KK-like relations of α' corrections to disk amplitudes

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Inspired by the definition of color-dressed amplitudes in string theory, we define analogous *color-dressed permutations* replacing the color-ordered string amplitudes by their corresponding permutations. Decomposing the color traces into symmetrized traces and structure constants, the color-dressed permutations define *BRST-invariant permutations*, which we show are elements of the inverse Solomon descent algebra and we find a closed formula for them. We then present evidence that these permutations encode KK-like relations among the different α' corrections to the disk amplitudes refined by their MZV content. In particular, the number of linearly independent amplitudes at a given α' order and MZV content is given by (sums of) Stirling cycle numbers. In addition, we show how the superfield expansion of BRST invariants of the pure spinor formalism corresponding to ${\alpha'}^2\zeta_2$ corrections is encoded in the descent algebra.

August 2021

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1. Introduction

It is well-known that superstring n-point color-ordered disk amplitudes satisfy monodromy relations which imply that the number of linearly independent amplitudes is $(n-3)!$, for all α' corrections [1,2]. These relations involve coefficients that depend on Mandelstam variables [3] and are famously related to the Bern-Carrasco-Johansson (BCJ) color-kinematics amplitude relations in the field-theory limit. In this paper we investigate a weaker set of relations, called KK-like relations [4], of higher α' corrections to disk amplitudes refined by their MZV content [5,6,7]. More precisely, writing the superstring color-ordered disk amplitude as a series of *MZV disk amplitudes*

$$A^{\text{string}}(1,2,\ldots,n) = A^{\text{SYM}}(1,2,\ldots,n) + \sum_{m=0}^{\infty}\sum_{M} A_{\zeta_2^m\zeta_M}(1,2,\ldots,n) \tag{1.1}$$

where M runs over all MZV basis elements [8] that do not contain factors of ζ_2, we will focus on relations of the form

$$\sum_{\sigma} c_\sigma A_{\zeta_2^m\zeta_M}(\sigma) = 0, \tag{1.2}$$

where the coefficients c_σ are rational numbers independent of Mandelstam invariants. It is well-known that (1.1) is cyclically symmetric and satisfies $A^{\text{string}}(1,2,\ldots,n) = (-1)^n A^{\text{string}}(n,n-1,\ldots,1)$, so the basis dimensions of all MZV disk amplitudes is at most $\frac{1}{2}(n-1)!$. When the superstring amplitude (1.1) is restricted to its field-theory limit A^{SYM}, the KK-like relations correspond to the famous Kleiss-Kuijf (KK) relations [9], under which there are only $(n-2)!$ independent amplitudes. However, it is not yet known in general the form of the KK-like relations and the corresponding basis dimensions for the $\zeta_2^m\zeta_M$ components of (1.1) with $m \geq 2$. For instance, a brute-force search indicates that the upper bound $\frac{1}{2}(n-1)!$ is saturated for $A_{\zeta_2^2\zeta_M}(1,2,\ldots,n)$ when $n = 4,5,6$ and 7, but the $n = 8$ case is different. In fact,

$$\#\big(A_{\zeta_2^2\zeta_M}(1,2,\ldots,8)\big) = 2519 = \frac{1}{2}\,7! - 1. \tag{1.3}$$

In this paper, this result and its generalization will be obtained as

$$\begin{aligned}
\#\Big(A_{\zeta_M}(1,2,\ldots,n)\Big) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = (n-2)! \\
\#\Big(A_{\zeta_2\zeta_M}(1,2,\ldots,n)\Big) &= \begin{bmatrix} n-1 \\ 3 \end{bmatrix} \\
\#\Big(A_{\zeta_2^m\zeta_M}(1,2,\ldots,n)\Big) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + \begin{bmatrix} n-1 \\ 3 \end{bmatrix} + \cdots + \begin{bmatrix} n-1 \\ 2m+1 \end{bmatrix}, \quad m \geq 2\,.
\end{aligned} \tag{1.4}$$

We will see that the general KK-like relations are closely related to the mathematical framework of the Solomon descent algebra [10,11,12,13,14,15,16]. To see this we will define the *color-dressed permutation*

$$P_n = \sum_{\sigma\in S_n,\sigma(1)=1} \mathrm{Tr}(T^\sigma)\,\sigma\,, \qquad T^\sigma := T^{\sigma(1)}T^{\sigma(2)}\cdots T^{\sigma(n)} \tag{1.5}$$

which is inspired by the expression of the color-dressed disk amplitudes, where $T^i := T^{a_i}$ denotes a Chan-Paton factor. When the closed formula from [17] for the color-trace decomposition [18] is plugged into (1.5), the permutations appearing as coefficients with respect to a basis of color factors define what we call *BRST invariant permutations* $\gamma_{1|P_1,\ldots,P_k}$ with $1 \leq k \leq n{-}1$. We conjecture the following closed formula

$$\gamma_{1|P_1,\ldots,P_k} = 1.\mathcal{E}(P_1)\sqcup\!\sqcup\mathcal{E}(P_2)\sqcup\!\sqcup\ldots\sqcup\!\sqcup\mathcal{E}(P_k) \tag{1.6}$$

where $\mathcal{E}(P)$ satisfying $\mathcal{E}(R\sqcup\!\sqcup S)=0$ for $R,S\neq\emptyset$ is the *Berends-Giele idempotent*, defined in section 2.2 from mapping the permutations of the Solomon idempotent [19] into their inverses. Then in section 3.1 we will find evidence that these BRST-invariant permutations encode the general KK-like relations as

$$\begin{aligned} A_{\zeta_M}(\gamma_{1|P_1,\ldots,P_k}) &= 0, \quad k\neq 1, \\ A_{\zeta_2\zeta_M}(\gamma_{1|P_1,\ldots,P_k}) &= 0, \quad k\neq 3 \\ A_{\zeta_2^m\zeta_M}(\gamma_{1|P_1,\ldots,P_k}) &= 0, \quad k\neq 1,3,5,\ldots,2m+1, \quad m\geq 2. \end{aligned} \tag{1.7}$$

Studying the color-dressed string disk amplitude we will obtain, following the results of [20], a correspondence between the above permutations and kinematics from the string disk amplitudes. More precisely, the duality from [20] is generalized to

$$A^{\mathrm{SYM}}(1,2,\ldots,n) \;\longleftrightarrow\; \gamma^{(1)}_{123\ldots n}\,, \qquad A_{\zeta_2}(1,2,\ldots,n) \;\longleftrightarrow\; \gamma^{(3)}_{123\ldots n}\,, \tag{1.8}$$

where $\gamma^{(1)}$ and $\gamma^{(3)}$ are orthogonal idempotents of the (inverse) descent algebra constructed from linear combinations of $\gamma_{1|P_1,\ldots,P_k}$ with $k=1$ and $k=3$ respectively. This interpretation is important because we can use a theorem from the work of Garsia and Reutenauer [11] to prove results conjectured in [20].

In section 3.3, the BRST-invariant permutations will be used to define higher-mass BRST-invariant superfields via

$$\gamma_{1|P_1,P_2,\ldots,P_k} \leftrightarrow A^{\mathrm{string}}(\gamma_{1|P_1,P_2,\ldots,P_k})\,, \qquad A_{\zeta_2^m\zeta_M}(\gamma_{1|P_1,\ldots,P_k}) = k!C^{\zeta_2^m\zeta_M}_{1|P_1,\ldots,P_k}\,. \tag{1.9}$$

As shown in section 3.4, the case for ζ_2 reduces to the BRST-invariants studied in [20,21]. In the appendices we review the descent algebra and collect various proofs and explicit expansions omitted from the main text.

1.0.1. Conventions

Words from the alphabet $\mathbb{N} = \{1, 2, \ldots\}$ will be denoted either by capital Latin letters or, especially when viewed as elements of the permutation group, by lower case Greek letters. The symmetric group S_n acts on words of length n via the right-action multiplication [16]

$$P \circ \sigma = p_{\sigma(1)} p_{\sigma(2)} \ldots p_{\sigma(n)}, \tag{1.10}$$

where p_i denotes the ith letter of P. For example $abcd \circ 3124 = cabd$. The inverse σ^{-1} of a permutation σ of length n is such that $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = 12 \ldots n$. For example, $(2314)^{-1} = 3124$. For typographical convenience, we write a permutation σ as W_σ.

2. The combinatorics of color-dressed permutations

2.1. Color-dressed permutations

In this section we will investigate the combinatorics of the *color-dressed permutations* P_n

$$P_n = \sum_{\sigma \in S_n, \sigma(1)=1} \mathrm{Tr}(T^\sigma)\, W_\sigma\,, \qquad T^\sigma := T^{\sigma(1)} T^{\sigma(2)} \cdots T^{\sigma(n)}\,, \tag{2.1}$$

arising from decomposing [18] the traces of color factors into symmetrized traces $d^{12\ldots k}$ and structure constants f^{abc} of the gauge group [17]

$$\mathrm{Tr}(T^0 T^1 \cdots T^{n-1}) = \sum_{S_{n-1} \ni \sigma = \sigma_1 \cdots \sigma_k} i^{n-1-k} \kappa_{\sigma_1} \cdots \kappa_{\sigma_k} d^{0a_1 \cdots a_k} F^{\sigma_1}_{a_1} \cdots F^{\sigma_k}_{a_k}. \tag{2.2}$$

where $\sigma = \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_k$ denotes the decreasing Lyndon factorization of σ to be defined below, F^σ_a for a word σ and a letter a is defined recursively by $F^{Pj}_a = F^P_b f^{bja}$ with base case $F^i_a = \delta^i_a$. The coefficients κ_σ are defined in (A.20). In addition, the symmetrized trace and the structure constant are given by

$$d^{12\ldots k} := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{Tr}(T^\sigma), \qquad d^{12} := \frac{1}{2}\delta^{12}, \quad [T^a, T^b] = i f^{abc} T^c. \tag{2.3}$$

The *decreasing Lyndon factorization* (dLf) of σ is defined as [22,11] $\sigma = \sigma_1.\sigma_2 \ldots \sigma_k$ representing the unique deconcatenation of σ into Lyndon subwords $\sigma_1, \ldots, \sigma_k$ such that $\sigma_1 > \cdots > \sigma_k$ in the lexicographical order of the alphabet $\mathbb{N} = \{1, 2, \ldots, \}$. Representing the concatenation by a dot to distinguish the subwords σ_i in the dLF factorization of σ, we have, for example $1432 = 1432$, $2134 = 2.134$, $54132 = 5.4.132$, $42671835 = 4.267.1835$.

Definition (BRST-invariant permutations). *The BRST-invariant permutations are the coefficients with respect to the basis of color factors* $d^{1a_1\ldots a_k}F^{\sigma_1}_{a_1}\ldots F^{\sigma_k}_{a_k}$*, where* $\sigma_1\cdots\sigma_k$ *is the decreasing Lyndon factorization of* σ*, in the color-dressed permutation (2.1), i.e.*

$$P_n=\sum_{\sigma\in S_{n-1}} i^{n-1-k}d^{1a_1\ldots a_k}F^{\sigma_1}_{a_1}\cdots F^{\sigma_k}_{a_k}\,\gamma_{1|\sigma_1,\ldots,\sigma_k}\,, \tag{2.4}$$

The reason for this terminology will become clear in (3.23) when $\gamma_{1|\sigma_1,\sigma_2,\sigma_3}$ will be associated to BRST invariants superfields in the pure spinor formalism. For example, plugging $\mathrm{Tr}(T^1T^2T^3)=d^{1a_1a_2}F^2_{a_1}F^3_{a_2}+\frac{i}{2}d^{1a}F^{23}_a$ into (2.1) yields

$$P_3=d^{1ab}F^2_aF^3_b\gamma_{1|2,3}+id^{1a}F^{23}_a\gamma_{1|23}\,, \tag{2.5}$$

with

$$\gamma_{1|2,3}=W_{123}+W_{132}\,,\quad \gamma_{1|23}=\frac{1}{2}W_{123}-\frac{1}{2}W_{132}\,. \tag{2.6}$$

Repeating the same exercise for $n=4$ using (2.2) we obtain

$$\begin{aligned} P_4&=d^{1abc}F^2_aF^3_bF^4_c\,\gamma_{1|2,3,4} \\ &+id^{1ab}F^{23}_aF^4_b\,\gamma_{1|23,4}+id^{1ab}F^{24}_aF^3_b\,\gamma_{1|24,3}+id^{1ab}F^2_aF^{34}_b\,\gamma_{1|2,34} \\ &+i^2d^{1a}F^{234}_a\,\gamma_{1|234}+i^2d^{1a}F^{243}_a\,\gamma_{1|243} \end{aligned} \tag{2.7}$$

where the BRST-invariant permutations are given by

$$\begin{aligned} \gamma_{1|2,3,4}&=W_{1234}+W_{1243}+W_{1324}+W_{1342}+W_{1423}+W_{1432}\,,\\ \gamma_{1|23,4}&=\frac{1}{2}W_{1234}+\frac{1}{2}W_{1243}-\frac{1}{2}W_{1324}-\frac{1}{2}W_{1342}+\frac{1}{2}W_{1423}-\frac{1}{2}W_{1432}\,,\\ \gamma_{1|2,34}&=\frac{1}{2}W_{1234}-\frac{1}{2}W_{1243}+\frac{1}{2}W_{1324}+\frac{1}{2}W_{1342}-\frac{1}{2}W_{1423}-\frac{1}{2}W_{1432}\,,\\ \gamma_{1|24,3}&=\frac{1}{2}W_{1234}+\frac{1}{2}W_{1243}+\frac{1}{2}W_{1324}-\frac{1}{2}W_{1342}-\frac{1}{2}W_{1423}-\frac{1}{2}W_{1432}\,,\\ \gamma_{1|234}&=\frac{1}{3}W_{1234}-\frac{1}{6}W_{1243}-\frac{1}{6}W_{1324}-\frac{1}{6}W_{1342}-\frac{1}{6}W_{1423}+\frac{1}{3}W_{1432}\,,\\ \gamma_{1|243}&=-\frac{1}{6}W_{1234}+\frac{1}{3}W_{1243}-\frac{1}{6}W_{1324}+\frac{1}{3}W_{1342}-\frac{1}{6}W_{1423}-\frac{1}{6}W_{1432}\,. \end{aligned} \tag{2.8}$$

For $n=5$ we obtain

$$\begin{aligned} P_5&=d^{1abcd}F^2_aF^3_bF^4_cF^5_d\,\gamma_{1|2,3,4,5} \\ &+id^{1abc}F^{23}_aF^4_bF^5_c\,\gamma_{1|23,4,5}+id^{1abc}F^{24}_aF^3_bF^5_c\,\gamma_{1|24,3,5}+id^{1abc}F^{25}_aF^3_bF^4_c\,\gamma_{1|25,3,4} \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&+ id^{1abc}F_a^2F_b^{34}F_c^5\,\gamma_{1|2,34,5} + id^{1abc}F_a^2F_b^{35}F_c^4\,\gamma_{1|2,35,4} + id^{1abc}F_a^2F_b^3F_c^{45}\,\gamma_{1|2,3,45}\\
&+ i^2d^{1ab}F_a^{234}F_b^5\,\gamma_{1|234,5} + i^2d^{1ab}F_a^{243}F_b^5\,\gamma_{1|243,5} + i^2d^{1ab}F_a^{235}F_b^4\,\gamma_{1|235,4}\\
&+ i^2d^{1ab}F_a^{245}F_b^3\,\gamma_{1|245,3} + i^2d^{1ab}F_a^{253}F_b^4\,\gamma_{1|253,4} + i^2d^{1ab}F_a^{254}F_b^3\,\gamma_{1|254,3}\\
&+ i^2d^{1ab}F_a^{345}F_b^2\,\gamma_{1|345,2} + i^2d^{1ab}F_a^{354}F_b^2\,\gamma_{1|354,2}\\
&+ i^2d^{1ab}F_a^{23}F_b^{45}\,\gamma_{1|23,45} + i^2d^{1ab}F_a^{24}F_b^{35}\,\gamma_{1|24,35} + i^2d^{1ab}F_a^{25}F_b^{34}\,\gamma_{1|25,34}\\
&+ i^3d^{1a}F_a^{2345}\,\gamma_{1|2345} + i^3d^{1a}F_a^{2354}\,\gamma_{1|2354} + i^3d^{1a}F_a^{2435}\,\gamma_{1|2435}\\
&+ i^3d^{1a}F_a^{2453}\,\gamma_{1|2453} + i^3d^{1a}F_a^{2534}\,\gamma_{1|2534} + i^3d^{1a}F_a^{2543}\,\gamma_{1|2543}
\end{aligned}
$$

where the various $\gamma_{1|A_1,A_2,\ldots,A_k}$ are listed in the appendix C.

2.1.1. Relating the BRST-invariant permutations with the descent algebra

The permutations in each $\gamma_{1|A_1,A_2,\ldots,A_k}$ turn out to be related to the descent algebra reviewed in the appendix A. To see this consider $\gamma_{1|23,4}$ from (2.8), relabel $i \to i-1$ and strip off the leading "0" (denoted by $\times$) to obtain

$$
\gamma_{\times|12,3} = \frac{1}{2}\underbrace{W_{123}}_{\in D_\emptyset} + \frac{1}{2}\underbrace{W_{132}}_{\in D_{\{2\}}} - \frac{1}{2}\underbrace{W_{213}}_{\in D_{\{1\}}} - \frac{1}{2}\underbrace{W_{231}}_{\in D_{\{2\}}} + \frac{1}{2}\underbrace{W_{312}}_{\in D_{\{1\}}} - \frac{1}{2}\underbrace{W_{321}}_{\in D_{\{1,2\}}}\,, \tag{2.10}
$$

The above permutations are *not* in the descent algebra $\mathcal{D}_3$ since permutations in the same descent class have different coefficients (see also proposition 2.1 from [23]). However, the inverse permutations in $\theta(\gamma_{\times|12,3})$ do belong to the same descent classes:

$$
\begin{aligned}
\theta\big(\gamma_{\times|12,3}\big) &= \frac{1}{2}\underbrace{W_{123}}_{\in D_\emptyset} + \frac{1}{2}\underbrace{W_{132}}_{\in D_{\{2\}}} - \frac{1}{2}\underbrace{W_{213}}_{\in D_{\{1\}}} - \frac{1}{2}\underbrace{W_{312}}_{\in D_{\{1\}}} + \frac{1}{2}\underbrace{W_{231}}_{\in D_{\{2\}}} - \frac{1}{2}\underbrace{W_{321}}_{\in D_{\{1,2\}}}\\
&= \frac{1}{2}D_\emptyset - \frac{1}{2}D_{\{1\}} + \frac{1}{2}D_{\{2\}} - \frac{1}{2}D_{\{1,2\}}\,.
\end{aligned} \tag{2.11}
$$

as can be verified using (A.3). This suggests that the BRST-invariant permutations belong to the *inverse* descent algebra $\mathcal{D}'_n := \theta(\mathcal{D}_n)$. To find an algorithm that generates these permutations, we will consider the inverse of the Eulerian idempotent (A.20).

2.2. The Berends-Giele idempotent

Define the *Berends-Giele idempotent*[1] $\mathcal{E}_n$ as the inverse of the Eulerian idempotent (A.20):

$$
\mathcal{E}_n = \sum_{\sigma\in S_n} \kappa_{\sigma^{-1}}\sigma\,, \qquad \mathcal{E}(P) = \mathcal{E}_P := P\circ\mathcal{E}_n\,, \quad |P| = n\,. \tag{2.12}
$$

[1] The inverse of an idempotent is also idempotent.

The reason for this terminology is the correspondence with the standard Berends-Giele current of Yang-Mills theory [24], see section 3.2. A few examples of (2.12) are

$$\mathcal{E}(1) = W_1, \qquad \mathcal{E}(12) = \frac{1}{2}(W_{12} - W_{21}), \tag{2.13}$$
$$\mathcal{E}(123) = \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321},$$

while the expansion of $\mathcal{E}(1234)$ can be found in the appendix C.1.

Proposition (Shuffle Symmetry). *The Berends-Giele idempotent (2.12) satisfies*

$$\mathcal{E}(R \sqcup\!\sqcup S) = 0. \tag{2.14}$$

Proof.[2] Since the sum in (2.12) is over all permutations we rename $P \circ \sigma = \tau$ and sum over τ. Notice that $\sigma^{-1} = \tau^{-1} \circ P$, so $\mathcal{E}(P) = \sum_\sigma \kappa_{\sigma^{-1}} P \circ \sigma = \sum_\tau \kappa_{(\tau^{-1} \circ P)} \tau$ and therefore

$$\mathcal{E}(R \sqcup\!\sqcup S) = \sum_\tau \kappa_{(\tau^{-1} \circ (R \sqcup\!\sqcup S))} \tau = \sum_\tau \kappa_{(\tau^{-1}(R) \sqcup\!\sqcup \tau^{-1}(S))} \tau = 0 \tag{2.15}$$

where the last equality follows from (A.22) and the crucial observation in (1.5) of [11] that $\sigma^{-1} \circ (R \sqcup\!\sqcup S) = \sigma^{-1}(R) \sqcup\!\sqcup \sigma^{-1}(S)$, where $\sigma^{-1}(R)$ denotes the word obtained by replacing each letter in R by its image under σ^{-1}. □

2.3. Inverse idempotent basis and BRST-invariant permutations

Following the realization in section 2.1.1 that the BRST-invariant permutations are related to the inverse of the descent algebra, we define the inverse of the idempotent basis I_p as

$$\mathcal{I}_{p_1 p_2 \ldots p_k}(P_1, P_2, \ldots, P_k) := \theta(I_{p_1 p_2 \ldots p_k})(P), \quad |P_i| = p_i, \quad P_1 P_2 \ldots P_k = P \tag{2.16}$$

where the map θ is defined in (A.14). For example $\mathcal{I}_{21}(12,3) = \frac{1}{2}\big(W_{123} + W_{132} - W_{213} - W_{231} + W_{312} - W_{321}\big)$. See (B.4) for the explicit permutations in $\mathcal{I}_{22}(12,34)$.

An alternative representation is proven in the appendix B

$$\mathcal{I}_{p_1 p_2 \ldots p_k}(P_1, P_2, \ldots, P_k) = \mathcal{E}(P_1) \sqcup\!\sqcup \mathcal{E}(P_2) \sqcup\!\sqcup \ldots \sqcup\!\sqcup \mathcal{E}(P_k), \quad |P_i| = p_i, \tag{2.17}$$

[2] We know from [25] that any Lie polynomial can be expanded as $\sum_\sigma M_\sigma \sigma$ with $M_{R \sqcup\!\sqcup S} = 0$ for nonempty R, S, so it follows that if Γ is a Lie polynomial then the word function $\mathcal{F}(P) := P \circ \theta(\Gamma)$ satisfies the shuffle symmetry $\mathcal{F}(R \sqcup\!\sqcup S) = 0$ for $R, S \neq \emptyset$.

from which it follows that ($\tilde{P}$ denotes the reversal[3] of P)

$$\mathcal{I}_{\dots p_i \dots}(\dots, R⧢S, \dots) = 0, \quad R, S \neq \emptyset, \quad |R| + |S| = p_i \,, \tag{2.18}$$

$$\mathcal{I}_{\dots p_i \dots p_j \dots}(\dots, P_i, \dots, P_j, \dots) = \mathcal{I}_{\dots p_j \dots p_i \dots}(\dots, P_j, \dots, P_i, \dots)\,,$$

$$\tilde{\mathcal{I}}_{p_1 \dots p_k}(P_1, \dots, P_k) = (-1)^{\#\mathrm{even}(p)} \mathcal{I}_{p_1 \dots p_k}(P_1, \dots, P_k)\,, \tag{2.19}$$

where $\#\mathrm{even}(p)$ denotes the number of even parts in the composition p. To see this note that if a function satisfies the shuffle symmetry or, in other words, belongs to the dual space of Lie polynomials [26], then $\tilde{F}(P) = (-1)^{|P|-1} F(P)$ and (2.19) follows from (2.17).

Finally, based on multiple examples we conjecture that

$$\gamma_{1|P_1,\dots,P_k} = 1.\mathcal{I}_{p_1 \dots p_k}(P_1, \dots, P_k) = 1.\mathcal{E}(P_1)⧢\mathcal{E}(P_2)⧢\dots⧢\mathcal{E}(P_k)\,, \quad p_i = |P_i|\,. \tag{2.20}$$

The shuffle symmetry of $\mathcal{E}(P_i)$ can be used to fix the first letter of P_i and the commutativity of the shuffle product implies total symmetry in word exchanges, so the basis dimension of $\gamma_{1|P_1,\dots,P_k}$ with n letters is given by the Stirling cycle numbers [27]

$$\#\big(\gamma_{1|P_1,P_2,\dots,P_k}\big) = \begin{bmatrix} n-1 \\ k \end{bmatrix}, \qquad \sum_{i=1}^{k} |P_i| = n-1\,, \tag{2.21}$$

which explains the total number of terms in P_n as $\sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} = (n-1)!$.

2.4. BRST-invariant permutations and orthogonal idempotents

Since the BRST-invariant permutations have been related to the idempotent basis of the (inverse) descent algebra in (2.20) we may construct orthogonal idempotents as in section A.3 by taking the inverse of the Reutenauer idempotents (A.32) $\gamma^{(i)}_{12\dots n} := 1.\theta\big(E^{(i)}\big)$, where the labels in $\theta\big(E^{(i)}\big)$ must be shifted as $i \to i+1$ prior to the left concatenation with the letter 1. Equivalently, from (A.30), (A.32), and (2.20) we obtain

$$\gamma^{(k)}_{12\dots n} = \sum_{23\dots n = P_1 \dots P_k} \frac{1}{k!} \gamma_{1|P_1,\dots,P_k}\,. \tag{2.22}$$

From the discussion of section A.3 it follows that (2.22) are orthogonal idempotents in the inverse descent algebra $\mathcal{D}'_n$ satisfying (δ^{ij} is the Kronecker delta)

$$\sum_{k=1}^{n-1} \gamma^{(k)}_{12\dots n} = W_{12\dots n}\,, \qquad \gamma^{(i)}_{12\dots n} \circ \gamma^{(j)}_{12\dots n} = \delta^{ij} \gamma^{(i)}_{12\dots n} \tag{2.23}$$

[3] If $Q = q_1 \dots q_n$ then its reversal is the word $\tilde{Q} := q_n \dots q_1$.

For example, from the BRST-invariant permutations in (2.6) we get

$$\gamma^{(1)}_{123} \equiv \gamma_{1|23} = \frac{1}{2}W_{123} - \frac{1}{2}W_{132}, \quad \gamma^{(2)}_{123} \equiv \frac{1}{2}\gamma_{1|2,3} = \frac{1}{2}W_{123} + \frac{1}{2}W_{132}\,, \tag{2.24}$$

which clearly satisfies (2.23). Similarly, at multiplicity four the definition (2.22)

$$\gamma^{(1)}_{1234} = \gamma_{1|234}\,, \qquad \gamma^{(2)}_{1234} = \frac{1}{2}\big(\gamma_{1|23,4} + \gamma_{1|2,34}\big)\,, \qquad \gamma^{(3)}_{1234} = \frac{1}{3!}\gamma_{1|2,3,4}\,, \tag{2.25}$$

yields

$$\begin{aligned}
\gamma^{(1)}_{1234} &= \frac{1}{3}W_{1234} - \frac{1}{6}W_{1243} - \frac{1}{6}W_{1324} - \frac{1}{6}W_{1342} - \frac{1}{6}W_{1423} + \frac{1}{3}W_{1432}\,, \\
\gamma^{(2)}_{1234} &= \frac{1}{2}W_{1234} - \frac{1}{2}W_{1432}\,, \\
\gamma^{(3)}_{1234} &= \frac{1}{6}W_{1234} + \frac{1}{6}W_{1243} + \frac{1}{6}W_{1324} + \frac{1}{6}W_{1342} + \frac{1}{6}W_{1423} + \frac{1}{6}W_{1432}\,.
\end{aligned} \tag{2.26}$$

It is straightforward but tedious [28] to check that the above satisfy (2.23). At multiplicity five, the orthogonal idempotents are given by

$$\begin{aligned}
\gamma^{(1)}_{12345} &= \gamma_{1|2345}\,, & \gamma^{(3)}_{12345} &= \frac{1}{3!}\big(\gamma_{1|23,4,5} + \gamma_{1|2,34,5} + \gamma_{1|2,3,45}\big)\,, \\
\gamma^{(2)}_{12345} &= \frac{1}{2}\big(\gamma_{1|234,5} + \gamma_{1|23,45} + \gamma_{1|2,345}\big)\,, & \gamma^{(4)}_{12345} &= \frac{1}{4!}\gamma_{1|2,3,4,5}\,,
\end{aligned} \tag{2.27}$$

whose expansions can be found in the appendix C and can be checked to obey (2.23).

3. The descent algebra and string disk amplitudes

3.1. KK-like relations of α' corrections to disk amplitudes

The color-dressed string disk amplitude

$$M_n = \sum_{\sigma\in S_{n-1}} \mathrm{Tr}\big(T^1 T^{\sigma(2)}\cdots T^{\sigma(n)}\big)\, A^{\rm string}(1,\sigma(2),\ldots,\sigma(n))\,, \tag{3.1}$$

is a sum over disk orderings of the open string color-ordered amplitude weighted by traces of Chan-Paton factors. The explicit form of the disk amplitudes is a linear combination of field-theory amplitudes $A^{\rm SYM}$ of ten-dimensional super-Yang-Mills [29] given by [30,31]

$$A^{\rm string}(P) = \sum_{Q,R\in S_{n-3}} Z(P|1,R,n,n{-}1)S[R|Q]_1 A^{\rm SYM}(1,Q,n{-}1,n) \tag{3.2}$$

where $S(P|Q)_1$ is the field-theory KLT kernel [32,33,34] conveniently computed recursively by $S[A,j|B,j,C]_i = (k_{iB} \cdot k_j) S[A|B,C]_i$, with $S[\varnothing|\varnothing]_i := 1$ [35,36,26]. In addition, $Z(P|Q)$ are the non-abelian Z-theory amplitudes of [37,31]. The color-ordered amplitudes are decomposed as

$$A^{\rm string}(1,2,\ldots,n) = A^{\rm SYM}(1,2,\ldots,n) + \sum_{m=0}^{\infty} \sum_{M} A_{\zeta_2^m \zeta_M}(1,2,\ldots,n) \tag{3.3}$$

where M runs over all MZV basis [8,38] elements[4] not containing factors of ζ_2, i.e. $\zeta_M = \{\zeta_3, \zeta_5, \zeta_7, \zeta_{3,5}, \ldots\}$. From now on we will use the shorthand *MZV amplitudes* for the $A_{\zeta_2^m \zeta_M}$ components of (3.3), and we will see below that components with the same α' but different MZV content satisfy different relations.

It is well-known that the color-ordered string disk amplitudes are cyclically symmetric and parity invariant [39]

$$\begin{aligned} A^{\rm string}(1,2,\ldots,n) &= A^{\rm string}(2,3,\ldots,n,1) \\ A^{\rm string}(1,2,\ldots,n) &= (-1)^n A^{\rm string}(n,n-1,\ldots,2,1)\,. \end{aligned} \tag{3.4}$$

These relations imply an upper bound on the number of linearly-independent amplitudes $A_{\zeta_2^m \zeta_M}(1,2,\ldots,n)$ of $\frac{1}{2}(n-1)!$. We want to know whether the different MZV amplitudes in (3.3) satisfy additional *KK-like* relations and their corresponding basis dimensions, where:

Definition (KK-like relation). *An identity of the form*

$$\sum_\sigma c_\sigma A_{\zeta_2^m \zeta_M}(\sigma) = 0, \tag{3.5}$$

where the coefficients c_σ are rational numbers is said to be KK-like [4].

For an example KK-like relation, one can verify that the ζ_2 amplitudes satisfy [20],

$$A_{\zeta_2}(1,2,3,4,5) + {\rm perm}(2,4,5) = 0\,. \tag{3.6}$$

A central observation of this paper is that the BRST-invariant permutations $\gamma_{1|P_1,\ldots,P_k}$ seem to encode all the KK-like relations among the α'-correction amplitudes. While the cyclicity relation in (3.4) was used in the definition (3.1) to fix the label in $\gamma_{1|\ldots}$, the parity

[4] This is the same organization of the motivic decomposition of the disk amplitudes [6]. The power of α' is equal to the weight of the associated MZV, e.g. both $A_{\zeta_2\zeta_3}$ and A_{ζ_5} contain ${\alpha'}^5$.

and cyclicity relations (3.4) are encoded by compositions of $n-1$ with even number of parts:

$$A^{\text{string}}(\gamma_{1|P_1,\ldots,P_k}) = 0, \quad k \text{ even}. \tag{3.7}$$

See the appendix D for a proof. The identity $\sum_{k \text{ even}} \left[{n-1 \atop k}\right] = \frac{1}{2}(n-1)!$ agrees with the counting of linearly independent amplitudes following from (3.4). Note that (3.7) implies the parity relation in (3.4) as

$$\sum_{k \text{ even}} \frac{1}{k!} A^{\text{string}}(\gamma_{1|P_1,\ldots,P_k}) = A^{\text{string}}(1,2,\ldots,n) - (-1)^n A^{\text{string}}(1,n,n-1,\ldots,2)\,. \tag{3.8}$$

Based on explicit computations, we find that MZV amplitudes with different ζ_2^m content[5] satisfy the following KK-like relations:

$$\begin{aligned} A_{\zeta_M}(\gamma_{1|P_1,\ldots,P_k}) &= 0, \quad k \neq 1, \\ A_{\zeta_2\zeta_M}(\gamma_{1|P_1,\ldots,P_k}) &= 0, \quad k \neq 3 \\ A_{\zeta_2^m\zeta_M}(\gamma_{1|P_1,\ldots,P_k}) &= 0, \quad k \neq 1,3,5,\ldots,2m+1, \quad m \geq 2, \end{aligned} \tag{3.9}$$

which constitute the *descent algebra decomposition of KK-like relations*. To count the basis dimensions implied by (3.9) we recall that $\#\big(\gamma_{1|P_1,\ldots,P_k}\big) = \left[{n-1 \atop k}\right]$ and $\sum_{k=1}^{n-1}\left[{n-1 \atop k}\right] = (n-1)!$. Thus, subtracting the dimensions of the BRST-invariant permutations from the number of cyclically symmetric n-point amplitudes[6] leads to

$$\#\big(A_{\zeta_M}(\gamma_{1|P_1,\ldots,P_k})\big) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = (n-2)!\,, \tag{3.10}$$

$$\#\big(A_{\zeta_2\zeta_M}(\gamma_{1|P_1,\ldots,P_k})\big) = \begin{bmatrix} n-1 \\ 3 \end{bmatrix}, \tag{3.11}$$

$$\#\big(A_{\zeta_2^m\zeta_M}(\gamma_{1|P_1,\ldots,P_k})\big) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + \begin{bmatrix} n-1 \\ 3 \end{bmatrix} + \cdots + \begin{bmatrix} n-1 \\ 2m+1 \end{bmatrix}. \tag{3.12}$$

The dimension (3.10) corresponds to the number of independent amplitudes under the Kleiss-Kuijf relations $A^{\text{SYM}}(P1Q,n) - (-1)^{|P|}A^{\text{SYM}}(1,\tilde{P}\sqcup Q,n) = 0$, which is also valid

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[5] The string monodromy relations [1,2] give rise to deformations of the field-theory BCJ relations by powers of $\alpha'^{2m}\zeta_2^m$ [35,5]. Since the BCJ-satisfying ζ_M components lead to the minimum $(n-3)!$ dimension, they are not expected to modify the dimensions within a given $\zeta_2^m\zeta_M$ class.

[6] Note that $\frac{1}{2}(n-1)!$ is equal to the sum over $\left[{n-1 \atop k}\right]$ with even k. Since k even is included in (3.9), the Stirling cycle numbers are subtracted from $(n-1)!$.

k	$A^{\text{string}}(\gamma_{1\|P_1,\ldots,P_k})=0$	$A^{\text{string}}(\gamma_{1\|P_1,\ldots,P_k})\neq 0$
7	$\zeta_7,\zeta_5,\zeta_3,\zeta_3^2,\zeta_2,\zeta_2\zeta_5,\zeta_2\zeta_3,\zeta_2^2,\zeta_2^2\zeta_3$	ζ_2^3
6	$\forall\zeta_M$	$\times$
5	$\zeta_7,\zeta_5,\zeta_3,\zeta_3^2,\zeta_2,\zeta_2\zeta_5,\zeta_2\zeta_3$	$\zeta_2^2,\zeta_2^2\zeta_3,\zeta_2^3$
4	$\forall\zeta_M$	$\times$
3	$\zeta_7,\zeta_5,\zeta_3,\zeta_3^2$	$\zeta_2,\zeta_2\zeta_5,\zeta_2\zeta_3,\zeta_2^2,\zeta_2^2\zeta_3,\zeta_2^3$
2	$\forall\zeta_M$	$\times$
1	$\zeta_2,\zeta_2\zeta_3,\zeta_2\zeta_5$	$\zeta_7,\zeta_5,\zeta_3,\zeta_3^2,\zeta_2^2,\zeta_2^2\zeta_3,\zeta_2^3$

Table 1. Overview of the descent algebra symmetries of higher α' corrections to string disk amplitudes of up to $n=8$ points displayed by their MZV content of weight $w\leq 7$. The entries depend only on the number of parts k of the composition of $n-1$. However, a partition with k parts cannot be probed by disk amplitudes with fewer than $k+1$ points.

for the BCJ-satisfying ζ_M corrections. The dimension (3.11) was obtained in [20] following a similar discussion of the all-plus one-loop amplitudes of [4] (see also [40]).

The basis dimension formula (3.12) and the corresponding amplitude relations in (3.9) are new. Interestingly, they imply that the basis dimension of the $\zeta_2^m\zeta_M$ components with $m\geq 2$ is given by $\frac{1}{2}(n-1)!$ only when $n\leq 2m+3$. More explicitly,

$$\#\big(A_{\zeta_2^m\zeta_M}(1,2,\ldots,n)\big)=\frac{1}{2}(n-1)!,\quad n=4,5,\ldots,2m+3\,. \tag{3.13}$$

For a deviation from the $\frac{1}{2}(n-1)!$ dimension, we have, for example with $m=2$

$$\#\big(A_{\zeta_2^2\zeta_M}(1,2,\ldots,8)\big)=\begin{bmatrix}7\\1\end{bmatrix}+\begin{bmatrix}7\\3\end{bmatrix}+\begin{bmatrix}7\\5\end{bmatrix}=2519=\frac{1}{2}7!-1\,, \tag{3.14}$$

which was confirmed by a long brute-force search using `FORM` [28]. The additional relation on top of (3.7) is seen to be $A_{\zeta_2^2}(\gamma_{1|2,3,4,5,6,7,8})=A_{\zeta_2^2}(1,2\sqcup\!\sqcup 3\sqcup\!\sqcup\ldots\sqcup\!\sqcup 8)=0$, in agreement with the analysis of [35,41]. At nine points and $m=2$, the formula (3.12) predicts a mismatch due to $\left[\begin{smallmatrix}8\\7\end{smallmatrix}\right]=28$ additional relations etc.

The evidence for the descent algebra decomposition of KK-like relations in (3.9) was collected from explicit calculations of $A_{\zeta_2^n\zeta_M}(\gamma_{1|P_1,\ldots,P_k})$ for various compositions $p=(p_1,\ldots,p_k)\models n-1$ with $|P_i|=p_i$. The results are summarized in Table 1[7]. The number of checks of $A_{\zeta_2^m\zeta_M}(\gamma_{1|P_1,\ldots,P_k})$ at n points would appear to grow exponentially, but luckily a vanishing outcome was observed to depend only on the number of parts k of the composition, independently of n (with data up to $n=8$). Thus it suffices to test the single

[7] The data in Table 1 was collected using the α' corrections to disk amplitudes obtained in [30,42,43,31] (see also [44,45,46,47] and references therein for earlier work),

case $k=n-1$ for each additional external leg. Since $\gamma_{1|2,3,\ldots,n} = 1.(2\shuffle 3\shuffle \ldots \shuffle n)$ by (2.20), we get a sum over all cyclic orderings of the n-point string disk amplitude (3.2), leading to the α'-corrected abelian Z-theory amplitudes $A^{\rm string}(\gamma_{1|2,3,\ldots,n}) \sim A^{\rm NLSM}(1,2,\ldots,n)$ of [35,41]. As a consistency check, the proof (D.1) implying that $A_{\zeta_2^m\zeta_M}(\gamma_{1|2,\ldots,k})$ vanishes when k is even (so n is odd) agrees with the vanishing of NLSM odd-point amplitudes.

3.2. The field-theory and α'^2 corrections

The SYM amplitudes are computed in pure spinor superspace from the expression M_1E_P, where E_P is a superfield satisfying the same shuffle symmetry $E_{R\shuffle S}=0$ for $R,S\neq\emptyset$ of the standard Berends-Giele current J^m_P [48,49]. The A_{ζ_2} amplitudes [50] can be computed in pure spinor superspace [20] using BRST-closed combinations of superfields $C_{1|X,Y,Z}$ symmetric under exchanges of any pairs $X\leftrightarrow Y,Z$ and satisfying $C_{1|R\shuffle S,Y,Z}=0$ for $R,S\neq\emptyset$. For convenience, define the BRST-closed combination $C_{1|P}:=M_1E_P$ so that[8]

$$A^{\rm SYM}(1,2,\ldots,n) = \langle C_{1|2\ldots n}\rangle\,, \tag{3.15}$$

$$A_{\zeta_2}(1,2,\ldots,n) = \sum_{2\ldots n}\langle C_{1|X,Y,Z}\rangle\,, \tag{3.16}$$

where $\sum_{2\ldots n}$ is a shorthand for the sum over the deconcatenations of $2\ldots n = XYZ$. In terms of these BRST invariants, the color-dressed amplitudes at four and five points can be written as [20]

$$\begin{aligned}
M_4 &= 6\zeta_2\alpha'^2 d^{1234}\,C_{1|2,3,4} \\
&\quad + i^2 d^{1a}F^{234}_a C_{1|234} + i^2 d^{1a}F^{243}_a C_{1|243} + \mathcal{O}(\alpha'^3)\,,\\
M_5 &= 6\zeta_2\alpha'^2\, i d^{1abc}\Big(F^{23}_aF^4_bF^5_c\,C_{1|23,4,5} + F^{24}_aF^3_bF^5_c\,C_{1|24,3,5} + F^{25}_aF^3_bF^4_c\,C_{1|25,3,4}\\
&\qquad + F^{2}_aF^{34}_bF^5_c\,C_{1|2,34,5} + F^{2}_aF^{35}_bF^4_c\,C_{1|2,35,4} + F^{2}_aF^{3}_bF^{45}_c\,C_{1|2,3,45}\Big)\\
&\quad + i^3d^{1a}F^{2345}_a\,C_{1|2345} + i^3d^{1a}F^{2354}_a\,C_{1|2354} + i^3d^{1a}F^{2435}_a\,C_{1|2435}\\
&\quad + i^3d^{1a}F^{2453}_a\,C_{1|2453} + i^3d^{1a}F^{2534}_a\,C_{1|2534} + i^3d^{1a}F^{2543}_a\,C_{1|2543} + \mathcal{O}(\alpha'^3)\,,
\end{aligned} \tag{3.17}$$

with similar expansions at higher points. Comparing (3.17) with the color-dressed permutations P_4 and P_5 in (2.7) and (2.9) implies the correspondences[9] at the given α' order,

$$C_{1|X} \;\longleftrightarrow\; \gamma_{1|X}\,, \qquad C_{1|X,Y,Z} \;\longleftrightarrow\; \frac{1}{6}\gamma_{1|X,Y,Z}\,. \tag{3.18}$$

[8] The angular brackets $\langle\ldots\rangle$ denotes the pure spinor zero-mode integration of [51], but it plays no role in the subsequent discussions.

[9] The first correspondence in (3.18) implies $M_1E_P\leftrightarrow 1\cdot\mathcal{E}_P$ and suggests the duality $E_P\leftrightarrow\mathcal{E}_P$ (E_P here is the superfield of [52], not the Eulerian idempotent). Since the superfield E_P is related to the Berends-Giele current M_P [52], this motivates the terminology of $\mathcal{E}_P$ in (2.12).

So the $A^{\rm SYM}$ and A_{ζ_2} amplitudes correspond to idempotents of the descent algebra

$$A^{\rm SYM}(1,2,\ldots,n) \;\longleftrightarrow\; \gamma^{(1)}_{123\ldots n}\,, \qquad A_{\zeta_2}(1,2,\ldots,n) \;\longleftrightarrow\; \gamma^{(3)}_{123\ldots n}\,. \tag{3.19}$$

where the deconcatenations (3.16) and (2.22) have been used in the last line. The correspondences in (3.19) can be used to justify the first two relations in (3.4) from the theorem 4.2 of [11], namely $E_\mu \circ I_p = 0$ if $\lambda(p) \neq \mu$, where E_μ for a partition μ is reviewed in (A.30), p is a composition and $\lambda(p)$ is its shape as defined in the beginning of section A.3. Under the dualities (3.19) this identity implies the relations[10]

$$A^{\rm SYM}\big(\gamma_{1|P_1,\ldots,P_k}\big) = 0\,, \quad k \neq 1\,, \qquad A_{\zeta_2}\big(\gamma_{1|P_1,\ldots,P_k}\big) = 0\,, \quad k \neq 3\,. \tag{3.20}$$

3.3. BRST-invariant permutations and BRST-invariant superfields

The linearity condition $M_n = A^{\rm string}(P_n)$ implies that BRST-invariant permutations in P_n are mapped to kinematics in M_n leading to a correspondence between the BRST-invariant permutations (2.4) and a series of MZV (or α') corrections from the string disk amplitude

$$\gamma_{1|P_1,P_2,\ldots,P_k} \leftrightarrow A^{\rm string}(\gamma_{1|P_1,P_2,\ldots,P_k})\,. \tag{3.21}$$

whose precise content follows from Table 1 and (3.2). For example,

$$\begin{aligned}
\gamma_{1|234} \leftrightarrow & A^{\rm SYM}(1,2,3,4) + \zeta_3 {\alpha'}^3 s_{12}s_{23}(s_{12}+s_{23})A^{\rm SYM}(1,2,3,4) + \cdots \\
\gamma_{1|2,3,4} \leftrightarrow & -6\zeta_2 {\alpha'}^2 s_{12}s_{23}A^{\rm SYM}(1,2,3,4) \\
& - 3\zeta_2^2 {\alpha'}^4 (s_{12}^3 s_{23} + s_{12}^2 s_{23}^2 + s_{12}s_{23}^3)A^{\rm SYM}(1,2,3,4) + \cdots
\end{aligned} \tag{3.22}$$

with similar expansions at higher points. The data presented in Table 1 and the discussion in section 3.2 suggest that the BRST-invariant permutations can be associated to a series of higher-mass BRST-invariant superfields by defining

$$A_{\zeta_2^m \zeta_M}(\gamma_{1|P_1,\ldots,P_k}) = k!\, C^{\zeta_2^m \zeta_M}_{1|P_1,\ldots,P_k}\,. \tag{3.23}$$

[10] To see this we use that $P\cdot(\sigma\circ\tau) = \big((P\cdot\sigma)\circ(P\cdot\tau)\big)$ if P has no common letters with σ and τ and $\theta(\sigma\circ\tau) = \theta(\tau)\circ\theta(\sigma)$ to show that the theorem 4.2 of [11] implies $(1\cdot\mathcal{I}_p)\circ(1\cdot E^\theta_\mu) = 0$ for $\lambda(p)\neq\mu$ leading to (3.20) since A_{ζ_2} corresponds to a sum of $\theta(E_\mu)$ with partitions with three parts $k(\mu)=3$ and $A^{\rm SYM}$ to $\theta(E_\mu)$ with $k(\mu)=1$.

The MZV amplitudes then follow from

$$A_{\zeta_2^m\zeta_M}(1,2,3,\ldots,n) = \sum_{k=1}^{2m+1}\sum_{23\ldots n} C^{\zeta_2^m\zeta_M}_{1|P_1,\ldots,P_k}, \tag{3.24}$$

where $\sum_{23\ldots n}$ is a shorthand notation for the deconcatenations of $P_1P_2\ldots P_k = 23\ldots n$. For example,

$$A_{\zeta_2^2}(1,2,3,4) = C^{\zeta_2^2}_{1|234} + C^{\zeta_2^2}_{1|2,3,4} = -\frac{2}{5}(s_{12}s_{23}^3 + \frac{1}{4}s_{12}^2s_{23}^2 + s_{12}^3s_{23})A^{\rm SYM}(1,2,3,4)\,, \tag{3.25}$$

where the BRST invariants $C^{\zeta_2^2}_{1|234} = 1/10(s_{12}s_{23}^3 + 4s_{12}^2s_{23}^2 + s_{12}^3s_{23})A^{\rm SYM}(1,2,3,4)$ and $C^{\zeta_2^2}_{1|2,3,4} = -1/2(s_{12}s_{23}^3+s_{12}^2s_{23}^2+s_{12}^3s_{23})A^{\rm SYM}(1,2,3,4)$ were used. The superfield representation of the above BRST invariants is known only for the two simplest cases, $C_{1|P} = M_1E_P$ [52] and $C^{\zeta_2}_{1|P_1,P_2,P_3}$ [21,53]. Note that $C_{1|P_1} = A^{\rm SYM}(1,P_1)$ while the S-map algorithm of [21] gives rise to a purely combinatorial translation between $C^{\zeta_2}_{1|P_1,P_2,P_3}$ and sums of $s_{ij}^2A^{\rm SYM}$. It remains to be seen whether there exists a general algorithm to rewrite $C^{\zeta_2^m\zeta_M}$ in terms of SYM amplitudes.

3.4. BRST invariants from A_{ζ_2}

Another consequence of the duality (3.18) is that the representation of A_{ζ_2} in terms of $C_{1|P_1,P_2,P_3}$ given in (3.16) is invertible, as argued indirectly in [20]. This follows from Theorem 4.2 of [11], $E_\mu \circ I_p = I_p$ if $\lambda(p) = \mu$ where $\lambda(p)$ is the shape of the composition p and E_μ is defined in (A.30). This implies $(1\cdot\mathcal{I}_p)\circ(1\cdot\theta(E_\mu)) = (1\cdot\mathcal{I}_p)$ for $\lambda(p)=\mu$ or, using the function interpretation of the right action $\sigma\circ F := F(\sigma)$ with a partition with three parts $k(\mu)=3$

$$A_{\zeta_2}\big(\gamma_{1|P_1,P_2,P_3}\big) = 6C_{1|P_1,P_2,P_3}\,,\quad |P_i| = p_i\,,\quad P_1P_2P_3 = 23\ldots n\,, \tag{3.26}$$

where we used the identifications (2.20), (2.22) and duality (3.19) on the left-hand side and the duality (3.18) on the right-hand side. For example, from $\gamma_{1|23,4,5}$ of (C.1) we get

$$\begin{aligned}6C_{1|23,4,5} = {}& A^{\zeta_2}_{12345} + A^{\zeta_2}_{12354} + A^{\zeta_2}_{12435} + A^{\zeta_2}_{12453} + A^{\zeta_2}_{12534} + A^{\zeta_2}_{12543}\\ &- A^{\zeta_2}_{13245} - A^{\zeta_2}_{13254} - A^{\zeta_2}_{13425} - A^{\zeta_2}_{13524} + A^{\zeta_2}_{14235} - A^{\zeta_2}_{14325}\,,\end{aligned} \tag{3.27}$$

where we used the parity relation (3.4) in the RHS.

3.5. The superfield expansion of $C_{1|P,Q,R}$ from BRST-invariant permutations

The so-called BRST invariants $C_{1|P,Q,R}$ of the pure spinor formalism [51] play an important role in the mapping between BRST-invariant permutations and kinematics, see section 3.3. They were firstly derived at low multiplicities in [20] and were subsequently studied in different contexts and given general recursive algorithms, see [21,53,54]. Their superfield expansions in terms of Berends-Giele currents follow from

$$C_{i|P,Q,R} = M_i M_{P,Q,R} + M_i \cdot \left[C_{p_1|p_2\ldots p_{|P|},Q,R} - C_{p_{|P|}|p_1\ldots p_{|P|-1},Q,R} + (P \leftrightarrow Q,R)\right] \quad (3.28)$$

starting from $C_{i|j,k,l} = M_i M_{j,k,l}$ with the dot representing concatenation, $M_i \cdot M_A := M_{iA}$. For example, the first few expansions are given by

$$\begin{aligned}
C_{1|2,3,4} &= M_1 M_{2,3,4}\,, \quad (3.29)\\
C_{1|23,4,5} &= M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}\,,\\
C_{1|234,5,6} &= M_1 M_{234,5,6} + M_{12} M_{34,5,6} + M_{123} M_{4,5,6} - M_{124} M_{3,5,6}\\
&\quad - M_{14} M_{23,5,6} - M_{142} M_{3,5,6} + M_{143} M_{2,5,6}\,,\\
C_{1|23,45,6} &= M_1 M_{23,45,6} + M_{12} M_{45,3,6} - M_{13} M_{45,2,6} + M_{14} M_{23,5,6} - M_{15} M_{23,4,6}\\
&\quad + M_{124} M_{3,5,6} - M_{134} M_{2,5,6} + M_{142} M_{3,5,6} - M_{152} M_{3,4,6}\\
&\quad - M_{125} M_{3,4,6} + M_{135} M_{2,4,6} - M_{143} M_{2,5,6} + M_{153} M_{2,4,6}\,.
\end{aligned}$$

These terms can be extracted from the permutations of the BRST-invariant permutations $\gamma_{1|P,Q,R}$ of the inverse descent algebra as follows:

1. Sum over the cyclic permutations of all permutations in $\gamma_{1|P,Q,R}$:
$$W_\sigma \to W_\sigma + \text{cyclic}(\sigma) \quad (3.30)$$
2. Decompose W_σ into all possible four-word deconcatenations:
$$W_\sigma = \sum_{XYZW=\sigma} W_X . W_Y . W_Z . W_W \quad (3.31)$$
3. Move label 1 to the front by repeatedly commuting $W_C . W_{A1B} = W_{A1B} . W_C$ if necessary and write the result in terms of Berends-Giele superfields:
$$W_{A1B} . W_C . W_D . W_E := \frac{1}{4!} M_{A1B} M_{C,D,E} \quad (3.32)$$

The resulting expressions have been explicitly checked[11] for all topologies of BRST invariants up to eight points. In addition, using the descent duality (3.18) one may also derive the change of basis identities for $C_{i\neq 1|\ldots} = \sum C_{1|\ldots}$ from [53,54] by choosing a different label to be singled-out in the color-dressed permutation (2.1) [55].

[11] The shuffle symmetry $AiB = (-1)^{|A|} i\tilde{A} \sqcup\!\sqcup B$ [23] is needed to rewrite words in a Lyndon basis.

4. Conclusion

In this paper we investigated the combinatorial properties of the permutations appearing in the *color-dressed permutations* (2.1) using the tools from the descent algebra. In particular, we defined BRST-invariant permutations, found a closed formula, and related them to orthogonal idempotents which sum to the identity permutation [12,56,11].

We then considered the color-dressed string disk amplitudes within this framework. This led to the discovery of the relations (3.9) obeyed by the α' corrections of disk amplitudes refined by their MZV content, dubbed the *descent algebra decomposition of KK-like relations*. The basis dimensions of linearly independent amplitudes at a given α' order and MZV content are given by sums of Stirling cycle numbers. These claims have been explicitly checked using various data points in string theory up to $n = 8$ and ${\alpha'}^7$.

Inspired by [20], we proposed a correspondence between the permutations from the (inverse) descent algebra and kinematics from the string disk amplitudes in terms of higher-mass BRST invariants. In the particular case of ${\alpha'}^2$, we exploited a theorem from the mathematics literature on descent algebra to prove certain claims in [20] and to systematically express BRST invariants as linear combinations of A_{ζ_2} corrections to disk amplitudes in (3.26).

And finally, we found an algorithm to extract the superfield content of the BRST invariants in the pure spinor formalism from the BRST-invariant permutations in the inverse descent algebra. It will be interesting to obtain superfield realizations of the higher-mass BRST invariants defined in section 3.3. They can probably be extracted from a perturbiner series of amplitudes at the appropriate mass level. For instance, the superfields in $C^{\rm SYM}_{1|P} = M_1 E_P$ are related to the series $\mathrm{Tr}(\mathbb{V}\mathbb{V}\mathbb{V})$ [57,29]. The superfields in $C^{\zeta_2}_{1|P_1,P_2,P_3}$ are related to the series $\mathrm{Tr}(\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}_{mn})$, while the superfields in $C^{\zeta_3}_{1|P}$ should follow from $\mathrm{Tr}\big((\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s\mathbb{W})\mathbb{F}_{mn}\mathbb{F}_{pq}\mathbb{F}_{rs}\big)$. It would also be interesting to find combinatorial algorithms to directly translate the higher-mass BRST invariants $C^{\zeta_2^m\zeta_M}_{1|P_1,\ldots,P_k}$ into linear combinations of super-Yang–Mills amplitudes and powers of Mandelstam invariants, generalizing the S-map algorithm of [21] for $C^{\zeta_2}_{1|P_1,P_2,P_3}$. This would give rise to a combinatorial description of the P_n and M_n matrices of [6].

Acknowledgements: I thank Ruggero Bandiera for discussions during an early stage of this project. I also thank Oliver Schlotterer for sharing notes containing intriguing observations about the permutations in the color-dressed amplitude that served as the motivation for this paper, for discussions, for collaboration on related topics, and for comments on the draft. CRM is supported by a University Research Fellowship from the Royal Society.

Appendix A. The Solomon descent algebra

We review the salient features of the Solomon descent algebra [10,11,12,13,14,15,16]. In particular, we discuss different bases and highlight the orthogonal idempotents discovered by Reutenauer, as they will be related to α' corrections to string amplitudes.

A.1. Descent classes and the Solomon descent algebra

The *descent set* $D(\sigma)$ and the and the *descent number* d_σ of a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ in S_n are defined by

$$D(\sigma) = \{i \in \{1,2,\ldots,n-1\} \mid \sigma_i > \sigma_{i+1}\}\,, \qquad d_\sigma = \#\big(D(\sigma)\big)\,. \tag{A.1}$$

For example, the permutation $\sigma = 546132$ has descent set $D(\sigma) = \{1,3,5\}$ and descent number $d_\sigma = 3$. The collection of permutations with a given descent set S is called a *descent class*,

$$D_S = \sum_{D(\sigma)=S} \sigma\,. \tag{A.2}$$

For example, the permutations in S_3 are distributed into four descent classes,

$$D_\emptyset = W_{123}, \quad D_{\{1\}} = W_{213} + W_{312}, \quad D_{\{2\}} = W_{132} + W_{231}, \quad D_{\{1,2\}} = W_{321}\,. \tag{A.3}$$

In general, the permutations of S_n decompose into 2^{n-1} distinct descent classes; all the subsets in the powerset of $\{1,2,\ldots,n-1\}$ since the last n-th position is never a descent. Solomon showed the remarkable property that descent classes are closed under the right action (1.10)

$$D_S \circ D_T = \sum_{U\subseteq\{1,2,\ldots,n-1\}} c_{S,T,U} D_U \tag{A.4}$$

where the coefficients $c_{S,T,U}$ are non-negative integers [10]. The descent classes therefore form a 2^{n-1} dimensional algebra, the so-called *Solomon's descent algebra* $\mathcal{D}_n$ [10,11,13,14,15,16].

As an example of (A.4), consider the permutations in S_4. Its 24 elements are organized into 8 descent classes as follows

$$\begin{aligned}
D_{\{\emptyset\}} &= W_{1234}\,, & D_{\{1,2\}} &= W_{3214} + W_{4213} + W_{4312}\,,\\
D_{\{1\}} &= W_{2134} + W_{3124} + W_{4123}\,, & D_{\{3\}} &= W_{1243} + W_{1342} + W_{2341}\,,\\
D_{\{2\}} &= W_{1324} + W_{1423} + W_{2314} + W_{2413} + W_{3412}\,, & D_{\{2,3\}} &= W_{1432} + W_{2431} + W_{3421}\,,\\
D_{\{1,3\}} &= W_{2143} + W_{3142} + W_{3241} + W_{4132} + W_{4231}\,, & D_{\{1,2,3\}} &= W_{4321}\,.
\end{aligned} \tag{A.5}$$

It is straightforward to multiply the permutations among these descent classes using the right-action of the symmetric group (1.10). For example,

$$\begin{aligned} D_{\{1\}} \circ D_{\{2\}} &= W_{1234} + W_{1243} + W_{1324} + W_{1342} + W_{1423} + W_{1432} + W_{2314} \\ &\quad + W_{2341} + W_{2413} + W_{2431} + W_{3214} + W_{3412} + W_{3421} + W_{4213} + W_{4312} \\ &= D_{\{\emptyset\}} + D_{\{1,2\}} + D_{\{2\}} + D_{\{2,3\}} + D_{\{3\}} \,, \end{aligned} \tag{A.6}$$

where the last line follows from the remarkable property (A.4) which ensures that the permutations in (A.6) are themselves a sum of descent classes.

A.2. Bases of the descent algebra

Apart from the descent classes D_S indexed by descent sets S, there are other convenient bases of the descent algebra [11].

A.2.1. Composition basis B_p

The composition p of n, denoted $p \models n$, is a k-tuple of positive integers with sum n,

$$p = (p_1, p_2, \ldots, p_k), \quad p_1 + p_2 + \cdots + p_k = n. \tag{A.7}$$

There is a bijection between compositions $p \models n$ and subsets S of $\{1, 2, \ldots, n-1\}$

$$p = (p_1, p_2, \ldots, p_k) \mapsto \{p_1,\, p_1 + p_2, \ldots,\, p_1 + p_2 + \cdots + p_{k-1}\} := S(p)\,, \tag{A.8}$$

$$S = \{i_1, i_2, \ldots, i_k\} \mapsto (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k) := C_n(S)\,. \tag{A.9}$$

Thus the total number of compositions of n is 2^{n-1}, the cardinality of the powerset of $\{1, 2, \ldots, n-1\}$. Note that the map $C_n(S) = p$ depends on the order n of the permutation group S_n for $C_3\big(\{1,2\}\big) = (1,1,1)$ but $C_4\big(\{1,2\}\big) = (1,1,2)$. In particular, $C_n(\emptyset) = (n)$.

The basis B_p is indexed by compositions p rather than subsets and is defined by [11]

$$B_p = D_{\subseteq S(p)}\,, \tag{A.10}$$

with $S(p)$ given by (A.8). For example, the D_S basis elements (A.5) become

$$\begin{aligned} B_{1111} &= D_\emptyset + D_{\{1\}} + D_{\{2\}} + D_{\{3\}} + D_{\{1,2\}} \\ &\quad + D_{\{1,3\}} + D_{\{2,3\}} + D_{\{1,2,3\}}\,, \\ B_{112} &= D_\emptyset + D_{\{1\}} + D_{\{2\}} + D_{\{1,2\}}\,, \\ B_{121} &= D_\emptyset + D_{\{1\}} + D_{\{3\}} + D_{\{1,3\}}\,, \\ B_{211} &= D_\emptyset + D_{\{2\}} + D_{\{3\}} + D_{\{2,3\}}\,, \end{aligned} \qquad \begin{aligned} B_{13} &= D_\emptyset + D_{\{1\}}\,, \\ B_{22} &= D_\emptyset + D_{\{2\}}\,, \\ B_{31} &= D_\emptyset + D_{\{3\}}\,, \\ B_4 &= D_\emptyset\,. \end{aligned} \tag{A.11}$$

The inverse of (A.10) is given by Lemma 8.18 in [16]

$$D_S = \sum_{T\subseteq S} (-1)^{|S|-|T|} D_{\subseteq T}\,. \tag{A.12}$$

For example (in S_4), $D_{\{1,2\}} = B_{112} - B_{13} - B_{22} + B_4$, $D_{\{1\}} = B_{13} - B_4$, $D_{\{2\}} = B_{22} - B_4$, and $D_\emptyset = B_4$, from which we verify that $D_\emptyset + D_{\{1\}} + D_{\{2\}} + D_{\{1,2\}} = B_{112}$.

The permutations within a basis element B_p can be found via [56,11]

$$B_{p_1p_2\ldots p_k} = \theta(X_1 \sqcup\!\sqcup X_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup X_k)\,, \quad 12\ldots n = X_1\ldots X_k, \quad |X_i| = p_i\,. \tag{A.13}$$

where the inverse map θ is given by

$$\theta(\sigma) \mapsto \sigma^{-1}\,. \tag{A.14}$$

For example, if $p = (1,1,2)$ then $X_1 = 1$, $X_2 = 2$ and $X_3 = 34$ and we get

$$\begin{aligned} B_{112} = \theta(1\sqcup\!\sqcup 2 \sqcup\!\sqcup 34) &= W_{1234} + W_{1324} + W_{1423} + W_{2134} + W_{2314} + W_{2413} \\ &\quad + W_{3124} + W_{3214} + W_{3412} + W_{4123} + W_{4213} + W_{4312}\,. \end{aligned} \tag{A.15}$$

A.2.2. Multiplication table for $B_p \circ B_q$

There is a closed formula for the multiplication of $B_p \circ B_q$ [14,11,58]. Let M be a matrix with non-negative integer entries m_{ij} whose *row sum* $r(M)$ and *column sum* $c(M)$ are vectors defined by

$$r(M)_i := \sum_j m_{ij}\,, \qquad c(M)_j := \sum_i m_{ij}\,. \tag{A.16}$$

Then

$$B_p \circ B_q = \sum_{\substack{c(M)=p \\ r(M)=q}} B_{\mathrm{co}(M)} \tag{A.17}$$

where $\mathrm{co}(M)$ denotes the composition obtained by reading the matrix M row by row from top to bottom while excluding the zero entries $m_{ij} = 0$. This product is associative and B_n is a multiplicative identity for compositions of n [58].

For example, let us recover the result (A.6) for $D_{\{1\}} \circ D_{\{2\}}$ using the above multiplication table (A.17) in S_4. Given that $D_{\{1\}} = B_{13} - B_4$ and $D_{\{2\}} = B_{22} - B_4$, the only non-trivial product we need is $B_{13} \circ B_{22}$ since B_4 is the identity for compositions of $n = 4$. The set of integer matrices M with $c(M) = (1,3)$ and $r(M) = (2,2)$ is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}. \tag{A.18}$$

Thus $B_{13} \circ B_{22} = B_{112} + B_{211}$ and $D_{\{1\}} \circ D_{\{2\}} = (B_{13} - B_4) \circ (B_{22} - B_4)$ implies

$$D_{\{1\}} \circ D_{\{2\}} = B_{112} + B_{211} - B_{13} - B_{22} + B_4 = D_\emptyset + D_{\{1,2\}} + D_{\{2\}} + D_{\{2,3\}} + D_{\{3\}} \tag{A.19}$$

where we used the conversions (A.11).

A.2.3. The Eulerian idempotent

The Eulerian (or Solomon) idempotent is defined by [19,12,56,59] (see also [60])

$$E_n = \sum_{\sigma\in S_n} \kappa_\sigma \sigma, \qquad \kappa_\sigma = \frac{(-1)^{d_\sigma}}{|\sigma|\binom{|\sigma|-1}{d_\sigma}} \tag{A.20}$$

where d_σ denotes the descent number (A.1) of the permutation σ. For example,

$$E_2 = \frac{1}{2}\big(W_{12} - W_{21}\big), \quad E_3 = \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321}\,. \tag{A.21}$$

Apart from being an idempotent satisfying $E_n \circ E_n = E_n$, the definition (A.20) is also a Lie polynomial [12]. Therefore its coefficients κ_σ must satisfy the shuffle symmetry [25]

$$\kappa_{R \sqcup\!\sqcup S} = 0, \quad R, S \neq \emptyset\,. \tag{A.22}$$

As usual, the definition (A.20) in terms of the fixed alphabet $\mathbb{N}$ in S_n can be turned into a *function* of an arbitrary word P by the right action (1.10) of the symmetric group [16,61],

$$E(P) = E^P := P \circ E_n, \quad |P| = n\,. \tag{A.23}$$

For example, $E(i,j,k) = ijk \circ E_3 = \frac{1}{3}W_{ijk} - \frac{1}{6}W_{ikj} - \frac{1}{6}W_{jik} - \frac{1}{6}W_{jki} - \frac{1}{6}W_{kij} + \frac{1}{3}W_{kji}$.

A.2.4. The idempotent basis I_p

The idempotent basis I_p of the descent algebra $\mathcal{D}_n$ satisfying $I_p \circ I_p = I_p$ was introduced in [11] and it is indexed by the compositions of n

$$I_{p_1 p_2 \ldots p_k}(P) = \sum_{\substack{X_1,\ldots,X_k \\ |X_i| = p_i}} \langle P, X_1 \sqcup\!\sqcup X_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup X_k \rangle E^{X_1} E^{X_2} \ldots E^{X_k}\,, \tag{A.24}$$

where the sum is constrained by the length of X_i being equal to the corresponding p_i in the composition p and E^{X_i} denote the Eulerian idempotent function (A.23). For example, with canonical $P = 12\ldots n$ we have

$$\begin{aligned}
I_{11} &= W_{12} + W_{21}\,, \qquad I_2 = \frac{1}{2}(W_{12} - W_{21})\,, \\
I_{111} &= W_{123} + W_{132} + W_{213} + W_{231} + W_{312} + W_{321}\,, \\
I_{21} &= \frac{1}{2}W_{123} + \frac{1}{2}W_{132} - \frac{1}{2}W_{213} + \frac{1}{2}W_{231} - \frac{1}{2}W_{312} - \frac{1}{2}W_{321}\,, \\
I_{12} &= \frac{1}{2}W_{123} - \frac{1}{2}W_{132} + \frac{1}{2}W_{213} - \frac{1}{2}W_{231} + \frac{1}{2}W_{312} - \frac{1}{2}W_{321}\,, \\
I_3 &= \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321}\,.
\end{aligned} \tag{A.25}$$

A.2.5. I_p to B_p

The idempotent basis elements I_p for $p = p_1 p_2 \dots p_k$ can be expanded in terms of compositions B_q using an algorithm discussed in [11]. First one defines *moments* e_m as a polynomial in non-commuting variables t_i for $i = 1, 2, \dots$ from the generating series

$$\sum x^n e_n = \log(1 + \sum t_i x^i) \tag{A.26}$$

where x is a commuting parameter. For example, from (A.26) it follows that

$$\begin{aligned} &e_1 = t_1, \quad e_2 = t_2 - \frac{1}{2}t_1^2, \quad e_3 = t_3 - \frac{1}{2}(t_1 t_2 + t_2 t_1) + \frac{1}{3}t_1^3 \\ &e_4 = -\frac{1}{4}t_1^4 + \frac{1}{3}t_1^2 t_2 + \frac{1}{3}t_1 t_2 t_1 - \frac{1}{2}t_1 t_3 + \frac{1}{3}t_2 t_1^2 - \frac{1}{2}t_2^2 - \frac{1}{2}t_3 t_1 + t_4 \end{aligned} \tag{A.27}$$

Then to convert the I_p basis elements to the composition basis B_q one uses [11]

$$I_p = \delta(e_{p_1} e_{p_2} \dots e_{p_k}), \text{ with } \delta(t_{i_1} t_{i_2} \dots t_{i_k}) := B_{i_1 i_2 \dots i_k}. \tag{A.28}$$

For example,

$$I_4 = -\frac{1}{4}B_{1111} + \frac{1}{3}B_{112} + \frac{1}{3}B_{121} - \frac{1}{2}B_{13} + \frac{1}{3}B_{211} - \frac{1}{2}B_{22} - \frac{1}{2}B_{31} + B_4\,. \tag{A.29}$$

A.3. Reutenauer orthogonal idempotents

A partition λ of n, denoted $\lambda \vdash n$, is a k-tuple of positive integers with sum n satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. If $p \models n$ is a composition of n, the *shape* $\lambda(p)$ of p is the partition of n obtained by rearranging the parts of p in decreasing order. Also, $k(p)$ is the *number of parts* of the composition p. For example, $p = (2, 3, 1, 2)$ implies $\lambda(p) = 3221$ and $k(p) = 4$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ into k parts, theorem 3.1 of [11] shows that

$$E_\lambda := \frac{1}{k!} \sum_{\lambda(p)=\lambda} I_p\,, \qquad \sum_{\lambda \vdash n} E_\lambda = W_{12\dots n}\,. \tag{A.30}$$

Note that when the partition λ of n has only one part, $E_\lambda = I_n$ coincides with the Eulerian idempotent E_n (A.20), so this notation is not ambiguous. For example, $E_1 = I_1$ and

$$\begin{aligned} &E_2 = I_2\,, \qquad E_3 = I_3\,, \qquad E_{111} = \frac{1}{3!}I_{111}\,, \\ &E_{11} = \frac{1}{2}I_{11}\,, \qquad E_{21} = \frac{1}{2}\big(I_{12} + I_{21}\big)\,, \qquad E_{211} = \frac{1}{3!}\big(I_{112} + I_{121} + I_{211}\big)\,. \end{aligned} \tag{A.31}$$

one can readily verify $E_3 + E_{21} + E_{111} = W_{123}$ using the expansions listed in the appendix C.

The Reutenauer idempotents $E^{(m)}$ are defined in the alphabet $\{1,2,\ldots\}$ as the sum over all permutations of E_λ from (A.30) such that λ is a partition of n with m parts, i.e.,

$$E^{(m)} = \sum_{\substack{\lambda\vdash n\\ k(\lambda)=m}} E_\lambda \tag{A.32}$$

For example,

$$\begin{aligned}
&n=2 \qquad E^{(1)}=E_2, \quad E^{(2)}=E_{11}\\
&n=3 \qquad E^{(1)}=E_3, \quad E^{(2)}=E_{21}, \quad E^{(3)}=E_{111}\\
&n=4 \qquad E^{(1)}=E_4, \quad E^{(2)}=E_{31}+E_{22}, \quad E^{(3)}=E_{211}, \quad E^{(4)}=E_{1111}
\end{aligned} \tag{A.33}$$

It was shown in [11,12] that (A.32) are orthogonal idempotents which sum to the identity permutation

$$\sum_{i=1}^{n} E^{(i)} = W_{123\ldots n}\,, \qquad E^{(i)}\circ E^{(j)} = \begin{cases} E^{(i)} & \text{if } i=j;\\ 0 & \text{otherwise.}\end{cases} \tag{A.34}$$

An alternative definition of the Reutenauer idempotents in terms of a generating function can be found in [16].

Appendix B. The inverse of the idempotent basis

In this appendix we will prove (2.17), that is:

Proposition. *The inverse of the idempotent basis (2.16) satisfies*

$$\mathcal{I}_{p_1p_2\ldots p_k}(P_1,P_2,\ldots,P_k) = \mathcal{E}(P_1)\sqcup\mathcal{E}(P_2)\sqcup\ldots\sqcup\mathcal{E}(P_k)\,, \quad |P_i| = p_i \tag{B.1}$$

where $P = P_1\ldots P_k$ is the factorization of P with P_i of length p_i.

Proof. The proof will be based on the following observations collected from [16], which should be consulted for more details as the equation numbers below refer to it. First, the adjoint of an arbitrary function $F(P) = P\circ F$ of a word P is given by $\theta(F)(P) = P\circ\theta(F)$, see (3.3.5). Second, the adjoint of $F_{p_1}\star F_{p_2}\ldots\star F_{p_k}$ is $\theta(F_{p_1})\star'\theta(F_{p_2})\ldots\star'\theta(F_{p_k})$ where $\star$ and $\star'$ are the convolution operators defined in (1.5.7) and (1.5.8) and $\theta(F_j)$ is the adjoint of F_j when viewed as a function by the right-action (1.10), see proof of Lemma 3.13. Third, for permutations F_{p_i} of length p_i one can show (by adapting the proof of Lemma 3.13)

$$\big(F_{p_1}\star'\ldots\star' F_{p_k}\big)(P) = F_{p_1}(P_1)\sqcup\ldots\sqcup F_{p_k}(P_k) \tag{B.2}$$

where the functions are defined via a right action as $F_{p_i}(P_i) := P_i \circ F_{p_i}$. The proof of (B.1) then follows from the observation by (1.5.4) and (1.5.7) that the idempotent basis I_p (A.24) can be rewritten as a convolution $I_{p_1\ldots p_k}(P) = \big(E_{p_1} \star \ldots \star E_{p_k}\big)\big(P\big)$ where E_p is the Eulerian idempotent (A.20). Therefore its adjoint $\theta(I_{p_1\ldots p_k})(P)$ is given by

$$\begin{aligned}\theta(I_{p_1\ldots p_k})(P) &= \theta(E_{p_1})(P_1) \star' \ldots \star' \theta(E_{p_k})(P_k), \quad P = P_1 \ldots P_k, \quad |P_i| = p_i \\ &= \mathcal{E}(P_1) ⧢ \ldots ⧢ \mathcal{E}(P_k) \end{aligned} \tag{B.3}$$

where we used (B.2) and $\mathcal{E}(P_i) = \theta(E_{p_i})(P_i)$. □

As a multiplicity-four example of the inverse idempotent basis of (2.16) we have

$$\begin{aligned}\mathcal{I}_{22}(12,34) = {} & \frac{1}{4}W_{1234} - \frac{1}{4}W_{1243} + \frac{1}{4}W_{1324} + \frac{1}{4}W_{1342} - \frac{1}{4}W_{1423} - \frac{1}{4}W_{1432} \\ & - \frac{1}{4}W_{2134} + \frac{1}{4}W_{2143} - \frac{1}{4}W_{2314} - \frac{1}{4}W_{2341} + \frac{1}{4}W_{2413} + \frac{1}{4}W_{2431} \\ & + \frac{1}{4}W_{3124} + \frac{1}{4}W_{3142} - \frac{1}{4}W_{3214} - \frac{1}{4}W_{3241} + \frac{1}{4}W_{3412} - \frac{1}{4}W_{3421} \\ & - \frac{1}{4}W_{4123} - \frac{1}{4}W_{4132} + \frac{1}{4}W_{4213} + \frac{1}{4}W_{4231} - \frac{1}{4}W_{4312} + \frac{1}{4}W_{4321} \,. \end{aligned} \tag{B.4}$$

Appendix C. Explicit permutations at low multiplicities

The multiplicity-five BRST-invariant permutations (defined in (2.4)) are given by

$$\begin{aligned}\gamma_{1|2,3,4,5} &= W_{1(2⧢3⧢4⧢5)} \\ \gamma_{1|23,4,5} &= \frac{1}{2}W_{12345} + \frac{1}{2}W_{12354} + \frac{1}{2}W_{12435} + \frac{1}{2}W_{12453} + \frac{1}{2}W_{12534} + \frac{1}{2}W_{12543} \\ & - \frac{1}{2}W_{13245} - \frac{1}{2}W_{13254} - \frac{1}{2}W_{13425} - \frac{1}{2}W_{13452} - \frac{1}{2}W_{13524} - \frac{1}{2}W_{13542} \\ & + \frac{1}{2}W_{14235} + \frac{1}{2}W_{14253} - \frac{1}{2}W_{14325} - \frac{1}{2}W_{14352} + \frac{1}{2}W_{14523} - \frac{1}{2}W_{14532} \\ & + \frac{1}{2}W_{15234} + \frac{1}{2}W_{15243} - \frac{1}{2}W_{15324} - \frac{1}{2}W_{15342} + \frac{1}{2}W_{15423} - \frac{1}{2}W_{15432} \\ \gamma_{1|234,5} &= \frac{1}{3}W_{12345} + \frac{1}{3}W_{12354} - \frac{1}{6}W_{12435} - \frac{1}{6}W_{12453} + \frac{1}{3}W_{12534} - \frac{1}{6}W_{12543} \\ & - \frac{1}{6}W_{13245} - \frac{1}{6}W_{13254} - \frac{1}{6}W_{13425} - \frac{1}{6}W_{13452} - \frac{1}{6}W_{13524} - \frac{1}{6}W_{13542} \\ & - \frac{1}{6}W_{14235} - \frac{1}{6}W_{14253} + \frac{1}{3}W_{14325} + \frac{1}{3}W_{14352} - \frac{1}{6}W_{14523} + \frac{1}{3}W_{14532} \\ & + \frac{1}{3}W_{15234} - \frac{1}{6}W_{15243} - \frac{1}{6}W_{15324} - \frac{1}{6}W_{15342} - \frac{1}{6}W_{15423} + \frac{1}{3}W_{15432} \end{aligned} \tag{C.1}$$

$$
\begin{aligned}
\gamma_{1|23,45} = {} & \frac{1}{4}W_{12345} - \frac{1}{4}W_{12354} + \frac{1}{4}W_{12435} + \frac{1}{4}W_{12453} - \frac{1}{4}W_{12534} - \frac{1}{4}W_{12543} \\
& - \frac{1}{4}W_{13245} + \frac{1}{4}W_{13254} - \frac{1}{4}W_{13425} - \frac{1}{4}W_{13452} + \frac{1}{4}W_{13524} + \frac{1}{4}W_{13542} \\
& + \frac{1}{4}W_{14235} + \frac{1}{4}W_{14253} - \frac{1}{4}W_{14325} - \frac{1}{4}W_{14352} + \frac{1}{4}W_{14523} - \frac{1}{4}W_{14532} \\
& - \frac{1}{4}W_{15234} - \frac{1}{4}W_{15243} + \frac{1}{4}W_{15324} + \frac{1}{4}W_{15342} - \frac{1}{4}W_{15423} + \frac{1}{4}W_{15432} \\
\gamma_{1|2345} = {} & \frac{1}{4}W_{12345} - \frac{1}{12}W_{12354} - \frac{1}{12}W_{12435} - \frac{1}{12}W_{12453} - \frac{1}{12}W_{12534} + \frac{1}{12}W_{12543} \\
& - \frac{1}{12}W_{13245} + \frac{1}{12}W_{13254} - \frac{1}{12}W_{13425} - \frac{1}{12}W_{13452} + \frac{1}{12}W_{13524} + \frac{1}{12}W_{13542} \\
& - \frac{1}{12}W_{14235} - \frac{1}{12}W_{14253} + \frac{1}{12}W_{14325} + \frac{1}{12}W_{14352} - \frac{1}{12}W_{14523} + \frac{1}{12}W_{14532} \\
& - \frac{1}{12}W_{15234} + \frac{1}{12}W_{15243} + \frac{1}{12}W_{15324} + \frac{1}{12}W_{15342} + \frac{1}{12}W_{15423} - \frac{1}{4}W_{15432}
\end{aligned}
$$

According to the deconcatenation (2.22) these BRST-invariant permutations give rise to the following orthogonal idempotents:

$$
\begin{aligned}
\gamma^{(1)}_{12345} = {} & \frac{1}{4}W_{12345} - \frac{1}{12}W_{12354} - \frac{1}{12}W_{12435} - \frac{1}{12}W_{12453} - \frac{1}{12}W_{12534} + \frac{1}{12}W_{12543} \\
& - \frac{1}{12}W_{13245} + \frac{1}{12}W_{13254} - \frac{1}{12}W_{13425} - \frac{1}{12}W_{13452} + \frac{1}{12}W_{13524} + \frac{1}{12}W_{13542} \\
& - \frac{1}{12}W_{14235} - \frac{1}{12}W_{14253} + \frac{1}{12}W_{14325} + \frac{1}{12}W_{14352} - \frac{1}{12}W_{14523} + \frac{1}{12}W_{14532} \\
& - \frac{1}{12}W_{15234} + \frac{1}{12}W_{15243} + \frac{1}{12}W_{15324} + \frac{1}{12}W_{15342} + \frac{1}{12}W_{15423} - \frac{1}{4}W_{15432} \\
\gamma^{(2)}_{12345} = {} & \frac{11}{24}W_{12345} - \frac{1}{24}W_{12354} - \frac{1}{24}W_{12435} - \frac{1}{24}W_{12453} - \frac{1}{24}W_{12534} - \frac{1}{24}W_{12543} \\
& - \frac{1}{24}W_{13245} - \frac{1}{24}W_{13254} - \frac{1}{24}W_{13425} - \frac{1}{24}W_{13452} - \frac{1}{24}W_{13524} - \frac{1}{24}W_{13542} \\
& - \frac{1}{24}W_{14235} - \frac{1}{24}W_{14253} - \frac{1}{24}W_{14325} - \frac{1}{24}W_{14352} - \frac{1}{24}W_{14523} - \frac{1}{24}W_{14532} \\
& - \frac{1}{24}W_{15234} - \frac{1}{24}W_{15243} - \frac{1}{24}W_{15324} - \frac{1}{24}W_{15342} - \frac{1}{24}W_{15423} + \frac{11}{24}W_{15432} \\
\gamma^{(3)}_{12345} = {} & \frac{1}{4}W_{12345} + \frac{1}{12}W_{12354} + \frac{1}{12}W_{12435} + \frac{1}{12}W_{12453} + \frac{1}{12}W_{12534} - \frac{1}{12}W_{12543} \\
& + \frac{1}{12}W_{13245} - \frac{1}{12}W_{13254} + \frac{1}{12}W_{13425} + \frac{1}{12}W_{13452} - \frac{1}{12}W_{13524} - \frac{1}{12}W_{13542} \\
& + \frac{1}{12}W_{14235} + \frac{1}{12}W_{14253} - \frac{1}{12}W_{14325} - \frac{1}{12}W_{14352} + \frac{1}{12}W_{14523} - \frac{1}{12}W_{14532} \\
& + \frac{1}{12}W_{15234} - \frac{1}{12}W_{15243} - \frac{1}{12}W_{15324} - \frac{1}{12}W_{15342} - \frac{1}{12}W_{15423} - \frac{1}{4}W_{15432} \\
\gamma^{(4)}_{12345} = {} & \frac{1}{24}W_{1(2\shuffle 3\shuffle 4\shuffle 5)}
\end{aligned}
$$

Here we list the first few expansions of E_λ defined in (A.30):

$$
\begin{aligned}
E_2 &= \frac{1}{2}W_{12} - \frac{1}{2}W_{21}\,, \qquad E_{11} = \frac{1}{2}W_{12} + \frac{1}{2}W_{21} \\
E_3 &= \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321} \\
E_{21} &= \frac{1}{2}W_{123} - \frac{1}{2}W_{321}\,, \qquad E_{111} = \frac{1}{6}W_{123} + \mathrm{perm}(1,2,3) \\
E_{211} &= \frac{1}{4}W_{1234} + \frac{1}{12}W_{1243} + \frac{1}{12}W_{1324} + \frac{1}{12}W_{1342} + \frac{1}{12}W_{1423} - \frac{1}{12}W_{1432} \\
&+ \frac{1}{12}W_{2134} - \frac{1}{12}W_{2143} + \frac{1}{12}W_{2314} + \frac{1}{12}W_{2341} + \frac{1}{12}W_{2413} - \frac{1}{12}W_{2431} \\
&+ \frac{1}{12}W_{3124} - \frac{1}{12}W_{3142} - \frac{1}{12}W_{3214} - \frac{1}{12}W_{3241} + \frac{1}{12}W_{3412} - \frac{1}{12}W_{3421} \\
&+ \frac{1}{12}W_{4123} - \frac{1}{12}W_{4132} - \frac{1}{12}W_{4213} - \frac{1}{12}W_{4231} - \frac{1}{12}W_{4312} - \frac{1}{4}W_{4321}
\end{aligned}
\tag{C.2}
$$

C.1. The Berends-Giele idempotents

The Berends-Giele idempotents $\mathcal{E}(P)$ are defined in section 2.2 as the inverse $\theta(E(P))$ of the Eulerian idempotent (A.20). Their expansions up to multiplicity three were given in (2.13) and now we write down the multiplicity four:

$$
\begin{aligned}
\mathcal{E}(1234) &= \frac{1}{4}W_{1234} - \frac{1}{12}W_{1243} - \frac{1}{12}W_{1324} - \frac{1}{12}W_{1342} - \frac{1}{12}W_{1423} + \frac{1}{12}W_{1432} \\
&- \frac{1}{12}W_{2134} + \frac{1}{12}W_{2143} - \frac{1}{12}W_{2314} - \frac{1}{12}W_{2341} + \frac{1}{12}W_{2413} + \frac{1}{12}W_{2431} \\
&- \frac{1}{12}W_{3124} - \frac{1}{12}W_{3142} + \frac{1}{12}W_{3214} + \frac{1}{12}W_{3241} - \frac{1}{12}W_{3412} + \frac{1}{12}W_{3421} \\
&- \frac{1}{12}W_{4123} + \frac{1}{12}W_{4132} + \frac{1}{12}W_{4213} + \frac{1}{12}W_{4231} + \frac{1}{12}W_{4312} - \frac{1}{4}W_{4321}
\end{aligned}
\tag{C.3}
$$

As a curiosity, noting that $E_4 = I_4$ one can derive these permutations using the conversion (A.28) together with (A.13) for the permutations in $\theta(B_p)$ (note $\theta^2 = 1$). So (A.29) yields the permutations in $\mathcal{E}_{1234} = \theta(I_4)$ as

$$
\begin{aligned}
\mathcal{E}(1234) = &-\frac{1}{4}1\sqcup\!\sqcup 2\sqcup\!\sqcup 3\sqcup\!\sqcup 4 + \frac{1}{3}1\sqcup\!\sqcup 2\sqcup\!\sqcup 34 + \frac{1}{3}1\sqcup\!\sqcup 23\sqcup\!\sqcup 4 - \frac{1}{2}1\sqcup\!\sqcup 234 \\
&+ \frac{1}{3}12\sqcup\!\sqcup 3\sqcup\!\sqcup 4 - \frac{1}{2}12\sqcup\!\sqcup 34 - \frac{1}{2}123\sqcup\!\sqcup 4 + 1234\,.
\end{aligned}
\tag{C.4}
$$

Appendix D. Parity of the disk amplitude and even partitions

Parity of the amplitude $A^{\rm string}(1,\ldots,n) = (-1)^n A^{\rm string}(n,\ldots,1)$ explains the vanishing of $A_{\zeta_2^m\zeta_M}(\gamma_{1|P_1,\ldots,P_k})$ for even k as observed in Table 1. A quick counting argument suggests why this is so as $\sum_k \left[{n-1 \atop 2k}\right] = \frac{1}{2}(n-1)!$ is the upper bound in the dimension of string disk amplitudes from properties of the string worldsheet alone [2]. More precisely:

Proposition. *If k is even then the n-point disk amplitude satisfies*

$$A^{\text{string}}(\gamma_{1|P_1,\ldots,P_k}) = 0, \tag{D.1}$$

where $\gamma_{1|P_1,\ldots,P_k}$ is the BRST-invariant permutation (2.4).

Proof. The parity of A^{string} at n points can be written as $A^{\text{string}}(1,\sigma) = (-1)^n A^{\text{string}}(1,\tilde{\sigma})$ by cyclicity. This means, by (2.20), that $A^{\text{string}}(\gamma_{1|P_1,\ldots,P_k})$ will vanish whenever the parity of A^{string} at n points is opposite to the parity of $\mathcal{I}_p$ for $p \models n-1$. To see why this is true consider the example of $A^{\text{string}}(\gamma_{1|23,4})$ with the expression for the BRST-invariant permutation in (2.8). The terms can be rearranged as

$$A^{\text{string}}(\gamma_{1|23,4}) = \frac{1}{2}\big(A^{\text{string}}_{1234} - A^{\text{string}}_{1432}\big) + \frac{1}{2}\big(A^{\text{string}}_{1243} - A^{\text{string}}_{1342}\big) + \frac{1}{2}\big(A^{\text{string}}_{1423} - A^{\text{string}}_{1324}\big) = 0 \tag{D.2}$$

which vanishes by parity $A^{\text{string}}_{1234} = A^{\text{string}}_{1432}$. Notice that this happens because the parity of $\mathcal{I}_{21}(23,4)$ from (2.19) is the opposite of the string disk amplitude; $\tilde{\mathcal{I}}_{21}(23,4) = -\mathcal{I}_{21}(23,4)$. The proposition can now be proven by considering the two cases when n is even or odd.

For n even the parity of the n-point disk amplitude is $+$ so $A^{\text{string}}(\gamma_{1|P_1,\ldots,P_k})$ vanishes if the parity of $\mathcal{I}_p$ is $-$ for a composition p of $n-1$. By (2.19) this means that there must be an odd number of even parts in the composition p (which sum to even). But since $n-1$ is odd, there must be an odd number of odd parts in p (which sum to odd). Therefore the number of parts $k(p)$ is even ($=$ odd $+$ odd). Similarly, when n is odd the number of parts $k(p)$ in the composition of p is also even (from even $+$ even). This finishes the proof. □

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