Self-similar propagation of parabolic pulses in normal-dispersion fiber amplifiers

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Pulse propagation in high-gain optical fiber amplifiers with normal group-velocity dispersion has been studied by self-similarity analysis of the nonlinear Schrödinger equation with gain. For an amplifier with a constant distributed gain, an exact asymptotic solution has been found that corresponds to a linearly chirped parabolic pulse that propagates self-similarly in the amplifier, subject to simple scaling rules. The evolution of an arbitrary input pulse to an asymptotic solution is associated with the development of low-amplitude wings on the parabolic pulse whose functional form has also been found by means of self-similarity analysis. These theoretical results have been confirmed with numerical simulations. A series of guidelines for the practical design of fiber amplifiers to operate in the asymptotic parabolic pulse regime has also been developed. © 2002 Optical Society of America

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1. INTRODUCTION

Self-similarity is a fundamental property of many physical systems and has been studied extensively in diverse areas of physics such as hydrodynamics, solid-state physics, and biophysics.1 In particular, the presence of self-similarity can be exploited to produce exact solutions to partial differential equations that describe a physical system by use of the mathematical technique of symmetry reduction to reduce the number of degrees of freedom. In optics, the use of such techniques has not been widespread, but some important results were obtained previously in the study of pattern formation,2 Hill grating growth,3 Raman scattering,4 the evolution of self-written waveguides,5 and the formation of Cantor set fractals in soliton systems.6,7 Recently, self-similarity methods were applied to the study of pulse propagation in optical fiber amplifiers with normal group-velocity dispersion (GVD), showing that linearly chirped parabolic pulses are self-similar solutions of the nonlinear Schrödinger equation (NLSE) with gain.8,9 These results have extended previous theoretical and numerical studies of parabolic pulse propagation.10,11 In addition to being of fundamental interest because they represent a new class of solution to the NLSE with gain, self-similar parabolic pulses also have wide-ranging practical significance because they can be propagated at high power without undergoing pulse distortion as a result of optical wave breaking, and their linear chirp leads to highly efficient pulse compression to the sub-100-fs domain.12

In the previously published theoretical description of self-similar pulse propagation in optical fiber amplifiers,9 it was assumed that pulse propagation occurs in the amplifier at very high intensities. It was thus possible to demonstrate parabolic pulse solutions for an amplifier with an arbitrary longitudinal gain profile, generalizing the results originally obtained by Anderson et al. for a normal GVD undoped fiber.10 However, although the numerical results obtained by Tamura and Nakazawa11 suggested that parabolic pulses are naturally generated in high-gain fiber amplifiers, such was not explicitly shown in Ref. 9. In this paper we consider this issue in detail for the particular case of a normal GVD fiber amplifier with constant distributed gain. For the associated NLSE with constant gain, self-similarity techniques are used to show that the linearly chirped parabolic pulse solution is an exact asymptotic solution toward which any arbitrary input pulse will evolve with sufficient propagation distance, giving a rigorous demonstration of the result stated in Ref. 9. In addition, we are also able to apply self-similarity techniques to describe the presence of the low-amplitude exponentially decaying wings that appear on the parabolic pulse as it evolves toward the asymptotic limit.

The paper is organized as follows: In Section 2 we describe the way in which the assumption of self-similarity simplifies the analysis of the NLSE, leading to the asymptotic linearly chirped parabolic pulse solution. Subsection 2.A describes the theory, Subsection 2.B presents results of numerical simulations that confirm the theoretical results obtained, and Subsection 2.C presents a series of guidelines for the practical design of fiber amplifiers to operate in the asymptotic parabolic pulse regime. One result of the simulations that will become apparent is the appearance of low-amplitude wings on the parabolic pulse that appear during propagation in the amplifier. In Section 3 we consider these wings in detail,
showing that they represent an intermediate asymptotic solution to the NLSE with gain that can also be described analytically by use of self-similarity techniques. Section 4 concludes the paper.

2. SELF-SIMILAR PARABOLIC PULSE SOLUTION TO THE NONLINEAR SCHRÖDINGER EQUATION WITH GAIN

A. Theory
We consider pulse evolution in a fiber amplifier in the absence of gain saturation and for incident pulses with spectral bandwidths less than the amplifier bandwidth. Such an analysis is well-suited to describe experiments that use high-gain broadband rare-earth fiber amplifiers, and, in this case, pulse propagation can be described by the NLSE with gain:

$$i \frac{\partial \Psi}{\partial z} = \frac{\beta_2}{2} \frac{\partial^2 \Psi}{\partial T^2} - \gamma |\Psi|^2 \Psi + i \frac{g}{2} \Psi. \tag{1}$$

Here $\Psi(z, T)$ is the slowly varying envelope of the pulse in a comoving frame, $\beta_2$ is the GVD parameter, $\gamma$ is the nonlinearity parameter, and $g$ is the distributed gain coefficient. The evolution of the pulse energy $U(z) = \int |\Psi(z, T)|^2 dT$ (from $z = 0$) in the amplifier satisfies the conservation integral

$$U(z) = U_{in} \exp(gz), \tag{2}$$

where $U_{in} = U(0)$.

From the previous research of Afanas'ev et al.\textsuperscript{2} and Anderson et al.\textsuperscript{10} we anticipate that for normal GVD there will exist a linearly chirped solution to Eq. (1) that, after some period of initial evolution, scales self-similarly as it propagates in $z$. To find such a self-similar solution, we expand the field in terms of a positive definite amplified in the asymptotic limit, allowing explicit analytic solutions for $F, f, \varphi, C$, and $C(z)$.

$$\begin{align*}
\frac{df}{dz} &= \beta_2 C_0 + \frac{g}{2} f, \tag{10} \\
\left(2\beta_2 C^2 - \frac{dC}{dz} \right) |f|^2 \exp(2gz) \frac{\partial^2 f}{\partial T^2} - \frac{1}{f^2} \frac{d\varphi}{dz} &= \beta_2 f^2 \frac{d^2 F}{d\vartheta^2} \exp(-2gz) - \gamma F^2. \tag{11}
\end{align*}$$

Thus, by reducing the number of degrees of freedom, we have reduced the original problem involving partial differential equations to one that involves a system of ordinary differential equations. These equations can be simplified in the asymptotic limit, allowing explicit analytic solutions for $F, f, \varphi, C$, and $C$ to be obtained. In particular, the term proportional to $d^2 F/d\vartheta^2$ in Eq. (11) can be neglected as $\vartheta \to \infty$. With this simplification, and making use of the fact that $F$ is a function of only one dependent variable, $\vartheta$, we obtain the following system of coupled equations for $f, \varphi, C, A$:

$$\begin{align*}
2\beta_2 C^2 - \frac{dC}{dz} |f|^2 \exp(2gz) \frac{\partial^2 f}{\partial T^2} - \frac{1}{f^2} \frac{d\varphi}{dz} &= \beta_2 f^2 \frac{d^2 F}{d\vartheta^2} \exp(-2gz) - \gamma F^2. \tag{11}
\end{align*}$$

It follows from Eqs. (11)–(13) that the solution for $F(\vartheta)$ has the form

$$F(\vartheta) = \sqrt{1 - a \vartheta^2}, \quad |\vartheta| < 1/\sqrt{a}, \tag{14}$$

and $F(\vartheta) = 0$ for $|\vartheta| > 1/\sqrt{a}$. Here $a$ is a constant that depends on the input parameters, as we shall see below, and $F$ obeys the normalization condition $F(0) = 1$. To find $f(z)$ we use Eq. (10) in Eq. (12) to obtain

$$\begin{align*}
\frac{d}{dz} \left( \frac{1}{f} \frac{df}{dz} \right) &= \frac{1}{f^2} \frac{df}{dz} - \frac{g^2}{f^2} + \beta_2 \gamma a f^6 \exp(-2gz) = 0, \tag{15}
\end{align*}$$
and it is straightforward to show that Eq. (15) has, as a solution,

\[ f(z) = A_0 \exp \left( \frac{g}{3} z \right), \]  

(16)

where \( A_0 \) is a constant (the peak amplitude) that depends on the parameters of the system (which we shall determine below), provided that \( a = g^2/18 \beta_2 \gamma A_0^5 \). Inasmuch as we consider only square-integrable solutions of Eq. (1), \( a \) must be positive, which implies that \( \gamma \beta_2 > 0 \), and thus we can consider two cases: 1, \( \beta_2 > 0 \), \( \gamma > 0 \) and 2, \( \beta_2 < 0 \), \( \gamma < 0 \). For current rare-earth-doped amplifiers we are restricted to \( \beta_2 > 0 \), but we note that the results below also apply to case 2. The form of \( f(z) \) given by Eq. (16) can now be substituted back into Eq. (7) to yield the explicit form of the self-similarity variable \( \theta \) as

\[ \theta = A_0^2 \exp \left( \frac{g}{3} z \right) T, \]  

(17)

which completes the definition of the function \( F(\theta) \).

Combining the solutions for \( f(z) \) and \( F(\theta) \) yields the asymptotic evolution of the amplitude, \( A(z, T) \), which we can write as

\[ A(z, T) = A_0 \exp \left( \frac{g}{3} z \right) \left[ 1 - \frac{T^2}{T_p^2(z)} \right]^{1/2}, \]  

(18)

at \( |T| \leq T_p(z) \), where \( A(z, T) = 0 \) for \( |T| > T_p(z) \) and \( T_p(z) \) defines the effective pulse width:

\[ T_p(z) = \frac{6(\gamma \beta_2)^{1/2} A_0}{g} \exp \left( \frac{g}{3} z \right). \]  

(19)

The solution therefore corresponds to a compactly supported pulse with a parabolic intensity profile whose zero-crossing point is given by \( T_p(z) \). To determine \( A_0 \), we use Eqs. (8) and (14) to obtain

\[ U(z) = \exp(gz) \int_{-\sqrt{\gamma} \theta}^{\sqrt{\gamma} \theta} (1 - a \theta^2) d\theta \]

\[ = \frac{8 \sqrt{\gamma} g^{1/2} A_0^3}{g} \exp(gz), \]  

(20)

which, on comparison with Eq. (2), yields

\[ A_0 = \frac{1}{2} \left( \frac{g U_{in}}{\sqrt{\gamma} \beta_2 g} \right)^{1/2}. \]  

(21)

It remains now to determine the general form of the phase of the self-similar pulse. Substituting the solution for \( f(z) \) in Eq. (16) into Eq. (10) yields

\[ C(z) = -\frac{g}{6 \beta_2}, \]  

(22)

which is, in fact, independent of \( z \). This then enables us to find \( \varphi(z) \) from Eqs. (13) and (16) to obtain the phase given by Eq. (6):

\[ \varphi(z, T) = \varphi_0 + \frac{3}{2g} A_0^2 \exp \left( \frac{2}{3} g^2 \right) - \frac{g}{6 \beta_2} T^2. \]  

(23)

at \( |T| \leq T_p(z) \), where \( \varphi_0 \) is an arbitrary constant. Such a form for the phase yields the following constant linear chirp \( \Omega_c \):

\[ \Omega_c(T) = -\frac{\partial \varphi}{\partial T} = \frac{g}{3 \beta_2} T, \quad |T| \leq T_p(z). \]  

(24)

The self-similar asymptotic pulse solution is thus \( \Psi(z, T) = A(z, T) \exp(\varphi(z, T)) \), with amplitude \( A(z, T) \) determined by Eqs. (18), (19), and (21) and phase \( \Phi(z, T) \) by Eq. (23). The asymptotic scaling predicted by these results is associated with the exponential growth of the pulse amplitude and width, with the asymptotic pulse characteristics determined only by the energy, and not the specific shape or width, of the initial pulse. The linear chirp across the asymptotic pulse is independent of \( z \), but, as the pulse broadens temporally, the leading and trailing edges become increasingly redshifted and blueshifted, respectively, and the pulse's spectral width also increases. Indeed, it is possible to obtain an analytical expression for the spectrum of this asymptotic parabolic pulse, defined by

\[ \tilde{\Psi}(z, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(z, T) \exp(i \omega T) dT, \]  

(25)

by the method of stationary phase,\(^{10,14}\) which yields

\[ |\tilde{\Psi}(z, \omega)|^2 = \frac{3 |\beta_2 A_0|^2}{g} \exp \left( \frac{2}{3} g^2 z \right) \left[ 1 - \frac{\omega^2}{\omega_p^2} \right] \]  

(26)

at \( |\omega| \leq \omega_p(z) \), and \( |\tilde{\Psi}(z, \omega)|^2 = 0 \) for \( |\omega| > \omega_p(z) \), which is also a parabolic function. Significantly, the scaling law for the increase in spectral width with propagation distance is also found to be exponential and is given by

\[ \omega_p(z) = \sqrt{\frac{2 \gamma}{\beta_2}} A_0 \exp \left( \frac{g}{3} z \right), \]  

(27)

where \( A_0 \) is defined by Eq. (21).

B. Numerical Simulations

The analytical results presented above have been confirmed by numerical simulation of the NLSE with gain by use of the standard split-step Fourier method. We first consider the evolution of a Gaussian input pulse in an amplifier with realistic parameters corresponding to those of currently available Yb:\(^3\):doped fiber.\(^8\) In this case we consider a 6-m amplifier with a total integrated gain of 50 dB, so \( g = 1.9 \ \text{m}^{-1}, \beta_2 = 25 \times 10^{-3} \ \text{ps}^2 \ \text{m}^{-1}, \) and \( \gamma = 5.8 \times 10^{-3} \ \text{W}^{-1} \ \text{m}^{-1}. \) The input pulse is chosen to have an energy \( U_{in} = 12 \ \text{pJ}, \) and \( \Delta T_0 = 200 \ \text{fs} \) (FWHM). Figure 1(a) shows the pulse evolution obtained from the simulations with a standard three-dimensional evolution plot that illustrates the expected increase in peak intensity and pulse width as well as the parabolic nature of the pulse in the latter stages of the amplifier. To illustrate clearly the various stages of pulse evolution, we have also included in the figure the pulse intensity profile obtained from simulations in 1-m increments, plotting [Fig. 1(b)] the intensity on a logarithmic axis and [Fig. 1(c)] the normalized pulse intensity on a linear scale.
The defining characteristic of a pulse with a parabolic intensity profile is the steepness of the slope on the pulse edges, which decay much faster than they would on either a Gaussian or a hyperbolic secant pulse. This property enables us to determine when a pulse is entering the parabolic regime because, as $T \to T_p(z)$, the edges of the intensity profile (plotted on a logarithmic scale) become vertical. From the curves in Fig. 1(b) we see that, for our parameters, the pulse first starts to exhibit these characteristic steep edges after ~4 m, and by 6 m the edges are nearly vertical over 10 orders of magnitude. Moreover, the plots in Fig. 1(c) show that the parabolic nature of the pulse is also apparent on a linear scale. [Note that the low-amplitude wings on the parabolic pulse region at power levels of $<10^{-5}$ W seen by use of the logarithmic plot in Fig. 1(b) are discussed in Section 3 below.]

Figure 2(a) shows the intensity $|\Psi(z, T)|^2$ and chirp $\Omega_c(T)/2\pi$ (in terahertz) for the parabolic output pulse obtained after 6 m of propagation, with the simulation results compared with the theoretical predictions of Eqs. (18), (19), and (21) for the intensity and Eq. (24) for the linear chirp. It is clear that there is very good agreement between simulations and theory over the central region of the pulse, confirming our analytical model of the parabolic pulse characteristics in the asymptotic regime. For completeness, Fig. 2(b) compares the spectrum obtained from simulations with that predicted by Eq. (26), which shows good agreement in the form of the spectral envelope. The simulations do, however, show oscillations in the spectrum that are not expected in the exact asymptotic limit. We have found that the corresponding time-domain feature associated with these spectral oscillations is the deviation of the chirp across the parabolic pulse from exact linearity. Indeed, imposing an exact linear chirp across the pulse profile obtained from simulations and shown in Fig. 2(a) gives an oscillation-free spectrum that is in excellent agreement with the asymptotic theoretical prediction.

An important prediction of the self-similarity analysis presented above is that, for a given amplifier, it is only the energy of the initial pulse, and not the pulse’s specific shape, that determines the asymptotic pulse characteristics. Thus we expect that, for a fixed energy, input pulses of a wide range of pulse durations and profiles should all evolve to a parabolic pulse with the same amplitude and width.

First, to determine the influence of the input pulse width, we consider the evolution of Gaussian input pulses with pulse durations that range from 100 fs to 5 ps (FWHM) but with fixed energy $U_{in} = 12$ pJ, again in a 6-m-long Yb$^{3+}$-doped fiber amplifier with parameters as above. To compare the evolution of the different input pulses, we show in Fig. 3 the evolution of the amplitude (top), and the effective width (bottom), obtained from simulation results, with analytic predictions for $A(z, 0)$ and $T_p(z)$ given by Eqs. (18), (19), and (21). (For this figure the effective width of the simulation results was estimated from a parabolic fit to the pulse intensity profiles.) It is clear from these results that, for a given amplifier, the rate at which a pulse evolves to the parabolic pulse solution depends strongly on the choice of the input pulse width. This is to be expected, because for a fixed energy a broader temporal width implies a lower peak power. We therefore expect that the initial nonlinear evolution will be substantially different for the various input pulses. Indeed, we notice that the pulse with the largest input pulse width (and hence the lowest peak power) is the slowest pulse to converge to the parabolic pulse solution. Nevertheless, after 5 m the different evolution maps overlap, indicating that the pulses have reached the asymptotic limit in all cases.
Amplifier and pulse parameters are given in the text.

Fig. 3. Evolution of pulse amplitude (top) and pulse width (bottom) as functions of propagation distance in the 6-m normal-GVD fiber amplifier. The theoretical predictions for the asymptotic parabolic pulse evolution (solid curves) are compared with the results obtained for simulations with Gaussian pulses of different pulse durations yet identical energy of 12 pJ. Amplifier parameters are given in the text.

Fig. 4. Pulse characteristics for input pulses with (a) Gaussian (leftmost column), (b) hyperbolic secant (middle column), and (c) super-Gaussian (rightmost column) profiles. The top figure in each column shows the input pulse; the middle figure shows the output pulse from simulations (curves) compared with asymptotic theoretical predictions (open circles), and the bottom figure shows the corresponding spectra for simulation results (curves) and asymptotic theoretical predictions (open circles). Amplifier and pulse parameters are given in the text.

Next, we consider additional simulations that involve pulses with the same energy but with different input profiles. In particular, we choose a Gaussian pulse, \( \Psi(0, T) = \sqrt{P_0} \exp[-1/2(T/T_0)^2] \); a hyperbolic secant pulse, \( \Psi(0, T) = \sqrt{P_0} \sech(T/T_0) \); and a super-Gaussian pulse, \( \Psi(0, T) = \sqrt{P_0} \exp[-1/2(T/T_0)^{2m}] \), where \( m = 3 \), for the same amplifier and input energy considered above. The initial FWHM pulse duration is 200 fs in all cases. The curves at the top of Fig. 4 show the input pulse profiles for these three cases, with the temporal and spectral characteristics of the output pulses shown in the lower two rows. The solid curves are the simulation results, and the open circles are the theoretical predictions. In all cases the general form of the intensity profile and the chirp are in good agreement with the analytical results. However, for the super-Gaussian input pulse, Fig. 4(c), we notice a strong oscillatory structure on the edges of the temporal intensity profile and on the chirp, which results in rapid oscillations over the spectral profile. However, additional simulations show that the energy content associated with these oscillations decreases relative to the energy of the parabolic pulse envelope if the propagation is continued over longer amplifier lengths and that the output intensity approaches the strictly parabolic profile expected from theory.

C. Design Criteria for Parabolic Pulse Amplifiers

Based on the preceding discussion, it is clear that different input pulses undergo significantly different evolution toward the asymptotic parabolic pulse solution. From an experimental point of view, these results are important, as they indicate that a careful choice of the input pulse and amplifier characteristics can greatly accelerate evolution to the parabolic pulse regime. This evolution has been investigated with an extensive series of simulations, which have clearly shown that certain initial values of the parameters \( \Delta T_0 \) and \( U_{in} \), in conjunction with the amplifier parameters \( \gamma, \beta_2 \), and \( g \), allow for the most efficient convergence to the parabolic pulse solution.

To find the optimum choice of \( \Delta T_0 \), we start from the equation for the effective width parameter \( T_p(z) \) given in Eq. (19). Given \( \beta_2 \), \( \gamma \), and \( A_0 \) [from Eq. (21)], we can calculate \( T_p(z) \) at \( z = 0 \) to obtain

\[
T_p(0) = \frac{6\sqrt{2A_o\gamma\beta_2}}{g},
\]  

which we might expect to be close to the initial pulse width that yields the fastest convergence to the asymptotic parabolic pulse solution. Although it is meaningless to refer to the effective width, \( T_{p(0)} \), of a non-parabolic input pulse, our simulations nonetheless suggest that the use of \( T_p(0) \) as the FWHM duration of an input pulse is associated with an evolution map that rapidly enters the asymptotic regime. We are now in a position to reinterpret Fig. 3, which showed the evolution for a variety of pulses of different input pulse durations. For this particular combination of amplifier parameters and input pulse energy, Eq. (28) yields \( T_p(0) = 0.19 \) ps, consistent with the results presented in the figure, where clearly the 0.2-ps FWHM input pulse was the fastest to converge to the parabolic pulse regime.

Experimentally, the fiber parameters \( \beta_2 \) and \( \gamma \) cannot generally be modified. However, it is often possible to control the fiber gain, as well as the pulse energy and duration, so Eq. (28) can be used for practical experimental design. For example, for a pulse of fixed energy, the appropriate pulse width may be able to be varied to ensure optimal evolution in an amplifier with a given gain. For this case, Fig. 5(a) shows a plot of \( T_p(0) \) as a function of \( g \) with \( \beta_2 = 25 \times 10^{-3} \) ps² m⁻¹, \( \gamma = 5.8 \times 10^{-3} \)
W$^{-1}$ m$^{-1}$, and an initial energy of 16.5 pJ. (Note: similar curves can be constructed for any choice of fiber parameters and input pulse energy.) Under these conditions, when $g = 2$ m$^{-1}$, a pulse with an initial FWHM width of $T_p(0) = 0.2$ ps gives the best convergence to the parabolic pulse solution. Experimentally, if it is difficult to control the duration, of the input pulse, the amplifier design may still be optimized if it is possible to vary the pulse energy over a wide range. Fixing $T_p(0)$ and rearranging Eq. (28), we can determine the value of the input pulse energy $U_{in}$ that gives the fastest convergence to the parabolic pulse solution:

$$U_{in} = \frac{2T_p^2(0)g^2}{27\gamma\beta_2}.$$  

Figure 5(b) shows a plot of $U_{in}$ as a function of $g$ with the same amplifier parameters as in Fig. 5(a) but with $T_p(0) = 0.2$ ps. Under these conditions, when $g = 2$ m$^{-1}$, $U_{in} = 16.5$ pJ is the initial energy, which converges most quickly to the parabolic pulse solution, as we would expect given the results of Fig. 5(a).

We now introduce a characteristic length that will describe the distance over which a pulse needs to propagate in the fiber amplifier to reach the parabolic regime. This is a useful guide for choosing the minimum amplifier length required for reaching the region in which the pulse propagates self-similarly. In Subsection 2.A we considered the limit $z \to \infty$, so we could neglect the term proportional to $d^2F/d\theta^2$ in Eq. (11) as it asymptotically tends to zero. This follows from Eqs. (11), (14), and (16) because the ratio

$$G = |\gamma F^2|/\left|\frac{\beta_2 d^2F}{2F} \frac{dF}{d\theta^2}f^2 \exp(-2g\theta)\right|$$  

 grows exponentially as $\exp(4g\theta/3)$ at $z \to \infty$ for fixed $\theta$. Introducing $N$ as $G|_{\theta=0} = N^2$, we define the characteristic propagation length $z_c(N)$:

$$z_c(N) = \frac{3}{2g} \ln\left(\frac{Ng}{6|\gamma A_0^2|}\right),$$

where we expect that the pulse will be close to the parabolic shape at $z \approx z_c(N)$ for some large value of $N$. Indeed, simulations indicate that values of $N > 100$ are associated with pulse propagation well into the parabolic regime, with characteristics similar to those shown in Fig. 6.

Fig. 6. Characteristic propagation distance $z_c(100)$ corresponding to an amplifier length sufficient for parabolic pulse characteristics to be observed, plotted as a function of amplifier gain. Other parameters are given in the text.

2. Figure 6 is a plot of $z_c(100)$ as a function of distributed gain $g$ with the parameters from Fig. 5(b). It shows that, for $g = 2$ m$^{-1}$, the pulse enters the parabolic regime at $\sim 3.5$ m.

3. INTERMEDIATE ASYMPTOTIC SOLUTION TO THE NONLINEAR SCHRODINGER EQUATION WITH GAIN

From the analysis above, we note that, although at $|T| = T_p(z)$ the amplitude of the parabolic solution in Eq. (18) has an infinite slope, a strictly parabolic pulse is reached only in the asymptotic limit $z \to \infty$. At intermediate propagation distances, the simulation results in Fig. 1(b) show the presence of low-amplitude wings on the parabolic pulse whose amplitude decreases with propagation distance and whose linearity when it is plotted on a logarithmic scale indicates that the wings are exponentially decaying functions of $T$.

In this context, we note that a well-known property of some self-similar systems is the presence of an intermediate asymptotic solution that describes the scaling of certain properties of the system, which is valid in an intermediate propagation regime between $z \to 0$ and $z \to \infty$. Establishing an intermediate asymptotic solution provides important information regarding the transition from the original non-self-similar solution to the final exact asymptotic solution. The simulation results above suggest that the development of exponentially decaying wings on the parabolic pulse is associated with the transition of an injected input pulse toward the asymptotic parabolic pulse solution, and we are thus motivated to search for an intermediate asymptotic solution that describes these wings.

We begin this analysis by reexamining the asymptotic approximation that the term in Eq. (11) proportional to $d^2F/d\theta^2$ can be neglected. Here we consider the more general case of a solution that consists of (i) a high-intensity region $|T| \leq T_p(z)$, where this approximation holds and the parabolic pulse solution above remains valid, and (ii) a low-intensity region in the wings $|T| > T_p(z)$, where this term can no longer be neglected. Obtaining the solution for the low-intensity wings is facilitated by the fact that in this case we can neglect the last term in Eq. (11) that is proportional to $F^2$, which corresponds to the limit $G \to 0$ in Eq. (30).

In particular, we assume that the solution in the wings has the same self-similar form as before:

$$A_w(z, T) = f_w(z) F_w(z, T) = f_w(z) F_w(\theta_w),$$  

where $f_w(z)$ and $F_w(\theta_w)$ are defined as before. For a given propagation length $L$, we can find $\theta_w$ such that $|T| = T_p(z)$, and then $f_w(z)$ is given by

$$f_w(z) = \frac{T_p(z)}{T_p(L)}.$$
\[ \Phi_w(z, T) = \varphi_w(z) + C_w(z)T^2, \]

where self-similarity variable \( \vartheta_w \) is given by

\[ \vartheta_w = f_w^{-2}(z) \exp(-gz)T, \]

although we have introduced the subscript \( w \) to denote the solution in the wings. Considering Eqs. (10) and (11), we obtain the following set of equations for \( F_w(\vartheta_w) \), \( f_w(z), \varphi_w(z), \) and \( C_w(z) \) in the limit \( G \to 0 \):

\[ \frac{df_w}{dz} = \beta_2 C_w f_w + \frac{g}{2} f_w' + \left( \frac{2 \beta_2 C_w^2}{dz} \right) f_w^{-8} \exp(4gz) \vartheta_w^2 = \frac{\beta_2}{2} \frac{d^2F_w}{dz}, \]

where the right-hand side of Eq. (36) is a function of only one variable, \( \vartheta \). From Eqs. (35) and (36) we obtain the following set of equations:

\[ \frac{d^2F_w}{dz} = \frac{1}{2} \frac{d\vartheta_w^2}{dz} = (b_0 - b \vartheta_w^2)F_w, \]

where \( b \) and \( b_0 \) are arbitrary constants.

We begin by considering the solution for \( F_w \), because Eq. (39) is readily seen to be a form of Weber's differential equation whose solutions are the well-known Weber parabolic cylinder functions. However, it can be demonstrated that physically meaningful (positive definite) solutions for \( F_w \) and \( f_w \) require that the arbitrary constant in this equation be \( b = 0 \). With this restriction, we obtain the solution for \( F_w(\vartheta_w) \) as

\[ F_w(\vartheta_w) = F_0(0) \exp(-\lambda |\vartheta_w|), \]

where \( \lambda = \sqrt{b_0} \). Here we also note that a physical solution requires the additional constraint that \( b_0 > 0 \) because, for \( b_0 < 0 \), \( F_w(\vartheta_w) \) is complex, and for \( b_0 = 0 \), \( F_w(\vartheta_w) \) is independent of \( T \). We can find the corresponding solution for \( f_w(z) \) at \( b = 0 \) by combining Eqs. (35) and (37) to obtain

\[ \frac{d}{dz} \left( \frac{1}{f_w} \frac{df_w}{dz} \right) = \frac{\beta_2}{2} \frac{d^2F_w}{dz} = 0. \]

When we use the same normalization condition as in Subsection 2.A that \( F_0 = 1, \) Eq. (41) leads to the solution for \( f_w(z) \):

\[ f_w(z) = \frac{B_w}{\sqrt{z}} \exp \left( \frac{g}{2} \frac{z}{2} \right), \]

where \( B_w \) is an amplitude parameter that we shall see depends on the input pulse and amplifier parameters. Given \( f_w(z) \) and Eq. (34), we obtain the new self-similarity variable \( \vartheta_w \):

\[ \vartheta_w = \frac{B_w^{-2}T}{z}, \]

so combining the solutions given by Eqs. (40)–(43), we obtain the amplitude \( A_w(z, T) \) in the region of the wings as

\[ A_w(z, T) = \frac{B_w}{\sqrt{z}} \exp \left( \frac{g}{2} \frac{z}{2} \right) \exp \left( -\lambda |T| \right), \]

at \( |T| > T_p(z) \), where we define \( \lambda = \lambda B_w^{-2} \). This solution implies that, for intermediate asymptotic distances, the wings of the parabolic pulse decay exponentially as a function of \( T \), confirming the simulation results in Figs. 1(b) and 2(a).

To completely determine the self-similar solution in the region \( |T| > T_p(z) \), we now consider the solution for the phase of the pulse. The form of the chirp parameter \( C_w \) follows from Eqs. (37) and (42) as

\[ C_w(z) = -\frac{1}{2\beta_2}, \]

Then, on solving Eq. (38) with Eq. (42), we find, given Eqs. (33) and (45), the form of the phase:

\[ \Phi_w(z, T) = \varphi_0 + \frac{\beta_2 \lambda^2}{2z} - \frac{T^2}{2\beta_2}, \]

where \( \varphi_0 \) is an arbitrary constant. The corresponding linear chirp \( \Omega_{w,c} \) is given by

\[ \Omega_{w,c}(z, T) = -\frac{\lambda \Phi_w}{\lambda T} = T \frac{\lambda}{\beta_2}, \]

This implies that the slope of the chirp in the wings of the pulse is fundamentally different from that in the parabolic region, as it depends on propagation distance \( z \). It is also easy to show that the energy in the wings goes to zero as \( z \to \infty \).

Equations (44) and (46) define the solution for the low-intensity regions of a pulse propagating in a normal-dispersion fiber amplifier under the influence of constant distributed gain, up to the undetermined parameters \( B_w \) and \( \Lambda \). Together with the results in Section 2, we now have a complete solution that describes the evolution of an arbitrary input pulse in a normal-dispersion fiber amplifier. However, unlike in Section 2, where all the parameters of the solution could be explicitly determined by use of conservation integral Eq. (2), there is no appropriate conservation integral in the region \( |T| > T_p(z) \). Such a solution is known as an intermediate asymptotic solution of the second kind, and the unknown parameters of the solution must be found by use of numerical simulations together with dimensional analysis to determine the particular dependence of \( B_w \) and \( \Lambda \) on the system parameters.

Although in principle it is possible to carry out such an analysis for any arbitrary input pulse, we have considered only pulses with Gaussian and hyperbolic secant pulse profiles with energy \( U_{in} \) and rms width \( \Delta \tau_{in} \). In this case, the results for an input Gaussian pulse are
tion is well described by the nonlinear Schro¨dinger equation in the normal-dispersion regime when the propagation is assumed to involve the combined action of self-phase modulation and dispersion. An asymptotic analytical form for a pulse generated under these conditions consists of an initial parabolic period of evolution.

4. CONCLUSIONS

The self-similarity analysis presented here yields an asymptotic analytical form for a pulse generated under the combined action of self-phase modulation and dispersion in the normal-dispersion regime when the propagation is well described by the nonlinear Schro¨dinger equation in the normal-dispersion regime when the propagation is assumed to involve the combined action of self-phase modulation and dispersion. An asymptotic analytical form for a pulse generated under these conditions consists of an initial parabolic period of evolution.

The existence of parabolic pulses has been verified both by numerical simulations and by the experiments previously described in Ref. 8, and the results presented here can be expected to have wide application, limited only by the restriction that the bandwidth of the generated parabolic pulses cannot exceed the gain bandwidth of the amplifier. An important extension of this research concerns the application of the asymptotic self-similarity analysis described here to the case of a fiber amplifier with an arbitrary longitudinal gain profile. Although the theoretical and numerical results for this case previously obtained in the high-intensity limit9 certainly suggest that the parabolic pulse profile is a universally stable shape that is always generated in fiber amplifiers, irrespective of the particular gain profile, a rigorous demonstration of this conclusion remains an open problem.

The final feature of these results concerns the asymptotic nature of the evolution, which ensures that the pulse shape is stable against perturbations. A formal analysis shows that the parabolic pulse solution and the wings are stable.16 Linearly chirped parabolic pulses (which can be effectively compressed) can thus be expected for any input pulse shape and in the presence of collisions and other perturbations within the amplifier. The final shape of the pulse is determined principally by the energy of the input pulses. For the case of a double-clad Yb3+-doped fiber amplifier (for which parabolic pulse shapes have already been demonstrated), the peak power and the pulse duration of the compressed pulses are comparable with those generated in a high-power mode-locked Ti:sapphire laser.

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