is the same as the first, and this forms a complete loop, isomorphic to L.

$$
\begin{tikzcd}
\boxed{\mathbf{u}_1^i} \arrow[r, dash, "\underline{\mathbf{s}}_1^{i-1}"'] & \boxed{\mathbf{u}_2^i} \arrow[r, dash, dashed] & \boxed{\mathbf{u}_{m-1}^i} \arrow[r, dash, "\underline{\mathbf{s}}_{m-1}^{i-1}"'] & \boxed{\mathbf{u}_m^i} \arrow[r, dash, "\underline{\mathbf{s}}_m^{i-1}"'] & \boxed{x_1^i \cdots x_m^i \cdot \mathbf{u}_1^i}
\end{tikzcd}
$$

FIGURE 2.1: Part of the descending link of X^i.

Each square is made up of two parallel copies of two types (note that by type of an edge we mean the corresponding RAAG generator label of the underlying Bestvina-Brady group) of edges, and each such square is contained in a unique plane made up of squares of only those two types of edges (see Section 12 in [Lea18]), thus each type of square can be labelled by an edge in the underlying RAAG graph corresponding to the Bestvina-Brady group. Given a vertex of positive height, we can consider all the squares which have this vertex at their top vertex, and form a pyramid by continuing each plane corresponding to each square down to the 0-level.

Vertex X^i, for positive i, contains the following vertices around the base of its corresponding pyramid at the 0-level: (see Lemma 14.3 in [Lea18] and the notion of 'shadow' there)

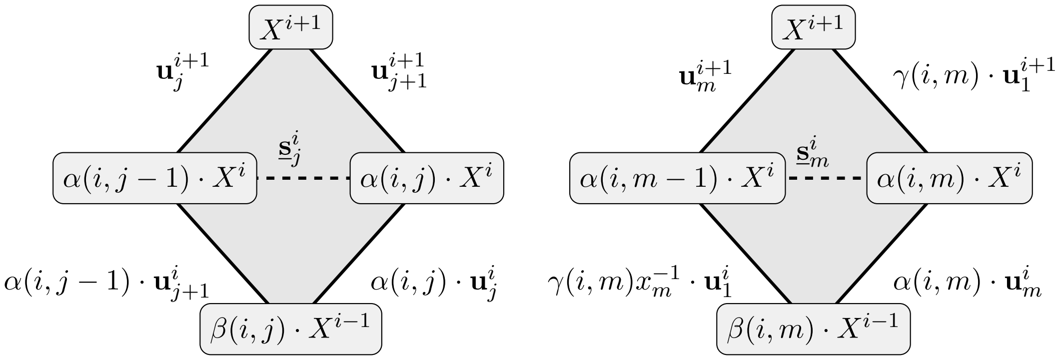
$$
\begin{array}{rrcr}
 & & & X^0, \\
x_1 \cdot X^0, & x_1^2 \cdot X^0, & \dots, & x_1^i \cdot X^0, \\
x_1^i x_2 \cdot X^0, & x_1^i x_2^2 \cdot X^0, & \dots, & x_1^i x_2^i \cdot X^0, \\
 & \vdots & & \\
x_1^i x_2^i \cdots x_{m-1}^i x_m \cdot X^0, & x_1^i x_2^i \cdots x_{m-1}^i x_m^2 \cdot X^0, & \dots, & x_1^i x_2^i \cdots x_m^i \cdot X^0
\end{array}
$$

in that order (when reading in an appropriate direction). We will refer to this as Fact (2.). Note that when $i \in k\mathbb{Z}$, then $x_1^i x_2^i \cdots x_m^i$ is the identity element, hence $x_1^i x_2^i \cdots x_m^i \cdot X^0 = X^0$ and the elements above lie along a complete loop and go around the entire base of the pyramid. We can do the same for vertices of negative height, by projecting upwards rather than downwards to the 0-level. Vertex X^i, for negative i, has the following vertices on its shadow (Fact (3.)):

$$
\begin{array}{rrcr}
 & & & X^0, \\
x_1^{-1} \cdot X^0, & x_1^{-2} \cdot X^0, & \dots, & x_1^i \cdot X^0, \\
x_1^i x_2^{-1} \cdot X^0, & x_1^i x_2^{-2} \cdot X^0, & \dots, & x_1^i x_2^i \cdot X^0, \\
 & \vdots & & \\
x_1^i x_2^i \cdots x_{m-1}^i x_m^{-1} \cdot X^0, & x_1^i x_2^i \cdots x_{m-1}^i x_m^{-2} \cdot X^0, & \dots, & x_1^i x_2^i \cdots x_m^i \cdot X^0.
\end{array}
$$

These can be thought of as the boundaries of the base of a pyramid with base at height zero and having apex vertex X^i. The faces of the pyramid are parts of embedded planes in X_m^k, consisting of two types of edges each. When $i \in S$, this pyramid will have m faces, and it will have mk faces otherwise.

Lemma 2.2. *Given the above notational conventions, the complex X_m^k and the action of G_m^k on it can be fully described in Figure 2.2.*



(A) $1 \leqslant j < m$ (B) $j = m$

FIGURE 2.2: G_m^k-orbit representatives of 2-cells in X_m^k in layer i.

In Figure 2.2, the following shorthand is used: (note that here, x_i for $i < 1$ should be thought of as the identity element, for example $\alpha(i,0)$ is just the identity element)

$$\alpha(i,j) := x_1^{i+1} \cdots x_{j-1}^{i+1}\ x_j\ x_{j-1}^{-i} \cdots x_1^{-i}$$

$$\beta(i,j) := x_1^{i+1} \cdots x_{j-1}^{i+1}\ x_j\ x_{j-1}^{1-i} \cdots x_1^{1-i}$$

$$\gamma(i,j) := x_1^{i+1} \cdots x_j^{i+1}$$

Proof. We proceed by induction on j.

Base case: ($j = 1$) Applying (2.) to vertex X^1 in combination with (1.) yields Figure 2.3.

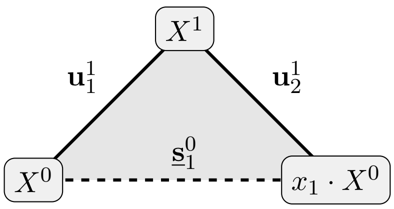


FIGURE 2.3: The top half of square $\underline{\mathbf{s}}_1^0$.

Applying (3.) to vertex X^{-1} and keeping in mind that edge $\mathbf{u}_1^0$ joins X^{-1} to X^0 yields Figure 2.4, for some $\mu, \lambda \in G_m^k$.

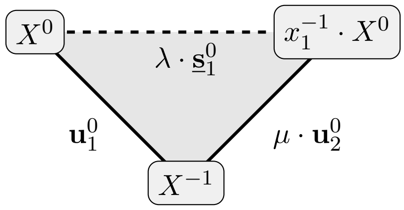


FIGURE 2.4: The bottom half of a square in the orbit of $\underline{\mathbf{s}}_1^0$.

Since 0 is not a branching layer, X^0 has a unique edge from $\mathrm{Orb}\left(\mathbf{u}_2^0\right)$ in its descending link. From (1.) we know that this is $\mathbf{u}_2^0$, which means that $\mu = x_1^{-1}$. This also means that $\lambda = x_1^{-1}$, since square $\underline{\mathbf{s}}_1^0$ has edge $\mathbf{u}_2^0$ attached underneath X^0. Hence square $\underline{\mathbf{s}}_1^0$ is as claimed in Lemma 2.2. We now proceed to proving the general form of square $\underline{\mathbf{s}}_1^i$ by induction, depending on whether i is positive or negative.

Case 1: (i positive) Claim: Square $\underline{\mathbf{s}}_1^i$ is as in Lemma 2.2 for $i > 0$.

Proof. Base case: ($i = 1$) Considering the pyramid with apex X^2 (specifically the $\mathbf{u}_1$-$\mathbf{u}_2$ face containing the line from X^0 to X^2) and applying (2.) along with (1.) gives Figure 2.5.

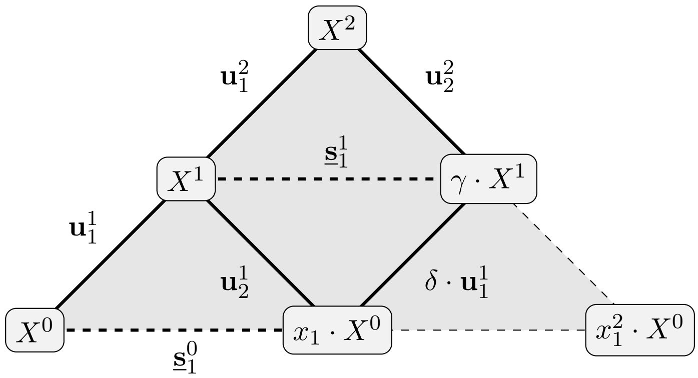


FIGURE 2.5: Figuring out the square $\underline{\mathbf{s}}_1^1$.

Now by considering which translate of the square $\underline{\mathbf{s}}_1^0$ fits in the bottom-right, we determine that $\delta = \gamma = x_1$. This completes the square $\underline{\mathbf{s}}_1^1$.

Inductive step: ($i > 1$) Assume that the claim has already been proven for all i up to and including $n \geqslant 1$. Consider the pyramid with apex X^{n+2}. Applying (1.) gives the top half of square $\underline{\mathbf{s}}_1^{n+1}$. Applying (2.) gives the vertices in layer 0 at the bottom of the face of the pyramid, and working upwards using the inductive hypothesis, keeping in mind that edges are in free orbit, completes square $\underline{\mathbf{s}}_1^{n+1}$, see Figure 2.6.

□

Case 2: (i negative) Claim: Square $\underline{\mathbf{s}}_1^{-i}$ is as in Lemma 2.2 for $i > 0$.

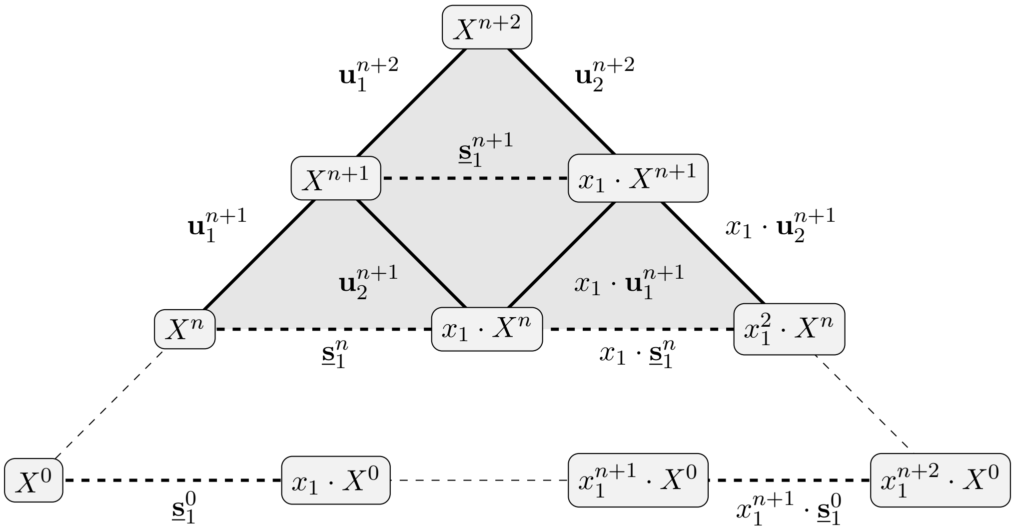


FIGURE 2.6: Figuring out the square $\underline{\mathbf{s}}_1^{n+1}$.

Proof. Base case: $(i = 1)$ Consider the pyramid with apex X^{-2} and apply (3.). Note that we know what the line from X^{-2} to X^0 (where two faces of the pyramid meet) looks like, because by convention this is a series of edges of type $\mathbf{u}_1$. By also considering which translates of $\underline{\mathbf{s}}_1^0$-squares go at the top, we get Figure 2.7.

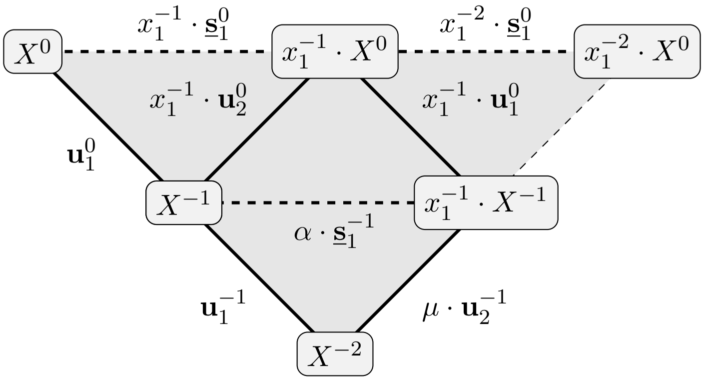


FIGURE 2.7: Figuring out the square $\underline{\mathbf{s}}_1^{-1}$.

Now by applying (1.) to vertex X^0, we can determine that $\alpha = x_1^{-1}$ by translation. In order to compute μ, we must take into account the full link of vertex X^{-1}. Using (1.) applied to vertex X^0, the square $x_1^{-1} \cdot \underline{\mathbf{s}}_1^0$ and Figure 2.7, we can deduce a part of the full link of vertex X^{-1}, shown in Figure 2.8. Note that labels on the edges in the link signify which squares these edges of the link are originating from.

From [Lea18] (Theorem 9.1), the full link of vertex X^{-1} is isomorphic to the spherical double of L. Therefore, there must be an edge between vertices $\mathbf{u}_1^0$ and $\mathbf{u}_2^{-1}$ in Figure 2.8. By translation, this means that in the full link of $x_1^{-1} \cdot X^{-1}$, the edge $x_1^{-1} \cdot \mathbf{u}_1^0$ is joined to the edge $x_1^{-1} \cdot \mathbf{u}_2^{-1}$. Thus, we deduce $\mu = x_1^{-1}$. This completes the square $\underline{\mathbf{s}}_1^{-1}$, after translating by x_1.

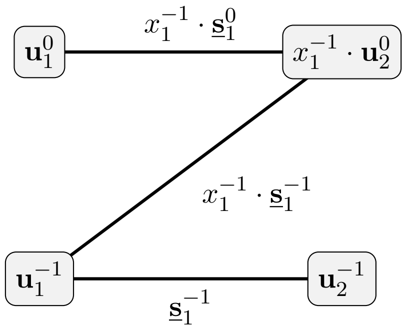


FIGURE 2.8: A part of the full link of vertex X^{-1}.

Inductive step: ($i > 1$) Assume that the claim has already been proven for all i up to and including $n \geqslant 1$. Consider the pyramid with apex $X^{-(n+2)}$. Applying (1.) gives the top half of square $\underline{\mathbf{s}}_1^{-(n+1)}$. Applying (3.) gives the vertices in layer 0 at the bottom of the face of the pyramid, and working downwards using the inductive hypothesis, along with a consideration of the full link of vertex $X^{-(n+1)}$ as above and translation, completes square $\underline{\mathbf{s}}_1^{-(n+1)}$.

□

This completes the proof of Lemma 2.2 for $j = 1$.

Inductive step: ($1 < j < m$) Suppose that Lemma 2.2 has already been proven for n up to and including $j - 1$. Then, using exactly the same method of proof as above, by using the fact that edges are in free orbit and that given a certain height and type (of square) there is a unique square containing a fixed edge, we can also deduce the general form of the square $\underline{\mathbf{s}}_{n+1}^i$. We use the inductive hypothesis to deduce what the lines from the corner on the 0-level to X^{-i} and X^i looks like, on the appropriate corners of the pyramid (where two faces of the pyramid meet). At each inductive step we move one more face around the pyramid, and finish off by translating the $\underline{\mathbf{s}}_{n+1}^i$-orbit square we get to have the identity coefficient on the square label (if not already), obtaining the vertex and edge labels in Figure 2.2.

Final step: ($j = m$) Once we have deduced what the squares $\underline{\mathbf{s}}_n^i$ for $1 \leqslant n \leqslant m-1$ look like, we can do exactly the same for the squares $\underline{\mathbf{s}}_m^i$. This time the coefficients break the pattern because of the new factor in (1.), resulting in a different form for the square $\underline{\mathbf{s}}_m^i$ in Figure 2.2. Note that because $x_1^i \cdots x_m^i \cdot X^i = X^i$, the vertex coefficients in any of the squares of the fundamental domain in Figure 2.2 are not necessarily unique. □

Much of the proof of Theorem 2.1 will involve translating around patches of the fundamental domain to understand the structure of hyperplanes, using the fact that edges or squares of a fixed height and type are in free orbit. Note that the above derivation did not depend on S, which will be useful in more general applications later (such as in Section 2.2). We obtain edges of adjacent type that osculate by taking two that touch at the corner of a square, and acting on one of them by a proper vertex stabiliser, so the resulting edges share a vertex but not a square. Every such osculation arises in this way, since two edges whose labels are adjacent in L will always meet at the corner of a square at a non-branching vertex. Similarly, every crossing of hyperplanes occurs in some corner of some square. We will use these facts to exhaust complete lists of possibilities for these events occurring in later proofs.

2.1.2 Hyperplane stabiliser images in a quotient

We show that G_m^k is virtually torsion-free by exhibiting an explicit surjection onto a finite group with torsion-free kernel as follows:

$$\begin{aligned} \phi_m^k : G_m^k &\to \bar{G}_m^k \cong \overbrace{C_k \times \cdots \times C_k}^{m \text{ copies}} \\ x_i &\mapsto (0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th position}}}{\sigma_i}, 0, \ldots, 0) \end{aligned} \tag{$\dagger$}$$

This is a homomorphism since each σ_i has order k. The only torsion elements come from point stabilisers of the action on X_m^k, which are conjugates of the torsion elements in the presentation (or their powers). Since these do not map to the identity (neither do their powers, as k is prime) in $\bar{G}_m^k$, we get that $\ker \phi_m^k$ is a torsion-free subgroup of G_m^k of index mk. Note that $\bar{G}_m^k$ is abelian. We could have used a smaller target group to get a torsion-free kernel, however this larger group will be useful for defining linear characters later. If k were not prime, this would be false, as then an intermediate power of a torsion element (which is a torsion element itself) would still lie in the kernel, due to some factorisation of k.

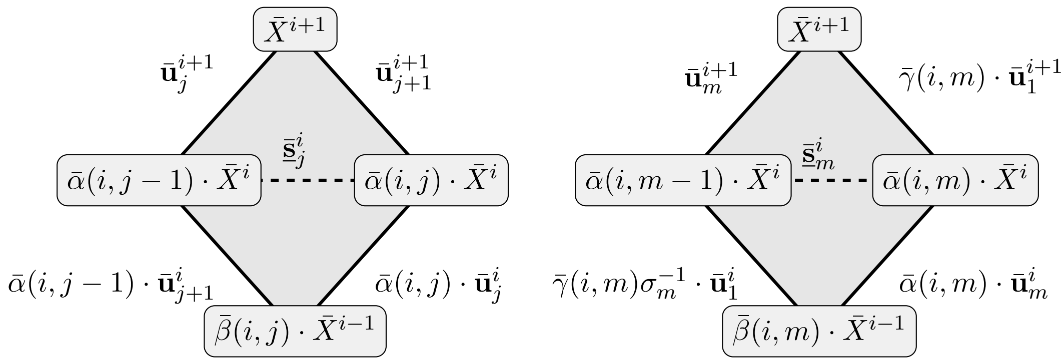
We can now turn our attention to the quotient complex $\bar{X}_m^k := X_m^k / \ker \phi_m^k$. We denote images by placing a bar over the notation. From Lemma 2.2, we obtain the new fundamental domain in Figure 2.9.

In Figure 2.9, the following shorthand is used:

$$\bar{\alpha}(i,j) := \phi_m^k\left(\alpha(i,j)\right) = \sigma_1 \cdots \sigma_j$$

$$\bar{\beta}(i,j) := \phi_m^k\left(\beta(i,j)\right) = \sigma_1^2 \cdots \sigma_{j-1}^2 \sigma_j$$

$$\bar{\gamma}(i,j) := \phi_m^k\left(\gamma(i,j)\right) = \sigma_1^{i+1} \cdots \sigma_j^{i+1}$$



(A) $1 \leqslant j < m$ (B) $j = m$

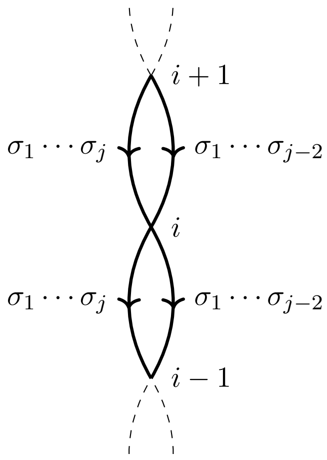
FIGURE 2.9: $\bar{G}_m^k$-orbit representatives of 2-cells in $\bar{X}_m^k$ in layer i.

Given a hyperplane $[\mathbf{u}]$, we define the image-hyperplane stabiliser as

$$\operatorname{Stab}([\mathbf{u}]) := \left\{ g \in \bar{G}_m^k \;\middle|\; g\cdot\mathbf{u} \sim \mathbf{u} \right\}.$$

We can compute the image-hyperplane stabilisers by observing which $\bar{G}_m^k$-coefficients of an edge type in a particular layer lie in the same hyperplane (in the image of the quotient). This is done by first observing that in order to “move” to a different layer and remain in the same hyperplane (for edge $\bar{\mathbf{u}}_j^i$, say), we must utilise squares $\bar{\mathbf{s}}_j$ and $\bar{\mathbf{s}}_{j-1}$ (using cyclic indexing), as they are the only ones to contain an edge of this type. We can think of the hyperplane stabilisers as ‘fundamental groups’ of loop spaces where the vertices represent heights and edge labels on the loop space represent how the coefficient changes when “jumping” across a square, as shown in Figure 2.10. These edge labels are calculated from the coefficients on the edges in Figure 2.9. The direction of the arrow shows the direction in which the edge label is multiplied (take the inverse for the opposite direction). For example, by looking at the top left and bottom right edges of square $g\cdot\bar{\mathbf{s}}_1^i$, we can deduce that using this square to drop down to layer i, the coefficient of edge $g\cdot\bar{\mathbf{u}}_1^{i+1}$ will be multiplied by $\bar{\alpha}(i,1)=\sigma_1$, resulting in the edge $g\sigma_1\cdot\bar{\mathbf{u}}_1^i$, which is still in the same hyperplane as $g\cdot\bar{\mathbf{u}}_1^{i+1}$. Note that this multiplication should be thought of as occurring on the right, but $\bar{G}_m^k$ is abelian, so we do not need to worry about this here.

From this, (using cyclic indexing) we can determine that the set of elements in the stabiliser is the set of elements you get from multiplying together the labels on edges in every loop based at i. But this is the same as if we took the set of all loops without backtrack, since inverses would immediately cancel out. From the fundamental group of the underlying space (which is a countably generated free group) we can see that this is a subgroup generated by loops γ_j for all integers j, where each γ_j is the loop that goes to level j using the right side of the diagram and returns using the left side. Thus, multiplying along those loops gives a generating set for the stabiliser. In this case, that simplifies to:



(A) $1 < j \leqslant m$ (B) $j = 1$

FIGURE 2.10: Moving between edges of different heights in the same hyperplane of type $\bar{\mathbf{u}}_j$.

$$\mathrm{Stab}\left([\bar{\mathbf{u}}_j^i]\right) = \langle \sigma_{j-1}\sigma_j \rangle .$$

Note that because squares of a fixed type are in free orbit and our group of coefficients is abelian, it does not matter which particular hyperplane we are in: the stabiliser will only depend on the type of edge. Coupled with the symmetry of the picture, we can ignore the coefficient or height of an edge when considering its hyperplane stabiliser, which will be mirrored in the notation from now on.

Lemma 2.3. *For integers a, b and elements $g, h \in \bar{G}_m^k$, we have:*

$$g \cdot \bar{\mathbf{u}}_j^a \sim h \cdot \bar{\mathbf{u}}_j^b \implies h \in g\,(\sigma_1 \cdots \sigma_j)^{a-b}\,\mathrm{Stab}\left([\bar{\mathbf{u}}_j]\right).$$

Proof. Consider which $\bar{\mathbf{u}}_j$-type edges lie on layer b and are part of the hyperplane $[g \cdot \bar{\mathbf{u}}_j^a]$. Their $\bar{G}_m^k$-coefficient will be g multiplied by $(\sigma_1 \cdots \sigma_j)^{a-b}$ from moving to layer b, by moving along the left side of the loop space in Figure 2.10, and then maybe also moved around in the layer by multiplication by an element from the stabiliser, giving the result. It did not matter that we chose to take the left side, as the stabiliser is the fundamental group of the loop space. The height of the edge does not matter because the labels are symmetrical at every level. □

2.1.3 Linear characters

The action of G_m^k on the CAT(0) cube complex X_m^k is interesting, because even though there are only finitely many G_m^k-orbits of hyperplanes, we get infinitely many $\operatorname{Stab}(\mathfrak{H})$-orbits of hyperplanes crossing a given hyperplane $\mathfrak{H}$. This makes it difficult to apply existing tools such as Theorem 4.1 in [HW10] for proving that the group is virtually special.

For an explicit example, consider the hyperplane $[\mathbf{u}_2^0]$. From the fundamental domain earlier, we have $\mathbf{u}_2^0 \sim \mathbf{u}_2^i \ \forall\ i \in \mathbb{Z}$ and also $\mathbf{u}_2^i \perp \mathbf{u}_3^i \ \forall\ i \in \mathbb{Z}$. Similarly to Lemma 2.5, we can compute the hyperplane stabilisers:

$$\operatorname{Stab}\left([\mathbf{u}_2^0]\right) = \langle x_1^a x_2^a \mid \forall\ a \in \mathbb{Z}\rangle\ ,\ \operatorname{Stab}\left([\mathbf{u}_3^0]\right) = \langle x_2^a x_3^a \mid \forall\ a \in \mathbb{Z}\rangle\ ,$$

as well as $\mathbf{u}_3^i \sim x_1^i \cdot \mathbf{u}_3^0$. This implies that the set of coefficients g for which $[\mathbf{u}_2^0]$ crosses $[g \cdot \mathbf{u}_3^0]$ includes $I = \{x_1^i \mid i \in \mathbb{Z}\}$. Two elements of I give the same hyperplane in the $\operatorname{Stab}\left([\mathbf{u}_2^0]\right)$-orbit of hyperplanes which cross $[\mathbf{u}_2^0]$ if one can get from one to the other by multiplying on the left by an element from $\operatorname{Stab}\left([\mathbf{u}_2^0]\right)$ and on the right by an element from $\operatorname{Stab}\left([\mathbf{u}_3^0]\right)$. If, after this identification, there would be only a finite list of elements left, then there would also be a finite list of such elements in the abelianisation. However, in ${G_m^k}^{ab}$ this consists of all elements of the form $x_1^{i+a} x_2^{a+b} x_3^b$ for some integers a, b, i. To remain within I, we require $b = a = 0$, so actually there are infinitely many elements in the abelianisation after identification.

This means that there are infinitely many $\operatorname{Stab}\left([\mathbf{u}_2^0]\right)$-orbits of hyperplanes crossing $[\mathbf{u}_2^0]$. We will use maps to finite groups in order to deal with this.

Fix a primitive kth root of unity μ in $\mathbb{C}$. For each $1 \leqslant j \leqslant m$, consider the homomorphism:

$$\begin{aligned} \mathfrak{D}(j) : \bar{G}_m^k &\to \mathbb{C} \\ \sigma_i &\mapsto \begin{cases} \mu & \text{if } \ i = j \\ 1 & \text{else} \end{cases} \end{aligned}$$

This defines a linear character. We will show that $\bar{X}_m^k$ is a special cube complex by working with $\operatorname{Ch}(\bar{G}_m^k)$, the abelian group of linear characters (under pointwise multiplication). In particular, we can multiply and take inverses. The main idea is to come up with characters constant on certain sets. Cyclic indexing will again be used in this section, so for example, $\mathfrak{D}(j-1)$ for $j = 1$ will refer to $\mathfrak{D}(m)$.

We will proceed by checking each of the 4 conditions for a special cube complex:

1. ($\sim$,$\perp$) Every square in $\bar{X}^k_m$ is such that every corner consists of two edges of different types meeting, hence a hyperplane can never cross itself. Note that this argument actually applies to all $G^M_L(S)$.

2. (2-sided) We choose an orientation for each edge by deciding to make each edge point upwards. Now each square takes positively oriented edges to positively oriented edges or negatively oriented edges to negatively oriented edges, so we can never have the situation $\bar{\mathbf{u}}^i_j \sim -\bar{\mathbf{u}}^i_j$. Note that this argument actually applies to all $G^M_L(S)$.

Remark: Since the above 2 arguments do not depend on L, M, S, we can deduce that any virtually torsion-free generalised Bestvina-Brady group of finite ramification has a finite-index subgroup which is the fundamental group of a cube complex with all hyperplanes 2-sided and no hyperplane self-intersections.

3. ($\sim$,$\circlearrowright$) Next, we investigate whether a hyperplane can self-osculate. In order for this to occur, we must have two distinct edges $g \cdot \bar{\mathbf{u}}^a_j$ and $h \cdot \bar{\mathbf{u}}^b_j$ for some integers a, b, some $1 \leqslant j \leqslant m$ and $g, h \in \bar{G}^k_m$, for which we have

$$g \cdot \bar{\mathbf{u}}^a_j \sim h \cdot \bar{\mathbf{u}}^b_j \quad \text{and also} \quad g \cdot \bar{\mathbf{u}}^a_j \circlearrowright h \cdot \bar{\mathbf{u}}^b_j.$$

We already know from Lemma 2.3 that the first condition implies that

$$h \in g\left(\sigma_1 \cdots \sigma_j\right)^{a-b} \operatorname{Stab}\left([\bar{\mathbf{u}}_j]\right).$$

Consider the second condition. Since the two edges in question osculate, they must share a common vertex. This implies that $b \in \{a-1, a, a+1\}$. We consider each in turn:

Case 1: ($b = a - 1$) From the fundamental domain in Figure 2.9, we know that edge $\sigma_1^{-1} \cdots \sigma_{j-1}^{-1} \cdot \bar{\mathbf{u}}^{i+1}_j$ is attached to vertex $\bar{X}^i$ and so is edge $\bar{\mathbf{u}}^i_j$. By translation, this means that to get at all the edges of type $\bar{\mathbf{u}}^{a-1}_j$ which osculate with $g \cdot \bar{\mathbf{u}}^a_j$, the coefficient is multiplied by $\sigma_1 \cdots \sigma_{j-1}$ and also possibly by some stabiliser of the vertex (note that since $\bar{G}^k_m$ is abelian, the stabiliser of a vertex is determined only by its height). This implies that

$$h = \left(\sigma_1^{a-1} \cdots \sigma_m^{a-1}\right)^c \sigma_1 \cdots \sigma_{j-1} g$$

for some integer c. In order for the self-osculation to be possible, it must be true that

$$\left(\sigma_1^{a-1} \cdots \sigma_m^{a-1}\right)^c \sigma_1 \cdots \sigma_{j-1} \in \left(\sigma_1 \cdots \sigma_j\right)^{a-b} \operatorname{Stab}\left([\bar{\mathbf{u}}_j]\right).$$

However, the character $\mathfrak{D}(j-1)\mathfrak{D}(j)^{-1}$ takes the value 1 on the set on the right and takes the value μ on the element on the left, hence this is not possible.

Case 2: $(b = a+1)$ Reasoning similarly to above, we have that, for some integer c,

$$h = (\sigma_1^a \cdots \sigma_m^a)^c \sigma_1^{-1} \cdots \sigma_{j-1}^{-1} g.$$

This implies that for self-osculation, we need

$$(\sigma_1^a \cdots \sigma_m^a)^c \sigma_1^{-1} \cdots \sigma_{j-1}^{-1} \in (\sigma_1 \cdots \sigma_j)^{a-b} \operatorname{Stab}([\bar{\mathbf{u}}_j]).$$

However, the character $\mathfrak{D}(j-1)\mathfrak{D}(j)^{-1}$ takes the value 1 on the set on the right and takes the value μ^{-1} on the element on the left, hence this is not possible.

Case 3: $(b = a)$ Here we can have two further possibilities: the two edges could be joined at a vertex of height a or $a-1$. Note that in either case, this requires this vertex to be a branching vertex, hence the respective heights are not divisible by k.

In the former case, we get h from g by multiplying by a stabiliser of a vertex of height a, so for some integer $c \notin k\mathbb{Z}$ (because we want the two edges to be distinct) we have:

$$h = (\sigma_1^a \cdots \sigma_m^a)^c g.$$

This implies that for self-osculation, we need

$$(\sigma_1^a \cdots \sigma_m^a)^c \in (\sigma_1 \cdots \sigma_j)^{a-b} \operatorname{Stab}([\bar{\mathbf{u}}_j]).$$

However, since $a = b$, this is the same as asking for $(\sigma_1^a \cdots \sigma_m^a)^c$ to be in $\langle \sigma_{j-1}\sigma_j \rangle$. Since $m > 2$, the character $\mathfrak{D}(j+1)$ evaluates to 1 on the stabiliser, however it takes the value μ^{ac} on $(\sigma_1^a \cdots \sigma_m^a)^c$. These do not agree, as k is prime and neither of a or c are divisible by k.

In the latter case, we get h from g by multiplying by a stabiliser of a vertex of height $a-1$. Similarly, we obtain, for some integer $c \notin k\mathbb{Z}$,

$$h = \left(\sigma_1^{a-1} \cdots \sigma_m^{a-1}\right)^c g.$$

Just as in the previous case, this is the same as asking for $\left(\sigma_1^{a-1} \cdots \sigma_m^{a-1}\right)^c$ to be in $\langle \sigma_{j-1}\sigma_j \rangle$. The character $\mathfrak{D}(j+1)$ evaluates to 1 on the stabiliser, but takes the value

$\mu^{c(a-1)}$ on the element. These again do not coincide, as k is prime and neither of c or $a-1$ are divisible by k in this case.

Therefore, we have shown that no hyperplane self-osculates in $\bar{X}_m^k$ (not even indirectly).

4. ($\perp$,$\circlearrowleft$) Finally, we investigate whether two hyperplanes $\mathfrak{H}$, $\mathfrak{H}'$ can inter-osculate. In order for this to occur, we must have two distinct hyperplanes which cross. This can only occur between one of type $[\bar{\mathbf{u}}_j]$ and one of type $[\bar{\mathbf{u}}_{j+1}]$ (using cyclic indexing). In fact, since an intersection of hyperplanes can only happen at a square, it does not matter which corner of the square or which edges of that square are considered, since each edge in a pair of parallel edges represents the same hyperplane. Hence without loss of generality we may assume that it is the top corner. Now we proceed according to the two ways this corner can be, according to the fundamental domain in Figure 2.9.

Case 1: ($1 \leqslant j < m$) In this situation, $\mathfrak{H} \perp \mathfrak{H}'$ comes from edges $g \cdot \bar{\mathbf{u}}_j^a$ and $g \cdot \bar{\mathbf{u}}_{j+1}^a$ for some integer a and $g \in \bar{G}_m^k$. As when dealing with the self-osculation, we now have 4 possible sources of osculation to deal with:

- **Sub-case 1.1:** (joined at the top vertex) From the top corner of the fundamental domain we can deduce that osculation comes from edges $h \cdot \bar{\mathbf{u}}_j^b$ and $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_{j+1}^b$, for some integers $b, c \notin k\mathbb{Z}$ (because we are attached at a branching vertex and the edges are not meeting at the corner of a square, respectively) and some $h \in \bar{G}_m^k$. We now need to check if $h \cdot \bar{\mathbf{u}}_j^b \sim g \cdot \bar{\mathbf{u}}_j^a$ and $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_{j+1}^b \sim g \cdot \bar{\mathbf{u}}_{j+1}^a$ are both possible simultaneously. By Lemma 2.3, this implies:

  $$g \in \left(h\left(\sigma_1 \cdots \sigma_j\right)^{b-a} \operatorname{Stab}\left([\bar{\mathbf{u}}_j]\right)\right) \cap \left(h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \left(\sigma_1 \cdots \sigma_{j+1}\right)^{b-a} \operatorname{Stab}\left([\bar{\mathbf{u}}_{j+1}]\right)\right),$$

  so we need

  $$\left(\left(\sigma_1 \cdots \sigma_j\right)^{b-a} \langle \sigma_{j-1}\sigma_j \rangle\right) \cap \left(\left(\sigma_1^b \cdots \sigma_m^b\right)^c \left(\sigma_1 \cdots \sigma_{j+1}\right)^{b-a} \langle \sigma_j \sigma_{j+1} \rangle\right)$$

  to not be empty, which is the same as

  $$\langle \sigma_{j-1}\sigma_j \rangle \cap \left(\sigma_1^b \cdots \sigma_m^b\right)^c \sigma_{j+1}^{b-a} \langle \sigma_j \sigma_{j+1} \rangle$$

  not being empty. However, as $m > 3$, the character $\mathfrak{D}(j+2)$ evaluates to 1 on the left set and to $\mu^{bc} \neq 1$ on the right set, so the intersection is empty.

- **Sub-case 1.2:** (joined at the bottom vertex) From the bottom corner of the fundamental domain we can deduce that osculation comes from edges $h\sigma_j \cdot \bar{\mathbf{u}}_j^{b+1}$ and $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_{j+1}^{b+1}$ for some integers $b, c \notin k\mathbb{Z}$ and some $h \in \bar{G}_m^k$. Similarly to above, this implies that

$$\sigma_j \langle \sigma_{j-1}\sigma_j \rangle \cap \left(\sigma_1^b \cdots \sigma_m^b\right)^c \sigma_{j+1}^{b-a+1} \langle \sigma_j \sigma_{j+1} \rangle$$

is not empty. However, as $m > 3$, the character $\mathfrak{D}(j+2)$ evaluates to 1 on the left set and to $\mu^{bc} \neq 1$ on the right set, so the intersection is empty.

- **Sub-case 1.3:** (j above $j+1$) From the left corner of the fundamental domain we can deduce that osculation comes from edges $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_j^{b+1}$ and $h\sigma_1 \cdots \sigma_{j-1} \cdot \bar{\mathbf{u}}_{j+1}^b$ for some integers $b, c \notin k\mathbb{Z}$ and some $h \in \bar{G}_m^k$. Similarly to above, this implies that

$$\left(\sigma_1^b \cdots \sigma_m^b\right)^c \sigma_j \langle \sigma_{j-1}\sigma_j \rangle \cap \sigma_{j+1}^{b-a} \langle \sigma_j \sigma_{j+1} \rangle$$

is not empty. However, as $m > 3$, the character $\mathfrak{D}(j+2)$ evaluates to 1 on the right set and to $\mu^{bc} \neq 1$ on the left set, so the intersection is empty.

- **Sub-case 1.4:** (j below $j+1$) From the right corner of the fundamental domain we can deduce that osculation comes from edges $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_{j+1}^{b+1}$ and $h\sigma_1 \cdots \sigma_j \cdot \bar{\mathbf{u}}_j^b$ for some integers $b, c \notin k\mathbb{Z}$ and some $h \in \bar{G}_m^k$. Similarly to above, this implies that

$$\left(\sigma_1^b \cdots \sigma_m^b\right)^c \sigma_{j+1}^{b-a+1} \langle \sigma_j \sigma_{j+1} \rangle \cap \langle \sigma_{j-1}\sigma_j \rangle$$

is not empty. However, as $m > 3$, the character $\mathfrak{D}(j+2)$ evaluates to 1 on the right set and to $\mu^{bc} \neq 1$ on the left set, so the intersection is empty.

Case 2: ($j = m$, "$j+1$" $= 1$) In this situation, $\mathfrak{H} \perp \mathfrak{H}'$ comes from edges $g \cdot \bar{\mathbf{u}}_m^a$ and $g\sigma_1^a \cdots \sigma_m^a \cdot \bar{\mathbf{u}}_1^a$ for some integer a and $g \in \bar{G}_m^k$. Once again we have 4 possible sources of osculation to deal with:

- **Sub-case 2.1:** (joined at the top vertex) From the top corner of the fundamental domain we can deduce that osculation comes from edges $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_m^b$ and $h\sigma_1^b \cdots \sigma_m^b \cdot \bar{\mathbf{u}}_1^b$ for some integers $b, c \notin k\mathbb{Z}$ and some $h \in \bar{G}_m^k$. From Lemma 2.3, in order for this to occur, we need both of

$$g \in h\left(\sigma_1^b \cdots \sigma_m^b\right)^c (\bar{x}_1 \cdots \sigma_m)^{b-a} \langle \sigma_{m-1}\sigma_m \rangle$$

and

$$g\sigma_1^a \cdots \sigma_m^a \in h\sigma_1^b \cdots \sigma_m^b \sigma_1^{b-a} \langle \sigma_1 \sigma_m \rangle$$

to hold simultaneously. This is the same as

$$\sigma_1^{b-a} \langle \sigma_1 \sigma_m \rangle \cap \left(\sigma_1^b \cdots \sigma_m^b\right)^c \langle \sigma_{m-1}\sigma_m \rangle$$

not being empty. However, since $m > 3$, the character $\mathfrak{D}(2)$ evaluates to 1 on the left set and to $\mu^{bc} \neq 1$ on the right set, so the intersection is empty.

- **Sub-case 2.2:** (joined at the bottom vertex) From the bottom corner of the fundamental domain we can deduce that osculation comes from edges

$$h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_m^{b+1}$$

and $h\sigma_1^{b+1} \cdots \sigma_{m-1}^{b+1}\sigma_m^b \cdot \bar{\mathbf{u}}_1^{b+1}$ for some integers $b, c \notin k\mathbb{Z}$ and some $h \in \bar{G}_m^k$. Similarly to above, this implies that

$$\left(\sigma_1^b \cdots \sigma_m^b\right)^c \sigma_m \left\langle \sigma_{m-1}\sigma_m \right\rangle \cap \sigma_1^{b-a+1} \left\langle \sigma_1\sigma_m \right\rangle$$

is not empty. However, since $m > 3$, the character $\mathfrak{D}(2)$ evaluates to 1 on the right set and to $\mu^{bc} \neq 1$ on the left set, so the intersection is empty.

- **Sub-case 2.3:** (1 above m) From the right corner of the fundamental domain we can deduce that osculation comes from edges $h\sigma_1^{b+1} \cdots \sigma_m^{b+1} \cdot \bar{\mathbf{u}}_1^{b+1}$ and $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \sigma_1 \cdots \sigma_m \cdot \bar{\mathbf{u}}_m^b$ for some integers $b, c \notin k\mathbb{Z}$ and some $h \in \bar{G}_m^k$. Similarly to above, this implies that

$$\sigma_1^{b-a+1} \left\langle \sigma_1\sigma_m \right\rangle \cap \left(\sigma_1^b \cdots \sigma_m^b\right)^c \left\langle \sigma_{m-1}\sigma_m \right\rangle$$

is not empty. However, since $m > 3$, the character $\mathfrak{D}(2)$ evaluates to 1 on the left set and to $\mu^{bc} \neq 1$ on the right set, so the intersection is empty.

- **Sub-case 2.4:** (1 below m) From the left corner of the fundamental domain we can deduce that osculation comes from edges $h\left(\sigma_1^b \cdots \sigma_m^b\right)^c \cdot \bar{\mathbf{u}}_m^{b+1}$ and $h\sigma_1^{b+1} \cdots \sigma_{m-1}^{b+1}\sigma_m^b \cdot \bar{\mathbf{u}}_1^b$ for some integers $b, c \notin k\mathbb{Z}$ and some $h \in \bar{G}_m^k$. Similarly to above, this implies that

$$\left(\sigma_1^b \cdots \sigma_m^b\right)^c \sigma_m \left\langle \sigma_{m-1}\sigma_m \right\rangle \cap \sigma_1^{b-a} \left\langle \sigma_1\sigma_m \right\rangle$$

is not empty. However, since $m > 3$, the character $\mathfrak{D}(2)$ evaluates to 1 on the right set and to $\mu^{bc} \neq 1$ on the left set, so the intersection is empty. Hence there is no inter-osculation between hyperplanes in $\bar{X}_m^k$.

Hence $\bar{X}_m^k$ is a special cube complex.

Proof of Theorem 2.1: We have that $\ker \phi_m^k$ is torsion-free and X_m^k is simply-connected, therefore $\ker \phi_m^k = \pi_1\left(\bar{X}_m^k\right)$. We also have that $\ker \phi_m^k$ is a finite-index subgroup of G_m^k, hence G_m^k is virtually special. By Theorem 1.7, $\ker \phi_m^k$ embeds into a finitely generated RAAG, so is residually finite. Finally, as $\ker \phi_m^k$ is a finite-index subgroup, G_m^k is also residually finite. □

2.1.4 Computing the matrices

Since the proof of Theorem 2.1 goes via showing that the groups are linear, it should be possible to go through the steps in the proof and extract the corresponding faithful matrix representation. Since, in general, this is cumbersome, we will do it just for the case of $k = 2$, $m = 4$. In the language of Chapter 3, this is the unique 2-pattern of length 2, namely $[2, 1]$ (see Definition 3.10 there).

Using the group presentation of Theorem 2.1, the group $G = G_4^2$ is generated by 4 elements, call them a, b, c, d here. We will compute matrices for these generators and check group relations on them. Since the group is not finitely presented, a rigorous approach would need to use induction to prove that all of the infinitely many relations are satisfied. However, since this does not provide us with further insight into the theory, and our main goal is a computation to see what the matrices may look like, we will satisfy ourselves with checking only a few consequences of the relations (in the process, reducing the amount of computation required). To satisfy the presentation, we need, for every integer n:

$$o(a^n b^n c^n d^n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{otherwise} \end{cases}$$

From $a^2b^2c^2d^2 = 1$, we have $d^2 = c^{-2}b^{-2}a^{-2}$, so if we set

$$x := c^{-2}b^{-2}a^{-2},$$

then for every integer n we should have

$$a^{2n}b^{2n}c^{2n}x^n = 1. \tag{$\star$}$$

We will compute the matrices for a, b and c, and check ($\star$) for several different integers n.

The group is linear because it is virtually special. So we have to induce a linear representation from a finite-index subgroup H. To do this, we use a finite transversal. From (†), we know that our finite-index subgroup has index 16, as the quotient is $(C_2)^4$. A suitable transversal is then:

$$\{e, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}.$$

Given a linear representation of H,

$$\rho : H \to \mathrm{Mat}_d(\mathbb{C}),$$

we can induce up to a representation of the overgroup G by using the transversal $\{t_i\}$ via the standard formula:

$$\rho\uparrow_H^G(g) = \left(\rho(t_i^{-1}gt_j)\right) = \begin{pmatrix} \rho(t_1^{-1}gt_1) & \rho(t_1^{-1}gt_2) & \cdots & \rho(t_1^{-1}gt_n) \\ \rho(t_2^{-1}gt_1) & \rho(t_2^{-1}gt_2) & \cdots & \rho(t_2^{-1}gt_n) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(t_n^{-1}gt_1) & \rho(t_n^{-1}gt_2) & \cdots & \rho(t_n^{-1}gt_n) \end{pmatrix},$$

where each $\rho(t_i^{-1}gt_j)$ is given by either the appropriate image matrix if $t_i^{-1}gt_j \in H$, or by a zero matrix otherwise. We are now in a position to determine what the matrices for the elements a, b, c look like in terms of the representation for H.

The matrix for a is given below:

$$\begin{pmatrix}
0 & a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b^{-1}a^2b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c^{-1}a^2c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}a^2d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^{-1}b^{-1}a^2bc & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}b^{-1}a^2bd & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}a^2cd & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}b^{-1}a^2bcd \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0
\end{pmatrix}$$

where an entry g corresponds to a matrix $\rho(g)$ and 0 corresponds to the zero matrix. The matrices for b, c, d are moved to the appendix to save space here (see A.3). Therefore to finish obtaining the matrices, it suffices to compute the representation ρ.

This is achieved by recalling that H is the fundamental group of a special cube complex, therefore by [HW08] there is a map to a Salvetti complex. We represent each element $h \in H$ by an edge loop γ_h in the cube complex $X_L^M(S)/H$, by picking the basepoint to be X^0 and insisting that each edge loop γ_h must be confined to consisting only of edges with height 1.

Following [HW08], each hyperplane maps onto one edge in the Salvetti complex corresponding to $X_L^M(S)/H$. Hence we can map each loop γ_h to a loop in this Salvetti complex. In order to do that, we need to identify the underlying RAAG. The cube complex has a periodic structure with period 2, and we have already seen above that a transversal is of size 16. There are in total 32 distinct hyperplanes in the cube complex. The underlying graph of the corresponding RAAG is hence on 32 vertices, with edges given by pairwise intersections between the hyperplanes. The matrix `M` encodes these intersections, and is given in the appendix (see A.4).

The hyperplane intersections were computed using Figure 2.9. Finally, the following GAP code implements the matrices for a, b, c, by combining all of the above. Note that the function `ToArtin`, which appears in the appendix under the code listing (see A.1), is used to get the linear representation of each Artin generator. It is based on [HW99] and the standard reflection representation for right-angled Coxeter groups (see e.g. [Bro89]).

```
One1:=One(RandomMat(64,64));
# a
# a^2 : u1 - v1 + a.u1 - a.v1 : 1 - 9 + 2 - 10
A1x2A2 := ToArtin(M,1)*ToArtin(M,9)^-1*ToArtin(M,2)*ToArtin(M,10)^-1;
# b^-1 a^2 b : 21 -9 3 -10 5 -9 9 -21
A3x6B1A2B := ToArtin(M,21)*ToArtin(M,9)^-1*ToArtin(M,3)*ToArtin(M,10)^-1*ToArtin(
    ↪ M,5)*ToArtin(M,21)^-1;
# c^-1 a^2 c : 32 -18 4 -11 6 -13 18 -32
A4x7C1A2C := ToArtin(M,32)*ToArtin(M,18)^-1*ToArtin(M,4)*ToArtin(M,11)^-1*ToArtin
    ↪ (M,6)*ToArtin(M,13)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# d^-1 a^2 d : 1 -29 2 -12 1 -14 29 -1
A5x8D1A2D := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,2)*ToArtin(M,12)^-1*ToArtin(
    ↪ M,1)*ToArtin(M,14)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# c^-1 b^-1 a^2 bc : 32 -18 18 -11 7 -13 8 -11 11 -18 18 -32
A9x12C1B1A2BC := ToArtin(M,32)*ToArtin(M,11)^-1*ToArtin(M,7)*ToArtin(M,13)^-1*
    ↪ ToArtin(M,8)*ToArtin(M,32)^-1;
# d^-1 b^-1 a^2 bd : 1 -29 24 -12 5 -14 3 -12 12 -24 29 -1
A10x13D1B1A2BD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,24)*ToArtin(M,12)^-1*
    ↪ ToArtin(M,5)*ToArtin(M,14)^-1*ToArtin(M,3)*ToArtin(M,24)^-1*ToArtin(M,29)*
    ↪ ToArtin(M,1)^-1;
# d^-1 c^-1 a^2 cd : 1 -29 29 -22 6 -15 4 -16 22 -29 29 -1
A11x14D1C1A2CD := ToArtin(M,1)*ToArtin(M,22)^-1*ToArtin(M,6)*ToArtin(M,15)^-1*
    ↪ ToArtin(M,4)*ToArtin(M,16)^-1*ToArtin(M,22)*ToArtin(M,1)^-1;
#  d^-1 c^-1 b^-1 a^2 bcd : 1 -29 29 -22 22 -15 4 -16 15 -22 22 -29 29 -1
A15x16D1C1B1A2BCD := ToArtin(M,1)*ToArtin(M,15)^-1*ToArtin(M,8)*ToArtin(M,16)^-1*
    ↪ ToArtin(M,7)*ToArtin(M,1)^-1;
Ablocks := BlockMatrix([[2,1,One1],[1,2,A1x2A2],[6,3,One1],[7,4,One1],[8,5,One1
    ↪ ],[4,7,A4x7C1A2C],[5,8,A5x8D1A2D],[9,12,A9x12C1B1A2BC],[3,6,A3x6B1A2B
    ↪ ],[10,13,A10x13D1B1A2BD],[11,14,A11x14D1C1A2CD],[15,16,A15x16D1C1B1A2BCD
    ↪ ],[12,9,One1],[13,10,One1],[14,11,One1],[16,15,One1]],16,16);
a:=MatrixByBlockMatrix(Ablocks);
# b
# b^2 : 10 -18 9 -21
B1x3B2 := ToArtin(M,10)*ToArtin(M,18)^-1*ToArtin(M,9)*ToArtin(M,21)^-1;
# a^-1 bab : 10 -2 9 -17 5 -9 9 -21
B2x6A1BAB := ToArtin(M,10)*ToArtin(M,2)^-1*ToArtin(M,9)*ToArtin(M,17)^-1*ToArtin(
    ↪ M,5)*ToArtin(M,21)^-1;
# c^-1 b^2 c : 32 -18 13 -21 11 -18 18 -32
B4x9C1B2C := ToArtin(M,32)*ToArtin(M,18)^-1*ToArtin(M,13)*ToArtin(M,21)^-1*
    ↪ ToArtin(M,11)*ToArtin(M,18)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# d^-1 b^2 d : 1 -29 14 -22 12 -24 29 -1
B5x10D1B2D := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,14)*ToArtin(M,22)^-1*
    ↪ ToArtin(M,12)*ToArtin(M,24)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
```

```
# b^-1 a^-1 ba : 21 -9 9 -5 10 -19 2 -10
B6x2B1A1BA := ToArtin(M,21)*ToArtin(M,9)^-1*ToArtin(M,9)*ToArtin(M,5)^-1*ToArtin(
    ↪ M,10)*ToArtin(M,19)^-1*ToArtin(M,2)*ToArtin(M,10)^-1;
# c^-1 a^-1 babc : 32 -18 13 -6 11 -19 8 -11 11 -18 18 -32
B7x12C1A1BABC := ToArtin(M,32)*ToArtin(M,18)^-1*ToArtin(M,13)*ToArtin(M,6)^-1*
    ↪ ToArtin(M,11)*ToArtin(M,19)^-1*ToArtin(M,8)*ToArtin(M,11)^-1*ToArtin(M,11)*
    ↪ ToArtin(M,18)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# d^-1 a^-1 babd : 1 -29 14 -1 12 -20 3 -12 12 -24 29 -1
B8x13D1A1BABD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,14)*ToArtin(M,1)^-1*
    ↪ ToArtin(M,12)*ToArtin(M,20)^-1*ToArtin(M,3)*ToArtin(M,12)^-1*ToArtin(M,12)*
    ↪ ToArtin(M,24)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# d^-1 c^-1 b^2 cd : 1 -29 29 -22 16 -24 15 -22 22 -29 29 -1
B11x15D1C1B2CD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,22)^-1*
    ↪ ToArtin(M,16)*ToArtin(M,24)^-1*ToArtin(M,15)*ToArtin(M,22)^-1*ToArtin(M,22)
    ↪ *ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# c^-1 b^-1 a^-1 bac : 32 -18 18 -11 11 -8 13 -17 6 -13 18 -32
B12x7C1B1A1BAC := ToArtin(M,32)*ToArtin(M,18)^-1*ToArtin(M,18)*ToArtin(M,11)^-1*
    ↪ ToArtin(M,11)*ToArtin(M,8)^-1*ToArtin(M,13)*ToArtin(M,17)^-1*ToArtin(M,6)*
    ↪ ToArtin(M,13)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# d^-1 b^-1 a^-1 bad : 1 -29 24 -12 12 -3 14 -23 1 -14 29 -1
B13x8D1B1A1BAD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,24)*ToArtin(M,12)^-1*
    ↪ ToArtin(M,12)*ToArtin(M,3)^-1*ToArtin(M,14)*ToArtin(M,23)^-1*ToArtin(M,1)*
    ↪ ToArtin(M,14)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# d^-1 c^-1 a^-1 babcd : 1 -29 29 -22 16 -4 15 -23 7 -15 15 -22 22 -29 29 -1
B14x16D1C1A1BABCD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,22)
    ↪ ^-1*ToArtin(M,16)*ToArtin(M,4)^-1*ToArtin(M,15)*ToArtin(M,23)^-1*ToArtin(M
    ↪ ,7)*ToArtin(M,15)^-1*ToArtin(M,15)*ToArtin(M,22)^-1*ToArtin(M,22)*ToArtin(M
    ↪ ,29)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# d^-1 c^-1 b^-1 a^-1 bacd : 1 -29 29 -22 22 -15 15 -7 16 -20 4 -16 22 -29 29 -1
B16x14D1C1B1A1BACD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,22)
    ↪ ^-1*ToArtin(M,22)*ToArtin(M,15)^-1*ToArtin(M,15)*ToArtin(M,7)^-1*ToArtin(M
    ↪ ,16)*ToArtin(M,20)^-1*ToArtin(M,4)*ToArtin(M,16)^-1*ToArtin(M,22)*ToArtin(M
    ↪ ,29)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
Bblocks := BlockMatrix([[1,3,B1x3B2],[2,6,B2x6A1BAB],[3,1,One1],[4,9,B4x9C1B2C
    ↪ ],[5,10,B5x10D1B2D],[6,2,B6x2B1A1BA],[7,12,B7x12C1A1BABC],[8,13,
    ↪ B8x13D1A1BABD],[9,4,One1],[10,5,One1],[11,15,B11x15D1C1B2CD],[12,7,
    ↪ B12x7C1B1A1BAC],[13,8,B13x8D1B1A1BAD],[14,16,B14x16D1C1A1BABCD],[15,11,One1
    ↪ ],[16,14,B16x14D1C1B1A1BACD]],16,16);
b:=MatrixByBlockMatrix(Bblocks);
# c
# c^2 : 21 -29 18 -32
C1x4C2 := ToArtin(M,21)*ToArtin(M,29)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# a^-1 cac : 10 -2 19 -27 6 -13 18 -32
C2x7A1CAC := ToArtin(M,10)*ToArtin(M,2)^-1*ToArtin(M,19)*ToArtin(M,27)^-1*ToArtin
    ↪ (M,6)*ToArtin(M,13)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# b^-1 cbc : 21 -9 18 -26 11 -18 18 -32
C3x9B1CBC := ToArtin(M,21)*ToArtin(M,9)^-1*ToArtin(M,18)*ToArtin(M,26)^-1*ToArtin
    ↪ (M,11)*ToArtin(M,18)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# d^-1 c^2 d : 1 -29 24 -32 22 -29 29 -1
C5x11D1C2D := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,24)*ToArtin(M,32)^-1*
    ↪ ToArtin(M,22)*ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# b^-1 a^-1 cabc : 21 -9 9 -5 17 -25 8 -11 11 -18 18 -32
C6x12B1A1CABC := ToArtin(M,21)*ToArtin(M,9)^-1*ToArtin(M,9)*ToArtin(M,5)^-1*
    ↪ ToArtin(M,17)*ToArtin(M,25)^-1*ToArtin(M,8)*ToArtin(M,11)^-1*ToArtin(M,11)*
    ↪ ToArtin(M,18)^-1*ToArtin(M,18)*ToArtin(M,32)^-1;
# c^-1 a^-1 ca : 32 -18 13 -6 17 -31 2 -10
C7x2C1A1CA := ToArtin(M,32)*ToArtin(M,18)^-1*ToArtin(M,13)*ToArtin(M,6)^-1*
    ↪ ToArtin(M,17)*ToArtin(M,31)^-1*ToArtin(M,2)*ToArtin(M,10)^-1;
# d^-1 a^-1 cacd : 1 -29 14 -1 23 -31 4 -16 22 -29 29 -1
```

```
C8x14D1A1CACD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,14)*ToArtin(M,1)^-1*
    ↪ ToArtin(M,23)*ToArtin(M,31)^-1*ToArtin(M,4)*ToArtin(M,16)^-1*ToArtin(M,22)*
    ↪ ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# c^-1 b^-1 cb : 32 -18 18 -11 21 -30 9 -21
C9x3C1B1CB := ToArtin(M,32)*ToArtin(M,18)^-1*ToArtin(M,18)*ToArtin(M,11)^-1*
    ↪ ToArtin(M,21)*ToArtin(M,30)^-1*ToArtin(M,9)*ToArtin(M,21)^-1;
# d^-1 b^-1 cbcd : 1 -29 24 -12 22 -30 15 -22 22 -29 29 -1
C10x15D1B1CBCD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,24)*ToArtin(M,12)^-1*
    ↪ ToArtin(M,22)*ToArtin(M,30)^-1*ToArtin(M,15)*ToArtin(M,22)^-1*ToArtin(M,22)
    ↪ *ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# c^-1 b^-1 a^-1 cab : 32 -18 18 -11 11 -8 19 -28 5 -9 9 -21
C12x6C1B1A1CAB := ToArtin(M,32)*ToArtin(M,18)^-1*ToArtin(M,18)*ToArtin(M,11)^-1*
    ↪ ToArtin(M,11)*ToArtin(M,8)^-1*ToArtin(M,19)*ToArtin(M,28)^-1*ToArtin(M,5)*
    ↪ ToArtin(M,9)^-1*ToArtin(M,9)*ToArtin(M,21)^-1;
# d^-1 b^-1 a^-1 cabcd : 1 -29 24 -12 12 -3 20 -28 7 -15 15 -22 22 -29 29 -1
C13x16D1B1A1CABCD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,24)*ToArtin(M,12)
    ↪ ^-1*ToArtin(M,12)*ToArtin(M,3)^-1*ToArtin(M,20)*ToArtin(M,28)^-1*ToArtin(M
    ↪ ,7)*ToArtin(M,15)^-1*ToArtin(M,15)*ToArtin(M,22)^-1*ToArtin(M,22)*ToArtin(M
    ↪ ,29)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# d^-1 c^-1 a^-1 cad : 1 -29 29 -22 16 -4 20 -27 1 -14 29 -1
C14x8D1C1A1CAD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,22)^-1*
    ↪ ToArtin(M,16)*ToArtin(M,4)^-1*ToArtin(M,20)*ToArtin(M,27)^-1*ToArtin(M,1)*
    ↪ ToArtin(M,14)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# d^-1 c^-1 b^-1 cbd : 1 -29 29 -22 22 -15 24 -26 12 -24 29 -1
C15x10D1C1B1CBD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,22)^-1*
    ↪ ToArtin(M,22)*ToArtin(M,15)^-1*ToArtin(M,24)*ToArtin(M,26)^-1*ToArtin(M,12)
    ↪ *ToArtin(M,24)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
# d^-1 c^-1 b^-1 a^-1 cabd : 1 -29 29 -22 22 -15 15 -7 23 -25 3 -12 12 -24 29 -1
C16x13D1C1B1A1CABD := ToArtin(M,1)*ToArtin(M,29)^-1*ToArtin(M,29)*ToArtin(M,22)
    ↪ ^-1*ToArtin(M,22)*ToArtin(M,15)^-1*ToArtin(M,15)*ToArtin(M,7)^-1*ToArtin(M
    ↪ ,23)*ToArtin(M,25)^-1*ToArtin(M,3)*ToArtin(M,12)^-1*ToArtin(M,12)*ToArtin(M
    ↪ ,24)^-1*ToArtin(M,29)*ToArtin(M,1)^-1;
Cblocks := BlockMatrix([[1,4,C1x4C2],[2,7,C2x7A1CAC],[3,9,C3x9B1CBC],[4,1,One1
    ↪ ],[5,11,C5x11D1C2D],[6,12,C6x12B1A1CABC],[7,2,C7x2C1A1CA],[8,14,
    ↪ C8x14D1A1CACD],[9,3,C9x3C1B1CB],[10,15,C10x15D1B1CBCD],[11,5,One1],[12,6,
    ↪ C12x6C1B1A1CAB],[13,16,C13x16D1B1A1CABCD],[14,8,C14x8D1C1A1CAD],[15,10,
    ↪ C15x10D1C1B1CBD],[16,13,C16x13D1C1B1A1CABD]],16,16);
c:=MatrixByBlockMatrix(Cblocks);
```

We are now in a position to test the matrices for a, b, c. The following is output directly from GAP. We satisfy ourselves with the following computations:

```
gap> x:=c^-2*b^-2*a^-2;;
gap> Order(a^4*b^4*c^4*x^2);
1
gap> Order(a^12*b^12*c^12*x^6);
1
gap> Order(a^-8*b^-8*c^-8*x^-4);
1
gap> Order(a^100*b^100*c^100*x^50);
1
gap> Size(a);
1024
```

LISTING 2.1: Testing the derived matrices.

Note that the size of the matrices comes from

$$16 \times 32 \times 2$$

due to subgroup index, number of hyperplanes, and the way we embed RAAGs into RACGs, respectively. It is likely to be the case that a faithful linear representation of smaller dimension exists, given that the matrices appear fairly sparse.

2.2 A family of virtually special groups via different S

We were able to use linear characters in Section 2.1 because the finite quotients with torsion-free kernel had the particularly nice form of being a direct product of cyclic groups. In order to allow this, we had to vary L (as various flag subdivisions of the circle) and adjust the branching set S accordingly, to allow the presentations to be satisfied. However, one may be interested in applications of the theory where there could be a requirement to keep L fixed. In general, a torsion-free finite-index subgroup could have a much more complicated structure for the corresponding finite quotient group (see Chapter 3). In this section, we examine a more involved case of keeping $M \to L$ fixed and obtaining a family of virtually special groups $G_L^M(S)$ by varying the branching set S. In particular, we prove:

Theorem 2.4. *Let L be the 12-vertex triangulation of the circle. For $M \to L$ a finite regular cover of degree k, the following are equivalent:*

- *$G_L^M(S)$ is virtually special.*
- *$G_L^M(S)$ is virtually torsion-free.*
- *S is periodic.*

Note that here, k can be any positive integer greater than 1 (with $k = 1$ being the case of the ordinary Bestvina-Brady group). The fundamental group $\pi_1(L)$ is $\mathbb{Z}$, which has a unique normal subgroup of index k, namely $k\mathbb{Z}$. The corresponding cover M is the $12k$-vertex triangulation of the circle. Both the triangulations of M and L are flag complexes. We shall denote the generalised Bestvina-Brady group by $G_{12}^k(S)$.

The proof will mimic that of Theorem 2.1 but using more general techniques, also relying on some results from Chapter 3. Note that said chapter deals with the equivalence of the last two bullet points, and the first bullet point implying the second is a well-known result. Thus, here we shall focus solely on showing that when S is periodic, we have a finite-index torsion-free subgroup which is the fundamental group of a special cube complex (that is, establishing that the third bullet point implies the first). We will attempt to generalise the idea of using linear characters outside of an abelian setting, by exploiting a direct product structure in this case. Projections onto different factors will play the role of picking out linear characters.

We use the same notation for the presentations of the groups as in Section 2.1, with the changes that we can vary S, and k does not necessarily have to be prime:

$$G_{12}^k(S) := \left\langle x_1, x_2, \ldots, x_{12} \;\middle|\; \begin{matrix} x_1^i x_2^i \cdots x_{12}^i & \text{for } i \in S \\ \left(x_1^i x_2^i \cdots x_{12}^i\right)^k & \text{for } i \notin S \end{matrix} \right\rangle.$$

2.2.1 Non-abelian tools

Note that Lemma 2.2 does not depend on S, so we can apply it in this subsection. However, the calculation of the hyperplane stabilisers, as well as Figure 2.10 and Lemma 2.3, were simplified due to the finite quotient group being abelian. We will derive more general versions of these in order to apply them to the non-abelian quotients needed for Theorem 2.4. We will use the same notation as defined in Section 2.1 for the action on the associated CAT(0) cube complex.

We begin with a version of Figure 2.10 for before taking a quotient. Reading the appropriate multiplication factors from Figure 2.2 of Lemma 2.2, we obtain Figure 2.11 due to similar reasoning as in Subsection 2.1.2. Note that since we are no longer in an abelian setting, it is important to remember that each multiplication of an edge coefficient when moving layers should be thought of as occurring on the right, whereas the group of deck transformations of the branched cover acts on the left.

In Figure 2.11, we use the same shorthand notation as in Subsection 2.1.1:

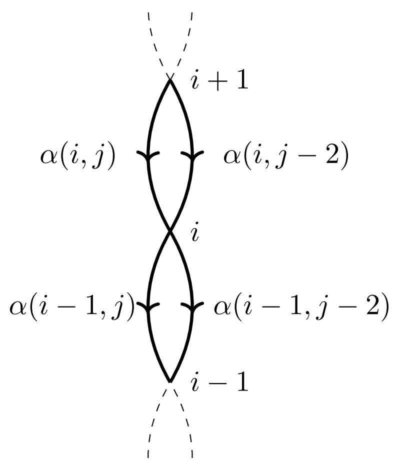
$$\alpha(i,j) := x_1^{i+1} \cdots x_{j-1}^{i+1}\, x_j\, x_{j-1}^{-i} \cdots x_1^{-i}$$

Note that after taking the appropriate quotient, we can recover Figure 2.10 from Figure 2.11.

We are now ready to generalise the computation of hyperplane stabilisers and Lemma 2.3. Note that we are once again using cyclic indexing.

Lemma 2.5. *For integers a, b and elements $g, h \in G_{12}^k(S)$, we have:*

$$g \cdot \mathbf{u}_j^a \sim h \cdot \mathbf{u}_j^b \implies h \in g \cdot x_1^a \cdots x_j^a \mathrm{Stab}\left([\mathbf{u}_j^0]\right) x_j^{-b} \cdots x_1^{-b},$$



(A) $1 < j \leqslant 12$ (B) $j = 1$

FIGURE 2.11: Moving between edges of different heights in the same hyperplane of type $\mathbf{u}_j$.

where

$$\mathrm{Stab}\left([\mathbf{u}_j^0]\right) = \left\langle x_{j-1}^n x_j^n \ \forall n \in \mathbb{Z} \right\rangle.$$

Proof. Using Figure 2.11, the derivation is exactly the same as in Subsection 2.1.2. Each hyperplane will have some edge representative in the 0-level set, so we use $i = 0$ to compute the particularly nice form for the stabilisers. Note that while previously, we had the stabiliser act on the left (it did not matter as everything was abelian), here is it important to make sure it acts on the right. In particular, it does not matter what the coefficient of an edge is, they will all have the same stabiliser according to just their type.

Moving along the left side of Figure 2.11, we get:

$$g \cdot \mathbf{u}_j^a \sim g x_1^a \cdots x_j^a \cdot \mathbf{u}_j^0$$

and

$$h \cdot \mathbf{u}_j^b \sim h x_1^b \cdots x_j^b \cdot \mathbf{u}_j^0.$$

If they are in the same hyperplane, then there exists some $d \in \mathrm{Stab}\left([\mathbf{u}_j^0]\right)$ such that:

$$g x_1^a \cdots x_j^a d = h x_1^b \cdots x_j^b,$$

and the result follows. $\square$

Note that after taking the appropriate quotient, we recover the same results as in Subsection 2.1.2.

2.2.2 A direct product of groups

Given a periodic S as above, let I be the corresponding period block starting from 1 (this will be called a k-pattern in Definition 3.10). Denote its length by m. We use the quotient with torsion-free kernel from Subsection 3.3.3 (referred to as the "solution" of this pattern there), and we will also use some of its properties established at the end of that subsection. Denoting this solution (the images of the elements that give a quotient with torsion-free kernel) by $a, b, c, d \in \bar{G}_I$, the homomorphism:

$$
\begin{aligned}
\psi : G^k_{12}(S) &\to \bar{G}_I \times \bar{G}_I \times \bar{G}_I \\
x_1 &\mapsto (a, e, e) \\
x_2 &\mapsto (e, a, e) \\
x_3 &\mapsto (e, e, a) \\
x_4 &\mapsto (b, e, e) \\
x_5 &\mapsto (e, b, e) \\
x_6 &\mapsto (e, e, b) \\
x_7 &\mapsto (c, e, e) \\
x_8 &\mapsto (e, c, e) \\
x_9 &\mapsto (e, e, c) \\
x_{10} &\mapsto (d, e, e) \\
x_{11} &\mapsto (e, d, e) \\
x_{12} &\mapsto (e, e, d)
\end{aligned}
$$

gives a torsion-free kernel. Hence $H = \ker \psi$ is the finite-index torsion-free subgroup we want. We will proceed to show that the corresponding quotient cube complex $X^M_L(S)/H$ is special. We will be observing what happens in each of the three factors of this direct product along the way. We may also abuse notation and refer to images of elements by the names of the elements themselves, if the context permits.

Definition 2.6. We say that a solution to a pattern is *separating* if no non-trivial proper power of an element from the solution (we call this a *solution element*) can ever equal a conjugate of a non-trivial point stabiliser. In other words, if $g^n = h^{-1}a^ib^ic^id^ih$, then $g^n = e$ for $g \in \{a, b, c, d\}$.

One of the key features of our construction is the fact that hyperplane stabilisers (in the image of the quotient) will take the form of powers of solution elements in each factor of the direct product $\bar{G}_I \times \bar{G}_I \times \bar{G}_I$ (this is how we mimic "linear characters" outside of an abelian setting). Hence, given a separating solution, they can never be conjugate to a proper (non-trivial) vertex stabiliser.

Definition 2.7. We say that the torsion-free kernel of a map from $G_L^M(S)$ to a finite group is *degenerate* if any of the squares from Lemma 2.2 collapse in the quotient complex (i.e. if any of the squares have an image with fewer than 4 distinct vertices).

Note that in order for a quotient to be degenerate,

$$\alpha(i, j-1) = \alpha(i, j)$$

must hold for some integers i, j. This, in turn, means that one of the solution elements x_j must map to the identity in the quotient. This does not happen in our solution, as each solution element is of order m.

Definition 2.8. We say that a solution to a pattern is *central* if the vertex stabilisers all lie in the centre of the solution group.

Note that our solution is also central when the period of S is length at least three (see the end of Subsection 3.3.3). We are now ready to check if the quotient cube complex is special.

Lemma 2.9 (Abelian stabilisers)**.** *Hyperplane stabilisers are abelian in the quotient defined above. Furthermore, they are simply powers of solutions elements (or identity) in each factor of the direct product.*

Proof. By construction, each solution element (using cyclic indexing) x_j commutes with each of $x_{j-2}, x_{j-1}, x_{j+1}, x_{j+2}$, since they lie in distinct factors of the direct product. The result now follows from the form of the stabilisers from Lemma 2.5. □

We will keep in mind that for all j, we have that x_j, x_{j-1} and x_{j+1} all pairwise commute.

Orientation and self-intersection: Exactly the same arguments as in Subsection 2.1.3 apply to show that every hyperplane is 2-sided and no hyperplane crosses itself.

Self-osculation: Suppose that some hyperplane self-osculates. That means that there are two edges which are in the same hyperplane that osculate. If they are both in the same layer, they must be attached at the top or bottom to the same vertex. Consequently, this means that a non-trivial element of the hyperplane stabiliser is also a vertex stabiliser. This is because only one edge of each type is attached above or below a non-branching vertex. Using Lemma 2.9 and passing to a factor of the direct product, we now get a proper vertex stabiliser equal to a power of a solution element. But by construction this cannot happen, as our solution is separating.

The only other possibility here is that the two edges that osculate are in different layers. However, if they are in the same hyperplane, consider some orientation of the edges. Both edges will either point upwards, or both point downwards. Therefore, even if there is some osculation, it will be indirect, which is allowed. Thus no direct self-osculation occurs in the complex.

Inter-osculation: We split the cases according to the possibilities from before, but we group them differently here. The main cases are about how the osculation can occur (according to Figure 2.2), so we can make thinking about the vertex stabilisers easier, by translation.

WLOG all crossings happen at the top corner of a square. Every osculation needs to involve a proper stabiliser, since otherwise the two edges would cross rather than osculate at a vertex. The sub-cases are split into the two cases in Figure 2.2. The strategy is the same in every case: write down equations that must hold and manipulate them to establish a power of a solution element being a conjugate of a proper stabiliser. The shorthand functions α, β, γ from before are utilised for brevity.

Case 1: Joined at the top vertex:

- **Sub-case 1.1**: ($1 \leqslant j < 12$) By translation, the osculation comes from edges $\mathbf{u}_j^a$ and $(x_1^a \cdots x_{12}^a)^c \cdot \mathbf{u}_{j+1}^a$ for some integers $c \notin k\mathbb{Z}$ and $a \notin S$. The intersection comes from edges $g \cdot \mathbf{u}_j^b$ and $g \cdot \mathbf{u}_{j+1}^b$ for some integer b. By Lemma 2.5, this means that there exist integers d, r such that:

  $$g = x_1^a \cdots x_j^a x_j^d x_{j-1}^d x_j^{-b} \cdots x_1^{-b}$$

  and

  $$g = (x_1^a \cdots x_{12}^a)^c\, x_1^a \cdots x_{j+1}^a x_{j+1}^r x_j^r x_{j+1}^{-b} \cdots x_1^{-b}.$$

  Passing to the factor of $\bar{G}_I \times \bar{G}_I \times \bar{G}_I$ on which x_{j-1} is not trivial, this re-arranges to:

  $$x_1^a \cdots x_{j-1}^a x_{j-1}^d = (x_1^a \cdots x_{12}^a)^c\, x_1^a \cdots x_{j-1}^a,$$

  which gives a power of x_{j-1} as a conjugate of a proper vertex stabiliser. This cannot happen, as our solution is separating.

- **Sub-case 1.2**: ($j = 12$, "$j+1$" $= 1$) The intersection comes from edges $g \cdot \mathbf{u}_{12}^b$ and $g x_1^b \cdots x_{12}^b \cdot \mathbf{u}_1^b$ for some integer b. The osculation comes from edges

$(x_1^a \cdots x_{12}^a)^c \cdot \mathbf{u}_{12}^a$ and $x_1^a \cdots x_{12}^a \cdot \mathbf{u}_1^a$, for some $a \notin S$ and $c \notin k\mathbb{Z}$. By Lemma 2.5, this means that there are integers d, r such that:

$$g = (x_1^a \cdots x_{12}^a)^c x_1^a \cdots x_{12}^a x_{11}^d x_{12}^d x_{12}^{-b} \cdots x_1^{-b}$$

and

$$g x_1^b \cdots x_{12}^b = x_1^a \cdots x_{12}^a x_1^a x_1^r x_{12}^r x_1^{-b}.$$

Passing to the x_{11}-factor and re-arranging yields a power of x_{11} as a conjugate of a proper vertex stabiliser, which contradicts the solution being separating.

Case 2: Joined at the bottom vertex:

- **Sub-case 2.1**: $(1 \leqslant j < 12)$ An osculation must come from some $(a-1) \notin S$, $c \notin k\mathbb{Z}$ with

$$\beta(a,j)^{-1}\alpha(a,j-1) \cdot \mathbf{u}_{j+1}^a \circlearrowleft \left(x_1^{a-1} \cdots x_{12}^{a-1}\right)^c \beta(a,j)^{-1}\alpha(a,j) \cdot \mathbf{u}_j^a,$$

  while a crossing must come from some $b \in \mathbb{Z}$, $g \in G_{12}^k(S)$ with

$$g \cdot \mathbf{u}_j^b \perp g \cdot \mathbf{u}_{j+1}^b.$$

  Similar reasoning to above and passing to the $(j+1)$-factor (where x_{j+1} does not vanish) yields a contradiction via the separating solution.

- **Sub-case 2.2**: $(j = 12$, "$j+1$" $= 1)$ An osculation must come from some $(a-1) \notin S$, $c \notin k\mathbb{Z}$ with

$$\beta(a,12)^{-1}\gamma(a,12)x_{12}^{-1} \cdot \mathbf{u}_1^a \circlearrowleft \left(x_1^{a-1} \cdots x_{12}^{a-1}\right)^c \beta(a,12)^{-1}\alpha(a,12) \cdot \mathbf{u}_{12}^a,$$

  while a crossing must come from some $b \in \mathbb{Z}$, $g \in G_{12}^k(S)$ with

$$\mathbf{u}_{12}^b \perp \gamma(b-1,12) \cdot \mathbf{u}_1^b.$$

  Similar reasoning to above and passing to the x_1-factor yields a contradiction via the separating solution.

Case 3: Joined at an intermediate vertex, j above $j+1$:

- **Sub-case 3.1**: $(1 \leqslant j < 12)$ An osculation must come from some $a \notin S$, $c \notin k\mathbb{Z}$ with

$$\alpha(a,j-1)^{-1} \cdot \mathbf{u}_j^{a+1} \circlearrowleft (x_1^a \cdots x_{12}^a)^c \cdot \mathbf{u}_{j+1}^a,$$

while a crossing must come from some $b \in \mathbb{Z}$, $g \in G^k_{12}(S)$ with

$$g \cdot \mathbf{u}^b_j \perp g \cdot \mathbf{u}^b_{j+1}.$$

Similar reasoning to above and passing to the $(j-1)$-factor (using cyclic indexing, so the x_{12}-factor for $j = 1$) yields a contradiction via the separating solution.

- **Sub-case 3.2**: ($j = 12$, "$j + 1$" $= 1$) An osculation must come from some $a \notin S$, $c \notin k\mathbb{Z}$ with

$$(x^a_1 \cdots x^a_{12})^c\, \alpha(a, 11)^{-1} \cdot \mathbf{u}^{a+1}_{12} \circlearrowright \alpha(a, 11)^{-1}\gamma(a, 12)x^{-1}_{12} \cdot \mathbf{u}^a_1,$$

while a crossing must come from some $b \in \mathbb{Z}$, $g \in G^k_{12}(S)$ with

$$g \cdot \mathbf{u}^b_{12} \perp g\gamma(b - 1, 12) \cdot \mathbf{u}^b_1.$$

Similar reasoning to above and passing to the x_{11}-factor yields a contradiction via the separating solution.

Case 4: Joined at an intermediate vertex, j below $j + 1$:

- **Sub-case 4.1**: ($1 \leqslant j < 12$) An osculation must come from some $a \notin S$, $c \notin k\mathbb{Z}$ with

$$(x^a_1 \cdots x^a_{12})^c \cdot \mathbf{u}^a_j \circlearrowright \alpha(a, j)^{-1} \cdot \mathbf{u}^{a+1}_{j+1},$$

while a crossing must come from some $b \in \mathbb{Z}$, $g \in G^k_{12}(S)$ with

$$g \cdot \mathbf{u}^b_j \perp g \cdot \mathbf{u}^b_{j+1}.$$

Similar reasoning to above and passing to the $(j-1)$-factor (using cyclic indexing, so the x_{12}-factor for $j = 1$) yields a contradiction via the separating solution.

- **Sub-case 4.2**: ($j = 12$, "$j + 1$" $= 1$) An osculation must come from some $a \notin S$, $c \notin k\mathbb{Z}$ with

$$(x^a_1 \cdots x^a_{12})^c \cdot \mathbf{u}^a_{12} \circlearrowright \alpha(a, 12)^{-1}\gamma(a, 12) \cdot \mathbf{u}^{a+1}_1,$$

while a crossing must come from some $b \in \mathbb{Z}$, $g \in G^k_{12}(S)$ with

$$g \cdot \mathbf{u}^b_{12} \perp g\gamma(b - 1, 12) \cdot \mathbf{u}^b_1.$$

Similar reasoning to above and passing to the x_{11}-factor yields a contradiction via the separating solution.

Proof of Theorem 2.4: The above shows that the third bullet point implies the first. The first bullet point implies the second because virtually special groups are linear over

$\mathbb{Z}$ [HW08], and thus are virtually torsion-free by Selberg's Lemma [Sel60]. The second bullet point implies the third by Theorem 3.1. Therefore the three bullet points are equivalent. □

Note that the key idea in the proof was using three copies of the same quotient with torsion-free kernel, stitched together via a direct product. The argument given here is minimal in the sense that 12 vertices for L is the minimum required to make this work (4 elements per quotient, at least 3 copies of the quotient). The exact same method works for any larger subdivision of l vertices of the circle for $l \geqslant 12$ by padding out with identity elements in the image.

2.2.3 Generalisations

Note that there was nothing special about the number 12 in the above proof, except the fact that when dealing with inter-osculations we had two L-adjacent edges (say, $\mathbf{u}_j$ and $\mathbf{u}_{j+1}$) and thus their respective stabilisers featured 3 different kinds of elements x_i. This meant that to properly separate the effect of the vertex stabilisers on the hyperplane structure we needed at least 3 factors to project to. Since our solution for general periodic sets S uses 4 elements, this led to $4 \times 3 = 12$ being the minimal number of variables needed. Exactly the same arguments apply for larger complexes.

The powerful idea here was constructing a quotient where hyperplane stabilisers are abelian, despite the original torsion-free finite-index kernel not having an abelian quotient group.

We are going to generalise the first part of Theorem 2.4 to any $G_L^M(S)$ of finite ramification in an upcoming paper with Ian Leary, provided that L is subdivided sufficiently [LV]. The method of proof is similar, passing to a larger quotient made with a direct product and featuring abelian stabilisers.

This begs the question of whether smaller flag triangulations yield virtually special groups. In particular, it is not yet known if the result holds for L being the 4-vertex (smallest possible flag) triangulation of the circle, even though we know $G_L^M(S)$ is virtually torsion-free for any suitable M and periodic S in this case.

Chapter 3

Torsion-free finite-index subgroups

It quickly becomes apparent that the key property underpinning everything, and what causes the most difficulty, is being virtually torsion-free. The source of torsion in the group is from finite ramification.

We begin to establish a connection between the branching set S and the property of being virtually torsion-free:

Theorem 3.1. *Assume that $G_L^M(S)$ is of finite ramification and that S is non-empty. Then it is virtually torsion-free only if S is periodic.*

Proof. We may assume that the group is not a Bestvina-Brady group (which, note, are torsion-free), since then the statement is true as $\mathbb{Z}$ is a periodic subset of itself with period 1. Assume that $G_L^M(S)$ is virtually torsion-free. Then there exists $H \dot{<} G_L^M(S)$ which is torsion-free, and in particular $H \neq G_L^M(S)$ because $S \neq \mathbb{Z}$, as we are assuming the group is not a Bestvina-Brady group. This implies that there is torsion in $G_L^M(S)$, coming from finite point stabilisers, as the group of deck transformations $M \to L$ is finite and non-trivial. Let $X = G_L^M(S)/H$ denote the set of left cosets of H in $G_L^M(S)$. There is a left $G_L^M(S)$-action on X:

$$G_L^M(S) \times X \to X \ , \ \ g' \cdot (gH) = (g'g)H \ , \ \ \forall\, g', g \in G_L^M(S).$$

Because H is of finite index in $G_L^M(S)$, we have $r = |X| < \infty$. The action gives a homomorphism

$$\phi : G_L^M(S) \to Sym_X \cong S_r.$$

Since $gH = H \implies g \in H$, we get $\ker \phi \leqslant H$.

Let p be a prime number dividing the order of the group of deck transformations $M \to L$ and let γ be a loop in L whose representative in this group has order p. Choose an edge loop $a_1, \ldots, a_m$ in L corresponding to γ.

Since S is non-empty, by translation of the height function we may assume without loss of generality that zero is in S. Utilising the presentation of Definition 1.1 in [Lea18], we get group generators from edges, so consider $\phi(a_1),\ \ldots,\ \phi(a_m)$. These are finite group elements in the finite group S_r. As ϕ is a homomorphism, denoting the order of an element g by $o(g)$,we get the infinite sequence J:

$$\ldots,\ o\left(\phi(a_1)^{-1}\cdots\phi(a_m)^{-1}\right),\ o\left(\phi(a_1)^{0}\cdots\phi(a_m)^{0}\right)=1,$$
$$o\left(\phi(a_1)\cdots\phi(a_m)\right),\ o\left(\phi(a_1)^{2}\cdots\phi(a_m)^{2}\right),\ldots$$

This is now a periodic sequence (because each $\phi(a_i)$ is an element in a finite group, thus the lowest common multiple of their finitely many orders is an upper bound on the period length) with each term in $\{1,p\}$ because of the long cycle relations corresponding to γ, with some finite period. Now H being torsion-free implies that $\ker\phi$ is also torsion-free. For $i\notin S$, $a_1^i\cdots a_m^i$ is a torsion element, and hence cannot belong to $\ker\phi$, therefore $o\left(\phi(a_1)^i\cdots\phi(a_m)^i\right)=p$, because of the homomorphism its order is either p or 1. But for $i\in S$, we have $a^i\cdots a_m^i=1$, and since ϕ is a homomorphism,

$$o\left(\phi(a_1)^i\cdots\phi(a_m)^i\right)=1$$

too. This means that if S is not periodic, then J is not periodic. Hence

$$G_L^M(S)\text{ virtually torsion-free}\implies J\text{ periodic}\implies S\text{ periodic.}$$

□

The hardest challenge is determining when the converse is true. Note that the above holds true for L of any dimension. We conjecture that the converse may always be true.

Note that Theorem 3.1 proves that the second bullet point implies the third bullet point in Theorem 2.4 as a special case.

Let $\mathfrak{R}$ be the set of generalised Bestvina-Brady groups of finite ramification which are virtually special, up to isomorphism.

Corollary 3.2. *The set $\mathfrak{R}$ is countably infinite.*

Proof. Since there are countably many periodic subsets of $\mathbb{Z}$, countably many finite groups and countably many finite connected flag complexes, there are at most countably many generalised Bestvina-Brady groups of finite ramification which are virtually torsion-free. Groups which are virtually special must be virtually torsion-free. Hence $\mathfrak{R}$ is at most countable. It is not finite because Theorem 2.1 provides infinitely many examples (up to isomorphism, see the remark following the statement of the theorem). □

Finally, note that one way to show that a group is not virtually torsion-free is to have the group be not residually finite, with torsion in its finite residual (intersection of all finite-index subgroups). The method above avoids this, thus allowing the possibility for not virtually torsion-free groups which are residually finite. Such examples are to be constructed in [LV]. Just as virtual torsion-freeness is related to periodicity, we conjecture residual finiteness to be similarly related to residual periodicity of S (closed in the profinite topology on $\mathbb{Z}$).

3.1 Application to hyperbolic groups

Note that the main property we used to prove Theorem 3.1 was the branching. Therefore, it is plausible to apply the ideas in settings outside of Bestvina-Brady groups, as long as some form of branching construction is present.

The following features material directly from [KV]. We will focus on whether certain groups are virtually torsion-free or not. First, we build up the relevant context.

We say that a group G acts *properly* on a metric space X if for all $r > 0$ and $x \in X$ there exists N such that $|\{g \in G \mid d(x, gx) \leq r\}| \leq N$. A group is said to be *hyperbolic* if it acts properly and cocompactly by isometries on a hyperbolic metric space. It is currently not known whether hyperbolic groups which are not virtually torsion-free exist. There have been many interesting generalisations of this notion, such as acylindrically hyperbolic groups or relatively hyperbolic groups. In this section, we will study the class of groups with uniformly proper actions considered in [CO19].

Definition 3.3. Let G be a group acting on a metric space X. We say that the action is *uniformly proper* if for every $r > 0$ there exists N such that for all $x \in X$,

$$|\{g \in G \mid d(x, gx) \leq r\}| \leq N.$$

One should note that N in this definition only depends on r and not on x. Otherwise, all groups admit a proper action a hyperbolic space, namely their combinatorial horoball [GM08].

The class of groups acting uniformly properly on hyperbolic spaces includes all subgroups of hyperbolic groups. In [CO19], the question of whether these two classes coincide is asked. The main result from [KV] is:

Theorem 3.4. *There exist uncountably many finitely generated groups H_W acting uniformly properly on hyperbolic spaces. Moreover, at most countably many of the groups obtained are virtually torsion-free.*

In particular, there exist uncountably many groups acting uniformly properly on hyperbolic spaces which are not subgroups of hyperbolic groups. We include here only a brief overview of the first part of this theorem, as it was joint work with Robert Kropholler. However, it serves as a foundation for the last part about torsion, which is more relevant to this chapter.

The construction starts by making a subgroup H of a hyperbolic group G that is finitely generated but not finitely presented. The presentation for H has relators contained in a set V which can be indexed by integers. By considering a subset $W \subset V$, we obtain a group H_W by replacing relators in W with their p-th powers.

The algebraic structure of the groups constructed is similar to that of subgroups of hyperbolic groups. Thus, it would be of interest to obtain a characterisation of when these groups embed in hyperbolic groups.

For example, if the set of relators for which p-th powers are taken is chosen in a periodic way, there is a $\mathbb{Z}$-action on the hyperbolic space. The group generated by H_W and $\mathbb{Z}$ then acts geometrically on the hyperbolic space, which makes that a hyperbolic group into which H_W embeds naturally. This yields a proper and cocompact action on a CAT(0) cube complex and so the hyperbolic group is virtually special by Agol's theorem [Ago13]. Hence, H_W is virtually torsion-free in this case.

One should compare this to Theorem 3.1, where it is shown that if the group is virtually torsion-free, then the set of relators for which powers are taken has to be periodic. Therefore we may conjecture that this is the case here as well.

Conjecture 3.5. *The group H_W is virtually torsion-free if and only if W is a periodic subset of V. This is the case if and only if H_W embeds in a hyperbolic group.*

If some H_W which is not virtually torsion-free does in fact embed into a hyperbolic group, then by [KW00] a non-residually finite hyperbolic groups exists.

It is interesting to note that it is the converse which is open in the setting of generalised Bestvina-Brady groups.

Now let us define the groups.

Let Γ be the graph with vertex set $A \sqcup B$, where $A = A^- \sqcup A^+$ and $B = B^- \sqcup B^+$, with $A^-, A^+, B^-, B^+ = \mathbb{Z}/9\mathbb{Z}$. There is an edge a to b if any of the following hold:

1. If $a \in A^+, b \in B^+$, then $a = b$ or $a = b + 1$.
2. If $a \in A^+, b \in B^-$, then $a = b$ or $a = b - 2$.
3. If $a \in A^-, b \in B^+$, then $a = b$ or $a = b + 2$.

4. If $a \in A^-, b \in B^-$, then $a = b + 1$ or $a = b + 2$.

Proposition 3.6. *Γ has no embedded loops of length < 5.*

The full subgraph of Γ spanned by $A^s \sqcup B^t$ is a loop of length 18, for all choices of s, t.

Proof. These conditions correspond to modularity conditions $\mod 9$. □

Let Λ_A be the graph with two vertices a^+, a^- and $|A|$ edges each running from a^- to a^+. Define Λ_B similarly. The squares in $\Lambda_A \times \Lambda_B$ are in one-to-one correspondence with $A \times B$.

Let $X_\Gamma \subset \Lambda_A \times \Lambda_B$ be the cubical subcomplex containing $(\Lambda_A \times \Lambda_B)^{(1)}$ and those squares (a, b) such that (a, b) is an edge of Γ.

Proposition 3.7. *The link of any vertex of X_Γ is Γ.*

Proof. There are 4 vertices in X_Γ but the definition permits a symmetry taking any vertex to any other. Thus we will focus on the case of the vertex $v = (a^+, b^+)$. Let $L = \mathrm{Lk}(v, X_\Gamma)$.

Since $(\Lambda_A \times \Lambda_B)^{(1)} \subset X_\Gamma$, we see that $V(L) = A \sqcup B$. There is an edge in L from the vertex a to the vertex b exactly when there is a square at v with edges a, b. We can see that this is exactly the case when (a, b) is an edge of Γ. □

Since Γ is triangle-free, we deduce that X_Γ is a non-positively curved cube complex. We can now apply a theorem of Moussong [Mou88]:

Theorem 3.8. *Let X be a 2-dimensional non-positively curved cube complex. Suppose that the link of each vertex does not contain an embedded loop of length 4. Then $\pi_1(X)$ is hyperbolic.*

We already know that Γ has no cycles of length less than 5, therefore we can conclude:

Corollary 3.9. *$\pi_1(X_\Gamma)$ is hyperbolic.*

Using Morse theory, we extract a finitely generated but not finitely presented subgroup H, on which we do the branching to obtain the groups H_W. This is done by using a map from X_Γ to the circle, where each edge has its endpoints identified and points between are mapped linearly. Bestvina-Brady Morse theory then applies to the kernel of the induced map on fundamental groups. An invariant from [Kro] shows that there are uncountably many groups up to isomorphism among all the choices of W.

We obtain a presentation by having vertices v of different heights and getting a relation r_v from each (these give a presentation for H), with it being a non-trivial word when

$v \in W$ (Lemma 4.2 in [KV]). The branching is controlled by a prime number p, so in the end the set of relators for H_W is:

$$\{r_v \mid v \notin W\} \cup \{r_v^p \mid v \in W\}.$$

The precise details can be found in [KV].

Using the above, we will now outline the proof that only countably many of the groups H_W are virtually torsion-free. Denote by $o(g)$ the order of a group element.

Let r_v be the sequence of words coming from v running over the possible vertices for the relators. We know that there is a generating set A such that H_W has the presentation $\langle A \mid R\rangle$, where $R = \{r_v \mid v \notin W\} \cup \{r_v^p \mid v \in W\}$. We know that r_v is not trivial if $v \in W$. Thus it has order p in this case.

Let G be a group and $\phi\colon H_W \to G$ be a homomorphism. If $v \notin W$, then $o(\phi(r_v)) = 1$. If $v \in W$, then $o(\phi(r_v)) \mid p$ and so is either 1 or p. Let O be the subset of V consisting of those r_v such that $o(\phi(r_v)) = p$.

Now suppose that H_W is virtually torsion-free. In this case, H_W contains a finite index subgroup F_W which is torsion-free. By considering the action of H_W on the cosets of F_W, we obtain a homomorphism $\phi\colon H_W \to S_n$, where $n = |H_W : F_W|$.

Since F_W is torsion-free, we can see that if $v \in W$, then $r_v \notin F_W$. Hence, we obtain $o(\psi(r_v)) = p$ for all $v \in W$. Thus $O = W$ in this case.

The homomorphism ϕ is determined by a map $A \to S_n$. There are only finitely many such maps for a fixed n. Therefore, only countably many such maps as n varies.

The set O is determined by the map $A \to S_n$. Thus there can only be countably many sets O picked out by this process. There are uncountably many groups H_W, with one for each $W \subset V$. This implies that only countably many of them can be virtually torsion-free.

This completes the proof of Theorem 3.4. Note that the strategy was similar to that used in Theorem 3.1, except that without explicit forms for the presentations, we had to resort to more abstract maps here.

3.2 Patterns

We know from Theorem 3.1 that the virtually torsion-free generalised Bestvina-Brady groups must have a periodic branching set. We will now lay the foundations for studying the converse of that statement: we will attempt to show that when the branching set is periodic, then the group is virtually torsion-free.

To help keep track of the problem, we introduce some combinatorial notation:

Definition 3.10 (Patterns)**.** Let $S \subset \mathbb{Z}$ be a non-empty periodic proper subset, with (minimal) period length m. Given an integer $n \geqslant 1$ (which we will refer to as the *ramification index*), define an indicator function f by:

$$f(i) := \begin{cases} 1 & \text{for } i \in S \\ n & \text{for } i \notin S \end{cases}$$

such that in particular, $f(0) = f(m)$. We say that

$$I := [f(1), f(2), \ldots, f(m)]$$

is the *pattern* corresponding to S, and we refer to such a pattern as an n-pattern of length m.

An *elementary* pattern is one for which

$$f(i) = \begin{cases} n & \text{for } i \equiv -1 \mod m \\ 1 & \text{otherwise} \end{cases}$$

For example, the elementary 2-pattern of length 2 is the pattern [2,1] (which corresponds to $S = 2\mathbb{Z}$) and the elementary 3-pattern of length 4 is [1,1,3,1]. Note that in general, the positions of the entries which are 1 are controlled by S. There is a natural action of $\mathbb{Z}/m\mathbb{Z}$ on the set of n-patterns of length m by cyclically permuting the entries of I. We say that two patterns are *equivalent* if they lie in the same orbit under this action, which is the same as the underlying sets S being the same up to translation. Note that there is a unique orbit of patterns featuring a single n, containing the elementary pattern. The precise choice of that particular orbit representative will become clear later, when reformulating the problem in terms of commutators.

The above definitions are motivated by the construction of $G_L^M(S)$: the patterns will be used to encode information about the long cycle relations. In particular, given a generalised Bestvina-Brady group of finite ramification, considering one particular loop γ in a set of loops Γ that normally generate $\pi_1(L)$, the ramification index n corresponds to the order of the element in $\pi_1(L, M)$ that is the image of γ, which will be the power in the corresponding long cycle relations in the presentation of the group.

3.2.1 Relationship with torsion-free subgroups

The above definitions were motivated by a search for finite-index torsion-free subgroups. The most important case is when L is a square. In particular:

Definition 3.11 (Solving patterns)**.** Given a pattern I, we say that this pattern can be *solved* if there exists a finite group $\bar{G}$ with elements $a, b, c, d \in \bar{G}$ such that:

$$o\left(a^i b^i c^i d^i\right) = f(i) \quad \forall i \in \mathbb{Z}$$

where f and S correspond to I as above. We say that the *solution* to the pattern I is the map ψ given by:

$$\begin{aligned} \psi : G_L^M(S) &\to \bar{G} \\ A &\mapsto a \\ B &\mapsto b \\ C &\mapsto c \\ D &\mapsto d \end{aligned}$$

and we say that the group $\bar{G}$ *solves* the pattern, where L is taken to be the 4-vertex flag triangulation of the circle; A, B, C, D are the corresponding group generators and M is the regular finite cover of degree n.

The reason for considering that specific choice of L, and in particular, four group elements, will become clearer towards the end of this subsection. Note that in the definition above, $\ker\psi \mathrel{\dot{\trianglelefteq}} G_L^M(S)$ is torsion-free. This is no coincidence:

Lemma 3.12 (Justification for the definition of patterns)**.** *The group $G_L^M(S)$ (as above) is virtually torsion-free if and only if S is periodic and the pattern corresponding to S can be solved.*

Proof. Suppose that the pattern can be solved. From [Lea18], we know that all torsion elements are conjugates of $A^i B^i C^i D^i$ with $i \notin S$, which do not lie in $\ker\psi$ because conjugation preserves group element order and $a^i b^i c^i d^i$ is never the identity element for $i \notin S$. Thus $G_L^M(S)$ is virtually torsion-free because $\ker\psi$ is a torsion-free finite-index subgroup.

The converse follows from Theorem 3.1: note particularly that from the proof of Theorem 3.1, it also follows that the finite-index torsion-free subgroup contains a normal finite-index torsion-free subgroup, and $\bar{G}$ here plays the role of the corresponding quotient. $\square$

Although patterns are combinatorial objects, we can now understand how they encode geometric information about $G_L^M(S)$:

Corollary 3.13 (Justification for pattern equivalence)**.** *If patterns I and J are equivalent and a 1 goes in the final entry of each, then I can be solved if and only if J can be solved.*

Proof. Equivalent patterns correspond to the same S up to translation, as the action of $\mathbb{Z}/m\mathbb{Z}$ on the patterns simply changes the repeating periodic block of S (by changing the initial and final entries of the block). In turn, this corresponds to the same group $G_L^M(S)$, because the corresponding presentation is for the same group, just with a shifted base point (the CAT(0) cube complex the group acts on is the same, but a different level without branching was chosen to be 0). This is compatible with our presentations, because we insisted that a 1 goes in the final entry of the block, i.e. by periodicity, the entry corresponding to the 0th position will be 1, which is the same as having $0 \in S$, so the presentations in [Lea18] apply. By Lemma 3.12, a pattern can be solved if and only if the corresponding group $G_L^M(S)$ is virtually torsion-free. Hence both the solubility of the pattern I and the solubility of the pattern J are equivalent to the corresponding same generalised Bestvina-Brady group being virtually torsion-free. □

The following lemma illustrates the importance of elementary patterns:

Lemma 3.14 (Diagonal trick, justification for the definition of elementary patterns)**.** *If the elementary n-pattern of length m can be solved, then any n-pattern of length m (having final entry 1 in its repeating block) can be solved.*

Proof. Let I be the elementary n-pattern of length m, with f, S corresponding to it as above. Let

$$J = [j_1, j_2, \ldots, j_m]$$

be any other n-pattern of length m where $j_m = 1$. Denote by σ the generator of $\mathbb{Z}/m\mathbb{Z}$ (from the action on the set of n-patterns of length m in Definition 3.10) which moves every entry one space to the left ("anti-clockwise"). We know from Corollary 3.13 that $\sigma^i(I)$ can be solved for all non-negative $i < m$, so denote by $\bar{G}_i, a_i, b_i, c_i, d_i$ their respective solutions. For a general group G, define maps $\delta_i : G \to G$ given by:

$$\delta_i(g) := \begin{cases} e_G & \text{if } \ j_i = 1 \\ g & \text{if } \ j_i = n, \end{cases}$$

then the direct product map

$$\psi : G_L^M(S) \to \prod_{i=1}^{n-1} \bar{G}_{i-1}$$

given by

$$A \mapsto \prod_{i=1}^{m-1} \delta_{m-i}(a_{i-1}),\ B \mapsto \prod_{i=1}^{m-1} \delta_{m-i}(b_{i-1}),\ C \mapsto \prod_{i=1}^{m-1} \delta_{m-i}(c_{i-1}),\ D \mapsto \prod_{i=1}^{m-1} \delta_{m-i}(d_{i-1})$$

solves the pattern J. □

To understand the significance of the number 4 in this context (as in Definition 3.11), observe that the smallest possible length of a non-nullhomotopic edge loop in a flag simplicial complex is 4. Further significance of this number will be explored in Section 4.2.

3.2.2 Commutator reformulation

Lemma 3.14 shows that the most important patterns to solve are the elementary patterns. Note that in the definition of elementary patterns, it did not matter where we chose to place the entry of n among the 1s, because of Corollary 3.13. Nonetheless, we will now examine an advantage of putting the n as far to the right as possible.

Suppose that a, b, c, d solve the elementary n-pattern of length m. Given m is large enough, we have:

$$abcd = 1 \implies d^{-1} = c^{-1}b^{-1}a^{-1},$$

$$a^2b^2c^2d^2 = 1 \implies a^2b^2c^2 \cdot c^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1} = 1$$

$$\implies ab^2cb^{-1}a^{-1}c^{-1}b^{-1} = ab \cdot bc \cdot b^{-1}a^{-1} \cdot c^{-1}b^{-1} = 1$$

$$\implies [(ab)^{-1}, (bc)^{-1}] = 1,$$

i.e. ab and bc commute. Keeping this in mind, we can continue, to obtain:

$$a^3b^3c^3d^3 = 1 \implies a^3b^3c^3 \cdot c^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1} = 1$$

$$\implies a^2b^2 \cdot bc \cdot b^{-1} \cdot bc \cdot b^{-1}a^{-1} \cdot c^{-1}b^{-1}a^{-1}c^{-1}b^{-1} = 1.$$

Now we use the fact that ab and bc commute to get:

$$a^2b^2 \cdot bc \cdot b^{-1} \cdot b^{-1}a^{-1} \cdot bc \cdot c^{-1}b^{-1}a^{-1}c^{-1}b^{-1} = 1$$

$$\implies a^2b^2 \cdot bc \cdot b^{-2}a^{-2} \cdot c^{-1}b^{-1} = 1,$$

i.e. the elements a^2b^2 and bc also commute. As long as there are sufficient 1s on the left of the pattern, we can keep going like this. Indeed, by induction, we have

$$a^kb^kc^kd^k = 1 \implies a^kb^kc^k\left(c^{-1}b^{-1}a^{-1}\right)^k = 1$$

$$\implies a^{k-1}b^kc^{k-1}b^{-1}a^{-1}\left(c^{-1}b^{-1}a^{-1}\right)^{k-2}c^{-1}b^{-1} = 1$$

$$\implies a^{k-1}b^{k-1}\cdot bc\cdot c^{k-2}\left(b^{-1}a^{-1}c^{-1}\right)^{k-2}b^{-1}a^{-1}\cdot c^{-1}b^{-1} = 1,$$

which given the appropriate inductive hypothesis yields

$$[a^{k-1}b^{k-1}, bc] = 1.$$

Finally, when we get to the place in the pattern where there is an n. Denoting the torsion element by g, we get:

$$a^{m-1}b^{m-1}c^{m-1}d^{m-1} = g$$

$$\implies a^{m-2}b^{m-1}c^{m-2}b^{-1}a^{-1}\left(c^{-1}b^{-1}a^{-1}\right)^{m-3}c^{-1}b^{-1} = a^{-1}\cdot g\cdot a$$

$$\implies g = a\cdot\left[a^{m-2}b^{m-2}, bc\right]\cdot a^{-1}.$$

Since conjugation preserves the order of an element, we can deduce that the commutator $\left[a^{m-2}b^{m-2}, bc\right]$ has order n. We will use this fact in Subsection 3.3. Note that if the pattern had an n further to the left, we would have extracted less information about what conditions these commutators satisfy. Therefore, the reason for the precise definition of elementary pattern is to deduce more facts about the elements a, b, c, d, to help find solutions. Summarising:

Lemma 3.15 (Towards a commutator reformulation)**.** *If a, b, c, d solve the elementary n-pattern of length m, then:*

- $d = c^{-1}b^{-1}a^{-1}$
- $\left[a^ib^i, bc\right]$ *is the identity element for* $i = 1, 2, \ldots, m-3$.
- $\left[a^{m-2}b^{m-2}, bc\right]$ *is an element of order* n.

The solution that follows shortly will essentially be a converse to this. Note that for $m > 2$ there is a condition of a commutator having non-trivial finite order. This means that if the corresponding pattern has a solution, then the finite group featuring in the solution cannot be abelian. This fact will be useful later, when searching for solutions.

Lemma 3.16 (The "converse" of Lemma 3.15)**.** *Suppose that there is a finite group with elements a, b, c satisfying:*

- $\left[a^i b^i, bc\right]$ *is the identity element for* $i = 1, 2, \ldots, m-3$.
- $\left[a^{m-2} b^{m-2}, bc\right]$ *is an element of order* n.
- $o(a) = o(b) = o(c) = m$
- $o(c^{-1}b^{-1}a^{-1}) = m$

Then a, b, c, d *solve the elementary* n*-pattern of length* m*, where* $d = c^{-1}b^{-1}a^{-1}$.

Proof. The algebraic deductions above can be done in reverse, giving $a^i b^i c^i d^i$ being the identity for i between 1 and $m-2$, and $a^{m-1}b^{m-1}c^{m-1}d^{m-1}$ being an order n element. The fact that each of a, b, c, d has order m then implies that $a^m b^m c^m d^m$ is trivial and that the pattern is periodic with period m, yielding the elementary n-pattern of length m. □

Note that we needed to add an extra assumption about the order of d to ensure periodicity in the converse.

We finish this subsection by stating a consequence for the derived subgroup of some of the $G_L^M(S)$ of finite ramification when S has sufficiently long period. First, we need a combinatorial lemma:

Lemma 3.17. *Let* $m > 2$*. Then any* n*-pattern of length* m *is equivalent to one of the following forms: (where the blanks can be filled in any way)*

- $[1, \ldots, 1]$ *(type I)*
- $[n, n, \ldots, 1]$ *(type II)*

Proof. Given an n-pattern of length m, consider it as a cyclic pattern (Corollary 3.13 allows us to do this, via the natural C_m-action). Note that since $m \neq 1$ and the pattern is supposed to be the minimal-length repeating block, it will consist of not all the same kind of entry. Look at the length of the longest block of the same number repeating. If this is of length 1, then as we don't consider patterns consisting entirely of only one number, the pattern must alternate

$$\ldots - 1 - n - 1 - n - 1 - \ldots$$

and hence is equivalent to the unique pattern of length 2, namely $[n,1]$. This can't happen, because by assumption we have $m > 2$.

Hence it must be the case that some number repeats at least twice in a row in the pattern. If this number is a 1, then we have

$$\ldots - 1 - 1 - \ldots$$

appearing in the pattern, and so we can rotate the cyclic pattern so it is equivalent to one of type I, by placing the first 1 in the block in the final position of the pattern.

Finally, if the number in the repeating block is n, then since the pattern is not entirely composed of n, the block must be preceded by a 1,

$$\ldots - 1 - n - n - \ldots$$

and so by placing this 1 at in the final position of the pattern by rotation of the cyclic pattern, it is equivalent to one of type II. ☐

Corollary 3.18. *There is a torsion element in the derived subgroup when $n = 2$.*

Proof. Using Lemma 3.17, if we are in the first case (type I), then using the commutators reformulation immediately gives us a way to write the torsion element corresponding to the first n in the pattern as a conjugate of a commutator, by the same computation that we get the commutator of order n in the proof of Lemma 3.15. However, when x is in the derived subgroup,

$$g^{-1} \cdot x \cdot g = x \cdot x^{-1} g^{-1} x g = x \cdot [x, g]$$

and so by closure, conjugates of commutators are in the derived subgroup. (Alternatively, we have already noted that conjugation preserves order.)

If we are in the second case (type II), we have $(abcd)^n = 1$ from the first n.

If we write this out as $abcd \ldots abcd = 1$, we can take the furthest-right copy of a and using the fact that $yx[x, y] = xy$ for any group elements x, y, move it to the left side of the expression by inserting a commutator. We can then repeat this with the next furthest-right a which is separated from the group of powers of a we are building up on the left, and keep going until the expression reads a^n on the left side.

Similarly, we can then gather up the copies of b next, so the expression then reads $a^n b^n$ on the left side.

If we keep going with also c and d, the equation eventually re-arranges to $a^n b^n c^n d^n \cdot g_1 \cdot \cdots \cdot g_i = 1$ for some elements g_j in the commutator subgroup.

Finally, the second $n = 2$ gives us $a^2 b^2 c^2 d^2$ being a torsion element, which by the above equation is in the derived subgroup. ☐

3.3 Solution for 1-dimensional L

The smallest flag simplicial complex with non-trivial fundamental group is the 4-vertex triangulation of the circle, which we have already encountered in Definition 3.11. Before we begin to prove in general things about $G_L^M(S)$ of finite ramification being virtually torsion-free for periodic S, it makes sense to study the case of this particular choice of L.

Lemma 3.14 tells us to focus only on the elementary patterns (as defined in the previous subsection), hence the groups we will first attempt to find torsion-free subgroups of finite index of will be:

$$G_{m,n} := \left\langle a, b, c, d \;\middle|\; \begin{array}{ll} \left(a^i b^i c^i d^i\right)^n & \text{for } i \equiv -1 \mod m \\ a^i b^i c^i d^i & \text{otherwise} \end{array} \right\rangle,$$

where $G_{m,n}$ corresponds to the elementary n-pattern of length m.

The smallest choice of ramification index is 2, and so if we are going to search for finite groups to solve the corresponding patterns, it makes sense to start with this, in an attempt to keep the finite groups involved in the solution as small as possible.

3.3.1 Computer search

A way to guarantee m-periodicity in the sequence of orders of products of elements is to have each of a, b, c, d be of order m. In general, this is an extra condition we are imposing, which may in practice be making it more difficult to find a solution. However it may also help to prove things about a potential solution later on, if it works.

Note that such an assumption is not completely unfounded, as given a solution to a pattern of length m, if a prime number p divides m, then it is necessary for at least one of the elements a, b, c, d to have order divisible by p. This is because in a finite group, the sequence

$$\left(o\left(a^i b^i c^i d^i\right)\right)_i$$

is naturally periodic, with period length being a factor of the lowest common multiple of

$$\{o(a), o(b), o(c), o(d)\}.$$

Similarly, if trying to find a solution to an n-pattern, the solution group must contain n-torsion, and so only groups with order divisible by n should be considered. For example, when n, m are distinct primes, then only groups with order divisible by mn should be considered.

Turning our attention to ramification index 2 now, the length 2 elementary pattern has a solution involving a symmetric group by sending one (or each) of the generators a, b, c, d to a (disjoint) 2-cycle. Following Lemma 3.15, this is the only case where an abelian solution group will be possible. In turn, knowing that we can ignore abelian groups for all the other patterns can speed up the search.

The following is GAP code designed to search for solutions. The code presented here is specifically to deal with the elementary 2-pattern of length 5, but adjusting the parameters can make the program look for solutions to other patterns.

```
Pattern:= [1,1,1,2,1];
for i in [1..2000] do
 if i mod 10 = 0 then
  for j in [1..NrSmallGroups(i)] do
   G:= SmallGroup(i,j);
   if not IsAbelian(G) then
    p:= 0;
    ACandidates:= [];
    BCCandidates:= [];
    cl:= ConjugacyClasses(G);
    for k in [1..Size(cl)] do
     if Order(Representative(cl[k])) = 5 then
      Add(ACandidates,Representative(cl[k]));
      Append(BCCandidates,AsList(cl[k]));
     fi;
    od;
    for a in ACandidates do
     for b in BCCandidates do
      for c in BCCandidates do
       p:= p+1;
       n:= 5;
       d:= c^-1 * b^-1 * a^-1;
       t:= 1;
       Testing:= true;
       while Testing do
        r:= t mod 5;
        if r = 0 then
         r:= 5;
        fi;
        if Order(a^t * b^t * c^t * d^t)=Pattern[r] then
         t:=t+1;
        else
         Testing:= false;
        fi;
        if t = n+1 then
         Testing:=false;
        fi;
       od;
       if t = n+1 then
        Print("Success! ",a," ",b," ",c," work in group ",j," of size ",i,"!\n");
        return;
       else
Print("Searching... Currently on ",j," of ",NrSmallGroups(i),", of size ",i,".
    ↪ Current group progress ",Float(100*(p/(Size(ACandidates)*Size(BCCandidates)
    ↪ ^2))),"%\r");
```

```
        fi;
       od;
      od;
     od;
    fi;
   od;
  fi;
od;
```

We summarise the findings of the above program in Table 3.1. Note that because of the way we searched, the solutions found were the smallest groups that the program came across that satisfied the required conditions.

Value of m	Solution `IdGroup`	Group structure	Time taken
3	[12,3]	A_4	16
4	[24,12]	S_4	235
5	[160,199]	$((C_2 \times Q_8) \rtimes C_2) \rtimes C_5$	7000
6	[192,201]	$\left((C_2^4 \rtimes C_2) \rtimes C_2\right) \rtimes C_3$	324828
7	[896,19309]	$((((C_2 \times Q_8) \rtimes C_2) \rtimes C_2) \rtimes C_2) \rtimes C_7$	4729032

TABLE 3.1: Finite groups solving the elementary 2-patterns of length m.

Note that we start with $m = 3$, because we are implementing the improvement of skipping out abelian groups, which applies only for patterns longer than 2. The `SmallGroups` library (which is standard even outside of GAP) is used to identify the finite groups appearing as solutions. The time taken is in milliseconds. It was measured on my laptop, using the `time` command in GAP, following a `Read` command of the file containing the code for the search. For the prime values of m, groups of order divisible by $2m$ were searched. Since 4 and 6 are even, for those values of m, groups of order divisible by 4 and 6 were searched, respectively. The structure of the groups was provided by the `StructureDescription` command in GAP, where A_n and S_n refer to the alternating and symmetric groups, respectively, and Q_8 is the quaternion group.

Note that $A_4 \cong C_2^2 \rtimes C_3$. Looking up the groups on the *GroupNames* website [Dok], we find that the groups for the cases of $m = 5, 6$ can also be written as $2_-^{1+4} \rtimes C_5$ and $2_+^{1+4} \rtimes C_6$, respectively. This begins to suggest a connection with extraspecial p-groups. Currently, the *GroupNames* database does not feature order 896, however the following GAP code:

```
gap> G:=ExtraspecialGroup(2^7,"+");
<pc group of size 128 with 7 generators>
```

```
gap> H:=AutomorphismGroup(G);
<group of size 2580480 with 8 generators>
gap> cl:=ConjugacyClasses(H);;
gap> g:=Group([Representative(cl[63])]);
<group with 1 generators>
gap> IsGroupOfAutomorphisms(g);
true
gap> IdGroup(SemidirectProduct(g,G));
[ 896, 19309 ]
gap> Size(g);
7
```

shows that the solution for $m = 7$ can also be written as $2_+^{1+6} \rtimes C_7$. Furthermore, by changing the starting point for the search, we can also use our program to determine that for these values of m, the elementary 2-pattern of length m has a solution in some group of size $2^m \cdot m$. Therefore we may conjecture that there will always be a solution of this size, having structure resembling an extension of an extraspecial p-group by a cyclic group.

3.3.2 Extraspecial groups

Let p be a prime number. Given the computational evidence above, it makes sense to focus our attention on extraspecial p-groups. This is a p-group G with centre Z of order p, such that G/Z is non-trivial elementary abelian. A p-group is called *special* if it is either elementary abelian, or if its centre, derived subgroup and Frattini subgroup (the intersection of all the maximal subgroups) coincide and are an elementary abelian group. The nonabelian special p-groups with centre of order p are precisely the extraspecial p-groups. Note that given the work on special cube complexes earlier, we must not confuse the two meanings of this word.

For every prime power order of the form p^{2m+1}, there are precisely two isomorphism classes of extraspecial p-group. It turns out that every such group is the central product of m nonabelian groups of order p^3. We can describe presentations for these groups in general as follows:

- For the extraspecial p-group G of order p^{2m+1}, denote the centre by Z. Let this be generated by an element z, such that $z^p = 1$.
- This group has generators $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ such that $\alpha_i^p \in Z$ and $\beta_i^p \in Z$ for all i.
- We have commutator relations $[\alpha_i, \beta_i] = z$ for all i.
- Every other pair of generators commutes.

For odd p, one isomorphism class has exponent p. We denote this by p_+^{1+2m}, where $\alpha_i^p = \beta_i^p = 1$ for all i.

The other isomorphism class for odd p is denoted by p_-^{1+2m}. This has exponent p^2. The relations that complete the group presentation are $\alpha_1^p = z$, $\beta_1^p = 1$ and $\alpha_i^p = \beta_i^p = 1$ for all other i in this case.

For $p = 2$, the extraspecial group of order 2^{2m+1} is isomorphic to the central product of m groups of order 8, each one being either a quaternion group or a dihedral group. If the number of quaternion groups in this central product is even (in particular, when it is 0, i.e. when it is a central product of just dihedral groups), then we denote this group by 2_+^{1+2m}. In this case we have relations $\alpha_i^2 = \beta_i^2 = 1$ for all i.

Finally, for a central product containing an odd number of quaternion groups, the group is denoted by 2_-^{1+2m}. In this case we have relations $\alpha_m^2 = \beta_m^2 = z$ and $\alpha_i^2 = \beta_i^2 = 1$ for all other i.

Note that for odd primes, the Heisenberg group of order p^3 is extraspecial. As such, one may think of extraspecial groups as generalised Heisenberg groups over finite fields.

In [Win72], the automorphism group of an extraspecial p-group is considered. Since we will aim to construct extensions of extraspecial groups, it will be useful for us to understand the automorphisms, so we can form semidirect products later.

We consider G/Z as a $2m$-dimensional vector space over the finite field of p elements. In particular, given $x, y \in G$, we know that $[x, y] \in Z$. Therefore there exists an integer a such that $[x, y] = z^a$. Denoting the images of x and y in G/Z by $\bar{x}$, $\bar{y}$, respectively, we can define

$$(\bar{x}, \bar{y}) := a \mod p,$$

which becomes a non-degenerate symplectic form.

Theorem 3.19 (D. Winter, [Win72])**.** *The symplectic group $Sp(2m, p)$ acts on G/Z and preserves the skew-symmetric bilinear form $(\bar{x}, \bar{y})$. Given the basis $\left\{\bar{\alpha}_i, \bar{\beta}_i\right\}$, a symplectic matrix with respect to this basis induces an automorphism of G if and only if the corresponding function on G satisfies $\phi(g)^p = g^p$ for all generators $g \in \{\alpha_i, \beta_i\}$.*

The function corresponding to a symplectic matrix is as follows:

- If we denote $x_i = \alpha_i$, $x_{m+i} = \beta_i$ for all i, then each element g of G can be uniquely expressed as $g = z^c \prod_{i=1}^{2m} x_i^{t_i}$.
- Given a matrix $M \in \mathrm{Sp}(2m, p)$ with entries (M_{ij}), the corresponding function $\phi : G \to G$ is defined by $\phi(g) := z^c \prod_{i=1}^{2m} \left(\prod_{j=1}^{2m} x_j^{M_{ij}}\right)^{t_i}$.

We are now ready to give the first general solution to a family of patterns of any length. Guided by the computational results from the case of elementary 2-patterns, we will construct semidirect products of extraspecial 2-groups.

Consider the group 2_+^{1+2m}. Denote its centre by $Z = \langle z \rangle$, with $o(z) = 2$. We use the appropriate generating set $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$ as above. All of the generators commute, except the commutator of α_i and β_i is z for each i. By [Win72], a mapping of its generators corresponding to a symplectic matrix over the field of two elements is an automorphism if the square of each element is equal to the square of its image under the mapping.

Define a mapping ϕ as follows:

- $\phi(\alpha_i) = \alpha_{i+1}$ for $i = 1, \ldots, m-1$.
- $\phi(\alpha_m) = \alpha_1$.
- $\phi(\beta_i) = \beta_{i+1}$ for $i = 1, \ldots, m-1$.
- $\phi(\beta_m) = \beta_1$.

This is a permutation, which defines a symplectic square matrix of dimension $2m$ by considering it as a permutation on the basis vectors. It is, in fact, an orthogonal matrix preserving the quadratic form

$$\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4 + \cdots + \epsilon_{2m-1}\epsilon_{2m},$$

so can be viewed as an element of $O^+(2m, 2)$. The square of each generator and its image are equal to the identity, therefore ϕ defines an automorphism of 2_+^{1+2m} by Theorem 3.19.

Since the order of ϕ is m, construct the semidirect product $2_+^{1+2m} \rtimes_\phi C_m$, with multiplication as follows:

$$\left(\phi^i, g\right) \cdot \left(\phi^j, h\right) = \left(\phi^{i+j}, \phi^j(g) \cdot h\right).$$

Denoting (ϕ^0, g) by g and (ϕ^i, e) by ϕ^i, as well as the conjugate of an element γ by λ as γ^λ, define the following elements:

- $a := \left(\phi^{-1}\right)^{\alpha_1}$,
- $b := \phi^{\beta_m}$,
- $c := \left(\phi^{-1}\right)^{\alpha_{m-1}\beta_m}$.

Now one can compute:

- $a^i = \left(\phi^{-i}, \alpha_1\alpha_{m+1-i}\right)$ for $i = 1, \ldots, m-1$, with a having order m.
- $b^i = \left(\phi^i, \beta_i\beta_m\right)$ for $i = 1, \ldots, m-1$, with b having order m.
- $a^i b^i = \left(\phi^0, \alpha_1\beta_i\alpha_{1+i}\beta_m\right)$ for $i = 1, \ldots, m-1$.
- $bc = \left(\phi^0, \alpha_{m-2}\alpha_{m-1}\right)$, and c has order m.

Finally, for $i = 1, 2, \ldots, m-1$,

$$\left[a^i b^i, bc\right] = \left(\phi^0, [\beta_i, \alpha_{m-2}\alpha_{m-1}]\right).$$

In particular, $a^i b^i$ commutes with bc for $i = 1, 2, \ldots, m-3$ and

$$\left[a^{m-2}b^{m-2}, bc\right] = \left(\phi^0, z\right),$$

which is of order 2. Finally, we compute that $c^{-1}b^{-1}a^{-1} = (\phi^1, \alpha_1\alpha_2\alpha_{m-1}\alpha_m)$, which is just the conjugate of ϕ by $\alpha_1\alpha_{m-1}$, thus it has order m too. Hence a, b, c satisfy the conditions of Lemma 3.16, and therefore the group $2_+^{1+2m} \rtimes_\phi C_m$ solves the elementary 2-pattern of length m.

Note that this provides a solution group of size $2^{2m+1} \cdot m$. This is larger than the conjectured possible size $2^{m+1} \cdot m$ from Subsection 3.3.1. However, we expect a solution of smaller size to utilise a more complicated sympletic matrix, and it not strictly needed. The main point is to determine which of the groups are virtually torsion-free, and the precise index of the subgroups is not important.

Note also that this solution proves that $\mathfrak{F}'(4) = \infty$, hence completing the computation of the entire function (as defined at the end of Subsection 4.2):

$$\mathfrak{F}'(k) = \begin{cases} 1 & \text{if } k = 1, 2 \\ 2 & \text{if } k = 3 \\ \infty & \text{otherwise} \end{cases}$$

By Lemma 3.14, we can now solve every 2-pattern (note that even though our method here does not cover length 2, we already discussed a solution for that earlier).

3.3.3 Solution for cyclic covers

We now turn our attention to p-patterns, for p an odd prime. In an attempt to mimic the above, we consider the extraspecial p-group p_+^{1+2m}, which has exponent p. We use the same notation for the group generators, and define the map ϕ in exactly the same way. Since it is a permutation of the generators, the conditions of Theorem 3.19 are satisfied, and we can form the semidirect product

$$p_+^{1+2m} \rtimes_\phi C_m$$

in the same way. Note that the calculations above were made easier by the fact that every element in the group had order 2. We define a, b, c in the same manner as before. This time, for the calculations we just also have to keep track of inverses:

- $a^i = \left(\phi^{-i}, \alpha_1 \alpha_{m+1-i}^{-1}\right)$
- $b^i = \left(\phi^i, \beta_i^{-1}\beta_m\right)$
- $a^i b^i = \left(\phi^0, \alpha_1^{-1}\alpha_{i+1}\beta_i^{-1}\beta_m\right)$
- $c = \left(\phi^{-1}, \beta_{m-1}^{-1}\alpha_{m-2}^{-1}\alpha_{m-1}\beta_m\right)$
- $bc = \left(\phi^0, \alpha_{m-2}^{-1}\alpha_{m-1}\right)$
- $\left[a^i b^i, bc\right] = \left(\phi^0, \left[\beta_i^{-1}, \alpha_{m-2}^{-1}\alpha_m\right]\right)$

The elements a, b, c each have order m (because they are conjugates of an element of order m), similarly for $d = c^{-1}b^{-1}a^{-1}$. For $i = m-2$, the commutator above is order p but vanishes for lower values (down to $i = 1$), thus for $m > 2$ this satisfies the conditions of Lemma 3.16. In order to be able to solve every p-pattern, it now suffices to solve the (unique elementary) p-pattern of length 2, namely $[p, 1]$.

For $m = 2$, keep the same formula for a as before. Setting $b = \phi$ and $c = d = e$ gives

$$o(a) = o(b) = 2,\ o(c) = o(d) = 1.$$

Hence we get 2-periodicity, and

$$abcd = \left(\phi^0, \alpha_1^{-1}\alpha_2\right),$$

which has order n, shows that the group $p_+^5 \rtimes_\phi C_2$ solves the elementary n-pattern of length 2. Therefore by Lemma 3.14, we can now solve every p-pattern for any prime p, of any length.

Note that if p, q are coprime, we can combine solutions. If we have a solution for a p-pattern of length m and a solution for the respective q-pattern of length m, where every instance of p is replaced by q, then by taking a direct product of the solutions, we obtain a solution to the corresponding pq-pattern of length m. Given the above results, this means that via direct products, we can already solve any n-pattern of any length, for n a square-free integer. We will now go on to remove the square-free assumption by generalising away from extraspecial groups while keeping some of the same properties, which we now understand to be crucial.

Note that in the calculation for the p-torsion solution, once the elements a, b, c were defined, the fact that p is prime was not used. We only used the fact it is a prime number by constructing the semidirect product, in particular we used the existence of the extraspecial p-group and Theorem 3.19 for the required automorphism.

Corollary 3.20. *If there exists a group $G(n, m)$ for integers $m, n > 2$ such that:*

- *It is generated by elements $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$, each of order n.*
- *The centre of the group is cyclic of order n, with generator z, such that $[\alpha_i, \beta_i] = z$ for all i.*
- *All of the other pairs of generators commute.*
- *There is an automorphism of the group which permutes the generators in the same way as ϕ above.*

Then every n-pattern of length m can be solved.

Proof. The existence of such a group allows a similar construction to the above with extraspecial groups. The corresponding function ϕ allows us to construct the semidirect product. The construction of the length 2 pattern for p-torsion also applies here. Therefore combining Lemma 3.14 and Lemma 3.16 gives the result. $\square$

We can show that such a group exists by exhibiting a faithful matrix representation over $\mathbb{Z}/n\mathbb{Z}$. Just as before, this group is a central product of groups of order n^3. In fact, we can immediately give the matrices for the semidirect product:

$$\alpha_1 := \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \ \beta_1 := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & 0 & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$\phi := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

These are each elements of $\mathrm{GL}_{m+2}(n)$. They generate the solution group, e.g. $\alpha_i = \alpha_1^{\phi^{i-1}}$.

Corollary 3.21. *Suppose that L is a flag triangulation of the circle. Then for any finite regular cover $M \to L$, the group $G_L^M(S)$ is virtually torsion-free for S periodic.*

Note that each element of the form $a^i b^i c^i d^i$ of this solution (and each conjugate) has the form $\left(\phi^0, g\right)$, whereas each of the elements a, b, c, d has a form (ϕ, g) or $\left(\phi^{-1}, g\right)$. Therefore for a solution of a pattern of length m, since $o(\phi) = m$, if we have

$$g^i = a^j b^j c^j d^j,$$

for $g \in \{a, b, c, d\}$, then both must be equal to the identity element, and $m|i$. Therefore the above solutions are separating, as in Definition 2.6. They are also central, as in Definition 2.8, except the special case when $m = 2$.

3.3.4 Graphs with sufficient independent edges

Note that the above constructions relied on writing group elements as commutators. We will now take the above idea and create a much larger finite group, to allow a homomorphism with torsion-free kernel for any group of deck transformations.

To write elements of an arbitrary finite group as commutators, we appeal to Ore's conjecture [Ore51]. We can embed any finite group into a symmetric group via the regular representation (also known as Cayley's theorem), and then embed the symmetric group S_n (WLOG $n > 3$) into a larger non-abelian alternating group A_{n+2} by sending even permutations to themselves and adding $(n+2, n+1)$ to odd permutations to make them even. This embeds any finite group G into a non-abelian simple group. By [LOST10], any element can now be written as a commutator.

We have already seen that to solve arbitrary patterns, at least 4 free variables are required. As such, to tackle general situations we will insist on having enough freedom.

Definition 3.22 (Independent edges)**.** We say that a connected simplicial complex L has *sufficient independent edges* if there exists a choice of vertex basepoint $v \in L$ and a set of distinct-up-to-homotopy edge-loops Γ based at v such that the following hold:

- The loops of Γ generate $\pi_1(L)$.
- Each loop in Γ contains at least 4 edges that are not contained in any other loop of Γ.

We call such a set a *good set of loops*. We will return to these ideas in the next section.

By only using the independent edges, we are now ready to resolve the case for any choice of finite group of deck transformations $M \to L$:

Theorem 3.23 (Solution for any finite group)**.** *Suppose that $G_L^M(S)$ is a generalised Bestvina-Brady group of finite ramification and L is a simplicial graph with sufficient independent edges. Then $G_L^M(S)$ is virtually torsion-free for S periodic.*

Proof. Denote by G the alternating group into which we embed the group of deck transformations $M \to L$. Let n be the period of the pattern corresponding to S. By Lemma 3.14, it suffices to solve the elementary n-pattern. We exhibit an explicit homomorphism

$$G_L^M(S') \to G^n \rtimes_\phi C_n$$

with torsion-free kernel, where ϕ is the exact same permutation automorphism as before and S' is amended to reflect the elementary pattern. Each edge of L which is not part of a good set of loops will be sent to the identity, and it remains to assign elements a, b, c, d for each directed loop $a - b - c - d$ of Γ (where all other edges of this loop were sent to the identity).

To do this, we use exactly the same form of elements as before:

$$a = \left(\phi^{-1}\right)^{\alpha_1}, \ b = \phi^{\beta_n}, \ c = \left(\phi^{-1}\right)^{\alpha_{n-1}\beta_n},$$

and d is given by $d = c^{-1}b^{-1}a^{-1}$. To finish, we just need to pick the appropriate $\alpha_i, \ \beta_i$.

We let α_i and β_i live in the i-th factor of G^n (by abuse of notation, their corresponding image in the semidirect product) and set each $\alpha_i = \alpha_1$ as well as each $\beta_i = \beta_1$. This ensures that for different i, they all commute, so the same computations from before go through and we have:

$$\left[a^{n-2}b^{n-2}, bc\right] = \left(\phi^0, [\alpha_1, \beta_1]\right).$$

All that remains is to choose $\alpha_1, \ \beta_1$ for this loop. Suppose that this loop corresponds to an element g in the group of deck transformations $M \to L$. This can be written as some commutator $[x, y]$. We set

$$\alpha_1 = x, \ \beta_1 = y.$$

$\square$

Chapter 4

Obstructions to extension problems

Throughout, S refers to a periodic subset of the integers which also contains 0. In the previous chapter, we showed that the group $G_\Gamma^M(S)$ is virtually torsion-free for M a finite regular cover of a suitable finite (flag) simplicial graph Γ with sufficient independent edges. In particualr, those assumptions allowed us to ignore the triangle relations in the presentation of the generalised Bestvina-Brady group. The aim of this chapter is to explore generalisations where we have the underlying flag complex be higher-dimensional and thus tackle the issue of the triangle relations (coming from the 2-simplices) having to hold in finite quotients.

In order to tackle the case of $G_L(S)$ for L being a finite flag simplicial complex having finite fundamental group (recall that this is the case where $M \to L$ is the universal cover), a starting point could be considering a 1-dimensional subcomplex $\Gamma \subset L$ such that each generator (or even element) of $\pi_1(L)$ is represented by a loop in Γ (for some choice of basepoint). Denoting by M the full pre-image of Γ in the universal cover, we know that M is connected (hence BB_M is finitely generated) and the group $G_\Gamma^M(S)$ is virtually torsion-free. The following lemma makes this an attractive approach:

Lemma 4.1 (Torsion conjugacy)**.** *Suppose that g is a torsion element in $G_L(S)$. Taking Γ and M as above, the group $G_\Gamma^M(S)$ is naturally a subgroup of $G_L(S)$. Then there exists $h \in G_L(S)$ such that*

$$h^{-1}gh \in G_\Gamma^M(S).$$

Proof. We pick a vertex X^0 in the 0-level set of the cube complex $X_L(S)$ to be the identity vertex when the level set is identified with the Cayley graph of $G_L(S)$. If we insist on this being also the identity vertex for $G_\Gamma^M(S)$, we have a natural embedding of $X_\Gamma^M(S)$ into $X_L(S)$, and $G_\Gamma^M(S)$ is precisely the subgroup of $G_L(S)$ which fixes $X_\Gamma^M(S)$. If g is a torsion element of $G_L(S)$, it must be the point stabiliser of some vertex of $X_L(S)$ with height not in S. We know that for a vertex with height not in S, the

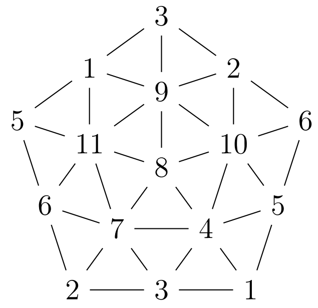


FIGURE 4.1: A minimal flag triangulation of $\mathbb{RP}^2$.

stabiliser is a subgroup isomorphic to $\pi_1(L)$, and stabilisers at the same height are conjugate. So there exists some h in $G_L(S)$ such that $h^{-1}gh$ stabilises a vertex in $X_\Gamma^M(S)$. Since the size of the stabiliser of this vertex is $|\pi_1(L)|$ in $G_\Gamma^M(S)$, we conclude that $h^{-1}gh$ in fact fixes $X_\Gamma^M(S)$, and so is an element of $G_\Gamma^M(S)$. $\square$

Being able to conjugate all torsion into a subgroup which we already know is virtually torsion-free seems somewhat useful to showing that the larger overgroup is virtually torsion-free also. This chapter will explore potential methods for expanding this from the subgroup, as well as highlighting some obstructions to this.

4.1 The real projective plane

We turn our attention to $G_L(S)$ for L a flag triangulation of $\mathbb{RP}^2$ (i.e. where M is the sphere). The strategy to find a torsion-free finite-index subgroup is to take the existing one-dimensional solution, use it for a loop that generates the fundamental group, then worry about how to fill in the rest. The only thing we need to think about here is how to assign group elements to the edges, such that the triangle relations are satisfied (to get a suitable quotient).

Lemma 4.2 (Direct approach)**.** *The two equations $abc = 1$ and $a^{-1}b^{-1}c^{-1} = 1$ both hold if and only if the two equations $c^{-1} = ab$ and $[a, b] = 1$ hold.*

Note that this means that given a triangle in L, when the triangle relations are satisfied, the edges of the triangle pairwise commute. Also going one way around the triangle from one point to another is the same as taking the other path, where you multiply together the two elements on the longer path (taking orientation into account).

For our flag triangulation of $\mathbb{RP}^2$, we use the 11-vertex one found in [BOW$^+$20], see Figure 4.1. The reason for this is that it is a minimal one, as such it cuts down the number of variables we have to deal with.

Proposition 4.3 (Solving $\mathbb{RP}^2$ by filling in triangles)**.** *Given a periodic set S, let the elements a, b, c, d solve the 2-pattern corresponding to S. Using the triangulation above, the group $G_{\mathbb{RP}^2}(S)$ is virtually torsion-free if there exists a group H with elements s, u, v, w, x, y, z such that $\langle a, b, c, d\rangle < H$ and the following are satisfied (where the centralisers are taken in H):*

- $s \in C(a) \cap C(b)$
- $u \in C(d) \cap C(as)$
- $v \in C(c) \cap C(du)$
- $b \in C(cv)$
- $w \in C(v^{-1}du) \cap C(u^{-1}as)$
- $x \in C(b^{-1}s) \cap C(w^{-1}u^{-1}as)$
- $b^{-1}sx \in C(c)$
- $y \in C(a) \cap C(d)$
- $yd^{-1} \in C(c^{-1}b^{-1}sx)$
- $z \in C(w^{-1}u^{-1}asx) \cap C(yd^{-1}c^{-1}b^{-1}sx)$
- $v^{-1}dasxz^{-1} = w^{-1}u^{-1}asxz^{-1}v^{-1}duw$
- $yd^{-1}c^{-1}b^{-1}sxz^{-1}ya = yayd^{-1}c^{-1}b^{-1}sxz^{-1}$
- $bcdasxz^{-1} = v^{-1}dasxz^{-1}bcv$

Proof. The fundamental group of $\mathbb{RP}^2$ is C_2, therefore we get 2-torsion at point stabilisers. These, and their conjugates, are all the torsion elements in the group. This means that the kernel of the quotient map from a loop to the solution group is torsion-free. Therefore we need to fix a loop that generates $\pi_1(\mathbb{RP}^2)$, write a, b, c, d on it along the edges in the chosen direction, and find a way to extend the quotient map to all the (directed) edges in the flag complex.

The following derivation is depicted in Figure 4.2:

We take the loop $(6-7-8-10-6)$ as the generator of the fundamental group. We assign the elements a, b, c, d to it, in that orientation. If we denote $(7-11)$ by s and $(2-6)$ by y, we get $s \in C(a) \cap C(b)$ and $y \in C(a) \cap C(d)$ (black section in the picture).

This now means that $(2-7)$ is ya, $(6-11)$ is as, $(8-11)$ is $b^{-1}s$ and $(2-10)$ is yd^{-1}. If we denote $(6-5)$ by u, we get $u \in C(d) \cap C(as)$ (red section in the picture).

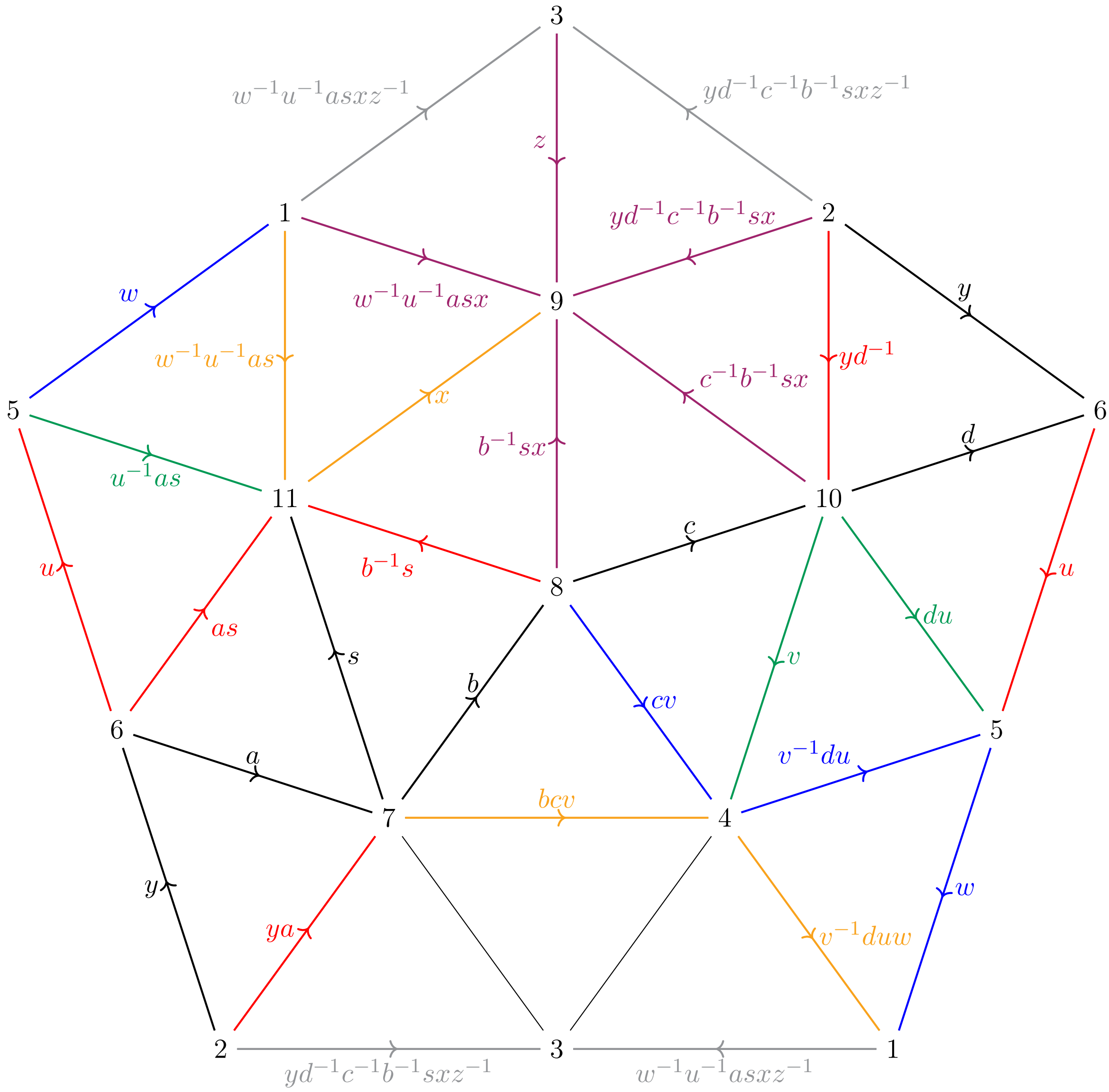


FIGURE 4.2: Deriving the conditions for triangle relations.

This now means that $(5-11)$ is $u^{-1}as$ and $(10-5)$ is du. If we denote $(10-4)$ by v, we get $v \in C(c) \cap C(du)$ (green section in the picture).

This now means that $(4-5)$ is $v^{-1}du$ and $(8-4)$ is cv. Due to triangle $(8-4-7)$, we must now also have $b \in C(cv)$. If we denote $(5-1)$ by w, we get $w \in C(v^{-1}du) \cap C(u^{-1}as)$ (blue section in the picture).

This now means that $(1-11)$ is $w^{-1}u^{-1}as$, $(7-4)$ is bcv and $(4-1)$ is $v^{-1}duw$. If we denote $(11-9)$ by x, we get $x \in C(b^{-1}s) \cap C(w^{-1}u^{-1}as)$ (yellow section in the picture).

This now means that $(1-9)$ is $w^{-1}u^{-1}asx$ and $(8-9)$ is $b^{-1}sx$. Due to triangle $(8-9-10)$, we must now also have $b^{-1}sx \in C(c)$. Which means that $(10-9)$ is $c^{-1}b^{-1}sx$. Due to triangle $(9-2-10)$, we must now also have $yd^{-1} \in C(c^{-1}b^{-1}sx)$.

Which means that $(2-9)$ is $yd^{-1}c^{-1}b^{-1}sx$. If we denote $(3-9)$ by z, we get $z \in C(w^{-1}u^{-1}asx) \cap C(yd^{-1}c^{-1}b^{-1}sx)$ (violet section in the picture).

This now means that $(1-3)$ is $w^{-1}u^{-1}asxz^{-1}$ and $(2-3)$ is $yd^{-1}c^{-1}b^{-1}sxz^{-1}$ (grey section in the picture). Due to triangle $(1-3-4)$, we must now also have that $v^{-1}duw$ and $w^{-1}u^{-1}asxz^{-1}$ commute. Due to triangle $(3-2-7)$, we must now also have that ya and $yd^{-1}c^{-1}b^{-1}sxz^{-1}$ commute.

Finally, we come to triangle $(4-3-7)$. We have that $(7-3)$ is $a^{-1}d^{-1}c^{-1}b^{-1}sxz^{-1}$ and that $(4-3)$ is $v^{-1}dasxz^{-1}$. The first triangle relation on this triangle tells us that we must have

$$bcv \cdot v^{-1}dasxz^{-1} = a^{-1}d^{-1}c^{-1}b^{-1}sxz^{-1},$$

but this is equivalent to $abcdabcd = 1$, which is always satisfied. Hence the only remaining condition we need is that bcv and $v^{-1}dasxz^{-1}$ commute. The conditions in the statement of the proposition are a summary of these.

□

Note that once we have a, b, c, d to solve the "1-dimensional" pattern, all of the conditions on the new elements are in the form of commutators. This means that if H is abelian, they will be satisfied by taking every new element to simply be the identity. The 2-pattern [2,1] is solved by sending a to the generator of C_2 and b, c, d to the identity. Hence this extends to a solution, and shows that $G_L(S)$ has an index 2 torsion-free subgroup in this case. Due to Lemma 3.15, any other pattern cannot be extended this way, since $\langle a, b, c, d \rangle$ cannot be abelian for a period longer than 2.

4.1.1 Search via GAP

We implement a search for a group H satisfying the conditions of Proposition 4.3 in GAP. The precise code can be found in the code listing at the end (see A.2).

We look for a finite-index torsion-free subgroup of $G_L(S)$ for $S = 3\mathbb{Z}$. The corresponding pattern is $[2, 2, 1]$ in this case. Note that since 2 and 3 are prime, any finite group solving the conditions must have order divisible by 6.

After running the program on and off for weeks, eventually a solution is found for a group having `SmallGroups` identifier $[576, 5129]$.

This is encouraging, since the pattern in this case has period length 3 but the fundamental group of L is order 2, which rules out any kind of obstructions in terms of the period length having to share prime factors with the size of the group.

No other pattern except the two above has been solved for this particular triangulation of $\mathbb{RP}^2$ as of the time of writing. This could be due to the smallest solution groups having very large size. The following section illustrates why it may be useful to study different triangulations.

4.2 Homomorphism extension

Since a finite-index subgroup always contains a normal finite-index subgroup, a torsion-free finite-index subgroup must contain a normal torsion-free finite-index subgroup. This means that when $G_L(S)$ is virtually torsion-free, there exists a homomorphism to some finite group G with torsion-free kernel.

Due to the nature of the presentation for $G_L(S)$, a homomorphism to G corresponds to a labelling of the directed edges of L with elements of G. As such, for appropriate Γ this restricts to a homomorphism from $G_\Gamma^M(S)$ to G, again with torsion-free kernel.

Despite every virtually torsion-free $G_L(S)$ yielding such a 1-dimensional restriction, there is no guarantee that every normal torsion-free finite-index subgroup of $G_\Gamma^M(S)$ arises in this way, nor can we be sure that it can always be directly extended in the simplest sense, as attempted in the above section. The aim of this section is to clarify the following:

Proposition 4.4 (Counter-example). *Let us insist that $\Gamma \subset L$ must be a 1-dimensional subcomplex which contains a representative loop for every element of $\pi_1(L)$ (after having chosen a fixed basepoint), with corresponding M as at the start of this chapter. It is possible to choose (L, Γ) such that there is a homomorphism $\phi : G_\Gamma^M(S) \to G$ for some finite group G with torsion-free kernel, but there is no homomorphism $\psi : G_L(S) \to H$ for a finite group H containing G as a subgroup, with $\ker\psi$ torsion-free and $\psi|_{G_\Gamma^M(S)} = \phi$. As a concrete example, one can take L to be a flag triangulation of $\mathbb{RP}^2$ and Γ to be a non-nullhomotopic loop.*

Thus one cannot take just any torsion-free finite-index subgroup H of $G_\Gamma^M(S)$ and hope to extend it to a torsion-free finite-index subgroup V of $G_L(S)$ with

$$V \cap G_\Gamma^M(S) = H, \tag{4.1}$$

since it may not exist. Nonetheless, Lemma 4.1 implies that if there is indeed a normal finite-index subgroup V of $G_L(S)$ satisfying (4.1), then $G_L(S)$ is virtually torsion-free.

Before we go into how to construct the counterexamples of Proposition 4.4, let us briefly return to the significance of the number 4, as pointed out earlier in Subsection 3.2.1.

Given a class of groups $\mathfrak{G}$, consider the following function $\mathfrak{F}_{\mathfrak{G}} : \mathbb{N} \to \mathbb{N} \cup \infty$, given by:

- Whenever $g_1, \dots, g_k$ are elements in a group $G \in \mathfrak{G}$, if there exists l such that

$$\prod_{i=1}^{k} g_i^j = e \text{ for } 1 \leqslant j \leqslant l \implies g_1^{l+1} \cdots g_k^{l+1} = e$$

  then we define the value of $\mathfrak{F}_{\mathfrak{G}}(k)$ to be equal to the minimum value of l for which the above holds (since the naturals are well-ordered, this is well-defined).

- If no such l exists, we set $\mathfrak{F}_{\mathfrak{G}}(k) = \infty$.

Note that the condition must hold for all groups $G \in \mathfrak{G}$. If we take $\mathfrak{G}$ to be all groups, denote the corresponding function simply by $\mathfrak{F}$.

Lemma 4.5. *For small k, we have that $\mathfrak{F}(k)$ is finite.*

Proof. In particular:

1. For $k = 1$, we already have at $j = 1$ that $g_1 = e$, hence $g_1^2 = e$, so $\mathfrak{F}(1) = 1$.
2. For $k = 2$, from $j = 1$ we get $g_1 = g_2^{-1}$. But then $g_1^2 g_2^2 = e$, hence $\mathfrak{F}(2) = 1$.
3. For $k = 3$, from $j = 1$ we obtain $g_1 = g_3^{-1} g_2^{-1}$. From $j = 2$,

$$g_1^2 g_2^2 g_3^2 = e \implies g_3^{-1} g_2^{-1} g_3^{-1} g_2^{-1} \cdot g_2^2 g_3^2 = e \implies g_3^{-1} g_2^{-1} g_3^{-1} g_2 g_3^2 = e$$

$$\implies g_2^{-1} g_3^{-1} g_2 g_3 = e \implies [g_2, g_3] = e,$$

   i.e. g_2 and g_3 commute, from which we get

$$g_1^3 g_2^3 g_3^3 = g_2^{-3} g_3^{-3} g_2^3 g_3^3 = e \implies \mathfrak{F}(3) = 2.$$

□

The function $\mathfrak{F}$ hence has consequences for $G_L^M(S)$ in terms of restricting the behaviour of normal finite-index torsion-free subgroups: the above shows that in general, a pattern featuring an initial segment of two consecutive entries which are 1, but followed by n, is impossible to solve using only three group element variables. Hence this shows that the full freedom of at least four group elements must be utilised to have any chance of solving every possible pattern, i.e. torsion-free subgroups in general are "complicated" in some sense (in particular, cyclic or few-generator quotients with torsion-free kernel are rare).

Note that for sufficiently large k, a small-cancellation theory argument shows that $\mathfrak{F}(k) = \infty$ (e.g. for $k > 12$). This is because given a finite initial string of such relations, we can form a finitely presented group, which will satisfy the $C'(1/6)$ condition. This is then hyperbolic, so Dehn's algorithm applies to show that the next word in the sequence is not trivial [LS01].

Note that $\mathfrak{F}$ is a non-decreasing function, because we can always pad out smaller sets of elements with simply adding on copies of the identity element. In particular,

$$\mathfrak{F}(k) = \infty \implies \mathfrak{F}(l) = \infty \;\; \forall\, l \geqslant k.$$

Generalised Bestvina-Brady groups, however, already show that $\mathfrak{F}(4) = \infty$, by taking L to be the 4-vertex triangulation of the circle. With application to finite-index subgroups, we are interested in $\mathfrak{G}$ being the class of finite groups. Denote $\mathfrak{F}_{\mathfrak{G}}$ by $\mathfrak{F}'$ in this case.

In order to be able to solve every possible pattern with k elements, it is necessary for $\mathfrak{F}'(k)$ to not be finite. This is because being able to solve any pattern with k elements implies that patterns starting with an arbitrarily long string of 1s are solvable. In Subsection 3.3.3, we showed that $\mathfrak{F}'(4) = \infty$.

We will exploit a geometric feature of the pair (L, Γ) to limit the freedom that we have for certain solution subgroups, using the above discussion of the function $\mathfrak{F}$.

Definition 4.6 (Shortcut-pair)**.** We say that (L, Γ) form a shortcut-pair if there is a non-nullhomotopic edge-loop γ in Γ which is at least 5 edges long and there is a loop α in L which is homotopic to γ, contains at most 3 edges outside Γ, but γ contains at least four independent edges which do not intersect α (call α the *shortcut loop*).

The idea is that since homotopic loops have to give rise to the same pattern, we can use the shortcut to force a solution of a more complicated pattern with fewer group elements. We are now in a position to prove Proposition 4.4:

Lemma 4.7 (Shortcut restriction)**.** *Suppose that (L, Γ) are as in Proposition 4.4. If (L, Γ) are a shortcut-pair, then the conclusion of Proposition 4.4 holds.*

Proof. Let $\gamma \subset \Gamma$ be a loop which satisfies the definition of short-cut pair for (L, Γ). We are most interested in elementary patterns. If we pick S to correspond to an elementary pattern of sufficiently long length, the pattern I corresponding to γ may only be solved using the full freedom of 4 group elements. Since the proof of Theorem 3.23 relies on 4 independent edges, we have a labelling of the edges of Γ which corresponds to a torsion-free finite-index subgroup of $G^M_\Gamma(S)$ by using these 4 edges for γ and setting the rest of the edges to the identity element. However now the short-cut loop α features all identity elements except for at most the 3 edges of it which lie outside Γ. Because $\mathfrak{F}'(3)$ is finite, this means that no labelling of edges of the short-cut loop by group elements from a finite group can ever solve the pattern I. This means that there is no labelling of the edges of L corresponding to a homomorphism to a finite group with torsion-free kernel that restricts to the chosen labels on Γ. $\square$

A long enough elementary pattern to make the above work would be any pattern of length at least 4. For an example of a shortcut-pair where L is a flag triangulation of $\mathbb{RP}^2$ and Γ is a non-nullhomotopic loop, we may take Figure 4.1, Γ as the outside boundary and $3-2-10-4-3$ as the shortcut.

The above informs us that if we wish to extend a torsion-free subgroup defined on Γ, we should ensure that there is enough flexibility for the unknown edges we are filling in with group elements. An easy way to get rid of (L, Γ) being a shortcut-pair would be to apply barycentric subdivision, while maintaining homotopy type. However, this still does not guarantee that a suitable extension will always be possible.

4.3 Simultaneous conjugacy

Note that of most importance are the elementary patterns (Definition 3.10), since their solution leads to the solution of any other periodic pattern. Putting the period-2 pattern aside for now, observe that every other elementary pattern begins with a 1. This means that in a labelling of L with finite group elements corresponding to a homomorphism to a finite group with torsion-free kernel, each directed edge-loop has the property that multiplying the elements corresponding to its directed edges (in the corresponding order) results in the identity. We can use this fact to present the quotient in a new way.

We will now proceed to define *weight diagrams*, a combinatorial way to encode a finite quotient of a generalised Bestvina-Brady group. This will consist of taking a copy of L and putting group elements on the vertices. The advantage of this approach is that, as we shall see shortly, we will be able to focus on the triangle relations in the presentation of the group (recall that these are the ones coming from directed edge-loops of length 3).

Because L is connected, if we place a *weight* on some basepoint vertex consisting of a group element, then insist that each directed edge is labelled by the difference of its endpoints, we get a unique solution for what all the other vertex weights must be, given a labelling of the edges. Recall that a labelling of the edges already naturally corresponds to a quotient because the edges of L are in bijection with a generating set for the generalised Bestvina-Brady group.

We make the convention that the group element assigned to directed edge going from vertex u to vertex v is given the value uv^{-1}.

Thus labelling the directed edges of L with elements of some finite group G is equivalent to labelling the vertices, up to the choice of weight of some base vertex.

If our goal is to extend labels on edges of Γ as in the above section, we could convert this to labels on vertices of Γ instead, and try to fill in the remaining vertices of L. The long-cycle relations are already taken care of in Γ, so the conditions that the new weights must satisfy come purely from the triangle relations of $G_L(S)$, see Figure 4.3.

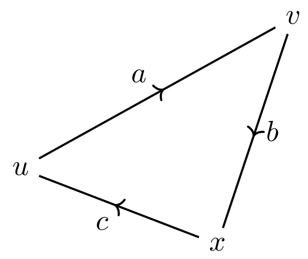


FIGURE 4.3: Deriving the conditions for triangle relations in terms of weights.

Suppose that adjacent weights u, v are already filled in and that they are part of a triangle with a vertex of weight x (see Figure 4.3). The corresponding group elements a, b and c, assigned to the directed edges around the boundary of this triangle, are given by:

$$a = u \cdot v^{-1}, \; b = v \cdot x^{-1}, \; c = x \cdot u^{-1}.$$

There are two triangle conditions which must be satisfied, the first of which automatically holds (abc being trivial). The second one is:

$$a^{-1}b^{-1}c^{-1} = vu^{-1}xv^{-1}ux^{-1}$$

being trivial. This is the same as:

$$x \cdot v^{-1}u \cdot x^{-1} = uv^{-1},$$

which means that x^{-1} conjugates $v^{-1}u$ to uv^{-1}. When u and v commute, this is the same as x commuting with $v^{-1}u$ $(= uv^{-1})$, i.e. with the edge label opposite x. This leads us to consider solving the conjugacy problem, to fill in new weights.

Definition 4.8 (Overgroup simultaneous conjugacy problem)**.** Given a group G and elements $(g_1, \ldots, g_n), (g'_1, \ldots, g'_n) \in G^n$, does there exist a group H such that $G \leqslant H$ and $\exists\; h \in H$ with $h^{-1}g_ih = g'_i$ for all i?

If we are interested in solving for just one particular triangle, this reduces to the case where $n = 1$.

Theorem 4.9 (Overgroup conjugacy)**.** *Given a finite group G and elements $a, b \in G$, there exists a finite group H such that $G \leqslant H$ and $\exists\; h \in H$ with $h^{-1}ah = b$ if and only if $o(a) = o(b)$.*

Proof. If a and b have different orders, they can never be conjugate in any overgroup, since conjugation preserves order of elements. Supposing that they do have the same order, then consider the regular representation of G, i.e. its (WLOG left) action on itself. This is the standard embedding of G into $Sym(|G|)$, where a and b will have the same cycle type of $|G|/o(a)$ many cycles of length $o(b)$. Hence they will be conjugate in this symmetric group, because conjugacy in symmetric groups is determined by cycle type. □

Since uv^{-1} and $v^{-1}u$ always have the same order, it follows that every triangle by itself can certainly be completed, if only missing one vertex weight. So now we consider the bigger picture, given some partially filled in weights diagram, is it possible to fill in missing weights? Semidirect products and automorphisms of G could be a good way to build overgroups in which conjugation happens, however it turns out that we cannot solely rely on such constructions:

Theorem 4.10 (Difficult diagrams)**.** *If we insist on the conjugating element h inducing an automorphism of G, then there is a diagram with 5 vertices and 7 edges where if we fill in 4 of the vertices with certain weights from S_5, there is no way to choose an element for the last vertex.*

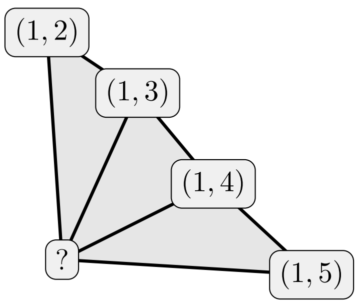


FIGURE 4.4: An example of a partial weight diagram.

Proof. We use the weights $(1, 2)$, $(1, 3)$, $(1, 4)$ and $(1, 5)$ (see Figure 4.4). In order for the 5th vertex to be filled in, an automorphism of A_5 must send $(1, 2, 3)$ to $(1, 3, 2)$ and $(1, 3, 4)$ to $(1, 4, 3)$, according to the first two triangles. Since $\mathrm{Aut}(A_5) = S_5$, the only automorphism which does this is the one that swaps 1 and 3 (i.e. corresponds to conjugation by $(1, 3)$ in S_5). However, the last triangle also tells us that the same automorphism must send $(1, 4, 5)$ to $(1, 5, 4)$, which does not happen under conjugation by $(1, 3)$ (it sends $(1, 4, 5)$ to $(3, 4, 5)$ instead). Thus no suitable automorphism of A_5 exists, and the final weight in the diagram cannot be filled in. □

In particular, we cannot always hope for the group G to be normal in the overgroup H. In practice, automorphisms will usually be induced, hence the suitability of solution groups will depend on the exact behaviour of the automorphism groups.

Note that the above counter-example consisted of three triangles put together side-by-side. It turns out that this is minimal, in the sense that the corresponding picture for just two triangles side-by-side can always be solved:

Lemma 4.11 (Easy triangles). *If a triangle has at least two of its vertices having the same weight, then it satisfies the triangle relations.*

Proof. We use the observation from earlier that when u, v commute, we just need the third vertex weight x to commute with their difference. However, if $u = v$, they certainly commute, and their difference is the identity, with which x also commutes. □

Thus if we have two triangles side-by-side, we can simply use the weight of the vertex in the middle between the triangles as the last unknown weight, giving two triangles which each have at most 2 different weights on their vertices.

While seemingly simple, the above lemma is actually useful in that it now allows us to prove that $G_L^N(S)$ is virtually torsion-free for a large class of complexes L.

Definition 4.12 (Graph retraction). We say that the pair (L, Γ), with L, Γ as before, *graph-retracts* if there exists a map

$$r : V(L) \to V(\Gamma)$$

from the vertices of L to the vertices of Γ such that $r(v) = v$ for all $v \in V(\Gamma)$, as well as if u and v are adjacent in L, we must have either $r(u) = r(v)$, or $r(u)$ is adjacent to $r(v)$.

This gives us one way to extend the 1-dimensional solution from Subsection 3.3.4 to higher-dimensional L.

Theorem 4.13 (Solution for retracts). *Suppose that L graph-retracts onto Γ with sufficiently many independent edges. Then if N is any finite regular cover of L, the group $G_L^N(S)$ is virtually torsion-free.*

Proof. We use the normal finite-index torsion-free subgroup we have for Γ to assign vertex weights to the vertices of Γ as above. Then we use the retraction r to assign weights to the remaining vertices of L, where a vertex v is assigned the weight that vertex $r(v)$ has. Since Γ is flag, there are no loops of length 3. Therefore every triangle in L will have its vertices labelled with at most 2 different weights. Thus by Lemma 4.11, the triangle relations are all satisfied and we have a corresponding normal finite-index torsion-free subgroup of $G_L^N(S)$. □

However, this technique cannot be used to tackle all possible L. For applications, the homotopy type of L is relevant, but even then, we cannot hope to always rely on this technique:

Proposition 4.14 (Topological obstruction)**.** *For finite groups G, if $\pi_1(L) = G$, then L cannot graph-retract to Γ with $\pi_1(\Gamma)$ surjecting onto $\pi_1(L)$.*

Proof. A graph Γ as above will have infinite free fundamental group. Since the complexes involved are all flag, we have induced maps on clique complexes. On the level of fundamental groups, if we compose the inclusion of Γ with the retraction map, we have an induced identity map on an infinite group which factors through a finite group (see e.g. [Hat02]). Since a finite group cannot surject onto an infinite group, this cannot happen. □

Note that the above argument actually shows that $\pi_1(L)$ must be free, if it is to satisfy the required hypotheses.

In order to extend the homomorphism from $G_\Gamma^M(S)$ to $G_L(S)$, we need the inclusion of Γ in L to be surjective on the level of fundamental groups, so Proposition 4.14 shows that Theorem 4.13 cannot apply to L with finite fundamental group.

Finally, we turn to attempting to prove that $G_L(S)$ is virtually torsion-free for when we only care about what $\pi_1(L)$ is. More precisely, we allow flexibility in choosing the flag triangulation. From the point of view of weight diagrams, this corresponds to fixing the boundary of a picture and assigning weights to it, then asking if it possible to subdivide the inside in any way such that weights on the vertices in the interior can be filled in. If it were possible to always do this, we could take L to be a presentation complex, Γ the 1-skeleton and then fill in each relation, to get virtually torsion-free $G_L(S)$ for any fundamental group of L that we choose. However, this may be too ambitious in general, given the following:

Theorem 4.15 (Impossible triangles)**.** *There exists a triangle with weights assigned to its vertices such that no matter how the interior of the triangle is subdivided, it is not possible to assign weights to the interior edges to satisfy the triangle relations.*

Proof. Suppose that every triangle can indeed be filled in as above. Let us take a loop of length 4, on the edges of which we write group elements that solve the pattern $[1, 1, 2, 1]$ (any non-trivial pattern which begins 1,1 will do). If we split this down the middle, we get two triangles that need filling in. By assumption, these can be filled in, and so in the end we get a subdivision L of the disk. Since the triangle relations are satisfied, this gives a quotient of BB_L. In particular, the pattern induced by two homotopic edge-loops is the same (see the proof in [DL99], consider the finitely many 2-simplices the loop has to cross during the homotopy and notice that crossing over each 2-simplex does not change the pattern). Since the disk is contractible, we can homotope the boundary down to a single triangle. But now this implies that the three elements written on the edges of this triangle solve the pattern we started with. This is impossible due to $\mathfrak{F}'(3) = 2$, which says that having only three elements forces all the entries in the pattern to be 1 once the first two are 1, but we are assuming the pattern begins [1,1,2]. Therefore one of the two triangles we made at the start cannot actually be filled in. $\square$

The above was an issue because the elements on the outer loop did not come from a quotient of a Bestvina-Brady group.

Conjecture 4.16 (Bestvina-Brady extension)**.** *Given a flag subdivision L of the circle and a finite quotient H of BB_L, there exists a flag subdivision of the disk M such that L is the boundary and the quotient H extends to a quotient of BB_M.*

4.4 Separability in Bestvina-Brady groups

The profinite topology on a group is the one when we take all finite-index normal subgroups as the closed base. We say that a subgroup is separable if it is closed in the profinite topology. Separable subgroups are also intersections of finite-index subgroups.

Proposition 4.17. *Given an inclusion of connected flag simplicial complexes $M \subset N$, the group BB_M naturally embeds into BB_N. Furthermore, BB_M is separable in BB_N.*

Proof. On the level of RAAGs, A_M is naturally a subgroup of A_N. We get

$$BB_M = BB_N \cap A_M,$$

and A_M is separable in A_N. $\square$

A group with all finitely generated subgroups separable is called subgroup separable, or LERF.

In [LR08], the property LERF and related ideas were used to extend homomorphisms from subgroups. In this subsection, we will explore if anything similar can occur for Bestvina-Brady groups. In particular, [LR08] makes use of virtual retractions of groups. Virtual retracts preserve the homological finiteness property FP_2, but when $G_L(S)$ is of type FP_2, the subgroup $G_\Gamma^M(S)$ cannot be, which shows that in general it is not a virtual retract. Therefore a new approach needs to be developed if separability is to be used.

Let L be a complex with sufficient independent edges (such as in Definition 3.22) and S be as previously. Given that S is a periodic subset of $\mathbb{Z}$, let n be the period of S. Let $G = \pi_1(L)$. Denote by N the universal cover of L, so we have a finite regular covering map:

$$p : N \to L.$$

Since L has sufficient independent edges, let

$$\Sigma := \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$$

be an appropriate good set of loops. Denote by Γ their union, which is naturally a one-dimensional simplicial subcomplex of L. Let:

$$M := p^{-1}(\Gamma).$$

Thus we get a commutative diagram of simplicial complexes as in Figure 4.5, where horizontal arrows are inclusions and vertical arrows are finite covers.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & N \arrow[d] \\ \Gamma \arrow[r, hook] & L \end{tikzcd}$$

FIGURE 4.5: The covers governing the branching in $G_L(S)$ and $G_\Gamma^M(S)$.

Since the loops in Γ generate $\pi_1(L)$, the simplicial graph M is connected. In particular, BB_M is finitely generated. Moreover, M resembles a Cayley graph of G with Σ as the generating set. The group $G_L(S)$ may also be denoted $G_L^N(S)$, consistent with our notation, but we tend to omit the N in order to avoid clutter.

For any loop $\gamma \subset M$, the image $p(\gamma)$ in L will be a nullhomotpic loop.

Fix some orientation for the edges of L, and denote them by x_i, where x_i^{-1} refers to the oppositely-oriented edge. For each loop $\gamma_i \in \Sigma$, let:

$$\gamma_i = \left(y_{i,1}, y_{i,2}, \ldots, y_{i,\sigma(i)}\right)$$

be the directed edges it consists of, in the appropriate order.

Let

$$\Delta' := \{(z_{1,1}, z_{1,2}, z_{1,3}), (z_{2,1}, z_{2,2}, z_{2,3}), \ldots, (z_{t,1}, z_{t,2}, z_{t,3})\}$$

be the set of all 2-cells in L, where for each one we have chosen a directed boundary triangle consisting of three appropriate directed edges to represent it.

Thinking of the abstract labels as a group alphabet for a group presentation, set:

$$\Delta := \bigcup_{k=1}^{t} \left\{ z_{k,1}z_{k,2}z_{k,3},\, z_{k,1}^{-1}z_{k,2}^{-1}z_{k,3}^{-1} \right\},$$

and:

$$Q := \bigcup_{k=1}^{l} \left\{ y_{k,1}^{i}y_{k,2}^{i}\cdots y_{k,\sigma(k)}^{i}\ \forall i \in S \right\}.$$

Then a presentation for $G_L(S)$ is given by:

$$G_L(S) = \left\langle x_i, x_i^{-1} \mid x_i x_i^{-1}, \Delta, Q \right\rangle,$$

or more simply:

$$G_L(S) = \langle x_i \mid \Delta, Q \rangle,$$

with inverses understood. We refer to the relations Δ, Q as *triangle relations* and *long cycle relations*, respectively. For i not in S, a relation of the form

$$\left(y_{j,1}^{i}y_{j,2}^{i}\cdots y_{j,\sigma(j)}^{i} \right)^{m}$$

for some positive integer m will arise due to the geometry of L, as a consequence of the triangle relations. Note that these will already be present in an ordinary Bestvina-Brady group (under a different labelling of the elements). This inspires comparing the presentation of BB_N and $G_L(S)$ to see how they are related. The presentation of BB_N has the edges of N as generators, and as relations we have the triangle relations already present. Recall that N is a universal cover.

There is a subset W of BB_N, such that setting

$$R := \langle\langle W \rangle\rangle_{BB_N},$$

we get

$$G_L(S) = BB_N/R.$$

The set of words W consists of words of BB_N of two types: those needed for the long cycle relations, and those which arise from the cover $N \to L$. This is essentially what is 'missing' in BB_N to form $G_L(S)$.

The words of the first kind are precisely the words in Q (we consider one base copy of L inside N and identify its edges by the same labels). The words of the second kind are there to deal with the edges of N which are outside the base copy of L:

$$C := \left\{ xy^{-1} \;\middle|\; x \text{ and } y \text{ map to the same edge in } L \right\},$$

thus $W = Q \cup C$. Note that since L is finite and we have a finite universal cover, the set C contains only finitely many words.

We consider BB_M as a natural subgroup of BB_N. This is induced by the natural inclusion of M into N. By fixing a chosen basepoint vertex X^0 in the 0-level set of the appropriate branched cover, we also have a naturally induced inclusion of CAT(0) cube complexes:

$$X_\Gamma^M(S) \hookrightarrow X_L(S).$$

Note that once again we avoid clutter by choosing to denote $X_L^N(S)$ by $X_L(S)$, which is still consistent with our notation, as N is a universal cover.

Following Subsection 3.3.4, we embed G into the derived subgroup of some finite group G'.

Since a presentation for BB_M can be understood as having generators corresponding to the directed edges of M, to specify a finite quotient of BB_M it suffices to assign elements of a finite group to them (such that the appropriate relations are satisfied). We will do so by labelling M with elements of the solution group derived in Theorem 3.23.

Denote by H the kernel of this homomorphism.

We will now try to extend the above construction (which only applies to M) to all of BB_N by "filling in" the triangle relations.

To this effect, we show that H is separable in BB_N. Note that while separability passes to finite-index over-groups, it does not in general pass to finite-index subgroups. Hence even though we know that BB_M is separable in BB_N, we need to do some work to establish the same for H. The connectivity of M will be the main geometric ingredient, with the rest following from more or less elementary, albeit convoluted group theory.

Firstly, observe that BB_M is finitely generated, because M is connected. This means that for a given positive integer n, there are only finitely many homomorphisms $BB_M \to S_n$. Hence by considering the action on cosets, there are only finitely many subgroups of index n in BB_M. Next, note that group isomorphisms preserve the index of finite-index subgroups. Therefore, if we consider:

$$K := \bigcap_{\psi \in \mathrm{Aut}(BB_M)} \psi(H),$$

we get that this is an intersection of finitely many finite-index subgroups of BB_M (each having the same finite index as H), hence K itself is a finite-index subgroup of BB_M (sometimes referred to as the characteristic core). Given an automorphism of BB_M, it will only permute the subgroups being intersected in the definition of K, therefore fixing K. In other words, K is a characteristic subgroup of BB_M and normal. Furthermore, setting ψ to be the identity automorphism in the definition of K, we see that K is a finite-index subgroup of H.

Given some $g \in A_M$, since BB_M is normal we have that conjugation by g in A_M induces an automorphism of BB_M. Because K is characteristic in BB_M, K will be invariant under this automorphism, and hence:

$$g^{-1}Kg = K$$

in A_M, i.e. K is normal in A_M. Now by a group isomorphism theorem we have:

$$(A_M/K) \, / \, (BB_M/K) \cong A_M/BB_M.$$

We know that A_M/BB_M is $\mathbb{Z}$, and since K is finite-index in BB_M, it follows that BB_M/K is a finite group. Therefore we can deduce that A_M/K is a finite-by-cyclic group. Since all such extensions split, A_M/K is in fact virtually cyclic, and hence residually finite.

Given some $g \in A_M$ which is not in K, its image $\bar{g} \in A_M/K$ will not be the identity element. By the residual finiteness of A_M/K, there is a normal finite-index subgroup $\bar{K}$ of A_M/K which does not contain $\bar{g}$. By a group isomorphism theorem, there exists a finite-index normal subgroup of A_M which contains K but not g, which shows that K is separable in A_M.

Now consider the fact that A_M is a retract of A_N. The retraction map ρ is continuous with respect to profinite topology, so the pre-image of a closed set is closed. Since K is closed in A_M, we have that $\rho^{-1}(K)$ is closed in A_N. However, viewed as a natural subgroup of A_N, we also have that

$$K = A_M \cap \rho^{-1}(K).$$

This is the intersection of two closed sets, hence is itself closed. Thus K is separable in A_N. This means that we can write

$$K = \bigcap_{i \in I} A_i$$

for some finite-index subgroups A_i of A_N for some index set I. Intersecting with BB_N, we get:

$$BB_N \cap K = K = \bigcap_{i \in I} (A_i \cap BB_N).$$

Each $B_i := A_i \cap BB_N$ is a finite-index subgroup of BB_N, hence we see that K is separable in BB_N. Let $\{g_1, g_2, \ldots, g_m\}$ be the finitely many coset representatives of K in H, such that

$$H = \bigcup_{i=1}^{m} g_i K.$$

Because each $g_j B_i$ is closed in the profinite topology on BB_N, this implies that H can be written as a finite union of intersections of closed sets, i.e. H itself is closed. Therefore H is separable in BB_N. Note that this would have followed immediately if BB_M has been a virtual retract of BB_N, which is not the case.

Let k denote the index of H in BB_M. Let $\{h_1, h_2, \ldots, h_{k-1}\}$ be the non-identity coset representatives of H in BB_M. Because H is separable in BB_N, for each $j \in \{1, 2, \ldots, k-1\}$ there exists a finite-index subgroup H_j of BB_N which contains H, but does not contain h_j. Define

$$V := \bigcap_{j=1}^{k-1} H_i.$$

Note that V has finite index in BB_N and has the property that:

$$V \cap BB_M = H.$$

Let V' be the normal core inside V with respect to BB_N. We have:

$$V' \cap BB_M \subseteq H.$$

Also, V' is still a finite-index subgroup of BB_N. We will now investigate the group $V'R/R$, which is a finite-index subgroup of BB_N. By an isomorphism theorem, this is isomorphic to a finite-index subgroup of $G_L(S)$.

Because both R and V' are normal in BB_N, so is the product $V'R$. Suppose there is some element $g \in V'R$ which maps under the quotient by R to some torsion element of $G_L(S)$. By the geometry of the cube complexes, there exists some $h \in BB_N$ such that it is conjugate to a torsion element of $G_\Gamma^M(S)$ (see Lemma 4.1), i.e.

$$\exists\, h \in BB_N : (hR)^{-1} (gR) (hR) \in G_\Gamma^M(S).$$

This means that the element $h^{-1}gh$ of BB_N maps to a torsion element in $G_\Gamma^M(S)$ under the quotient by R. But as $V'R$ is normal, this means that $h^{-1}gh \in V'R$.

But by an isomorphism theorem, $V' \cap R$ is normal in V' and:

$$V'R/R \cong V'/\left(V' \cap R\right).$$

Thus $V'/\left(V' \cap R\right)$ contains an entire conjugacy class of torsion too. Since:

$$V' \cap BB_M \subseteq H,$$

this only leaves the possibility that torsion in $V'/\left(V' \cap R\right)$ comes from some element which is in HR but not in H.

So in order to deduce that the bigger group is virtually torsion-free, just having H be separable in BB_N is not sufficient. However, it would be sufficient for HR to be separable in BB_N.

This is plausible, since we can cover a lot of HR with closed subsets of BB_N in the profinite topology: H is closed, and there are only finitely many words of W which are outside H (see the set C from earlier). By [Min12], each of their conjugacy classes is closed, so their finite union with H is also closed. The main missing elements of HR which remain are BB_N-conjugates of elements of H which lie outside H, and some products.

Determining the virtual torsion-freeness of $G_L(S)$ for L having finite fundamental group is still work in progress at the time of writing. Note that the two-dimensional case would already be good enough to allow applications where the fundamental group of L plays a role.

Appendix A

Code listings

GAP programs mentioned previously, which were not included as part of the main text, appear here.

A.1 The GAP function ToArtin

```
ToArtin := function(M,n)
local i,j,n1,r1,n2,r2,v,k,ired,jred,D;
D := 2*Size(M);
#r-
n1 := 2*n-1;
r1 := [];
for i in [1..D] do
        if i mod 2 = 0 then
                ired := i/2;
        else
                ired := (i+1)/2;
        fi;
        v := [];
        for j in [1..D] do
                if j mod 2 = 0 then
                        jred := j/2;
                else
                        jred := (j+1)/2;
                fi;
                if i=j then
                        if i=n1 then
                                k := -1;
                        else
                                k := 1;
                        fi;
                else
                        if i=n1 then
                                if M[ired][jred] = 0 then
                                        k := 2;
                                else
                                        k := 0;
                                fi;
```

```
                        else
                                k := 0;
                        fi;
                fi;
                Add(v,k);
        od;
        Add(r1,ShallowCopy(v));
od;
#r+
n2 := 2*n;
r2 := [];
for i in [1..D] do
        if i mod 2 = 0 then
                ired := i/2;
        else
                ired := (i+1)/2;
        fi;
        v := [];
        for j in [1..D] do
                if j mod 2 = 0 then
                        jred := j/2;
                else
                        jred := (j+1)/2;
                fi;
                if i=j then
                        if i=n2 then
                                k := -1;
                        else
                                k := 1;
                        fi;
                else
                        if i=n2 then
                                if M[ired][jred] = 0 then
                                        k := 2;
                                else
                                        k := 0;
                                fi;
                        else
                                k := 0;
                        fi;
                fi;
                Add(v,k);
        od;
        Add(r2,ShallowCopy(v));
od;
return r1*r2;
end;;
```

A.2 *RP2* search in GAP

```
Pattern:= [2,2,1];
for i in [576..2000] do
 if i mod 6 = 0 then
  for j in [5129..NrSmallGroups(i)] do
   G:= SmallGroup(i,j);
   ACandidates:= [];
```

```
BCandidates:= [];
CDCandidates:= [];
cl:=ConjugacyClasses(G);
for k in [1..Size(cl)] do
 if Order(Representative(cl[k])) = 3 then
  Add(ACandidates,Representative(cl[k]));
  Append(CDCandidates,AsList(cl[k]));
  Append(BCandidates,AsList(cl[k]));
 fi;
od;
p:= 0;
TotalCases:= (Size(ACandidates)*Size(BCandidates)*Size(CDCandidates)^2);
for a in ACandidates do
 for b in BCandidates do
  for c in CDCandidates do
   for d in CDCandidates do
    p:= p + 1;
    Print("Searching... Currently on ",j," of ",NrSmallGroups(i),", of size ",
        ↪ i,". Stage 0. Current group progress ",Float(100*(p/TotalCases)),"%\
        ↪ r");
    n:= 3;
    t:= 1;
    Testing:= true;
    while Testing do
     r:= t mod 3;
     if r = 0 then
      r:= 3;
     fi;
     if Order(a^t * b^t * c^t * d^t) = Pattern[r] then
      t:= t + 1;
     else
      Testing:= false;
     fi;
     if t = n + 1 then
      Testing:= false;
     fi;
    od;
    if t = n + 1 then
     h:= 0;
     SInt1:= Intersection(Centralizer(G,a),Centralizer(G,b));
     SIntsiz1:= Size(SInt1);
     for s in SInt1 do
      e:= 0;
      h:= h + 1;
      Print("Searching... Currently on ",j," of ",NrSmallGroups(i),", of size
          ↪ ",i,". Stage 1. Current group progress ",Float(100*((p+(h/SIntsiz1
          ↪ ))/TotalCases)),"%\r");
      SInt2:= Intersection(Centralizer(G,d),Centralizer(G,s*a));
      SIntsiz2:= Size(SInt2);
      for u in SInt2 do
       f:= 0;
       e:= e + 1;
       Print("Searching... Currently on ",j," of ",NrSmallGroups(i),", of size
           ↪  ",i,". Stage 1. Current group progress ",Float(100*((p+(h+(e/
           ↪ SIntsiz2)/SIntsiz1))/TotalCases)),"%\r");
       SInt3:= Intersection(Centralizer(G,c),Centralizer(G,d*u));
       SIntsiz3:= Size(SInt3);
       for v in SInt3 do
        f:= f + 1;
```

```
Print("Searching... Currently on ",j," of ",NrSmallGroups(i),", of
    ↪ size ",i,". Stage 1. Current group progress ",Float(100*((p+(h+(
    ↪ e+(f/SIntsiz3)/SIntsiz2)/SIntsiz1))/TotalCases)),"%\r");
 if b*c*v = c*v*b then
  #Print("Success! ",a," ",b," ",c," ",d," reach Stage 2 in group ",j
     ↪ ," of size ",i,"!\n");
  l:= 0;
  SInt4:= Intersection(Centralizer(G,v^-1*d*u),Centralizer(G,u^-1*a*s)
     ↪ );
  SIntsiz4:= Size(SInt4);
  for w in SInt4 do
   m:= 0;
   l:= l + 1;
   Print("Searching... Currently on ",j," of ",NrSmallGroups(i),", of
      ↪ size ",i,". Stage 2. Current group progress ",Float(100*((p+(
      ↪ h+(e+(f+(l/SIntsiz4)/SIntsiz3)/SIntsiz2)/SIntsiz1))/
      ↪ TotalCases)),"%\r");
   SInt5:= Intersection(Centralizer(G,b^-1*s),Centralizer(G,w^-1*u^-1*
      ↪ a*s));
   SIntsiz5:= Size(SInt5);
   for x in SInt5 do
    m:= m + 1;
    Print("Searching... Currently on ",j," of ",NrSmallGroups(i),", of
       ↪  size ",i,". Stage 2. Current group progress ",Float(100*((p
       ↪ +(h+(e+(f+(l+(m/SIntsiz5)/SIntsiz4)/SIntsiz3)/SIntsiz2)/
       ↪ SIntsiz1))/TotalCases)),"%\r");
    if b^-1*s*x*c = c*b^-1*s*x then
     #Print("Success! ",a," ",b," ",c," ",d," reach Stage 3 in group
        ↪ ",j," of size ",i,"!\n");
     o:= 0;
     SInt6:= Intersection(Centralizer(G,d),Centralizer(G,a));
     SIntsiz6:= Size(SInt6);
     for y in SInt6 do
      o:= o + 1;
      Print("Searching... Currently on ",j," of ",NrSmallGroups(i),",
         ↪ of size ",i,". Stage 2. Current group progress ",Float
         ↪ (100*((p+(h+(e+(f+(l+(m+(o/SIntsiz6)/SIntsiz5)/SIntsiz4)/
         ↪ SIntsiz3)/SIntsiz2)/SIntsiz1))/TotalCases)),"%\r");
      if y*d^-1*c^-1*b^-1*s*x = c^-1*b^-1*s*x*y*d^-1 then
       #Print("Success! ",a," ",b," ",c," ",d," reach Stage 4 in group
          ↪  ",j," of size ",i,"!\n");
       q:= 0;
       SInt7:= Intersection(Centralizer(G,w^-1*u^-1*a*s*x),Centralizer
          ↪ (G,y*d^-1*c^-1*b^-1*s*x));
       SIntsiz7:= Size(SInt7);
       for z in SInt7 do
        q:= q + 1;
        Print("Searching... Currently on ",j," of ",NrSmallGroups(i)
           ↪ ,", of size ",i,". Stage 2. Current group progress ",
           ↪ Float(100*((p+(h+(e+(f+(l+(m+(o+(q/SIntsiz7)/SIntsiz6)/
           ↪ SIntsiz5)/SIntsiz4)/SIntsiz3)/SIntsiz2)/SIntsiz1))/
           ↪ TotalCases)),"%\r");
        if v^-1*d*a*s*x*z^-1 = w^-1*u^-1*a*s*x*z^-1*v^-1*d*u*w then
         if y*d^-1*c^-1*b^-1*s*x*z^-1*y*a = y*a*y*d^-1*c^-1*b^-1*s*x*z
            ↪ ^-1 then
          if b*c*d*a*s*x*z^-1 = v^-1*d*a*s*x*z^-1*b*c*v then
           Print("Success! ",a," ",b," ",c," ",d," reach Stage 5 in
              ↪ group ",j," of size ",i,"!\n");
           return;
```

```
                     fi;
                    fi;
                   fi;
                  od;
                 fi;
                od;
               fi;
              od;
             od;
            fi;
          od;
         od;
        od;
       fi;
      od;
     od;
    od;
   od;
  od;
 fi;
od;
```

A.3 Representation matrices

b:

$$\begin{pmatrix}
0 & 0 & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a^{-1}bab & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^{-1}b^2c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}b^2d & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b^{-1}a^{-1}ba & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^{-1}a^{-1}babc & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}a^{-1}babd & 0 & 0 & 0 \\
0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}b^2cd & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c^{-1}b^{-1}a^{-1}bac & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}b^{-1}a^{-1}bad & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}a^{-1}babcd \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}b^{-1}a^{-1}bacd & 0 & 0
\end{pmatrix}$$

c:

$$\begin{pmatrix}
0 & 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a^{-1}cac & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^{-1}cbc & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^2d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^{-1}a^{-1}cabc & 0 & 0 & 0 & 0 \\
0 & c^{-1}a^{-1}ca & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}a^{-1}cacd & 0 & 0 \\
0 & 0 & c^{-1}b^{-1}cb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}b^{-1}cbcd & 0 \\
0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c^{-1}b^{-1}a^{-1}cab & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}b^{-1}a^{-1}cabcd \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}a^{-1}cad & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}b^{-1}cbd & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^{-1}c^{-1}b^{-1}a^{-1}cabd & 0 & 0 & 0
\end{pmatrix}$$

A.4 The matrix M

```
M:=[
[0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1],
#u1 1
[0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1],
#a.u1 2
[0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0],
#b.u1 3
[0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1],
#c.u1 4
[0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0],
#ab.u1 5
[0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1],
#ac.u1 6
[0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0],
#bc.u1 7
[0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,1,1,0,1,0,1,0,0],
#abc.u1 8
[1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0],
#v1 9
[1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0],
#a.v1 10
[0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0],
#c.v1 11
[1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0],
#d.v1 12
[0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0],
#ac.v1 13
[1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0],
#ad.v1 14
[0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0],
#cd.v1 15
[0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0],
#acd.v1 16
```

```
[0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0],
#w1 17
[0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1],
#a.w1 18
[0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0],
#b.w1 19
[0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0],
#d.w1 20
[0,0,0,0,0,0,0,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1],
#ab.w1 21
[0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1],
#ad.w1 22
[0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0],
#bd.w1 23
[0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1],
#abd.w1 24
[0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0,0,0,0,0,0,0,0,0],
#x1 25
[0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1,0,0,0,0,0,0,0,0],
#a.x1 26
[1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0,0,0,0,0,0,0,0,0],
#b.x1 27
[0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0,0,0,0,0,0,0,0,0],
#c.x1 28
[1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1,0,0,0,0,0,0,0,0],
#ab.x1 29
[0,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1,0,0,0,0,0,0,0,0],
#ac.x1 30
[1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,1,0,0,0,0,0,0,0,0,0],
#bc.x1 31
[1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,1,0,0,0,0,0,0,0,0],
#abc.x1 32
];
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