Time-domain metric reconstruction using the Hertz potential

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Abstract

Historically the Teukolsky equation corresponding to gravitational perturbations, such as a point particle in a background Kerr spacetime, is solved for Weyl-like scalars. However, reconstructing the metric from these scalars involves solving a fourth-order PDE to obtain the Hertz potential and then acting with a second-order differential operator to construct the metric perturbation. Solving the (adjoint) Teukolsky equation for the Hertz potential directly simplifies the metric reconstruction procedure to applying the second-order operator. We discuss using this method in the time domain with a point particle source and its foreseeable application in self-force calculations.

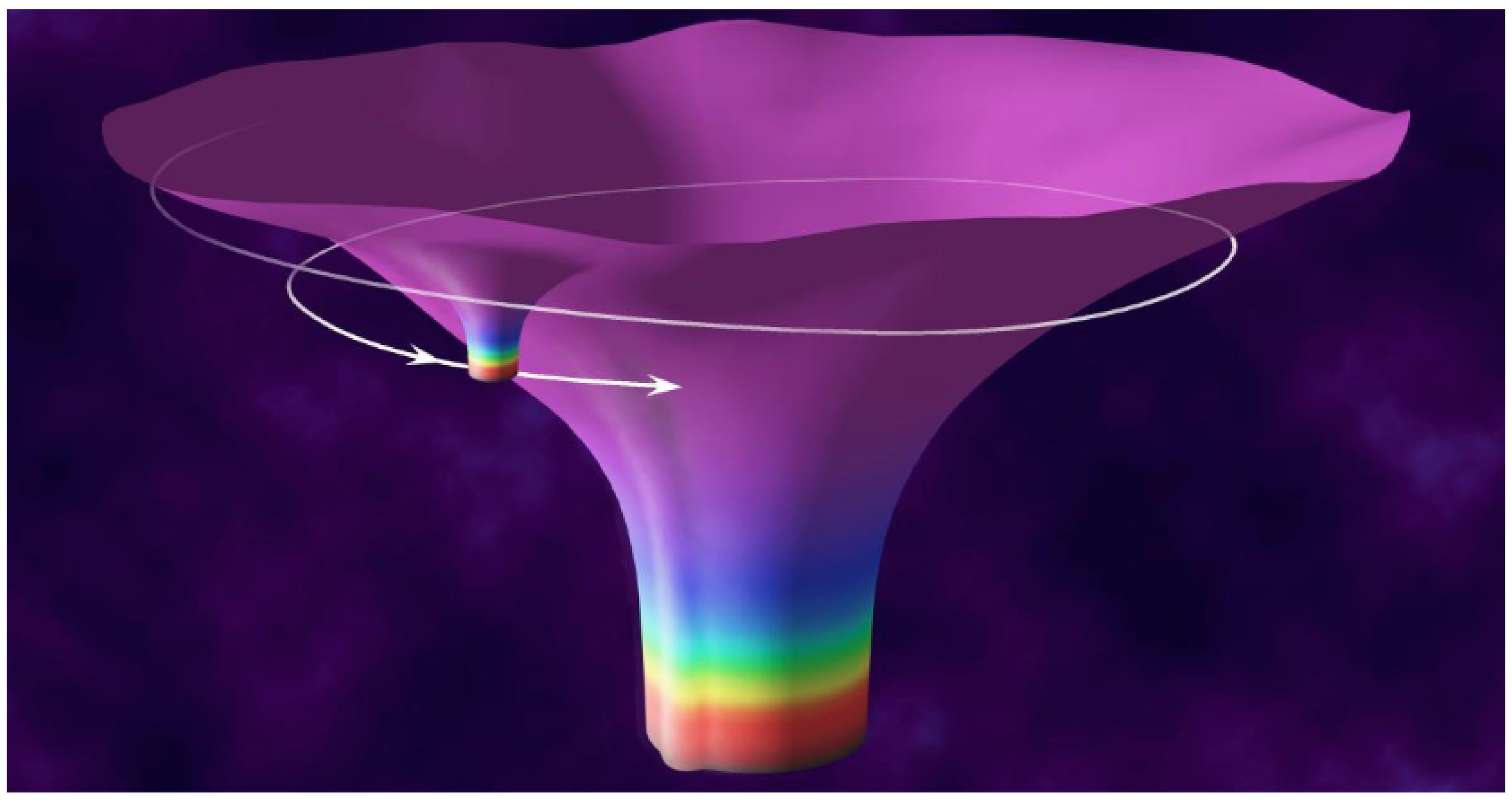

Figure 1: An artist's impression of the spacetime of an extreme-mass-ratio inspiral. Credit: NASA

Vacuum Metric Reconstruction

Consider a metric perturbation h such that it obeys $\hat{\mathsf{E}}h = 0$ where the operator $\hat{\mathsf{E}}$ is the linearised Einstein operator. There exists Weyl-like curvature scalars $\Psi_\pm$ which satisfy $\hat{\mathcal{O}}_\pm \Psi_\pm = 0$, where $\hat{\mathcal{O}}_\pm$ is the ± 2 Teukolsky operator. These scalars can be obtained from h using second-order differential operators $\hat{\mathsf{T}}_\pm$ such that $\hat{\mathsf{T}}_\pm h = \Psi_\pm$.

However, the conditions above do not constrain the solutions such that h is the physical metric perturbation. The Hertz potential $\Phi_\pm$ is introduced such that the field solves the adjoint[1] Teukolsky equation $\hat{\mathcal{O}}^\dagger_\pm \Phi_\pm = \hat{\mathcal{O}}_\mp \Phi_\pm = 0$.

Let $\hat{\mathsf{S}}_\pm$ be the operators that take the linearised Einstein's equations into the Teukolsky equations with spin ± 2 such that $\hat{\mathsf{S}}_\pm \hat{\mathsf{E}} h = \hat{\mathcal{O}}_\pm \Psi_\pm$. It can be shown that

$$h^\pm := \hat{\mathsf{S}}^\dagger_\pm \Phi_\pm \tag{1}$$

are two solutions of the linearised Einstein equations. The form of $\hat{\mathsf{S}}^\dagger_\pm$ returns h_+ and h_- in the ingoing radiation gauge (IRG) and outgoing radiation gauge (ORG) respectively. h can be determined from $h^\pm$ by adding known gauge and completion pieces [1]. In order for $h^\pm$ to give the original perturbation h, it still has to be true that $\hat{\mathsf{T}}_\pm h^+ = \Psi_\pm$ (IRG) and $\hat{\mathsf{T}}_\pm h^- = \Psi_\pm$ (ORG). This leads to the relations

$$\hat{\mathsf{T}}_\pm \hat{\mathsf{S}}^\dagger_+ \Phi_+ = \Psi_\pm \quad \text{(IRG)}, \tag{2}$$

$$\hat{\mathsf{T}}_\pm \hat{\mathsf{S}}^\dagger_- \Phi_- = \Psi_\pm \quad \text{(ORG)}. \tag{3}$$

These are relations between the Hertz potential and the Weyl-like curvature scalars.

Given a Weyl-like curvature scalar in IRG (or ORG) it is now possible to reconstruct the metric perturbation. First (2) (or (3)) is used to construct Φ_+ (or Φ_-) then (1) is used to construct h^+ (or h^-). The gauge and completion pieces can then be added to give h.

[1] For a linear operator $\hat{\mathsf{L}}$ taking an n-rank tensor field ϕ to an m-rank tensor field ψ, the adjoint $\hat{\mathsf{L}}^\dagger$ takes ψ to ϕ and is defined via $\psi(\hat{\mathsf{L}}\phi) - (\hat{\mathsf{L}}^\dagger\psi)\phi = s^\alpha{}_{;\alpha}$, where s^α is an arbitrary vector field.

Point-Particle No-String Solution

If a source is present in the spacetime, the above vacuum procedure can fail to return a valid metric perturbation solution. If a point particle is present, the reconstructed metric contains a radial string-like singularity from the particle [2]. These solutions can be radially inward or outwards ('half-string' solutions) or both ('full-string' solutions). The singularities can be removed by combining two half-string solutions as shown in Figure 2. The regular parts of both domains are taken and combined to give a no-string solution which is regular except for at the particle's radius $r = r_{\mathrm{p}}$ where there are gauge discontinuities.

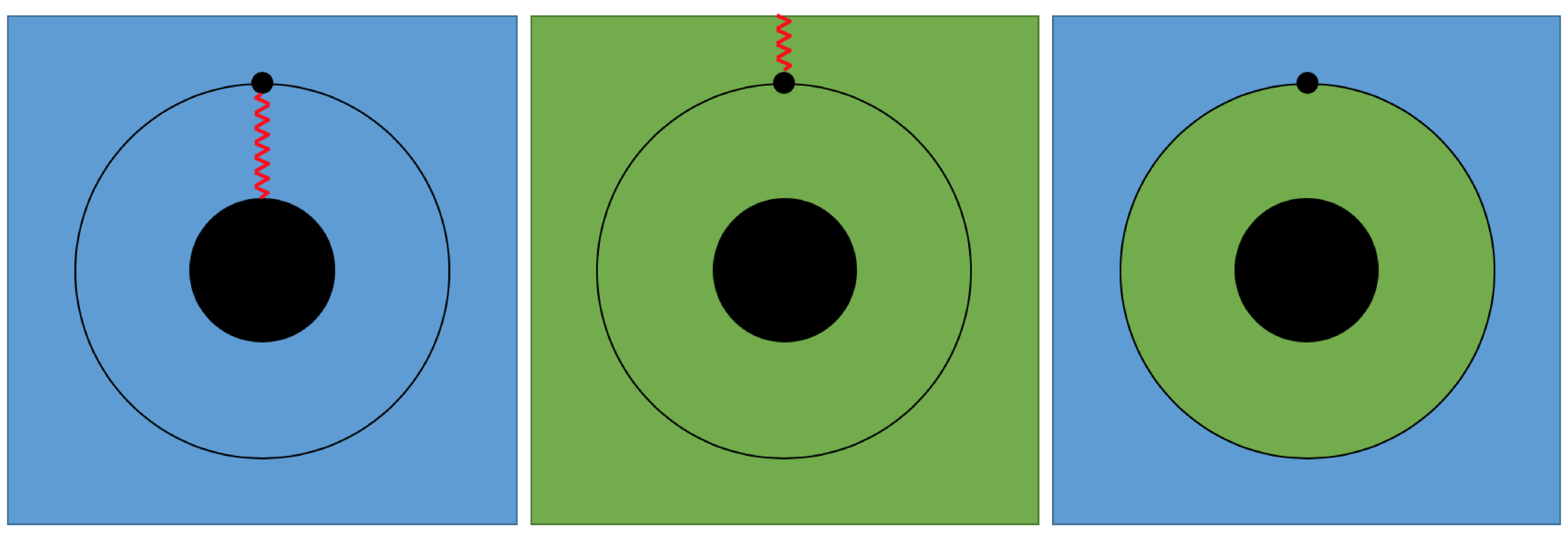

Figure 2: Pictorial representations of the two half-string (left and centre) and no-string (right) solutions. The radial string singularities in the reconstructed metric are represented by the red jagged line.

Solving for the Hertz Potential

Solving for the Hertz potential is advantageous as the metric reconstruction procedure only requires one second-order differential operator whereas the Weyl-like scalars require an extra fourth-order PDE to be solved.

In Schwarzschild, the Teukolsky equation can be separated into modes using the spin-weighted spherical harmonics ${}_sY_{lm}$ which contain all the angular dependence of the equation. The time-radial part of the field ϕ^{lm} can then be solved mode by mode as the equation has the form

$$\phi^{lm}_{,uv} + U(r)\phi^{lm}_{,u} + V(r)\phi^{lm}_{,v} + W(l,r)\phi^{lm} = 0, \tag{4}$$

where u and v are the double-null Eddington-Finkelstein coordinates [3]. The Kerr case is more complicated as there is an added coupling term between the different l modes.

The Hertz potential solutions are uniquely determined by the jump conditions across the particle's trajectory. The jump conditions are derived from the source of the Teukolsky equation by considering the differences in the field on either side of $r = r_{\mathrm{p}}$.

Numerical Implementation

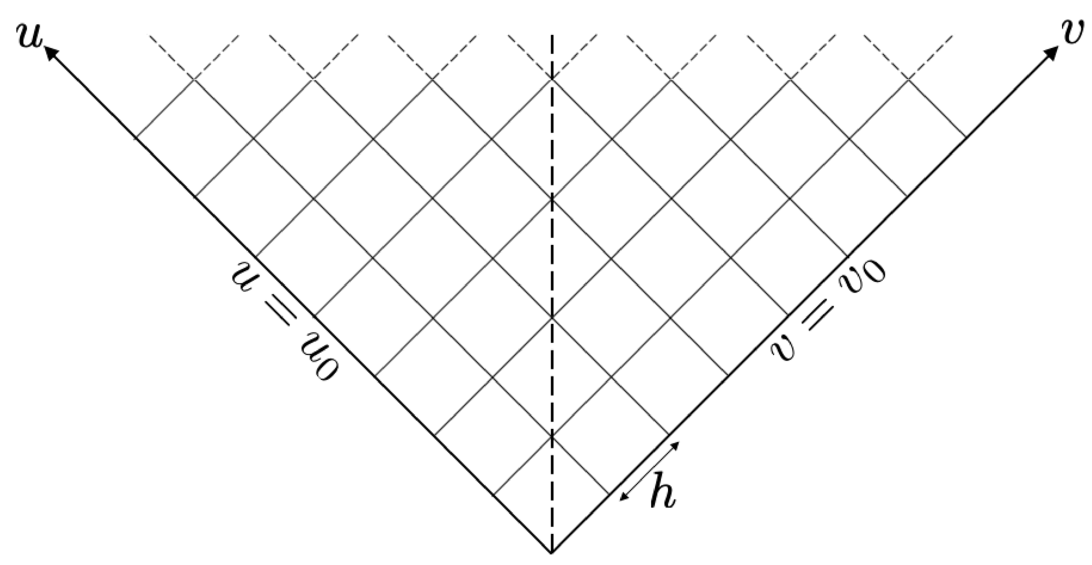

Figure 3: The double-null grid in (u, v) coordinates. The vertical dashed line represents the particle's worldline for a circular orbit. Initial conditions are present on the slices $u = u_0$ and $v = v_0$.

A numerical code has been implemented in the circular orbit case to solve the spin -2 Teukolsky equation for the Hertz potential on a Schwarzschild background. The code uses a finite difference scheme on slices of constant u and v as shown in Figure 3. The initial conditions used are arbitrary as any response from the finite difference scheme to the data decays sufficiently quickly such that the so-called junk initial data can just be discarded. The majority of the points in the null grid solve for the vacuum Teukolsky equation but points near the orbit of the particle require a more careful treatment due to the no-string solution combining fields from different gauges.

Sample Results

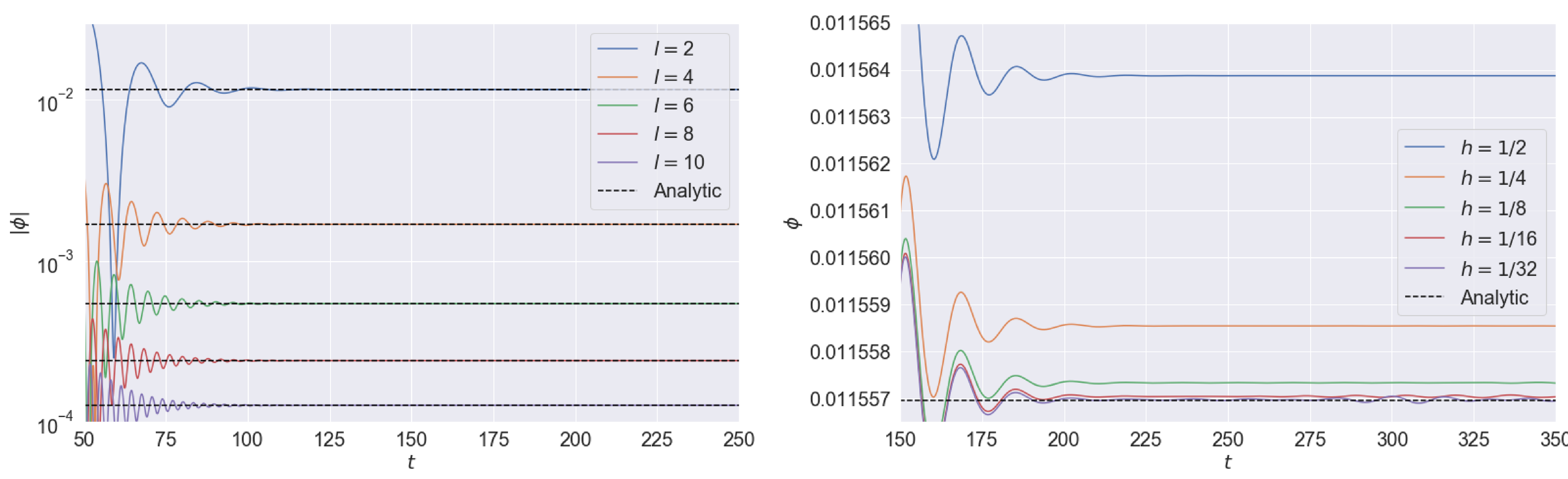

Figure 4: Plots comparing the results of the numerical code for static modes of circular orbits in Schwarzschild. Left: Data from different l modes with the same grid spacing. Right: Data from the $l = 2$ mode with varying grid spacing.

The left plot of Figure 4 shows plots of the field values for static modes ($m = 0$) with different values of l. In each case the numerical data converges to the analytic solution after the decay of the junk data caused by the arbitrary initial conditions. The right plot of Figure 4 shows the results for numerical runs with different values for the distance between the grid points h. As h is decreased, there are more points in the grid which means the calculated value should converge to the true value as shown in the plot.

Future Applications

The code is to be extended to calculate the Hertz potential for generic orbits in Schwarzschild with an emphasis on scattering orbits. The ultimate goal is to calculate scattering angles with self-force corrections in Schwarzschild. After this, the method will be extended to Kerr.

References

[1] C. Merlin *et al.*, "Completion of metric reconstruction for a particle orbiting a Kerr black hole," *Physical Review D*, vol. 94, November 2016.

[2] L. Barack and A. Ori, "Gravitational self-force and gauge transformations," *Physical Review D*, vol. 64, October 2001.

[3] L. Barack and P. Giudice, "Time-domain metric reconstruction for self-force applications," *Physical Review D*, vol. 95, May 2017.