A NOTE ON TORSION LENGTH AND TORSION SUBGROUPS

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ABSTRACT. Answering Questions 19.23 and 19.24 from the Kourovka notebook we construct poly-cyclic groups with arbitrary torsion lengths and give examples of finitely presented groups whose quotients by their torsion subgroups are not finitely presented.

1. INTRODUCTION

Given a group G, the *torsion subgroup*, $\mathrm{Tor}(G)$, is the normal subgroup of G generated by all elements of finite order. The torsion length of G is defined as follows. Let $G_0 = G$ and $G_{n+1} = G_n/\mathrm{Tor}(G_n)$, for $n \geq 0$. The *torsion length* of G, $\mathrm{TorLen}(G)$, is the smallest $n \geq 0$ such that G_n is torsion-free. If no such n exists then we define $\mathrm{TorLen}(G) = \omega$. Torsion length was defined and studied by Chiodo and Vyas in [3], and in [4] the same authors constructed finitely presented groups of arbitrary torsion length.

Chiodo and Vyas asked in [6, Questions 19.23 and 19.24] whether there exist finitely presented soluble groups of torsion length greater than 2 and whether there exists a finitely presented group G such that $G/\mathrm{Tor}(G)$ is not finitely presented. Versions of these questions also appeared in [3, Subsection 3.3] and [4, Question 1]. In this note we answer both questions affirmatively.

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2. POLYCYCLIC GROUPS OF ARBITRARILY LARGE TORSION LENGTH

2.1. **Neutral homomorphisms and automorphisms of semidirect products.** In this subsection we develop a few tools that will be used in the main construction.

Definition 2.1. Let K be a group and let $\beta : K \to K$ be an endomorphism.
- We define the map $\Delta_\beta : K \to K$ by $\Delta_\beta(k) = k^{-1}\beta(k)$, for all $k \in K$.
- We denote by $\mathcal{N}(\beta) \lhd K$ the normal closure of the subset $\Delta_\beta(K) = \{k^{-1}\beta(k) \mid k \in K\}$.
- If C is another group and $\phi : K \to C$ is a homomorphism, we say that ϕ is *β-neutral* if $\phi \circ \beta = \phi$, i.e., if $\phi(\beta(k)) = \phi(k)$ for all $k \in K$.

The map Δ_β can be thought of as a (non-abelian) 1-cocycle from K to K, where K acts on itself by conjugation (cf. [8, Section I.5.1]).

Lemma 2.2. *Suppose that K is a group generated by a set X and $\beta \in \mathrm{End}(K)$.*
- *(i) If $\phi : K \to C$ is a group homomorphism then ϕ is β-neutral if and only if $\mathcal{N}(\beta) \subseteq \ker\phi$ if and only if $\Delta_\beta(X) \subseteq \ker\phi$.*
- *(ii) $\mathcal{N}(\beta)$ is the normal closure in K of the subset $\Delta_\beta(X) = \{x^{-1}\beta(x) \mid x \in X\}$.*

Proof. Claim (i) follows from the observation that for any $k \in K$, $\phi(\beta(k)) = \phi(k)$ is equivalent to $\phi(k^{-1}\beta(k)) = 1$ and the fact that two homomorphisms from G to C are equal if and only if they agree on X.

For claim (ii), let $C = K/N$, where N is the normal closure of $\Delta_\beta(X)$ in K, and let $\phi : K \to C$ be the natural epimorphism. Then, according to claim (i), ϕ is β-neutral and $\mathcal{N}(\beta) \subseteq \ker \phi = N$. Clearly $N \subseteq \mathcal{N}(\beta)$, by definition, hence $\mathcal{N}(\beta) = N$. $\qquad \square$

Definition 2.3. Let K, H be two groups, let $\beta \in \mathrm{Aut}(K)$ be an automorphism of K and let $\phi : K \to \mathrm{Aut}(H)$ be a homomorphism. Denote by A the semidirect product $H \rtimes_\phi K$. We define the *extension of β to A* as the map $\beta_A : A \to A$, given by

$$\beta_A((h,k)) = (h, \beta(k)), \text{ for all } h \in H \text{ and all } k \in K.$$

In this paper we regard the semidirect product $A = H \rtimes_\phi K$ as the set $H \times K$, with the multiplication $(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1)(h_2), k_1 k_2)$, for all $(h_1, k_1), (h_2, k_2) \in A$.

Lemma 2.4. *Using the notation of Definition 2.3, suppose that ϕ is β-neutral. Then the following properties hold:*

(a) the map β_A is an automorphism of A which has the same order as β;
(b) $\mathcal{N}(\beta_A) = (1, \mathcal{N}(\beta))$ in A;
(c) the quotient $A/\mathcal{N}(\beta_A)$ is naturally isomorphic to the semidirect product $H \rtimes_{\bar\phi} (K/\mathcal{N}(\beta))$, where $\bar\phi : K/\mathcal{N}(\beta) \to \mathrm{Aut}(H)$ is the homomorphism induced by ϕ.

Proof. (a) Evidently β_A is a bijection with the same order as β. Consider any $(h_1, k_1), (h_2, k_2) \in A$, where $h_1, h_2 \in H$ and $k_1, k_2 \in K$, and observe that

$$\beta_A((h_1, k_1)(h_2, k_2)) = \beta_A((h_1\phi(k_1)(h_2), k_1 k_2)) = (h_1\phi(k_1)(h_2), \beta(k_1)\beta(k_2)).$$

On the other hand,

$$\beta_A((h_1, k_1))\beta_A((h_2, k_2)) = (h_1, \beta(k_1))(h_2, \beta(k_2)) = (h_1\phi(\beta(k_1))(h_2), \beta(k_1)\beta(k_2)).$$

Since ϕ is β-neutral, we see that $\beta_A((h_1, k_1)(h_2, k_2)) = \beta_A((h_1, k_1))\beta_A((h_2, k_2))$, hence β_A is an automorphism of A.

(b) By Lemma 2.2(i), $\mathcal{N}(\beta) \subseteq \ker \phi$, which means that the subgroup $(1, \mathcal{N}(\beta))$ centralizes the subgroup $(H, 1)$ in A. Since $\mathcal{N}(\beta) \lhd K$ and A is generated by $(H, 1) \cup (1, K)$, it follows that $(1, \mathcal{N}(\beta)) \lhd A$. Now, according to Lemma 2.2(ii), $\mathcal{N}(\beta_A)$ is the normal closure in A of the subset

$$\Delta_{\beta_A}((H, 1)) \cup \Delta_{\beta_A}((1, K)) = \{(1, 1)\} \cup (1, \Delta_\beta(K)) = (1, \Delta_\beta(K)).$$

Therefore $\mathcal{N}(\beta_A) \subseteq (1, \mathcal{N}(\beta)) \subseteq \mathcal{N}(\beta_A)$, i.e., $\mathcal{N}(\beta_A) = (1, \mathcal{N}(\beta))$.

Claim (c) is a consequence of claim (b) and the fact that $\mathcal{N}(\beta) \subseteq \ker \phi$. $\qquad \square$

2.2. **Exact automorphisms.** Throughout this subsection we assume that F is a group and $\beta \in \mathrm{Aut}(F)$ is an automorphism of finite order $m \in \mathbb{N}$. We define the map $\Sigma_\beta : F \to F$ by

$$\Sigma_\beta(f) = f\beta(f)\ldots\beta^{m-1}(f), \text{ for all } f \in F.$$

Remark 2.5. It is easy to check that the image $\mathrm{Im}\,\Delta_\beta$ is always contained in the kernel $\ker \Sigma_\beta$.

The above remark naturally motivates the following terminology, which was originally proposed to us by the referee.

Definition 2.6. We will say that the automorphism $\beta \in \mathrm{Aut}(F)$ is *exact* if $\mathrm{Im}\,\Delta_\beta = \ker \Sigma_\beta$.

For example, the order 2 automorphism of $\mathbb{Z}^2$, which interchanges the standard generators, is exact. More generally, if F is a direct product of n copies of a group K and β is an automorphism of F cyclically permuting the factors then β is exact. On the other hand, the automorphism of $\mathbb{Z}$ sending 1 to -1 is not exact.

We will use exact automorphisms to control torsion in extensions by β.

Lemma 2.7. *Let G be a group with a torsion-free normal subgroup $F \lhd G$ and an element $x \in G$ of finite order $n \in \mathbb{N}$, such that $G = \langle F, x \rangle$. Let $\beta \in \mathrm{Aut}(F)$ denote the automorphism induced by conjugation by x. The following are equivalent:*

(i) β is an exact automorphism of F;

(ii) if an element of the form fx, for some $f \in F$, has finite order in G then it is conjugate to x in G;

(iii) G contains exactly one conjugacy class of cyclic subgroups of order n.

Proof. By the assumptions, β must have order $m \in \mathbb{N}$ in $\mathrm{Aut}(F)$, where $n = ml$, for some $l \in \mathbb{N}$. Therefore, for any $f \in F$ we have

$$(1) \qquad (fx)^n = f(xfx^{-1})(x^2 fx^{-2})\ldots(x^{n-1}fx^{-n+1})x^n = f\beta(f)\beta^2(f)\ldots\beta^{n-1}(f) = (\Sigma_\beta(f))^l.$$

Let us show that (i) implies (ii). Suppose that β is an exact automorphism of F and $fx \in G$ is an element of finite order, for some $f \in F$. Equation (1) yields that $\Sigma_\beta(f)$ has finite order in F, and since F is torsion-free we can conclude that $\Sigma_\beta(f) = 1$. Hence $f = \Delta_\beta(h)$, for some $h \in F$, by exactness. Therefore

$$fx = h^{-1}\beta(h)x = h^{-1}xhx^{-1}x = h^{-1}xh,$$

so (ii) holds.

To show that (ii) implies (i), suppose that $f \in \ker \Sigma_\beta$ in F. Then $(fx)^n = 1$ by (1), so, in view of (ii), there must exist an element $g \in G$ such that $fx = g^{-1}xg$. Now, $g = x^t h$, for some $t \in \mathbb{Z}$ and some $h \in F$, because G is generated by F and x and $F \lhd G$. Hence

$$fx = g^{-1}xg = h^{-1}xh = h^{-1}\beta(h)x,$$

so $f = \Delta_\beta(h)$. Thus $\beta \in \mathrm{Aut}(F)$ is exact, as required.

We have shown that (i) is equivalent to (ii). The equivalence between (ii) and (iii) is an easy consequence of the observation that every cyclic subgroup of order n in G contains precisely one element of the form fx, for $f \in F$, which will generate this subgroup. $\qquad\square$

The equivalence between (i) and (iii) in the previous lemma gives the following corollary.

Corollary 2.8. *Suppose that F is a torsion-free group with an automorphism β of order $m \in \mathbb{N}$. If β is exact then so is $\beta^p \in \mathrm{Aut}(F)$, for any integer p coprime to m.*

Proof. Let $G = F \rtimes \langle \beta \rangle$ be the natural semidirect product of F with $\langle \beta \rangle \cong C_m$. We identify F with its image in G and let $x \in G$ denote the image of β. Then $F \lhd G$, $G = \langle F, x \rangle$ and $xfx^{-1} = \beta(f)$, for all $f \in F$.

Take any $p \in \mathbb{Z}$, coprime to m. Since x has order m in G, x^p also has order m, so $G = \langle F, x^p \rangle$ and conjugation by x^p induces the automorphism β^p on F. If $\beta \in \mathrm{Aut}(F)$ is exact then G has exactly one conjugacy class of cyclic subgroups of order m, by Lemma 2.7. Therefore, the same lemma shows that $\beta^p \in \mathrm{Aut}(F)$ is also exact. $\qquad\square$

Remark 2.9. The results of this subsection can be expressed in terms of the first non-abelian cohomology group of the cyclic group C_n with coefficients in a group F, which we compute using the presentation 2-complex associated to $\langle \beta \mid \beta^n \rangle$ rather than the general approach described in [8, Section I.5.1]. In this setting, the kernel of Σ_β is the set of 1-cocycles, and the image of Δ_β is the 1-coboundaries. The automorphism β of F is exact if and only if the pointed set $H^1(C_n; F)$ has just one element, and Lemma 2.7 is closely related to [8, Exercise 1 in Section I.5.1].

The final lemma in this subsection verifies that extensions of exact automorphisms to semidirect products, as discussed in Subsection 2.1, are again exact.

Lemma 2.10. *Let K, H be groups, let $\beta \in \operatorname{Aut}(K)$ be a finite order automorphism. Suppose that $\phi : K \to \operatorname{Aut}(H)$ is a β-neutral homomorphism and $A = H \rtimes_\phi K$ is the corresponding semidirect product. If H is torsion-free and β is an exact automorphism of K then its extension β_A, given by Definition 2.3, is an exact automorphism of A.*

Proof. Suppose that β has order $m \in \mathbb{N}$ in $\operatorname{Aut}(K)$. Then $\beta_A \in \operatorname{Aut}(A)$ and it has the same order m, by Lemma 2.4(a). Observe that for any $h \in H$ and $k \in K$ we have

$$
\Sigma_{\beta_A}\big((h,k)\big) = \big(h, k\big)\big(h, \beta(k)\big)\ldots\big(h, \beta^{m-1}(k)\big)
$$

(2)
$$
= \Big(h \cdot \phi\big(k\big)(h) \cdot \phi\big(k\beta(k)\big)(h) \cdots \phi\big(k\beta(k)\ldots\beta^{m-2}(k)\big)(h), k\beta(k)\ldots\beta^{m-1}(k)\Big)
$$

$$
= \Big(h \cdot \phi\big(k\big)(h) \cdot \phi\big(k^2\big)(h) \cdots \phi\big(k^{m-1}\big)(h), \Sigma_\beta(k)\Big),
$$

where in the last equality we used the fact that ϕ is β-neutral.

Now, assume that $\Sigma_{\beta_A}((h,k)) = (1,1)$. Then $\Sigma_\beta(k) = 1$ by (2), so $k = \Delta_\beta(u)$, for some $u \in K$, by exactness. Lemma 2.2(i) implies that $\phi(k) = \operatorname{Id}_H$, so (2) shows that

$$
\Sigma_{\beta_A}\big((h,k)\big) = \big(h^m, 1\big) = (1,1).
$$

Thus $h^m = 1$, whence $h = 1$ as H is torsion-free. Consequently,

$$
(h,k) = (1, u^{-1}\beta(u)) = (1,u)^{-1}\,\beta_A((1,u)) \in \Delta_{\beta_A}(A).
$$

Therefore $\ker \Sigma_{\beta_A} = \Delta_{\beta_A}(A)$, as required. $\qquad\square$

2.3. The main construction.

The building block for our examples will be the group K, given by the presentation

(3)
$$
K = \langle a, s, b, t \mid a^s = a^{-1},\ b^t = b^{-1},\ [a,b] = [a,t] = [s,b] = [s,t] = 1 \rangle,
$$

where we write $a^s = sas^{-1}$ and $[a,b] = aba^{-1}b^{-1}$.

Clearly $K \cong \pi \times \pi$, where $\pi = \langle a, s \mid a^s = a^{-1} \rangle$ is the fundamental group of the Klein bottle. In particular, K is torsion-free, metabelian and polycyclic of Hirsch length 4.

We define $\beta \in \operatorname{Aut}(K)$ to be the order 2 automorphism interchanging the two copies of π, i.e., $\beta(a) = b$, $\beta(s) = t$, $\beta(b) = a$ and $\beta(t) = s$.

Let C_2 be the cyclic group of order 2, generated by an element α, and let $\phi : K \to C_2$ be the homomorphism sending a and b to α and s and t to 1. Verification of the following elementary properties is left for the reader (Lemma 2.2(ii) may be useful for claim (ii)).

Lemma 2.11. *The group K, its automorphism β and the homomorphism ϕ enjoy the following properties.*

 (i) *$\phi \circ \beta = \phi$, i.e., ϕ is β-neutral.*
 (ii) *The quotient $K/\mathcal{N}(\beta)$ is isomorphic to $C_2 \times C_\infty$, where C_2 is generated by the image of a and C_∞ is generated by the image of s.*
 (iii) *The automorphism β of K is exact.*

We now construct auxiliary torsion-free polycyclic groups H_n as iterated semidirect products of K with itself n times. These extensions will be defined by induction on n.

Construction 2.12. Set $H_0 = \{1\}$ and let $\beta_0 \in \operatorname{Aut}(H_0)$ be the identity automorphism; set $H_1 = K$ and let $\beta_1 \in \operatorname{Aut}(H_1)$ be the automorphism β defined above. Now suppose that the group H_{n-1} and an order 2 automorphism $\beta_{n-1} \in \operatorname{Aut}(H_{n-1})$ have already been constructed, for some $n \geq 2$. We define the group H_n as the semidirect product

$$
H_n = H_{n-1} \rtimes_{\phi_{n-1}} K,
$$

where $\phi_{n-1} : K \to \mathrm{Aut}(H_{n-1})$ is essentially the homomorphism ϕ defined above, whose image is the cyclic subgroup $\langle \beta_{n-1} \rangle$.

By Lemma 2.11(i), ϕ_{n-1} is β-neutral, so we can use Definition 2.3 and Lemma 2.4 to define an automorphism $\beta_n \in \mathrm{Aut}(H_n)$ as the extension β_{H_n}, of β to H_n. Thus

$$\beta_n((h,k)) = (h, \beta(k)), \text{ for all } h \in H_{n-1} \text{ and all } k \in K.$$

Proposition 2.13. *For all $n \in \mathbb{N}$, the group H_n satisfies the following properties:*

  (i) H_n is torsion-free;
  (ii) the quotient $H_n/\mathcal{N}(\beta_n)$ is isomorphic to $(H_{n-1} \rtimes \langle \beta_{n-1} \rangle) \times C_\infty$, where $\langle \beta_{n-1} \rangle \leqslant \mathrm{Aut}(H_{n-1})$ is the cyclic subgroup of order 2 generated by β_{n-1};
  (iii) β_n is an exact automorphism of H_n.

Proof. When $n = 1$ the claims follow from the definition of H_1 and Lemma 2.11. For $n \geq 2$, (i) holds because the extension of a torsion-free group by a torsion-free group is again torsion-free, (ii) is an easy consequence of Lemma 2.4(c) and the definition of ϕ_{n-1}, and claim (iii) follows from Lemma 2.10. $\square$

Theorem 2.14. *For every integer $n \geq 1$ there exists a polycyclic group G_n of torsion length n.*

Proof. For each $n \geq 0$ we define the group G_{n+1} as follows:

$$(4) \qquad G_{n+1} = \langle H_n, x \mid x^2 = 1, \ xgx^{-1} = \beta_n(g), \text{ for all } g \in H_n \rangle \cong H_n \rtimes \langle \beta_n \rangle,$$

where H_n and $\beta_n \in \mathrm{Aut}(H_n)$ are provided by Construction 2.12. Thus H_n is a subgroup of G_{n+1} of index 2. Note that H_n is a polycyclic group of Hirsch length $4n$, by definition, hence so is G_{n+1}.

Let us show that the torsion length of G_{n+1} is $n+1$ by induction on n. When $n = 0$, $G_1 \cong C_2$ has torsion length 1, thus we can further assume that $n \geq 1$.

Recall that H_n is torsion-free and $\beta_n \in \mathrm{Aut}(H_n)$ is an exact automorphism by Proposition 2.13. Therefore, according to Lemma 2.7, every non-trivial finite order element of G_{n+1} must be conjugate to x. Thus $\mathrm{Tor}(G_{n+1})$ is the normal closure of x in G_{n+1}. The definition of G_{n+1}, given in (4), implies that the quotient $G_{n+1}/\mathrm{Tor}(G_{n+1})$ is clearly isomorphic to the quotient of H_n by $\mathcal{N}(\beta_n)$. So, Proposition 2.13.(ii) yields that

$$G_{n+1}/\mathrm{Tor}(G_{n+1}) \cong (H_{n-1} \rtimes \langle \beta_{n-1} \rangle) \times C_\infty \cong G_n \times C_\infty.$$

Evidently, the torsion length of $G_n \times C_\infty$ is the same as the torsion length of G_n, which equals n, by induction. Therefore the torsion length of G_{n+1} is $n+1$, as claimed. $\square$

Remark 2.15. It is not difficult to show that the groups H_n and G_n, constructed above, are, in fact, virtually abelian. A more careful analysis reveals that for every $n \geq 1$ the group G_n contains a free abelian normal subgroup F_n such that G_n/F_n is isomorphic to the direct power $(C_2 \wr C_2)^{n-1}$. It follows that, for each $n \in \mathbb{N}$, G_n is soluble of derived length 3, hence the (restricted) direct product $\times_{n=1}^{\infty} G_n$ is a countable soluble group of derived length 3 with infinite torsion length.

Remark 2.16. It is easy to see that the torsion length of a polycyclic group G is at most $h(G) + 1$, where $h(G)$ denotes the Hirsch length of G. In particular, the torsion length of any polycyclic group is finite.

In [3, Subsection 3.3] Chiodo and Vyas asked whether there exists a finitely generated soluble group with infinite torsion length. This interesting question remains open. As Remark 2.16 shows, such a group cannot be polycyclic.

3. Finitely presented groups G with $G/\mathrm{Tor}(G)$ not finitely presented

In [4, Question 1] and [6, 19.24] it was asked whether there is a finitely presented group G so that $G/\mathrm{Tor}(G)$ is recursively presented but not finitely presented. We give two different constructions of such a group, one of which is soluble and the other of which is virtually torsion-free.

Construction 3.1. First we construct a soluble group G. In this case one starts with Abels' group H which is a finitely presented torsion-free soluble group of 4×4 upper-triangular matrices over $\mathbb{Z}[1/p]$ for a fixed prime p, such that the centre $Z = Z(H)$ is isomorphic to the additive group $\mathbb{Z}[1/p]$ [1]. Let C be an infinite cyclic subgroup with $C \leq Z$, and let $G = H/C$. The matrix representation of H gives rise to a sequence $Z = H_1 < H_2 < H_3 < H_4 = H$ of normal subgroups of H with each H_{i+1}/H_i torsion-free abelian, and it follows that H/Z is torsion-free. Hence the subgroup $\mathrm{Tor}(G)$ is equal to $Z/C \cong \mathbb{Z}[1/p]/\mathbb{Z}$. This subgroup of H/C is central and not finitely generated. It follows that $G/\mathrm{Tor}(G) \cong H/Z$ is a non-finitely presented soluble group. Since Z is clearly recursively generated, H/Z is recursively presented.

The virtually torsion-free examples rely on Bestvina-Brady groups [2], so we start by recalling these groups. Let L be a finite flag simplicial complex, i.e., a clique complex. The right-angled Artin group A_L is the group with generators the vertices of L subject only to the relations that the ends of each edge of L commute. There is a homomorphism $f_L : A_L \to \mathbb{Z}$ that sends each generator to $1 \in \mathbb{Z}$, and the kernel of f is the Bestvina-Brady group BB_L. Bestvina-Brady show that BB_L is finitely generated if and only if L is connected, and finitely presented if and only if L is simply-connected [2].

Construction 3.2. Let L be a finite connected flag complex with non-trivial finite fundamental group π, and let K be the universal cover of L. The action of π on K by deck transformations induces an action on A_K and on BB_K. The group G will be the semidirect product $G = BB_K \rtimes \pi$. Let $\overline{G} = A_K \rtimes \pi$, and note that G is a normal subgroup of $\overline{G}$ with $\overline{G}/G \cong \mathbb{Z}$. It follows that every torsion element of $\overline{G}$ is in G. It is easy to see that $\overline{G}/\mathrm{Tor}(\overline{G}) = A_L$, and hence $G/\mathrm{Tor}(G) = BB_L$, a non-finitely presented subgroup of the group A_L. Thus the group G contains infinitely many conjugacy classes of subgroups isomorphic to π (see also [7, Thm. 3] for another proof of this fact). An explicit presentation for $G/\mathrm{Tor}(G) = BB_L$ with generators the directed edges of L is given in [5].

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