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A new projection-type method with nondecreasing adaptive step-sizes for pseudo-monotone variational inequalities

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Abstract We propose a new projection-type method with inertial extrapolation for solving pseudo-monotone and Lipschitz continuous variational inequalities in Hilbert spaces. The proposed method does not require the knowledge of the Lipschitz constant as well as the sequential weak continuity of the corresponding operator. We introduce a self-adaptive procedure, which generates dynamic step-sizes converging to a positive constant. It is proved that the sequence generated by the proposed method converges weakly to a solution of the considered variational inequality with the nonasymptotic $O(1/n)$ convergence rate. Moreover, the linear convergence is established under strong pseudo-monotonicity and Lipschitz continuity assumptions. Numerical examples for solving a class of Nash–Cournot oligopolistic market equilibrium model and a network equilibrium flow problem are given illustrating the efficiency of the proposed method.

Keywords Variational inequality · projection methods · pseudo-monotonicity · Lipschitz continuity · convergence rate

Mathematics Subject Classification (2000) 65Y05 · 65K15 · 68W10 · 47H09 · 47J25

1 Introduction

Variational inequalities (VIs) are fundamental in a broad range of mathematical and applied sciences, such as economics, engineering mechanics, transportation, and many more, see for

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example, [1,5,13]. The VIs theoretical, algorithmic foundations and applications have been extensively studied in the literature. For the current state-of-the-art results, see for instance [13,24] and the extensive list of references therein. In order to solve a VI, many solution methods have been proposed. Among them, projection-type methods are simple in form and useful in practice, provided that the projection is easy to calculate [13]. Various projection methods such as basic projection, extragradient projection, and hyperplane projection methods, have been designed to solve different class of VIs [1,13,24,22]. In principle, each projection method is confined to certain class of VIs so that the convergence of the algorithm can be guaranteed.

Recently, when monotonicity is replaced by pseudo-monotonicity, the extragradient method has been considered for solving VIs in infinite dimensional Hilbert spaces [45,46]. It is proved that the iterative sequence generated by the extragradient method converges weakly to a solution. However, it is known that the extragradient method requires two projections onto the feasible set at each iteration. If the feasible set is a general closed convex set, then this might seriously affect the efficiency of the algorithm. So, a natural question which raises is: *Can we reduce the number of projections in the extragradient method for solving pseudomonotone VIs in real Hilbert spaces?* Some first attempts were considered in [7,38], where a modified forward-backward-forward method was analyzed, see also [19,37, 42] for weak and strong convergence with some applications.

We continue this research direction by considering a new projection-type method with inertial effect. Inertial type algorithms recently becomes a new research direction which attracts interests of many mathematicians. The technique is based upon a discrete version of a second order dissipative dynamical system, see, for example, [3,4] and can be regarded as a procedure of speeding up the convergence properties, see, e.g., [2,27,33]. These results and other related ones analyzed the weak convergence properties of inertial extrapolation type algorithms and demonstrated their improved performance numerically on some imaging and data analysis problems.

In this paper, we consider a new method, which is a combination of projection and contraction method [8,10,18,36] for solving pseudo-monotone VIs with inertial extrapolation. It is proved that the iterative sequence generated by the proposed method converges weakly to a solution of the VI considered. While we still require the operator to be Lipschitz continuous, the knowledge of the Lipschitz constant is not necessary. Moreover, we introduce a self-adaptive procedure, in the spirit of [47], which generates a sequence of step-sizes converging monotonically to a constant dominating the small step-size in the gradient projection method [17,22]. Comparing with extragradient-type methods [20,40,45], it requires only one projection instead of two. In addition, we further weaken the sequential weak continuity assumption required in recent research [7,38,45,46]. While this paper was under reviewed, we are aware of another algorithm in [37] which proposes a similar idea, i.e. to incorporate the inertial technique into the projection and contraction method for solving monotone VIs. Nevertheless, there are several key differences between our method and [37, Algorithm 1]. While the range of the inertial factor in [37, Algorithm 1] is more relaxed than ours, the new iteration in [37, Algorithm 1] must be combined linearly with the previous iteration with a factor larger than $1/2$ (see Remark 2 for more detail), which may cause slow convergence (see numerical experiments in Section 5). In addition, our stepsizes can be increase during the course of iterations, instead of monotonically decreasing as in [37, Algorithm 1]. Moreover, while [37, Algorithm 1] is for solving *monotone and global Lipschitz continu-*

ous VIs, our proposed Algorithm solves a broader class VIs, namely *pseudo-monotone and local Lipschitz continuous* VIs. We also provided a linear convergence analysis, which was not provided in [37].

A special class of pseudo-monotonicity VIs is the class of strongly pseudomonotone VIs, which has attracted a lot of attentions in recent years, see e.g. [11,12,17,20,22,25, 40,44]. The existence and uniqueness as well as stability of this problem were studied in [25]. In [22], the authors proved that the gradient projection method converges linearly to the unique solution provided that the step-size is sufficiently small, depending on the strong pseudo-monotonicity and Lipschitz continuity constants of the considered operator. Some modifications of the gradient projection method were recently considered in [17], where diminishing step-sizes were required but the Lipschitz continuity was relaxed to continuity. Variants of the extragradient method, where two projections per iteration are needed, were studied in [23]. Under additional strong pseudo-monotonicity, our proposed algorithm converges linearly to the uniques solution of the VI problem.

The paper is organized as follows. We first recall some basic definitions and results in Section 2. Our algorithm is presented and analyzed in Section 3. Section 4 considers the linear convergence analysis when pseudo-monotonicity is replaced by strong pseudo-monotonicity. We provide some examples which demonstrate the performance of the proposed algorithm by comparing it with some related algorithms in Section 5. Final remarks and conclusions are given in the last Section.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $F : H \to H$ be a single-valued continuous mapping. We consider a classical variational inequality VI(F,C) in the sense of Fichera [14] and Stampacchia [35] (see also Kinderlehrer and Stampacchia [24]) which is formulated as follows: Find a point $\varphi^* \in C$ such that

$$\langle F\varphi^*, \varphi - \varphi^* \rangle \geq 0 \ \ \forall \varphi \in C. \tag{1}$$

We denote by Ω the solution set of the VI(F,C) (1), which is assumed to be nonempty.

Definition 1 [32] Let $\{x_n\}$ be a sequence in H.
i) $\{x_n\}$ is said to converge R-linearly to x^* with rate $\rho \in [0,1)$ if there is a constant $c > 0$ such that

$$\|x_n - x^*\| \leq c\rho^n \ \ \forall n \in \mathbb{N}.$$

ii) $\{x_n\}$ is said to converge Q-linearly to x^* with rate $\rho \in [0,1)$ if

$$\|x_{n+1} - x^*\| \leq \rho \|x_n - x^*\| \ \ \forall n \in \mathbb{N}.$$

Definition 2 [21] Let $F : H \to H$ be an operator. Then F is called

1. L-Lipschitz continuous with constant $L > 0$ if

$$\|Fx - Fy\| \leq L\|x - y\| \quad \forall x, y \in H;$$

2. monotone if

$$\langle Fx - Fy, x - y \rangle \geq 0 \quad \forall x, y \in H;$$

3. pseudo-monotone if

$$\langle Fx, y-x\rangle \geq 0 \Longrightarrow \langle Fy, y-x\rangle \geq 0 \quad \forall x,y \in H;$$

4. κ-strongly pseudo-monotone if there exists a constant $\kappa > 0$ such that

$$\langle Fx, y-x\rangle \geq 0 \Longrightarrow \langle Fy, y-x\rangle \geq \kappa\|x-y\|^2 \quad \forall x,y \in H.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by P_Cx such that $\|x-P_Cx\| \leq \|x-y\|\ \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive. For properties of the metric projection, the interested reader could be referred to Section 3 in [15].

Lemma 1 ([15, Sec. 3]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_Cx \Longleftrightarrow \langle x-z, z-y\rangle \geq 0\ \ \forall y \in C$. Moreover,*

$$\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x-y\rangle\ \forall x,y \in C.$$

Lemma 2 ([2]) *Let $\{\varphi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0,+\infty)$ such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n\ \forall n \geq 1, \sum_{n=1}^{+\infty}\delta_n < +\infty,$$

and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:

i) $\sum_{n=1}^{+\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$, where $[t]_+ := \max\{t,0\}$;

ii) there exists $\varphi^* \in [0,+\infty)$ such that $\lim_{n\to+\infty}\varphi_n = \varphi^*$.

Lemma 3 ([31]) *Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

i) for every $x \in C$, $\lim_{n\to\infty}\|x_n - x\|$ exists;

ii) every sequential weak cluster point of $\{x_n\}$ is in C.

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 4 ([9, Lemma 2.1]) *Consider the problem $VI(F,C)$ with C being a nonempty, closed, convex subset of a real Hilbert space H and $F: C \to H$ being pseudo-monotone and continuous. Then, φ^* is a solution of $VI(F,C)$ if and only if*

$$\langle F\varphi, \varphi - \varphi^*\rangle \geq 0\ \ \forall \varphi \in C.$$

The following simple identity will be used repeatedly in the sequel

$$\|\alpha x + \beta y\|^2 = \alpha(\alpha+\beta)\|x\|^2 + \beta(\alpha+\beta)\|y\|^2 - \alpha\beta\|x-y\|^2 \quad \forall x,y \in H, \forall \alpha,\beta \in \mathbb{R}. \tag{2}$$

3 The Algorithm and Convergence Analysis

For solving (1), we propose the following algorithm:

Algorithm 1

Step 0. *Given* $\nu_1 > 0, \mu \in (0,1), \theta \in \left[0, \frac{1}{3}\right)$. *Let* $\varphi_0, \varphi_1 \in H$ *be arbitrary and* $\{\alpha_n\}$ *be a nonnegative real numbers sequence such that* $\sum_{n=1}^{\infty} \alpha_n < +\infty$.

Step 1. *Set* $t_n = \varphi_n + \theta(\varphi_n - \varphi_{n-1})$. *Compute*

$$s_n = P_C(t_n - \nu_n F t_n).$$

If $t_n = s_n$ *then stop and* s_n *is a solution of* $VI(F,C)$. *Otherwise go to* ***Step 2***.
Step 2. *Compute*

$$\varphi_{n+1} = t_n - \eta_n \Omega_n,$$

where

$$\Omega_n = t_n - s_n - \nu_n(Ft_n - Fs_n) \text{ and } \eta_n = \begin{cases} \dfrac{\langle t_n - s_n, \Omega_n \rangle}{\|\Omega_n\|^2} & \text{if } \Omega_n \neq 0, \\ 0 & \text{if } \Omega_n = 0 \end{cases}$$

and update

$$\nu_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu \|t_n - s_n\|}{\|Ft_n - Fs_n\|}, \nu_n + \alpha_n \right\} & \text{if } Ft_n \neq Fs_n, \\ \nu_n + \alpha_n & \text{otherwise}. \end{cases} \quad (3)$$

Set $n := n+1$ *and go to* ***Step 1***

Remark 1 As noted in [26] the sequence $\{\nu_n\}$ generated by (3) is allowed to be increase from iteration to iteration. In particular, it needs to emphasize here that our results in this work are proved under the condition that the mapping F is locally Lipschitz. Hence, our results in this paper are different from the studied results in [18,26,36,37,47].

Remark 2 A similar idea was considered recently in [37, Algorithm 1] for solving *monotone and global Lipschitz continuous* VIs, where the inertial factor θ can be taken in $[0,1)$. Nevertheless, the new iteration in [37, Algorithm 1] is updated as

$$\varphi_{n+1} = (1-\rho_n)\varphi_n + \rho_n(t_n - \eta_n \Omega_n),$$

where $\rho_n \in (0, \frac{1}{2})$ for all n, i.e. the new iteration φ_{n+1} must be linearly combined with the previous iteration φ_n with a factor $1-\rho_n > 1/2$. The stepsizes $\{\nu_n\}$ in [37, Algorithm 1] are chosen as

$$\nu_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu \|t_n - s_n\|}{\|Ft_n - Fs_n\|}, \nu_n \right\} & \text{if } Ft_n \neq Fs_n, \\ \nu_n & \text{otherwise}. \end{cases}$$

which is monotonically decreasing. They are different with Step 2 in our Algorithm 1. In addition, our proposed algorithm solves a broader class of VIs, i.e., *pseudo-monotone and local Lipschitz continuous* VIs.

Lemma 5 *Assume that F is pseudo-monotone on C. Then for every solution φ^* of VI(F,C) it holds*

$$\|\varphi_{n+1}-\varphi^*\|^2 \leq \|t_n-\varphi^*\|^2 - \|\varphi_{n+1}-t_n\|^2.$$

Proof Let φ^* be a solution of $VI(F,C)$. We first prove that

$$\langle t_n-\varphi^*, \Omega_n\rangle \geq \langle t_n-s_n, \Omega_n\rangle.$$

Indeed, we have

$$\langle t_n-\varphi^*, \Omega_n\rangle = \langle t_n-s_n, \Omega_n\rangle + \langle s_n-\varphi^*, t_n-s_n-\nu_n(Ft_n-Fs_n)\rangle.$$

On the other hand since $s_n = P_C(t_n-\nu_n Ft_n)$, we get

$$\langle t_n-s_n-\nu_n Ft_n, s_n-\varphi^*\rangle \geq 0.$$

By the pseudomonotonicity of F it holds

$$\langle \nu_n Fs_n, s_n-\varphi^*\rangle \geq 0.$$

Thus

$$\langle t_n-s_n-\nu_n(Ft_n-Fs_n), s_n-\varphi^*\rangle \geq 0.$$

Therefore

$$\langle t_n-\varphi^*, \Omega_n\rangle \geq \langle t_n-s_n, \Omega_n\rangle.$$

Hence

$$\begin{aligned}
\|\varphi_{n+1}-\varphi^*\|^2 =& \|t_n-\eta_n\Omega_n-\varphi^*\|^2\\
=& \|t_n-\varphi^*\|^2 - 2\eta_n\langle t_n-\varphi^*, \Omega_n\rangle + \eta_n^2\|\Omega_n\|^2\\
\leq& \|t_n-\varphi^*\|^2 - 2\eta_n\langle t_n-s_n, \Omega_n\rangle + \eta_n.\eta_n\|\Omega_n\|^2\\
=& \|t_n-\varphi^*\|^2 - \eta_n\langle t_n-s_n, \Omega_n\rangle\\
=& \|t_n-\varphi^*\|^2 - \eta_n^2\|\Omega_n\|^2\\
=& \|t_n-\varphi^*\|^2 - \|\varphi_{n+1}-t_n\|^2.
\end{aligned}$$

Lemma 6 *Assume that F is pseudo-monotone on C then for every solution φ^* of VI(F,C) it holds that*

(i) the sequence $\{\Sigma_n\}$ defined by

$$\Sigma_n := \|\varphi_n-\varphi^*\|^2 - \theta\|\varphi_{n-1}-\varphi^*\|^2 + (1-\theta)\|\varphi_n-\varphi_{n-1}\|^2.$$

is non negative and non-increasing;

(ii)

$$\sum_{n=1}^{\infty}\|\varphi_n-\varphi_{n-1}\|^2 < +\infty.$$

(iii) the sequence $\{\|\varphi_n-\varphi^*\|\}$ is convergent.

Proof (i) and (ii): For every $\varphi^* \in C$ we have from (2) that

$$\begin{aligned}\|t_n-\varphi^*\|^2 &= \|\varphi_n+\theta(\varphi_n-\varphi_{n-1})-\varphi^*\|^2\\ &= \|(1+\theta)(\varphi_n-\varphi^*)-\theta(\varphi_{n-1}-\varphi^*)\|^2\\ &= (1+\theta)\|\varphi_n-\varphi^*\|^2-\theta\|\varphi_{n-1}-\varphi^*\|^2+\theta(1+\theta)\|\varphi_n-\varphi_{n-1}\|^2, \end{aligned}\tag{4}$$

and from the Cauchy-Schwarz inequality that

$$\begin{aligned}\|\varphi_{n+1}-t_n\|^2 &= \|\varphi_{n+1}-\varphi_n-\theta(\varphi_n-\varphi_{n-1})\|^2\\ &= \|\varphi_{n+1}-\varphi_n\|^2+\theta^2\|\varphi_n-\varphi_{n-1}\|^2-2\theta\langle\varphi_{n+1}-\varphi_n,\varphi_n-\varphi_{n-1}\rangle\\ &\geq \|\varphi_{n+1}-\varphi_n\|^2+\theta^2\|\varphi_n-\varphi_{n-1}\|^2-2\theta\|\varphi_{n+1}-\varphi_n\|\|\varphi_n-\varphi_{n-1}\|\\ &\geq (1-\theta)\|\varphi_{n+1}-\varphi_n\|^2+\left(\theta^2-\theta\right)\|\varphi_n-\varphi_{n-1}\|^2. \end{aligned}\tag{5}$$

It follows from (4), (5) and Lemma 5 that

$$\|\varphi_{n+1}-\varphi^*\|^2 \leq (1+\theta)\|\varphi_n-\varphi^*\|^2-\theta\|\varphi_{n-1}-\varphi^*\|^2+2\theta\|\varphi_n-\varphi_{n-1}\|^2-(1-\theta)\|\varphi_{n+1}-\varphi_n\|^2, \tag{6}$$

or equivalently

$$\begin{aligned}\|\varphi_{n+1}-\varphi^*\|^2-\theta\|\varphi_n-\varphi^*\|^2+(1-\theta)\|\varphi_{n+1}-\varphi_n\|^2 \leq &\|\varphi_n-\varphi^*\|^2-\theta\|\varphi_{n-1}-\varphi^*\|^2\\ &+(1-\theta)\|\varphi_n-\varphi_{n-1}\|^2-(1-3\theta)\|\varphi_n-\varphi_{n-1}\|^2. \end{aligned}\tag{7}$$

Let

$$\Sigma_n := \|\varphi_n-\varphi^*\|^2-\theta\|\varphi_{n-1}-\varphi^*\|^2+(1-\theta)\|\varphi_n-\varphi_{n-1}\|^2.$$

Now, we show that $\Sigma_n \geq 0$ for all n. Indeed, we have

$$\begin{aligned}\|\varphi_{n-1}-\varphi^*\|^2 &= \|\varphi_{n-1}-\varphi_n+\varphi_n-\varphi^*\|^2 = \|\varphi_{n-1}-\varphi_n\|^2+\|\varphi_n-\varphi^*\|^2+2\langle\varphi_{n-1}-\varphi_n,\varphi_n-\varphi^*\rangle\\ &\leq \|\varphi_{n-1}-\varphi_n\|^2+\|\varphi_n-\varphi^*\|^2+2\|\varphi_{n-1}-\varphi_n\|\|\varphi_n-\varphi^*\|\\ &\leq \|\varphi_{n-1}-\varphi_n\|^2+\|\varphi_n-\varphi^*\|^2+k\|\varphi_{n-1}-\varphi_n\|^2+\frac{1}{k}\|\varphi_n-\varphi^*\|^2\\ &= (1+k)\|\varphi_{n-1}-\varphi_n\|^2+\left(1+\frac{1}{k}\right)\|\varphi_n-\varphi^*\|^2, \end{aligned}\tag{8}$$

for all $k>0$. Substituting (8) into Σ_n, we get

$$\Sigma_n \geq \left[1-\left(1+\frac{1}{k}\right)\theta\right]\|\varphi_n-\varphi^*\|^2+\left[1-(2+k)\theta\right]\|\varphi_n-\varphi_{n-1}\|^2.$$

Choosing $k=1$ we obtain

$$\Sigma_n \geq (1-2\theta)\|\varphi_n-\varphi^*\|^2+(1-3\theta)\|\varphi_n-\varphi_{n-1}\|^2 \geq 0$$

for all $\theta \in (0,1/3)$. From (7) we obtain

$$\Sigma_{n+1}-\Sigma_n \leq -(1-3\theta)\|\varphi_n-\varphi_{n-1}\|^2,$$

or equivalently

$$(1-3\theta)\|\varphi_n-\varphi_{n-1}\|^2 \leq \Sigma_n-\Sigma_{n+1}, \tag{9}$$

which implies (i). This follows from (9) that

$$\sum_{n=1}^{\infty}\|\varphi_n-\varphi_{n-1}\|^2<+\infty,$$

which is (ii).
(iii) It follows from (6) that

$$\|\varphi_{n+1}-\varphi^*\|^2\le\|\varphi_n-\varphi^*\|^2+\theta\left(\|\varphi_n-\varphi^*\|^2-\|\varphi_{n-1}-\varphi^*\|^2\right)+2\theta\|\varphi_n-\varphi_{n-1}\|^2.$$

Hence from Lemma 2 and (ii), we deduce that $\lim_{n\to\infty}\|\varphi_n-\varphi^*\|^2$ exists for any solution φ^* of $VI(F,C)$.

Corollary 1 *Assume that F is pseudo-monotone on C and L-Lipschitz continuous on any bounded subset of H. Then the sequences $\{\varphi_n\},\{t_n\}$ and $\{s_n\}$ are bounded.*

Proof Fixing $\varphi^*\in\Omega$ we have $\varphi^*=P_C(\varphi^*-\nu_nF\varphi^*)$ for all $\nu_n>0$. Since the sequence $\{\|\varphi_n-\varphi^*\|^2\}$ is convergent, there exists $r_1>0$ such that $\|\varphi_n-\varphi^*\|\le r_1$ for all $n\ge0$, i.e., $\{\varphi_n\}$ is bounded. Moreover

$$\|t_n-\varphi^*\|=\|(1+\theta)(\varphi_n-\varphi^*)-\theta(\varphi_{n-1}-\varphi^*)\|\le(1+2\theta)r_1=:r_2,$$

which implies that $\{t_n\}$ is bounded. Using the Lipschitz continuity of F on any bounded subset of H, there exists $L_1>0$ such that $\|Ft_n-F\varphi^*\|\le L_1\|t_n-\varphi^*\|$ for all $n>0$. From the nonexpansiveness of P_C we deduce

$$\begin{aligned}\|s_n-\varphi^*\|&=\|P_C(t_n-\nu_nFt_n)-P_C(\varphi^*-\nu_nF\varphi^*)\|\\&\le\|t_n-\varphi^*-\nu_n(Ft_n-F\varphi^*)\|\\&\le\|t_n-\varphi^*\|+\nu_n\|Ft_n-F\varphi^*\|\\&\le(1+\nu_nL_1)\|t_n-\varphi^*\|\\&\le(1+\nu_0L_1)\|t_n-\varphi^*\|\le(1+\nu_0L_1)r_2=:r_3,\end{aligned}$$

which implies that $\{s_n\}$ is bounded.

Remark 3 It is clear from Corollary 1 that each element of the sequences $\{\varphi_n\},\{t_n\}$ and $\{s_n\}$ belongs to the closed ball $\mathbb{B}[\varphi^*,r_3]$. We denote by L the Lipschitz constant of F on $\mathbb{B}[\varphi^*,r_3]$, i.e.

$$\|Fx-Fy\|\le L\|x-y\|\quad\forall x,y\in\mathbb{B}[\varphi^*,r_3].$$

From now on we will use this L as the Lipschitz constant of F.

Lemma 7 ([26]) *Assume that F is Lipschitz continuous on any bounded subset of H. Let $\{\nu_n\}$ be the sequence computed in* (3). *Then*

$$\lim_{n\to\infty}\nu_n=\nu \text{ with } \nu\in\left[\min\left\{\nu_0,\frac{\mu}{L}\right\},\nu_0+\alpha\right],$$

where $\alpha=\sum_{n=1}^{\infty}\alpha_n$. Moreover

$$\|Ft_n-Fs_n\|\le\frac{\mu}{\nu_{n+1}}\|t_n-s_n\|. \tag{10}$$

Proof Since F is Lipschitz continuous on any bounded subset of H and $\{t_n\}$, $\{s_n\}$ are bounded, we get $\|Ft_n - Fs_n\| \le L\|t_n - s_n\|$ with L defined in Corollary 1. Consequently

$$\frac{\mu\|t_n - s_n\|}{\|Ft_n - Fs_n\|} \ge \frac{\mu}{L} \quad \text{if} \quad Ft_n \neq Fs_n.$$

The rest of the proof is similar to the Lemma 3.1 in [26], so we omit it.

Lemma 8 *Assume that F is Lipschitz continuous on any bounded subset of H. Then there exists $n_0 \in \mathbb{N}$ such that*

$$\|\varphi_{n+1} - t_n\|^2 \ge \frac{\left(1 - \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1 + \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}\|t_n - s_n\|^2 \quad \forall n \ge n_0.$$

Proof From Lemma 7, we can deduce that

$$\lim_{n\to\infty}\left(1 - \mu\frac{\nu_n}{\nu_{n+1}}\right) = 1 - \mu > 0.$$

Hence there exists $n_0 > 0$ such that

$$1 - \mu\frac{\nu_n}{\nu_{n+1}} > 0 \quad \forall n \ge n_0.$$

We have

$$\begin{aligned}
\eta_n\|\Omega_n\|^2 &= \langle t_n - s_n, \Omega_n\rangle \\
&= \langle t_n - s_n, t_n - s_n - \nu_n(Ft_n - Fs_n)\rangle \\
&= \|t_n - s_n\|^2 - \nu_n\langle t_n - s_n, Ft_n - Fs_n\rangle \\
&\ge \|t_n - s_n\|^2 - \nu_n\|t_n - s_n\|\|Ft_n - Fs_n\| \\
&\ge \left(1 - \mu\frac{\nu_n}{\nu_{n+1}}\right)\|t_n - s_n\|^2,
\end{aligned}$$

where we have used (10) in the last inequality.
We also have

$$\begin{aligned}
\|\Omega_n\| &= \|t_n - s_n - \nu_n(Ft_n - Fs_n)\| \\
&\le \|t_n - s_n\| + \nu_n\|Ft_n - Fs_n\| \\
&\le \left(1 + \mu\frac{\nu_n}{\nu_{n+1}}\right)\|t_n - s_n\|,
\end{aligned}$$

which implies

$$\frac{1}{\|\Omega_n\|^2} \ge \frac{1}{\left(1 + \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}\cdot\frac{1}{\|t_n - s_n\|^2}.$$

Therefore, we obtain

$$\eta_n = \frac{\langle t_n - s_n, \Omega_n\rangle}{\|\Omega_n\|^2} \ge \frac{\left(1 - \mu\dfrac{\nu_n}{\nu_{n+1}}\right)}{\left(1 + \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}.$$

Hence for all $n \geq n_0$

$$\|\varphi_{n+1} - t_n\|^2 = \eta_n^2 \|\Omega_n\|^2 \geq \frac{\left(1 - \mu \dfrac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1 + \mu \dfrac{\nu_n}{\nu_{n+1}}\right)^2} \|t_n - s_n\|^2. \tag{11}$$

The main result of this section can be stated as follows, where we employ the technique developed originally in [45], see also [34, 38, 46].

Theorem 1 *Assume that F is pseudo-monotone and Lipschitz continuous on any bounded subset of H. If in addition, the functional $f(x) := \|F(x)\|$ is weakly lower semicontinuous, then the sequence $\{\varphi_n\}$ generated by Algorithm 1 converges weakly to a solution of $VI(F,C)$.*

Proof Since $\{\varphi_n\}$ is bounded, we assume that $\{\varphi_{n_k}\}$ is a subsequence of $\{\varphi_n\}$ such that $\varphi_{n_k} \rightharpoonup z$. Moreover, we get

$$\|t_n - \varphi_n\| = \theta \|\varphi_n - \varphi_{n-1}\| \to 0 \text{ as } n \to \infty$$

and

$$\|\varphi_{n+1} - t_n\| \leq \|\varphi_{n+1} - \varphi_n\| + \|\varphi_n - \varphi_{n-1}\| \to 0 \text{ as } n \to \infty.$$

From Lemma 8 and $\|\varphi_{n+1} - t_n\| \to 0$ we get $\|t_n - s_n\| \to 0$. Since $\varphi_{n_k} \rightharpoonup z$, we have $s_{n_k} \rightharpoonup z$. Moreover, $\{s_n\} \subset C$ implies $z \in C$. We will show that z is a solution of $VI(F,C)$. Indeed, since $s_{n_k} = P_C(t_{n_k} - \nu_{n_k} F t_{n_k})$, we obtain

$$\langle t_{n_k} - \nu_{n_k} F t_{n_k} - s_{n_k}, x - s_{n_k} \rangle \leq 0 \quad \forall x \in C,$$

or equivalently

$$\frac{1}{\nu_{n_k}} \langle t_{n_k} - s_{n_k}, x - s_{n_k} \rangle \leq \langle F t_{n_k}, x - s_{n_k} \rangle \quad \forall x \in C.$$

Consequently

$$\frac{1}{\nu_{n_k}} \langle t_{n_k} - s_{n_k}, x - s_{n_k} \rangle + \langle F t_{n_k}, s_{n_k} - t_{n_k} \rangle \leq \langle F t_{n_k}, x - t_{n_k} \rangle \quad \forall x \in C. \tag{12}$$

Being weakly convergent, $\{t_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of F, $\{F t_{n_k}\}$ is bounded. As $\|t_{n_k} - s_{n_k}\| \to 0$, $\{s_{n_k}\}$ is bounded and $\nu_{n_k} \geq \min\left\{\nu_1, \dfrac{\mu}{L}\right\}$, passing (12) to limit as $k \to \infty$, we get

$$\liminf_{k \to \infty} \langle F t_{n_k}, x - t_{n_k} \rangle \geq 0 \quad \forall x \in C. \tag{13}$$

Moreover, we have

$$\langle F s_{n_k}, x - s_{n_k} \rangle = \langle F s_{n_k} - F t_{n_k}, x - t_{n_k} \rangle + \langle F t_{n_k}, x - t_{n_k} \rangle + \langle F s_{n_k}, t_{n_k} - s_{n_k} \rangle. \tag{14}$$

Since $\lim_{k \to \infty} \|t_{n_k} - s_{n_k}\| = 0$ and F is Lipschitz continuous we get

$$\lim_{k \to \infty} \|F t_{n_k} - F s_{n_k}\| = 0$$

which, together with (13) and (14) implies that

$$\liminf_{k \to \infty} \langle F s_{n_k}, x - s_{n_k} \rangle \geq 0.$$

We choose a sequence $\{\varepsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle Fs_{n_j}, x - s_{n_j}\rangle + \varepsilon_k \geq 0 \ \ \forall j \geq N_k. \tag{15}$$

Since $\{\varepsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{s_{N_k}\} \subset C$ we can suppose $Fs_{N_k} \neq 0$ (otherwise, s_{N_k} is a solution) and, setting

$$v_{N_k} = \frac{Fs_{N_k}}{\|Fs_{N_k}\|^2},$$

we have $\langle Fs_{N_k}, v_{N_k}\rangle = 1$ for each k. Now, we can deduce from (15) that for each k

$$\langle Fs_{N_k}, x + \varepsilon_k v_{N_k} - s_{N_k}\rangle \geq 0.$$

Since F is pseudomonotone, we get

$$\langle F(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - s_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Fx, x - s_{N_k}\rangle \geq \langle Fx - F(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - s_{N_k}\rangle - \varepsilon_k \langle Fx, v_{N_k}\rangle. \tag{16}$$

We claim that $\lim_{k\to\infty} \varepsilon_k v_{N_k} = 0$. Indeed, since $t_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|t_{n_k} - s_{n_k}\| = 0$, we obtain $s_{N_k} \rightharpoonup z$ as $k \to \infty$. By $\{s_n\} \subset C$, we obtain $z \in C$. Suppose that $0 < \|Fz\|$, otherwise z is a solution. By the weak lower semicontinuity of the functional $\|F(x)\|$, we have

$$0 < \|Fz\| \leq \liminf_{k\to\infty} \|Fs_{n_k}\|.$$

Since $\{s_{N_k}\} \subset \{s_{n_k}\}$ and $\varepsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\varepsilon_k v_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\varepsilon_k}{\|Fs_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} \|Fs_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \varepsilon_k v_{N_k} = 0$.

Now, letting $k \to \infty$, then the right hand side of (16) tends to zero since F is Lipschitz continuous, $\{s_{N_k}\}, \{v_{N_k}\}$ are bounded and $\lim_{k\to\infty} \varepsilon_k v_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty} \langle Fx, x - s_{N_k}\rangle \geq 0.$$

Hence, for all $x \in C$ we have

$$\langle Fx, x - z\rangle = \lim_{k\to\infty} \langle Fx, x - s_{N_k}\rangle = \liminf_{k\to\infty} \langle Fx, x - s_{N_k}\rangle \geq 0.$$

By Lemma 4, $z \in VI(F,C)$ and applying Lemma 3 the proof is completed.

Remark 4 1. In recent works on pseudomonotone variational inequalities, see, [7,19,38, 45,46], the sequential weak continuity of the mapping F is often required. Clearly, if F is sequentially weakly continuous then the functional $\|F(x)\|$ is weakly lower semicontinuous. The following example shows that the converse statement is not true. Let H be a Hilbert space with an orthonomal sequence $\{e_n\}$, C be a closed ball centered at 0 and with a radius $r \geq 2$, and let $F(x) := \|x\|x$. Let $C \ni \varphi_n \rightharpoonup z$. Due to the weak lower semicontinuity of the norm, one has $\|z\| \leq \liminf_{n\to\infty} \|\varphi_n\|$, hence,

$$\|F(z)\| = \|z\|^2 \leq (\liminf_{n\to\infty} \|\varphi_n\|)^2 \leq \liminf_{n\to\infty} \|\varphi_n\|^2 = \liminf_{n\to\infty} \|F(\varphi_n)\|,$$

which means the weak lower semicontinuity of $\|F(x)\|$ on C. On the other hand, let $C \ni s_n = e_n + e_1$. Then $s_n \rightharpoonup e_1$. For $n > 1$, $F(s_n) = \sqrt{2}(e_n + e_1) \rightharpoonup \sqrt{2}e_1$ and $F(e_1) = e_1$. Thus, $F(x)$ is not sequential weakly continuous. Besides, it is obvious that the mapping F is Lipschitz continuous with constant $L = 2r$ on C.

2. The imposed sequential weak lower semicontinuity of $\|F(x)\|$ can be omitted in one of the following cases: either F is monotone (see, [11,45]), or F is strongly pseudomonotone (see next Theorem 3).

Next, we establish the nonasymptotic $O(1/n)$ convergence rate of the weak convergence Algorithm 1. The $O(1/n)$ convergence rate of projection-contraction method for monotone VIs was studied in [8]. This convergence rate result has been recently obtained for a different projection method in [34].

Theorem 2 *Assume that F is pseudo-monotone and Lipschitz continuous on any bounded subset of H. Let the sequence $\{\varphi_n\}$ be generated by Algorithm 1. Then for any $\varphi^* \in VI(F,C)$ there exists constants $M, \gamma > 0$ and $n_0 > 0$ such that the following estimate holds*

$$\min_{n_0 \leq k \leq n} \|s_k - t_k\|^2 \leq \frac{\frac{1}{\gamma}(\|\varphi_{n_0} - \varphi^*\|^2 + \frac{1}{1-\theta}[\|\varphi_{n_0} - \varphi^*\|^2 - \|\varphi_{n_0-1} - \varphi^*\|^2]_+ + \frac{1}{1-\theta}M)}{n - n_0 + 1}.$$

Proof Combining Lemma 5 and Lemma 6, we get

$$\|\varphi_{n+1} - \varphi^*\|^2 \leq \|t_n - \varphi^*\|^2 - \frac{\left(1 - \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1 + \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}\|s_n - t_n\|^2. \tag{17}$$

Let $\gamma < \dfrac{(1-\mu)^2}{(1+\mu)^2}$, then we have $\lim_{n\to\infty} \dfrac{\left(1 - \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1 + \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2} = \dfrac{(1-\mu)^2}{(1+\mu)^2} > \gamma$.

Hence, there exists $n_0 \in \mathbb{N}$ such that $\dfrac{\left(1 - \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1 + \mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2} > \gamma \ \forall n \geq n_0$, which together with (17) implies

$$\|\varphi_{n+1} - \varphi^*\|^2 \leq \|t_n - \varphi^*\|^2 - \gamma\|s_n - t_n\|^2,$$

or equivalently

$$\gamma\|s_n - t_n\|^2 \leq \|t_n - \varphi^*\|^2 - \|\varphi_{n+1} - \varphi^*\|^2. \tag{18}$$

Substituting (4) into (18) we get

$$\begin{aligned}
\gamma\|s_n - t_n\|^2 \leq& (1+\theta)\|\varphi_n - \varphi^*\|^2 - \theta\|\varphi_{n-1} - \varphi^*\|^2 + (1+\theta)\theta\|\varphi_n - \varphi_{n-1}\|^2 - \|\varphi_{n+1} - \varphi^*\|^2 \\
=& \|\varphi_n - \varphi^*\|^2 - \|\varphi_{n+1} - \varphi^*\|^2 + \theta(\|\varphi_n - \varphi^*\|^2 - \|\varphi_{n-1} - \varphi^*\|^2) \\
&+ (1+\theta)\theta\|\varphi_n - \varphi_{n-1}\|^2 \\
\leq& \|\varphi_n - \varphi^*\|^2 - \|\varphi_{n+1} - \varphi^*\|^2 + \theta(\|\varphi_n - \varphi^*\|^2 - \|\varphi_{n-1} - \varphi^*\|^2) \\
&+ 2\theta\|\varphi_n - \varphi_{n-1}\|^2.
\end{aligned}$$

Setting $\rho_n := \|\varphi_n - \varphi^*\|^2$, $\gamma_n := \rho_n - \rho_{n-1}$ and $\delta_n := 2\theta\|\varphi_n - \varphi_{n-1}\|^2$ and $[t]_+ := \max\{0, t\}$, we obtain

$$\begin{aligned}\gamma\|s_n - t_n\|^2 &\leq \rho_n - \rho_{n+1} + \theta\gamma_n + \delta_n \\ &\leq \rho_n - \rho_{n+1} + \theta[\gamma_n]_+ + \delta_n. \end{aligned} \tag{19}$$

From (6), we get

$$\|\varphi_{n+1} - \varphi^*\|^2 \leq (1+\theta)\|\varphi_n - \varphi^*\|^2 - \theta\|\varphi_{n-1} - \varphi^*\|^2 + 2\theta\|\varphi_n - \varphi_{n-1}\|^2 \quad \forall n \geq n_0,$$

or equivalently

$$\gamma_{n+1} \leq \theta\gamma_n + \delta_n \leq \theta[\gamma_n]_+ + \delta_n.$$

This follows that

$$\begin{aligned}[\gamma_{n+1}]_+ &\leq \theta[\gamma_n]_+ + \delta_n \\ &\leq \theta^{n-n_0+1}[\gamma_{n_0}]_+ + \sum_{k=1}^{n-n_0+1} \theta^{k-1}\delta_{n+1-k}.\end{aligned}$$

Hence

$$\sum_{n=n_0}^{\infty} [\gamma_{n+1}]_+ \leq \sum_{n=n_0}^{\infty} \theta^{n-n_0+1}[\gamma_{n_0}]_+ + \sum_{n=n_0}^{\infty}\sum_{k=1}^{n-n_0+1} \theta^{k-1}\delta_{n+1-k}.$$

Using Lemma 2, we get

$$\begin{aligned}\sum_{n=n_0}^{\infty} [\gamma_{n+1}]_+ &\leq \frac{\theta}{1-\theta}[\gamma_{n_0}]_+ + \frac{1}{1-\theta}\sum_{n=n_0}^{\infty}\delta_n \\ &\leq \frac{\theta}{1-\theta}[\gamma_{n_0}]_+ + \frac{1}{1-\theta}M,\end{aligned}$$

for some $M > 0$. From (19), we can deduce

$$\begin{aligned}\gamma\sum_{k=n_0}^{n}\|s_k - t_k\|^2 &\leq \rho_{n_0} - \rho_{n+1} + \theta\sum_{k=n_0}^{n}[\gamma_k]_+ + \sum_{k=n_0}^{n}\delta_k \\ &\leq \rho_{n_0} + [\gamma_{n_0}]_+ + \theta\sum_{k=n_0}^{n}[\gamma_{k+1}]_+ + \sum_{k=n_0}^{n}\delta_k \\ &\leq \rho_{n_0} + [\gamma_{n_0}]_+ + \frac{\theta^2}{1-\theta}[\gamma_{n_0}]_+ + \frac{\theta}{1-\theta}M + M \\ &\leq \rho_{n_0} + [\gamma_{n_0}]_+ + \frac{\theta}{1-\theta}[\gamma_{n_0}]_+ + \frac{1}{1-\theta}M \\ &\leq \rho_{n_0} + \frac{1}{1-\theta}[\gamma_{n_0}]_+ + \frac{1}{1-\theta}M.\end{aligned}$$

This follows that

$$\gamma\sum_{k=n_0}^{n}\|s_k - t_k\|^2 \leq \|\varphi_{n_0} - \varphi^*\|^2 + \frac{1}{1-\theta}[\|\varphi_{n_0} - \varphi^*\|^2 - \|\varphi_{n_0-1} - \varphi^*\|^2]_+ + \frac{1}{1-\theta}M,$$

which implies

$$\min_{n_0 \leq k \leq n}\|s_k - t_k\|^2 \leq \frac{\frac{1}{\gamma}(\|\varphi_{n_0} - \varphi^*\|^2 + \frac{1}{1-\theta}[\|\varphi_{n_0} - \varphi^*\|^2 - \|\varphi_{n_0-1} - \varphi^*\|^2]_+ + \frac{1}{1-\theta}M)}{n - n_0 + 1}.$$

The proof is completed.

4 A Linear Convergence Analysis

Lemma 9 *Assume that $F: H \to H$ is L-Lipschitz continuous on any bounded subset of H and κ-strongly pseudo-monotone on C. Let φ^* be the unique solution of $VI(F,C)$. Then for all $\xi \in (0,1)$ there exists $n_1 > 0$ and $\rho \in (0,1)$ such that*

$$\|\varphi_{n+1} - \varphi^*\|^2 \leq \rho \|t_n - \varphi^*\|^2 - \xi \|\varphi_{n+1} - t_n\|^2 \quad \forall n \geq n_1, \tag{20}$$

where $\rho := 1 - \beta \in (0,1)$ with

$$\beta := \min\left\{\frac{(1-\xi)(1-\mu)}{4(1+\mu)}, \frac{\kappa\nu}{2}\right\} \text{ and } \nu = \lim_{n\to\infty} \nu_n,$$

Proof Since $s_n = P_C(t_n - \nu_n F t_n)$ and $\varphi^* \in C$, we get

$$\langle t_n - \nu_n F t_n - s_n, s_n - \varphi^* \rangle \geq 0.$$

Since $\langle F\varphi^*, s_n - \varphi^* \rangle \geq 0$, by the strong pseudomonotonicity of F it holds

$$\nu_n \langle F s_n, s_n - \varphi^* \rangle \geq \nu_n \kappa \|s_n - \varphi^*\|^2.$$

Hence

$$\langle \Omega_n, s_n - \varphi^* \rangle = \langle t_n - s_n - \nu_n (F t_n - F s_n), s_n - \varphi^* \rangle \geq \nu_n \kappa \|s_n - \varphi^*\|^2.$$

Therefore

$$\begin{aligned}
\|\varphi_{n+1} - \varphi^*\|^2 =& \|t_n - \eta_n \Omega_n - \varphi^*\|^2 \\
=& \|t_n - \varphi^*\|^2 - 2\eta_n \langle t_n - \varphi^*, \Omega_n \rangle + \eta_n^2 \|\Omega_n\|^2 \\
\leq& \|t_n - \varphi^*\|^2 - 2\eta_n \langle t_n - s_n, \Omega_n \rangle - 2\nu_n \kappa \|s_n - \varphi^*\|^2 + \eta_n . \eta_n \|\Omega_n\|^2 \\
=& \|t_n - \varphi^*\|^2 - \eta_n \langle t_n - s_n, \Omega_n \rangle - 2\nu_n \kappa \|s_n - \varphi^*\|^2 \\
=& \|t_n - \varphi^*\|^2 - \eta_n^2 \|\Omega_n\|^2 - 2\nu_n \kappa \|s_n - \varphi^*\|^2 \\
=& \|t_n - \varphi^*\|^2 - \|\varphi_{n+1} - t_n\|^2 - 2\nu_n \kappa \|s_n - \varphi^*\|^2.
\end{aligned}$$

Combining this inequality with (11) we have for all $n \geq n_0$ that

$$\|\varphi_{n+1} - \varphi^*\|^2 \leq \|t_n - \varphi^*\|^2 - \xi \|\varphi_{n+1} - t_n\|^2 - (1-\xi) \frac{\left(1 - \mu \frac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1 + \mu \frac{\nu_n}{\nu_{n+1}}\right)^2} \|t_n - s_n\|^2 - 2\nu_n \kappa \|s_n - \varphi^*\|^2.$$

Setting

$$\beta := \min\left\{\frac{(1-\xi)(1-\mu)}{4(1+\mu)}, \frac{\kappa\nu}{2}\right\} \text{ where } \nu = \lim_{n\to\infty} \nu_n,$$

we have

$$1 > \lim_{n\to\infty} \frac{(1-\xi)\left(1 - \mu \frac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1 + \mu \frac{\nu_n}{\nu_{n+1}}\right)^2} = \frac{(1-\xi)(1-\mu)}{(1+\mu)} \geq 4\beta$$

and

$$\lim_{n\to\infty} \nu_n \kappa = \nu\kappa \geq 2\beta.$$

Hence we can choose $n_1 > 0$ such that for all $n \geq n_1$

$$\frac{(1-\xi)\left(1-\mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2}{\left(1+\mu\dfrac{\nu_n}{\nu_{n+1}}\right)^2} \geq 2\beta$$

and

$$\nu_n\kappa \geq \beta.$$

Hence for all $n \geq n_1$

$$\begin{aligned}\|\varphi_{n+1}-\varphi^*\|^2 \leq &\|t_n-\varphi^*\|^2 - \xi\|\varphi_{n+1}-t_n\|^2 - 2\beta\|t_n-s_n\|^2 - 2\beta\|s_n-\varphi^*\|^2\\ \leq &(1-\beta)\|t_n-\varphi^*\|^2 - \xi\|\varphi_{n+1}-t_n\|^2\\ = &\rho\|t_n-\varphi^*\|^2 - \xi\|\varphi_{n+1}-t_n\|^2,\end{aligned}$$

where $\rho := 1-\beta \in (0,1)$.

We are now in a position to establish the main result of this section. The argument technique is adapted from [39].

Theorem 3 *Assume that $F : H \to H$ is L-Lipschitz continuous on any bounded subset of H and κ-strongly pseudo-monotone on C. Let $\xi, \delta \in (0,1)$ and θ be such that*

$$0 \leq \theta \leq \min\left\{\frac{\xi}{2+\xi}, \frac{\sqrt{(1+\delta\xi)^2+4\delta\xi}-(1+\delta\xi)}{2}, (1-\delta)\left(1-\frac{(1-\xi)(1-\mu)}{2(1+\mu)}\right)\right\} \tag{21}$$

Then the sequence $\{\varphi_n\}$ generated by Algorithm 1 converges linearly to the unique solution φ^* of $VI(F,C)$.

Proof We have from (2) that

$$\begin{aligned}\|t_n-\varphi^*\|^2 &= \|(1+\theta)(\varphi_n-\varphi^*)-\theta(\varphi_{n-1}-\varphi^*)\|^2\\ &= (1+\theta)\|\varphi_n-\varphi^*\|^2 - \theta\|\varphi_{n-1}-\varphi^*\|^2 + \theta(1+\theta)\|\varphi_n-\varphi_{n-1}\|^2\end{aligned}$$

and from the Cauchy-Schwarz inequality that

$$\begin{aligned}\|\varphi_{n+1}-t_n\|^2 &= \|\varphi_{n+1}-\varphi_n-\theta(\varphi_n-\varphi_{n-1})\|^2\\ &= \|\varphi_{n+1}-\varphi_n\|^2 + \theta^2\|\varphi_n-\varphi_{n-1}\|^2 - 2\theta\langle\varphi_{n+1}-\varphi_n, \varphi_n-\varphi_{n-1}\rangle\\ &\geq \|\varphi_{n+1}-\varphi_n\|^2 + \theta^2\|\varphi_n-\varphi_{n-1}\|^2 - 2\theta\|\varphi_{n+1}-\varphi_n\|\|\varphi_n-\varphi_{n-1}\|\\ &\geq \|\varphi_{n+1}-\varphi_n\|^2 + \theta^2\|\varphi_n-\varphi_{n-1}\|^2 - \theta\|\varphi_{n+1}-\varphi_n\|^2 - \theta\|\varphi_n-\varphi_{n-1}\|^2\\ &\geq (1-\theta)\|\varphi_{n+1}-\varphi_n\|^2 - \theta(1-\theta)\|\varphi_n-\varphi_{n-1}\|^2.\end{aligned}$$

Combining these inequalities with (20) we obtain for all $n \geq n_1$ that

$$\begin{aligned}\|\varphi_{n+1}-\varphi^*\|^2 \leq \rho(1+\theta)\|\varphi_n-\varphi^*\|^2 - \rho\theta\|\varphi_{n-1}-\varphi^*\|^2 + \rho\theta(1+\theta)\|\varphi_n-\varphi_{n-1}\|^2\\ - \xi(1-\theta)\|\varphi_{n+1}-\varphi_n\|^2 + \xi\theta(1-\theta)\|\varphi_n-\varphi_{n-1}\|^2,\end{aligned}$$

or equivalently

$$\begin{aligned}&\|\varphi_{n+1}-\varphi^*\|^2-\rho\theta\|\varphi_n-\varphi^*\|^2+\xi(1-\theta)\|\varphi_{n+1}-\varphi_n\|^2\\ \le&\rho\left[\|\varphi_n-\varphi^*\|^2-\theta\|\varphi_{n-1}-\varphi^*\|^2+\xi(1-\theta)\|\varphi_n-\varphi_{n-1}\|^2\right]\\ &-(\rho\xi(1-\theta)-\rho\theta(1+\theta)-\xi\theta(1-\theta))\|\varphi_n-\varphi_{n-1}\|^2.\end{aligned}$$

Setting

$$a_n:=\|\varphi_n-\varphi^*\|^2-\theta\|\varphi_{n-1}-\varphi^*\|^2+\xi(1-\theta)\|\varphi_n-\varphi_{n-1}\|^2,$$

since $\rho\in(0,1)$ we can write

$$\begin{aligned}a_{n+1}\le&\|\varphi_{n+1}-\varphi^*\|^2-\rho\theta\|\varphi_n-\varphi^*\|^2+\xi(1-\theta)\|\varphi_{n+1}-\varphi_n\|^2\\ \le&\rho a_n-(\rho\xi(1-\theta)-\rho\theta(1+\theta)-\xi\theta(1-\theta))\|\varphi_n-\varphi_{n-1}\|^2.\end{aligned}$$

Note that from (21) and Lemma 9 we have

$$\begin{aligned}\theta\le&(1-\delta)\left(1-\frac{(1-\xi)(1-\mu)}{2(1+\mu)}\right)\\ \le&(1-\delta)(1-\beta)=(1-\delta)\rho,\end{aligned}$$

which implies

$$\xi\theta(1-\theta)\le(1-\delta)\rho\xi(1-\theta). \tag{22}$$

Since

$$\theta\le\frac{\sqrt{(1+\delta\xi)^2+4\delta\xi}-(1+\delta\xi)}{2}$$

it holds

$$\theta^2+(1+\delta\xi)\theta-\delta\xi\le 0,$$

or equivalently

$$\theta(1+\theta)\le\delta\xi(1-\theta).$$

Hence

$$\rho\theta(1+\theta)\le\delta\rho\xi(1-\theta). \tag{23}$$

From (22) and (23) we deduce

$$\rho\xi(1-\theta)-\rho\theta(1+\theta)-\xi\theta(1-\theta)\ge 0.$$

Moreover, since $\theta\le\frac{\xi}{2+\xi}$, we have $\theta\le\frac{\xi(1-\theta)}{2}$, which implies

$$\begin{aligned}a_n=&(1-\xi(1-\theta))\|\varphi_n-\varphi^*\|^2+\xi(1-\theta)\left(\|\varphi_n-\varphi^*\|^2+\|\varphi_n-\varphi_{n-1}\|^2\right)-\theta\|\varphi_{n-1}-\varphi^*\|^2\\ \ge&(1-\xi(1-\theta))\|\varphi_n-\varphi^*\|^2+\frac{\xi(1-\theta)}{2}\|\varphi_{n-1}-\varphi^*\|^2-\theta\|\varphi_{n-1}-\varphi^*\|^2\\ \ge&(1-\xi(1-\theta))\|\varphi_n-\varphi^*\|^2\ge 0.\end{aligned}$$

Hence for all $n\ge n_1$ it holds

$$a_{n+1}\le\rho a_n\le\ldots\le\rho^{n-n_1+1}a_{n_1}.$$

This follows that

$$\|\varphi_n-\varphi^*\|^2\le\frac{a_{n_1}}{\rho^{n_1}(1-\xi(1-\theta))}\rho^n,$$

which implies that $\{\varphi_n\}$ converges R-linearly to φ^*.

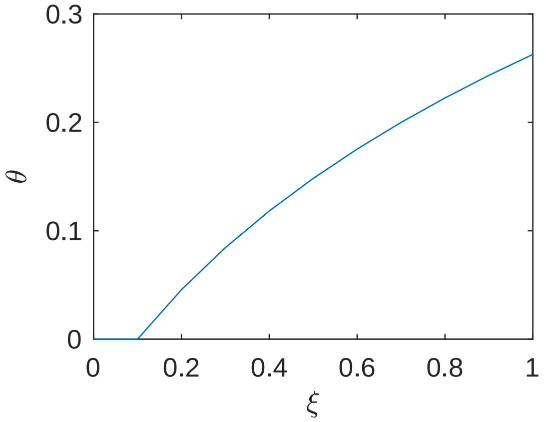

Fig. 1: Inertial effect θ as a function of ξ when $\delta = \mu = 1/2$

Remark 5 Let us emphasize that (21) allows some freedom in choosing the inertial effect θ. For example, if $\delta = \mu = 1/2$ then $\theta \in [0, 0.27]$ depending on ξ, see figure 1.

As a strongly monotone operator is obviously strongly pseudo-monotone, we have immediately the following corollary.

Corollary 2 *Assume that $F : H \to H$ is L-Lipschitz continuous on H and κ-strongly monotone on C. Then, any sequence $\{\varphi_n\}$ generated by Algorithm 1 converges linearly to the unique solution φ^* of* $\mathrm{VI}(F,C)$.

5 Numerical Illustrations

In this section, we present some numerical examples to illustrate the main results obtained in Section 3 and 4. All codes are implemented in Matlab 2019b and performed on a Windows PC Desktop Intel(R) Core(TM) i5-2400CPU @ 3.1Hz, 8GB RAM. In the experiments, we choose

$$\mu = 0.5, \nu_0 = 1, \alpha_n = \frac{1}{n\sqrt{n}}$$

and stopting conditions is Residual $:= \|t_n - s_n\| \leq 10^{-10}$. We perform Algorithm 1 with two values of inertial parameters, $\theta = 0.1$ (PCiFBF1) and $\theta = 0.2$ (PCiFBF2). We compare the performance of (PCiFBF1) and (PCiFBF2) with the algorithm proposed by Shehu et. al. [37, Algorithm 1] with $\theta = 0.9$ and $\rho_n = \rho = 0.4$ (Shehu et. al.).

5.1 Nash–Cournot oligopolistic market equilibrium

In this example, we study an important Nash–Cournot oligopolistic market equilibrium model, which proposed orginally by Murphy et. al. [29] as a convex optimization problem. Later, Harker reformulated it as a monotone vaiational inequality in [16]. We provide only a short description of the problem, for more details we refer to [13, 16, 29]. Consider N firms, each of them supplies a homogeneous product in a non-cooperative fashion. Let

Table 1: Parameters for experiment 1

firm i	c_i	L_i	β_i
1	10	5	1.2
2	8	5	1.1
3	6	5	1.0
4	4	5	0.9
5	2	5	0.8

$q_i \geq 0$ be the ith firm's supply at cost $f_i(q_i)$. Let $p(Q)$ be the inverse demand curve, where $Q \geq 0$ is the total supply in the market, i.e., $Q = \sum_{i=1}^{N} q_i$. A Nash equilibrium solution for the market defined above is a set of nonnegative output levels $(q_1^*, q_2^*, ...q_N^*)$ such that q_i^* is an optimal solution to the following problem for all $i = 1, 2.....N$:

$$\max_{q_i \geq 0} q_i p(q_i + Q_i^*) - f_i(q_i) \tag{24}$$

where

$$Q_i^* = \sum_{j=1, j\neq i}^{N} q_j^*.$$

A variational inequality that corresponds to (24) is (see [16])

$$\text{find} \quad (q_1^*, q_2^*, ...q_N^*) \in \mathbb{R}_+^N \quad \text{such that} \quad \langle F(q^*), q - q^* \rangle \geq 0 \quad \forall q \in \mathbb{R}_+^N, \tag{25}$$

where $F(q^*) = (F_1(q^*), F_2(q^*), ...F_N(q^*))$ and

$$F_i(q^*) = f_i'(q_i^*) - p\left(\sum_{j=1}^{N} q_j^*\right) - q_i^* p'\left(\sum_{j=1}^{N} q_j^*\right)$$

As in the classical example of the Nash-Cournot equilibrium [16,29], we assume that the inverse demand function p and the cost function f_i take the form

$$p(Q) = 5000^{1/1.1} Q^{-1/1.1} \quad \text{and} \quad f_i(q_i) = c_i q_i + \frac{\beta_i}{\beta_i + 1} L_i^{\frac{1}{\beta_i}} q_i^{\frac{\beta_i+1}{\beta_i}}$$

with some constants that are defined as follow.

In the first experiment, we consider $N = 5$ and the parameters c_i, L_i, β_i as in [29], see Table 1. We perform Algorithm 1 with two values of inertial parameters, $\theta = 0.1$ (PCiFBF1) and $\theta = 0.2$ (PCiFBF2). Figure 2 compares the performance of (PCiFBF1) and (PCiFBF2) with the algorithm proposed by Shehu et. al. [37, Algorithm 1] with $\theta = 0.9$ and $\rho_n = \rho = 0.4$ (Shehu et. al.). It can be seen that (PCiFBF1) and (PCiFBF2) are comparable and both outperform Shehu et. al..

In the second experiment, to make the problem even more challenging, we take $N = 1000$ and generate our data randomly. Each entry of c_i, L_i and β_i are drawn independently from the uniform distributions with the following parameters

$$c_i \sim \mathscr{U}(1, 100), \quad L_i \sim \mathscr{U}(0.5, 5), \quad \beta_i \sim \mathscr{U}(0.5, 2)$$

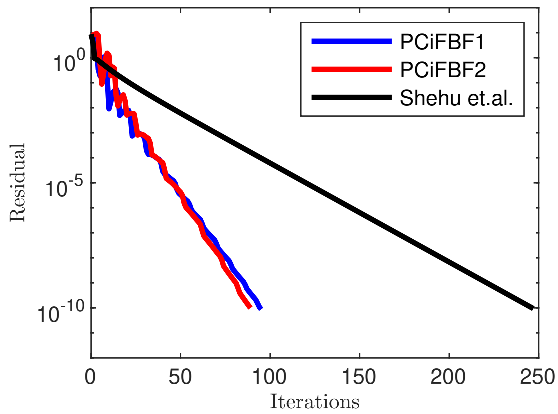

Fig. 2: Performance of Algorithm 1 with different inertial parameters and comparison with Shehu et. al. in the first experiment.

and the other parameters are chosen as in the first experiments. In Figure 3, we also compare the performance of (PCiFBF1) and (PCiFBF2) with Shehu et. al. [37, Algorithm 1] with $\theta = 0.9$ and $\rho_n = \rho = 0.4$. Again it is clear that both (PCiFBF1) and (PCiFBF2) outperform Shehu et. al., although the difference is not significant as in experiment 1.

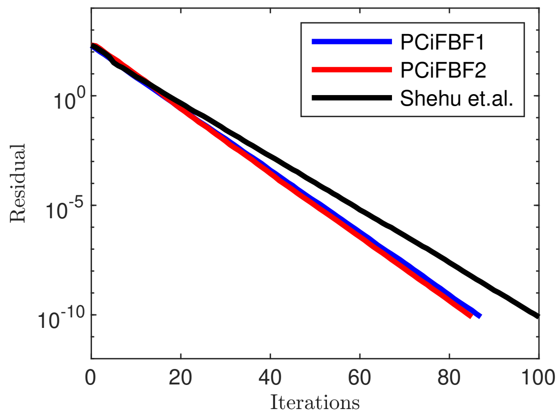

Fig. 3: Performance of Algorithm 1 with different inertial parameters and comparison with Shehu et. al. in the second experiment.

5.2 Network equilibrium flow

In this example, we consider a variational model for one of the most important problems in traffic networks, namely, the network equilibrium flow which is characterized by minimum cost flow traditionally in the context of operations research. This model was formulated by means of a suitable variational inequality [28,30]. We provide only a short description of the problem, for more details we refer to [28,30]. We will use the following notations:

- f_i is the flow on the arc $A_i := (r,s)$ and $f := (f_1,...,f_n)^T$ is the vector of the flows on all arcs;
- we assume that each arc A_i is associated with an upper bound d_i on its capacity, $d := (d_1,...,d_n)$.
- $c_i(f)$ is the cost-variation on the arc A_i as function of the flows, $\forall i = 1,...,n$ and $c(f) := (c_1(f),...,c_n(f))^T$; we assume that $c(f) \geq 0$;
- q_j is the balance at the node $j, j = 1,...,p$ and $q := (q_1,...,q_p)^T$;
- $\Gamma = (\gamma_{ij}) \in \mathbb{R}^p \times \mathbb{R}^n$ is the node-arc incidence matrix whose elements are

$$\gamma_{ij} = \begin{cases} -1, & \text{if } i \text{ is the initial node of the arc } A_j, \\ +1, & \text{if } i \text{ is the final node of the arc } A_j, \\ 0, & \text{otherwise.} \end{cases}$$

A flow f is a variational equilibrium flow for the capacitated model if and only if it solves the following VI (see [28])

$$\text{find} \quad f^* \in K_f \quad \text{such that} \quad \langle c(f^*), f - f^* \rangle \geq 0 \quad \forall f \in K_f, \tag{26}$$

where

$$K_f = \{f \in \mathbb{R}^n, \quad \Gamma f = q,\ 0 \leq f \leq d\}.$$

The problem (26) collapses to the minimal-cost network-flow problem when the function $c(f)$ is independent of f, namely, $c(f) := (c_{ij}, (i,j) \in A)$. In the numerical experiment, as in [28, Example 2.1] we take

$$\Gamma = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix};$$

$$q = (-2,0,0,0,0,2)^T, \quad d = (2,1,1,1,1,1,2,2)^T.$$

The cost function is defined by $c(f) := Cf$ where $C = \text{diag}(D)$ with

$$D = (5.5,1,2,3,4,50,3.5,1.5).$$

The solution of this network flow problem is given by

$$f^* = (1.000,1.000,0.1575,0.8425,0.885,0.115,1.0425,0.9575)^T.$$

In the experiment, we choose $f_0 = f_1 = (1,1,1,...1)^T$. We perform Algorithm 1 with two values of inertial parameters, $\theta = 0.1$ (PCiFBF1) and $\theta = 0.2$ (PCiFBF2). We compare the performance of (PCiFBF1) and (PCiFBF2) with the algorithm proposed by Shehu et. al. [37, Algorithm 1] with $\theta = 0.9$ and $\rho_n = \rho = 0.4$ (Shehu et. al.) in Figure 4, which shows that (PCiFBF1) and (PCiFBF2) are comparable and both outperform Shehu et. al..

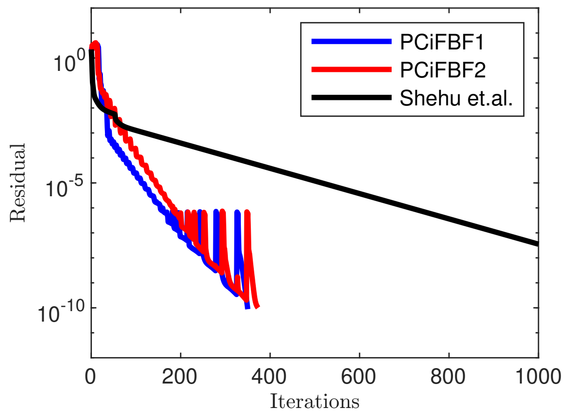

Fig. 4: Performance of Algorithm 1 with different inertial parameters and comparison with Shehu et. al. in the network flow experiment.

6 Conclusions

In this work, we have introduced new a projection and contraction method with inertial effect for solving variational inequalities in real Hilbert spaces. The weak convergence is proved under pseudomonotonicity and Lipschitz continuity of the mapping F. The linear convergence is obtained under strong pseudomonotonicity and Lipschitz continuity assumptions. The advantage of our algorithm is that it requires only one projection onto the feasible set C per iteration and does not require the knowledge of Lipschitz constant as well as the weak continuity of the mapping F.

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