Hole operations on Hurwitz maps[*]

Gábor Gévay[†] [iD]

Bolyai Institute, University of Szeged
H-6720 Szeged, Hungary

Gareth A. Jones [iD]

School of Mathematics, University of Southampton
Southampton SO17 1BJ, UK

Dedicated to Marston Conder in celebration of his 65th birthday

Abstract

For a given group G the orientably regular maps with orientation-preserving automorphism group G are used as the vertices of a graph $\mathcal{O}(G)$, with undirected and directed edges showing the effect of duality and hole operations on these maps. Some examples of these graphs are given, including several for small Hurwitz groups. For some G, such as the affine groups $\mathrm{AGL}_1(2^e)$, the graph $\mathcal{O}(G)$ is connected, whereas for some other infinite families, such as the alternating and symmetric groups, the number of connected components is unbounded.

Keywords: Regular map, orientably regular, duality, hole operation, Hurwitz group.

Math. Subj. Class.: 05C10 (primary), 20B25 (secondary)

1 Introduction

For any group G let $\mathcal{O}(G)$ denote the set of (isomorphism classes of) orientably regular maps $\mathcal{M}$ with orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{M}$ isomorphic to G. For a given finite group G this set is finite since its elements correspond bijectively to the orbits of $\mathrm{Aut}\, G$ on certain generating sets for G. These maps are related to each other

---

by the application of certain operations, described by Wilson in [40], such as duality D and the hole operations H_j for j coprime to the valency of the vertices, since all of these preserve the group G. One way of understanding how the various maps associated with G are related to each other is to regard this set $\mathcal{O}(G)$ as the vertex set of a graph, with directed or undirected edges indicating the actions of these operations. For example, one can consider whether or not this graph is connected, and if not, what invariants might be used to distinguish maps in different components. We will describe the maps $\mathcal{M}$ and graphs $\mathcal{O}(G)$ arising for some particularly interesting groups, including several of the smallest Hurwitz groups. We will show that for the symmetric and alternating groups, the number of connected components of $\mathcal{O}(G)$ is unbounded, building on work of Marston Conder [5] on alternating groups as Hurwitz groups in the latter case.

In addition to topological graph theory and the theory of Riemann surfaces, Grothendieck's dessins d'enfants provide motivation for this study, as a preparation for an investigation of the relationship between the actions of these operations and that of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ on finite oriented maps, regarded as algebraic curves defined over number fields (see [26], for example). We also see this paper as a first step towards a study of a wider class of operations, including Petrie duality, on the set $\mathcal{R}(G)$ of all regular maps with a given automorphism group G. Constraints of space and time oblige us to defer these extensions to a later date.

2   Orientably regular maps

If $\mathcal{M}$ is an orientably regular map then its arcs (directed edges) can be identified with the elements of a group G, so that its orientation-preserving monodromy and automorphism groups are identified with the right and left regular representations of G.

Let $\mathcal{M}$ have type $\{p,q\}$, meaning that its faces are p-gons and its vertices have valency q. (The term *Schläfli type*, or *Schläfli symbol*, is also used [10]). We note that later on in this paper, we shall use the same symbols p and q for denoting a prime and a prime power, respectively; to avoid confusion, where it is not clear from the context, the actual meaning of these symbols will be emphasized.)

Then G has generators x, y, z satisfying

$$x^q = y^2 = z^p = xyz = 1,$$

where x and y act (as monodromy permutations) by rotating all arcs around their incident vertices, following the orientation, and by reversing them, so that z rotates them around faces. Conversely any such generating triple for G defines a map $\mathcal{M}$ of this type, with isomorphic maps corresponding to triples equivalent under $\mathrm{Aut}\,G$. Of course the triple, and hence the map, is uniquely determined by any two of its members, so we will often restrict attention to the generating pair x, y.

Map duality D, transposing vertices and faces, replaces the triple (x, y, z) with $(z, y, (zy)^{-1} = x^y)$, corresponding to the dual map $D(\mathcal{M})$. Then D^2 sends (x, y, z) to (x^y, y, z^y) $= (x, y, z)^y$, giving a map isomorphic to $\mathcal{M}$.

If j is coprime to q then another orientably regular map $\mathcal{M}_j = H_j(\mathcal{M})$, of type $\{p', q\}$ for some p' and with the same orientation-preserving automorphism group G as $\mathcal{M}$, can be found by applying the hole operation H_j, described by Wilson in [40], to $\mathcal{M}$. This map embeds the same graph as $\mathcal{M}$, but the rotation x of arcs around each vertex is replaced with x^j, so that the faces, and hence the underlying surface, may be changed. In terms of group

theory, $H_j(\mathcal{M})$ corresponds to the generating triple (x^j, y, z_j) for G, where $z_j := (x^j y)^{-1}$. Thus the boundary of each face of $H_j(\mathcal{M})$, given the orientation of the underlying surface, follows a *j-hole* of $\mathcal{M}$, at each vertex v taking the j-th incident edge, following the reverse orientation around v, rather than the first; its length is the order of the element z_j. (Note that a j-hole, taken in the reverse direction, becomes an $(q-j)$-hole.)

Example 2.1. Figure 1 shows a 3-hole $(16, 5, 6, 20, 16)$ of length 4 in Klein's map $\mathcal{K}$ of type $\{3, 7\}$ and genus 3 with $G = \mathrm{PSL}_2(7)$ (see Section 7 for details).
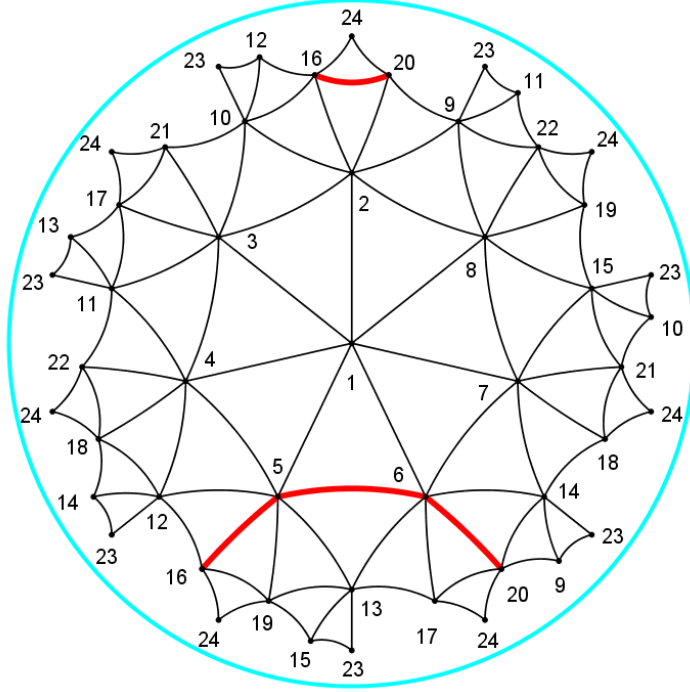

Figure 1: Klein's map $\mathcal{K}$ of type $\{3, 7\}$ and genus 3 represented in Poincaré's disc model of the hyperbolic plane. A 3-hole $(16, 5, 6, 20, 16)$ is highlighted.

Clearly $H_j = H_k$ whenever $j \equiv k \bmod (q)$; since $H_j \circ H_k = H_{jk}$ for all j and k coprime to q, these operations give a representation of the group U_q of units mod (q). The operation H_{-1} sends each map to its mirror-image, so an orientably regular map $\mathcal{M}$ is regular (has a flag-transitive full automorphism group $\mathrm{Aut}\,\mathcal{M}$) if and only if $H_{-1}(\mathcal{M}) \cong \mathcal{M}$; thus the action on regular maps gives a representation of the group $U_q/\{\pm 1\}$.

One can also apply H_j to $\mathcal{M}$ when $k := \gcd(j, q) > 1$. However, in this case the operation is not invertible, and it does not preserve the embedded graph, since each vertex is replaced with k vertices of valency q/k. Moreover, the orientation-preserving automorphism group may also change: if $G_0 := \langle x^j, y \rangle$ is a proper subgroup of index n in G, then $H_j(\mathcal{M})$ is the disjoint union of n isomorphic connected orientably regular maps $\mathcal{M}_0$, each with $\mathrm{Aut}^+\mathcal{M}_0 \cong G_0$, and $\mathrm{Aut}^+ H_j(\mathcal{M})$ is isomorphic to the wreath product $G_0 \wr \mathrm{S}_n$. Since our aim is to study those maps with a given orientation-preserving automorphism group G, we will avoid such complications by concentrating on cases where $k = 1$.

A further reason for doing this is that we eventually hope to compare hole operations with cyclotomic Galois operations, and these act by raising qth roots of unity to their jth powers where $\gcd(j, q) = 1$.

Another useful map operation is Petrie duality. A *Petrie polygon* in a map $\mathcal{M}$ is a closed zig-zag path turning alternately first right and first left at successive vertices. The *Petrie dual* $P(\mathcal{M})$ of $\mathcal{M}$ embeds the same graph as $\mathcal{M}$, but the faces of $\mathcal{M}$ are replaced with new faces bounded by the Petrie polygons. If $\mathcal{M}$ is orientably regular their length r (the *Petrie length* of $\mathcal{M}$) is twice the order of the commutator $[x, y] = x^{-1}y^{-1}xy = x^{-1}yxy$, so that $P(\mathcal{M})$ has type $\{r, q\}$. As in [11, 12], we will often refer to the *extended type* $\{p, q\}_r$ of $\mathcal{M}$. For instance, in Figure 1 the 3-holes $(16, 5, 6, 20, 16)$ and $(24, 17, 13, 19, 24)$ in $\mathcal{K}$ enclose a Petrie polygon of length $r = 8$. (More generally, whenever $p = 3$ each Petrie polygon is enclosed by two 3-holes of length $r/2$ in this way.)

Given any group G, one can regard $\mathcal{O}(G)$ as the vertex set of a graph (also denoted by $\mathcal{O}(G)$) by adding edges showing the actions of the operations D and H_j (for selected j); they will be undirected for cycles of length 2, e.g. for involutions such as D, and directed in the case of longer cycles for hole operations H_j. Loops, corresponding to invariant maps, will be omitted. In particular, we will use dashed and dotted undirected edges for D and H_{-1}, and unbroken edges for other hole operations. Although the operation P preserves the full automorphism group of a map, it has the disadvantage from our current point of view of not always preserving orientability. For example, if $\mathcal{M}$ is the tetrahedral map on the sphere, then $P(\mathcal{M})$ is the antipodal quotient of the cube, on the real projective plane; similarly, $P(\mathcal{K})$ is a non-orientable regular map of genus 41. In this paper we will therefore concentrate mainly on the operations D and H_j, which preserve automorphism groups and orientability.

In general we will not draw edges labelled H_j for all j coprime to the relevant valencies q, but merely enough to identify the connected components of $\mathcal{O}(G)$ by showing how various maps can be tranformed into each other. For example, if U_q (or $U_q/\{\pm 1\}$ in the case of regular maps) is cyclic it is sufficient to take j to be a generator.

Provided its genus g is neither too large nor less than 2, each map $\mathcal{M} \in \mathcal{O}(G)$ can be located in Conder's computer-generated lists of regular or chiral orientable maps in [7]. Each entry there has the form R$g.n$ or C$g.n$ respectively, where g denotes the genus, and n denotes the nth entry in the list of maps of that genus, ordered lexicographically by their type $\{p, q\}$, with $p \leq q$. Within each list, each entry refers either to a dual pair of maps of types $\{p, q\}$ and $\{q, p\}$ or to a single self-dual map with $p = q$. If $p < q$ we will denote the maps of type $\{p, q\}$ and $\{q, p\}$ by X$g.n$a and X$g.n$b respectively, where X is R or C; if $p = q$ and the entry denotes a non-isomorphic dual pair we will assign the labels X$g.n$a and X$g.n$b arbitrarily, whereas a single self-dual map will be denoted simply by X$g.n$. Similar conventions will apply to the list of non-orientable regular maps, denoted by N$g.n$ in [7].

2.1   Some simple examples of graphs $\mathcal{O}(G)$

If G is the alternating group A_4 then $\mathcal{O}(G)$ contains only the tetrahedral map $\{3, 3\}$ of genus 0.

If G is a dihedral group D_m for some $m > 2$ then $\mathcal{O}(G)$ consists of the dual pair of maps $\{2, m\}$ and $\{m, 2\}$ of genus 0, whereas if $m = 2$, so that $G = D_2 \cong V_4$, then $\mathcal{O}(G)$ contains only the self-dual map $\{2, 2\}$ of genus 0. In either case the operations H_j for j coprime to m act trivially on $\mathcal{O}(G)$.

If $G = S_4$ then the involution y must be a transposition: the double transpositions lie in the normal subgroup $K \cong V_4$ with $G/K \cong S_3$ non-cyclic, so they cannot be members of generating pairs. One can take x to be any 3-cycle or 4-cycle not inverted by y, giving the cube $\{4, 3\}$ or its dual, the octahedron $\{3, 4\}$. The operations H_j act trivially on $\mathcal{O}(G)$.

The graphs $\mathcal{O}(G)$ corresponding to these groups G are shown in Figure 2.

$$\{3,3\} \qquad \{2,n\} \qquad \{n,2\} \qquad \{2,2\} \qquad \{3,4\} \qquad \{4,3\}$$

$$\mathcal{O}(A_4) \qquad\qquad \mathcal{O}(D_n), n > 2 \qquad\qquad \mathcal{O}(D_2) \qquad\qquad \mathcal{O}(S_4)$$
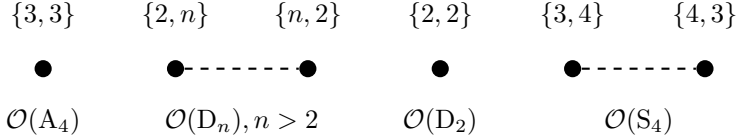
Figure 2: The graphs $\mathcal{O}(A_4)$, $\mathcal{O}(D_n)$ and $\mathcal{O}(S_4)$

If G is the alternating group A_5 (also isomorphic to $PSL_2(4)$ and $PSL_2(5)$) then $\mathcal{O}(G)$ consists of three maps, namely the dodecahedron $\{5, 3\}$, the icosahedron $\{3, 5\}$ and the great dodecahedron $\{5, 5\}_6$ of genus 4 (the self-dual map R4.6 in [7]); duality D transposes the first two, while H_2 transposes the last two, so $\mathcal{O}(G)$ is connected (see Figure 3).
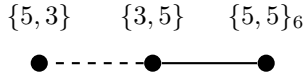
$$\{5,3\} \qquad \{3,5\} \qquad \{5,5\}_6$$

Figure 3: The graph $\mathcal{O}(A_5)$

3   Isotactic polygons

Let $\mathcal{M}$ be an orientably regular map of type $\{p, q\}$. An *isotactic polygon* P of *type* $(d_1, d_2, \ldots, d_m)^n$ in $\mathcal{M}$ is a closed path in the underlying graph of $\mathcal{M}$ formed by successively taking the d_i-th edge at the i-th vertex v_i visited, following the local orientation around v_i, using the sequence $d_1, d_2, \ldots, d_m \in \mathbb{Z}_q$ repeated n times. We call d_i the *right degree* of P at v_i, and $l = mn$ the *length* of P. By orientable regularity, if such a polygon P exists, then it does so starting at any directed edge in $\mathcal{M}$, since it is equivalent to a relation

$$(x^{d_1} y x^{d_2} y \ldots x^{d_m} y)^n = 1$$

in the monodromy group $G = \langle x, y \rangle$ of $\mathcal{M}$.

This concept is a common generalisation of the classical notions of *face*, *Petrie polygon* and *hole*. In fact,

- an isotactic polygon of type $(1)^p$ or $(-1)^p$ is the boundary of a p-valent face;

- more generally, an isotactic polygon of type $(j)^l$ with j coprime to q is a j-hole of length l;

- an isotactic polygon of type $(1, -1)^{r/2}$ is a Petrie polygon of length r (provided this length is even);

- more generally, an isotactic polygon of type $(j, -j)^{r/2}$ is a Petrie polygon of order j, where r is its (even) length.

These well-known examples of isotactic polygons are distinguished by their roles in the Petrie and hole operations, but other types also occur in various roles. For example, hexagons of type $(2,4)^3$ occur in the polyhedral realisation of Klein's map $\mathcal{K}$ of genus 3 (see Section 7) given by Schulte and Wills in [31]. On the other hand, the heuristic role of hexagons of type $(2,4,4)^2$ is emphasised in the investigation of 7-fold rotational symmetry of the Fricke–Macbeath map $\mathcal{F}$ of genus 7 (see Section 8) in [3, 4]. Moreover, in this map the 3-holes, together with three suitable triangular faces, form generalised Petersen graphs of type $\mathrm{GP}(9,3)$ (see Figure 4); the presence of such subgraphs in the underlying graph of $\mathcal{F}$ confirms that this map has no polyhedral embedding in $\mathbb{E}^3$ with 9-fold symmetry [4].
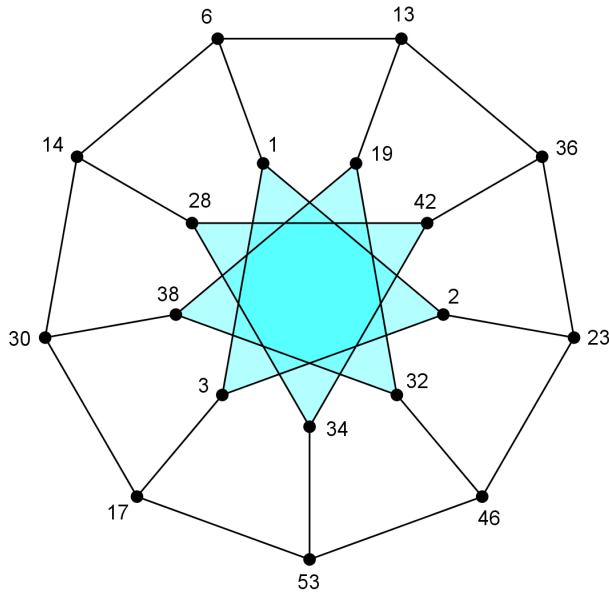

Figure 4: A generalized Petersen graph of type GP(9,3) in the Fricke–Macbeath map $\mathcal{F}$ of type $\{3,7\}$ and genus 7 (vertex labels correspond to those given in Figure 9 below).

4 Regularity

An orientably regular map $\mathcal{M}$, corresponding to a generating pair x, y of G, is regular if and only if there is an automorphism α of G inverting x and y, or equivalently inverting x and centralising y. In this case the full automorphism group $\operatorname{Aut}\mathcal{M}$ is a semidirect product $G \rtimes \langle \alpha \rangle$ of G and C_2, with α acting as a reflection of $\mathcal{M}$. For example, the maps in $\mathcal{O}(G)$ for $G = \mathrm{A}_4$, D_n and S_4 are all regular, with $\operatorname{Aut}\mathcal{M} \cong \mathrm{S}_4$, $\mathrm{D}_n \times \mathrm{C}_2$ and $\mathrm{S}_4 \times \mathrm{C}_2$ respectively.

We will say that a regular map $\mathcal{M}$ is *inner* or *outer regular* as α is or is not an inner automorphism, that is, conjugation by some element $c \in G$. If α is inner then c^2 is in the centre $Z(G)$ of G, so if $Z(G) = 1$ (as is the case with all the groups considered here, apart from D_n for n even), then $c^2 = 1$ and $\operatorname{Aut}\mathcal{M} \cong G \times \mathrm{C}_2$ with $\alpha c = \alpha c^{-1}$ generating the second direct factor. In this case $\mathcal{M}$ has a non-orientable regular quotient map $\overline{\mathcal{M}} = \mathcal{M}/\mathrm{C}_2$, of genus $g+1$ where $\mathcal{M}$ has genus g, with automorphism group G; then $\mathcal{M}$ is the canonical orientable double cover of $\overline{\mathcal{M}}$.

It is easy to check that regularity, inner regularity and $\text{Aut } \mathcal{M}$ are preserved by the operations D and H_j, so they are constant throughout each connected component of $\mathcal{O}(G)$. In particular, maps in $\mathcal{O}(G)$ with different regularity properties must lie in different connected components of the graph.

4.1 Non-orientable regular maps

Although the emphasis of this paper is on orientably regular maps $\mathcal{M}$ with a given group $\text{Aut}^+\mathcal{M} \cong G$, non-orientable regular maps $\mathcal{N}$ with $\text{Aut } \mathcal{N} \cong G$ will also be detected by the construction of $\mathcal{O}(G)$. (Note that if G has no subgroup of index 2, then every map in $\mathcal{R}(G)$ is non-orientable.) Each such map $\mathcal{N}$ has a canonical orientable double cover $\mathcal{M}$, an inner regular map with $\text{Aut } \mathcal{M} \cong G \times Z$ where $Z \cong \text{C}_2$, and with $\text{Aut}^+\mathcal{M} \cong G$, so that $\mathcal{N}$ arises as the central quotient $\overline{\mathcal{M}} = \mathcal{M}/Z$ of a map $\mathcal{M} \in \mathcal{O}(G)$.

Conversely, each inner regular map $\mathcal{M} \in \mathcal{O}(G)$ has full automorphism group $\text{Aut } \mathcal{M} = G \times Z$, with $Z \cong \text{C}_2$ reversing orientation, so it has a non-orientable regular quotient $\mathcal{N} = \overline{\mathcal{M}} = \mathcal{M}/Z$ with automorphism group G. This gives a bijection $\mathcal{M} \mapsto \mathcal{N}$ between the subsets of $\mathcal{O}^\times(G) \subseteq \mathcal{O}(G)$ and $\mathcal{R}^-(G) \subseteq \mathcal{R}(G)$ consisting of their inner regular and non-orientable maps; here $\mathcal{N}$ has the same type as $\mathcal{M}$ (but half its Petrie length), and has genus $g + 1$ if $\mathcal{M}$ has genus g. Since this bijection commutes with the operations D and H_j, it induces an isomorphism $\mathcal{O}^\times(G) \to \mathcal{R}^-(G)$ of directed graphs.

4.2 $\mathcal{O}(G)$ for $G = \text{S}_5$

The earlier examples were very straightforward, involving familiar maps. This case needs rather more thought.

If $G = \text{S}_5$ then the involution y can be a transposition or a double transposition. In the first case, there are possible generators x of orders 4, 5 and 6, giving rise to two dual pairs of regular maps, the dual pair R4.2 of types $\{4, 5\}_6$ and $\{5, 4\}_6$ on Bring's curve of genus 4 (see [38], for example), and the dual pair R9.16 of types $\{5, 6\}_4$ and $\{6, 5\}_4$, with H_2 transposing R4.2a and R9.16b; the component of $\mathcal{O}(G)$ containing these four maps is a path graph, with three edges labelled D, H_2, D.

In the second case x, which must be odd, can have order 4 or 6, giving the dual pair R6.2 of types $\{4, 6\}_{10}$ and $\{6, 4\}_{10}$, together with the self-dual map R11.5 of type $\{6, 6\}_6$ (for this last map one can take $x = (1, 5, 3)(2, 4), y = (1, 2)(3, 4)$); all operations H_j act trivially on these maps, so the pair R6.2 and the map R11.5 form two more connected components of $\mathcal{O}(G)$. The graph is shown in Figure 5: the vertices are labelled with the extended types and our extension of the notation in [7] for the corresponding maps. This example shows that the involution y (or more precisely its orbit under $\text{Aut } G$) is insufficient to characterise a component of $\mathcal{O}(G)$.

The maps $\mathcal{M} \in \mathcal{O}(G)$ are all regular, and since $\text{Out } \text{S}_5$ and $Z(\text{S}_5)$ are both trivial they are inner regular, with automorphism group $\text{S}_5 \times \text{C}_2$; their non-orientable quotients $\overline{\mathcal{M}}$, reading Figure 5 from left to right, are the dual pairs N5.1, N10.4, N7.1 and the self-dual map N12.3, all with automorphism group S_5.

We note that R6.2b occurs first as one of Coxeter's regular skew polyhedra [10]. It is realised as a subcomplex of the boundary complex of the dual of the 4-polytope $t_{1,2}\{3, 3, 3\}$. Its faces form the faces of 10 equal Archimedean truncated tetrahedra (the facets of the polytope). The non-identity element of C_2 is realised in this case as a central inversion (i.e. reflection in a point) in $\mathbb{E}^4$ interchanging two disjoint 10-tuples of hexagons.

$\{5,4\}_6$ $\{4,5\}_6$ $\{6,5\}_4$ $\{5,6\}_4$ $\{4,6\}_{10}$ $\{6,4\}_{10}$ $\{6,6\}_6$

R4.2b   R4.2a   R9.16b   R9.16a   R6.2a   R6.2b   R11.5
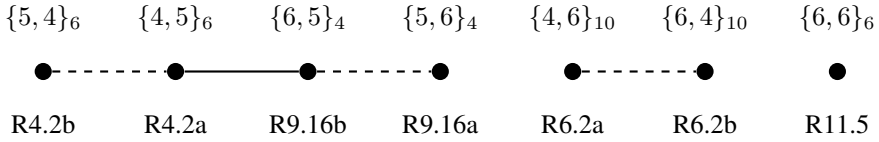
Figure 5: The graph $\mathcal{O}(S_5)$.

Two triples of hexagons sharing a hole of length 3 belong to the same such 10-tuples; in Figure 6 both the hexagonal faces and the triangular holes can clearly be seen.
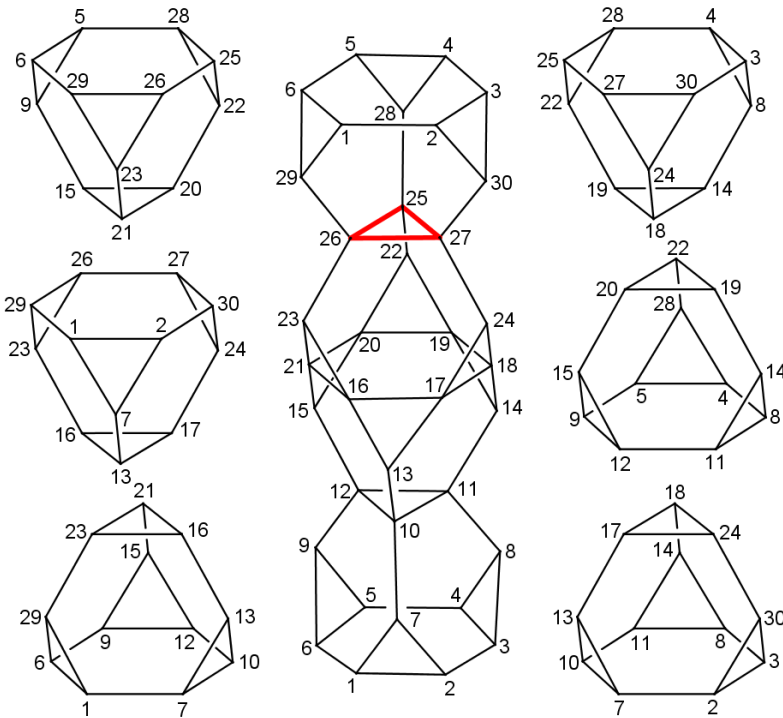

Figure 6: The combinatorial structure of R6.2b [18], with one hole of length 3 highlighted.

R4.2a goes back even earlier, in fact to Gordan's work [19]. Both the dual pairs R4.2a, R4.2b and R6.2a, R6.2b have polyhedral realisations in Euclidean 3-space [32, 33].

5   Hurwitz surfaces, groups and maps

Many of our chosen examples of automorphism groups are Hurwitz groups. This is partly because these groups exhibit interesting phenomena, and partly because they and their associated maps and surfaces have been intensively studied. Here we summarise some of their important properties.

Hurwitz [23] showed that the automorphism group $G = \operatorname{Aut} \mathcal{S}$ of a compact Riemann surface $\mathcal{S}$ of genus $g \geq 2$ has order at most $84(g-1)$, attained if and only if $\mathcal{S} \cong \mathbb{H}/K$, where $\mathbb{H}$ is the hyperbolic plane and K is a normal subgroup of finite index in the triangle group

$$\Delta = \Delta(7,2,3) = \langle X, Y, Z \mid X^7 = Y^2 = Z^3 = XYZ = 1 \rangle$$

with $G \cong \Delta/K$. Such surfaces and groups attaining this upper bound are called *Hurwitz surfaces* and *Hurwitz groups*. In each case, $\mathcal{S}$ carries an orientably regular map $\mathcal{M}$ of type $\{3, 7\}$, called a *Hurwitz map*, with orientation-preserving automorphism group G. This map is regular if and only if K is normal in the extended triangle group $\Delta[7,2,3]$ which contains Δ with index 2, in which case the full automorphism group of $\mathcal{M}$ is isomorphic to $\Delta[7,2,3]/K$; this is equivalent to G having an automorphism inverting two of its standard generators x, y and z. Conder has written some very useful surveys on Hurwitz groups in [6, 8].

Every Hurwitz group is perfect, since Δ is, so it is a covering of a non-abelian finite simple group, which is itself a Hurwitz group. Two important infinite classes of simple Hurwitz groups are given by the following theorems [30, 5]:

Theorem 5.1 (Macbeath, 1969). *The group* $\operatorname{PSL}_2(q)$ *is a Hurwitz group if and only if*

(1) $q = 7$, *with a unique Hurwitz surface and map, or*

(2) $q = p$ *for some prime* $p \equiv \pm 1 \bmod (7)$, *with three Hurwitz surfaces and maps for each q, or*

(3) $q = p^3$ *for some prime* $p \equiv \pm 2$ *or* $\pm 3 \bmod (7)$, *with one Hurwitz surface and map for each q.*

(Note that here we use the standard group notation, so that q denotes a prime power.)

Theorem 5.2 (Conder, 1980). *The alternating group* A_n *is a Hurwitz group for each* $n \geq 168$.

In [5], Conder also determined which alternating groups of degree $n < 168$ are Hurwitz groups; the smallest is A_{15}.

6   Maps with $G = \operatorname{PSL}_2(q)$

6.1   Properties of $\operatorname{PSL}_2(q)$

Later in this paper we will construct the graphs $\mathcal{O}(G)$ for some specific groups $G = \operatorname{PSL}_2(q)$. Here we briefly summarise a few of the properties of these groups which we will need; see [13, Ch. XII] or [22, II.8] for full details.

The group $G = \operatorname{PSL}_2(q) = \operatorname{SL}_2(q)/\{\pm I\}$, $q = p^e$, has order $q(q^2-1)$ or $q(q^2-1)/2$ as $p = 2$ or $p > 2$. It is simple for all $q \geq 4$.

One useful property of $G = \operatorname{PSL}_2(q)$ is that conjugacy classes of non-identity elements $g \in G$ are determined uniquely by their traces (more precisely, trace-pairs) $\pm \operatorname{tr}(g) \in \mathbb{F}_q$. For example, the elements of orders 2, 3 and p are the non-identity elements with traces 0, ± 1 and ± 2 respectively.

6.2   Consistent choice of y

For any group G, since the generator y is invariant (up to automorphisms) under the opera-
tions D and H_j, it makes sense to use the same element for each map in a given component
of $\mathcal{O}(G)$, or even for each map in $\mathcal{O}(G)$ when this is possible.

For example, if $G = \mathrm{PSL}_2(q)$ then all involutions in G are conjugate to

$$y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(Here we identify each matrix $A \in \mathrm{SL}_2(q)$ with $-A$.) With this as y, its zero entries make
it easier to see the traces of words such as $x^j y$ and $[x, y]$, and hence to find their orders,
giving the extended types of various maps.

For instance, if y is as above and

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad xy = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}, \quad \text{so} \quad z = (xy)^{-1} = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix},$$

with trace $\pm(c - b)$. We will call $\tau = \pm(a + d)$ and $\tau' = \pm(c - b)$ the *trace* and *cotrace*
of the element x and of the triple (x, y, z). If this triple generates G then it corresponds to
a map $\mathcal{M} \in \mathcal{O}(G)$ of Schläfli type $\{p, q\}$, where the pair consisting of the trace $\pm(a + d)$
and cotrace $\pm(c - b)$ (unique up to the action of $\mathrm{Gal}\,\mathbb{F}_q$) determine the orders q and p of x
and z and thus the type $\{p, q\}$ of $\mathcal{M}$, together with its genus

$$g = 1 + \frac{|G|}{2(\frac{1}{2} - \frac{1}{p} - \frac{1}{q})}$$

by the Riemann–Hurwitz formula. We also have

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}yxy = \begin{pmatrix} -b^2 - d^2 & ab + cd \\ ab + cd & -a^2 - c^2 \end{pmatrix}$$

with trace $\sigma = \pm(a^2 + b^2 + c^2 + d^2)$, giving the Petrie length r (twice the order of $[x, y]$)
and hence the extended type $\{p, q\}_r$ of $\mathcal{M}$.

Once y is chosen, for example as above, all of this data for a map $\mathcal{M}$ is uniquely
determined by the generating element x, so instead of working with triples (x, y, z) it is
more efficient to work with the single elements x when applying operations such as D and
H_j.

For example, if $\tau = \pm(a + d)$ is the trace of x then

$$x^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix}$$

has trace $\pm(a^2 + 2bc + d^2) = \pm(a^2 + 2(ad - 1) + d^2) = \pm(\tau^2 - 2)$, and

$$(x^2 y)^{-1} = \begin{pmatrix} c(a + d) & -a^2 - bc \\ d^2 + bc & -b(a + d) \end{pmatrix}$$

has trace $\pm(a + d)(c - b) = \pm\tau\tau'$, giving the cotrace and hence the type and genus of
$H_2(\mathcal{M})$. Similarly, its Petrie length is determined by the trace

$$\pm((a^2 + bc)^2 + (b(a + d))^2 + (c(a + d))^2 + (d^2 + bc)^2)$$

of $[x^2, y]$. Applying H_{-1} or D is achieved by replacing x with

$$x^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{or} \quad z = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix}.$$

Of course, these leave r and g invariant, while D transposes p and q.

By an observation of Singerman [35], if $G = \mathrm{PSL}_2(q)$ then all maps $\mathcal{M} \in \mathcal{O}(G)$ are regular, so $H_{-1}(\mathcal{M}) \cong \mathcal{M}$ and there is no need to apply H_{-1}. However, this allows us, if we wish, to replace x with

$$z^{-1} = xy = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$$

instead of z when applying D.

Of course, many of the above expressions for matrices and traces are a little simpler if q is a power of 2.

7 $\quad G = \mathrm{PSL}_2(7)$

There is a unique Hurwitz surface of genus 3, Klein's quartic curve (see [27, 28]) given in projective coordinates by $x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0$. The corresponding Hurwitz map $\mathcal{K}$ of type $\{3, 7\}$, *Klein's map* (see Figure 1), has orientation-preserving automorphism group $G \cong \mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$, a simple group of order 168, and has full automorphism group $\mathrm{PGL}_2(7)$. This is the map R7.1 in [7], discussed as a polyhedron by Schulte and Wills in [31].

The non-identity elements of $\mathrm{PSL}_2(7)$ have trace $0, \pm 1, \pm 2$ or ± 3 as they have order 2, 3, 7 or 4 respectively. For a Hurwitz triple we require x and z to have orders 7 and 3, that is, $\tau = \pm(a + d) = \pm 2$, $\tau' = \pm(c - b) = \pm 1$ and of course $ad - bc = 1$, so we may take $a = 1$, $b = 1$, $c = 0$, $d = 1$; the resulting generating triple for $\mathcal{K}$ is

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Since $\sigma = a^2 + b^2 + c^2 + d^2 = 3$, $[x, y]$ has order 4, so $\mathcal{K}$ has Petrie length 8 and extended type $\{3, 7\}_8$.

Squaring x to apply H_2, we obtain

$$x^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

corresponding to the map $H_2(\mathcal{K})$, which has extended type $\{7, 7\}_6$ and genus 19. Squaring again, we obtain

$$x^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix},$$

corresponding to the map $H_4(\mathcal{K}) = H_{-3}(\mathcal{K}) = H_3(\mathcal{K})$ of type $\{4, 7\}_8$ and genus 10. Since $2^3 \equiv 1 \mod (7)$, applying H_2 again simply returns us to $\mathcal{K}$.

For $D(\mathcal{K})$, of type $\{7, 3\}_8$ and genus 3, we can replace x with

$$z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

and for $D(H_4(\mathcal{K}))$, of type $\{7,4\}_8$ and genus 10, we can replace x^4 with

$$\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.$$

However, $H_2(\mathcal{K})$ is self-dual, that is, $DH_2(\mathcal{K}) \cong H_2(\mathcal{K})$; this follows immediately from the following result:

Proposition 7.1. *There is a unique orientably regular map of type $\{7,7\}$ with orientation-preserving automorphism group $G \cong \mathrm{PSL}_2(7)$.*

Proof. Any such map $\mathcal{M}$ corresponds to a generating triple (x,y,z) for G of type $(7,2,7)$. We will use the Frobenius triple-counting formula [17], which states that if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are conjugacy classes in any finite group G, then the number of triples $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$ with $xyz = 1$ is equal to

$$\frac{|\mathcal{X}| \cdot |\mathcal{Y}| \cdot |\mathcal{Z}|}{|G|} \sum_\chi \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}, \tag{7.1}$$

where the sum is over all irreducible complex characters χ of G.

In ATLAS notation [9], the conjugacy classes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ of $G = \mathrm{PSL}_2(7)$ containing x, y and z are respectively 7A or 7B, 2A, and 7A or 7B. In all cases

$$\frac{|\mathcal{X}| \cdot |\mathcal{Y}| \cdot |\mathcal{Z}|}{|G|} = \frac{(2^3 \cdot 3) \cdot (3 \cdot 7) \cdot (2^3 \cdot 3)}{2^3 \cdot 3 \cdot 7} = 2^3 \cdot 3^2 = 72.$$

If $\mathcal{X} \neq \mathcal{Z}$ then by using the character values in [9] we find that the character sum in (7.1) is

$$1 + 2 \cdot \frac{(-1) \cdot (-1 + i\sqrt{7})/2 \cdot (-1 - i\sqrt{7})/2}{3} + \frac{(-1) \cdot 2 \cdot (-1)}{6} = 0.$$

Thus there are no such triples in G and hence there are no corresponding maps. If $\mathcal{X} = \mathcal{Z} = $ 7A or 7B then the character sum is

$$1 + \frac{(-1) \cdot ((-1 + i\sqrt{7})/2)^2}{3} + \frac{(-1) \cdot ((-1 - i\sqrt{7})/2)^2}{3} + \frac{(-1) \cdot 2 \cdot (-1)}{6} = \frac{7}{3},$$

so, adding these contributions, the total number of such triples is

$$2 \cdot 72 \cdot \frac{7}{3} = 336.$$

These triples all generate G, since no proper subgroup of G has order divisible by 14. The triples are permuted by $\mathrm{Aut}\, G = \mathrm{PGL}_2(7)$, semi-regularly since only the identity automorphism can fix a generating set. Since $|\mathrm{PGL}_2(7)| = 336$, $\mathrm{Aut}\, G$ is transitive on the triples, so $\Delta(7,2,7)$ has one normal subgroup with quotient G, and hence there is one orientably regular map of type $\{7,7\}$ with orientation-preserving automorphism group G. $\qquad\square$

By the Riemann–Hurwitz formula, the above map $\mathcal{M}$ has genus 19. By its uniqueness, it is regular and self-dual, and is isomorphic to $H_2(\mathcal{K})$ and to the map R19.23 of extended
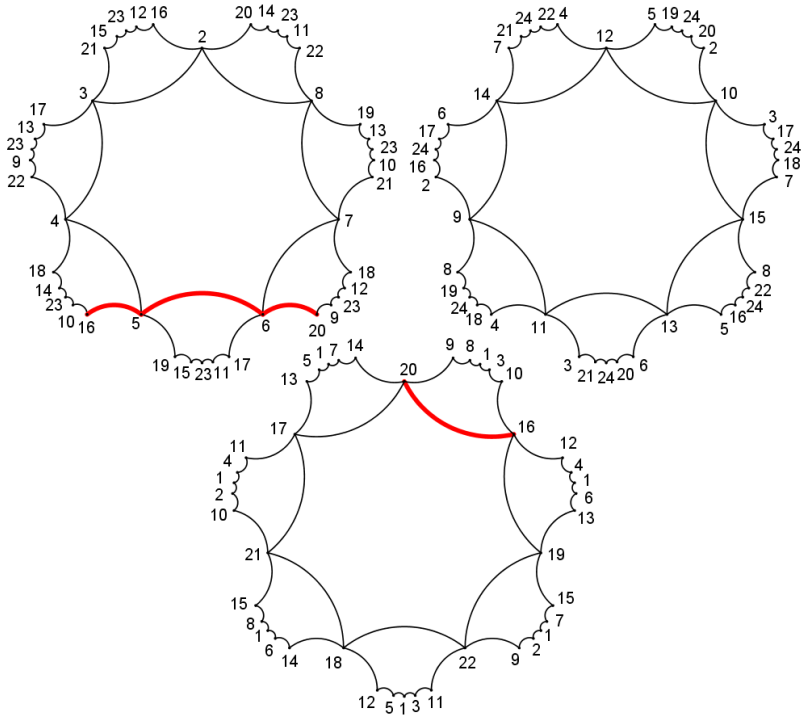
Figure 7: The map $H_2(\mathcal{K})$ of type $\{7,7\}_6$ and genus 19. It is trisected, and each part is represented in the disc model of the hyperbolic plane. A 2-hole is highlighted; this is the same as the 3-hole in $\mathcal{K}$ highlighted in Figure 1.

type $\{7,7\}_6$, the only regular orientable map of this genus and type in [7]. This map is shown in Figure 7.

Our other claims for uniqueness of maps, or enumerations of maps of a given genus and type, can be proved in a similar way. However, in future we will not give such full details unless there are special aspects of the calculation which need to be mentioned.

In fact the maps $\mathcal{K}$, $D(\mathcal{K})$, $H_2(\mathcal{K})$, $H_4(K)$ and $DH_4(\mathcal{K})$ are the only maps in $\mathcal{O}(G)$. To see this, note that the face- and vertex-valencies p and q for any such map must be orders of non-identity elements of G, so they must take the values $2, 3, 4$ or 7. Since $p^{-1} + q^{-1} < 1/2$ the only possible types (up to duality) are $\{3,7\}$, $\{7,3\}$, $\{4,7\}$, $\{7,4\}$ and $\{7,7\}$; as in the case of $\{7,7\}$ it can be verified by using the Frobenius triple-counting formula (or by inspection of [7]) that there is only one map of each type in $\mathcal{O}(G)$.

The graph $\mathcal{O}(G)$ is therefore as shown in Figure 8, where the actions of D and H_2 are represented by undirected broken and directed unbroken edges; that of H_3 is given by reversing the directed edges for H_2.
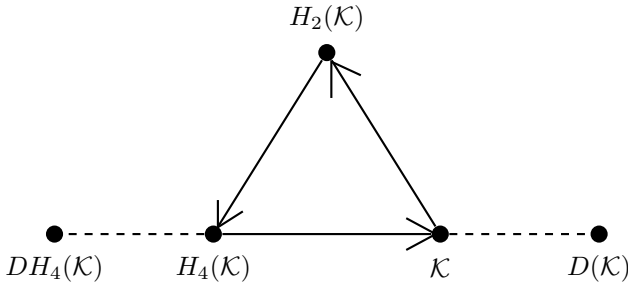
Figure 8: The graph $\mathcal{O}(\mathrm{PSL}_2(7))$

In Table 1 we list the maps $\mathcal{M} \in \mathcal{O}(\mathrm{PSL}_2(7))$. The first column shows the corresponding entry in [7], with letters 'a' and 'b' assigned as explained earlier. The second column gives the extended type $\{p, q\}_r$ of $\mathcal{M}$, with p, q and r denoting the face- and vertex-valencies and the Petrie length. The third column shows how $\mathcal{M}$ may be obtained from Klein's map $\mathcal{K}$ by applying duality and hole operations. The final column gives the effect of applying the hole operation H_2 to $\mathcal{M}$; this is left blank if $\mathcal{M}$ is unchanged (as when $q = 3$, for example) or q is even. Of course, duality D transposes pairs R$g.n$a and R$g.n$b, while leaving maps R$g.n$ invariant. Since the graph $\mathcal{O}(G)$ is connected, and $\mathcal{K}$ is regular with full automorphism group $\mathrm{PGL}_2(7)$, the same applies to every map $\mathcal{M}$ in $\mathcal{O}(G)$. These maps are all outer regular, so we obtain no non-orientable regular maps with automorphism group G.

Entry in [7]	Type	Relationship to $\mathcal{K}$	$H_2(\mathcal{M})$
R3.1a	$\{3, 7\}_8$	$\mathcal{K}$	R19.23
R3.1b	$\{7, 3\}_8$	$D(\mathcal{K})$	
R10.9a	$\{4, 7\}_8$	$H_4(\mathcal{K})$	R3.1a
R10.9b	$\{7, 4\}_8$	$DH_4(\mathcal{K})$	
R19.23	$\{7, 7\}_6$	$H_2(\mathcal{K})$	R10.9a

Table 1: The maps $\mathcal{M}$ in $\mathcal{O}(\mathrm{PSL}_2(7))$.

Example 7.2. Taking subscripts mod (7) and using regularity, we see that

$$H_3(\mathcal{K}) \cong H_{-3}(\mathcal{K}) \cong H_4(\mathcal{K}) \cong H_2^2(\mathcal{K}) \cong \mathrm{R}10.9\mathrm{a},$$

of type $\{4, 7\}$; Figure 1 confirms this, showing that the 3-holes of $\mathcal{K}$, giving the faces of $H_3(\mathcal{K})$, have length 4.

8 $G = \mathrm{PSL}_2(8) = \mathrm{SL}_2(8)$

There is a unique Hurwitz surface $\mathcal{S}$ of genus 7, described by Fricke [16] in 1899 and rediscovered by Macbeath [29] in 1965; its automorphism group G, isomorphic to the simple group $\mathrm{PSL}_2(8) = \mathrm{SL}_2(8)$ of order 504, is the orientation-preserving automorphism group of a regular map $\mathcal{F}$ of type $\{3, 7\}$ on $\mathcal{S}$, the *Hurwitz map* of genus 7, with full automorphism group $G \times C_2$.

The combinatorial structure of this map is shown in Figure 9. We note that because of the large size of the map, this representation is more suitable for studying its geometric properties, in comparison with the examples in Figures 1 and 7 where Poincaré's disc model is used. For other representations, including various topological embeddings by Carlo Sequin and by Jarke van Wijk, as well as for topological embeddings of regular maps of large genus given by Polthier and Razafindrazaka, see [3] and the references therein. For polyhedral realisations of the Hurwitz surface $\mathcal{S}$ of genus 7, see [2, 3, 4]. A detailed summary of all the known polyhedral realizations of regular maps of genus $g \geq 2$ can be found in [4].
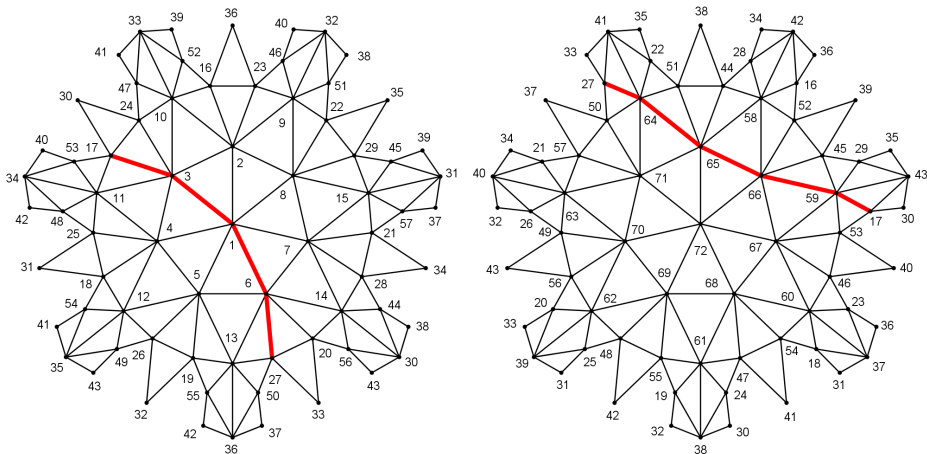

Figure 9: Combinatorial scheme of the Hurwitz map $\mathcal{F}$ of genus 7 (with vertex labels taken from [4]). A 3-hole is highlighted.

8.1 The group G

In order to work with this group G, let us represent the field $\mathbb{F}_8$ of order 8 as $\mathbb{F}_2[t]/(t^3 + t + 1)$, with its elements represented as polynomials in $\mathbb{F}_2[t]$ of degree at most 2. Then the multiplicative group of $\mathbb{F}_8$ consists of

$$t, \ t^2, \ t^3 = t + 1, \ t^4 = t^2 + t, \ t^5 = t^2 + t + 1, \ t^6 = t^2 + 1, \ t^7 = 1.$$

The non-identity elements of G form the following conjugacy classes:

- one class of elements of order 2, with trace 0;

- one class of elements of order 3, with trace 1;

- three classes of elements of order 7, with traces $t + 1, t^2 + 1, t^2 + t + 1$;

- three classes of elements of order 9, with traces $t, t^2, t^2 + t$.

Each class is inverse-closed. The outer automorphism group of G, isomorphic to C_3, is induced by the Galois group of $\mathbb{F}_8$ which is generated by the Frobenius automorphism $t \mapsto t^2$. This group permutes the three classes of elements of order 7 in a single cycle, and likewise for those of order 9.

Since $\operatorname{Out} G$ has odd order, and $Z(G)$ is trivial, all maps in $\mathcal{O}(G)$ are inner regular, with full automorphism group $G \times C_2$.

8.2   Construction of $\mathcal{O}(G)$

Since all involutions in G are conjugate, in forming $\mathcal{O}(G)$ we can take

$$y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

using the fact that $\mathbb{F}_8$ has characteristic 2 to eliminate minus signs. If

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad z = \begin{pmatrix} c & a \\ d & b \end{pmatrix},$$

with the trace $\tau = a + d$ and the cotrace $\tau' = b + c$ giving the type and hence the genus of $\mathcal{M}$. Also the trace $\sigma = a^2 + b^2 + c^2 + d^2$ of $[x, y]$ gives its Petrie length and hence its extended type. Then

$$x^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix}$$

has trace $a^2 + d^2 = \tau^2$, and the trace of

$$(x^2 y)^{-1} = \begin{pmatrix} c(a + d) & a^2 + bc \\ d^2 + bc & b(a + d) \end{pmatrix}$$

gives the cotrace $(a + d)(b + c) = \tau\tau'$, giving the type and genus of $H_2(\mathcal{M})$. Similarly, its Petrie length is determined by the trace

$$(a^2 + bc)^2 + (b(a + d))^2 + (c(a + d))^2 + (d^2 + bc)^2$$

of $[x^2, y]$. Applying H_{-1} or D is achieved by replacing x with

$$x^{-1} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \quad \text{or} \quad z = \begin{pmatrix} c & a \\ d & b \end{pmatrix}.$$

Alternatively, one can replace x with

$$z^{-1} = xy = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

instead of z when applying D.

The Frobenius triple-counting formula shows that $\mathcal{F}$ is the only map of type $\{3, 7\}$ in $\mathcal{O}(G)$. For x to correspond to such a map we require $ad + bc = 1$, $a + d = t + 1$, $t^2 + 1$ or $t^2 + t + 1$ and $b + c = 1$. Without loss of generality we can use the solution

$$a = t, \quad b = 1, \quad c = 0, \quad d = t^2 + 1,$$

so that $\mathcal{F}$ is represented by the matrix

$$x = \begin{pmatrix} t & 1 \\ 0 & t^2 + 1 \end{pmatrix}$$

with trace $\tau = t^2 + t + 1$ and cotrace $\tau' = 1$, confirming that the type is $\{3, 7\}$ and the genus is 7. Moreover, since $\sigma = t^2 + 1^2 + 0^2 + (t^2 + 1)^2 = t$, the Petrie length is 18, so the extended type is $\{3, 7\}_{18}$. This is therefore the map R7.1a in [7]. The dual map $D(\mathcal{F}) = $ R7.1b of extended type $\{7, 3\}_{18}$ is represented by the matrix

$$z = \begin{pmatrix} 0 & t \\ t^2 + 1 & 1 \end{pmatrix}.$$

The map $H_2(\mathcal{F})$ is represented by the matrix

$$\begin{pmatrix} t^2 & t^2 + t + 1 \\ 0 & t^2 + t + 1 \end{pmatrix}$$

with trace $\tau^2 = t + 1$, cotrace $\tau\tau' = t^2 + t + 1$ and sum of squares $t^2 + t$, so it has extended type $\{7, 7\}_{18}$ and genus 55. It must therefore be one of the dual pair R55.32 in [7]; we will denote it by R55.32a, and its dual map $DH_2(\mathcal{F})$, corresponding to the matrix

$$\begin{pmatrix} 0 & t^2 \\ t^2 + t + 1 & t^2 + t + 1 \end{pmatrix}$$

and also of type $\{7, 7\}_{18}$, by R55.32b. (See Proposition 8.1 for a proof that there are just two maps of type $\{7, 7\}$ in $\mathcal{O}(G)$; the fact that the matrix x^2 for $H_2(\mathcal{F})$ has distinct trace and cotrace shows that they are not self-dual.)

Iterating this process, we find that $H_2^2(\mathcal{F}) = H_4(\mathcal{F}) = H_3(\mathcal{F})$ is represented by the matrix

$$\begin{pmatrix} t^2 + t & t \\ 0 & t + 1 \end{pmatrix}$$

with trace $t^2 + 1$, cotrace t and sum of squares $t^2 + 1$, so it has extended type $\{9, 7\}_{14}$ and genus 63. It is therefore either R63.6b or R63.7b, since R63.6 and R63.7 are the only entries of this genus and extended type in [7]. One can verify that it is R63.7b by checking that the matrices x (given above) and z corresponding to this map satisfy the defining relations for the full automorphism group of R63.7 given in [7], but not those for R63.6. Specifically, the defining relations for R63.7, with generators R and S corresponding to our x and z, and a third orientation-reversing generator T inverting them both, are given as

$$R^{-7} = S^{-9} = (RS)^2 = (S^{-1}R)^3 = (RS^{-3}R^2)^2 = 1,$$

$$T^2 = (RT)^2 = (ST)^2 = 1.$$

One can check that the matrices

$$x = \begin{pmatrix} t^2 + t & t \\ 0 & t + 1 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} 0 & t^2 + t \\ t + 1 & t \end{pmatrix}$$

corresponding to $H_3(\mathcal{F})$ satisfy these relations when substituted for R and S.

For example,

$$z^{-1}x = \begin{pmatrix} t & t^2+t \\ t+1 & 0 \end{pmatrix} \begin{pmatrix} t^2+t & t \\ 0 & t+1 \end{pmatrix} = \begin{pmatrix} t^2+t+1 & t^2+1 \\ 1 & t^2+t \end{pmatrix}$$

has trace 1, so $(z^{-1}x)^3 = 1$. On the other hand, the relations for R65.6 include $(RS^{-2}R)^2 = 1$, and since

$$xz^{-2}x = \begin{pmatrix} t^2+1 & t \\ t^2+t & t^2+t+1 \end{pmatrix}$$

has trace $t \neq 0$ we have $(xz^{-2}x)^2 \neq 1$. Thus $H_3(\mathcal{F})$ is R63.7b.

Proposition 8.1. *There are two orientably regular maps of type $\{7,7\}$ with orientation-preserving automorphism group $G = \mathrm{SL}_2(8)$. They form a dual pair.*

Proof. We will use the Frobenius triple-counting formula (7.1), as in the proof of Proposition 7.1, to count triples of type $(7,2,7)$ in G. There are three choices for the conjugacy classes $\mathcal{X}$, $\mathcal{Z}$ of elements of order 7 in G, each containing $2^3 \cdot 3^2$ elements, and one class $\mathcal{Y}$ of $3^2 \cdot 7$ involutions. Using the character values in [9], we find that for each of the six choices of $\mathcal{X} \neq \mathcal{Z}$ the number of triples (x, y, z) of type $(7,2,7)$ in G, with $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, is

$$\frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \left(1 + \frac{1}{9}\sum_{j=0}^{2}(\zeta^{2^j} + \zeta^{-2^j})(\zeta^{2^{j+1}} + \zeta^{-2^{j+1}})\right)$$

$$= \frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \cdot \frac{7}{9} = 2^3 \cdot 3^2 \cdot 7,$$

where $\zeta = \exp(2\pi i/7)$, so the total number of such triples is

$$2^4 \cdot 3^3 \cdot 7 = 2|\mathrm{Aut}\, G|.$$

These triples all generate G, since no maximal subgroup of G contains such a triple with non-conjugate x and z, so there are two corresponding maps.

There are also three choices of classes $\mathcal{X} = \mathcal{Z}$, and the total number of corresponding triples in G is

$$3 \cdot \frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \left(1 + \frac{1}{9}\sum_{j=0}^{2}(\zeta^{2^j} + \zeta^{-2^j})^2\right)$$

$$= 3 \cdot \frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \cdot \frac{14}{9} = 2^4 \cdot 3^3 \cdot 7.$$

Now G has nine Sylow 2-subgroups T, each with normaliser $N_G(T) \cong \mathrm{AGL}_1(8) \cong V_8 \rtimes C_7$ generated by $48 \cdot 7$ triples of type $(7,2,7)$, all with conjugate x and z (see Section 10 for details). Since

$$9 \cdot 48 \cdot 7 = 2^4 \cdot 3^3 \cdot 7,$$

these account for all the triples of type $(7,2,7)$ in G with x and z conjugate; thus no such triples generate G, so they do not correspond to maps in $\mathcal{O}(G)$.

We have shown that there are two maps of type $\{7,7\}$ in $\mathcal{O}(G)$. One map corresponds to the pair $(t+1, t^2+1)$ of traces for x and z, together with its Galois conjugates (t^2+1, t^2+t+1) and $(t^2+t+1, t+1)$, while the other map corresponds to the pairs $(t+1, t^2+t+1)$, $(t^2+1, t+1)$ and (t^2+t+1, t^2+1). Since the first three pairs differ from the last three by transposition of x and z, these two maps form a dual pair.   □

8.3   Summary of results for $\mathcal{O}(G)$

Continuing in this way, using a similar criterion to identify the other maps of extended type $\{9,7\}_{14}$ or $\{7,9\}_{14}$, one eventually finds that there are fourteen maps $\mathcal{M} \in \mathcal{O}(G)$; they are described in Table 2, where the first four columns are analogues of those in Table 1 for $\mathrm{PSL}_2(7)$, with the Fricke–Macbeath map $\mathcal{F}$ replacing Klein's map $\mathcal{K}$. The final column gives the entries in [7] corresponding to the non-orientable quotients $\mathcal{M}/\mathrm{C}_2$ by the centre C_2 of the full automorphism group $G \times \mathrm{C}_2$ of $\mathcal{M}$ (see Section 8.4).

Entry in [7]	Type	Relationship to $\mathcal{F}$	$H_2(\mathcal{M})$	$\mathcal{M}/\mathrm{C}_2$
R7.1a	$\{3,7\}_{18}$	$\mathcal{F}$	R55.32a	N8.1a
R7.1b	$\{7,3\}_{18}$	$D(\mathcal{F})$		N8.1b
R15.1a	$\{3,9\}_{14}$	$H_2DH_3(\mathcal{F})$	R71.15a	N16.1a
R15.1b	$\{9,3\}_{14}$	$DH_2DH_3(\mathcal{F})$		N16.1b
R55.32a	$\{7,7\}_{18}$	$H_2(\mathcal{F})$	R63.7b	N56.5a
R55.32b	$\{7,7\}_{18}$	$DH_2(\mathcal{F})$	R63.5b	N56.5b
R63.5a	$\{7,9\}_6$	$DH_2DH_2(\mathcal{F})$	R71.15b	N64.3a
R63.5b	$\{9,7\}_6$	$H_2DH_2(\mathcal{F})$	R63.6b	N64.3b
R63.6a	$\{7,9\}_{14}$	$DH_3DH_2(\mathcal{F})$	R63.5a	N64.4a
R63.6b	$\{9,7\}_{14}$	$H_3DH_2(\mathcal{F})$	R55.32b	N64.4b
R63.7a	$\{7,9\}_{14}$	$DH_3(\mathcal{F})$	R15.1a	N64.5a
R63.7b	$\{9,7\}_{14}$	$H_3(\mathcal{F}) = H_2^2(\mathcal{F})$	R7.1a	N64.5b
R71.15a	$\{9,9\}_{18}$	$H_4DH_3(\mathcal{F})$	R63.7a	N72.9a
R71.15b	$\{9,9\}_{18}$	$DH_4DH_3(\mathcal{F})$	R63.6a	N72.9b

Table 2: The maps in $\mathcal{O}(\mathrm{SL}_2(8))$.
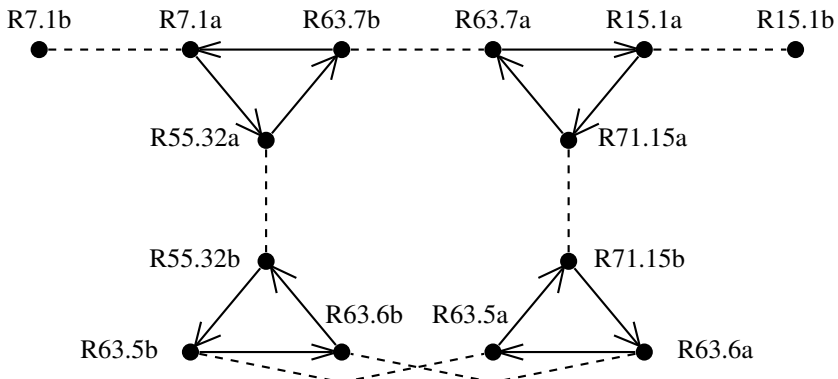

Figure 10: The graph $\mathcal{O}(\mathrm{SL}_2(8))$, with names of maps.

Figure 10 shows the graph $\mathcal{O}(G)$, with broken and unbroken edges representing the actions of D and H_2. The bilateral symmetry reveals an interesting duality between the parameters 7 and 9 appearing in the types of the maps in $\mathcal{O}(G)$ (see Figure 11). However, this does not extend consistently to the Petrie lengths: for example, R7.1b of extended type $\{7,3\}_{18}$ is paired with R15.1b of extended type $\{9,3\}_{14}$, whereas R63.7b, of extended type $\{9,7\}_{14}$, is paired with R63.7a, of extended type $\{7,9\}_{14}$.

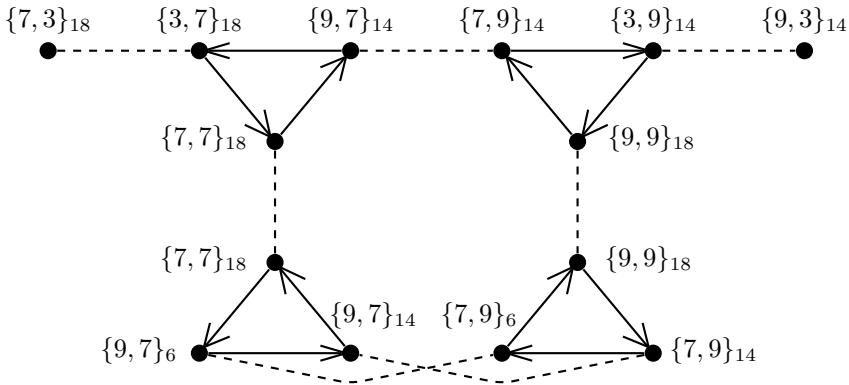Figure 11: The graph $\mathcal{O}(\mathrm{SL}_2(8))$, with types of maps.

Example 8.2. $H_3(\mathcal{F}) = H_2^2(\mathcal{F}) \cong$ R63.7b, of type $\{9,7\}$; this is confirmed by Figure 9, which shows that the 3-holes of $\mathcal{F}$ have length $4 + 5 = 9$.

Note that for this group G an operation H_j with $\gcd(j,q) > 1$ also transforms some maps in $\mathcal{O}(G)$ into others in $\mathcal{O}(G)$, namely H_3 for maps with $q = 9$, since one can check that if $x^9 = 1$ and $\langle x, y \rangle = G$ then $\langle x^3, y \rangle = G$ also. However, since $\mathcal{O}(G)$ is already connected, adding the resulting extra directed edges to the graph would not change its connectivity properties.

8.4 Non-orientable quotients

Each map $\mathcal{M} \in \mathcal{O}(G)$ is inner regular, with full automorphism group $G \times C_2$: this is true for $\mathcal{M} = \mathcal{F}$ since the corresponding generators x and y are both inverted by the matrix

$$\begin{pmatrix} t+1 & t \\ t & t+1 \end{pmatrix} \in G;$$

now the graph $\mathcal{O}(G)$ is connected, and $\mathrm{Aut}\,\mathcal{M}$ is preserved by D and H_j, so it is true for all $\mathcal{M} \in \mathcal{O}(G)$. Since G, being simple, has no subgroups of index 2, every map in $\mathcal{R}(G)$ is non-orientable, so as explained in Section 4.1, we obtain an isomorphism $\mathcal{O}(G) \to \mathcal{R}(G), \mathcal{M} \mapsto \mathcal{N} = \mathcal{M}/C_2$ of directed graphs. Using their extended types to identify the maps $\mathcal{N}$ in the list of non-orientable regular maps in [7] gives the final column of Table 2.

9 Some other Hurwitz groups

We have seen that if $G = \mathrm{PSL}_2(7)$ or $\mathrm{SL}_2(8)$ then $\mathcal{O}(G)$ is connected, consisting of maps which are respectively all outer or all inner regular, with automorphism groups isomorphic to $\mathrm{PGL}_2(7)$ or $\mathrm{SL}_2(8) \times C_2$.

In fact, a group G can have both inner and outer regular maps in $\mathcal{O}(G)$, necessarily in different components. For instance, if $G = \mathrm{PSL}_2(13)$ then $\mathcal{O}(G)$ contains three regular Hurwitz maps R14.1, R14.2 and R14.3, of genus 14 and extended types $\{3,7\}_{12}$, $\{3,7\}_{26}$ and $\{3,7\}_{14}$. Calculations similar to those given earlier show that R14.2 is inner regular, with automorphism group $G \times C_2$ and non-orientable regular quotient N15.1 of type $\{3,7\}_{13}$, while the other two maps are outer regular, with automorphism group $\mathrm{PGL}_2(13)$. This is interesting because, as shown by Streit [37], for each prime $p \equiv \pm 1$ mod (7) the three Hurwitz dessins in $\mathcal{O}(\mathrm{PSL}_2(p))$ are Galois conjugate, under the Galois group C_3 of the real cyclotomic field $\mathbb{Q}(\cos 2\pi i/7)$, so although the orientation-preserving automorphism group of a dessin is a Galois invariant, the full automorphism group is not.

More generally, Wendy Hall [21] has shown that a Hurwitz map $\mathcal{M}$ with $\mathrm{Aut}^+\mathcal{M} \cong \mathrm{PSL}_2(q)$ is inner regular if and only if $3 - \tau^2$ is a square in $\mathbb{F}_q$, where τ is the trace of the canonical generator x of order 7. For primes $q = p \equiv \pm 1$ mod (7), where there are three Hurwitz maps, her examples $p = 167, 13, 43$ and 181 show that none, one, two or all three of them can be inner regular.

In order to realise alternating groups A_n as Hurwitz groups (Theorem 5.2), Conder [5] constructed a sequence of planar maps $\mathcal{M}_n$ of type $\{7,3\}$ which have Hurwitz maps $\mathcal{H}_n$ with $\mathrm{Aut}^+\mathcal{H}_n \cong A_n$ as orientably regular covers. (Each $\mathcal{M}_n$ is presented as a coset diagram for a subgroup $A_{n-1} < A_n = \langle x, y \rangle$, but by shrinking the triangles for x to points it can be interpreted as a cubic map with monodromy group A_n.) The maps $\mathcal{H}_n$ are all outer regular: indeed, the bilateral symmetry in the construction of the maps $\mathcal{M}_n$ was designed to show that each map $\mathcal{H}_n$ has full automorphism group S_n.

Example 9.1. Among the alternating groups, the smallest Hurwitz group is A_{15}. There are three Hurwitz maps $\mathcal{H}$ with $\mathrm{Aut}^+\mathcal{H} \cong A_{15}$. Instead of drawing them (they have genus 7 783 776 001), we show their planar quotients $\mathcal{M} = \mathcal{H}/A_{14}$ in Figure 12. On the left is the map $\mathcal{M}_{15}$, corresponding to Conder's diagram B from his set of basic coset diagrams $A, \ldots, N$ in [5]; it is covered by the outer regular Hurwitz map $\mathcal{H}_{15}$. The other two maps are covered by a chiral pair of Hurwitz maps $\mathcal{H}$.
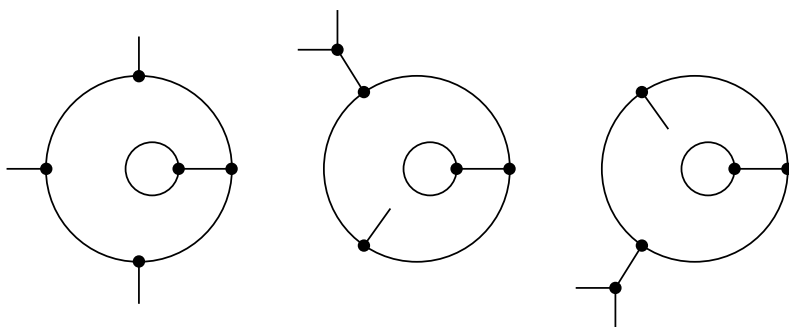

Figure 12: Three maps of type $\{7,3\}$ with monodromy group A_{15}.

By contrast, the simple Ree groups $\mathrm{Re}(q) = {}^2G_2(q)$ ($q = 3^e$ for odd $e \geq 3$) are also Hurwitz groups, but as shown in [25] the Hurwitz maps associated with them are all chiral (the generator x of order 3 is not inverted by any automorphism). In the next section we will consider a more straightforward example of this last phenomenon, but this time not involving Hurwitz maps.

10 $\mathrm{AGL}_1(q), q = 2^e$

In this section, instead of considering individual groups such as $\mathrm{PSL}_2(7)$ and $\mathrm{SL}_2(8)$, we will consider an infinite family of groups G for which $\mathcal{O}(G)$ exhibits uniform behaviour.

Let G be the 1-dimensional affine group $\mathrm{AGL}_1(q)$ for $q = 2^e$, consisting of the affine transformations

$$t \mapsto at + b, \quad (a, b \in \mathbb{F}_q, \, a \neq 0)$$

of the field $\mathbb{F}_q$. This is a semidirect product $T \rtimes S$ of an elementary abelian normal subgroup

$$T \cong (\mathbb{F}_q, +) \cong \mathrm{V}_q = (\mathrm{C}_2)^e$$

consisting of the translations $t \mapsto t + b$, by a complement

$$S \cong (\mathbb{F}_q^*, \times) \cong \mathrm{C}_{q-1},$$

consisting of the transformations $t \mapsto at, a \neq 0$. To avoid trivial cases, we will assume from now on that $e \geq 2$. (Note that these groups G, being solvable, are not perfect, so they are not Hurwitz groups.)

Theorem 10.1. (a) *If $e > 2$ there are $\phi(q-1)/e$ maps in $\mathcal{M} \in \mathcal{O}(G)$, all of extended type $\{q-1, q-1\}_4$ and genus $(q-1)(q-4)/4$; they are all chiral, and satisfy $D(\mathcal{M}) \cong H_{-1}(\mathcal{M})$ and $H_2(\mathcal{M}) \cong \mathcal{M}$.*

(b) *If $e = 2$ there is a single map $\mathcal{M} \in \mathcal{O}(G)$, the tetrahedral map $\{3, 3\}$ of genus 0, which is outer regular with $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{A\Gamma L}_1(4) \cong \mathrm{S}_4$.*

(c) *For each $e \geq 2$ the graph $\mathcal{O}(G)$ is connected.*

Proof. The $q - 1$ involutions in T are all conjugate in G, and the remaining non-identity elements, all of order dividing $q-1$, form $q-2$ conjugacy classes of size q (the non-identity cosets of T in G). These fuse into $(q-2)/e$ orbits of size qe under the action of

$$\mathrm{Aut}\,G = \mathrm{A\Gamma L}_1(q) \cong G \rtimes \mathrm{Gal}\,\mathbb{F}_q \cong G \rtimes \mathrm{C}_e.$$

Any map $\mathcal{M} \in \mathcal{O}(G)$ corresponds to a generating triple (x, y, z) for G, where y has order 2 and xT generates G/T. There are $\phi(q-1)q$ choices for x and $q-1$ choices for y, giving $\phi(q-1)q(q-1) = \phi(q-1)|\mathrm{Aut}\,G|/e$ triples, so there are $\phi(q-1)/e$ maps $\mathcal{M}$ in $\mathcal{O}(G)$. Since x and z have order $q - 1$ these maps have type $\{q-1, q-1\}$ and hence have genus $(q-1)(q-4)/4$. Since $[x, y]$ is a non-identity element of T it has order 2, so the Petrie length is 4. In particular, if $e = 2$ there is a single map $\mathcal{M} \in \mathcal{O}(G)$; this is the tetrahedral map $\{3, 3\}$, which is outer regular with $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{A\Gamma L}_1(4) \cong \mathrm{S}_4$. If $e > 2$ then no element of $\mathrm{Gal}\,\mathbb{F}_q$ inverts $\mathbb{F}_q^*$, so the maps $\mathcal{M} \in \mathcal{O}(G)$ are all chiral; since z is conjugate to x^{-1} they satisfy $D(\mathcal{M}) \cong H_{-1}(\mathcal{M})$. Since $x \mapsto x^2$ is an automorphism of $\mathbb{F}_q$ they are all invariant under H_2. Since any two generators of S are powers of each other, any two maps $\mathcal{M}, \mathcal{M}' \in \mathcal{O}(G)$ satisfy $\mathcal{M}' = H_j(\mathcal{M})$ for some j coprime to $q - 1$, so $\mathcal{O}(G)$ is connected. (It is, in fact, a quotient of a Cayley graph for the group of units U_{q-1}.) □

The involutions in G all lie in the proper subgroup T, so $\mathcal{R}(G)$ is empty. These maps $\mathcal{M}$ are instances of the orientably regular embeddings of complete graphs K_q constructed by Biggs in [1], where he showed that K_q has such an embedding if and only if q is a prime power; see [24] for the classification of such maps.

10.1 Examples

The maps $\mathcal{M}$ arising for $e = 3$ are the chiral pair of Edmonds maps of type $\{7,7\}_4$, corresponding to the entry C7.2 in [7]; these both lie on the Fricke–Macbeath surface $\mathcal{S}$ of genus 7 realising the Hurwitz group $\mathrm{SL}_2(8)$. The affine group $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{AGL}_1(8)$ is a subgroup of index 9 in $\mathrm{Aut}\,\mathcal{S} = \mathrm{Aut}^+\mathcal{F} \cong \mathrm{SL}_2(8)$: it is the stabiliser of ∞ in the natural representation of $\mathrm{SL}_2(8)$ on the projective line $\mathbb{P}^1(\mathbb{F}_8)$, and also the normaliser of a Sylow 2-subgroup. This inclusion lifts to an index 9 inclusion $\Delta(7,2,7) < \Delta = \Delta(7,2,3)$ of triangle groups, namely item (B) in Singerman's list of triangle group inclusions [34].

For $e = 4$ we have the chiral pair C45.2 of type $\{15,15\}_2$, and for $e = 5$ we have the three chiral pairs C217.45–47 of type $\{31,31\}_4$. The index 9 inclusions of automorphism groups and triangle groups mentioned above for $e = 3$ do not generalise to higher powers of 2.

When $e = 5$, if we take a to be a generator of $\mathbb{F}_{32}^* = \langle a \mid a^{31} = 1 \rangle \cong \mathrm{C}_{31}$, then the chiral pairs of maps $\mathcal{M}_{\pm i}$ $(i = 1, 3, 5)$ correspond to the following mutually inverse pairs of orbits $\Omega_{\pm i}$ of $\mathrm{Gal}\,\mathbb{F}_{32} = \langle t \mapsto t^2 \rangle \cong \mathrm{C}_5$ on $\mathbb{F}_{32}^*$:

$$\Omega_1 = \{a^i \mid i = 1, 2, 4, 8, 16\}, \quad \Omega_{-1} = \{a^i \mid i = 15, 30 \equiv -1, 29, 27, 23\};$$
$$\Omega_3 = \{a^i \mid i = 3, 6, 12, 24, 17\}, \quad \Omega_{-3} = \{a^i \mid i = 7, 14, 28 \equiv -3, 25, 19\};$$
$$\Omega_5 = \{a^i \mid i = 5, 10, 20, 9, 18\}, \quad \Omega_{-5} = \{a^i \mid i = 11, 22, 13, 26 \equiv -5, 21\}.$$

Then $D(\mathcal{M}_i) = H_{-1}(\mathcal{M}_i) = \mathcal{M}_{-i}$ and $H_2(\mathcal{M}_i) = \mathcal{M}_i$ for all i, while H_3 induces a 6-cycle $(1, 3, 5, -1, -3, -5)$ on the subscripts i. The graph $\mathcal{O}(\mathrm{AGL}_1(32))$ is shown in Figure 13, with directed edges representing the action of H_3, and undirected dashed and dotted edges the actions of D and H_{-1}. The identification of these maps with the entries C217.45–47 in [7] depends on the choice of a, or more precisely that of its minimal polynomial in the action on the additive group of $\mathbb{F}_{32}$, one of the six irreducible factors of the cyclotomic polynomial $\Phi_{31}(t) = t^{30} + t^{29} + \cdots + t + 1$ in $\mathbb{F}_2[t]$ (see [24]).
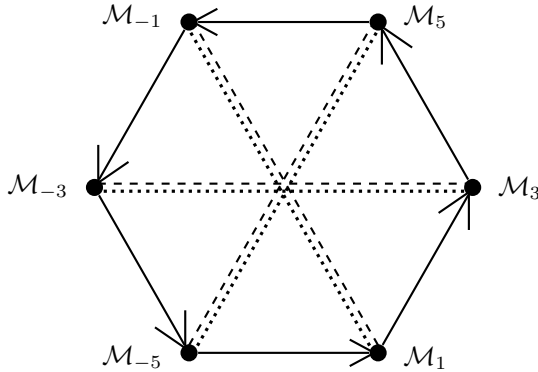

Figure 13: The graph $\mathcal{O}(\mathrm{AGL}_1(32))$

11 Groups G for which $\mathcal{O}(G)$ has many components

In contrast with the groups $G = \mathrm{AGL}_1(2^e)$, for which $\mathcal{O}(G)$ is connected for all e, we will now consider some families of groups G for which $\mathcal{O}(G)$ has an unbounded number of connected components.

The map operations D and H_j on orientably regular maps preserve the involution $y \in G$ in generating triples, up to the action of $\operatorname{Aut} G$, so if $\operatorname{Aut} G$ has $i = i(G)$ orbits on useful involutions $y \in G$, where 'useful' means 'member of a generating pair for G', then the number $c = c(G)$ of connected components of $\mathcal{O}(G)$ satisfies $c \geq i$. The example $G = \mathrm{S}_5$, with $i = 2$ but $c = 3$ (see Section 4.2), shows that c can exceed i; the following example shows that both can be arbitrarily large.

Lemma 11.1. *If $G = \mathrm{S}_n$ with $n \geq 5$ then each involution $y \in G$ is useful.*

Proof. It is sufficient to show that at least one involution in each conjugacy class of G is useful. The result is straightforward if y is a transposition (i, j), since one can then take x to be an n-cycle with $ix = j$, so we may assume that y consists of t transpositions and $n - 2t$ fixed points for some $t \geq 2$.

First we will deal with the case $n \geq 8$. Let $m = \lfloor n/2 \rfloor$, so that $m > 3$. By the Bertrand–Chebyshev Theorem (see [36], for example) there is a prime p such that $m < p < 2m - 2$, so $n/2 < p \leq n - 3$. Let us take x to have cycles of length p and $n - p$ if n is odd, and $p, n - p - 1$ and 1 if n is even, so that x is odd in either case. Now x has at most three cycles, so given any $t \geq 2$ we can choose the involution y, which moves $2t \geq 4$ points, so that $H := \langle x, y \rangle$ is transitive. It follows that H must be primitive, for otherwise H is contained in a wreath product $\mathrm{S}_a \wr \mathrm{S}_b$ for some proper factorisation $n = ab$ of n, which is impossible since x has order divisible by p whereas $\mathrm{S}_a \wr \mathrm{S}_b$ has order $(a!)^b b!$ coprime to p. Thus H is a primitive group containing a p-cycle (a suitable power of x) with $n - p \geq 3$ fixed points, so by a classic theorem of Jordan (see [39, Theorem 13.9]) $H \geq \mathrm{A}_n$. Since H contains the odd permutation x we must have $H = \mathrm{S}_n$, as required.

The case $n = 5$ was considered in Section 4.2, while the cases $n = 6$ and $n = 7$ can easily be dealt with by hand or by using GAP. $\qquad\square$

The restriction $n \geq 5$ is required in this lemma: see Section 2.1.

Corollary 11.2. *For each $m \in \mathbb{N}$ there exists $N_m \in \mathbb{N}$ such that if $n \geq N_m$ then $\mathcal{O}(\mathrm{S}_n)$ has at least m connected components.*

Proof. By Lemma 11.1 the involutions in $G = \mathrm{S}_n$ are all useful for $n \geq 5$. They have $\lfloor n/2 \rfloor$ possible cycle-structures, so if $n \neq 6$ then $\operatorname{Aut} G\ (= G)$ has $\lfloor n/2 \rfloor$ orbits on them. Thus $\mathcal{O}(G)$ has $c(G) \geq i(G) = \lfloor n/2 \rfloor$ connected components for all $n \geq 7$. We may therefore take $N_m = \max\{2m, 7\}$. $\qquad\square$

By adapting Conder's proof in [5] of Theorem 5.2 one can also prove:

Theorem 11.3. *For each $m \in \mathbb{N}$ there exists $N'_m \in \mathbb{N}$ such that if $n \geq N'_m$ then $\mathcal{O}(\mathrm{A}_n)$ has at least m connected components containing Hurwitz maps.*

Outline of proof. In [5] Conder proved that A_n is a Hurwitz group for each $n \geq 168$ (and also for some smaller n) by constructing coset diagrams for subgroups of index n in Δ, and showing that the induced permutation group on the cosets is A_n. These diagrams are constructed by joining copies of 14 basic cosets diagrams $A, B, \ldots, N$ of degrees between 14 and 108. An *i-handle* in a coset diagram for Δ is a pair (a, b) of fixed points of y with $a = bz^i$, where $i = 1, 2$ or 3. (In this paper we have transposed Conder's notation in [5] for the generators x and y of Δ.) Two coset diagrams of degrees d and d' with i-handles (a, b) and (a', b') can be joined by an *i-join* to create a diagram of degree $d + d'$ by replacing a and a' with a transposition (a, a') for y, and similarly for b and b'.

Conder's construction (simplified a little here) is as follows. His diagram G of degree 42 has three 1-handles. First use (1)-joins to form a chain of k copies of G for some $k \geq 1$. For each of the 42 congruence classes $[c] = [0], \ldots [41] \bmod (42)$, join two specified combinations of basic diagrams, of total degree $d_c \equiv c \bmod (42)$, to the ends of the chain, to give a coset diagram of degree $n = 42k + d_c \equiv c \bmod (42)$. By taking $k = 1, 2, \ldots$ this realises A_n for each sufficiently large $n \in [c]$ as a quotient of Δ and hence as a Hurwitz group.

Each of the k copies of G in the chain has an unused (1)-handle, giving two fixed points of y. By using these to make i further joins, for any $i \leq \lfloor k/2 \rfloor$, one can reduce the total number of fixed points of y by $4i$; this does not change the fact that the resulting permutation group is A_n since Conder's proof of this still applies. We thus obtain $\lfloor k/2 \rfloor + 1$ Hurwitz maps in $\mathcal{O}(A_n)$, with mutually distinct cycle-structures for y and hence in distinct components of this graph. Taking k sufficiently large proves the result. □

12 The order of $\mathcal{O}(G)$

One obvious problem which we have not yet addressed is to determine $|\mathcal{O}(G)|$ for any finite group G; however, a method for solving this is already known. The maps in $\mathcal{O}(G)$ correspond bijectively to the orbits of $\operatorname{Aut} G$ on generating pairs x, y for G satisfying $y^2 = 1$. Since $\operatorname{Aut} G$ acts semiregularly (i.e. fixed-point-freely) on all generating sets for G, we have

$$|\mathcal{O}(G)| = \phi(G)/|\operatorname{Aut} G|,$$

where $\phi(G)$ is the number of such generating pairs x, y for G. If $\sigma(G)$ denotes the total number of pairs $x, y \in G$ with $y^2 = 1$, then by applying Philip Hall's technique of Möbius inversion in groups [20] to the obvious equation $\sigma(G) = \sum_{H \leq G} \phi(H)$ we obtain

$$\phi(G) = \sum_{H \leq G} \mu(H)\sigma(H), \tag{12.1}$$

where μ is the Möbius function for the subgroup lattice of G, defined by

$$\mu(G) = 1 \quad \text{and} \quad \sum_{K \geq H} \mu(K) = 0 \quad \text{for all} \quad H < G.$$

This applies to generating sets satisfying any given set of relations, but in our case, with $y^2 = 1$ the only relation, we have

$$\sigma(H) = |H|(|H|_2 + 1),$$

where $|H|_2$ is the number of involutions in H.

In [20] Hall determined the function μ for many groups, including $\mathrm{PSL}_2(p)$ for primes p. Downs [14] extended this to $\mathrm{PSL}_2(q)$ for prime powers $q = p^e$. Their results for $p > 2$ are complicated but $|\mathcal{O}(\mathrm{PSL}_2(13))| = 33$ is a simple example. If $q = 2^e$ then $|\operatorname{Aut} G| = e|G|$ and as in [15] we have

$$|\mathcal{O}(G)| = \frac{1}{e} \sum_{f | e} \mu\left(\frac{e}{f}\right)(2^f - 1)(2^f - 2),$$

where μ now denotes the Möbius function of elementary number theory (see Sections 2.1 and 8 for the cases $e = 2$ and $e = 3$). For any q the sum in (12.1) is dominated by the summand with $H = G$, so that for $G = \mathrm{PSL}_2(q)$ we have

$$|\mathcal{O}(G)| \sim \frac{|G|(|G|_2 + 1)}{|\mathrm{Aut}\, G|} \sim \frac{q^2}{e} \quad \text{or} \quad \frac{q^2}{4e} \quad \text{as} \quad q \to \infty,$$

where q is respectively even or odd.

We close with a question: if $G = \mathrm{PSL}_2(q)$, how does the number $c(G)$ of connected components of $\mathcal{O}(G)$ behave as $q \to \infty$?

ORCID iDs

Gábor Gévay https://orcid.org/0000-0002-5469-5165
Gareth A. Jones https://orcid.org/0000-0002-7082-7025

References

[1] N. Biggs, Classification of complete maps on orientable surfaces, *Rend. Mat., VI. Ser.* **4** (1972), 645–655.

[2] J. Bokowski, A Kepler–Poinsot-type polyhedron of the genus 7 Hurwitz surface, *Art Discrete Appl. Math.* **4** (2021), P3.10, 5 pp., doi:10.26493/2590-9770.1394.9e7.

[3] J. Bokowski and M. Cuntz, Hurwitz's regular map $(3, 7)$ of genus 7: A polyhedral realization, *Art Discrete Appl. Math.* **1** (2018), P1.02, 17 pp., doi:10.26493/2590-9770.1186.258.

[4] J. Bokowski and G. Gévay, On polyhedral realizations of Hurwitz's regular map $\{3, 7\}_{18}$ of genus 7 with geometric symmetries, *Art Discrete Appl. Math.* **4** (2021), P3.09, 26 pp., doi: 10.26493/2590-9770.1362.6f4.

[5] M. D. E. Conder, Generators for alternating and symmetric groups, *J. Lond. Math. Soc., II. Ser.* **22** (1980), 75–86, doi:10.1112/jlms/s2-22.1.75.

[6] M. D. E. Conder, Hurwitz groups: A brief survey, *Bull. Am. Math. Soc., New Ser.* **23** (1990), 359–370, doi:10.1090/s0273-0979-1990-15933-6.

[7] M. D. E. Conder, Regular maps and hypermaps of Euler characteristic -1 to -200, *J. Comb. Theory, Ser. B* **99** (2009), 455–459, doi:10.1016/j.jctb.2008.09.003.

[8] M. D. E. Conder, An update on Hurwitz groups., *Groups Complex. Cryptol.* **2** (2010), 35–49, doi:10.1515/gcc.2010.002.

[9] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[10] H. S. M. Coxeter, Regular skew polyhedra in three and four dimensions and their topological analogues, *Proc. Lond. Math. Soc. (2)* **43** (1937), 33–62, doi:10.1112/plms/s2-43.1.33.

[11] H. S. M. Coxeter, The abstract groups $G^{m,n,p}$, *Trans. Amer. Math. Soc.* **45** (1939), 73–150, https://www.ams.org/journals/tran/1939-045-01/.

[12] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, volume 14 of *Ergeb. Math. Grenzgeb.*, Springer-Verlag, Berlin, 1980, doi:10.1007/978-3-662-21943-0.

[13] L. E. Dickson, *Linear Groups*, Dover, New York, 1958.

[14] M. Downs, The Möbius function of $\mathrm{PSL}_2(q)$, with application to the maximal normal subgroups of the modular group, *J. Lond. Math. Soc., II. Ser.* **43** (1991), 61–75, doi:10.1112/jlms/s2-43.1.61.

[15] M. L. N. Downs and G. A. Jones, Enumerating regular objects with a given automorphism group, *Discrete Math.* **64** (1987), 299–302, doi:10.1016/0012-365x(87)90199-3.

[16] R. Fricke, Ueber eine einfache Gruppe von 504 Operationen, *Math. Ann.* **52** (1899), 321–339, doi:10.1007/bf01476163.

[17] G. Frobenius, Über Gruppencharaktere, *Berl. Ber.* **1896** (1896), 985–1021.

[18] G. Gévay, A class of cellulated spheres with non-polytopal symmetries, *Can. Math. Bull.* **52** (2009), 366–379, doi:10.4153/cmb-2009-040-7.

[19] P. Gordan, Über die Auflösung der Gleichungen vom fünften Grade, *Math. Ann.* **13** (1878), 375–404.

[20] P. Hall, The Eulerian functions of a group, *Q. J. Math., Oxf. Ser.* **7** (1936), 134–151, doi:10.1093/qmath/os-7.1.134.

[21] W. Hall, *Automorphisms and coverings of Klein surfaces*, Ph.D. thesis, University of Southampton, 1977, https://eprints.soton.ac.uk/259986/.

[22] B. Huppert, *Endliche Gruppen I. Nachdr. d. 1. Aufl*, volume 134 of *Grundlehren Math. Wiss.*, Springer, Cham, 1979.

[23] A. Hurwitz, Über algebraische Gebilde mit eindeutigen Transformationen in sich, *Math. Ann.* **41** (1893), 403–442, doi:10.1007/bf01443420.

[24] L. D. James and G. A. Jones, Regular orientable imbeddings of complete graphs, *J. Comb. Theory, Ser. B* **39** (1985), 353–367, doi:10.1016/0095-8956(85)90060-7.

[25] G. A. Jones, Ree groups and Riemann surfaces, *J. Algebra* **165** (1994), 41–62, doi:10.1006/jabr.1994.1097.

[26] G. A. Jones and J. Wolfart, *Dessins d'Enfants on Riemann Surfaces*, Springer Monogr. Math., Cham: Springer, 2016, doi:10.1007/978-3-319-24711-3.

[27] F. Klein, Über die Transformationen siebenter Ordnung der elliptischen Funktionen, *Math. Ann.* **14** (1878), 428–471, doi:10.1007/bf01677143.

[28] S. Levy (ed.), *The Eightfold Way. The Beauty of Klein's Quartic Curve*, volume 35 of *Math. Sci. Res. Inst. Publ.*, Cambridge: Cambridge University Press, 2001, www.msri.org/communications/books/Book35/.

[29] A. M. Macbeath, On a curve of genus 7, *Proc. London Math. Soc.* **s3-15** (1965), 527–542, doi:10.1112/plms/s3-15.1.527.

[30] A. M. Macbeath, Generators of the linear fractional groups, in: *Number Theory, Proc. Sympos. Pure Math. 14-32.*, volume 12, 1969 pp. 14–32.

[31] E. Schulte and J. M. Wills, A polyhedral realization of Felix Klein's map $\{3, 7\}_8$ on a Riemann surface of genus 3, *J. Lond. Math. Soc., II. Ser.* **32** (1985), 539–547, doi:10.1112/jlms/s2-32.3.539.

[32] E. Schulte and J. M. Wills, On Coxeter's regular skew polyhedra, *Discrete Math.* **60** (1986), 253–262, doi:10.1016/0012-365x(86)90017-8.

[33] E. Schulte and J. M. Wills, Kepler-Poinsot-type realization of regular maps of Klein, Fricke, Gordan and Sherk, *Can. Math. Bull.* **30** (1987), 155–164, doi:10.4153/cmb-1987-023-9.

[34] D. Singerman, Finitely maximal Fuchsian groups, *J. Lond. Math. Soc., II. Ser.* **6** (1972), 29–38, doi:10.1112/jlms/s2-6.1.29.

[35] D. Singerman, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* **210** (1974), 17–32, doi:10.1007/bf01344543.

[36] J. Sondow and E. W. Weisstein, *Bertrand's Postulate*, MathWorld–A Wolfram Web Resource, https://mathworld.wolfram.com/BertrandsPostulate.html.

[37] M. Streit, Field of definition and Galois orbits for the Macbeath–Hurwitz curves, *Arch. Math.* **74** (2000), 342–349, doi:10.1007/s000130050453.

[38] M. Weber, Kepler's small stellated dodecahedron as a Riemann surface, *Pac. J. Math.* **220** (2005), 167–182, doi:10.2140/pjm.2005.220.167.

[39] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964, https://www.sciencedirect.com/book/9780127496504/finite-permutation-groups.

[40] S. E. Wilson, Operators over regular maps, *Pac. J. Math.* **81** (1979), 559–568, doi:10.2140/pjm.1979.81.559.