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Dyson's Equations for Quantum Gravity in the Hartree–Fock Approximation

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Abstract: Unlike scalar and gauge field theories in four dimensions, gravity is not perturbatively renormalizable and as a result perturbation theory is badly divergent. Often the method of choice for investigating nonperturbative effects has been the lattice formulation, and in the case of gravity the Regge–Wheeler lattice path integral lends itself well for that purpose. Nevertheless, lattice methods ultimately rely on extensive numerical calculations, leaving a desire for alternate methods that can be pursued analytically. In this work, we outline the Hartree–Fock approximation to quantum gravity, along lines which are analogous to what is done for scalar fields and gauge theories. The starting point is Dyson's equations, a closed set of integral equations which relate various physical amplitudes involving graviton propagators, vertex functions, and proper self-energies. Such equations are in general difficult to solve, and as a result they are not very useful in practice, but nevertheless provide a basis for subsequent approximations. This is where the Hartree–Fock approximation comes in, whereby lowest order diagrams get partially dressed by the use of fully interacting Green's function and self-energies, which then lead to a set of self-consistent integral equations. The resulting nonlinear equations for the graviton self-energy show some remarkable features that clearly distinguish it from the scalar and gauge theory cases. Specifically, for quantum gravity one finds a nontrivial ultraviolet fixed point in Newton's constant G for spacetime dimensions greater than two, and nontrivial scaling dimensions between $d = 2$ and $d = 4$, above which one obtains Gaussian exponents. In addition, the Hartree–Fock approximation gives an explicit analytic expression for the renormalization group running of Newton's constant, suggesting gravitational antiscreening with Newton's constant slowly increasing on cosmological scales.

Keywords: quantum gravity; quantum cosmology; quantum field theory

1. Introduction

The traditional approach to quantum field theory is generally based on Feynman diagrams, which involve a perturbative expansion in some suitable small coupling constant. In cases like QED this procedure works rather well, as the vacuum about which one is expanding is rather close, in its physical features, to the physical vacuum (weakly coupled electrons and photons). In other cases, such as QCD, subtleties arise because of effects which are non-analytic in the coupling constant and cannot therefore be seen to any order in perturbation theory. Indeed, in QCD it is well known that the perturbative ground state describes free quarks and gluons, and as result the fundamental property of asymptotic freedom is easily derived in perturbation theory. Nevertheless, important features such as gluon condensation, quark confinement, and chiral symmetry breaking remain invisible to any order in perturbation theory. However, the fact remains that the formulation of quantum field theory based on the Feynman path integral is generally not linked in any way to a perturbative expansion in terms of diagrams. As a result, the path integral generally makes sense even in a genuinely nonperturbative context, provided that it is properly defined and regulated, say via a spacetime lattice and a suitable Wick rotation. Already at the level of nonrelativistic quantum mechanics, examples abound of physical system for which perturbation theory entirely fails to capture the essential properties of

the model. Leading to the reason why well-established nonperturbative methods such as the variational trial wave function method, numerical methods, or the Hartree–Fock (HF) self-consistent method were developed early on in the history of quantum mechanics, motivated by the need to go beyond the—often uncertain—predictions of naive perturbation theory. In addition, many of these genuinely nonperturbative methods are relatively easy to implement, and in many cases lead to significantly improved physical predictions in atomic, molecular, and nuclear physics. Going one step further, examples abound also in relativistic quantum field theory and many-body theory, where naive perturbation theory clearly fails to give the correct answer. Interesting, and physically very relevant, cases of the latter include the theory of superconductivity, superfluidity, screening in a Coulomb gas, turbulent fluid flow, and generally many important aspects of critical (or cooperative) phenomena in statistical physics. A common thread within many of these widely different systems is the emergence of coherent quantum mechanical behavior, which is not easily revealed by the use of perturbation theory about some non-interacting ground state. However, some of the approximate methods mentioned above seem, at first, to be intimately tied to the existence of a Hamiltonian formulation (as in the case of the variational method), and are thus not readily available if one focuses on the Feynman path integral approach to quantum field theory.

Nevertheless, it is possible, in some cases, to cleverly modify perturbation theory in such a way as to make the diagrammatic perturbative approach still viable and useful. One such method was pioneered by Wilson, who suggested that in some theories a double expansion in both the coupling constant and the dimensionality (the so-called ϵ-expansion) could be performed [1–3]. These expansion methods turn out to be quite powerful, but they also generally lead to series that are known to be asymptotic, and in some cases (such as the nonlinear sigma model) can have rather poor convergence properties, mainly due to the existence of renormalon singularities along the positive real Borel transform axis. Nevertheless, these investigations lead eventually to the deep insight that perturbatively non-renormalizable theories might be renormalizable after all, if one is willing to go beyond the narrow context perturbation theory [4–6]. In addition, lately direct numerical methods have acquired a more prominent position, largely because of the increasing availability of very fast computers, allowing many of the foregoing ideas to be rigorously tested. A small subset of the vast literature on the subject of field theory methods combined with modern renormalization group ideas, as applied mostly to Euclidean quantum field theory and statistical physics, can be found in [7–11].

Turning to the gravity context, it is not difficult to see that the case perhaps closest to quantum gravity is non-Abelian gauge theories, and specifically QCD. In the latter case one finds that, besides the established Feynman diagram perturbative approach in the gauge coupling g, there are very few additional approximate methods available, mostly due to the crucial need of preserving exact local gauge invariance. It is not surprising therefore that in recent years, for these theories, the method of choice has become the spacetime lattice formulation, which attempts to evaluate the Feynman path integral directly and exactly by numerical methods. The above procedure then leads to answers which, at least in principle, can be improved arbitrarily given a fine enough lattice subdivision and sufficient amounts of computer time. More recently, lattice methods have also been applied to the case of quantum gravity, in the framework of the simplicial Regge–Wheeler discretization [12–14], where they have opened the door to accurate determinations of critical points and nontrivial scaling dimensions.

It would seem nevertheless desirable to be able to derive certain basic results in quantum gravity using perhaps approximate, but largely analytical methods. Unfortunately, in the case of gravity perturbation theory, Newton's G is even less useful than for non-Abelian gauge theories and QCD, as the theory is unequivocally not perturbatively renormalizable in four spacetime dimensions [15–29]. To some extent it is possible to partially surmount this rather serious issue, by developing Wilson's $2 + \epsilon$ diagrammatic expansion [3] for gravity, with $\epsilon = d - 2$. Such an expansion is constructed around the two-dimensional

theory, which, apart from being largely topological in nature, also leads in the end to the need to set $\epsilon = 2$ (clearly not a small value) in order to finally reach the physical theory in $d = 4$ [30–32]. Consequently, one finds significant quantitative uncertainties regarding the interpretation of the results in four spacetime dimensions, especially given the yet unknown convergence properties of the asymptotic series in ϵ. Lately, approximate truncated renormalization group (RG) methods have been applied directly in four dimensions, which thus bypass the need for Wilson's dimensional expansion. These nevertheless can lead to some significant uncertainties in trying to pin down the resulting truncation errors, in part due to what are often rather drastic renormalization group truncation procedures, which in turn generally severely limit the space of operators used in constructing the renormalization group flows.

In this work, we will focus instead on the development of the Hartree–Fock approximation for quantum gravity. The Hartree–Fock method is of course well known in non-relativistic quantum mechanics and quantum chemistry, where it has been used for decades as a very successful tool for describing correctly (qualitatively, and often even quantitatively to a high degree) properties of atoms and molecules, including, among other things, the origin of the periodic table [33]. In the Hartree–Fock approach, the direct interaction of one particle (say an electron in an atom) with all the other particles is represented by some sort of average field, where the average field itself is determined self-consistently via the solution of a nonlinear, but single-particle Schroedinger-like equation. The latter in turn then determines, via the single-particle wave functions, the average field exerted on the original single particle introduced at the very start of the procedure. As such, it is often referred to as a self-consistent method, where an explicit solution to the effective nonlinear coupled wave equations is obtained by successive (and generally rapidly convergent) iterations.

It has been known for a while that the Hartree–Fock method has a clear extension to a quantum field theory (QFT) context. In quantum field theory (and more generally in many-body theory), it is possible to derive the leading order Hartree–Fock approximation [33–44] as a truncation of the Dyson–Schwinger (DS) equations [45–47]. For the latter, a detailed, modern exposition can be found in the classic quantum field theory book [48], as well as (for non-relativistic applications) in [38]. The Hartree–Fock method is in fact quite popular to this day in many-body theory, where it is used, for example, as a way of deriving the gap equation for a superconductor in the Bardeen–Cooper–Schrieffer (BCS) theory [34–36]. There the Hartree–Fock method was originally used to show that the opening of a gap Δ at the Fermi surface is directly related to the existence of a Cooper pair condensate. Many more applications of the Hartree–Fock method in statistical physics can be found in [37,38]. Later on, the successful use of the Hartree–Fock approach in many-body theory spawned many useful applications to relativistic quantum field theory [39–44], including one of the earliest investigations of dynamical chiral symmetry breaking. There is also a general understanding that the Hartree–Fock method, as it is usually based on the notion of a (self-consistently determined) average field, cannot properly account for the regime where large fluctuations and correlations are important. The precursor to these kind of reasonings is the general validity of mean field theory for spin systems and ferromagnet above four dimensions (as described by the Landau theory), but not in low dimensions where fluctuations become increasingly important.

In statistical field theory, the Hartree–Fock approximation is seen to be closely related to the large-N expansion, as described for example in detail in [44]. There one introduces N copies for the fields in question (the simplest case is for a scalar field, but it can be applied to Fermions as well), and then expands the resulting Green's functions in powers of $1/N$. The expansion thus generally starts out from some sort of extended symmetry group theory, which is taken to be $O(N)$- or $SU(N)$-invariant. Then, it is easy to see that the leading order in the $1/N$ expansion reproduces what one obtains to lowest order in the Hartree–Fock approximation for the same theory. Furthermore, the next orders in the $1/N$ expansion generally correspond to higher order corrections within the Hartree–Fock

approximation. The large N expansion can be developed for $SU(N)$ gauge theories as well, and to leading order it gives rise to the 'tHooft planar diagram expansion. In view of the previous discussion, it would seem therefore rather natural to consider the analog of the Hartree–Fock approximation for QCD as well [49,50]. Nevertheless, this avenue of inquiry has not been very popular lately, due mainly to its rather crude nature, and because of the increasing power (and much greater reliability) of ab initio numerical lattice approaches to QCD.

Unfortunately, in the case of quantum gravity there is no obvious large N expansion framework. This can be seen from the fact that one starts out with clearly only one spacetime metric, and introducing multiple N copies of the same metric seems rather unnatural. Such multiple metric formulation would cause some rather serious confusion on which of the N metrics should be used to determine the spacetime interval between events, and thus provide the needed spacetime arguments for gravitational n-point functions. In addition, and unlike QCD, there is no natural global (or local) symmetry group that would relate these different copies of the metric to each other. As will be shown below, it is nevertheless possible (and quite natural) to develop the Hartree–Fock approximation for quantum gravity, in a way that is rather similar to the procedure followed in the case of the nonlinear sigma model, gauge theories and general many-body theories. In the gravitational case one starts out, in close analogy to what is done for these other theories, from the Dyson–Schwinger equations, and then truncates them to the lowest order terms, replacing the bare propagators by dressed ones, the bare vertices by dressed ones etc. Thus, the Hartree–Fock approximation for quantum gravity can be written out without any reference to a $1/N$ expansion, which as discussed above is often the underlying justification for other non-gravity theories. In addition, the leading Hartree–Fock result represents just the lowest order term in a modified diagrammatic expansion, which in principle can be carried out systematically to higher order.

As a reference, for the scalar $\lambda\phi^4$ theory the Hartree–Fock approximation was already outlined some time ago in [44], where its relationship to the $1/N$ expansion was laid out in detail. One important aspect of both the Hartree–Fock approximation and the $1/N$ expansion discussed in the references given above is the fact that it can usually be applied in any dimensions, and is therefore not restricted in any way to the spacetime (or space) dimension in which the theory is found to be perturbatively renormalizable. It therefore represents a genuinely nonperturbative method, of potentially widespread application. In the scalar field case, one can furthermore clearly establish, based on the comparison with other methods, to what extent the Hartree–Fock approximation is able to capture the key physical properties of the underlying theory, such as correlation functions and universal scaling exponents. These investigations in turn provide a partial insight into what physical aspects the Hartree–Fock approximation can correctly reproduce, and where it fails. Generally, the conclusion appears to be that, while quantitatively not exceedingly accurate, the Hartree–Fock approximation and the $1/N$ expansion tend to reproduce correctly at least the qualitative features of the theory in various dimensions, including, for example, the existence of an upper and a lower critical dimension, the appearance of nontrivial fixed points of the renormalization group, as well as the dependence of nontrivial scaling exponents on the dimension of spacetime.

In this work, we first review the properties of the Hartree–Fock approximation, as it applies initially to some of the simpler field theories. In the case of the scalar field theory, with the scalar field having N components, one can see that the Hartree–Fock approximation is easily obtained from the large N saddle point [44]. The resulting gap equation is generally valid in any dimension, and once solved provides an explicit expression for the mass gap (the inverse correlation length) as a function of the bare coupling constant. Solutions to the gap equation show some significant sensitivity to the number of dimensions. For the scalar field a clear transition is seen between the lower critical dimension $d = 2$ (for $N > 2$) and the upper critical dimension $d = 4$, above which the critical exponents attain the Landau theory values. Indeed, the Hartree–Fock approximation reproduces correctly the fact that

for $d \leq 2$ the $O(N)$ symmetric nonlinear sigma model for $N > 2$ exhibits a non-vanishing gap for bare coupling $g > 0$, and that for $d \geq 4$ it reduces to a non-interacting (Gaussian) theory in the long-distance limit (also known as the triviality of $\lambda \phi^4$ in $d \geq 4$). Then, as a warm-up to the gravity case, it will pay here to discuss briefly the $SU(N)$ gauge theory case. Here again one can see that the Hartree–Fock result correctly reproduces the asymptotic freedom running of the gauge coupling g, and furthermore shows that $d = 4$ is the lower critical dimension for $SU(N)$ gauge theories. This last statement is meant to refer to the fact that above four dimensions non-Abelian gauge theories are known to have a nontrivial ultraviolet fixed point at some g_c, separating a weak coupling Coulomb-like phase from a strong coupling confining phase.

Finally, in the gravitational case, one starts out by following a procedure which closely parallels the gauge theory case. Nevertheless, at the same time one is faced with the fact that the gravitational case is considerably more difficult, for a number of (mostly well known) reasons which we proceed to enumerate here. Unlike the scalar field and gauge theory case in four dimensions, gravity is not perturbatively renormalizable in four dimensions, and as a result perturbation theory is badly divergent, and thus not very useful. Furthermore, as mentioned previously, there is no sensible way of formulating a large N expansion for gravity. That leaves as the method of choice, at least for investigating nonperturbative properties of the theory, the lattice formulation, and specifically the elegant simplicial lattice formulation due to Regge and Wheeler [12,13] and its Euclidean Feynman path integral extension. One disadvantage is that the lattice methods nevertheless ultimately rely on extensive numerical calculations, leaving a desire for alternate methods that can be pursued analytically. As mentioned previously, it is also possible to develop for quantum gravity the $2 + \epsilon$ dimensional expansion [30–32], in close analogy to Wilson's $4 - \epsilon$ and $2 + \epsilon$ dimensional expansion for scalar field theories [3]; the ability to perform such an expansion for gravity hinges on the fact that Einstein's theory becomes formally perturbatively renormalizable in two dimensions. However, there are significant problems associated with this expansion, notably the fact that $\epsilon = d - 2$ has to be set equal to two at the end of the calculation. One more possible approach is via the Wheeler–DeWitt equations, which provide a useful Hamiltonian formulation for gravity, both in the continuum and on the lattice, see in [51] and references therein. Nevertheless, while it has been possible to obtain some exact results on the lattice in $2 + 1$ dimensions, the $3 + 1$ case still appears rather difficult. As a result, generally the number of reasonably well-tested nonperturbative methods available to investigate the vacuum structure of quantum gravity seem rather limited. It is therefore quite remarkable that it is possible to develop a Hartree–Fock approximation to quantum gravity, in a way that in the end is completely analogous to what is done in scalar field theories and gauge theories. In the following we briefly outline the features of the Hartree–Fock method as applied to gravity.

For any field theory it is possible to write down a closed set of integral equations, which relate various physical amplitudes involving particle propagators, vertex functions, proper self-energies, etc. These equations follow in a rather direct way from their respective definitions in terms of the Feynman path integral, and the closely connected generating function $Z[J]$. One particularly elegant and economic way to obtain these expressions is to apply suitable functional derivatives to the generating function in the presence of external classical sources. The resulting exact relationship between n-point functions, known as the set of Dyson–Schwinger equations, are, by virtue of their derivation, not reliant in any way on perturbation theory, and thus genuinely nonperturbative. Nevertheless, the set of coupled Dyson–Schwinger equations for the propagators, self-energies, and vertex functions are in general very difficult to solve, and as a result not very useful in practice. This is where the Hartree–Fock approximation comes in, whereby lowest order diagrams get partially dressed by the use of fully interacting Green's function and self-energies. As the resulting coupled equations involve fully dressed propagators and full proper self-energies to some extent, they are referred to as a set of self-consistent equations. In principle the Hartree–Fock approximation to the Dyson–Schwinger equations can be carried out to

arbitrarily high order, by examining the contribution of increasingly complicated diagrams involving dressed propagators, dressed self-energies, etc. Of particular interest here will be therefore the lowest order Hartree–Fock approximation for quantum gravity. As in the scalar and gauge theory case, it will be necessary, in order to derive the Hartree–Fock equations, to write out the lowest order contributions to the graviton propagator, to the gravitational self-energy, and to the gravitational vertex function, and for which the lowest order perturbation theory expressions are known as well. Indeed, one remarkable aspect of most Hartree–Fock approximations lies in the fact that ultimately the calculations can be shown to reduce to the evaluation of a single one-loop integral, involving a self-energy contribution, which is then determined self-consistently.

Nevertheless, quite generally the resulting diagrammatic expressions are still ultraviolet divergent, and the procedure thus requires the introduction of an ultraviolet regularization, such as the one provided by dimensional regularization, an explicit momentum cutoff, or a lattice as in the Regge–Wheeler lattice formulation of gravity [12,13]. Once this is done, one then finds a number of significant similarities of the Hartree–Fock result with the (equally perturbatively non-renormalizable in $d > 2$) nonlinear sigma model. Specifically, in the quantum gravity case one finds that the ultraviolet fixed point in Newton's constant G is at $G = 0$ in two spacetime dimensions, and moves to a non-zero value above $d = 2$. Nontrivial scaling dimensions are found between $d = 2$ and $d = 4$, above which one obtains Gaussian (free field) exponents, thus suggesting that, at least in the Hartree–Fock approximation, the upper critical dimension for gravity is $d = 4$. One useful and interesting aspect of the above calculations is that the Hartree–Fock results are not too far off from what one obtains by either numerical investigations within the Regge–Wheeler lattice formulation, or via the $2 + \epsilon$ dimensional expansion, or by truncated renormalization group methods. Furthermore, as will be shown below, the Hartree–Fock approximation provides an explicit expression for the renormalization group (RG) running of Newton's constant, just like the Hartree–Fock leads to similar results for the nonlinear sigma model. As in the lattice case, the Hartree–Fock results suggest a two-phase structure, with gravitational screening for $G < Gc$, and gravitational antiscreening for $G > Gc$ in four dimensions. Due to the known nonperturbative instability of the weak coupling phase of $G < G_c$, only the $G > G_c$ phase will need to be considered further, which then leads to a gravitational coupling $G(k)$ that slowly increases with distance, with a momentum dependence that can be obtained explicitly within the Hartree–Fock approximation.

The paper is organized as follows. In Section 2, we recall the main features of Dyson's equations and how they lead to the Hartree–Fock approximation. We point out the important and general result that Dyson's equations can be easily derived using the elegant machinery of the Feynman path integral combined with functional methods. The Hartree–Fock approximation then follows from considering a specific set of sub-diagrams with suitably dressed propagators, all to be determined later via the solution of a set of self-consistent equations. In Section 3, we discuss in detail, as a first application and as preamble to the quantum gravity case, the case of the $O(N)$-symmetric nonlinear sigma model, initially here from the perturbative perspective of Wilson's $2 + \epsilon$ expansion. A summary of the basic results will turn out to be useful when comparing later to the Hartree–Fock approximation. In Section 4, we briefly recall the main features of the nonlinear sigma model in the large-N expansion, which can be carried out in any dimension. Again, the discussion here is with an eye towards the later, more complex discussion for gravity. In Section 5, we discuss the main features of the Hartree–Fock result for the $O(N)$ nonlinear sigma model in general dimensions $2 \leq d \leq 4$, and point out its close relationship to the large-N results described earlier. The section ends with a comparison of the three approximations, and how they favorably relate to each other and to other results, such as the lattice formulation and actual experiments. In Section 6 we proceed to a more difficult case, namely, the discussion of some basic results for the Hartree–Fock approximation for non-Abelian gauge theories in general spacetime dimensions $4 \leq d \leq 6$. The basic starting point is again a general gap equation for a nonperturbative, dynamically generated mass

scale. Here, again we will note that the Hartree–Fock results capture many of the basic ingredients of nonperturbative physics, while still missing out on some (such as confinement). In Section 7, we move on to the gravitational case, and discuss in detail the results of the Hartree–Fock approximation in general spacetime dimensions $2 \leq d \leq 4$. For the gravity case, the basic starting point will be again a gap equation for the nonperturbatively generated mass scale, whose solution will lead to a number of explicit results, including an explicit expression for the renormalization group running of Newton's constant G. In Section 8, we then present a sample application of the Hartree–Fock results to cosmology, where the running of G is compared to current cosmological data, specifically here the Planck-18 Cosmic Background Radiation (CMB) data. Finally, Section 9 summarizes the key points of our study and concludes the paper.

2. Dyson's Equations and the Hartree–Fock Approximation

One of the earliest applications of the Hartree–Fock approximation to solving Dyson's equations for propagators and vertex functions was in the context of the BCS theory for superconductors [34–36]. A few years later it was applied to the (perturbatively nonrenormalizable) relativistic theory of a self-coupled Fermion, where it provided the first convincing evidence for a dynamical breaking of chiral symmetry and the emergence of Nambu–Goldstone bosons [39,40]. A necessary preliminary step involved in deriving the Hartee–Fock approximation to a given theory is writing down Dyson's equations (often referred to as the Schwinger–Dyson equations) for the field, or fields, in question. Dyson's equations have a nice, almost self-explanatory, form when written in terms of Feynman diagrams for various propagators, self-energies, and vertex functions associated with the particles appearing in the theory, such as for example electrons and photons in QED. These generally describe a set of coupled integral equations for the various dressed, fully interacting, n-point functions, but whose solution is generally rather difficult, if not impossible. As a result, they are usually only discussed as a method for deriving exact Ward identities that follow from local gauge invariance, with their practical application then usually restricted in most cases to perturbation theory only. Nevertheless, as will be shown below, they play a key role in deriving the Hartree–Fock approximations for the fields in question, thus providing a method that is essentially nonperturbative, involving a partial re-summation of an infinite class of diagrams.

In non-relativistic many body theory, Dyson's equation takes on a simple form when describing the static electromagnetic interaction between electrons [38] (It is often customary in many body theory to describe the interaction of electrons close to the Fermi surface in terms of electrons and holes situated around the conduction band, in analogy to the treatment of relativistic electrons in the framework of the Dirac equation, with the positively charged holes playing the role of positrons in the Dirac theory). In terms of the electron propagator $G(p)$, one has at first the decomposition

$$G(p) \; = \; G_0(p) \; + \; G_0(p) \, \Sigma(p) \, G_0(p) \; , \tag{1}$$

where $G_0(p)$ is the bare (tree level) propagator, $G(p)$ the dressed one, and $\Sigma(p)$ the electron self-energy. The above should really be regarded as a matrix equation, with indices appropriate for the particle being exchanged intended, and not written out in this case. If one regards the self-energy Σ as being constructed out of repeated iterations of a proper self-energy $\Sigma^*(p)$, then Dyson's equation takes on the form

$$G(p) \; = \; G_0(p) \; + \; G_0(p) \, \Sigma^*(p) \, G(p) \tag{2}$$

with solution

$$G(p) \; = \; \frac{1}{G_0^{-1}(p) \, - \, \Sigma^*(p)} \; . \tag{3}$$

As, generally, $G(p)$ is regarded as a matrix with spin indices, the above should be interpreted as the matrix inverse. This then leads to the identification of $\Sigma^*(0)$ as an effective

mass correction for the electron, induced, perturbatively or nonperturbatively, by radiative corrections.

Similarly, the boson (photon or phonon, depending of the interaction being considered) exchange between electrons leads to the expansion for the interaction term

$$U(p) = U_0(p) + U_0(p)\,\Pi(p)\,U_0(p) \tag{4}$$

where $U_0(p)$ is the bare (tree level) contribution, $U(p)$ the dressed one and $\Pi(p)$ the self-energy contribution. Again, the above should be regarded as a matrix equation, with indices appropriate for the particle being exchanged. If one considers the above self-energy $\Pi(p)$ as being constructed out of repeated iterations of a proper self-energy $\Pi^*(p)$, then Dyson's equation in this case takes on the form

$$U(p) = U_0(p) + U_0(p)\,\Pi^*(p)\,U(p) \tag{5}$$

with solution

$$U(p) = \frac{1}{U_0^{-1}(p) - \Pi^*(p)} \tag{6}$$

again intended as a proper matrix inverse. If the interaction is spin-independent, as in the Coulomb case, then the above matrices can all be regarded as diagonal. This last result then leads to the identification of $\Pi^*(0)$ as the effective mass (or inverse range) associated with the interaction described by $U(p)$. The ratio $\kappa(p) = U_0(p)/U(p)$ is referred to, in the electromagnetic case, as an effective generalized dielectric function [38].

Alternatively, it is possible to derive Dyson's equations directly from the Feynman path integral using the rather elegant machinery of functional differentiation of the generating function $Z[J]$ with respect to a suitable source terms $J(x)$. It is this latter approach that we follow here, as it generalizes rather straightforwardly to the case of quantum gravity. For a scalar field one has the statement that the integral of a derivative is zero for suitable boundary conditions, or

$$\int [d\,\phi]\,\frac{\delta}{\delta\phi(x)}\,e^{i\int dx(\mathcal{L}+J\phi)} = 0 \tag{7}$$

where $\mathcal{L}$ is the Lagrangian density for the scalar field in question. As a consequence

$$\int [d\,\phi]\left[\frac{\partial\mathcal{L}}{\partial\phi(x)} + J(x)\right]e^{i\int dx(\mathcal{L}+J\phi)} = 0\,. \tag{8}$$

If the scalar field potential is generally denoted by $V(\phi)$, then the previous expression can be recast, defining

$$Z[J] = \int [d\,\phi]\,e^{i\int dx(\mathcal{L}+J\phi)} \tag{9}$$

as

$$\left[(\Box_x + m^2)\,\frac{\delta}{i\,\delta J(x)} + V'\left(\frac{\delta}{i\,\delta J(x)}\right) - J(x)\right]\,Z[J] = 0 \tag{10}$$

The latter then plays the role of the quantum Heisenberg equations of motion. Successive applications of functional derivatives with respect to $J(x)$ then leads to a sequence of simple identities at zero source

$$\int [d\,\phi]\left[(\Box_x + m^2)\,\phi(x) + V'(\phi(x))\right]e^{i\int dx\,\mathcal{L}} = 0 \tag{11}$$

$$\int [d\,\phi]\left[(\Box_x + m^2)\,\phi(x)\cdot\phi(x_1) + V'(\phi(x))\cdot\phi(x_1)\right]e^{i\int dx\,\mathcal{L}} = 0 \tag{12}$$

and so on. More generally, for n-point functions one obtains

$$
\begin{aligned}
\int [d\,\phi] \quad & \left[(\Box_x + m^2)\phi(x) + V'(\phi(x)) \right] \phi(x_1)\,\phi(x_2)\dots\phi(x_n)\,e^{i\int dx\,\mathcal{L}} \\
+ \quad & \int [d\,\phi]\,\delta(x - x_1)\,\phi(x_2)\dots\phi(x_n)\,e^{i\int dx\,\mathcal{L}} \\
+ \quad & \cdots + \int [d\,\phi]\,\phi(x_1)\,\phi(x_2)\dots\delta(x - x_n)\,e^{i\int dx\,\mathcal{L}} = 0
\end{aligned} \tag{13}
$$

Within the current discussion of Dyson's equations, a step up in difficulty is clearly represented by the gauge theory case, with QED as the simplest example. As the resulting equations share some similarity with quantum gravity, it will be useful here to recall briefly the main aspects. Compared to scalar field theory, the first new ingredient is the presence of multiple fields, for QED in the form of photon and matter fields. The generating function is

$$
Z(J,\eta,\bar\eta) = e^{G(J,\eta,\bar\eta)} = \int [dA][d\psi][d\bar\psi]\,e^{iS} \tag{14}
$$

with

$$
S = I + \int dx (J \cdot A + \bar\eta\,\psi + \bar\psi\,\eta) \tag{15}
$$

where $I(A,\psi,\bar\psi)$ is the QED action. Again, for suitable boundary conditions the integral of a derivative is zero, which leads to the identity

$$
\int [dA][d\psi][d\bar\psi]\,\frac{\delta}{\delta A^\mu(x)}\,e^{iS} = 0 \;, \tag{16}
$$

and consequently

$$
\int [dA][d\psi][d\bar\psi]\left(\frac{\delta I}{\delta A^\mu(x)} + J_\mu \right) e^{iS} = 0 \;. \tag{17}
$$

For the case of the QED action one has

$$
\frac{\delta I}{\delta A^\mu(x)} = \left[\Box g_{\mu\nu} - (1-\lambda)\partial_\mu\partial_\nu \right] A^\nu - e\,\bar\psi\gamma_\mu\,\psi \;, \tag{18}
$$

where λ is the gauge parameter for a gauge fixing term $\frac{1}{2}\lambda(\partial \cdot A)^2$, and $g_{\mu\nu}$ here the flat Lorentz metric. The resulting equation

$$
J_\mu + \left[\Box g_{\mu\nu} - (1-\lambda)\partial_\mu\partial_\nu \right]\frac{\delta G}{i\,\delta J^\nu} - e\,\frac{\delta G}{\delta\eta}\,\gamma_\mu\,\frac{\delta G}{\delta\bar\eta} - e\,\frac{\delta}{\delta\eta}\left(\gamma_\mu\,\frac{\delta G}{\delta\bar\eta} \right) = 0 \tag{19}
$$

can then be regarded as the quantum version of Maxwell's equations [48]. A graphical illustration of Dyson's equation for QED is given in Figure 1. In QED and gauge theories in general it is often useful to introduce the generating function Γ of one-particle-irreducible (1PI) Green's functions, defined as the Legendre transform of G,

$$
G(J,\eta,\bar\eta) = i\,\Gamma(A,\psi,\bar\psi) + i\int dx\,(J \cdot A + \bar\eta\,\psi + \bar\psi\,\eta) \;. \tag{20}
$$

Then, the introduction of the classical fields A_μ, ψ, and $\bar\psi$ via the definitions

$$
\frac{\delta G}{i\delta J^\mu} = A_\mu \qquad -\frac{\delta G}{i\delta\eta} = \bar\psi \qquad \frac{\delta G}{i\delta\bar\eta} = \psi \tag{21}
$$

and conversely, as a consequence of the definition of Γ,

$$
\frac{\delta\Gamma}{\delta A^\mu} = -J_\mu \qquad \frac{\delta\Gamma}{\delta\psi} = \bar\eta \qquad \frac{\delta\Gamma}{\delta\bar\psi} = -\eta \tag{22}
$$

which allows one to rewrite Equation (19) simply as

$$J_\mu + \left[\Box g_{\mu\nu} - (1-\lambda)\partial_\mu \partial_\nu \right] A^\nu - e\,\bar{\psi}\,\gamma_\mu\,\psi - ie\,\frac{\delta}{\delta\eta}\left(\gamma_\mu\,\psi\right) = 0 \ . \tag{23}$$

Equivalently, one has in terms of the generating function Γ exclusively

$$\frac{\delta\Gamma}{\delta A^\mu} = \left[\Box g_{\mu\nu} - (1-\lambda)\partial_\mu \partial_\nu \right] A^\nu - e\,\bar{\psi}\,\gamma_\mu\,\psi + ie\,\text{Tr}\left[\gamma_\mu\left(\frac{\delta^2\Gamma}{\delta\psi\,\delta\bar{\psi}}\right)^{-1}\right] \ . \tag{24}$$

A very similar procedure can be carried out for non-Abelian gauge theories and gravity; nevertheless, the complexity increases greatly due to the proliferation of Lorentz indices, color indices, and a multitude of interaction vertices. We will skip here almost entirely a detailed discussion of gauge theories, and only refer to the relevant diagrams and corresponding expressions, where appropriate. Moreover, for ease of exposition, the quantum gravity case will be postponed until the Hartree–Fock approximation for gravity is developed.
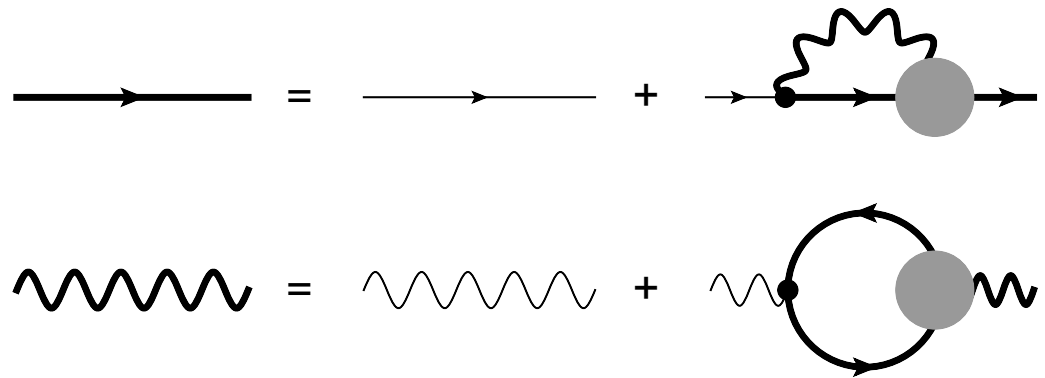
Figure 1. Dyson's equations in QED relating the dressed fermion propagator, the dressed photon propagator and the dressed vertex function. Thin lines represent bare propagators, while thicker lines represent the dressed ones.

3. The Case of the Nonlinear Sigma Model

The question then arises, are there any field theories where the standard perturbative treatment fails, yet for which one can find alternative methods and from them develop consistent predictions? The answer seems unequivocally yes [4–6]. Indeed, outside the quantum gravity framework, there are two notable examples of field theories: the nonlinear sigma model and the self-coupled fermion model. Both are not perturbatively renormalizable for $d > 2$, and yet lead to consistent and, in some instances, precision-testable predictions above $d = 2$. The key ingredient to all of these results is, as recognized originally by Wilson, the existence of a nontrivial ultraviolet fixed point (also known as a phase transition in the statistical field theory context) with nontrivial universal scaling dimensions [1–3,52]. Furthermore, several different calculational approaches are by now available for comparing predictions: the $2 + \epsilon$ expansion, the large-N limit, the functional renormalization group, the conformal bootstrap, and a variety of lattice approaches. From within the lattice approach, several additional techniques are available: the Wilson (block-spin) renormalization group approach, the strong coupling expansion, the weak coupling expansion, and the numerically exact evaluation of the lattice path integral.

The $O(N)$-symmetric nonlinear sigma model provides an instructive (and rich) example of a theory which, above two dimensions, is not perturbatively renormalizable in the traditional sense, and yet can be studied in a controlled way via Wilson's $2 + \epsilon$ expansion [53–60]. Such a framework provides a consistent way to calculate nontrivial scaling properties of the theory in those dimensions where it is not perturbatively renormalizable

(e.g., $d = 3$ and $d = 4$), which can then be compared to nonperturbative results based on the lattice theory. In addition, the model can be solved exactly in the large N limit for any d, without any reliance on the $2 + \epsilon$ expansion. In all three approaches, the model exhibits a nontrivial ultraviolet fixed point at some coupling g_c (a phase transition in statistical mechanics language), separating a weak coupling massless ordered phase from a massive strong coupling phase. Finally, the results can then be compared to experiments, since in $d = 3$ the model describes either a ferromagnet or superfluid helium in the vicinity of its critical point.

The nonlinear sigma model is described by an N-component scalar field ϕ_a satisfying a unit constraint $\phi^2(x) = 1$, with functional integral given by

$$
\begin{aligned}
Z[J] \;\; &= \;\; \int [\, d\phi \,] \prod_x \delta[\phi(x) \cdot \phi(x) - 1] \\
&\times \exp\left(- \frac{\Lambda^{d-2}}{g^2} S(\phi) + \int d^d x \, J(x) \cdot \phi(x) \right) \; .
\end{aligned}
\tag{25}
$$

The action is taken to be $O(N)$-invariant

$$
S(\phi) \;\; = \;\; \tfrac{1}{2} \int d^d x \, \partial_\mu \phi(x) \cdot \partial_\mu \phi(x) \; .
\tag{26}
$$

Here, Λ is an ultraviolet cut-off (as provided for example by a lattice), and g the bare dimensionless coupling at the cut-off scale Λ; in a statistical field theory context the coupling g^2 plays the role of a temperature.

In perturbation theory one can eliminate one ϕ field by introducing a convenient parametrization for the unit sphere, $\phi(x) = \{\sigma(x), \pi(x)\}$ where π_a is an $N - 1$-component field, and then solving locally for $\sigma(x)$

$$
\sigma(x) \;\; = \;\; [\, 1 - \pi^2(x) \,]^{1/2} \; .
\tag{27}
$$

In the framework of perturbation theory in g the constraint $|\pi(x)| < 1$ is not important as one is restricting the fluctuations to be small. Accordingly the π integrations are extended from $-\infty$ to $+\infty$, which reduces the development of the perturbative expansion to a sequence of Gaussian integrals. Values of $\pi(x) \sim 1$ give exponentially small contributions of order $\exp(-\text{const.}/g)$ which are considered negligible to any finite order in perturbation theory. In term of the π field the original action S becomes

$$
S(\pi) \;\; = \;\; \tfrac{1}{2} \int d^d x \left[(\partial_\mu \pi)^2 + \frac{(\pi \cdot \partial_\mu \pi)^2}{1 - \pi^2} \right] \; .
\tag{28}
$$

The change of variables from $\phi(x)$ to $\pi(x)$ gives rise to a Jacobian

$$
\prod_x \left[1 - \pi^2 \right]^{-1/2} \; \sim \; \exp\left[-\tfrac{1}{2} \delta^d(0) \int d^d x \, \log(1 - \pi^2) \right] \; ,
\tag{29}
$$

which is necessary for the cancellation of spurious tadpole divergences. In the presence of an explicit ultraviolet cut-off $\delta(0) \simeq (\Lambda/\pi)^d$. The combined functional integral for the unconstrained π field is then given by

$$
Z[J] = \int [\, d\pi \,] \exp\left(- \frac{\Lambda^{d-2}}{g^2} S_0(\pi) + \int d^d x \, J(x) \cdot \pi(x) \right)
\tag{30}
$$

with

$$
S_0(\pi) \;\; = \;\; \tfrac{1}{2} \int d^d x \left[(\partial_\mu \pi)^2 + \frac{(\pi \cdot \partial_\mu \pi)^2}{1 - \pi^2} \right] + \tfrac{1}{2} \delta^d(0) \int d^d x \, \log(1 - \pi^2) \; .
\tag{31}
$$

In perturbation theory, the above action is then expanded out in powers of π, and the propagator for the π field can be read off from the quadratic part of the action,

$$\Delta_{ab}(k^2) = \frac{\delta_{ab}}{k^2} \ .$$ (32)

In the weak coupling limit, the π fields correspond to the Goldstone modes of the spontaneously broken $O(N)$ symmetry, the latter broken spontaneously in the ordered phase by a non-vanishing vacuum expectation value $\langle \pi \rangle \neq 0$. As the π field has mass dimension $\frac{1}{2}(d-2)$, and thus the interaction $\partial^2 \pi^{2n}$ consequently has dimension $n(d-2)+2$, one finds that the theory is perturbatively renormalizable in $d = 2$, and perturbatively non-renormalizable above $d = 2$. Potential infrared problems due to massless propagators are handled by introducing an external h-field term for the original composite σ field, which then can be seen to act as a regulating mass term for the π field.

One can write down the same field theory on a lattice, where it corresponds to the $O(N)$-symmetric classical Heisenberg model at a finite temperature $T \sim g^2$. The simplest procedure is to introduce a hypercubic lattice of spacing a, with sites labeled by integers $\mathbf{n} = (n_1 \ldots n_d)$, which introduces an ultraviolet cutoff $\Lambda \sim \pi/a$. On the lattice, field derivatives are replaced by finite differences

$$\partial_\mu \phi(x) \ \rightarrow \ \Delta_\mu \phi(\mathbf{n}) \equiv \frac{\phi(\mathbf{n}+\mu) - \phi(\mathbf{n})}{a}$$ (33)

and the discretized path integral then reads

$$
\begin{aligned}
Z[J] \ &= \ \int \prod_{\mathbf{n}} d\phi(\mathbf{n}) \, \delta[\phi^2(\mathbf{n}) - 1] \\
&\times \ \exp\left[-\frac{a^{2-d}}{2g^2} \sum_{\mathbf{n},\mu} (\Delta_\mu \phi(\mathbf{n}))^2 + \sum_{\mathbf{n}} J(\mathbf{n}) \cdot \phi(\mathbf{n}) \right] \ .
\end{aligned}
$$ (34)

The above expression is recognized as the partition function for a ferromagnetic $O(N)$-symmetric lattice spin system at finite temperature. Besides ferromagnets, it can be used to describe systems which are related to it by universality, such as superconductors and superfluid helium transitions.

In two dimensions one can compute the renormalization of the coupling g from the action of Equation (30) and one finds after a short calculation [55,56] for small g

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2} + \frac{N-2}{8\pi} \log \frac{\mu^2}{\Lambda^2} + \ldots$$ (35)

where μ is an arbitrary momentum scale. Physically, one can view the origin of the factor of $N-2$ in the fact that there are $N-2$ directions in which the spin can experience rapid small fluctuations perpendicular to its average slow motion on the unit sphere, and that only these fluctuations contribute to leading order. In two dimensions, the quantum correction (the second term on the r.h.s.) increases the value of the effective coupling at low momenta (large distances), unless $N = 2$ in which case the correction vanishes. There the quantum correction can be shown to vanish to all orders in this case; nevertheless, the vanishing of the β-function in two dimensions for the $O(2)$ model is true only in perturbation theory. For sufficiently strong coupling a phase transition appears, driven by the unbinding of vortex pairs [61]. For $N > 2$ as $g(\mu)$ flows toward increasingly strong coupling it eventually leaves the regime where perturbation theory can be considered reliable.

Above two dimensions, $d - 2 = \epsilon > 0$, one can redo the same type of perturbative calculation to determine the coupling constant renormalization. There one finds [55,56] for the Callan–Symanzik β-function for g

$$\mu \, \frac{\partial \, g^2}{\partial \, \mu} \;=\; \beta(g) \;=\; \epsilon \, g^2 - \frac{N-2}{2\pi} g^4 + O\!\left(g^6, \epsilon g^4\right) \; . \tag{36}$$

The latter determines the scale dependence of g for an arbitrary momentum scale μ, and from the differential equation $\mu \frac{\partial g^2}{\partial \mu} = \beta(g(\mu))$ one determines how $g(\mu)$ flows as a function of momentum scale μ. The scale dependence of $g(\mu)$ is such that if the initial g is less than the ultraviolet fixed point value g_c, with

$$g_c^2 \;=\; \frac{2\pi\epsilon}{N-2} + \cdots \tag{37}$$

then the coupling will flow towards the Gaussian fixed point at $g = 0$. The new phase that appears when $\epsilon > 0$ and corresponds to a low temperature, spontaneously broken phase with non-vanishing order parameter. On the other hand, if $g > g_c$ then the coupling $g(\mu)$ flows towards increasingly strong coupling, and eventually out of reach of perturbation theory.

The one-loop running of g as a function of a sliding momentum scale $\mu = k$ and $\epsilon > 0$ are obtained by integrating Equation (36), and one finds

$$g^2(\mu) \;=\; \frac{g_c^2}{1 \pm a_0 \, (m^2/\mu^2)^{(d-2)/2}} \tag{38}$$

with a_0 a positive constant and m a mass scale. The choice of $+$ or $-$ sign is determined from whether one is to the left $(+)$, or to right $(-)$ of g_c, in which case $g(\mu)$ decreases or, respectively, increases as one flows away from the ultraviolet fixed point. The renormalization group invariant mass scale $\sim m$ arises here as an arbitrary integration constant of the renormalization group equations. One can integrate the β-function equation in Equation (36) to obtain the renormalization group invariant quantity

$$\xi^{-1}(g) = m(g) = \text{const.} \; \Lambda \, \exp\!\left(-\int^g \frac{dg'}{\beta(g')}\right) \tag{39}$$

which is identified with the correlation length appearing in n-point functions. The multiplicative constant in front of the expression on the right hand side arises as an integration constant, and cannot be determined from perturbation theory in g. The quantity $m = 1/\xi$ is usually referred to as the mass gap of the theory, that is the energy difference between the ground state (vacuum) and the first excited state.

In the vicinity of the fixed point at g_c one can do the integral in Equation (39), using the linearized expression for the β-function in the vicinity of the ultraviolet fixed point,

$$\beta(g) \;\underset{g \to g_c}{\sim}\; \beta'(g_c) \, (g - g_c) + \cdots \tag{40}$$

and one has for the inverse correlation length

$$\xi^{-1}(g) = m(g) \propto \Lambda \, |\, g - g_c \,|^{\nu} \;, \tag{41}$$

with correlation length exponent $\nu = -1/\beta'(g_c) \sim 1/(d-2) + \cdots$.

4. Nonlinear Sigma Model in the Large-N Limit

A rather fortunate circumstance is provided by the fact that in the large N limit the nonlinear sigma model can be solved exactly [62–67]. This allows an independent verification of the correctness of the general ideas developed in the previous section, as well as a direct comparison of explicit results for universal quantities. The starting point is the functional integral of Equation (25),

$$Z = \int [d\phi(x)] \prod_x \delta\left[\phi^2(x) - 1\right] \exp(-S(\phi)) \tag{42}$$

with

$$S(\phi) = \frac{1}{2T} \int d^d x \, \partial_\mu \phi(x) \cdot \partial_\mu \phi(x) \tag{43}$$

and $T \equiv g^2 / \Lambda^{d-2}$, with g dimensionless and Λ the ultraviolet cut-off. The constraint on the ϕ field is implemented via an auxiliary Lagrange multiplier field $\alpha(x)$. One has

$$Z = \int [d\phi(x)] [d\alpha(x)] \exp(-S(\phi, \alpha)) \tag{44}$$

with

$$S(\phi, \alpha) = \frac{1}{2T} \int d^d x \left[(\partial_\mu \phi(x))^2 + \alpha(x)(\phi^2(x) - 1)\right] . \tag{45}$$

As the action is now quadratic in $\phi(x)$ one can integrate over $N - 1$ ϕ-fields (denoted previously by π). The resulting determinant is then re-exponentiated, and one is left with a functional integral over the remaining first field $\phi_1(x) \equiv \sigma(x)$, as well as the Lagrange multiplier field $\alpha(x)$,

$$Z = \int [d\sigma(x)] [d\alpha(x)] \exp(-S_N(\phi, \alpha)) \tag{46}$$

with now

$$S_N(\phi, \alpha) = \frac{1}{2T} \int d^d x \left[(\partial_\mu \sigma)^2 + \alpha(\sigma^2 - 1)\right] + \tfrac{1}{2}(N-1) \, \mathrm{tr} \log[-\partial^2 + \alpha] . \tag{47}$$

In the large N limit one neglects, to leading order, fluctuations in the α and σ fields. For a constant α field, $\langle \alpha(x) \rangle = m^2$, the last (trace) term can be written in momentum space as

$$\tfrac{1}{2}(N-1) \int^\Lambda \frac{d^d k}{(2\pi)^d} \log(k^2 + m^2) \tag{48}$$

which makes the evaluation of the trace straightforward. As should be clear from Equation (45), the parameter m can be interpreted as the mass of the ϕ field. The functional integral in Equation (46) can be evaluated by the saddle point method, with saddle point conditions

$$\begin{aligned} \sigma^2 &= 1 - T(N-1)\Omega_d(m) \\ m^2\sigma &= 0 \end{aligned} \tag{49}$$

and the function $\Omega_d(m)$ given by the integral

$$\Omega_d(m) = \int^\Lambda \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} . \tag{50}$$

The above integral can be evaluated explicitly in terms of hypergeometric functions,

$$\Omega_d(m) = \frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \frac{\Lambda^d}{m^2 d} \, {}_2F_1\left[1, \frac{d}{2}; 1 + \frac{d}{2}; -\frac{\Lambda^2}{m^2}\right] . \tag{51}$$

One only needs the large cut-off limit, $m \ll \Lambda$, in which case one finds the more useful expression

$$\Omega_d(m) - \Omega_d(0) = m^2 \left[c_1 \, m^{d-4} + c_2 \, \Lambda^{d-4} + O(m^2 \Lambda^{d-6})\right] \tag{52}$$

with c_1 and c_2 some d-dependent coefficients, given below. From Equation (49) one notices that at weak coupling and for $d > 2$ a non-vanishing σ-field expectation value implies that

m, the mass of the $\boldsymbol{\pi}$ field, is zero. If one sets $(N-1)\,\Omega_d(0) = 1/T_c$, one can then write the first expression in Equation (49) as

$$\sigma(T) \;=\; \pm\,[1 - T/T_c]^{1/2} \tag{53}$$

which shows that T_c is the critical coupling at which the order parameter σ vanishes. Above T_c the order parameter σ vanishes, and $m(T)$ is obtained from Equation (49) by solving the nonlinear gap equation

$$\frac{1}{T} = (N-1) \int^{\Lambda} \frac{d^d k}{(2\pi)^d}\,\frac{1}{k^2 + m^2} \quad . \tag{54}$$

Using the definition of the critical coupling T_c, one can now write, in the interval $2 < d < 4$, for the common mass of the σ and π fields

$$m(T) \underset{m \ll \Lambda}{\sim} \left(\frac{1}{T_c} - \frac{1}{T}\right)^{1/(d-2)} \quad . \tag{55}$$

This then gives for the correlation length exponent the non-gaussian value $\nu = 1/(d-2)$, with the gaussian (Wilson–Fischer) value $\nu = 1/2$ being recovered as expected in $d = 4$. Note that in the large N limit the constant of proportionality in Equation (55) is completely determined by the explicit expression for $\Omega_d(m)$. Perhaps one of the most striking aspects of the nonlinear sigma model above two dimensions is that all particles are massless in perturbation theory, yet they all become massive in the strong coupling phase $T > T_c$, with masses proportional to the nonperturbative scale m.

Again, one can perform a renormalization group analysis, as was done in the previous section within the context of the $2 + \epsilon$ expansion. To this end one defines dimensionless quantities $g = \Lambda^{d-2} T$ and $g_c = \Lambda^{d-2} T_c$ as was done in Equation (25). Then the nonperturbative result of Equation (55) becomes

$$m(g) \;\simeq\; c_d \cdot \Lambda \left(\frac{1}{g_c} - \frac{1}{g}\right)^{1/(d-2)} \tag{56}$$

with the numerical coefficient given by $c_d = [\frac{1}{2}(d-2)\pi\,|\csc\!\left(\frac{d\pi}{2}\right)|]^{\frac{1}{d-2}}$. Furthermore, one finds for the $O(N)$ β-function in the large N limit

$$\beta(g) = (d-2)\,g^2\,(1 - g^2/g_c^2) \quad . \tag{57}$$

The latter is valid again in the vicinity of the fixed point at g_c, due to the assumption, used in Equation (55), that $m \ll \Lambda$. Then, Equation (57) gives the momentum dependence of the coupling at fixed cut-off, and upon integration one finds

$$\frac{g^2(\mu)}{g_c^2} = \frac{1}{1 - c\,(\mu_0/\mu)^{d-2}} \approx 1 + c\,(\mu_0/\mu)^{d-2} + \dots \tag{58}$$

with $c\,\mu_0^{d-2}$ a dimensionful integration constant. The sign of c then depends on whether one is on the right ($c > 0$) or on the left ($c < 0$) of the ultraviolet fixed point at g_c. At the fixed point g_c, the β-function vanishes and the theory becomes scale invariant. Moreover, one can check again that $\nu = -1/\beta'(g_c)$ where ν is the exponent in Equation (55).

5. Hartree–Fock Method for the Nonlinear Sigma Model

The large N saddle point equation of Equation (54) is in fact identical to the Hartree–Fock approximation for the self-energy, where the bare propagator in the loop is replaced by the dressed one, with a mass parameter m to be determined by the Hartree–Fock self-

consistency conditions. Figure 2 shows the typical perturbative expansion for the proper self-energy of a self-interacting scalar field.
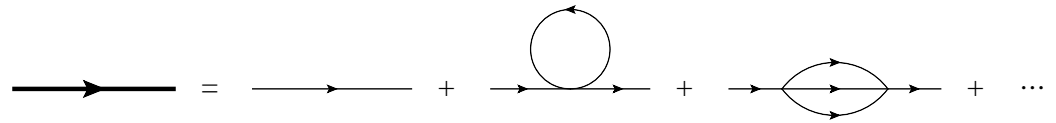

Figure 2. Perturbative expansion for the dressed propagator of a self-interacting scalar field.

To derive the Hartree–Fock approximation, one can first write down Dyson's equations for the dressed propagator and the dressed vertices, as shown in Figure 3. The Hartree–Fock approximation to the self-energy is then obtained by replacing the bare propagator with a dressed one in the lowest order loop diagram, as shown in Figure 4. There the dressed scalar propagator is to be determined self-consistently from the solution of the nonlinear Hartree–Fock equations. It should be noted that in the lowest order approximation the loop integrals involve the dressed scalar propagator $\propto (k^2 + m^2)^{-1}$, while the scalar vertex is still the bare one. This is a characteristic feature of the Hartree–Fock approximation both in many-body theories [38] and in QCD [49,50].
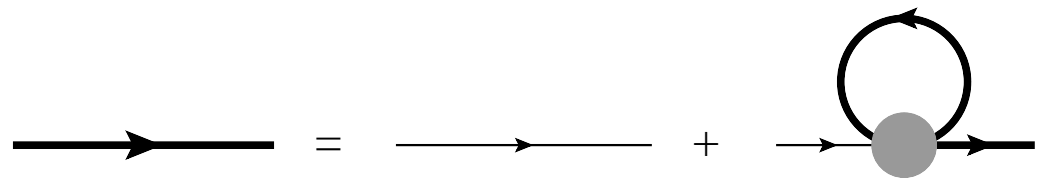

Figure 3. Dyson's equation for the scalar field propagator.

One important aspect that needs to be brought up at this stage is the dependence of the results on the specific choice of ultraviolet cut-off. In the foregoing discussion the momentum integrals were cut off, in magnitude, at some large virtual momentum Λ, which resulted in an explicit expression for the gap equation which reflected, at low momenta, the original spherical symmetry of the underlying cutoff. The latter is nevertheless not the only possibility. An alternative procedure would be the introduction of an underlying hypercubic (or other) lattice, with lattice spacing a and for which then the momentum cut-off is $\Lambda = \pi/a$. One more possibility would be the use of dimensional regularization, where ultraviolet divergences reappear as simple poles in the dimension d. The detailed choice of ultraviolet cutoff would then affect the form of the Λ terms in the gap equation. It is nevertheless well known, and expected on the basis of very general and well-established renormalization group arguments (based mainly on the nature of what are referred to as irrelevant operators), that the main effect would go into a shift of the non-universal critical coupling T_c, leaving the universal exponents and their dependence on the dimension d and on the N of $O(N)$ unchanged. See, for example, in [1–6], the comprehensive monographs [7–11], and the many more references therein.

In view of the later discussion on quantum gravity, it will be useful here to next look explicitly at a few individual cases, mainly as far as the dimension d is concerned. These follow from the general expression for the loop integral given above

$$S_d \int_0^\Lambda dk\, k^{d-1}\, \frac{1}{k^2 + m^2} \tag{59}$$

with $S_d = 1/2^{d-1}\, \pi^{d/2}\, \Gamma(d/2)$. One finds for the integral itself in general dimensions

$$I(\Lambda, m, d) \equiv \int_0^\Lambda dk\, k^{d-1}\, \frac{1}{k^2 + m^2} = \frac{\Lambda^d}{d\, m^2}\, {}_2F_1(1, \frac{d}{2}, \frac{2+d}{2}, -\frac{\Lambda^2}{m^2}) \tag{60}$$

with $_2F_1(a, b, c, z)$ the generalized hypergeometric function. The general nonlinear gap equation for m in d dimensions then reads

$$(N - 1) \, T \cdot S_d \cdot \frac{\Lambda^d}{d \, m^2} \, _2F_1(1, \frac{d}{2}, \frac{2+d}{2}, -\frac{\Lambda^2}{m^2}) - 1 \; = \; 0 \; . \tag{61}$$

Here, one recognizes that the only remnant of the large N expansion is the overall coefficient $\beta_0 = N - 1$; perhaps a better, physically motivated choice would have been a factor of $N - 2$, as $N = 2$ is known to be the special case for which the mass gap disappears in the weak coupling, spin wave phase in $d = 2$. (The self-consistent Hartree–Fock gap equation for m in the nonlinear sigma model shares some analogies with the gap equation for a superconductor. There one finds, in the Hartree–Fock approximation, the following nonlinear gap equation [37,38],

$$g \, N(0) \int_0^{\hbar \omega_D} \frac{d\xi}{(\xi^2 + \Delta^2)^{\frac{1}{2}}} \, \tanh \frac{(\xi^2 + \Delta^2)^{\frac{1}{2}}}{2 k_B \, T} = 1 \tag{62}$$

where g is the electron-phonon coupling constant, ω_D the upper phonon Debye (cut-off) frequency, Δ is the electron energy gap at the Fermi surface, T the temperature and ξ a shifted energy variable defined as

$$\xi \; = \; \frac{\hbar^2 \, k^2}{2 \, m} - \mu \; . \tag{63}$$

In the above expression, μ is the chemical potential, and the quantity $N(0) = m \, k_F / 2\pi^2 \hbar^3$ indicates the density of states for one spin projection near the Fermi surface, with k_F the Fermi wavevector. The above equation then leads, in close analogy to what is done in the nonlinear sigma model, to an estimate for the critical temperature at which the electron gap Δ vanishes

$$k_B \, T_c \; = \; \frac{2 e^\gamma}{\pi} \, \hbar \, \omega_D \, e^{-1/N(0)g} \; . \tag{64}$$

Note that the role of the ultraviolet cutoff here is played by ω_D, and that the critical temperature is non-analytic in the electron–phonon coupling g. Furthermore, the role of the mass gap m in the nonlinear sigma model is played here by the energy gap at the Fermi surface Δ. The condensed spin zero electron Cooper pairs then act as massless Goldstone bosons, and later induce a dynamical Higgs mechanism in the presence of an external electromagnetic field, commonly referred to as the Meissner effect.)
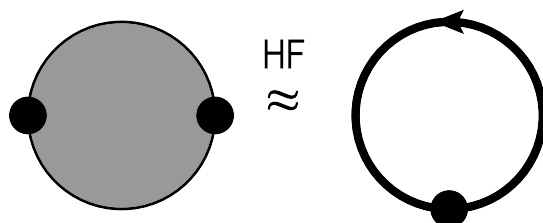

Figure 4. Self-consistent Hartree–Fock approximation for the self-energy of the scalar field.

It now pays to look at some specific dimensions individually, just as will be done later in the case of quantum gravity. Then, specifically, for dimension $d = 2$, one has

$$\int_0^\Lambda dk \, k \, \frac{1}{k^2 + m^2} \; = \; \frac{1}{2} \, \log \left[\frac{m^2 + \Lambda^2}{m^2} \right] \tag{65}$$

leading to the $d = 2$ gap equation

$$(N-1) \frac{T}{4\pi} \log\left(\frac{m^2 + \Lambda^2}{m^2}\right) = 1 \ . \tag{66}$$

The solution is given by

$$m = \Lambda \left[e^{\frac{4\pi}{(N-1)T}} - 1\right]^{-\frac{1}{2}} = \Lambda \, e^{-\frac{2\pi}{(N-1)T}} \left(1 + \tfrac{1}{2} e^{-\frac{4\pi}{(N-1)T}} + \cdots\right) \tag{67}$$

so that here the constant of proportionality between the mass gap m and the scaling parameter $\Lambda \, e^{-\frac{2\pi}{(N-1)T}}$ is exactly one for weak coupling $T = g^2 \ll 1$. Then the above result corresponds to a correlation length exponent $\nu = \infty$ in $d = 2$. On the other hand, in the opposite strong coupling limit $T = g^2 \gg 1$ one has

$$m \simeq \frac{g \sqrt{N-1}}{2 \sqrt{\pi}} \Lambda \tag{68}$$

so that, as expected, the correlation length $\xi = 1/m$ approaches zero in this limit.

Moving up one dimension, in three dimensions, $d = 3$, one has for the basic integral

$$\int_0^\Lambda dk \, k^2 \, \frac{1}{k^2 + m^2} = \Lambda - m \arctan\left(\frac{\Lambda}{m}\right) \tag{69}$$

leading to the $d = 3$ gap equation

$$(N-1) \frac{T}{2\pi^2} \left[\Lambda - m \arctan\left(\frac{\Lambda}{m}\right)\right] = 1 \tag{70}$$

with critical point at

$$T_c = \frac{2\pi^2}{(N-1)\Lambda} \tag{71}$$

for which the mass gap $m = 0$. In the vicinity of the critical point one can solve explicitly for the mass gap m

$$m = \frac{4\pi}{(N-1) T_c^2} |T - T_c| + \frac{4(\pi^2 - 4)}{(N-1)\pi T_c^3} |T - T_c|^2 + \cdots \tag{72}$$

so that in $d = 3$ the universal correlation length exponent is $\nu = 1$. Equivalently, in terms of the dimensionless coupling g, for which $T = g^2/\Lambda$, one has $g_c = \sqrt{2}\,\pi/\sqrt{N-1}$, and therefore

$$m = \Lambda \frac{8\pi}{(N-1) g_c^3} |g - g_c| + \cdots \ . \tag{73}$$

Again, moving up in dimensions, one finds that four dimensions ($d = 4$) represents a marginal case and one obtains

$$\int_0^\Lambda dk \, k^3 \, \frac{1}{k^2 + m^2} = \tfrac{1}{2} \left[\Lambda^2 + m^2 \log\left(\frac{m^2}{m^2 + \Lambda^2}\right)\right] \tag{74}$$

leading to the $d = 4$ gap equation

$$(N-1) \frac{T}{16\pi^2} \left[\Lambda^2 + m^2 \log\left(\frac{m^2}{m^2 + \Lambda^2}\right)\right] = 1 \tag{75}$$

with critical point

$$T_c = \frac{16\pi^2}{(N-1)\Lambda^2} \,, \tag{76}$$

and therefore $g_c = 4\pi/\sqrt{N-1}$ where $m = 0$. The four-dimensional case is slightly more complex, and here one has to determine the solution to the gap equation recursively. In the vicinity of the critical point one finds for the mass gap m

$$m = \Lambda \frac{4\sqrt{2}\pi}{(N-1)^{1/2} g_c^{3/2}} \frac{|g - g_c|^{1/2}}{(-\log|g - g_c|)^{1/2}} + \cdots \,, \tag{77}$$

so that in $d = 4$ the universal correlation length exponent is given by the Landau theory value $\nu = 1/2$, up to logarithmic corrections. This last result is in agreement with triviality arguments for scalar field theories in and above four dimensions [1–3]. Moreover, the logarithmic correction has the right form with the correct power $-1/2$, in agreement with the exact universal result for the Ising model (given here by the $N=1$ case) in four dimensions [68].

Going up one more dimension, in five dimensions ($d = 5$) one has for the integral

$$\int_0^\Lambda dk\, k^4 \frac{1}{k^2 + m^2} = \tfrac{1}{3}\left[\Lambda^3 - 3m^2\Lambda + 3m^3 \arctan\left(\frac{\Lambda}{m}\right)\right] \tag{78}$$

leading to the $d = 5$ gap equation

$$(N-1)\frac{T}{36\pi^3}\left[\Lambda^3 - 3m^2\Lambda + 3m^3 \arctan\left(\frac{\Lambda}{m}\right)\right] = 1 \tag{79}$$

with critical point at

$$T_c = \frac{36\pi^3}{(N-1)\Lambda^3} \,, \tag{80}$$

and therefore $g_c = 6\pi^{3/2}/\sqrt{N-1}$, at which point again $m = 0$. In the vicinity of the critical point one finds for the mass gap m

$$m = \Lambda \frac{2\,6^{1/6}\,\pi}{(N-1)^{2/3} g_c^{7/6}} |g - g_c|^{1/2} + \cdots \tag{81}$$

so that in $d = 5$ (and above) the universal correlation length exponent stays at $\nu = 1/2$, again in agreement with the Landau theory prediction for all scalar field theories above $d = 4$ [1–3].

More generally, the last set of results agree with the general formula for the critical point T_c, valid for both $d < 4$ and $d \geq 4$,

$$T_c = \frac{(d-2)\,2^{d-1}\,\pi^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)}{N-1}\,\Lambda^{2-d} \tag{82}$$

and similarly for the dimensionless critical coupling $g_c = \sqrt{T_c}/\Lambda^{(2-d)/2}$. A dimensionless critical amplitude A_m can be defined by

$$m = \underset{g \to g_c}{\sim}\; A_m\,\Lambda\,|g - g_c|^{1/(d-2)} + \cdots \tag{83}$$

and is given for $d < 4$ explicitly by the expression

$$A_m = \left(\frac{2^{d+1}\,\pi^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{g_c^3\,(N-1)}\right)^{\frac{1}{d-2}} \,. \tag{84}$$

For $d > 4$, the amplitude can be computed explicitly as well, and is given instead by

$$A_m = 2^{\frac{3d-4}{2(d-2)}} (d-2)^{\frac{4-d}{2(d-2)}} (d-4)^{\frac{1}{2}} g_c^{\frac{d+2}{2(2-d)}} n^{-\frac{1}{d-2}} \pi^{\frac{d}{2(d-2)}} \Gamma(\frac{d}{2})^{\frac{1}{d-2}} \tag{85}$$

which shows again that $d = 4$ is indeed a special case, and needs to be treated with some care. One also notes that both the $d < 4$ and the $d > 4$ amplitudes vanish as one approaches $d = 4$ due to the infrared divergence in this case, which leads to the *log* correction described above.

The previous, rather detailed, discussion shows that the Hartee–Fock approximation (and, in this case, the equivalent large-N limit), based on a single one loop tadpole diagram, leads to a number of basically correct and nontrivial analytical results in various dimensions, which we proceed to enumerate here. The first one is the fact that it correctly predicts the absence of a phase transition, and thus asymptotic freedom in the coupling g, for any $N > 1$ in two dimensions. This result is indeed known to be correct up to $N = 2$, the latter representing the special case of the Kosterlitz–Thouless vortex unbinding transition in the two dimensional planar model. The beta function in this case becomes then exactly correct for weak coupling, if one replaces the saddle point value $N - 1$ by $N - 2$ in the relevant expressions.

Second, in three dimensions the theory is known not to be perturbatively renormalizable. Nevertheless, the Hartree–Fock approximation correctly predicts the existence of a critical point (a phase transition) at a finite T_c, and the correspondingly modified scaling relations. In renormalization group language, a critical point corresponds to a nontrivial fixed point associated with the renormalization group trajectories, here in the dimensionless relevant coupling g. The critical exponent $\nu = 1$ is somewhat higher than the accepted value for the Heisenberg model $N = 3$ ($\nu = 0.710$) [54], nevertheless not too far off, given the crudeness of the approximation, based on a single loop diagram.

Furthermore, for dimension d between two (lower critical) and four (upper critical), one has

$$\nu = \frac{1}{d-2} \qquad \eta = 0 \qquad \gamma = \frac{2}{d-2} \tag{86}$$

where η is the field (or propagator) anomalous dimension, $G(p) \sim 1/p^{2-\eta}$ in the vicinity of the nontrivial fixed point, and γ the zero-field magnetic susceptibility exponent, $\chi \sim \xi^{\gamma/\nu}$, again in the vicinity of the nontrivial fixed point at g_c. In four dimensions the Hartree–Fock approximation correctly reproduces the expectation of mean field exponents ($\nu = 1/2$ here) up to a logarithmic correction. This is not surprising, as some sort of mean field theory is incorporated into the very nature of the Hartree–Fock approximation. Above four dimensions the results again reflect, correctly, the expectation that mean field theory exponents, and in particular $\nu = 1/2$, should apply to all $d \geq 4$, and give an explicit analytic expression for the mass gap m, the critical coupling T_c and the corresponding amplitude A_m as a function of d. Thus, the Hartree–Fock approximation as discussed here gives correctly for the upper critical dimension of the $O(N)$ nonlinear sigma model $d = 4$ independent of N, whereas the lower critical dimension here is correctly $d = 2$, at least for $N > 2$.

To conclude this section, a few words should be spent on the physical interpretation of the mass gap parameter m. It is possible to describe its properties either within the context of Lorentzian quantum field theory with its generating functional and vacuum expectation values, or alternatively in the context of statistical mechanics and Euclidean quantum field theory, with its correlation functions and thermodynamic averages. The two descriptions are of course expected to be equivalent and complemetary, related to each other by a Wick rotation. Therefore, when one refers to the mass gap, it means the gap in energy between the first excited stated and the ground state, $m = E_1 - E_0$ as derived from the quantum Hamiltonian or transfer matrix describing, in this case, the nonlinear sigma model. Its relationship with the Euclidean correlation length ξ is provided by the Lehman representation (i.e., completeness) for the two-point function, for

example, which then gives $m = 1/\xi$. In addition, ξ and thus m are related by scaling to a multitude of other observables, such as the order parameter or magnetization $\langle \sigma \rangle$ in the low temperature phase. As an example, in the nonlinear sigma model $\langle \sigma \rangle \sim 1/\xi^{\beta/\nu}$ by renormalization group scaling, where β is the magnetic exponent ($\approx 1/3$ in $d = 3$) and ν the correlation length exponent. This shows more generally that the mass gap is in some way directly related to the vacuum expectation value of the field, at least in the ordered phase. For Gaussian fields $\langle \sigma \rangle \sim 1/\xi$ in four dimensions in the ordered phase, simply by dimensional arguments.

6. Gauge Theories

Before tackling the most complex case of quantum gravity, it will be useful to first explore the implications of these ideas to a new and slightly more complex, but still familiar case—the $SU(N)$ gauge theory. In the case of $SU(N)$ Yang–Mills theories, one can follow a similar line of analysis to develop the Hartree–Fock approximation. Overall, the strategy is again to rely on well-known results for one-loop diagrams, suitably modified by the insertion of an effective, dynamically generated mass parameter. The latter is then determined self-consistently by a suitable nonlinear gap equation. There are a number of aspects which are quite similar to the nonlinear sigma model case, which will be helpful here in vastly streamlining the discussion. In particular, in gauge theories the gluon propagator is modified by the gauge and matter vacuum polarization contribution $\delta^{ab}\Pi_{\mu\nu}(p)$ with

$$\Pi_{\mu\nu}(p) = (p^2 \eta_{\mu\nu} - p_\mu p_\nu) \Pi(p^2) \tag{87}$$

with $p_\mu \Pi^{\mu\nu}(p) = 0$. In the Fermi–Feynman gauge, the gluon propagator then becomes

$$D_{\mu\nu}^{ab}(p) = \frac{\delta^{ab} \eta_{\mu\nu}}{p^2 [1 + \Pi(p^2)]} . \tag{88}$$

In perturbation theory, and to all orders, the gluon stays massless as a consequence of gauge invariance, with $p^2 \Pi(p^2) \sim_{p\to 0} 0$ (here we will assume no spontaneous symmetry breaking via the Higgs mechanism). In more detail, the relevant one loop integral has the form

$$\int \frac{d^d k}{(2\pi)^d} \frac{V(p, -k-p, k) \cdot V(-p, -k, k+p)}{k^2 (k+p)^2} \tag{89}$$

where the dot here indicates a dot product over the relevant Lorentz indices associated with the three-gluon, matter, or ghost vertices. Additional factors involve a symmetry factor $\frac{1}{2}$ for the diagram, the gauge coupling g^2 weight, and an overall group theory color factor. Figure 5 shows the lowest order Feynman diagrams contribution to the vacuum polarization tensor in gauge theories, Figure 6 the equation relating the bare gluon propagator to the dressed one via the proper vacuum polarization contribution.
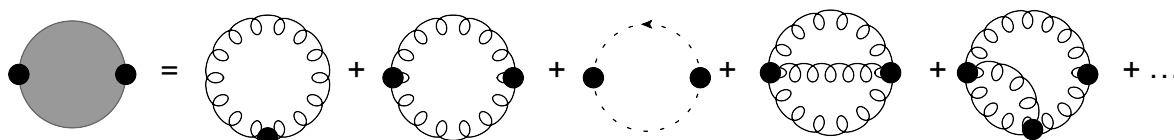
Figure 5. Pure Yang–Mills vacuum polarization contributions to lowest order in perturbation theory. The dashed line represents the ghost contribution.

Figure 7 then represents Dyson's equation for the dressed gluon propagator (as written out in terms of the bare propagator and bare and dressed vertices), while Figure 8 illustrates Dyson's equation for the proper gluon vacuum polarization term.
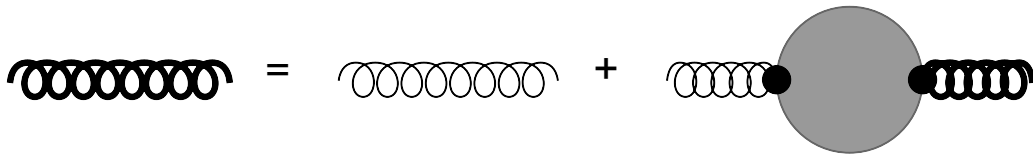
Figure 6. Proper vacuum polarization in Yang–Mills theories (the large gray shaded blob).
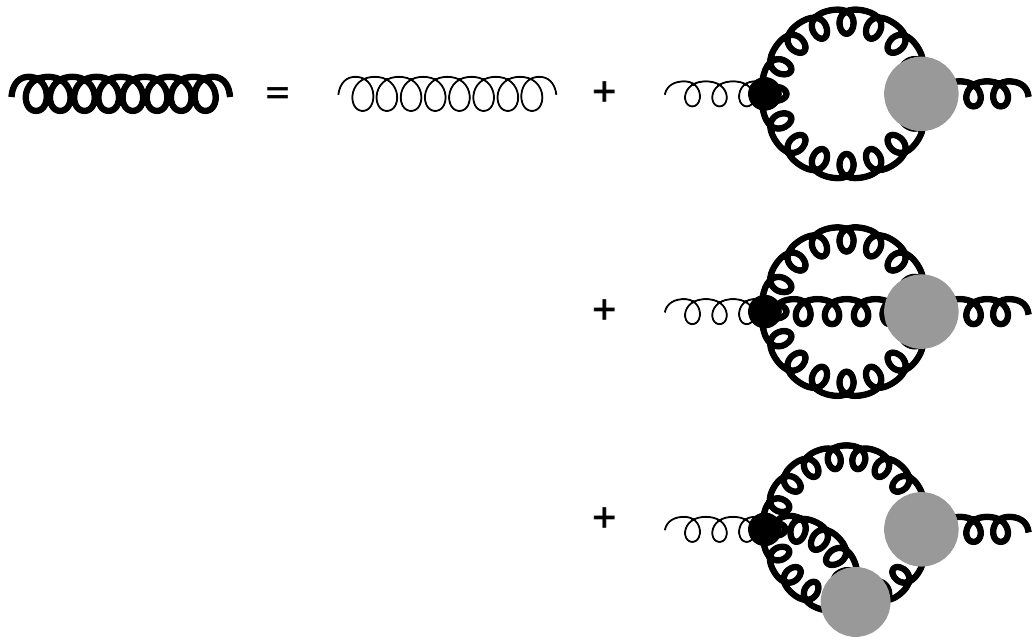

Figure 7. Dyson's equations for the dressed Yang–Mills boson propagator. Ghost contributions have been omitted from the diagrams.
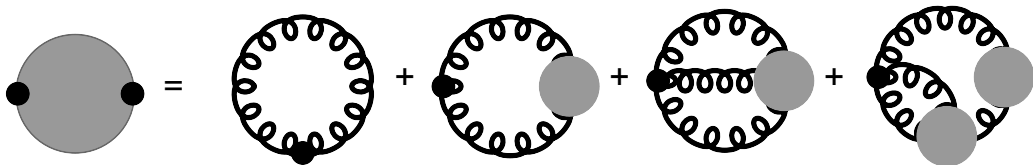

Figure 8. Dyson's equations for the Yang–Mills gauge boson vacuum polarization tensor.

Finally, Figure 9 illustrates the Hartree–Fock equation for gauge theories. Consequently, in the Hartree–Fock approximation (and in close analogy to the treatment of the nonlinear sigma model described previously, specifically in Section 5) one is left with the evaluation of the following one-loop integral,

$$S_d \int_0^\Lambda dk\, k^{d-1}\, \frac{k^{2\sigma}}{(k^2+m^2)^2} \quad, \tag{90}$$

with again $S_d = 1/2^{d-1}\,\pi^{d/2}\,\Gamma(d/2)$ and $\sigma = 0$ in the gauge theory case. Here, Λ is an ultraviolet cut-off, such as the one implemented in Wilson's lattice gauge theory formulation, for which $\Lambda \sim \pi/a$ with a the lattice spacing. Note the insertion of a mass parameter m, to be determined later self-consistently by the gap equation given below.

One finds for the integral itself in general dimensions the explicit result

$$\int_0^\Lambda dk\, k^{d-1}\, \frac{1}{(k^2+m^2)^2} = \frac{\Lambda^d}{2\,d\,m^2\,(m^2+\Lambda^2)} + \left(\frac{1}{d} - \frac{1}{2}\right)\frac{\Lambda^d}{d\,m^4}\, {}_2F_1(1,\frac{d}{2},1+\frac{d}{2},-\frac{\Lambda^2}{m^2}) \tag{91}$$

with $_2F_1(a, b, c, z)$ the generalized hypergeometric function. The general gap equation for the dynamically generated mass scale m then reads

$$g^2 \, \beta_0 \, S_d \left[\frac{\Lambda^d}{2 \, d \, m^2 \, (m^2 + \Lambda^2)} + \left(\frac{1}{d} - \frac{1}{2} \right) \frac{\Lambda^d}{d \, m^4} \, {}_2F_1(1, \frac{d}{2}, \frac{1+d}{2}, -\frac{\Lambda^2}{m^2}) \right] = 1 \, , \quad (92)$$

with g the gauge coupling and β_0 an overall N-dependent numerical coefficient resulting from combined group theory weights, Lorentz traces and diagram symmetry factors.
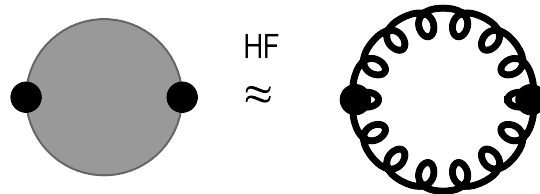
Figure 9. Self-consistent Hartree–Fock approximation for the gauge boson proper self-energy.

As in the case of the nonlinear sigma model discussed previously, it will be instructive to look in detail at individual dimensions. In four dimensions ($d = 4$) the gap equation reduces to

$$\frac{1}{2} \, g^2 \, \beta_0 \, S_4 \left[\log \left(1 + \frac{\Lambda^2}{m^2} \right) - \frac{\Lambda^2}{m^2 + \Lambda^2} \right] = 1 \, . \quad (93)$$

The solution to the above equation is

$$m = \Lambda \, e^{-\frac{1}{2}} \, e^{-\frac{1}{2 \beta_0 g^2}} = \Lambda \, e^{-\frac{1}{2}} \, e^{-\frac{24 \pi^2}{11 N g^2}} \, . \quad (94)$$

Here, at the end the correct expression $\beta_0 = 11N/3$ for $SU(N)$ gauge theories was inserted, as obtained from one loop perturbation theory. The above explicit results then implies, within the current Hartree–Fock approximation, that the mass gap m is by a factor $e^{-\frac{1}{2}} = 0.6065$ smaller than the scaling violation parameter $\Lambda \exp(-1/2 \, \beta_0 \, g^2)$, with the latter appearing on the r.h.s. of Equation (94). Note that one could have obtained the overall coefficient of the one loop contribution, as used here in the Hartree–Fock approximation, from the Nielsen–Hughes formula [69–71]. There the one-loop β-function contribution coefficient arising from a particle of spin s running around the loop is given generally by $\beta = -(-1)^{2s} \, [(2s)^2 - \frac{1}{3}]$, with the two individual competing contributions arising from a generalization of Pauli diamagnetism and Landau paramagnetism in four spacetime dimensions. For spin $s = 1$ the above formula then gives indeed for the pure $SU(N)$ gauge theory in four dimensions the expected result, $\beta_0 = 11N/3$ (with, in this case, an overall negative sign for the beta function, correctly taken into account previously).

The renormalization group running of the gauge coupling $g(\mu)$ again follows from the requirement that the mass gap m be scale-independent,

$$\mu \frac{d \, m}{d \, \mu} = 0 = \mu \frac{d}{d \, \mu} \left(\mu \, e^{-\frac{1}{2}} \, e^{-\frac{1}{2 \beta_0 \, g(\mu)^2}} \right) \, , \quad (95)$$

which then gives the expected asymptotic freedom running of $g(\mu)$ in the vicinity of the ultraviolet fixed point at $g = 0$,

$$g^2(\mu) = \frac{1}{2 \, \beta_0 \, [\log(\mu/m) - 1]} \, . \quad (96)$$

One observes that the quantity m here plays a role analogous to the $\Lambda_{\overline{MS}} \approx 330$ MeV parameter used, for example, to describe deep inelastic scattering in QCD. For short enough distances the result of Equation (96) implies that the static potential between quarks behaves as $V(r) \sim 1/r \, log(r)$, a Coulomb potential with a weak quantum logarithmic correction,

of course a well-known result already from perturbation theory. It should be noted also that so far the gauge theory result in $d = 4$ looks rather similar to the $d = 2$ result for the nonlinear sigma model (for $N > 2$), in the sense that both models have a mass gap that vanishes only at $g = 0$, and also exhibit asymptotic freedom in the relevant coupling. This fact is of course already well known from the simple, direct application of perturbation theory; nevertheless, it is encouraging that both of those features are correctly reproduced by the lowest order Hartree–Fock approximation.

Gauge theories are mostly relevant in four spacetime dimensions. As a purely academic exercise one can nevertheless look at $d = 5$, where Yang–Mill theories—just like the nonlinear sigma model for $d > 2$ - are not perturbatively renormalizable. In this case, one needs the integral

$$\int_0^{\Lambda} dk\, k^5 \, \frac{1}{(k^2 + m^2)^2} \; = \; \Lambda + \frac{m^2 \, \Lambda}{2\,(m^2 + \Lambda^2)} - \frac{3}{2}\, m \, \arctan\left(\frac{\Lambda}{m}\right) \; . \tag{97}$$

For small mass gap parameter m, the gap equation in five dimensions then reads

$$g^2 \, \beta_0 \, S_5 \left[\Lambda - \frac{3\pi}{4}\, m \right] \; = \; 1 \; . \tag{98}$$

The solution for $g > g_c$ is then given by

$$m \; = \; \Lambda \cdot \frac{8}{3\pi g_c} \, (g - g_c) + \mathcal{O}\left((g - g_c)^2 \right) \; . \tag{99}$$

There is a nontrivial fixed point at $g_c^2 = 12\pi^3/\Lambda$, and the correlation length exponent is $\nu = 1$, just like for the $d = 3$ nonlinear sigma model discussed previously. For the gauge theory case one expects that the critical point at g_c separates a weak coupling Coulomb phase from a confining strong coupling case. Then, as d approaches four from above, the weak coupling massless gluon Coulomb phase disappears.

The renormalization group running of the gauge coupling $g(\mu)$ close to the nontrivial ultraviolet fixed point then follows again from the requirement that the mass gap m be scale independent, $\mu \frac{d\,m}{d\,\mu} = 0$. In the vicinity of the five-dimensional ultraviolet fixed point, and within the strong coupling phase ($g > g_c$) one obtains

$$g(\mu) \; = \; g_c \left[1 + \frac{3\pi}{8} \cdot \left(\frac{m}{\mu}\right) + \mathcal{O}\left(\left(\frac{m}{\mu}\right)^2 \right) \right] \; . \tag{100}$$

Note that, in this case, the power associated with the running in momentum space is determined again by the exponent $1/\nu = 1$, and that the coefficient of the running term here is in fact independent of β_0. The result of Equation (100) in turn implies that in real space the effective coupling constant initially grows linearly with distance in the strong coupling phase, $g(r) \sim r/\xi$. At this stage one would conclude that the gauge theory result in $d = 5$ bears some similarity to the $d = 3$ result for the nonlinear sigma model, in the sense that both models have a critical point located at some finite value of the bare coupling, and have an exponent $\nu = 1$, at least within the Hartee–Fock approximation.

In fact, it is possible to get a general yet simple analytical result from the Hartree–Fock result for larger d as well, based on a general expression for the momentum integral. For a large enough dimension, $d > 6$, one has

$$g_c^2 \; = \; \frac{(d-4)\, 2^{d-1}\, \pi^{d/2}\, \Gamma(d/2)}{\beta_0 \, \Lambda^{d-4}} \; . \tag{101}$$

The solution for m is then, for $g > g_c$ and $d > 6$,

$$m = \Lambda \left(\frac{d-6}{(d-4)\,g_c} \right)^{1/2} (g - g_c)^{1/2} + \mathcal{O}\left((g - g_c)^{3/2} \right) , \tag{102}$$

which implies $\nu = 1/2$ for any $d > 6$, just like the case $d > 4$ for the nonlinear sigma model discussed earlier. Note that six dimensions is the borderline case here, with logarithmic corrections to a pure power law behavior, in analogy to the $d = 4$ case for the nonlinear sigma model, as given in Equation (77).

Again, the renormalization group running of $g(\mu)$ follows from the requirement that the mass gap m be scale independent, and in the vicinity of the ultraviolet fixed point, and in the strong coupling phase $g > g_c$, one finds here

$$g(\mu) = g_c \left[1 + \frac{d-4}{d-6} \left(\frac{m}{\mu} \right)^2 + \mathcal{O}\left(\left(\frac{m}{\mu} \right)^4 \right) \right] . \tag{103}$$

It follows that the power associated with the running in momentum space is always given by the exponent $1/\nu = 2$ for $d > 6$. The result of Equation (103) in turn implies that in real space the effective coupling constant initially grows with the square of the distance, above $d = 6$ and in the strong coupling phase, $g(r) \sim (r/\xi)^2$. Note that, when written in this form, the coefficient of the running term is in fact independent of the magnitude of β_0.

Overall, the general conclusion here is that the gauge theory result in $d > 6$ looks rather similar to the $d > 4$ result for the nonlinear sigma model, in the sense that both models have a critical point at some value of the bare coupling, and have exponent $\nu = 1/2$, at least within the Hartee–Fock approximation used here. In other words, the Hartree–Fock approximation gives for the upper critical dimension of gauge theories $d = 6$ independent of N, whereas the lower critical dimension here is $d = 4$, again independent of $N > 1$. As a comparison, we note that some time ago it was suggested that the upper critical dimension for gauge theories is either $d = 8$ based on the theory of random surfaces [72], or possibly $d = \infty$ [73] when considering the strong coupling expansion of lattice gauge theories at large d. If one denotes by d_H the fractal (Hausdorff) dimension of Wiener paths contributing to the Feynman path integral, then in the Hartree–Fock approximation discussed here an upper critical dimension of six would imply $d_H = 3$, if the arguments in [74–76] are followed here as well. On the other hand, for a Gaussian scalar field (a free particle with no spin) it is known that $d_H = 2$, as for regular Brownian motion [77–79].

A few words should be spent here on the physical interpretation of m in gauge theories. When one refers to the mass gap, it means a gap in energy between the ground state and the first excited stated, $m = E_1 - E_0$, as derived from the quantum Hamiltonian (or transfer matrix) describing, in this case, the nonlinear sigma model. It's relationship with the Euclidean correlation length ξ is provided by the Lehman representation (i.e., completeness) for the two point function (as an example), which then gives $m = 1/\xi$. In addition, ξ, and thus m, are related by scaling to a multitude of other observables, such as the order parameter or magnetization $\langle \sigma \rangle$ in the low temperature phase for spin system, and the gluon condensate $\langle F_{\mu\nu}^2 \rangle$ for gauge theories. In the gauge theory case one has a fundamental relationship between the nonperturbative scale $\xi = m^{-1}$ and a nonvanishing vacuum expectation value for the gluon field [80,81].

$$\langle F_{\mu\nu}^2 \rangle \simeq \frac{1}{\xi^4} . \tag{104}$$

In QCD, this last result is obtained from purely dimensional arguments, once the existence of a fundamental correlation length ξ (inversely related to the mass gap) is established. Actual physical values for the QCD condensates are well known; current lattice and phenomenological estimates cluster around $\langle \frac{\alpha_S}{\pi} F_{\mu\nu}^2 \rangle \simeq (440\,\text{MeV})^4$ [82,83]. In gauge theories an additional physically related quantity is provided by the quark field condensate

$$\langle \bar{\psi}\psi \rangle \simeq \frac{1}{\zeta^3} \tag{105}$$

whose physical value is estimated at $\langle \bar{\psi}\psi \rangle \simeq (280\,\text{MeV})^3$ [84]. Again, the power of ζ here is fixed by the canonical dimension of the corresponding fermion field.

A possible physical explicit value for the mass gap parameter m in N=3 QCD is provided by the $J^{PC} = 0^{++}$ spin zero 500 MeV σ or $f_0(500)$ glueball state, a very broad resonance seen in $\pi\pi$ and $\gamma\gamma$ scattering [85]. If this particle is roughly considered as a spin-zero bound state of two (effectively massive) spin-one gluons with antiparallel spins, then that would give for the mass gap parameter $m \approx 250\,\text{MeV}$. For the scaling violation parameter on the r.h.s. of Equation (94) one then obtains a value $e^{\frac{1}{2}} = 1.649$ larger, or around 410 MeV. The latter is not too far off from the experimentally well-established value of $\Lambda_{\overline{MS}} \approx 330\,\text{MeV}$ for QCD (with three light quark flavors). Conversely, if one uses the current world average of $\Lambda_{\overline{MS}}^{(3)} \approx (332 \pm 17)\,\text{MeV}$ [86] for three flavor QCD, then one obtains for the mass of the σ / f_0 meson $m_\sigma \approx 403\,\text{MeV}$. These results seem to suggest that quantitatively the lowest order Hartree–Fock result is not expected to be better than a factor or two or so. Many of these considerations will be useful later when the gravity case is discussed, due to the many deep analogies between gauge theories and quantum gravity.

7. The Quantum Gravity Case

Upon gaining confidence in this approach and technique, one can again perform a similar type of analysis in the case of quantum gravity, as outlined in detail in the preceding sections. In the end, one major result which makes the calculation feasible is the reliance on well-known results for simple one loop diagrams. This important insight makes it possible to avoid lengthy and complex one-loop gravity calculations, and rely instead, as in all the previous cases, on suitably modified known diagrams via the insertion of an effective, dynamically generated mass parameter m. The latter is then determined self-consistently via the solution of the resulting nonlinear gap equation, as discussed in detail for the nonlinear sigma model (in Section 5), and for gauge theories (in Section 6).

A number of new ingredients arise in the quantum gravity calculation, which we proceed to enumerate here. The first component is the (gauge choice dependent) graviton propagator

$$D_{\alpha\beta,\mu\nu}(p) \;, \tag{106}$$

with the above expression denoting in general the dressed propagator, and $D^0_{\alpha\beta,\mu\nu}(p)$ the bare (tree-level) one. An explicit form for the tree-level graviton propagator in covariant gauges, as well as the three-graviton and four-graviton vertex, can be found in the Feynman rules section in [14] and the references therein, and for brevity will not be reproduced here. Occasionally, in the following it will be convenient to suppress Lorentz indices altogether in order to avoid unnecessary extensive cluttering. Via loop corrections, the tree-level graviton propagator is then modified by gauge and matter vacuum polarization contributions $\Pi_{\mu\nu,\rho\sigma}(p)$, with the latter written as

$$\Pi_{\alpha\beta,\mu\nu}(p) \;=\; (p^2\,\eta_{\alpha\beta} - p_\alpha\,p_\beta)\,(p^2\,\eta_{\mu\nu} - p_\mu\,p_\nu)\,\Pi(p^2) \tag{107}$$

in terms of the scalar quantity $\Pi(p^2)$. Here, η denotes the flat spacetime metric. The vacuum polarization contribution in quantum gravity is shown pictorially in Figure 10. By virtue of energy-momentum conservation, one then has the transversality condition $p_\alpha\,\Pi^{\alpha\beta,\mu\nu}(p) = p_\mu\,\Pi^{\alpha\beta,\mu\nu}(p) = 0$, and the graviton propagator (in the harmonic gauge [14,21]) can be written as

$$D_{\alpha\beta,\mu\nu}(p) \;=\; \frac{I_{\alpha\beta,\mu\nu}}{p^2\,[\,1 + \Pi(p^2)\,]} \tag{108}$$

with $I_{\alpha\beta,\mu\nu} = \eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\mu\nu}\eta_{\alpha\beta}$. As a consequence of gauge invariance (here more properly described as general coordinate, or diffeomorphism, invariance) in perturbation theory, and to all orders, the graviton is expected to stay massless, $p^2\,\Pi(p^2)\ \sim_{p\to 0}\ 0$. Here, we will find it quite useful that an explicit form for the vacuum polarization contribution $\Pi_{\alpha\beta,\mu\nu}(p)$ in the context of perturbative quantum gravity was given some time ago in [22], and such an explicit form will be used below.
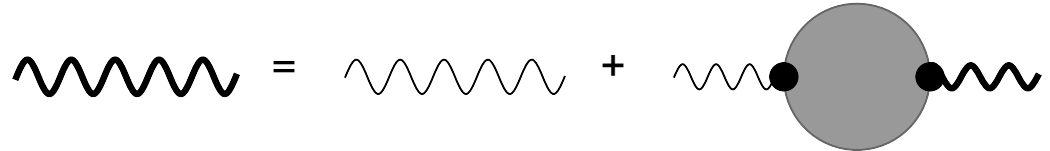
Figure 10. Dressed graviton propagator $D_{\alpha\beta,\mu\nu}(p)$, with a graviton proper vacuum polarization insertion $\Pi_{\alpha\beta,\mu\nu}(p)$. The thin wavy line denotes the tree-level graviton propagator $D^0_{\alpha\beta,\mu\nu}(p)$.

In addition, one has the dressed three-graviton vertex (shown here in Figure 11)

$$V^{(3)}_{\alpha\beta,\mu\nu,\rho\sigma}(p,q,r) \tag{109}$$

and the dressed four-graviton vertex (shown here in Figure 12)

$$V^{(4)}_{\alpha\beta,\gamma\delta,\mu\nu,\rho\sigma}(p,q,r,s)\ , \tag{110}$$

and similarly for the $\mathcal{O}(h^n)$ higher order graviton vertices, as they arise in the weak field expansion of the Einstein–Hilbert action about flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Recall that for the Einstein–Hilbert action $\int \sqrt{g}\,R$ written in terms of the weak field metric field $h_{\mu\nu}$ one has

$$\sqrt{g} = 1 + \tfrac{1}{2}h - \tfrac{1}{4}h_{\mu\nu}h^{\mu\nu} + \tfrac{1}{8}h^2 + O(h^3) \tag{111}$$

with trace $h = h^\mu{}_\mu$, and consequently (up to total derivatives) one obtains schematically

$$I_{EH} = \frac{1}{G}\int d^d x\,\sqrt{1+h}\,\partial^2 h \ \sim\ \frac{1}{G}\int d^d x\left[\tfrac{1}{2}h\,\square\,h - \tfrac{1}{8}h^2\,\square\,h + \mathcal{O}(h^4)\right]. \tag{112}$$

The latter shows the origin of the trilinear (h^3) vertex, with weight proportional to momenta squared, $\sim k^2$ in Fourier space, as well as the appearance of the (infinitely many) higher order momentum-dependent vertices $\mathcal{O}(h^n)$.
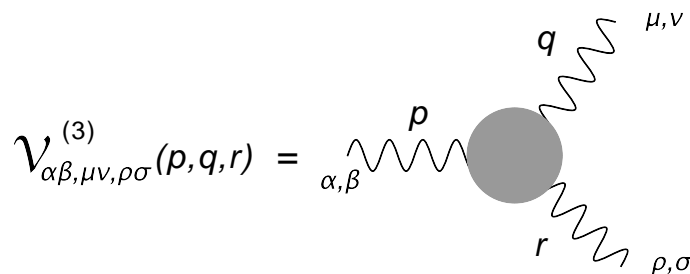
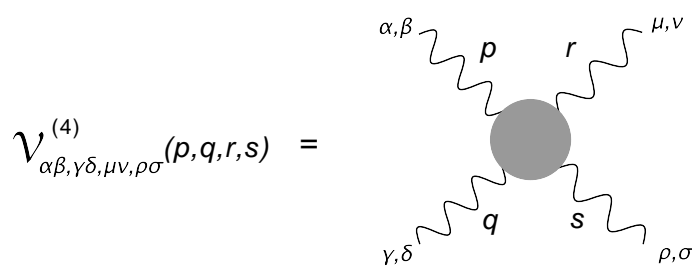**Figure 11.** Graviton three-point vertex $V^{(3)}_{\alpha\beta,\mu\nu,\rho\sigma}(p,q,r)$.

Figure 12. Graviton four-point vertex $V^{(4)}_{\alpha\beta,\gamma\delta,\mu\nu,\rho\sigma}(p,q,r,s)$.

Two additional steps are needed in order to systematically develop the Hartree–Fock approximation for quantum gravity. First, and in close analogy to the previous discussion of the nonlinear sigma model and gauge theories, one needs to write down Dyson's coupled equations for the graviton propagator, vacuum polarization, and proper vertices. The lowest order Feynman diagrams (in a systematic perturbative expansion in Newton's constant G) as they contribute to the gravitational vacuum polarization are shown, as an illustration, in Figure 13. These generally contain contributions from graviton, matter, and ghost loops. To derive the full set of Dyson's equations one then follows the same procedure as in QED and Yang–Mills theories, as outlined earlier in this paper in Section 2. First, one writes down the full Feynman path integral (including gauge fixing and ghost terms), including a classical source term $g_{\mu\nu}(x) J^{\mu\nu}(x)$ for the gravitational field, as well as any additional fields present such as ghost and matter contributions. One then applies suitable combinations of functional derivatives of the generating function $Z[J]$ (as in Equations (16)–(18)) to obtain the full set of coupled integro-differential equations for the gravitational n-point functions, including the quantum version of Einstein's field equations (in close analogy to the QED result of Equations (19), (23) and (24)). As an example, Figure 10 illustrates the relationship between the tree-level graviton propagator to the dressed one via the proper gravity vacuum polarization contribution, while Figure 13 shows the lowest order Feynman diagram contributions to the vacuum polarization tensor for gravity.
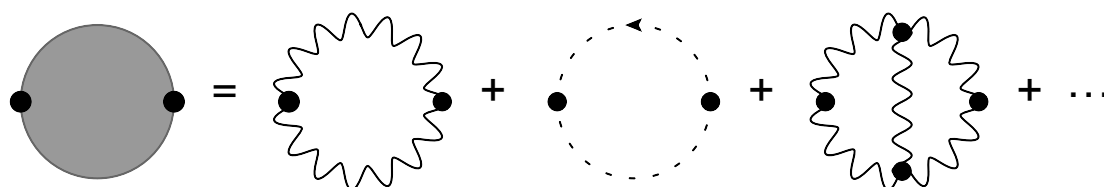

Figure 13. Quantum gravity lowest order vacuum polarization $\Pi_{\alpha\beta,\mu\nu}(p)$ diagrams. A thin wavy lines denotes a bare graviton propagator, and small sized dots stand for bare graviton vertices. A dashed line appears in the ghost loop. Dots indicate higher order corrections.

Next, we come to the actual form of Dyson's equations for quantum gravity. Figure 14 shows Dyson's equation for the dressed graviton propagator (as written out in terms of the bare propagator and bare, as well as dressed, vertices), while Figure 15 illustrates Dyson's equation for the proper gravity vacuum polarization term. Here we give explicitly Dyson's equation for the dressed graviton propagator $D(p)$, in terms of the bare propagator $D^0(p)$ and the trilinear bare $V^{(3),0}$ and dressed $V^{(3)}$, as well as the quadrilinear $V^{(4),0}$ and $V^{(4)}$ vertices, with the dots indicating contributions from higher order vertices $V^{(n),0}$ and $V^{(n)}$ with $n > 4$. Written out explicitly, the equation reads

$$D(p) = D^0(p)$$

$$+ \frac{1}{2} \; D^0(p) \cdot \int \frac{d^d q}{(2\pi)^d} \; V^{(3),0}(p, q, -p-q) \cdot D(p+q) \cdot V^{(3)}(p+q, -q, p) \cdot D(q) \cdot D(p)$$

$$+ \frac{1}{3!} \; D^0(p) \cdot \int \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}} \; V^{(4),0}(p, -q_1, q_2, -p-q_2-q_1) \cdot D(p+q_2-q_1)$$
$$\cdot V^{(3)}(p+q_2-q_1, q_1-q_2, -p) \cdot D(q_1-q_2) \cdot V^{(3)}(q_1, -q_2, -q_1+q_2) \cdot D(q_2) \qquad (113)$$
$$\cdot D(q_1) \cdot D(p)$$

$$+ \frac{1}{3!} \; D^0(p) \cdot \int \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}} \; V^{(4),0}(p, -q_2, q_1, -p-q_1+q_2) \cdot D(p+q_1-q_2)$$
$$\cdot V^{(4)}(-q_1, q_2, -p, p+q_1-q_2) \cdot D(q_1) \cdot D(q_2) \cdot D(p)$$

$$+ \cdots$$

The next step is the derivation of the Hartree–Fock approximation for quantum gravity. As should be clear from the various cases discussed earlier, namely, the nonlinear sigma model (discussed in Section 5) and gauge theories (discussed in Section 6), one needs to focus on just the lowest order loop diagrams. Nevertheless, these will have bare graviton propagators replaced by dressed ones, later to be determined self-consistently. Figure 16 illustrates the Hartree–Fock approximation for gravity, carried out to the lowest order by just considering the lowest order graviton loop diagram, the subject of the current investigation. On the other hand, Figure 17 illustrates the next order Hartree–Fock approximation for gravity, which would include both the lowest order graviton loop diagram, as well as the next order graviton two-loop diagrams. These will not be considered further here, but could in the future provide an improved answer, as well as useful quantitative estimates for the overall uncertainty.
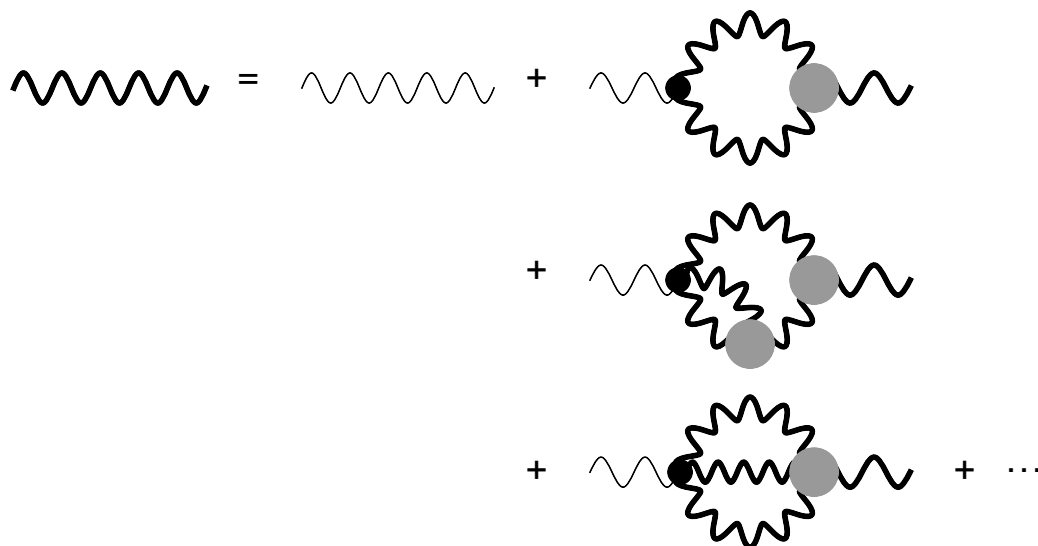

Figure 14. Dyson's equations for the dressed graviton propagator $D_{\alpha\beta,\mu\nu}(p)$. Small dots denote bare graviton vertices, while larger dots stand for dressed vertices. Thin wavy lines denote bare graviton propagators, while thicker lines stand for dressed propagators.
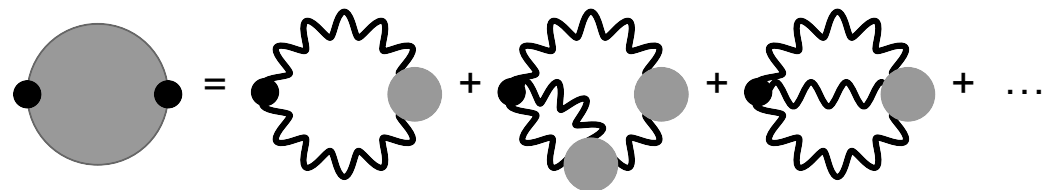
Figure 15. Dyson's equations for the proper graviton vacuum polarization insertion $\Pi_{\alpha\beta,\mu\nu}(p)$. Small dots denote bare graviton vertices, while larger dots stand for dressed vertices. A thin wavy lines denotes a bare graviton propagator, while thicker lines stand for dressed propagators.
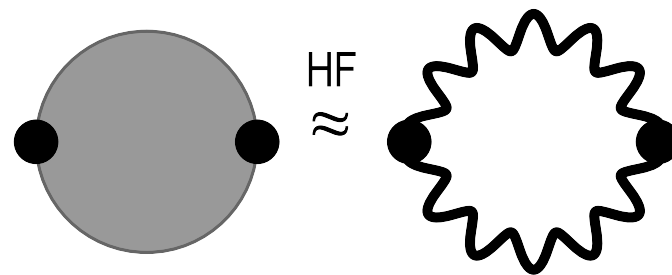
Figure 16. Self-consistent Hartree–Fock approximation for the graviton proper vacuum polarization tensor. Small dots denote bare graviton vertices, and the thick wavy line indicates a dressed graviton propagator, to be determined self-consistently.
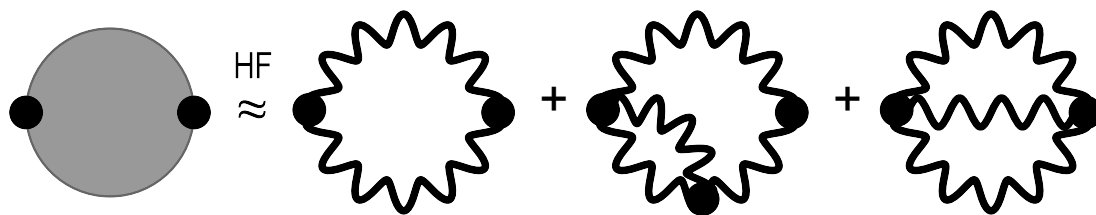
Figure 17. Next higher order terms in the Hartree–Fock approximation to the graviton proper vacuum polarization tensor. Small dots denote bare graviton vertices, and the thick wavy line indicates a dressed graviton propagator, to be determined later self-consistently.

In practice, for quantum gravity the relevant one loop integral for the vacuum polarization has the form

$$\int \frac{d^d k}{(2\pi)^d} \frac{V(p, -k - p, k) \cdot V(-p, -k, k + p)}{k^2 (k + p)^2} \quad , \tag{114}$$

where the dot here indicates a generalized dot product over the relevant multitude of Lorentz indices associated with the three-graviton, matter, and ghost vertices. Additional ingredients involve a symmetry factor $\frac{1}{2}$ for the diagram, and of course the gravitational coupling G. In spite of the complexity of the one-loop expression for the graviton vacuum polarization contribution, the actual integral that needs to be evaluated here has a rather simple form. In the Hartree–Fock approximation, and in close analogy to the treatment of the nonlinear sigma model and gauge theories described previously, one is generally left with an evaluation of the following integral,

$$S_d \int_0^\Lambda dk \, k^{d-1} \frac{k^{2\sigma}}{(k^2 + m^2)^2} \tag{115}$$

with a combined spacetime volume factor $S_d = 1/2^{d-1} \pi^{d/2} \Gamma(d/2)$, but now with $\sigma = 2$, specific to the gravitational case. The latter steep momentum dependence is of course at the root of the perturbative non-renormalizability of quantum gravity in four dimen-

sion [15–29]: the significant weight of the high momentum region in loop integrals renders perturbation theory in Newton's *G* badly divergent.

Going back to the required integral of Equation (115), one finds for the momentum integral itself, in general dimension *d*,

$$\int_0^\Lambda dk\, k^{d-1} \frac{k^{2\sigma}}{(k^2+m^2)^2} = \frac{\Lambda^{d+2}\,\sigma}{2\,m^2\,(m^2+\Lambda^2)} - \frac{d+2\,\sigma-2}{d+2\,\sigma}\,\frac{\Lambda^{d+2\sigma}}{2\,m^4}\,{}_2F_1(1,\frac{d}{2}+\sigma,1+\frac{d}{2}+\sigma,-\frac{\Lambda^2}{m^2})\ ,\tag{116}$$

with ${}_2F_1(a,b,c,z)$ the generalized hypergeometric function. Here, again Λ is the ultraviolet cut-off, such as one implemented in the Regge–Wheeler lattice formulation of gravity [12,13], for which $\Lambda \sim \pi/a$, with a defined as an average lattice spacing $a^2 \equiv \langle l^2 \rangle$. Indeed, as in all the previous examples, an explicit ultraviolet cutoff is required in order to be able to write down a meaningful form of the Hartree–Fock equation. Generally, for any quantum mechanical system a lattice cut-off is required to make the Feynman path integral well defined, as discussed already in great detail some time ago by the inventor himself [77], and in some more recent monographs [78,79]. The root cause of this situation is that the dominant quantum paths contributing to the path integral are in general nowhere differentiable, a key realization that motivated Regge and Wheeler to develop a lattice formulation for gravity[12,13]. For more details on the Regge–Wheeler lattice theory of gravity see the work in [14], and further references therein. The use of an explicit momentum cutoff leads to some subtle differences in the treatment of ultraviolet divergences, some of which are ordinarily sanctioned to be zero by the formal rules of dimensional regularization, such as

$$\int \frac{d^d k}{(2\,\pi)^d}\,\frac{1}{k^n} = 0\ .\tag{117}$$

These nevertheless can be non-zero (and possibly highly divergent) when evaluated explicitly with both an infrared (μ) and an ultraviolet (Λ) cutoff [14,18].

In quantum gravity another subtlety arises from the fact that one can easily reabsorb some troublesome ultraviolet divergences by a simple rescaling of the metric [14]. Consider the metric field redefinition

$$g_{\mu\nu} = \omega\, g'_{\mu\nu}\tag{118}$$

with ω a constant. For pure gravity with a cosmological constant term proportional to λ, one writes for the Lagrangean

$$\mathcal{L} = -\frac{1}{16\pi G}\,\sqrt{g}\,R + \lambda\sqrt{g}\ .\tag{119}$$

Under a rescaling of the metric as in Equation (118), one obtains

$$\mathcal{L} = -\frac{1}{16\pi G}\,\omega^{d/2-1}\,\sqrt{g'}\,R' + \lambda\,\omega^{d/2}\,\sqrt{g'}\ ,\tag{120}$$

which just amounts to a rescaling of Newton's constant and the cosmological constant

$$G \to \omega^{-d/2+1} G\ ,\quad \lambda \to \lambda\,\omega^{d/2}\ ,\tag{121}$$

leaving the dimensionless combination $G^d \lambda^{d-2}$ unchanged. As a consequence, it seems physically meaningless to discuss separately the renormalization properties of *G* and λ, as they are both individually gauge-dependent in the sense just illustrated. To some extent these arguments should contribute to clarifying why in the following it will be sufficient to focus on the renormalization properties of Newton's constant *G*. In addition, one issue we have not touched here at all is the role played by the gravitational functional measure (normally taken to be the De Witt one, see [14] and references therein) in canceling spurious divergences. The issue is a rather subtle one, whose importance was already emphasized in the earlier discussion of the nonlinear sigma model, where the role of the functional measure in canceling spurious tadpole divergences was pointed out in Equation (29). In the

gravity case, the important role of the gravitational functional measure contribution in canceling spurious ultraviolet divergences can be seen from the following simple argument given in [14]. Consider the following covariant (DeWitt) gravitational measure,

$$\int [d\, g_{\mu\nu}] \equiv \int \prod_x [g(x)]^\sigma \prod_{\mu \geq \nu} d\, g_{\mu\nu}(x) \qquad (122)$$

with σ a real (positive or negative) parameter. Then, one has

$$\prod_x [g(x)]^\sigma \prod_{\mu \geq \nu} d\, g_{\mu\nu}(x) = \left(\prod_x \prod_{\mu \geq \nu} d\, g_{\mu\nu}(x) \right) \exp\left\{ \sigma\, \delta^d(0) \int d^d x\, \log g(x) \right\} \; . \qquad (123)$$

Given a proper lattice regularization (such as the Regge–Wheeler one), with ultraviolet cut-off Λ, one has $\delta^d(0) = (\Lambda/\pi)^d$. The above argument then shows that a proper choice of functional measure can play an essential role in canceling spurious Λ^d divergences in d spacetime dimensions.

From the integral in Equation (116), one then obtains the general nonlinear gap equation for the parameter m in d dimensions, which reads

$$G\, \beta_0\, S_d \left[\frac{d+2}{2\,(d+4)} \frac{\Lambda^{d+4}}{m^4} \, {}_2F_1(1, \frac{d}{2}+2, \frac{d}{2}+3, -\frac{\Lambda^2}{m^2}) - \frac{\Lambda^{d+2}}{m^2\,(m^2+\Lambda^2)} \right] = 1 \; . \qquad (124)$$

Here, G is Newton's constant and β_0 an overall numerical coefficient resulting from Lorentz index traces, diagram multiplicities, and individual diagram symmetry factors. In the above equation G is dimensionful, with mass dimension $d-2$, $G \sim \Lambda^{2-d}$. The next step is to discuss the solution to this nonlinear equation in various dimensions, both to draw parallels to the nonlinear sigma model and gauge theory cases outlined earlier, and to explore the sensitivity of the results to the dimension d.

It will be useful here to digress very briefly about the physics of quantum gravity in $d < 4$. In general it is possible at least in principle to define quantum gravity in any $d > 2$. There are $d(d+1)/2$ independent components of the metric in d dimensions, and the same number of algebraically independent components of the Ricci tensor appearing in the field equations. The contracted Bianchi identities reduce the count by d, and so does general coordinate invariance, leaving $d(d-3)/2$ physical gravitational degrees of freedom in d dimensions. At the same time, four spacetime dimensions is known to be the lowest dimension for which Ricci flatness does not imply the vanishing of the gravitational field, $R_{\mu\nu\lambda\sigma} = 0$, and therefore the first dimension to allow for gravitational waves and their quantum counterparts, gravitons. Note that in a general dimension d, the position space tree-level graviton propagator obtained from the linearized theory, given earlier in k-space in Equation (108), is obtained by Fourier transform. It is proportional to the integral

$$\int d^d k\, \frac{1}{k^2}\, e^{i\,k \cdot x} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\,\pi^{d/2}\,(x^2)^{d/2-1}} \; , \qquad (125)$$

while the static gravitational potential involves just the spatial Fourier transform

$$V(r) \propto \int d^{d-1}\mathbf{k}\, \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2} \sim \frac{1}{r^{d-3}} \; . \qquad (126)$$

In two spacetime dimensions there are no genuine gravitational (transverse-traceless) degrees of freedom, the Einstein–Hilbert action gives a topological invariant proportional to the Euler characteristic of the manifold. The only surviving degree of freedom is the conformal mode, which in two dimensions enters the gravitational action as a total derivative. In the absence of matter, the only residual interaction is then generated by the non-local Fadeev–Popov determinant associated with the choice of conformal gauge.

Nevertheless, it is possible to consider, as was done some time ago in [30–32] and later in [87–92], two dimensions as the $\epsilon \equiv d - 2 \to 0$ limit for gravity formulated in $2 + \epsilon$ dimensions, with ϵ considered a small quantity. This followed a similar procedure, the dimensional expansion, pioneered earlier by Wilson for scalar fields [1–3]. It was also realized early on that in the case of gravity this limit is rather subtle, due to confluent kinematic singularities as $d \to 2$.

In two spacetime dimensions ($d = 2$), the gravitational Hartree–Fock gap equation of Equation (124) reduces to

$$\frac{G \beta_0}{2\pi} \left[2 \log\left(\frac{\Lambda}{m}\right) - \tfrac{1}{2} \right] = 1 \;. \tag{127}$$

A spurious quadratic divergence in the integral given above needs to be subtracted, in order to achieve consistency with the known, more recent perturbative result for gravity in $2 + \epsilon$ dimensions given in [88–92], discussed in greater detail further below. The critical point (or ultraviolet fixed point) in this case is located at the origin

$$G_c = 0 \;. \tag{128}$$

The solution to the above $d = 2$ gap equation then has the form

$$m = \Lambda \, e^{-\frac{1}{4}} \, e^{-\frac{\pi}{\beta_0 G}} = \Lambda \, e^{-\frac{1}{4}} \, e^{-\frac{3\pi}{25 G}} \;, \tag{129}$$

where at the end the correct expression $\beta_0 = 25/3$ for pure gravity in $d = 2$ derived in [92] has been inserted. Here $G \equiv G(\Lambda)$ is the bare coupling at the cutoff scale Λ. The above results then implies, within the current approximation, that the dynamically induced gravitational mass gap m is by a factor $e^{-\frac{1}{4}} = 0.7788$ smaller than the gravitational scaling parameter $\Lambda \exp(-\pi/\beta_0 G)$.

The above result then leads to some rather immediate consequences, again very much in line with the previous discussion for scalar fields in Section 5, and for gauge theories in Section 6. Thus, the renormalization group running of the gravitational coupling $G(\mu)$ follows again from the requirement that the mass gap parameter m be scale-independent,

$$\mu \, \frac{d \, m}{d \, \mu} = 0 = \mu \, \frac{d}{d \, \mu} \left(\mu \, e^{-\frac{1}{4}} \, e^{-\frac{\pi}{\beta_0 \, G(\mu)}} \right) . \tag{130}$$

This then gives the expected asymptotic freedom-like running of the gravitational coupling $G(\mu)$ in the vicinity of the $d = 2$ ultraviolet fixed point located at $G = 0$, namely

$$G(\mu) = \frac{\pi}{\beta_0 \left[\log(\mu/m) - \tfrac{1}{4} \right]} \;. \tag{131}$$

One notes here that the quantity m plays a role quite analogous to the $\Lambda_{\overline{MS}}$ of QCD and, more generally, $SU(N)$ gauge theories, as discussed here earlier in Section 6. Indeed, the quantum gravity result in $d = 2$ looks rather similar to the result for the two-dimensional nonlinear sigma model (for $N > 2$) and to the gauge theory result in $d = 4$, in the sense that all three models have a mass gap that vanishes exponentially at zero coupling. This last fact is of course already well known from the simplest application of perturbation theory. Nevertheless, the gravity result in two dimensions can be regarded as of very limited physical interest, given that the Einstein–Hilbert action is just a topological invariant in $d = 2$. It is also worthy of note here that, overall, the main results follow from the structure of the one loop integral—and from the sign of β_0—but are otherwise universal, in the sense that a specific value for the magnitude of β_0 does not play much of a role in the scaling laws, i.e., quite generally the mass gap has an essential singularity at $G = 0$ in $d = 2$.

Before proceeding to discuss the Hartree–Fock results in higher dimensions, it will be useful here to recall the main results of the $2 + \epsilon$ expansion for gravity [88–92]. To first order in G one finds

$$\mu \frac{\partial}{\partial \mu} G \equiv \beta(G) = \epsilon\, G - \beta_0\, G^2 + O(G^3, G^2 \epsilon) \tag{132}$$

with $\beta_0 = \frac{2}{3} \cdot (25 - c)$ where c is the central charge for the massless matter fields. Later the original one-loop calculation was laboriously extended to two loops [92], with the result

$$\mu \frac{\partial}{\partial \mu} G \equiv \beta(G) = \epsilon\, G - \frac{2}{3}\, (25 - c)\, G^2 - \frac{20}{3}\, (25 - c)\, G^3 + \dots \;, \tag{133}$$

again for c massless real scalar fields minimally coupled to gravity. At the nontrivial ultraviolet fixed point, for which $\beta(G_c) = 0$,

$$G_c = \frac{3}{2\,(25 - c)}\, \epsilon - \frac{45}{2(25 - c)^2}\, \epsilon^2 + \dots \tag{134}$$

the theory becomes scale invariant by definition. In statistical field theory language, the fixed point corresponds to a phase transition where the correlation length $\xi = 1/m$ diverges in units of the cutoff. If one defines the correlation length exponent ν in the usual way by

$$\beta'(G_c) = -1/\nu \tag{135}$$

then one finds the expansion in ϵ

$$\nu^{-1} = \epsilon + \frac{15}{25 - c}\, \epsilon^2 + \dots \tag{136}$$

This then gives, for pure gravity without matter ($c = 0$) in four dimensions ($d = 4$), to lowest order $\nu^{-1} = 2$, and $\nu^{-1} \approx 4.4$ at the next order. One can integrate the β-function equation in Equation (133) to obtain the renormalization group invariant quantity

$$m = \xi^{-1} = \text{const. } \Lambda \exp\left(-\int^G \frac{dG'}{\beta(G')} \right) \tag{137}$$

which is identified with the mass gap m, or equivalently the inverse correlation length ξ. The multiplicative constant in front of the last expression on the right hand side arises as an integration constant of the renormalization group equation, and thus cannot be determined from perturbation theory in G alone. From Equation (133), in the $2 + \epsilon$ expansion the running of Newton's G in the strong coupling (antiscreening) phase with $\epsilon > 0$ and $G > G_c$ is given by

$$G(\mu) \simeq G_c \left[1 + c_0 \left(\frac{m^2}{\mu^2} \right)^{(d-2)/2} + \dots \right] \tag{138}$$

where the dots indicate higher order radiative corrections. Again, it is noteworthy here that the quantum amplitude c_0 arises as an integration constant of the renormalization group equations, and thus (unlike the Hartree–Fock result) remains undetermined in perturbation theory.

Next, consider the case of quantum gravity in a spacetime dimension $d = 3$. Even in three dimensions it is known that there are no genuine gravitational (transverse-traceless) degrees of freedom, with the only dynamical surviving remnant being the conformal mode [93]. In $d = 3$ the gravitational gap equation of Equation (124) for small m reduces to

$$\frac{G\,\beta_0}{2\pi^2} \left[2\,\Lambda - \frac{5\,\pi}{4}\, m - \frac{3m^2}{\Lambda} \right] = 1 \;. \tag{139}$$

Again, a spurious cubic divergence in the integral needs to be subtracted, as discussed earlier, in order to achieve consistency with the known perturbative results for gravity in $2 + \epsilon$ dimensions. The solution to the above equation then has the form

$$m = \Lambda \frac{8}{5\,G_c}\,(G - G_c) + \mathcal{O}\big((G - G_c)^2\big) \ . \tag{140}$$

What is new here is the appearance of a critical point in G, located at

$$G_c = \frac{\pi^2}{\beta_0\,\Lambda} \ , \tag{141}$$

with a mass gap exponent (the power on the r.h.s. of Equation (140)) $\nu = 1$. This last result is seen here to be universal, in the sense that it does not depend on the magnitude of β_0, Λ or G_c. One can roughly estimate the magnitude of β_0 in $d = 3$ by interpolating between the value in $d = 2$ and the value in $d = 4$ given later below. By this procedure one obtains approximately $\beta_0 \simeq (25 + 47)/6 = 12$ and thus $G_c \simeq 0.822/\Lambda$, not too far off from the $d = 3$ lattice gravity value $G_c \approx 0.505/\Lambda$ given some time ago in [94].

The relationship between m and $G(\Lambda)$ in Equation (140) can then be inverted, to produce the general renormalization group running of $G(\mu)$. The latter follows, as before, from the requirement that the mass gap m be scale-independent $\mu \frac{d\,m}{d\,\mu} = 0$. One then finds for $G(\mu)$ in the vicinity of the $d = 3$ gravity nontrivial ultraviolet fixed point

$$G(\mu) = G_c\left[1 + \frac{5\,\pi}{8}\frac{m}{\mu} + \mathcal{O}\left(\left(\frac{m}{\mu}\right)^2\right)\right] \ . \tag{142}$$

Note again that the coefficient β_0 does not affect the scaling directly; it only enters indirectly through the value of G_c, which in any case is affected by the choice of the ultraviolet cutoff Λ. More importantly, the ultraviolet cut-off Λ has disappeared entirely from the expression of Equation (142) for $G(\mu)$ (it is hidden in the physical parameter m, and in G_c). The main result here follows again, quite generally, from the structure of the one loop integral—and from the sign of β_0—but is otherwise universal, in the sense that a specific value for the magnitude of β_0 does not play much of a role in the scaling laws: the mass gap in $d = 3$ vanishes linearly at G_c within the Hartree–Fock approximation. This last result, which implies a mass gap exponent $\nu = 1$, can be compared to the numerical lattice gravity result in $d = 3$ of [94] which gave $\nu = 0.59(2)$, as well as to the exact solution of the lattice Wheeler–DeWitt equation in $2 + 1$ dimension, which gives $\nu = 2/3$ [95–97]. Note again here the similarity with the corresponding results for the $d = 3$ nonlinear sigma model, and for gauge theories in $d = 5$ discussed earlier.

Next we come to the physical case of quantum gravity in four spacetime dimensions ($d = 4$). The gravitational gap equation of Equation (124) for small m reduces to

$$\frac{G\,\beta_0}{8\pi^2}\left[\Lambda^2 + \tfrac{1}{2}\,m^2 - 3\,m^2\,\log\left(\frac{\Lambda}{m}\right)\right] = 1 \ . \tag{143}$$

Again, a spurious quartic divergence in the integral needs to be subtracted in order to achieve consistency with the known perturbative results for gravity in $2 + \epsilon$ dimensions; as mentioned previously, in the $2 + \epsilon$ expansion the latter is removed by a suitable rescaling of the metric. Furthermore, one would need to consider here the role played by the gravitational functional measure (normally taken to be the De Witt one, see the work in [14] and references therein) in canceling spurious $\delta(0) \simeq (\Lambda/\pi)^d$ tadpole divergences, as in Equation (123). The measure's effect was already emphasized earlier in the discussion of the nonlinear sigma model, where its role in canceling spurious tadpole divergences was pointed out in the discussion surrounding Equation (29).

The solution to the above $d = 4$ gap equation then has the form

$$m = \Lambda \sqrt{\frac{2}{3\,G_c}} \, |G - G_c|^{\frac{1}{2}} \left[-\log \left| \frac{G - G_c}{G_c} \right| \right]^{-\frac{1}{2}} + \mathcal{O}(G - G_c) \ . \tag{144}$$

Note the appearance of a new ingredient, a logarithmic correction similar to the nonlinear sigma model result in $d = 4$, Equation (77). Here again $G \equiv G(\Lambda)$ is the bare coupling at the cut-off scale Λ. In four dimensions one finds that the critical point in Newton's G is located at

$$G_c = \frac{8\,\pi^2}{\beta_0\,\Lambda^2} \ , \tag{145}$$

and for the mass gap one now has an exponent $\nu = 1/2$. This last result is again universal, in the sense that it does not depend on the magnitude of β_0, nor on the value of G_c.

The above $d = 4$ gravity results share some significant similarity with the corresponding results for the $d = 4$ nonlinear sigma model, and for gauge theories in $d = 6$. Recall that the main result here follows generally from the structure of the one loop integral—and from the sign of β_0—but is otherwise universal, in the sense that a specific value for the magnitude of β_0 does not play any role in the scaling laws: the mass gap in $d = 4$ goes to zero with a square root singularity (up to log corrections) at G_c, at least within the Hartree–Fock approximation.

For gravity one can use again the Nielsen–Hughes formula [69–71] to provide an estimate for the numerical coefficient β_0 in $d = 4$. The one-loop β-function contribution coefficient arising from a particle of spin s running around the loop is

$$\beta = -(-1)^{2s} \left[(2s)^2 - \tfrac{1}{3} \right] \ . \tag{146}$$

For spin $s = 2$ the above formula then gives for pure gravity (no matter or radiation fields) in four dimensions $\beta_0 = 47/3$, quite a bit larger than the gauge theory case (and with an overall negative sign in the beta function, correctly taken into account previously). This then gives from Equation (146) $G_c = 5.035/\Lambda^2$, compared to the lattice gravity value in $d = 4$ $G_c = 1.069/\Lambda^2$ given in [98]. A significant quantitative difference between the two results is not surprising, given the entirely different nature of the ultraviolet cutoff in the two cases (momentum cutoff vs. spacetime lattice cut-off).

The relationship between m and $G(\Lambda)$ in Equation (144) can be inverted to give the renormalization group running of Newton's $G(\mu)$, as was done earlier, and it follows from the requirement that the mass gap m be scale independent, $\mu \frac{d\,m}{d\,\mu} = 0$. One then finds for $G(\mu)$ in the vicinity of the $d = 4$ gravity nontrivial renormalization group ultraviolet fixed point the explicit result

$$G(\mu) = G_c \left[1 - \frac{3}{2} \left(\frac{m}{\mu} \right)^2 \log \left(\frac{3\,m^2}{2\,\mu^2} \right) + \mathcal{O} \left(\left(\frac{m}{\mu} \right)^4 \right) \right] \ . \tag{147}$$

In principle, there are here two solutions for $G > G_c$ and $G < G_c$, which differ by the sign of the quantum correction. Here, we have chosen, in accordance with the results of the lattice calculations of [98], the sign that corresponds to gravitational antiscreening, as the weak coupling gravitational screening phase is nonperturbatively unstable. It is noteworthy here that the coefficient β_0 does not enter renormalization group running directly, it only enters indirectly through the value of m and G_c, and the latter is in any case affected by the choice of ultraviolet cutoff. Physically the result of Equation (147) implies that for $d = 4$ and in real space the quantum correction to Newton's constant initially grows quadratically (up to the slowly varying log correction) with distance in the strong coupling phase, $\delta G(r) \sim (r/\xi)^2$, with $\xi = 1/m$. Perhaps one of the most interesting aspects of the above result is the value of the universal quantum amplitude $c_0 = 3/2$ of the m^2/μ^2 correction, as well as the weak *log* correction appearing in four dimensions. Note that the power of m/μ appearing in

the formula for $G(\mu)$ is given generally (by definition) by $1/\nu$, so that here one has in the Hartree–Fock approximation a power $\nu = 1/2$. Note that the quantity G_c (the fixed point value for Newton's constant) can be entirely eliminated if one compares the strength of the gravitational coupling on two different momentum scales μ_1 and μ_2,

$$\frac{G(\mu_1)}{G(\mu_2)} = \frac{1 - \frac{3}{2}\left(\frac{m}{\mu_1}\right)^2 \log\left(\frac{3\,m^2}{2\,\mu_2^2}\right) + \cdots}{1 - \frac{3}{2}\left(\frac{m}{\mu_2}\right)^2 \log\left(\frac{3\,m^2}{2\,\mu_2^2}\right) + \cdots} \,, \tag{148}$$

again in the vicinity of the ultraviolet fixed point, and thus up to higher order corrections in m/μ.

Next, consider the universal scaling exponent ν, which is $\nu = \frac{1}{2}$ for the Hartree–Fock approximation in four dimensions, as can be seen from Equation (144). As a direct consequence, one finds a power of $1/\nu = 2$ in the momentum dependence of $G(\mu)$ in Equation (147). The value of ν can then be compared to the lattice gravity estimate, where one finds in four dimensions $\nu \simeq 1/3$, which then implies a larger power of 3 for the running of $G(\mu)$ (as given below). Indeed it is useful at this stage to compare the Hartree–Fock result for $G(\mu)$ to the corresponding result in the Regge–Wheeler lattice theory of gravity [98], where one finds for the running of Newton's G

$$G_{Latt}(\mu) = G_c\left[1 + c_0\left(\frac{m}{\mu}\right)^3 + \mathcal{O}\left(\left(\frac{m}{\mu}\right)^6\right)\right] . \tag{149}$$

One notes that the structure of the running is similar to the Hartree–Fock result, nevertheless the powers are different (two versus three) and there is some difference in the numerical value of the amplitude as well. The amplitude in the lattice case, $c_0 \simeq 16.0$ [98], is thus somewhat larger (by a factor of five) than the Hartree–Fock value $c_0 = 3/2$ given above. A recent comparison of the running of G with current cosmological observations, to be discussed later here, leads to useful observational constraints on the amplitude c_0, suggesting a value $c_0 \approx 2.29$ and thus smaller than the current lattice estimate, and more in line with the above Hartree–Fock result.

It pays to look briefly at a few cases of dimension greater than four. In five spacetime dimensions ($d = 5$) the gravitational gap equation of Equation (124) for small m reduces to

$$\frac{G\,\beta_0}{12\pi^3}\left[\frac{2}{3}\Lambda^3 - 3\,m^2\,\Lambda\right] = 1 \,, \tag{150}$$

again with the spurious leading Λ^5 divergence subtracted in order to achieve consistency with the known perturbative result for gravity in $2 + \epsilon$ dimensions. The solution to the above equation then has the form

$$m = \Lambda\sqrt{\frac{2}{3\,G_c}}\sqrt{G - G_c} + \mathcal{O}(G - G_c) \,. \tag{151}$$

The critical point in Newton's G is now located at

$$G_c = \frac{18\,\pi^3}{\beta_0\,\Lambda^3} \,. \tag{152}$$

For the mass gap exponent one has again the universal value $\nu = 1/2$, independent of β_0 or G_c. As before, the above relation between m and bare G can be inverted to give the renormalization group running of $G(\mu)$, from $\mu\frac{d\,m}{d\,\mu} = 0$. This then gives for $G(\mu)$ in the vicinity of the $d = 5$ gravity ultraviolet fixed point

$$G(\mu) = G_c \left[1 + \frac{9}{2} \left(\frac{m}{\mu} \right)^2 + \mathcal{O}\left(\left(\frac{m}{\mu} \right)^4 \right) \right] , \tag{153}$$

with, in this case, no log correction. As expected the power in the running is $1/\nu = 2$. In turn the result of Equation (153) implies that here in real space the quantum correction to Newton's constant initially grows quadratically with distance in the strong coupling phase, $\delta G(r) \sim (r/\xi)^2$, with $\xi = 1/m$. Here, the quantum amplitude coefficient in $d = 5$ is $c_0 = 9/2 = 4.5$ in the Hartree–Fock approximation, which is about one-fourth the value found on the lattice in $d = 4$, $c_0 \simeq 16.0$, see in [98,99] and references therein. We will return to this point later on.

Overall, the gravity results here share some similarity with the corresponding results for the $d > 4$ nonlinear sigma model, and for gauge theories in $d > 6$. They also suggest that the upper critical dimension for gravity is $d_c = 4$, as in the case of scalar (spin zero) field theories. Moreover, if one denotes by d_H the fractal (Hausdorff) dimension of Wiener paths contributing to the Feynman path integral, then in the Hartree–Fock approximation an upper critical dimension of four would imply $d_H = 2$ if the arguments in [74–76] are followed here as well. In comparison, for a Gaussian scalar field (a free particle with no spin) it has been known that $d_H = 2$, as for regular Brownian motion [77–79]. Indeed, a large-d dimensional expansion was pursued for lattice gravity in [100,101]. Nevertheless, one finds that in this limit the lowest order results are not particularly illuminating regarding to what might happen in four dimensions. We should mention that a similar large-d expansions were pursued for scalar field theories in [102–104] and for gauge theories in [73], and in both cases the theory was formulated on the lattice. Some additional papers that also looked at quantum gravity in the continuum from the large d point of view can be found in [105,106].

It is possible to derive a general analytical expression for the mass gap m in general dimensions. In d dimensions, with $d > 4$, one finds from Equation (124) for the critical point

$$G_c = \frac{(d-2)\, 2^{d-2}\, \pi^{\frac{d}{2}}\, \Gamma(\frac{d}{2})}{\beta_0} \Lambda^{2-d} , \tag{154}$$

and for the mass gap

$$m = \Lambda \sqrt{\frac{2(d-4)}{3(d-2)\, G_c}} \sqrt{G - G_c} + \mathcal{O}(G - G_c) . \tag{155}$$

This then leads in $d > 4$ to the renormalization group running of Newton's constant

$$G(\mu) = G_c \left[1 + \frac{3(d-2)}{2(d-4)} \left(\frac{m}{\mu} \right)^2 + \mathcal{O}\left(\left(\frac{m}{\mu} \right)^4 \right) \right] . \tag{156}$$

The dimensionless quantum amplitude is given here in general by $c_0 = 3(d-2)/2(d-4)$ for $d > 4$, and the pole in $d = 4$ is then seen as the cause of the log correction seen earlier in Equation (144). Conversely, for $d < 4$ the general d result from Equation (124) is that $\nu = 1/(d-2)$. The running of Newton's constant for this case is then given by

$$G(\mu) = G_c \left[1 - \frac{(d^2 - 4)\, \pi\, \csc(\pi d/2)}{8} \left(\frac{m}{\mu} \right)^{d-2} + \mathcal{O}\left(\left(\frac{m}{\mu} \right)^{2(d-2)} \right) \right] . \tag{157}$$

The previous result of Equation (156) then implies that for $d > 4$ in real space the quantum correction to Newton's constant always grows quadratically with distance as $\delta G(r) \sim (r/\xi)^2$, with $\xi = 1/m$ (in agreement with the results of Equations (147) and (153)), while for $d < 4$ in real space the same quantum correction grows instead more slowly as $\delta G(r) \sim (r/\xi)^{d-2}$ (in agreement with the earlier results of Equations (142) and (131)).

Figure 18 gives a visual comparison of the Regge–Wheeler lattice gravity vs. Hartree–Fock results for the universal exponent ν as a function of dimension. It is noteworthy that the qualitative behavior is similar, although there are some notable differences, and perhaps, more importantly, the value of ν in $d = 4$.
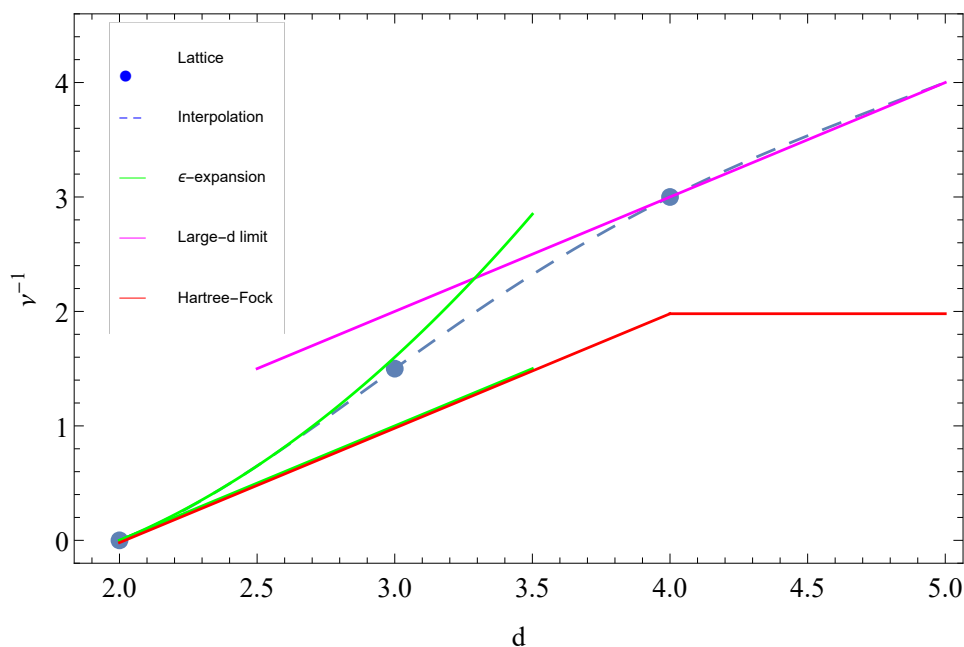
Figure 18. Universal correlation length scaling exponent ν as a function of spacetime dimension d. Shown are the $2 + \epsilon$ expansion result to one (lower curve) and two (upper curve) loops [92], the value in $2 + 1$ dimensions obtained from the exact solution of the Wheeler–DeWitt equation [95,97], the numerical lattice result in four spacetime dimensions [98], the large d result $\nu^{-1} \simeq d - 1$ [100], and the Hartree–Fock approximation results discussed in the text.

The previous discussion left largely open the question of what the physical interpretation of the (self-consistently determined) mass parameter m is. Recall that it describes a quantity which is zero to all orders in perturbation theory, nevertheless it is nonzero when an infinite number of diagrams are re-summed, as in the nonlinear gap equation of Equation (124). Define the correlation length $\xi = 1/m$, which on account of its definition as the inverse of the mass gap and its relationship, via the Lehmann representation of invariant two-point functions, describes the large distance exponential decay of n-point functions. The discussion of the previous sections on the nonlinear sigma model in Section 5, and on gauge theories in Section 6 shows that the correlation length ξ has a deep relationship with the vacuum condensates of quantum fields. For a scalar field in the ordered phase in four dimensions one has the non-vanishing vacuum condensate (or magnetization for a spin system)

$$\langle \phi \rangle \simeq \frac{1}{\xi} \; . \tag{158}$$

On the other hand, for non-Abelian gauge theories in four dimensions one has the corresponding result, describing a non-vanishing vacuum expectation value for the chromo-electric and chromo-magnetic fields [80–83]:

$$\langle F^a_{\mu\nu} F^{a \, \mu\nu} \rangle \simeq \frac{1}{\xi^4} \; . \tag{159}$$

In QCD, this last result can be obtained from purely dimensional arguments, once the existence of a fundamental correlation length ξ (which for QCD is given by the inverse mass of the lowest spin zero glueball) is established. Note that the latter result is completely

gauge invariant, in spite of the appearance of the quantity ξ, which, by virtue of the Hartree–Fock gap equation and its dependence on the gauge-fixed gluon propagator, could have inherited some gauge dependence. The power of ξ in the two cases above is thus determined largely by the canonical dimension of the primary fields ϕ or A_μ^a. The close analogy between non-Abelian gauge theories and gravity then suggests that a similar identification should be true in gravity for the Ricci scalar,

$$\langle\, g_{\mu\nu}\, R^{\mu\nu}\,\rangle \;\simeq\; \frac{6}{\xi^2}\;, \tag{160}$$

as argued in [99,107,108], and reference therein. Via the classical (or quantum Heisenberg) vacuum field equations $R = 4\lambda$ one can then relate the above quantities to the observed (scaled) cosmological constant $\lambda_{\rm obs}$ (up to a numerical constant of proportionality, expected to be of order unity). One obtains

$$\tfrac{1}{3}\,\lambda_{\rm obs} \;\simeq\; \frac{1}{\xi^2}\;. \tag{161}$$

In this picture, the latter quantity is regarded as the quantum gravitational condensate, a measure of the vacuum energy, and thus of the intrinsic curvature of the vacuum [107,108]. While it is zero to all orders in perturbation theory, it is eventually nonzero as a result of nonperturbative graviton condensation, in close analogy to what happens in non-Abelian gauge theories.

The above considerations can help in providing a quantitative handle on the physical magnitude of the new genuinely nonperturbative scale ξ. From the observed value of the cosmological constant (see, for example, the recent 2018 Planck satellite data [109]) one obtains a rough estimate for the absolute magnitude of the scale ξ,

$$\xi \;\simeq\; \sqrt{3/\lambda_{\rm obs}} \;\approx\; 5320\,{\rm Mpc}\;. \tag{162}$$

Irrespective of the specific value of ξ, this would indicate that generally the recovery of classical General Relativity (GR) results takes place at distances much smaller than the correlation length ξ so that coherent quantum gravity effects become negligible on distance scales $r \ll \xi$ (The actual value for $\lambda_{\rm obs}$, and therefore ξ, relies on a multitude of current cosmological observations, which nowadays is usually analyzed in the framework of the standard ΛCDM model. Included in the usual assumptions is the fact that Newton's G does not run with scale. If such an assumption were to be relaxed, it could affect a number of cosmological parameters, including $\lambda_{\rm obs}$, whose value could then perhaps change significantly. In the following, the estimate of Equation (162) is used as a sensible starting point). In particular, the static Newtonian potential is expected to acquire a tiny quantum correction from the running of G (see Equations (147) and (149)), $V(r) \sim -\, G(r)/r$.

Figure 19 provides a direct comparison between the continuum analytical self-consistent Hartee–Fock quantum gravity result for $G(q)$ in Equation (147),

$$G_{HF}(q) \;=\; G_c \left[1 - \frac{3}{2\,q^2\,\xi^2} \log\!\left(\frac{3}{2\,q^2\,\xi^2} \right) + \cdots \right]\;, \tag{163}$$

and the Regge–Wheeler lattice gravity running of $G(q)$ of Equation (149) as given in [98],

$$G_{Latt}(q) \;=\; G_c \left[1 + c_0 \left(\frac{1}{q\,\xi} \right)^3 + \mathcal{O}\!\left(\left(\frac{1}{q\,\xi} \right)^6 \right) \right]\;. \tag{164}$$

Figure 20 later provides a direct comparison between the two results, but in real space. In generating this last graph, we have simply substituted $\mu \to 1/r$. This was done in order to avoid at this early stage the complexities of having to deal, in a general coordinate space, with a renormalization group running Newton's $G(\Box)$ via the covariant substitution
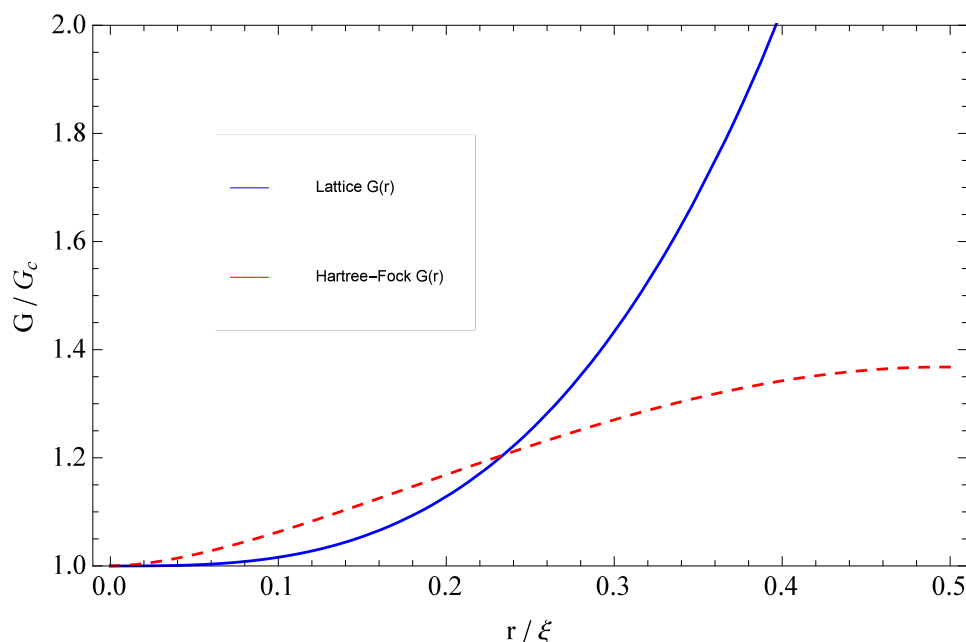
$$-q^2 \rightarrow \Box \; , \qquad G(q) \rightarrow G(\Box) \tag{165}$$

with $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$, as discussed in detail elsewhere, see for example [110,111].
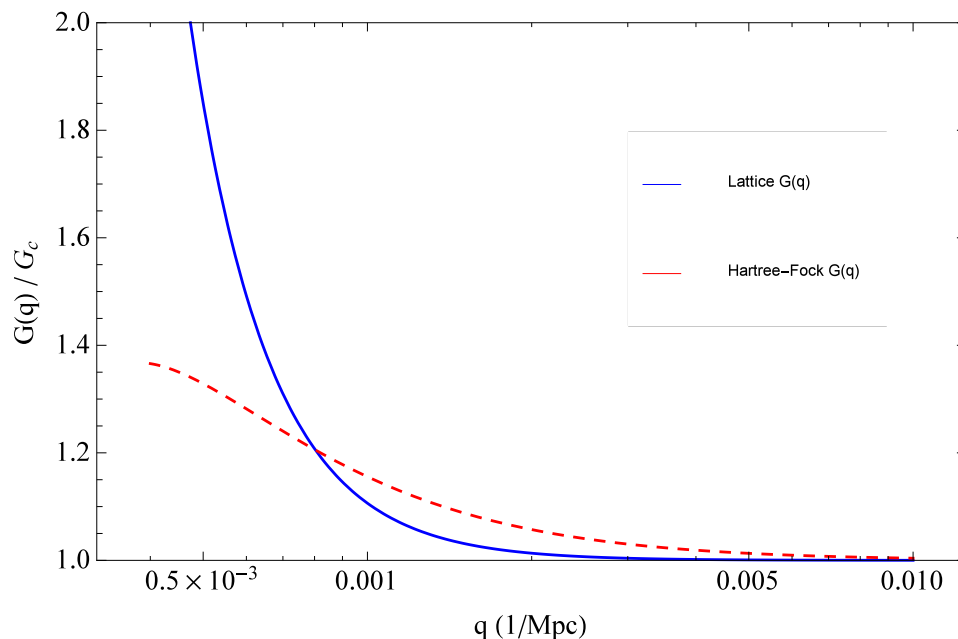
Figure 19. Comparison of Newton's running $G(q)$, as given in Equations (163) (the Hartree–Fock result) and (164) (the Regge–Wheeler lattice result). The nonperturbative reference scale ξ is directly to the gravitational vacuum condensate, as in Equations (160) and (161).

Figure 20. Comparison of the gravitational running coupling $G(r)$ versus r, obtained from $G(q)$ in Equations (163) (the Hartree–Fock result) and (164) (the Regge–Wheeler lattice result) by setting $q \sim 1/r$. Note that the approximate Hartree–Fock analytical result of Equation (163) (red line) initially rises more rapidly for small r. The nonperturbative scale ξ used here is related to the gravitational vacuum condensate via Equations (160) and (161).

To conclude this section, one can raise the legitimate concern of how these results are changed by quantum fluctuations of various matter fields coupled to gravity (scalars, fermions, vector bosons, spin-3/2 fields, etc.). These would enter in the vacuum polarization loop diagrams containing these fields. Their contribution will appear through many additional loops, and will therefore affect the value of the coefficient β_0 appearing in Equation (124). Nevertheless, one would expect that significant changes to the result of Equations (147) will arise from matter fields which are light enough to compete with gravity, and whose Compton wavelength is therefore comparable to the scale of the gravitational vacuum condensate, or equivalently the observed cosmological constant, so that for these fields $m^{-1} \sim 1/\sqrt{\lambda/3}$. At present the number of candidate fields that could fall into this category is rather limited, with the photon and a near-massless gravitino belonging to this category. Note that in the $2 + \epsilon$ perturbative expansion for quantum gravity, one encounters factors of $25 - c$ in the renormalization groups β function, where c is the central charge associated with the (massless) matter fields [91,92]. In four dimensions, similar factors involve $48 - c$ [69,70], which would again lend support to the argument that such effects should be rather small in four dimensions. This would suggest again that matter loop and radiation corrections are indeed rather small, unless one has very many massless matter fields. Nevertheless, at first one might worry that this would ultimately affect for example the value of ν, and thus the power in the running of Newton's constant G, as in Equation (147). However, this is not possible, at least within the lowest-order Hartree–Fock approximation discussed here. The reason here is the fact that, remarkably, both the power and the amplitude of the quantum correction to $G(\mu)$ in Equations (147), (156), and (163) do not depend on the amplitude β_0. The latter only enters in the (non-universal) value of G_c.

8. A Sample Application to Cosmology

The results presented so far show how the Hartree–Fock approximation applied to quantum gravity implies a renormalization group running of Newton's constant, given explicitly in Equation (163). The reference scale ξ appearing in the above expression is given in Equations (160) and (161), and is seen here as related to the current value of the gravitational vacuum condensate, or equivalently to the observed cosmological constant. In the previous section a comparison of the Hartree–Fock result was made to the numerical Regge–Wheeler lattice result of Equation (164), which is rather similar in form but nevertheless has both a larger power and slightly larger amplitude. In either case, the corrections to a classical (scale-independent) Newton's constant start to appear only at very large, cosmological distance, on account of the phenomenologically huge value for ξ in Equation (162).

On cosmological scales, a running of G leads to a multitude of physically observable effects, ranging from modifications to early cosmological evolution, the primordial growth of matter perturbations, and the cosmic background radiation (CMB). We chose here to focus on the latter for two main reasons. The first one is the sensitivity of the CMB data to extremely large scales, clearly comparable in magnitude to the value for ξ given in Equation (162). The second rather obvious reason is the availability of very accurate recent satellite data presented in the Planck15 and Planck18 surveys [109].

Here, we will focus almost exclusively on one well-studied physical observable, the matter power spectrum $P_m(k)$, for which detailed graphs have been produced by the aforementioned Planck collaboration. The matter power spectrum relates to the cosmologically observed matter density correlation

$$G_\rho(r; t, t') \equiv \big\langle\, \delta_m(\mathbf{x}, t)\, \delta_m(\mathbf{y}, t') \,\big\rangle = \frac{1}{V} \int_V d^3\mathbf{z}\, \delta_m(\mathbf{x} + \mathbf{z}, t)\, \delta_m(\mathbf{y} + \mathbf{z}, t)\ , \qquad (166)$$

where $r = |\mathbf{x} - \mathbf{y}|$, and

$$\delta_m(\mathbf{x}, t) \equiv \frac{\delta\rho(\mathbf{x}, t)}{\bar{\rho}(t)} = \frac{\rho(\mathbf{x}, t) - \bar{\rho}(t)}{\bar{\rho}(t)} \quad . \tag{167}$$

is the matter density contrast, which measures the fractional overdensity of matter density ρ above an average background density $\bar{\rho}$. In the literature, this correlation is more often studied in wavenumber-space, $G_\rho(\mathbf{k}; t, t') \equiv \langle \delta(\mathbf{k}, t) \delta(-\mathbf{k}, t') \rangle$, via a Fourier transform. It is also common to bring these measurements to the same time, say t_0, so that one can compare density fluctuations of different scales as they are measured and appear today. The resultant object $P_m(k)$ is referred to as the matter power spectrum,

$$P_m(k) \equiv (2\pi)^3 \langle |\delta(\mathbf{k}, t_0)|^2 \rangle = (2\pi)^3 F(t_0)^2 \langle |\Delta(\mathbf{k}, t_0)|^2 \rangle \,, \tag{168}$$

where $\delta(\mathbf{k}, t) \equiv F(t) \Delta(\mathbf{k}, t_0)$. The factor $F(t)$ then simply follows the standard General Relativity evolution formulas, as governed by the Friedmann–Robertson–Walker (FRW) metric. As a result, $P_m(k)$ can be related to, and extracted from, the real-space measurements via the inverse transform

$$G_\rho(r; t, t') = \int \frac{d^3k}{(2\pi)^3} G_\rho(k; t, t') e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{169}$$

$$= \frac{1}{2\pi^2} \frac{F(t)F(t')}{F(t_0)^2} \int_0^\infty dk \, k^2 \, P_m(k) \frac{\sin(kr)}{kr} \quad . \tag{170}$$

It is common to parameterize these correlators by a so-called scale-invariant spectrum, which includes an amplitude and a scaling index, conventionally written as

$$G_\rho(r; t_0, t_0) = \left(\frac{r_0}{r}\right)^\gamma \,, \tag{171}$$

or equivalently

$$P_m(k) = \frac{a_0}{k^s} \,, \tag{172}$$

with the powers related by $s = (d-1) - \gamma = 3 - \gamma$ via the Fourier transform. Nevertheless, such simple parametrizations generally only apply to specific regions in wave-vector space (specifically the galaxy regime at larger k, or the primordial CMB regime at smaller k), in spite of the fact that different regimes are known to be related to each other by standard classical cosmological evolution, via the so-called transfer function.

In the matter-dominated era, such as the one where galaxies and clusters are formed, the energy momentum tensor follows a perfect pressureless fluid to first approximation. Therefore, the trace equation reads

$$R - 4\lambda = -8\pi G \, T \quad . \tag{173}$$

(For any perfect fluid the trace gives $T = 3p - \rho$, and thus $T \simeq -\rho$ for a non-relativistic fluid.) As λ is a constant, the curvature and matter variations, and thus their correlations, are directly related

$$\langle \delta R(x) \delta R(y) \rangle = (8\pi G)^2 \langle \delta\rho(x) \delta\rho(y) \rangle \quad . \tag{174}$$

A number of subtleties arise here, such as the fact that to extend beyond the linear matter dominated regime, the trace equation alone becomes insufficient (as the trace of the energy momentum tensor for radiation vanishes), and the full tensor field equations have to be used. Furthermore, in a real universe with multiple fluid components, interactions and transient behaviors have to be taken into account, which are governed by coupled non-linear Boltzmann equations. However, these classical procedures have been fully worked out in standard cosmology texts [112,113], as well as in currently popular sophisticated

computer codes, such as CAMB and MGCAMB [114–116], CLASS and MGCLASS [117,118], ISiTGR [119–121] and COSMOMC [121] (for a detailed comparison of features in various codes see in [122]).

Quite generally, the matter power spectrum can be decomposed as two parts—an initial condition known as the primordial spectrum $\mathcal{R}_k^o$, and an interpolating function between the domains known as a transfer function $\mathcal{T}(k)$ [112]. As a result, the full matter power spectrum $P_m(k)$ beyond the galaxy (larger k) domain will take the form

$$P_m(k) \,=\, C_0 \, (\mathcal{R}_k^o)^2 k^4 \, [\mathcal{T}(\kappa)]^2 \quad, \tag{175}$$

where $C_0 \equiv 4(2\pi)^2 \, C^2(\Omega_\Lambda/\Omega_M)/25 \, \Omega_M^2 H_0^4$ is a combined constant of cosmological parameters, and the k^4 factor is purely for convenience. The transfer function is usually written in terms of $\kappa \equiv \sqrt{2}k/k_{eq}$, a scaled dimensionless wavenumber, with k_{eq} being the wavenumber at matter–radiation equality. With this decomposition, the transfer function is a fully classical solution of the set of Friedmann and Boltzmann equations, capturing the nonlinear dynamics. This leaves the initial primordial function, which is usually parameterized as a scale-invariant spectrum

$$(\mathcal{R}_k^0)^2 \,=\, N^2 \, \frac{1}{k^3} \left(\frac{k}{k_\mathcal{R}}\right)^{n_s-1} \quad, \tag{176}$$

involving an overall amplitude N^2 and a spectral index n_s. $k_\mathcal{R}$ is commonly referred to as the "pivot scale", a reference scale conventionally taken to be $k_\mathcal{R} = 0.05 \, \mathrm{Mpc}^{-1}$ in cosmology. While the transfer function $\mathcal{T}(\kappa)$— the solution to the coupled and nonlinear set of Friedmann, Boltzmann, and continuity differential equations—is difficult to obtain as an explicit function, it is nevertheless in principle fully determined from classical dynamics. Moreover, assuming standard ΛCDM cosmology dynamics and evolution, a semi-analytical interpolating formula for $\mathcal{T}(\kappa)$ [112] is known. As a result, if the initial spectrum $\mathcal{R}_k^o$, or more specifically the parameters N and n_s, are fixed, then $P_m(k)$ is fully determined.

Here, in this section, the main concern will be the inclusion of the effects of a running of Newton's G, as given in Equations (163), (164), and (161) in the matter power spectrum, as measured for example in current CMB data. Modifications to the matter power spectrum $P_m(k)$ at small k originating in a running $G(k)$ can be done either analytically using the transfer function [123,124] or by relying on more comprehensive numerical programs [125]. Analytically, the effect of a running Newton's constant (Equation (163)) can be included via dimensional analysis for the correct factors of G to include, $P_m(k) \rightarrow (G_0/G(k))^2 P_m(k)$ [124]. Here, we will focus instead on the more comprehensive, and more accurate, results that can be obtained by consistently incorporating the running of G in all relevant cosmological equations, and in many more cosmological observables, such as temperature and polarization correlations [125].

The key resultant prediction for the matter power spectrum $P_m(k)$ is found here in Figure 21, showing (not surprisingly) still an almost perfect fit to all observational data for $k \gg m \simeq 2.8 \times 10^{-4} \, h\mathrm{Mpc}^{-1}$. Nevertheless, for scales of k comparable to m, additional quantum effects become significant due to the $G(k)$, enough to cause significant deviations from the classical ΛCDM result for $P_m(k)$. In Figure 21, the middle solid orange curve shows the Hartree–Fock expression for the running of Newton's constant G, while the bottom dashed green curve and the top blue dotted curve show the lattice result of a running Newton's constant G (with the lattice coefficient $c_0 = 16.0$), as well as the result for no running respectively for reference. It seems that the Hartree–Fock running of G is in reasonably good consistency with the lattice expression, except for the eventual unwieldly upturn below $k < 2 \times 10^{-4} \, h\,\mathrm{Mpc}^{-1}$. However, this upturn is most likely an artifact from the Hartree–Fock expression being just a first-order analytical approximation after all (as discussed earlier, the lowest order Hartree–Fock approximation can be extended to higher order, by including increasingly complex higher loop diagrams, with dressed propagators and vertices still determined self-consistently by a truncated version of the

Schwinger–Dyson equations). Nevertheless, the Hartree–Fock approximation shows good consistency with both the latest available observational data sets, as well as with the lattice result. The fact that it exhibits a gentler dip at small k perhaps also provides support for a potentially smaller lattice running coefficient of Equation (164) of approximate value $c_0 \approx 2.29$. More details for the most recent constraints on this lattice parameter c_0 are discussed in [124,125].
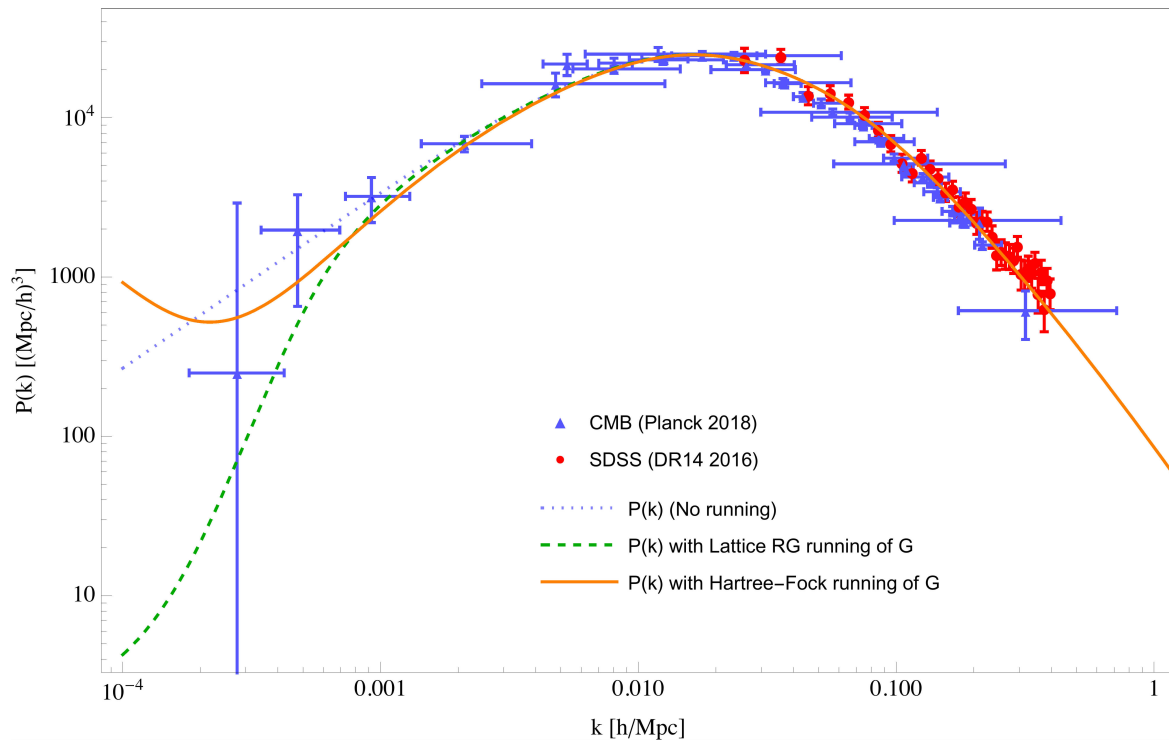
Figure 21. Matter power spectrum $P(k)$ with lattice versus Hartree–Fock running of Newton's constant G. The middle solid (orange) curve shows $P(k)$ implemented with the Hartree–Fock running of Newton's constant G factor as given in Equation (163). The lower dashed (green) curve shows the original lattice RG running of Newton's constant G (with the original lattice coefficient of Equation (164) equal to $c_0 = 16.0$) for comparison. The original spectrum with no running is also displayed by the top dotted (blue) curve for reference.

Following similar analysis to determining $P_m(k)$, other spectra such as the angular temperature spectrum C_l^{TT}, should be fully derivable from the primordial function $\mathcal{R}_k^o$, or specifically n_s, which is set by the scaling of gravitational curvature fluctuations ν.

9. Conclusions

In this work, we have shown how the Hartree–Fock approximation to quantum gravity can be carried out, by considering what ultimately reduces to a set of relatively easy to evaluate one-loop integrals. The main feature of such an approach is that the complete analytic expression for the one-loop integrals allows one to write down a gap equation for the dynamically generated mass scale in the strongly coupled phase and, more importantly, for any spacetime dimension. This result can be viewed as analogous to the situation in the nonlinear sigma model, where a nonlinear gap equation is obtained explicitly, and can then in principle be solved for any dimension. Due to the inherent approximation of a mean-field type approach the calculation is expected to be rather crude, but nevertheless provides interesting insights, and can furthermore provide a useful starting point for improved, higher loop self-consistent calculations.

There are a number of interesting features that arise from the results presented earlier, which will be summarized in the following paragraphs. The first notable aspect is the

appearance of a nontrivial renormalization group fixed point (referred to as a critical point in statistical field theory language) G_c in Newton's constant, for any spacetime dimension d greater than two. As a consequence, above two dimensions the theory exhibits two phases, a Coulomb-like phase with gravitational screening for $G < G_c$, and a strongly coupled phase with gravitational antiscreening for $G > G_c$. As the Hartree–Fock method retains some vestiges of a perturbative diagrammatic calculation, the above results do not allow one to determine whether both phases are indeed physical. On the other hand current lattice calculations, which are genuinely nonperturbative, suggest that gravitational screening is impossible (no stable ground state exists $G < G_c$) and that, as in Yang Mills theories, only the gravitational antiscreening phase for $G > G_c$ is physically realized in this theory.

For the latter phase the Hartree–Fock calculation gives, once suitable renormalization group arguments are applied, an explicit expression for the running of the gravitational Newton's constant as a function of scale. The latter can be expressed as a $G(k)$ in wavevector space, or, if one wishes, more generally as a covariantly formulated $G(\Box)$, with covariant d'Alembertian $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. Unlike perturbatively renormalizable theories such as QCD, the RG running of G in the gravitational case is not logarithmic, but instead follows a power law in the relevant scale. In addition, the Hartree–Fock results suggest that for quantum gravity the lower critical dimension is two (as expected based on naive dimensional arguments, and also on the basis of the lattice results, and initially from Wilson's $2 + \epsilon$ gravity perturbative expansion as well), and furthermore that the upper critical dimension is six. This would correlate with the fact that, within the Hartree–Fock approximation, above six spacetime dimensions scaling dimensions and critical exponents flow into their Gaussian, free field value. How that would explicitly reflect on the nature of various (local and non-local) gravitational correlations is still not entirely clear at this point.

More importantly, in the physically relevant case of four spacetime dimensions one finds that the gravitational coupling grows like a distance squared, up to logarithmic corrections (between $d = 2$ and $d = 4$ the relevant power varies as $1/(d - 2)$, and stays constant above that). This result is similar to, but nevertheless still quantitatively different from, the Regge–Wheeler lattice result, which gives a renormalization group growth of the gravitational coupling G with the cube of the distance, see in [98,99] and references therein. Nevertheless, in either case the reference scale for the growth of Newton's constant as a function of distance is determined by a new, genuinely nonperturbative quantity $\xi = 1/m$. The latter is non-analytic in the bare couplings, and arises naturally as an integration constant of the renormalization group equations. It was argued elsewhere [99,107] that this new nonperturbative scale, specific to quantum gravity, should be identified with the gravitational vacuum condensate $\langle R \rangle = 2\lambda$ via $\lambda = 3/\xi^2$, where λ is the observed (tiny) scaled cosmological constant, in this picture more properly referred to as the gravitational vacuum energy. The latter would then provide the needed reference scale for the RG running of the gravitational coupling G. As ξ is very large (from current astrophysical observations of λ [109]) one has $\xi \sim 5300$ Mpc), which would then lead to a very slow rise of G, observable only on very large, cosmological scales.

It is of some interest here to point out that the Hartree–Fock approximate calculation does not just give values for exponents and scaling dimensions, it also provides an explicit analytic expression for quantum amplitudes. Such as, for example, the overall amplitude of the quantum correction to $G(k)$ when referred to the new nonperturbative scale ξ. This is quite different from ordinary perturbation theory say in the $2 + \epsilon$ expansion, where no information can ever be gained on the quantum amplitudes themselves (only on some ratios of critical amplitudes, as is already the case in the nonlinear sigma model).

What is the physical interpretation of the self-consistent Hartree–Fock type calculation? For suggestions, one can look at the analogous results in condensed matter physics, or from the nonlinear sigma model, to try to gain some insight into what type of physical process is taken into account in the Hartree–Fock self-consistent, mean field-type approximation. Perturbation theory initially only takes into account single graviton exchange

(at the tree level), then to higher order two graviton exchanges in loop diagrams, then three graviton exchanges, etc. On the other hand, in the Hartree–Fock approximation all these multi-graviton effects are included into one single, self-consistently determined, diagrammatic contribution. The latter sums up infinitely many perturbative diagrams, and thus can lead to entirely novel features such as a dynamically generated nonperturbative scale. The lowest order Hartree–Fock self-consistent approximation should therefore be viewed as a (physically motivated) re-summation of a certain subset of infinitely many diagrams. The next order Hartree–Fock correction then self-consistently re-sums a second set of topologically distinct more complex diagrams, leading presumably to a further improvement over the lowest order Hartree–Fock result. It is therefore noteworthy here that the emergence of the new nonperturbative scale ξ can only arise if infinitely many graviton loop diagrams are resummed. The various loop diagrams then all add up coherently, and give rise to the new phenomenon of a nonvanishing gravitational vacuum condensate. Nevertheless, in the Hartree–Fock approximation the connection between the nonperturbative mass scale $m = 1/\xi$ and the gravitational field vacuum condensate $\langle R \rangle$ is not immediate. One way to justify such as result is to consider the fact that the renormalized, effective cosmological λ constant (related to the vacuum expectation value of the scalar curvature via the Heisenberg equations of motion) clearly appears as a mass-like term in the weak field expansion of the gravitational action. Nevertheless, in the gravity case one has the well-known subtlety that such a mass-like term remains fully consistent with general covariance (and thus entirely independent of gauge choice), as the λ term in the gravitational action retains that property.

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