Energy flow characteristics of periodical orbits of nonlinear dynamical systems

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Summary. Based on energy flow theory, it is revealed that a necessary sufficient condition for nonlinear dynamical systems (NDS) $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$ to have periodical orbits is that there exist a non-zero spin matrix and one closed orbit with a corresponding period *T*, along which the time averaged flows of generalised potential energy (GPE) and generalised kinetic energy (GKE) vanish. For autonomous systems, a necessary condition to have periodical orbits is the energy flow characteristic factors (EFCFs) must not be semi-positive or semi-negative. Three examples are given to support the above revealed characteristics.
Keywords: Nonlinear dynamics, Periodical orbits, Energy flow matrices, Spin matrices, Generalised potential / kinetic energies.

Introduction

NDS investigated herein are generally sufficient to regard a second order differential equation with its initial conditions in a non-dimensional form, as discussed by [1-5], which can be transformed into the first order differential equation

$$d\mathbf{y}/dt \equiv \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{1}$$

We consider that $\mathbf{y} = \mathbf{y}(t) \in R^n$ is a vector valued function of an independent variable $t \in I = (t_1, t_2) \subseteq R$ and $\mathbf{f}: U \to R^n$ is a smooth function of variable t and vector $\mathbf{y}$ defined on some subset $U \subseteq R^n$, an *n*-dimensional phase space, and we seek a solution $\boldsymbol{\varphi}(\mathbf{y}_0, t)$ such that

$$\boldsymbol{\varphi}(\mathbf{y}_0, 0) = \mathbf{y}_0. \tag{2}$$

The solution $\boldsymbol{\varphi}(\mathbf{y}_0,\cdot): I \to R^n$ defines a solution curve, trajectory or orbit of Eq.1 based at $\mathbf{y}_0$ as shown by Fig. 1(a). According to the basic local existence and uniqueness theorem [6], there exist no intersections of the trajectories of Eq.1 in the solution space except at its fixed points. The solution curves $\boldsymbol{\varphi}_t(U)$ generate the flow shown by Fig. 1(b).To investigate the behaviors of NDS, Xing [5] developed an energy flow theory, which has discovered that i) GPE automatically plays a role of Lyapunov function for stability at fixed points of NDS, ii) generalised energy conservation law of chaotic motions [5,7], iii) behaviors of friction-induced vibrations [8]. This paper aims to tackle the periodical solutions of NDS to reveal its energy flow characteristics.

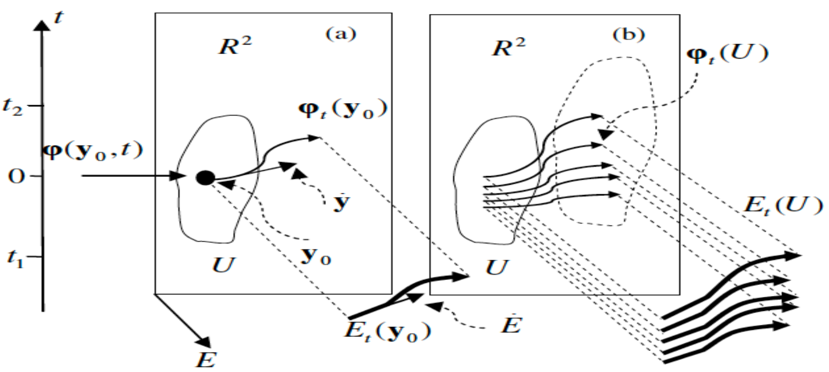

Figure 1: (a) a solution curve $\boldsymbol{\varphi}_t(\mathbf{y}_0)$ with its energy flow curve $E_t(\mathbf{y}_0)$, of which their tangent vectors at a point $\mathbf{y}$ are $\dot{\mathbf{y}} = \mathbf{f}$ and $\dot{E}$, respectively; (b) the flow $\boldsymbol{\varphi}_t(U)$ and the energy flow $E_t(U)$ in R^n (Xing [5, 7]).

Fundamentals of energy flow analysis

Energy flow variables, matrices, and equations

In the energy flow theory, the following energy flow variables based on Eq.1 in phase space are defined,

$$\text{GPE:}\ \ E = 0.5\mathbf{y}^T\mathbf{y}; \qquad \text{GKE:}\ \ K = 0.5\dot{\mathbf{y}}^T\dot{\mathbf{y}}; \quad \text{Force power:}\ \ P = \mathbf{y}^T\mathbf{f}, \tag{3}$$

of which the time averaged GPE and GKE as well as the time averaged GPE-flow (GPEF) and the time averaged GKE-flow (GKEF) during a period $(0, T)$ are respectively defined by

$$\langle E \rangle = \int_0^T 0.5\mathbf{y}^T\mathbf{y}\, dt/T, \qquad \langle K \rangle = \int_0^T 0.5\dot{\mathbf{y}}^T\dot{\mathbf{y}}\, dt/T, \qquad \langle P \rangle = \int_0^T \mathbf{y}^T\mathbf{f}\, dt/T,$$
$$\langle \dot{E} \rangle = \int_0^T \dot{E}\, dt/T = [E(T) - E(0)]/T = \int_0^T P\, dt/T = \langle P \rangle, \tag{4}$$

$$\langle \dot{K} \rangle = \int_0^T \dot{K}\, dt/T = [K(T) - K(0)]/T = \int_0^T \dot{\mathbf{y}}^T \dot{\mathbf{f}}\, dt/T\,.$$

The corresponding energy flow equilibrium equations take the forms

$$\dot{E} = P, \qquad E_0 = 0.5\mathbf{y}_0^T \mathbf{y}_0, \quad \mathbf{J} = \partial \mathbf{f}/\partial \mathbf{y}^T, \qquad \mathbf{E} = 0.5(\mathbf{J} + \mathbf{J}^T), \qquad \mathbf{U} = 0.5(\mathbf{J} - \mathbf{J}^T),$$

$$\dot{K} = \dot{\mathbf{y}}^T \ddot{\mathbf{y}} = \dot{\mathbf{y}}^T \dot{\mathbf{f}} = \dot{\mathbf{y}}^T (\partial \mathbf{f}/\partial t + \mathbf{J}\dot{\mathbf{y}}) = \dot{\mathbf{y}}^T\, \partial \mathbf{f}/\partial t + \dot{\mathbf{y}}^T \mathbf{E}\dot{\mathbf{y}} = \partial K/\partial t + \dot{\mathbf{y}}^T \mathbf{E}\dot{\mathbf{y}}, \tag{5}$$

where $\mathbf{J}$ is *Jacobian* matrix, the partial derivative of vector $\mathbf{f}$ with respect to vector $\mathbf{y}^T$ [5]. For autonomous NDS from Eq. 5 we have

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}), \qquad \dot{K} = \dot{\mathbf{y}}^T \mathbf{E}\dot{\mathbf{y}}, \qquad \langle \dot{K} \rangle = \frac{1}{T}\int_0^T \dot{\mathbf{y}}^T \mathbf{E}\dot{\mathbf{y}}\, dt, \tag{6}$$

since $\partial K/\partial t = 0$. Here $\mathbf{E}$ is a real symmetrical *energy flow matrice* while $\mathbf{U}$ is a real skew-symmetrical *spin matrix*, so that $\mathbf{y}^T\mathbf{U}\mathbf{y} = \mathbf{0} = \dot{\mathbf{y}}^T\mathbf{U}\dot{\mathbf{y}}$. Geometrically, GPE relates the position $\mathbf{y}$ of a point in phase space, while GKE involves the velocity or tangent vector $\dot{\mathbf{y}}$ of solution curve, and the generalised force power P gives the energy flow, the time change rate of GPE.

Energy flow characteristic factors

The energy flow matrix $\mathbf{E}$ is a real symmetrical matrix, of which the eigenvalues and the corresponding eigenvectors are real, and its characteristic equation

$$\mathbf{E}\boldsymbol{\varepsilon} = \lambda\boldsymbol{\varepsilon}, \qquad |\mathbf{E} - \lambda\mathbf{I}| = 0, \tag{7}$$

can give the eigenvalue λ_I and its corresponding eigenvector $\boldsymbol{\psi}_I$ satisfying the orthogonal relationships

$$\boldsymbol{\Psi}^T\mathbf{E}\boldsymbol{\Psi} = \boldsymbol{\Lambda} = \mathbf{diag}(\lambda_I), \qquad \boldsymbol{\Psi}^T\boldsymbol{\Psi} = \mathbf{I}, \qquad \boldsymbol{\Psi} = [\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2 \quad \cdots \quad \boldsymbol{\psi}_n]. \tag{8}$$

Normally, the eigenvectors with different eigenvalues span a complete subspace in the neighbor of the point where the matrix $\mathbf{E}$ defined, so that the vector $\boldsymbol{\varepsilon}$ can be represented as

$$\boldsymbol{\varepsilon} = \boldsymbol{\Psi}\boldsymbol{\zeta}, \tag{9}$$

which, when substituted into the term $\boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon}$ and by using the orthogonal relationships in Eq.8, gives

$$\boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon} = \boldsymbol{\zeta}^T\boldsymbol{\Psi}^T\mathbf{E}\boldsymbol{\Psi}\boldsymbol{\zeta} = \boldsymbol{\zeta}^T\boldsymbol{\Lambda}\boldsymbol{\zeta} = \textstyle\sum_{I=1}^n \lambda_I\, \zeta_I^2. \tag{10}$$

This result implies that the value of $\boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon}$ about a point is totally determined by the eigenvalues and eigenvectors of energy flow matrix. We respectively call λ_I and $\boldsymbol{\psi}_I$ as the energy flow characteristic factors (EFCFs) and the energy flow characteristic vectors (EFCVs) of NDS.

Spin matrix

The spin matrix $\mathbf{U}$ is a real skew-symmetric matrix and therefore its non-zero eigenvalues κ_I must be conjugate purely imaginaries with the complex eigenvector matrix $\mathbf{Y}$ satisfying the following orthogonal relationships

$$\mathbf{Y}^{*T}\mathbf{U}\mathbf{Y} = \mathbf{diag}(\kappa_I), \qquad \mathbf{Y}^{*T}\mathbf{Y} = \mathbf{I}, \tag{11}$$

Here * denotes a conjugate of complex number.

Curl of a vector field

The curl of a vector field $\mathbf{f}$, denoted by curl$\mathbf{f}$, or $\nabla \times \mathbf{f}$, at a point O is defined in terms of its projection onto various lines through the point. As shown in Fig.2, if $\mathbf{v}$ is a unit vector, the projection of the curl$\mathbf{f}$ onto $\mathbf{v}$ is defined as a limited value of a closed-curve integral in a plane orthogonal to $\mathbf{v}$, divided by the area A enclosed by the closed curve. Here, the path C of integration is constructed around the point O, so that, when Eqs. 1 and 3 are introduced, we have

$$(\nabla \times \mathbf{f})_\nu = \lim_{A\to 0}\left\{\oint_C \mathbf{f}\cdot d\mathbf{y}/A\right\} = \lim_{A\to 0}\left\{\oint_C \dot{\mathbf{y}}\cdot\dot{\mathbf{y}}/A\right\} = \lim_{A\to 0}\left\{\oint_C 2Kdt/A\right\}. \tag{12}$$

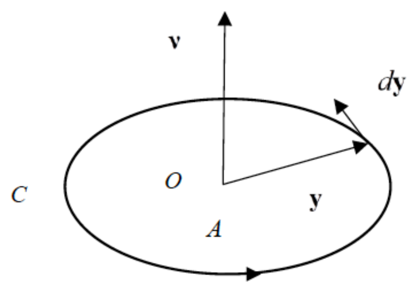

Figure 2: Circulation integration of path C with its positive direction obeying the right-hand rule to define the curl of vector field $\mathbf{f}$.

For 3-dimensional space, the curl$\mathbf{f}$ can be denoted in a tensor form

$$(\nabla \times \mathbf{f})_i = e_{ijk} f_{k,j}, \tag{13}$$

where e_{ijk} is the permutation tensor [9,10]. The curl$\mathbf{f}$ is a dual vector of a skew-symmetrical matrix $\mathbf{U}$, spin matrix, satisfying the following relationship

$$(\mathbf{U})_{ij} = U_{ij} = -0.5 e_{ijk} (\nabla \times \mathbf{f})_k = -0.5 e_{ijk} e_{krs} f_{s,r} = -0.5(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) f_{s,r} = 0.5(f_{i,j} - f_{j,i}) = 0.5(\mathbf{J} - \mathbf{J}^T)_{ij}. \tag{14}$$

Periodical orbits

For NDS, a periodical orbit is defined as a closed path governed by Eq. 1 in phase space, along which the phase point $\mathbf{y}(t)$ with its velocity $\dot{\mathbf{y}}(t)$, starting from a position $\mathbf{y}(\hat{t})$ at time $\hat{t}$, moves to the same position $\mathbf{y}(\hat{t}+T) = \mathbf{y}(\hat{t})$ with the same velocity $\dot{\mathbf{y}}(\hat{t}+T) = \dot{\mathbf{y}}(\hat{t})$ after a period T, and the motion repeats again, such as $\hat{t} = 0$ shown in Fig.3. If we assume that dS denotes a differential line element with its unit outside normal vector ν_i and unit tangent vector τ_i at a point on the closed curve in Fig.3, so that

$$\boldsymbol{\tau} dS = \dot{\mathbf{y}} |d\mathbf{y}| / |\dot{\mathbf{y}}| = \dot{\mathbf{y}} |\dot{\mathbf{y}} dt| / |\dot{\mathbf{y}}| = \dot{\mathbf{y}} dt, \tag{15}$$

based on which the integrals along the curve in the next section hold.

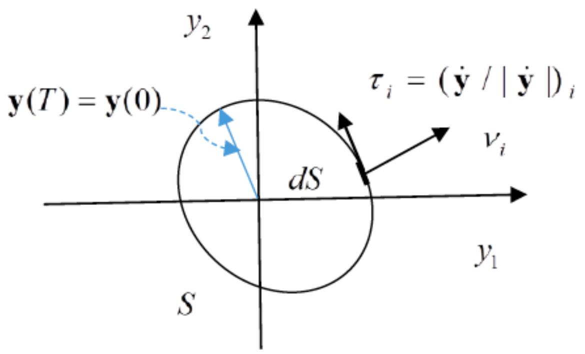

Figure 3: The periodical orbit and the unit normal / tangent vectors of line element dS on orbit.

This definition implies that for a periodical orbit, its position vector should be a periodical function of time. Generally, this periodical function can be represented by a Fourier series of period T. For simplifying mathematic formulations but not losing generality, as an example, we consider this function is a sinusoidal function

$$\mathbf{y}(t) = \hat{\mathbf{y}} \sin(\omega t + \hat{\phi}), \qquad \dot{\mathbf{y}}(t) = \hat{\mathbf{y}} \omega \cos(\omega t + \hat{\phi}), \qquad \omega = \frac{2\pi}{T}, \tag{16}$$

where $\hat{\mathbf{y}}$ and $\hat{\phi}$ denote the amplitude and phase angle, ω is a frequency corresponding to the period. From Eq. 16, it follows that GPE and GKE respectively are

$$E = \frac{1}{2}\sin^2(\omega t + \hat{\phi})\, \hat{\mathbf{y}}^T \hat{\mathbf{y}} = 2\hat{E} \sin^2(\omega t + \hat{\phi}), \qquad \hat{E} = \frac{\hat{\mathbf{y}}^T \hat{\mathbf{y}}}{4}, \tag{17}$$
$$K = \frac{1}{2}\cos^2(\omega t + \hat{\phi})\, \hat{\mathbf{y}}^T \hat{\mathbf{y}} \omega^2 = 2\hat{K} \sin^2(\omega t + \hat{\phi}), \qquad \hat{K} = \frac{\hat{\mathbf{y}}^T \hat{\mathbf{y}}}{4} \omega^2 = \omega^2 \hat{E}.$$

Energy flow characteristics of periodical orbits of NDS

Based on the definition of periodical orbits governed by Eq. 1, if there exists a periodical orbit, the following energy flow characteristics must hold.

Time averaged GPE

$$\langle E \rangle = \frac{1}{T} \oint_S E dt = \hat{E}, \tag{18}$$

where $\hat{E}$ is a positive constant representing the averaged distance of phase points on the orbit to the origin since E is positive and the motion repeats along the closed orbit.

Time averaged GPEF

$$\langle \dot{E} \rangle = \frac{1}{T} \int_{\hat{t}}^{T+\hat{t}} \dot{E}\, dt = \frac{E(T+\hat{t}) - E(\hat{t})}{T} = 0 = \frac{1}{T} \int_{\hat{t}}^{T+\hat{t}} \mathbf{y}^T \mathbf{f}\, dt, \tag{19-1}$$

since for the periodical orbit S, $\mathbf{y}(\hat{t}+T) = \mathbf{y}(\hat{t})$. If the vector field $\mathbf{f}$ takes a form

$$\mathbf{f} = \mathbf{B}(\mathbf{y})\mathbf{y} = (\bar{\mathbf{E}} + \bar{\mathbf{U}})\mathbf{y}, \quad \bar{\mathbf{E}} = (\mathbf{B} + \mathbf{B}^T)/2, \qquad \bar{\mathbf{U}} = (\mathbf{B} - \mathbf{B}^T)/2, \tag{19-2}$$

Eq. 19-1 requires the EFCFs of matrix $\bar{\mathbf{E}}$ not always being semi-positive or semi-negative in the period *T*.

Time averaged GKE

$$\langle K \rangle = \frac{1}{T}\oint_S \; K dt = \widehat{K} = \frac{1}{2T}\oint_S \; \dot{\mathbf{y}} \cdot \dot{\mathbf{y}} dt = \frac{1}{2T}\oint_S \; \dot{\mathbf{y}} \cdot \boldsymbol{\tau} dt, \tag{20}$$

where $\widehat{K}$ is a positive constant, sine GKE is not negative. From this result, when Eq.12 noticed, it follows that curl**f** must not vanish, so that Eq. 14 implies

$$\mathbf{U} \neq 0. \tag{21}$$

In a reverse case if Eq.21 holds, then Eq. 20 holds. Therefore, Eq. 21 is a necessary and sufficient condition for Eq. 20.

Time averaged GKEF

$$\langle \dot{K} \rangle = \frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} \dot{K}\, dt = \frac{K(T+\hat{t})-K(\hat{t})}{T} = 0 = \frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} (\frac{\partial K}{\partial t} + \dot{\mathbf{y}}^T \mathbf{E} \dot{\mathbf{y}})\, dt, \tag{22-1}$$

For autonomous NDS, $\partial K/\partial t = 0$, from Eq. 10, it follows

$$\langle \dot{K} \rangle = \frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} (\dot{\mathbf{y}}^T \mathbf{E} \dot{\mathbf{y}})\, dt = \frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} (\boldsymbol{\zeta}^T \boldsymbol{\Lambda} \boldsymbol{\zeta})\, dt = \frac{1}{T}\int_{\hat{t}}^{T+\hat{t}} (\textstyle\sum_{I=1}^{n} \lambda_I\, \zeta_I^2)\, dt = 0, \tag{22-2}$$

holds, implying that the EFCFs of NDS must not always be semi-positive or semi-negative in the period *T*.

Above equations are valid if exsisting periodical orbits of NDS, so that they are necessary conditions. Considering GPE geometrically involves the distance of a phase point to the origin, we can confirm that for a periodical oribit curve, of which each point has its positive distance to the origin, so that Eq. 18 always is valid. Moreover, as discussed above, the condition in Eq. 21 can replace the Eq. 20, which implies that non-zero spin matrix of NDS is a necessary condition for its periodical orbits.

Theorem A necessary sufficient condition for NDS, governed by Eq. 1, having periodical orbits is that its spin matrix $\mathbf{U} \neq 0$ and there exists at least one closed curve with a corresponding period *T*, along which the time averaged GPEF and GKEF vanish. For autonomous NDS, the condition of time averaged GKEF can be replaced by that its EFCFs of energy flow matrix **E** are always not semi-positive or semi-negative in the period *T*.

Examples

A linear system

As an example, we consider a system with one degree of freedom governed by equation

$$\ddot{x} + \alpha\dot{x} + x = f\cos t, \tag{23-1}$$

which can be rewritten in the form of phase space

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ f\cos t \end{bmatrix}, \tag{23-2}$$

with its Jacobian, energy flow, spin matrices and energy flow equation respectively as follows

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix} = \mathbf{B}, \bar{\mathbf{E}} = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha \end{bmatrix}, \bar{\mathbf{U}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dot{E} = -\alpha y^2 + yf\cos t\,,\ \dot{K} = \frac{\partial K}{\partial t} - \alpha y^2 = -\alpha y^2 - \dot{y} f\sin t. \tag{23-3}$$

The time averaged GPEF and GKEF of the system are respectively given by

$$\langle \dot{E} \rangle = \frac{1}{2\pi}\int_0^{2\pi} (-\alpha y^2 + yf\cos t) dt, \qquad \langle \dot{K} \rangle = \frac{1}{2\pi}\int_0^{2\pi} (-\alpha y^2 - \dot{y} f\sin t) dt. \tag{23-4}$$

It is not difficult to obtain the EFCFs of the system

$$\bar{\lambda}_1 = -\alpha, \qquad \bar{\lambda}_2 = 0, \tag{23-5}$$

implying they are semi-positive for $\alpha < 0$, semi-negative for $\alpha > 0$, and vanish when $\alpha = 0$. Based on the above results, we discuss its two cases as follows.

Non-forced case $f = 0$

For this case, the time averaged GPEF and GKEF have the values

$$\langle \dot{E} \rangle = \langle \dot{K} \rangle = \begin{cases} > 0, & \alpha < 0, \\ < 0, & \alpha > 0, \\ = 0, & \alpha = 0. \end{cases} \tag{23-6}$$

As shown in Fig. 4, from this result it follows that the system is a divergence one when $\alpha < 0$, due to time averaged GPEF and GKEF are always increasing and the orbit tends to infinite; while it is a converged one when $\alpha > 0$, the orbit tends to the origin of phase space. There are no periodical orbits for non-zero values of parameter α although the spin matrix of the system is not zero. Also, EFCFs are semi-positive or semi-negative which do not satisfy the conditions in

Theorem. When $\alpha = 0$ the system has its periodical orbit of radius $\rho = \sqrt{x_0^2 + y_0^2}$ depending on the initial condition (x_0, y_0) shown in Fig.4.

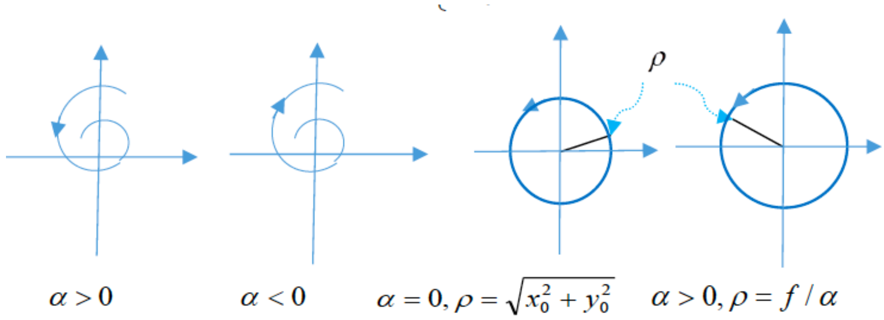

Figure 4: The orbits of 1-DOF system affected by the damping parameter α and external force *f*.

Forced case $f \neq 0, \alpha > 0$

For this case, since the system is linear, we know the solution of system is

$$x = \rho \sin t, \quad y = \rho \cos t, \tag{23-7}$$

so that its time averaged GPEF and GKEF calculated by using Eq. 23-4 are

$$\langle \dot{E} \rangle = \frac{1}{2\pi}\int_0^{2\pi}(-\alpha\rho^2\cos^2 t + \rho f \cos^2 t)dt = \frac{-\alpha\rho^2 + \rho f}{2} = \langle \dot{K} \rangle = \frac{1}{2\pi}\int_0^{2\pi}(-\alpha\rho^2\cos^2 t + \rho f \sin^2 t)dt. \tag{23-8}$$

Therefore, both GPEF and GKEF vanish when $\rho = f/\alpha$, in which the work done by the external force is dissipated by the damping of the system and the system undergoes a periodical motion with radius $\rho = f/\alpha$ shown in Fig. 4.

Van der Pol's equation

Van der Pol's equation provides an example of an oscillation with nonlinear damping, its energy dissipated at large amplitude but generated at low amplitude. The governing equation of Van der Pol's system is given by

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \mathbf{B}\begin{bmatrix}x\\ y\end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix}0 & 1\\ -1-2xy & -(x^2-1)\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0 & 1\\ -1 & -(x^2-1)\end{bmatrix}, \tag{24-1}$$

from which, when Eq. 5 used, it follows

$$\mathbf{E} = \begin{bmatrix}0 & -xy\\ -xy & -(x^2-1)\end{bmatrix}, \ \mathbf{U} = \begin{bmatrix}0 & 1+xy\\ -1-xy & 0\end{bmatrix}, \ \dot{E} = -(x^2-1)y^2, \ \dot{K} = \dot{\mathbf{y}}^T\mathbf{E}\dot{\mathbf{y}} = -2xy\dot{x}\dot{y} - \dot{y}^2(x^2-1). \tag{24-2}$$

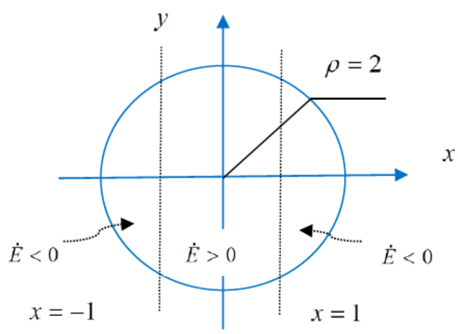

Figure 5: the periodical orbit of Van der Pol's equation**.**

To check if the time averaged GPEF and GKEF vanish in a possible closed orbit, we assume that

$$x = \rho \sin t, \quad y = \rho \cos t, \tag{24-3}$$

so that we obtain

$$\langle \dot{E} \rangle = \frac{1}{2\pi}\int_0^{2\pi}[-(\rho^2\cos^2 t - 1)\rho^2\sin^2 t)dt = -\frac{\rho^2}{2}\left(\frac{\rho^2}{4} - 1\right), \tag{24-4}$$

$$\langle \dot{K} \rangle = \frac{1}{2\pi}\int_0^{2\pi}[2\rho^4\sin^2 t\cos^2 t - \rho^2\sin^2 t(\rho^2\sin^2 t - 1)]\,dt = -\frac{\rho^2}{2}\left(\frac{\rho^2}{4} - 1\right).$$

Therefore, when $\rho = 2$, the GPEF and GKEF vanish, which gives a periodical orbit. For matrix **E**, its EFCFs are

$$\lambda_{1,2} = \left\{-(x^2-1) \pm \sqrt{(x^2-1) + 4x^2y^2}\right\}/2, \tag{24-5}$$

of which one positive and another negative. These results satisfy the conditions in theorem. As shown in Fig. 5, the periodical orbit is a circle of radius $\rho = 2$, along which on the domain with $|x| > 1$, the energy flow $\dot{E} < 0$, while on the domain with $|x| < 1$,the energy flow $\dot{E} > 0$, and on the full circle the averaged energy flow vanishes.

A planar system

We investigate a planar system governed by

$$\dot{x} = xy, \qquad \dot{y} = -y + x^2/2, \tag{25-1}$$

of which its Jacobian, energy flow and spin matrices respectively as

$$\mathbf{J} = \begin{bmatrix} y & x \\ x & -1 \end{bmatrix} = \mathbf{E}, \qquad \mathbf{U} = 0, \tag{25-2}$$

so that there are no periodical orbits due to spin matrix vanishes. We can calculate the energy flow of the system, i.e.

$$\dot{E} = 3yx^2/2 - y^2, \tag{25-3}$$

of which its zero energy flow curves are

$$y = 0, \qquad y = 3x^2/2\,, \tag{25-4}$$

as shown in Fig.6. Using the Eq. 24-3, the time averaged GPEF is calculated as follows

$$\langle \dot{E} \rangle = \frac{\rho^2}{2\pi}\int_0^{2\pi}\left(\frac{3\rho}{2}\cos^2 t \sin t - \sin^2 t\right)dt = -\frac{\rho^2}{2} \neq 0, \tag{25-5}$$

which indicates periodical orbits are impossible.

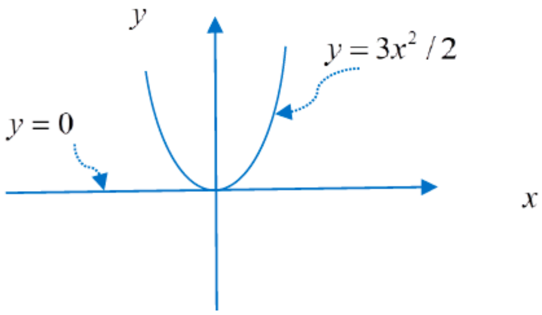

Figure 6: The zero energy flow curves of system shown by Eq. 25-1.

Conclusions

Energy flow theory with two scalars, GPE and GKE as well as the real symmetrical energy flow matrix **E** and the real skew-symmetrical spin matrix **U** is effectively used to investigate NDS in phase space. It is revealed that a necessary sufficient condition for NDS $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ having periodical orbits is that there exist a non-zero spin matrix and a closed orbit with a corresponding period *T*, along which the time averaged GPEF and GKEF vanish. For autonomous systems with energy flow matrix **E** the necessary condition for periodical orbits is its EFCFs not being semi-positive or semi-negative. Three examples, a damping / forced linear system, the Van der Pol's system, and a planar one, are presented to illustrate the revealed characteristics. The developed energy flow theory provides an important means to explore dynamic characteristics of various NDS.

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