MEASURED ASYMPTOTIC EXPANDERS AND RIGIDITY FOR ROE ALGEBRAS

KANG LI, JÁN ŠPAKULA, AND JIAWEN ZHANG

Abstract. In this paper, we give a new geometric condition in terms of measured asymptotic expanders to ensure rigidity of Roe algebras. Consequently, we obtain the rigidity for all bounded geometry spaces which coarsely embed into some L^p-space for $p \in [1, \infty)$. Moreover, we also verify rigidity for the box spaces constructed by Arzhantseva-Tessera and Delabie-Khukhro even though they do *not* coarsely embed into any L^p-space.

The key step in our proof of rigidity is showing that a block-rank-one (ghost) projection on a sparse space X belongs to the Roe algebra $C^*(X)$ if and only if X consists of (ghostly) measured asymptotic expanders. As a by-product, we also deduce that ghostly measured asymptotic expanders are new sources of counterexamples to the coarse Baum-Connes conjecture.

Mathematics Subject Classification (2020): 46H35, 20F65, 05C81. Secondary: 19K56.
Keywords: Measured asymptotic expanders, Measured weak embeddings, Rigidity of Roe algebras, Coarse Baum-Connes conjecture, Quasi-locality.

1. Introduction

(Uniform) Roe algebras are C^*-algebras associated to metric spaces, which reflect and encode the coarse (or large-scale) geometry of the underlying metric spaces. They have been well-studied and have fruitful applications, among which the most important ones would be the coarse Baum-Connes conjecture, the Novikov conjecture on higher signatures and the Gromov–Lawson conjecture on positive scalar curvature (see e.g. [42] for details).

The main purpose of this paper is to provide a new geometric condition in terms of measured asymptotic expanders to guarantee the rigidity for (uniform) Roe algebras associated to metric spaces. In recent years, the study of the rigidity for those C^*-algebras has gained considerable attention (for instance [8, 9, 11, 10, 12, 15, 39, 41]). Let us recall the general setting: let (X, d) be a metric space with bounded geometry, and denote by $C^*_u(X)$ the uniform Roe algebra of X, defined to be the norm closure of all bounded operators on $\ell^2(X)$ with finite propagation. Similarly the Roe algebra of X, denoted by $C^*(X)$, is defined to be the norm closure of all locally compact bounded operators on $\ell^2(X; \mathcal{H}_0)$ with finite propagation,

Date: September 12, 2022.

KL was supported by the Internal KU Leuven BOF project C14/19/088. KL has also received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 677120-INDEX).

JŠ was partially supported by Marie Curie FP7-PEOPLE-2013-CIG Coarse Analysis (631945), and by EPSRC EP/V002899/1.

JZ was supported by NSFC11871342.

where $\mathcal{H}_0$ is an infinite-dimensional separable Hilbert space. It is well-known that if X and Y are coarsely equivalent metric spaces with bounded geometry, then their (uniform) Roe algebras are (stably) $*$-isomorphic. The rigidity problem refers to the converse implication:

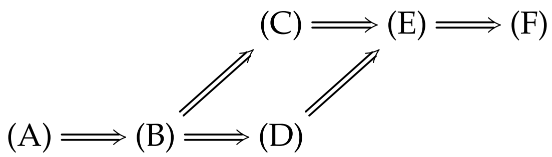
Problem 1.1 (Rigidity Problem)**.** Let X and Y be two metric spaces with bounded geometry.

(1) If $C_u^*(X)$ and $C_u^*(Y)$ are stably $*$-isomorphic, are X and Y coarsely equivalent?
(2) If $C^*(X)$ and $C^*(Y)$ are $*$-isomorphic, are X and Y coarsely equivalent?

To briefly summarise the current status of Rigidity Problem (at the time of submission of the present paper[1]), and some of the main results of this paper, let us consider the following properties of a metric space with bounded geometry:

(A) Property (A).
(B) Coarse embeddability into a Hilbert space.
(C) Coarse embeddability into some L^p-space for $p \in [1, \infty)$.
(D) "All sparse subspaces yield only compact ghost projections in their (uniform) Roe algebras." (This technical condition in the uniform case was first introduced in [9, 11, 10]; the general case in [8].)
(E) The space contains no sparse subspaces consisting of ghostly measured asymptotic expanders. (This geometric condition is introduced in the present paper, see below for precise definitions.)
(F) A positive answer to Rigidity Problem 1.1.

All of (A) – (E) imply (F): The implication (A)$\Rightarrow$(F) is the main result in [39] (see Theorem 1.4 and 1.8 therein) where the Rigidity Problem was first posed. The next substantial progress was then made with the help of (D) by Braga and Farah in [9, Corollary 1.3] (for $*$-isomorphic uniform Roe algebras), but the full generality of (D)$\Rightarrow$(F) was given in [8, Theorem 1.3]. One of the main results of this paper is to establish (E)$\Rightarrow$(F) (see Theorem A below). One has the following implications:



Here (A)$\Rightarrow$(B) is due to Yu [44, Theorem 2.2]. The implication (B)$\Rightarrow$(C) holds trivially[2], however the converse is, to the best of our knowledge, still open for bounded geometry metric spaces. Next, (B)$\Rightarrow$(D) is essentially due to Finn-Sell [19, Proposition 35], made precise in [8, Theorem 5.2]. The remaining implications (C)$\Rightarrow$(E), (D)$\Rightarrow$(E) and (E)$\Rightarrow$(F) are all established in this paper (see Corollary 7.2, Corollary C and Theorem A respectively).

[1] About half a year after this paper was first announced, an unconditional positive answer to the rigidity problem for *uniform* Roe algebras is given in [6]. However, the method in [6] does not immediately apply to Roe algebras. To the best of the authors' knowledge, the rigidity problem for Roe algebras is still open in general.

[2] In fact (B) implies coarse embeddability into *any* L^p-space for $p \in [1, \infty)$, but L^p-spaces for $p > 2$ do not coarsely embed into a Hilbert space, cf. [32, Theorem 1.11], and [33, 23].

Some of the above implications are known to be *not* reversible: (B)$\not\Rightarrow$(A) by [3, Theorem 1.1], (D)$\not\Rightarrow$(B) by [19, Proposition 35] and [17, Corollary 1.2]. Furthermore, (E)$\not\Rightarrow$(C) by the results in this paper (see Theorem F and Lemma 2.8 below). Finally, we do not know whether (D)$\Rightarrow$(E) is reversible, and thus whether our new rigidity results here are actually (rather than formally) stronger than those in [8]. Nevertheless, the condition (E) is geometric and easier to check than the analytic condition (D), which allows us to provide new examples of rigid spaces in Section 7, and to establish the condition (C) as another geometric criterion for Rigidity (see Corollary E).

To explain our results and approach in more detail, we start with measured asymptotic expanders. They were introduced in [27, Definition 6.1] and can be naturally constructed from strongly ergodic or asymptotically expanding actions (see [27, Theorem 6.16]).

Definition 1.2. A sequence of finite probability-measured metric spaces $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is called a sequence of *measured asymptotic expanders* if for any $\alpha \in (0, \frac{1}{2}]$, there exist $c_\alpha \in (0, 1)$ and $R_\alpha > 0$ such that for any $n \in \mathbb{N}$ and $A \subseteq X_n$ with $\alpha \leq m_n(A) \leq \frac{1}{2}$, we have $m_n(\partial_{R_\alpha} A) > c_\alpha \cdot m_n(A)$, where $\partial_{R_\alpha} A := \{x \in X_n \backslash A : d_n(x, A) \leq R_\alpha\}$.

Note that when all probability measures m_n are taken to be normalised counting measures, then we recover the notion of asymptotic expanders introduced in [25].

Definition 1.3. Let (X, d) be a metric space.

(1) X is called *sparse*[3] if there exists a disjoint partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ such that
- $0 < |X_n| < \infty$ for all $n \in \mathbb{N}$, and
- $d(X_n, X_m) \to \infty$ as $n + m \to \infty$ and $n \neq m$.

(2) We say that a sparse space X *consists of ghostly measured asymptotic expanders* if there exist a decomposition $X = \bigsqcup_n X_n$ as in (1) and a sequence of probability measures m_n on X_n with $\lim_{n\to\infty} \sup_{x\in X_n} m_n(x) = 0$ such that $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ forms a sequence of measured asymptotic expanders, where d_n is the restriction of d to X_n.

We are ready to state our main rigidity theorem:

Theorem A (Theorem 6.13). *Let X and Y be metric spaces with bounded geometry. Assume that either X or Y contains no sparse subspaces consisting of ghostly measured asymptotic expanders. Then the following are equivalent:*

(1) X is coarsely equivalent to Y;
(2) $C^*_u(X)$ is stably $*$-isomorphic to $C^*_u(Y)$;
(3) $C^*(X)$ is $*$-isomorphic to $C^*(Y)$.

Before we discuss examples where the above theorem applies, let us outline our approach to its proof. The first step is to show that Problem 1.1 has a positive answer under the condition that all sparse subspaces contain no block-rank-one ghost projections in their Roe algebras (see Proposition 3.9)[4] in the following sense:

[3] In this case, we also say that X is a *coarse disjoint union* of X_n.

[4] We would like to point out that our assumptions in Proposition 3.9 are formally weaker than the technical condition in [8, Theorem 1.3].

Definition 1.4. Let $(X,d) = \bigsqcup_{n\in\mathbb{N}}(X_n,d_n)$ be a sparse space and $\mathcal{H}_0$ be a Hilbert space. We say that an operator $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ is a *block-rank-one projection* with respect to $\{(X_n,d_n)\}_{n\in\mathbb{N}}$ if it is a projection of the form

$$P := \bigoplus_{n\in\mathbb{N}} P_n \in \mathfrak{B}\left(\bigoplus_{n\in\mathbb{N}} \ell^2(X_n;\mathcal{H}_0)\right) = \mathfrak{B}(\ell^2(X;\mathcal{H}_0)),$$

where each P_n is a rank-one projection in $\mathfrak{B}(\ell^2(X_n;\mathcal{H}_0))$.

As each P_n has the form $P_n(\eta) = \langle \eta, \xi_n\rangle \xi_n$ for all $\eta \in \ell^2(X_n;\mathcal{H}_0)$ where ξ_n is a unit vector in $\ell^2(X_n;\mathcal{H}_0)$, we define for each $n \in \mathbb{N}$ the *associated probability measure* m_n on X_n by $m_n(x) := \|\xi_n(x)\|^2$ for $x \in X_n$.

Finally, we say that the block-rank-one projection P is a *ghost*[5] if the sequence of associated probability measures satisfies $\lim_{n\to\infty} \sup_{x\in X_n} m_n(x) = 0$.

If each P_n is the rank-one projection onto constant functions on X_n, then the resulting projection P is the so-called *averaging projection*. Its relation to asymptotic expanders has been extensively studied in [24, 25]. The reader is cautioned that the notion of block-rank-one projections depends on the choice of the decomposition of X as a coarse disjoint union of finite metric spaces $X = \bigsqcup_{n\in\mathbb{N}} X_n$, not only on the bijectively coarse equivalence type of X (see [25, Remark 3.5] for details).

The second step is to show that a block-rank-one (ghost) projection P on a sparse space $\bigsqcup_{n\in\mathbb{N}}(X_n,d_n)$ belongs to the Roe algebra $C^*(X)$ if and only if $\{(X_n,d_n,m_n)\}_{n\in\mathbb{N}}$ forms a (ghostly) sequence of measured asymptotic expanders, where each m_n denotes the associated probability measure on X_n. To this end, we prove the following theorem, which extends [24, Theorem 6.1] from asymptotic expanders to the measured case:

Theorem B (Proposition 4.8, Theorem 6.1 and Theorem 6.6)**.** *Let $(X,d) = \bigsqcup_{n\in\mathbb{N}}(X_n,d_n)$ be a sparse space with bounded geometry and $\mathcal{H}_0$ be a Hilbert space. Let $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ be a block-rank-one projection with respect to $\{(X_n,d_n)\}_{n\in\mathbb{N}}$ and m_n be the associated probability measure on X_n. Then the following hold:*

- *If $\mathcal{H}_0 = \ell^2(\mathbb{N})$, then P belongs to the Roe algebra $C^*(X)$ if and only if $\{(X_n,d_n,m_n)\}_{n\in\mathbb{N}}$ forms a sequence of measured asymptotic expanders.*
- *If $\mathcal{H}_0 = \mathbb{C}$, then P belongs to the uniform Roe algebra $C^*_u(X)$ if and only if $\{(X_n,d_n,m_n)\}_{n\in\mathbb{N}}$ forms a sequence of measured asymptotic expanders.*

Theorem B is the fundamental theorem of this paper, and its proof is rather technical. To briefly summarise, we start with a structure theorem inspired by [24, Theorem 3.5] that measured asymptotic expanders admit a uniform exhaustion by measured expander graphs with bounded measure ratios (see Corollary 4.21)[6]. Next, we show that certain Laplacians associated to measured expanders with bounded measure ratios have spectral gap (see Proposition 5.4). Unfortunately, the measures involved in measured expanders may not come from reversible random walks on graphs, hence we cannot directly follow the classical argument from the theory of expanders to obtain the required spectral gap. We overcome this issue by building auxiliary random walks, whose stationary measures uniformly control the original measures. Such control is guaranteed when we deal

[5]See Definition 2.11 for general ghost operators.

[6]A dynamical counterpart of structure theorem can be found in [26, 27].

with measured expanders with bounded measure ratios. We refer the reader to Lemma 5.2 for the precise statement.

As outlined above, Theorem A can be deduced from our fundamental theorem (Theorem B). Additionally, Theorem B has two further substantial consequences. The first one is to provide both analytic and geometric characterisations for our rigidity condition. Namely, we have the following corollary (see also [10, Problem 8.6]):

Corollary C (Corollary 6.12)**.** *Let X be a metric space with bounded geometry. Then the following coarse properties are equivalent:*

(1) all sparse subspaces of X contain no block-rank-one ghost projections in their Roe algebras;
(2) all sparse subspaces of X contain no block-rank-one ghost projections in their uniform Roe algebras;
(3) X contains no sparse subspaces consisting of ghostly measured asymptotic expanders.
(4) X coarsely contains no sparse spaces consisting of ghostly measured asymptotic expanders with uniformly bounded geometry.

Secondly, Theorem B possibly gives rise to a new source of counterexamples to the coarse Baum-Connes conjecture. More precisely, we obtain a measured analogue of [24, Theorem D]:

Corollary D (Corollary 6.11)**.** *Let X be a sparse space consisting of ghostly measured asymptotic expanders. If X has bounded geometry and admits a fibred coarse embedding into a Hilbert space, then it does not satisfy the coarse Baum-Connes conjecture.*

The remainder of the paper concentrates on the study of measured (asymptotic) expanders in order to provide sufficient conditions and examples for Corollary C (and thus also Theorem A). In Section 5.2, we obtain an L^p-Poincaré inequality for measured expanders (see Corollary 5.3). Using the structure theorem for measured asymptotic expanders (Corollary 4.21), we conclude that any sparse space consisting of ghostly measured asymptotic expanders with uniformly bounded geometry cannot coarsely embed into any L^p-space for $p \in [1, \infty)$ (see Corollary 7.2). Thus, we obtain the following corollary of Theorem A:

Corollary E. *Let X and Y be metric spaces with bounded geometry such that Y coarsely embeds into some L^p-space for $p \in [1, \infty)$. Then the following are equivalent:*

(1) X is coarsely equivalent to Y;
(2) $C^*_u(X)$ is stably $*$-isomorphic to $C^*_u(Y)$;
(3) $C^*(X)$ is $*$-isomorphic to $C^*(Y)$.

Apart from Corollary E, we provide concrete examples of spaces which *are* rigid, but *do not* coarsely embed into any L^p-space. These examples are *not* covered by previously existing results at least for Roe algebras.

To this end, we come up with the concept of a *measured weak embedding* (see Definition 2.6) and extend the main results in [4, 16] to the measured setting. More precisely, the structure result (Corollary 4.21) together with the L^p-Poincaré inequality for measured expanders (Corollary 5.3) leads to Proposition 7.3, which is a measured analogue of [4, Proposition 2]. By virtue of Proposition 7.3, the same idea in [4, 16] can be easily adapted to show the following theorem:

Theorem F (Theorem 7.5 and Theorem 7.7). *There exists a box space X of a finitely generated residually finite group Γ such that X does not coarsely embed into any L^p-space for $1 \leq p < \infty$, but X does not measured weakly contain any measured asymptotic expanders with uniformly bounded geometry. In fact, the group Γ may also be chosen to be the free group F_3 of rank three.*

As the box space in Theorem F obviously cannot contain sparse subspaces consisting of ghostly measured asymptotic expanders (see Lemma 2.8), we can apply Theorem A to produce new rigid spaces:

Corollary G. *There exist box spaces with bounded geometry that do not coarsely embed into any L^p-space for $1 \leq p < \infty$, but affirmatively answer Problem 1.1.*

In [5, Thereom 1.2 and Thereom 1.3], Arzhantseva and Tessera constructed two examples of finitely generated groups as split extensions of two groups coarsely embeddable into Hilbert space but which themselves do not coarsely embed into Hilbert space. Hence Problem 1.1 also has a positive answer for these two finitely generated groups, as they contain no sparse subspaces consisting of ghostly measured asymptotic expanders by Proposition 7.3. However, the rigidity for these groups is *not* new at all and has already been verified in [8]. Indeed, we simply apply [8, Theorem 1.5] because these two groups satisfy the coarse Baum-Connes conjecture *with coefficients* (see [8, Remark 2.10 and Proposition 2.11]).

We end the introduction by asking the following two questions:

Question 1.5. Are ghostly measured asymptotic expanders obstacles to Problem 1.1 (2)?

Question 1.6. In the last decade, a theory of "high dimensional expanders" has begun to emerge (see [30] for a detailed survey). Therefore, it is natural to ask: can we define *"measured asymptotic expanders in high dimensions"* such that the corresponding Theorem B holds for all block-rank-n projections for $n \in \mathbb{N}$?

Convention. Throughout the paper, all metric spaces are *non-empty and discrete*, and graphs are *connected and undirected*.

2. Preliminaries

In this section, we introduce notation for metric spaces, and recall several basic definitions and results from coarse geometry.

2.1. Metric spaces and expander graphs.

Let (X, d) be a metric space, $x \in X$ and $R > 0$. Then $B(x, R)$ denotes the *closed ball* with radius R and centre at x. For any subset $A \subseteq X$, $|A|$ denotes the *cardinality* of A; $\mathrm{diam}(A)$ denotes the *diameter* of A; $\mathcal{N}_R(A) = \{x \in X : d(x, A) \leq R\}$ denotes the *R-neighbourhood of A*; and $\partial_R A = \{x \in X \backslash A : d(x, A) \leq R\}$ denotes the *(outer) R-boundary of A*. For a subspace $Y \subseteq X$ and a subset $A \subseteq Y$, we may also consider the *relative* neighbourhood and boundary defined as $\mathcal{N}_R^Y(A) = \mathcal{N}_R(A) \cap Y$ and $\partial_R^Y A = (\partial_R A) \cap Y$.

Moreover, we say that a metric space (X, d) has *bounded geometry* if $\sup_{x \in X} |B(x, R)|$ is finite for each $R > 0$. Any metric space with bounded geometry is automatically countable and discrete. More generally, a sequence of metric spaces $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ has *uniformly bounded geometry* if for all $R > 0$, the number $N_R := \sup_{n \in \mathbb{N}} \sup_{x \in X_n} |B(x, R)|$ is finite.

In this paper, we are particularly interested in a certain type of metric spaces arising from graphs, the so-called expander graphs.

For a (connected) graph $\mathcal{G} = (V, E)$ with the vertex set V and the edge set E, we endow V with the *edge-path* metric, which is defined to be the length (*i.e.*, the number of edges) in a shortest path connecting given two vertices. For $A \subseteq V$, $\partial^V A$ (or just ∂A) denotes its 1-boundary, which is also called the *vertex boundary* of A, and its *edge boundary* $\partial^E A$ is defined to be the set of all edges in E with one endpoint in A and the other one in $V \setminus A$.

We say that two vertices v and w in V are *adjacent* if there is an edge in E connecting them (in symbols $v \sim_E w$ or just $v \sim w$). For a vertex $v \in V$, its *valency* is defined to be the number of vertices adjacent to v. It is clear that the edge-path metric on V has bounded geometry if and only if the graph has *bounded valency* in the sense that there exists some $K \geq 0$ such that each vertex has valency bounded by K. More generally, a sequence of graphs $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is said to have *uniformly bounded valency* if there exists some $K \geq 0$ such that each vertex in $\mathcal{G}_n$ has valency bounded by K for all n. Hence, a sequence of graphs has uniformly bounded valency if and only if it has uniformly bounded geometry with respect to the edge-path metrics.

For a finite graph $\mathcal{G} = (V, E)$, its *Cheeger constant* is defined to be

$$c(\mathcal{G}) := \min\Big\{\frac{|\partial A|}{|A|} : A \subseteq V,\ 0 < |A| \leq \frac{|V|}{2}\Big\},$$

where ∂A is the vertex boundary of A. Now we recall the definition of expander graphs, which are highly connected but sparse at the same time.

Definition 2.1. A sequence of finite (connected)[7] graphs $\{\mathcal{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ is called a sequence of *expander graphs* if it has uniformly bounded valency, $|V_n| \to \infty$ as $n \to \infty$, and $\inf_{n\in\mathbb{N}} c(\mathcal{G}_n) > 0$.

We can always make a sequence of finite metric spaces into a single metric space, which is sparse in the sense of Definition 1.3:

Definition 2.2. Let $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ be a sequence of finite metric spaces. A *coarse disjoint union* of $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ is a metric space (X, d), where $X = \bigsqcup_{n\in\mathbb{N}} X_n$ is the disjoint union of $\{X_n\}$ as a set and d is a metric on X such that

- the restriction of d on each X_n coincides with d_n;
- $d(X_n, X_m) \to \infty$ as $n + m \to \infty$ and $n \neq m$.

If we need a precise choice of d, we can choose d as follows: $d(x, y) = n + m + \mathrm{diam}(X_n) + \mathrm{diam}(X_m)$ for all $x \in X_n$ and $y \in X_m$, whenever m and n are distinct natural numbers. Moreover, any two of such metrics are coarsely equivalent (see Section 2.2 for the precise meaning).

Clearly, a sequence of finite metric spaces $\{X_n\}_{n\in\mathbb{N}}$ has uniformly bounded geometry *if and only if* its coarse disjoint union has bounded geometry.

[7]We note that the expanding condition already implies that all the graphs $\mathcal{G}_n$ in an expander sequence are connected.

2.2. **Metric embeddings.** Given two metric spaces (X, d_X) and (Y, d_Y), a map $f : X \to Y$ is called a *coarse embedding* if there exist non-decreasing unbounded functions $\rho_\pm : [0, \infty) \to [0, \infty)$ such that

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$

for any $x, y \in X$. In this case, we simply say that Y *coarsely contains* X. If $f : X \to Y$ is a coarse embedding and $\mathcal{N}_C(f(X)) = Y$ for some constant $C > 0$, then we say that f is a *coarse equivalence* between X and Y. In this case, we also say that X and Y are *coarsely equivalent*.

More generally, let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of metric spaces and let Y be another metric space. A sequence of maps $f_n : X_n \to Y$ is called a *coarse embedding of* $\{X_n\}_{n\in\mathbb{N}}$ *into* Y if all f_n are coarse embeddings with the same control functions $\rho_\pm$. One can easily check that a sequence of *finite* metric spaces admits a coarse embedding into a non-trivial Banach space Y if and only if its coarse disjoint union as a single metric space admits a coarse embedding into Y.

Following an idea in [20][8], we say that a sequence of finite graphs $\{X_n\}_{n\in\mathbb{N}}$ admits a *weak embedding* into a metric space Y if there exist $L > 0$ and L-Lipschitz maps $f_n \colon X_n \to Y$ such that for every $R > 0$,

$$\lim_{n\to\infty} \sup_{x\in X_n} \frac{|f_n^{-1}(B(f_n(x), R))|}{|X_n|} = 0. \tag{2.1}$$

In this case, we also say that Y *weakly contains* the sequence $\{X_n\}_{n\in\mathbb{N}}$.

Remark 2.3. If the target space Y has bounded geometry, then (2.1) is equivalent to the following (see [21, Definition 5.2]):

$$\lim_{n\to\infty} \sup_{x\in X_n} \frac{|f_n^{-1}(f_n(x))|}{|X_n|} = 0.$$

Thus, a coarse embedding of a sequence of finite graphs $\{X_n\}_{n\in\mathbb{N}}$ with uniformly bounded valency into a metric space Y is a weak embedding provided that $|X_n| \to \infty$ as $n \to \infty$.

For the purpose of this paper, we would also like to consider measured weak embeddings of ghostly sequences of finite measured metric spaces.

Definition 2.4. We say that $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of *finite measured metric spaces* if each (X_n, d_n) is a finite metric space, and each m_n is a non-trivial and finite measure defined on the σ-algebra of all subsets of X_n. Moreover, the sequence $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is called *ghostly* if

$$\lim_{n\to\infty} \sup_{x\in X_n} \frac{m_n(x)}{m_n(X_n)} = 0. \tag{2.2}$$

[8]Here we refer to the technical need in [20] to use something weaker than quasi–isometric embeddings of expanders into groups in his construction of finitely generated groups that do not coarsely embed into a Hilbert space: compare condition (pr_1) on page 139, "Remark" on page 140, the italicised paragraph and the discussion following it on page 141, and item (2) on page 142 of [20]. The notion of a weak embedding in the present form was formalised in [22, Section 7]; see also [21, Definition 5.2].

Remark 2.5. If a sequence of finite measured metric spaces $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is ghostly, then $|\mathrm{supp}(m_n)| \to \infty$ as $n \to \infty$. However, it is clear that the converse may fail in general. Moreover, if each m_n is the counting measure on X_n then $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is ghostly if and only if $|X_n| \to \infty$ as $n \to \infty$.

Definition 2.6. We say that a sequence of finite measured metric spaces $\{(X_n, d_n, m_n)\}_n$ admits a *measured weak embedding* into a metric space Y if there exists a sequence of maps $f_n\colon X_n \to Y$ such that

(1) the sequence $\{f_n\}_{n\in\mathbb{N}}$ is *coarse*[9], *i.e.*, there exists a non-decreasing unbounded function $\rho_+ : [0,\infty) \to [0,\infty)$ such that for every $n \in \mathbb{N}$ and $x, y \in X_n$, we have $d_Y(f_n(x), f_n(y)) \le \rho_+(d_{X_n}(x, y))$;
(2) for every $R > 0$,

$$\lim_{n\to\infty} \sup_{x\in X_n} \frac{m_n(f_n^{-1}(B(f_n(x), R)))}{m_n(X_n)} = 0. \tag{2.3}$$

In this case, we also say that Y *measured weakly contains* the sequence $\{(X_n, d_n, m_n)\}_n$.

Remark 2.7. When all X_n in Definition 2.6 are finite graphs, condition (1) is equivalent to that all maps $f_n : X_n \to Y$ are L-Lipschitz for some $L > 0$. Moreover, if each m_n is the counting measure on X_n, then we recover the usual notion of weak embedding.

Similar to Remark 2.3, if the target space Y has bounded geometry then (2.3) is equivalent to the following:

$$\lim_{n\to\infty} \sup_{x\in X_n} \frac{m_n(f_n^{-1}(f_n(x)))}{m_n(X_n)} = 0.$$

At this point, the reader might want to compare this expression with (2.2).

Recall that a map $f : X \to Y$ between two metric spaces is called *finite-to-one* if the preimage $f^{-1}(y)$ is finite for every $y \in Y$. Moreover, a sequence of maps $\{f_n : X_n \to Y\}_{n\in\mathbb{N}}$ between metric spaces is called *uniformly finite-to-one* if there exists some $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and any $y \in Y$ we have $|f_n^{-1}(y)| \le N$. In particular, we say that a map $f : X \to Y$ between two metric spaces is *uniformly finite-to-one* if the constant sequence $\{f_n = f\}_{n\in\mathbb{N}}$ is uniformly finite-to-one. Notice that if X has bounded geometry, then any coarse embedding $f : X \to Y$ is uniformly finite-to-one.

The proof of the following lemma is straightforward from the definitions, thus omitted.

Lemma 2.8. *Let $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ be a ghostly sequence of finite measured metric spaces, and Y be a metric space. Then the following hold:*

- *If $\{(X_n, d_n)\}_n$ has uniformly bounded geometry, then every coarse embedding of the sequence $\{(X_n, d_n, m_n)\}_n$ into Y is a measured weak embedding.*
- *If Y has bounded geometry, then every uniformly finite-to-one and coarse sequence of maps from $\{(X_n, d_n, m_n)\}_n$ into Y is a measured weak embedding.*

[9]We follow the terminology from [4]; other sources use *bornologous* or *uniformly expansive* for "coarse", and further *uniformly coarse* or *equi–coarse* when used for families for maps.

2.3. **Rigidity for Roe-like algebras.** For a Hilbert space $\mathcal{H}$, we denote the closed unit ball of $\mathcal{H}$ by $(\mathcal{H})_1$. Moreover, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{K}(\mathcal{H})$ denote the spaces of bounded and compact operators on $\mathcal{H}$, respectively.

For a discrete metric space (X, d), we denote by χ_A the characteristic function on a subset $A \subseteq X$ and $\delta_x = \chi_{\{x\}}$ for $x \in X$. Fixing a Hilbert space $\mathcal{H}_0$, we consider the Hilbert space $\ell^2(X;\mathcal{H}_0) \cong \ell^2(X)\otimes\mathcal{H}_0$. When $\mathcal{H}_0 = \mathbb{C}$, we also write $\ell^2(X) = \ell^2(X;\mathbb{C})$. There is a natural $*$-representation $\ell^\infty(X) \to \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ induced by the canonical multiplication representation of $\ell^\infty(X)$ on $\ell^2(X)$.

We can always regard any operator $T \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ as an X-by-X matrix $(T_{x,y})_{x,y\in X}$ with entries in $\mathfrak{B}(\mathcal{H}_0)$. More precisely, $T_{x,y} \in \mathfrak{B}(\mathcal{H}_0)$ is defined by

$$T_{x,y}\xi = \Big(T(\delta_y \otimes \xi)\Big)(x)$$

for $\xi \in \mathcal{H}_0$ and $x, y \in X$. (Here $\delta_y \otimes \xi \in \ell^2(X) \otimes \mathcal{H}_0 \cong \ell^2(X;\mathcal{H}_0)$ represents the function from X to $\mathcal{H}_0$ taking the value ξ at y and 0 otherwise.) The *propagation of* T is defined to be

$$\mathrm{ppg}(T) := \sup\{d(x, y) : T_{x,y} \neq 0\}.$$

We say that $T \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ has *finite propagation* if $\mathrm{ppg}(T)$ is finite, and T is *locally compact* if $T_{x,y} \in \mathfrak{K}(\mathcal{H}_0)$ for all $x, y \in X$.

In this paper, we will deal with the rigidity problem for the following variants of Roe algebras:

Definition 2.9. Let X be a metric space with bounded geometry and $\mathcal{H}_0$ be an infinite-dimensional separable Hilbert space.

(1) The *Roe algebra of* X, denoted by $C^*(X)$, is defined to be the norm closure of all finite propagation locally compact operators in $\mathfrak{B}(\ell^2(X;\mathcal{H}_0))$.
(2) The *uniform algebra of* X, denoted by $UC^*(X)$, is defined to be the norm closure of all finite propagation operators $T \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ such that there exists some $N \in \mathbb{N}$ satisfying $\mathrm{rank}(T_{x,y}) \leq N$ for all $x, y \in X$.
(3) The *stable Roe algebra of* X, denoted by $C^*_s(X)$, is defined to be the norm closure of all finite propagation operators $T \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ such that there exists a finite-dimensional subspace $H \subseteq \mathcal{H}_0$ satisfying $T_{x,y} \in \mathfrak{B}(H)$ for all $x, y \in X$.
(4) The *uniform Roe algebra of* X, denoted by $C^*_u(X)$, is defined to be the norm closure of all finite propagation operators $T \in \mathfrak{B}(\ell^2(X))$.

It is known that $C^*_u(X) \hookrightarrow C^*_s(X) \subseteq UC^*(X) \subseteq C^*(X)$, where the last two inclusions are canonical. Moreover, we have $C^*_u(X)\otimes\mathfrak{K}(\mathcal{H}_0) \cong C^*_s(X)$. We also use the following simplified terminology:

Definition 2.10 ([8, Definition 2.3], but see also [9])**.** Let X be a metric space with bounded geometry. A C^*-subalgebra $A \subseteq C^*(X)$ is called *Roe-like* if $C^*_s(X) \subseteq A$.

The rigidity for Roe-like algebras has already been extensively studied in the literature (*e.g.*, [9, 10, 8, 39]). In order to formulate the most up-to-date rigidity result, we need to recall several concepts:

Definition 2.11 (G. Yu[10])**.** Let X be a discrete metric space, $\mathcal{H}_0$ be a Hilbert space and $T \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$. We say that T is a *ghost* if for any $\varepsilon > 0$, there exists a bounded subset $F \subseteq X$ such that for any $(x, y) \notin F \times F$ we have $\|T_{x,y}\| < \varepsilon$.

[10] Attributed to G. Yu in [36, 11.5.2].

The following lemma clarifies Definition 1.4 and Definition 2.4. Since its proof is elementary, we leave the details to the reader:

Lemma 2.12. *Let $(X,d) = \bigsqcup_{n\in\mathbb{N}}(X_n, d_n)$ be a sparse space and $\mathcal{H}_0$ be a Hilbert space. If $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ is a block-rank-one projection with respect to $\{(X_n,d_n)\}_{n\in\mathbb{N}}$, then P is a ghost* if and only if *the associated sequence of finite measured metric spaces $\{(X_n,d_n,m_n)\}_n$ is ghostly.*

Definition 2.13. Let X be a metric space with bounded geometry. We say that *all sparse subspaces of X yield only compact ghost projections in their Roe algebras* if for any sparse subspace $X' \subseteq X$, all ghost projections in $C^*(X')$ are compact.

Theorem 2.14 ([8, Theorem 1.3])**.** [11] *Let X and Y be metric spaces with bounded geometry. If all sparse subspaces of Y yield only compact ghost projections in their Roe algebras, then the following are equivalent:*

(1) X is coarsely equivalent to Y;
(2) $C_u^*(X)$ is stably $*$-isomorphic to $C_u^*(Y)$;
(3) $C_s^*(X)$ is $*$-isomorphic to $C_s^*(Y)$;
(4) $UC^*(X)$ is $*$-isomorphic to $UC^*(Y)$;
(5) $C^*(X)$ is $*$-isomorphic to $C^*(Y)$.

As shown in [8, Theorem 5.3], all sparse subspaces of a metric space Y yield only compact ghost projections in their Roe algebras if Y satisfies various forms of the Baum-Connes conjecture. In particular, rigidity holds for those spaces.

3. Rigidity

In this section, we introduce new analytic conditions which guarantee rigidity for Roe-like algebras. More precisely, we define the following:

Definition 3.1. Let X be a metric space with bounded geometry. We say that *all sparse subspaces of X contain no block-rank-one ghost projections in their Roe algebras (or in their uniform Roe algebras)* if $C^*(X')$ (or $C_u^*(X')$) contains no block-rank-one ghost projections for every sparse subspace $X' \subseteq X$.

In the remainder of this section, we will follow an approach as in [8] to show that the above new analytic condition concerning Roe algebras in Definition 3.1 is already sufficient for rigidity. First of all, we need the following lemma which is a combination of [9, Theorem 7.4] and [8, Lemma 3.1].

Lemma 3.2. *Let (Y,d) be a metric space with bounded geometry, and assume that all sparse subspaces of Y contain no block-rank-one ghost projections in their Roe algebras. Let $(p_n)_n$ be an orthogonal sequence of rank-one projections such that $\sum_{n\in M} p_n$ converges in the strong operator topology to an element in $C^*(Y)$ for all $M \subseteq \mathbb{N}$. Then*

$$\inf_{n\in\mathbb{N}} \sup\left\{\|p_n\delta_y \otimes v\| : y \in Y,\ v \in (\mathcal{H}_0)_1\right\} > 0.$$

Proof. The proof is almost the same as the one for [8, Lemma 3.1], except that we need the following claim instead of Claim 3.3 therein:

[11] We also refer the reader to [9, Corollary 1.3].

Claim. By going to a subsequence of $(p_n)_n$, there exists a sequence $(Y_n)_n$ of disjoint finite non-empty subsets of Y and a sequence of rank-one projections $(q_n)_n$ in $C^*(Y)$ such that

(1) $d(Y_k, Y_m) \to \infty$ as $k + m \to \infty$ and $k \neq m$;
(2) $\|p_n - q_n\| < 2^{-n}$.

The proof of this claim is very similar to Claim 3.3 in [8, Lemma 3.1], except that we require each q_n to be rank-one rather than finite rank. But this follows automatically from the functional calculus construction $q_n = f(a)$ therein, where a is a rank-one self-adjoint operator and f is a continuous function on the spectrum of a with $f(0) = 0$. In particular, the rank of q_n is bounded by one and exactly equals to one by (2). For more details we refer the reader to [8, Lemma 3.1]. □

As a direct consequence of Lemma 3.2, we obtain the following corollary (*cf.* [8, Corollary 3.5]):

Corollary 3.3. *Let X and Y be metric spaces with bounded geometry, and assume that all sparse subspaces of Y contain no block-rank-one ghost projections in their Roe algebras. If $A \subseteq C^*(X)$ is a Roe-like C^*-subalgebra, then every strongly continuous rank-preserving $*$-homomorphism $\Phi : A \to C^*(Y)$ is a* rigid $*$-homomorphism *in the following sense:*

$$\sup_{u\in(\mathcal{H}_0)_1} \inf_{x\in X} \sup_{y\in Y,\ v\in(\mathcal{H}_0)_1} \|\Phi(e_{(x,u),(x,u)})\delta_y \otimes v\| > 0.$$

The next proposition slightly extends [8, Theorem 4.5].

Proposition 3.4. *Let X and Y be metric spaces with bounded geometry, and assume that all sparse subspaces of Y contain no block-rank-one ghost projections in their Roe algebras. Let $A \subseteq C^*(X)$ and $B \subseteq C^*(Y)$ be Roe-like C^*-algebras such that either $B = C^*_s(Y)$ or $\mathrm{UC}^*(Y) \subseteq B$. If A embeds onto a hereditary C^*-subalgebra of B, then X coarsely embeds into Y.*

Proof. Let $\Phi : A \to B$ be an embedding onto a hereditary C^*-subalgebra of B. By [8, Lemma 4.1], Φ is strongly continuous and rank-preserving. By Corollary 3.3, Φ is a rigid $*$-homomorphism. It follows from [8, Lemma 4.2 and Lemma 4.4] that Φ induces a coarse embedding from X into Y (see also the proof of [8, Theorem 4.5] for more details). □

We also need the following key proposition:

Proposition 3.5. *Let $f : (X, d_X) \to (Y, d_Y)$ be a finite-to-one coarse map between metric spaces with bounded geometry and $\mathcal{H}_0$ be an infinite-dimensional separable Hilbert space. Assume that there exist a sparse subspace $X_0 = \bigsqcup_{n\in\mathbb{N}} X_n$ of X and a block-rank-one projection $P = \bigoplus_{n\in\mathbb{N}} P_n \in \mathfrak{B}(\ell^2(X_0; \mathcal{H}_0))$ with respect to $\{X_n\}_{n\in\mathbb{N}}$ such that $P \in C^*(X_0)$.*

Let m_n be the associated probability measure on X_n given by $m_n(x) = \|P_n\delta_x\|^2$ for $x \in X_n$. Denote $f_n := f|_{X_n}$ and $d_n := d_X|_{X_n}$. If the sequence $\{f_n : (X_n, d_n, m_n) \to Y\}_{n\in\mathbb{N}}$ is a measured weak embedding, then there exist a sparse subspace $Y' \subseteq Y$ and a block-rank-one ghost projection in $C^*(Y')$.

Remark 3.6. We do not know whether the conclusion holds also in the case when $\mathcal{H}_0$ is finite-dimensional.

Proof. Since the map f is finite-to-one and Y has bounded geometry, it is easy to show that there is a subsequence $\{X_n\}_{n\in M}$ such that $d_Y(f(X_n), f(X_m)) \to \infty$ as $n+m \to \infty$ for $n \neq m$, where M is an infinite subset of $\mathbb{N}$. If we take $Y_n := f(X_n)$ for each $n \in M$ and $Y' := \bigsqcup_{n\in M} Y_n$ to be their disjoint union, then Y' is a sparse subspace of Y. We consider the sparse subspace $X' := \bigsqcup_{n\in M} X_n$ and the block-rank-one projection $P' := \bigoplus_{n\in M} P_n \in \mathfrak{B}(\ell^2(X'; \mathcal{H}_0))$. Since $P' = \chi_{X'} P \chi_{X'}$ and $\chi_{X'}$ has finite propagation, we deduce that $P' \in C^*(X')$.

Let $\mathcal{H}_{X'} := \ell^2(X'; \mathcal{H}_0) \cong \ell^2(X') \otimes \mathcal{H}_0$ and $\mathcal{H}_{Y'} := \ell^2(Y'; \mathcal{H}_{X'}) \cong \ell^2(Y') \otimes \ell^2(X') \otimes \mathcal{H}_0$. Similarly to the proof of [24, Proposition 6.4], we construct an isometry $V : \mathcal{H}_{X'} \to \mathcal{H}_{Y'}$ covering the coarse map $f' := f|_{X'} : X' \to Y'$ in the sense that $\operatorname{supp}(V) \subseteq \{(f'(x), x) \in Y' \times X' \mid x \in X'\}$, where $\operatorname{supp}(V) := \{(y,x) \in Y' \times X' \mid V_{y,x} \neq 0\}$.

Indeed, for each $y \in Y'$ we define an isometry $V_y : \ell^2(f'^{-1}(y)) \otimes \mathcal{H}_0 \to \mathbb{C}\delta_y \otimes \ell^2(X') \otimes \mathcal{H}_0$ by the formula

$$\delta_z \otimes \xi \mapsto \delta_{f'(z)} \otimes \delta_z \otimes \xi = \delta_y \otimes \delta_z \otimes \xi, \text{ for } z \in f'^{-1}(y) \text{ and } \xi \in \mathcal{H}_0.$$

Since $\mathcal{H}_{X'} = \bigoplus_{y\in Y'} \left(\ell^2(f'^{-1}(y)) \otimes \mathcal{H}_0\right)$ and $\mathcal{H}_{Y'} = \bigoplus_{y\in Y'} \left(\mathbb{C}\delta_y \otimes \ell^2(X') \otimes \mathcal{H}_0\right)$, we can define $V := \bigoplus_{y\in Y'} V_y : \mathcal{H}_{X'} \longrightarrow \mathcal{H}_{Y'}$. By the construction of V, we clearly have that $\operatorname{supp}(V) \subseteq \{(f'(x), x) \in Y' \times X' \mid x \in X'\}$.

As V is a covering isometry for the coarse map f', it follows that the isometry V induces a $*$-homomorphism $\mathrm{Ad}_V : C^*(X') \to C^*(Y')$ given by $T \mapsto VTV^*$ (see also [42, Lemma 5.1.12 and Remark 5.1.13]). Hence, $Q := VP'V^*$ is a projection in the Roe algebra $C^*(Y')$. As $f' = \bigsqcup_{n\in M} f_n$, we have that $V = \bigoplus_{n\in M} V_n$ where

$$V_n := \bigoplus_{y\in Y_n} V_y : \ell^2(X_n; \mathcal{H}_0) \to \ell^2(Y_n; \mathcal{H}_{X'}).$$

Thus $Q_n := V_n P_n V_n^*$ is a rank-one projection in $\mathfrak{B}(\ell^2(Y_n; \mathcal{H}_{X'}))$ and $Q = \bigoplus_{n\in M} Q_n$ is block-rank-one with respect to $\{Y_n\}_{n\in M}$.

Finally, we verify that Q is a ghost. A direct calculation shows that for any $y, z \in Y_n$, we have that $(Q_n)_{y,z} = \chi_{f_n^{-1}(y)} P_n \chi_{f_n^{-1}(z)}$. Moreover,

$$\|(Q_n)_{y,z}\| = \sqrt{m_n(f_n^{-1}(y)) \cdot m_n(f_n^{-1}(z))}. \tag{3.1}$$

Thus, it follows easily that Q is a ghost as $\{f_n : (X_n, d_n, m_n) \to Y\}_{n\in M}$ forms a measured weak embedding. □

Combining Proposition 3.5 with Lemma 2.8 and Lemma 2.12, we obtain the following (*cf.* [10, Theorem 7.6 and Remark 7.8]):

Corollary 3.7. *Let X and Y be metric spaces with bounded geometry, and let $f : X \to Y$ be a uniformly finite-to-one coarse map. If all sparse subspaces of Y contain no block-rank-one ghost projections in their Roe algebras, then the same holds for X.*

As every coarse embedding from a metric space with bounded geometry into another metric space is uniformly finite-to-one, Proposition 3.4 and Corollary 3.7 together give rise to the following corollary (*cf.* [8, Corollary 4.6]):

Corollary 3.8. *Let X and Y be metric spaces with bounded geometry, and assume that all sparse subspaces of Y contain no block-rank-one ghost projections in their Roe algebras. Let $A \subseteq C^*(X)$ and $B \subseteq C^*(Y)$ be Roe-like C^*-algebras such that either $B = C^*_s(Y)$ or*

$UC^*(Y) \subseteq B$. If A embeds onto a hereditary C^*-subalgebra of B, then all sparse subspaces of X contain no block-rank-one ghost projections in their Roe algebras.

Finally, we reach the main result of this section.

Proposition 3.9. *Let X and Y be metric spaces with bounded geometry. Assume that all sparse subspaces of Y contain no block-rank-one ghost projections in their Roe algebras. Then the following are equivalent:*

(1) X is coarsely equivalent to Y;
(2) $C^*_u(X)$ is stably $*$-isomorphic to $C^*_u(Y)$;
(3) $C^*_s(X)$ is $*$-isomorphic to $C^*_s(Y)$;
(4) $UC^*(X)$ is $*$-isomorphic to $UC^*(Y)$;
(5) $C^*(X)$ is $*$-isomorphic to $C^*(Y)$.

Proof. We follow exactly the same proof of [8, Theorem 1.3] except we use Corollary 3.8 instead of [8, Corollary 4.6] and use Corollary 3.3 instead of [8, Corollary 3.5]. Hence, we decide not to repeat the proof word for word. □

Remark 3.10. For a metric space of bounded geometry, it is clear that if all sparse subspaces yield only compact ghost projections in their Roe algebras, then they also do not contain any block-rank-one ghost projection in their Roe algebras. Thus, we have formally generalised [8, Theorem 1.3], and all of the four cases in [8, Theorem 5.3] can be included in the setting of Proposition 3.9.

An advantage of using block-rank-one ghost projections (rather than arbitrary non-compact ghost projections) is that we have a geometric characterisation proved in Corollary 6.12, which formulates in terms of measured asymptotic expanders. More precisely, we will show that both conditions concerning Roe algebras and uniform Roe algebras in Definition 3.1 are equivalent to that X contains no sparse subspaces consisting of ghostly measured asymptotic expanders. Studying measured asymptotic expanders allows us to produce new examples of rigid spaces (see Section 7 for details).

4. Measured asymptotic expanders

In this section, we begin to characterise quasi-locality of a block-rank-one projection by means of measured asymptotic expanders, which were introduced in [27, Definition 6.1]. The precise statement can be found in Proposition 4.8, which generalises [25, Theorem 3.11] from counting measures to general probability measures. Afterwards, we also establish a structure theorem for measured asymptotic expanders (Corollary 4.21), which is our main technical tool to prove Theorem B.

4.1. **Quasi-locality for block-rank-one projections.**

In this subsection, we connect measured asymptotic expanders with the quasi-locality of block-rank-one projections. In [24, 25], we studied quasi-locality of averaging projections, which led to introducing asymptotic expanders, and also showed that an averaging projection is quasi-local *if and only if* it belongs to the uniform Roe algebra (see [24, Theorem 6.1]).

Recall that bounded operators which can be approximated by operators with finite propagation are always quasi-local in the following sense:

Definition 4.1. Let X be a discrete metric space, $\mathcal{H}_0$ be a Hilbert space and $T \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$. We say that T is *quasi-local*[12] if for any $\varepsilon > 0$ there exists some $R > 0$ such that for any $A, B \subseteq X$ with $d(A,B) > R$, we have $\|\chi_A T \chi_B\| < \varepsilon$.

Throughout this subsection, we fix a sequence of finite metric spaces $\{(X_n, d_n)\}_{n\in\mathbb{N}}$. Let (X,d) be their coarse disjoint union and $\mathcal{H}_0$ be a Hilbert space. Recall from Definition 1.4 that a block-rank-one projection with respect to $\{X_n\}_{n\in\mathbb{N}}$ is an infinite rank projection of the form

$$P = \bigoplus_{n\in\mathbb{N}} P_n \in \mathfrak{B}\left(\bigoplus_{n\in\mathbb{N}} \ell^2(X_n;\mathcal{H}_0)\right) = \mathfrak{B}(\ell^2(X;\mathcal{H}_0)),$$

where each P_n is a rank-one projection in $\mathfrak{B}(\ell^2(X_n;\mathcal{H}_0))$. Note that each P_n has the form $P_n(\eta) = \langle \eta, \xi_n\rangle \xi_n$ for all $\eta \in \ell^2(X_n;\mathcal{H}_0)$, where ξ_n is a unit vector in $\ell^2(X_n;\mathcal{H}_0)$. In this case, we also say that $P = \bigoplus_{n\in\mathbb{N}} P_n$ is the *block-rank-one projection associated to the unit vectors* $\{\xi_n\}_{n\in\mathbb{N}}$. Then the associated probability measure m_n on X_n is given by $m_n(x) = \|\xi_n(x)\|^2$.

Remark 4.2. When $\mathcal{H}_0 = \mathbb{C}$ and ξ_n is the unit vector $\frac{1}{\sqrt{|X_n|}}\chi_{X_n} \in \ell^2(X_n)$, the corresponding block-rank-one projection P is the so-called *averaging projection*. It has been well-studied in [24, 25], and the most significant result is the fact that P is quasi-local *if and only if* P belongs to the uniform Roe algebra $C_u^*(X)$ *if and only if* $\{X_n\}_{n\in\mathbb{N}}$ forms a sequence of asymptotic expanders in the sense of Definition 4.7 (see [24, Theorem 6.1]).

Similarly to [25, Lemma 3.8 and Proposition 3.9] it is straightforward to obtain the following two results:

Lemma 4.3. *For each $n \in \mathbb{N}$ and any $A, B \subseteq X_n$, we have*

$$\|\chi_A P_n \chi_B\| = \|\chi_A \xi_n\| \cdot \|\chi_B \xi_n\| = \sqrt{m_n(A)\cdot m_n(B)}.$$

Proposition 4.4. *A block-rank-one projection $P = \bigoplus_{n\in\mathbb{N}} P_n \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ with respect to $\{X_n\}_{n\in\mathbb{N}}$ is quasi-local* if and only if

$$0 = \lim_{R\to+\infty} \sup\Big\{m_n(A)\cdot m_n(B) : n \in \mathbb{N},\ A, B \subseteq X_n,\ d(A,B) \geq R\Big\}.$$

The following observations will allow us to reduce the proof of Theorem B to the case that all associated probability measures m_n have full support:

For each $n \in \mathbb{N}$, let $Z_n \subseteq X_n$ be the support of m_n and $Z = \bigsqcup_n Z_n$ be the sparse subspace of X. Let $Q_n : \ell^2(X_n;\mathcal{H}_0) \to \ell^2(Z_n;\mathcal{H}_0)$ be the orthogonal projection, and

$$Q := \bigoplus_{n\in\mathbb{N}} Q_n \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0), \ell^2(Z;\mathcal{H}_0)).$$

Note that each $Q_n P_n Q_n^* \in \mathfrak{B}(\ell^2(Z_n;\mathcal{H}_0))$ is the orthogonal projection onto the one-dimensional subspace spanned by $\xi_n|_{Z_n}$ in $\ell^2(Z_n;\mathcal{H}_0)$, and $Z_n = \mathrm{supp}(\xi_n)$. It follows that $QPQ^* \in \mathfrak{B}(\ell^2(Z;\mathcal{H}_0))$ is a block-rank-one projection with respect to $\{Z_n\}_{n\in\mathbb{N}}$. The following is obvious:

Lemma 4.5. *A block-rank-one projection $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ with respect to $\{X_n\}_{n\in\mathbb{N}}$ is quasi-local* if and only if $QPQ^* \in \mathfrak{B}(\ell^2(Z;\mathcal{H}_0))$ *is quasi-local.*

[12]The notion of quasi-locality was first introduced by Roe in [34, Part I, Section 5] and [35, Remark on page 20], and we refer readers to [25, 29, 38, 40] for more details.

On the other hand, we also have the following observation:

Lemma 4.6. *Let $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ be a block-rank-one projection with respect to $\{X_n\}_{n\in\mathbb{N}}$. Then the following hold:*

- *When $\mathcal{H}_0 = \mathbb{C}$, then P belongs to $C_u^*(X)$* if and only if *QPQ^* belongs to $C_u^*(Z)$.*
- *When $\mathcal{H}_0$ is a separable infinite-dimensional Hilbert space, then P belongs to $C^*(X)$* if and only if *QPQ^* belongs to $C^*(Z)$.*

Proof. This follows from the fact that Q can be also regarded as an element in $\mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ with propagation zero, and $P = Q^*(QPQ^*)Q$. □

When $\mathcal{H}_0 = \mathbb{C}$ and P is the averaging projection with respect to $\{X_n, d_n\}_{n\in\mathbb{N}}$, then we know from [25, Theorem B] that P is quasi-local if and only if $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ forms a sequence of asymptotic expanders in the following sense:

Definition 4.7 ([25])**.** A sequence of finite metric spaces $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ is called a sequence of *asymptotic expanders*[13] if for any $\alpha \in (0, \frac{1}{2}]$ there exist $c_\alpha \in (0,1)$ and $R_\alpha > 0$ such that for any $n \in \mathbb{N}$ and $A \subseteq X_n$ with $\alpha|X_n| \leq |A| \leq \frac{1}{2}|X_n|$, we have $|\partial_{R_\alpha} A| > c_\alpha |A|$.

The following result is one of the main motivations for us to study measured asymptotic expanders, and it is also the first step to attack the fundamental theorem (Theorem B). Since its proof is essentially a repetition of the arguments used to prove [25, Theorem 3.11], we omit further details.

Proposition 4.8. *Let (X,d) be a coarse disjoint union of a sequence of finite metric spaces $\{(X_n, d_n)\}_{n\in\mathbb{N}}$, and $P = \bigoplus_{n\in\mathbb{N}} P_n \in \mathfrak{B}(\ell^2(X))$ be a block-rank-one projection with respect to $\{X_n\}_{n\in\mathbb{N}}$. If m_n are the probability measures associated to P, then P is quasi-local* if and only if *$\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ forms a sequence of measured asymptotic expanders.*

For later purposes, we extend the notion of measured asymptotic expanders to general *finite measured metric spaces* defined in Definition 2.4 as follows:

Definition 4.9 ([27, Definition 6.1])**.** A sequence of finite measured metric spaces $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is called a sequence of *measured asymptotic expanders* if for any $\alpha \in (0, \frac{1}{2}]$ there exist $c_\alpha \in (0,1)$ and $R_\alpha > 0$ such that for any $n \in \mathbb{N}$ and $A \subseteq X_n$ with $\alpha \cdot m_n(X_n) \leq m_n(A) \leq \frac{1}{2} m_n(X_n)$, we have $m_n(\partial_{R_\alpha} A) > c_\alpha \cdot m_n(A)$.

In this case, we call functions $\underline{c} : \alpha \mapsto c_\alpha$ and $\underline{R} : \alpha \mapsto R_\alpha$ from $(0, \frac{1}{2}]$ to $(0, \infty)$ *parameter functions* of $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$, and $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is called a sequence of *measured $(\underline{c}, \underline{R})$-asymptotic expanders*.

Remark 4.10. The notion of asymptotic expanders introduced and studied in [24, 25] is exactly the notion of *ghostly* measured asymptotic expanders with respect to counting measures (see Remark 2.5).

Remark 4.11. Recall that in [27, Theorem 6.16], the authors established a connection between measured asymptotic expanders and asymptotic expansion in measure for continuous measure-class-preserving actions by means of measured approximating spaces. As a consequence, it provides an efficient and unified way to construct measured asymptotic expanders from strongly ergodic actions (see [27, Proposition 3.5]).

[13] In this paper, we do not require the condition $|X_n| \to \infty$ as $n \to \infty$ as a part of the definition.

We need three auxiliary lemmas in order to prove the structure theorem for measured asymptotic expanders in the next subsection. The first one allows us to handle subsets $A \subseteq X_n$ with $m_n(A) \geq \frac{1}{2} m_n(X_n)$. Although its proof is already given in [27], we still include it here for the sake of completeness.

Lemma 4.12. (*see* [27, Lemma 6.3]) *Let $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of measured $(\underline{c}, \underline{R})$-asymptotic expanders. Then for any $\beta \in [\frac{1}{2}, 1)$, there exist $\tilde{c} > 0$ and $\tilde{R} > 0$ depending only on the parameter functions $\underline{c}$ and $\underline{R}$ such that for every $n \in \mathbb{N}$ and $A \subseteq X_n$ with $\frac{1}{2} m_n(X_n) \leq m_n(A) \leq \beta \cdot m_n(X_n)$, we have $m_n(\partial_{\tilde{R}} A) > \tilde{c} \cdot m_n(A)$.*

Furthermore, if $\underline{c} \equiv c$ and $\underline{R} \equiv R$ are constant functions, we can choose $\tilde{R} = R$ and $\tilde{c} = \frac{1-\beta}{2\beta} c$.

Proof. Replacing m_n by $\frac{1}{m_n(X_n)} m_n$, we may assume that each m_n is a probability measure. Now we fix $\beta \in [\frac{1}{2}, 1)$ and set $\alpha' := \frac{1-\beta}{2} \in (0, \frac{1}{4}]$. By the hypothesis, there exist $c_{\alpha'} \in (0,1)$ and $R_{\alpha'} > 0$ such that for any $n \in \mathbb{N}$ and $A' \subseteq X_n$ with $\alpha' \leq m_n(A') \leq \frac{1}{2}$, we have $m_n(\partial_{R_{\alpha'}} A') > c_{\alpha'} \cdot m_n(A')$.

Given $A \subseteq X_n$ with $\frac{1}{2} \leq m_n(A) \leq \beta$, let $B := X_n \setminus (\mathcal{N}_{R_{\alpha'}}(A))$. Then $m_n(B) \leq m_n(X_n \setminus A) \leq \frac{1}{2}$. If $m_n(B) < \frac{1-\beta}{2}$, then $m_n(\mathcal{N}_{R_{\alpha'}}(A)) > \frac{1+\beta}{2} \geq \frac{1+\beta}{2\beta} m_n(A)$. Since $\mathcal{N}_{R_{\alpha'}}(A) = A \sqcup \partial_{R_{\alpha'}} A$, we deduce that $m_n(\partial_{R_{\alpha'}} A) > c_{\alpha'} \frac{1-\beta}{2\beta} m_n(A)$. On the other hand, if $m_n(B) \geq \frac{1-\beta}{2} = \alpha'$ then we have $m_n(\partial_{R_{\alpha'}} B) > c_{\alpha'} \cdot m_n(B)$. Since $\partial_{R_{\alpha'}} B \subseteq \partial_{R_{\alpha'}} A$, we conclude that

$$m_n(\partial_{R_{\alpha'}} A) \geq m_n(\partial_{R_{\alpha'}} B) > c_{\alpha'} \cdot m_n(B) \geq c_{\alpha'} \frac{1-\beta}{2\beta} m_n(A).$$

Thus, we have completed the proof. □

In the next lemma, we show that subspaces of measured asymptotic expanders are themselves measured asymptotic expanders in a uniform way provided that their measures are uniformly bounded below. Although elementary, this lemma plays a crucial role to prove the structure theorem for measured asymptotic expanders, and the corresponding lemma is *not* necessary in the case of asymptotic expanders as in [24].

Lemma 4.13. *Let $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of measured asymptotic expanders. For any $\beta \in (0,1]$, $\alpha \in (0, \frac{1}{2}]$ and $c \in (0,1)$ there exists $R > 0$ such that for any sequence $\{Y_n \subseteq X_n\}_{n\in\mathbb{N}}$ with $m_n(Y_n) \geq \beta m_n(X_n)$ and any $A \subseteq Y_n$ with $\alpha \cdot m_n(Y_n) \leq m_n(A) \leq \frac{1}{2} m_n(Y_n)$, we have $m_n(\partial_R^{Y_n} A) > c \cdot m_n(A)$.*

Proof. Replacing m_n by $\frac{1}{m_n(X_n)} m_n$, we may assume that each m_n is a probability measure. From Proposition 4.4 and Proposition 4.8, we know that for any $\varepsilon > 0$, there exists some $R_\varepsilon > 0$ such that for any $n \in \mathbb{N}$ and $A, B \subseteq X_n$ with $d(A,B) \geq R_\varepsilon$ we have $m_n(A) \cdot m_n(B) < \varepsilon$. In particular, it holds for $\varepsilon := \frac{\alpha\beta^2(1-c)}{2} > 0$.

Fix an arbitrary sequence $\{Y_n \subseteq X_n\}_{n\in\mathbb{N}}$ with $m_n(Y_n) \geq \beta$, and any $A \subseteq Y_n$ with $\alpha \cdot m_n(Y_n) \leq m_n(A) \leq \frac{1}{2} \cdot m_n(Y_n)$. Suppose that $m_n(\partial_{R_\varepsilon}^{Y_n} A) \leq c \cdot m_n(A)$. Then due to the decomposition

$$Y_n = (Y_n \setminus \mathcal{N}_{R_\varepsilon}^{Y_n}(A)) \sqcup A \sqcup \partial_{R_\varepsilon}^{Y_n} A,$$

we have that

$$m_n(Y_n \setminus \mathcal{N}^{Y_n}_{R_\varepsilon}(A)) = m_n(Y_n) - m_n(A) - m_n(\partial^{Y_n}_{R_\varepsilon} A) \geq \frac{1-c}{2} m_n(Y_n) \geq \frac{1-c}{2}\beta.$$

Hence, it follows that

$$m_n(A) \cdot m_n(Y_n \setminus \mathcal{N}^{Y_n}_{R_\varepsilon}(A)) \geq \alpha \cdot \beta \cdot \frac{1-c}{2}\beta = \varepsilon.$$

Since $d(A, Y_n \setminus \mathcal{N}^{Y_n}_{R_\varepsilon}(A)) \geq R_\epsilon$, we have reached a contradiction. □

We finish this subsection by showing the existence of subspaces whose measure is more "balanced" in the lemma below. Note that every counting measure is automatically "balanced" so that the next lemma is redundant in the study of asymptotic expanders. However, it is crucial for us to be able to apply Proposition 5.4 in proving our fundamental theorem (Theorem B).

In order to formulate the lemma in a concise way, we here introduce the following notation: *given two non-negative numbers $a, b \in [0, \infty)$, we denote $a \sim_s b$ for some $s \in (0, 1)$ if $sa \leq b \leq \frac{a}{s}$.*

Lemma 4.14. *Let $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of finite measured metric spaces with uniformly bounded geometry. Given $R > 0$ and $s \in (0, 1)$, there exists a subspace $X^{R,s}_n \subseteq X_n$ for each $n \in \mathbb{N}$ satisfying the following:*

(1) For any $x, y \in X^{R,s}_n$ with $d_n(x, y) \leq R$, we have $m_n(x) \sim_s m_n(y)$;
(2) $m_n(X_n \setminus X^{R,s}_n) \leq s \cdot N_R \cdot m_n(X^{R,s}_n)$, where $N_R := \sup_{n\in\mathbb{N}} \sup_{x\in X_n} |B(x, R)|$.

Proof. We will recursively construct the desired $X^{R,s}_n \subseteq X_n$ for each $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$, and choose any $x_0 \in X_n$ such that $m_n(x_0) = \max_{x\in X_n} m_n(x)$. Define

$$X_{n,1} := X_n \setminus \{y \in B(x_0, R) : m_n(y) < s \cdot m_n(x_0)\}.$$

Since $s < 1$, we see that $x_0 \in X_{n,1}$. Thereafter we choose any $x_1 \in X_{n,1} \setminus \{x_0\}$ so that $m_n(x_1) = \max_{x\in X_{n,1}\setminus\{x_0\}} m_n(x)$, and define

$$X_{n,2} := X_{n,1} \setminus \{y \in X_{n,1} \cap B(x_1, R) : m_n(y) < s \cdot m_n(x_1)\}.$$

Recursively, we choose $x_k \in X_{n,k} \setminus \{x_{k-1}, \ldots, x_0\}$ so that $m_n(x_k) = \max_{x\in X_{n,k}\setminus\{x_{k-1},\ldots,x_0\}} m_n(x)$, and define

$$X_{n,k+1} := X_{n,k} \setminus \{y \in X_{n,k} \cap B(x_k, R) : m_n(y) < s \cdot m_n(x_k)\}.$$

By convention, $X_{n,0} := X_n$. Since X_n is finite, the recursive process will finish after finitely many steps, when $X_{n,k_0} = \{x_0, x_1, \ldots, x_{k_0}\}$ for some $k_0 \in \mathbb{N} \cup \{0\}$. Note that $m_n(x_0) \geq m_n(x_1) \geq \ldots \geq m_n(x_{k_0})$. We denote the resulting subspace by $X^{R,s}_n = X_{n,k_0}$. Clearly, $X^{R,s}_n$ satisfies condition (1).

Let $A_i := \{y \in X_{n,i} \cap B(x_i, R) : m_n(y) < s \cdot m_n(x_i)\}$ for $i = 0, 1, \ldots, k_0$. Then

$$X_n \setminus X^{R,s}_n = A_{k_0} \sqcup A_{k_0-1} \sqcup \ldots \sqcup A_0.$$

Since $m_n(A_i) \leq s \cdot N_R \cdot m_n(x_i)$ and $X^{R,s}_n = \{x_0, x_1, \ldots, x_{k_0}\}$, it is easily seen that $X^{R,s}_n$ also satisfies condition (2), as required. □

4.2. **Structure theorem for measured asymptotic expanders.** Recall that the authors in [24, Theorem 3.7] proved a structure theorem for asymptotic expanders. Here we extend the structure result to the case of measured asymptotic expanders by a slightly different and improved argument.

Theorem 4.15. *Let $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of finite measured metric spaces with uniformly bounded geometry. Then the following are equivalent:*

(1) $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of measured asymptotic expanders;
(2) for any $c \in (0,1)$, there exists a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ in $(0,1)$ with $\alpha_k \to 0$, and a positive sequence $\{R_k\}_{k\in\mathbb{N}}$ such that for each fixed $n \in \mathbb{N}$ there exists a sequence of subspaces $\{Y_{n,k}\}_{k\in\mathbb{N}}$ in X_n satisfying the following for every $k \in \mathbb{N}$:
  (i) $Y_{n,k} \subseteq \mathrm{supp}(m_n)$ and $m_n(Y_{n,k}) \geq (1-\alpha_k)\cdot m_n(X_n)$;
  (ii) for each $A \subseteq Y_{n,k}$ with $0 < m_n(A) \leq \frac{1}{2}m_n(Y_{n,k})$, then $m_n(\partial_{R_k}^{Y_{n,k}} A) > c\cdot m_n(A)$;
  (iii) there exists $s_k \in (0,1)$ such that for any $x, y \in Y_{n,k}$ with $d_n(x,y) \leq R_k$, we have $m_n(x) \sim_{s_k} m_n(y)$.

The hypothesis of (iii) is not *needed for the implication "(2) ⇒ (1)".*

Proof. "(1) ⇒ (2)": Fix any $c \in (0,1)$. For each $\alpha \in (0, \frac{1}{2}]$, let R_α satisfy the conclusion in Lemma 4.13 for $\beta = \frac{1}{2}$. Without loss of generality, we may assume that $R_\alpha > 1$ and $\lim_{\alpha\to 0} R_\alpha = \infty$. Let $N_{R_\alpha} := \sup_{n\in\mathbb{N}} \sup_{x\in X_n} |B(x, R_\alpha)|$ so that $1 \leq N_{R_\alpha} < \infty$. If we set $s_\alpha := \frac{1}{R_\alpha \cdot N_{R_\alpha}} \in (0,1)$, then Lemma 4.14 provides a subspace $X_{n,\alpha} := X_n^{R_\alpha, s_\alpha}$ in X_n for each $n \in \mathbb{N}$ satisfying the following:

- For any $x, y \in X_{n,\alpha}$ with $d_n(x,y) \leq R_\alpha$, we have $m_n(x) \sim_{s_\alpha} m_n(y)$;
- $m_n(X_n \setminus X_{n,\alpha}) \leq s_\alpha \cdot N_{R_\alpha} \cdot m_n(X_{n,\alpha}) = \frac{1}{R_\alpha}\cdot m_n(X_{n,\alpha})$.

From the second condition, we obtain that

$$m_n(X_{n,\alpha}) \geq \frac{R_\alpha}{R_\alpha + 1} m_n(X_n) \geq \frac{1}{2} m_n(X_n) \text{ for every } n \in \mathbb{N}.$$

So we know from Lemma 4.13 that for any $A \subseteq X_{n,\alpha}$ with $\alpha \cdot m_n(X_{n,\alpha}) \leq m_n(A) \leq \frac{1}{2} m_n(X_{n,\alpha})$, we have $m_n(\partial_{R_\alpha}^{X_{n,\alpha}} A) > c \cdot m_n(A)$.

Now we assume that $\alpha \in (0, \frac{c}{4+2c}]$. For each $n \in \mathbb{N}$, we consider a family $\mathcal{F}_{n,\alpha}$ of subsets in $X_{n,\alpha}$ given by

$$\mathcal{F}_{n,\alpha} := \Big\{ A \subseteq X_{n,\alpha} : m_n(A) \leq \frac{1}{2} m_n(X_{n,\alpha}) \text{ and } m_n(\partial_{R_\alpha}^{X_{n,\alpha}} A) \leq c \cdot m_n(A) \Big\}.$$

We see that $\mathcal{F}_{n,\alpha}$ always contains the empty set $\emptyset$ so that it admits a maximal element directed by the inclusion. Let $F_{n,\alpha}$ be a maximal element in $\mathcal{F}_{n,\alpha}$, and set $Y_{n,\alpha} := X_{n,\alpha} \setminus F_{n,\alpha}$. By construction, we have $m_n(F_{n,\alpha}) < \alpha \cdot m_n(X_{n,\alpha})$ for each $n \in \mathbb{N}$. Thus, it follows that

$$m_n(Y_{n,\alpha}) > (1-\alpha) m_n(X_{n,\alpha}) \geq \frac{(1-\alpha)\cdot R_\alpha}{R_\alpha + 1} m_n(X_n) \text{ for every } n \in \mathbb{N}.$$

In the following, we divide into two cases:

Case I. Let $A \subseteq Y_{n,\alpha}$ satisfy $0 < m_n(A) \leq \frac{1}{2} m_n(X_{n,\alpha}) - m_n(F_{n,\alpha})$. In particular, $m_n(A) \leq \frac{1}{2} m_n(Y_{n,\alpha})$ and $m_n(A \sqcup F_{n,\alpha}) \leq \frac{1}{2} m_n(X_{n,\alpha})$. Since $\partial_{R_\alpha}^{X_{n,\alpha}}(A \sqcup F_{n,\alpha}) \subseteq (\partial_{R_\alpha}^{X_{n,\alpha}} F_{n,\alpha}) \cup (\partial_{R_\alpha}^{Y_{n,\alpha}} A)$, we deduce that

$$m_n\Big(\partial_{R_\alpha}^{X_{n,\alpha}}(A \sqcup F_{n,\alpha})\Big) \leq m_n\Big(\partial_{R_\alpha}^{X_{n,\alpha}} F_{n,\alpha}\Big) + m_n\Big(\partial_{R_\alpha}^{Y_{n,\alpha}} A\Big) \leq c \cdot m_n(F_{n,\alpha}) + m_n\Big(\partial_{R_\alpha}^{Y_{n,\alpha}} A\Big).$$

On the other hand, since $F_{n,\alpha}$ is maximal in $\mathcal{F}_{n,\alpha}$ and $A \neq \emptyset$ we have that

$$m_n\Big(\partial_{R_\alpha}^{X_{n,\alpha}}(A \sqcup F_{n,\alpha})\Big) > c \cdot m_n(A \sqcup F_{n,\alpha}) = c \cdot m_n(A) + c \cdot m_n(F_{n,\alpha}).$$

Combining them together, we conclude that $m_n(\partial_{R_\alpha}^{Y_{n,\alpha}} A) > c \cdot m_n(A)$.

Case II. Let $A \subseteq Y_{n,\alpha}$ satisfy $\frac{1}{2}m_n(X_{n,\alpha}) - m_n(F_{n,\alpha}) < m_n(A) \leq \frac{1}{2}m_n(Y_{n,\alpha})$. Since $\alpha \leq \frac{1}{4}$ and $m_n(F_{n,\alpha}) < \alpha \cdot m_n(X_{n,\alpha})$, we have that

$$\frac{1}{2}m_n(X_{n,\alpha}) \geq m_n(A) > \frac{1}{2}m_n(X_{n,\alpha}) - m_n(F_{n,\alpha}) > \Big(\frac{1}{2} - \alpha\Big) \cdot m_n(X_{n,\alpha}) \geq \alpha \cdot m_n(X_{n,\alpha}).$$

It follows that $m_n(\partial_{R_\alpha}^{X_{n,\alpha}} A) > c \cdot m_n(A)$. As $\partial_{R_\alpha}^{X_{n,\alpha}} A \subseteq (\partial_{R_\alpha}^{Y_{n,\alpha}} A) \sqcup F_{n,\alpha}$, we see that

$$m_n(\partial_{R_\alpha}^{Y_{n,\alpha}} A) \geq m_n(\partial_{R_\alpha}^{X_{n,\alpha}} A) - m_n(F_{n,\alpha}) > c \cdot m_n(A) - m_n(F_{n,\alpha}).$$

Combining the facts that $\alpha \leq \frac{c}{4+2c}$, $m_n(A) > \frac{1}{2}m_n(X_{n,\alpha}) - m_n(F_{n,\alpha})$ and $m_n(F_{n,\alpha}) < \alpha \cdot m_n(X_{n,\alpha})$, we deduce that

$$m_n(A) > \frac{1}{c+2} \cdot m_n(X_{n,\alpha}),$$

which further implies that

$$c \cdot m_n(A) - m_n(F_{n,\alpha}) > \frac{c}{2} \cdot m_n(A).$$

Hence, we obtain $m_n(\partial_{R_\alpha}^{Y_{n,\alpha}} A) > \frac{c}{2} \cdot m_n(A)$ in this case.

In conclusion, for a given $c \in (0,1)$ and any $\alpha \in (0, \frac{c}{4+2c}]$ there exist $R_\alpha > 1$ with $\lim_{\alpha \to 0} R_\alpha = \infty$, $s_\alpha \in (0,1)$, and a sequence of subspaces $\{Y_{n,\alpha} \subseteq X_n\}_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ and $\alpha \in (0, \frac{c}{4+2c}]$:

- $m_n(Y_{n,\alpha}) > \frac{(1-\alpha) \cdot R_\alpha}{R_\alpha + 1} m_n(X_n)$;
- for each $A \subseteq Y_{n,\alpha}$ with $0 < m_n(A) \leq \frac{1}{2}m_n(Y_{n,\alpha})$ we have $m_n(\partial_{R_\alpha}^{Y_{n,\alpha}} A) > \frac{c}{2} \cdot m_n(A)$;
- for any $x, y \in Y_{n,\alpha}$ with $d_n(x,y) \leq R_\alpha$ we have $m_n(x) \sim_{s_\alpha} m_n(y)$.

Now we choose a sequence $\{\tilde{\alpha}_k\}_{k \in \mathbb{N}}$ in $(0, \frac{c}{4+2c}]$ such that $\tilde{\alpha}_k \to 0$, and set $R_k := R_{\tilde{\alpha}_k}$, $s_k := s_{\tilde{\alpha}_k}$, and $Y_{n,k} := Y_{n,\tilde{\alpha}_k} \cap \mathrm{supp}(m_n)$. Since $\lim_{k \to \infty} \frac{(1-\tilde{\alpha}_k)R_k}{R_k+1} = 1$, we complete the proof by letting $\alpha_k := 1 - \frac{(1-\tilde{\alpha}_k)R_k}{R_k+1}$.

"(2) ⇒ (1)": We assume that condition (2) holds with the constants therein. Given $\alpha \in (0, \frac{1}{2}]$, we take a $k \in \mathbb{N}$ such that $\alpha_k \leq \frac{\alpha}{8}$. For any $n \in \mathbb{N}$ and $A \subseteq X_n$ with $\alpha \cdot m_n(X_n) \leq m_n(A) \leq \frac{1}{2}m_n(X_n)$, we observe that

$$m_n(A \cap Y_{n,k}) \geq m_n(A) - m_n(X_n \setminus Y_{n,k}) \geq m_n(A) - \alpha_k \cdot m_n(X_n) \geq m_n(A) - \frac{\alpha}{2} \cdot m_n(X_n) \geq \frac{1}{2}m_n(A).$$

In the following, we divide into two cases:

Case I. When $m_n(A \cap Y_{n,k}) \leq \frac{1}{2}m_n(Y_{n,k})$: By the hypothesis of *(ii)*, we obtain that

$$m_n(\partial_{R_k}^{X_n} A) \geq m_n(\partial_{R_k}^{Y_{n,k}}(A \cap Y_{n,k})) > c \cdot m_n(A \cap Y_{n,k}) \geq \frac{c}{2} \cdot m_n(A).$$

Case II. When $m_n(A \cap Y_{n,k}) > \frac{1}{2}m_n(Y_{n,k})$: Since $m_n(Y_{n,k}) \geq (1-\alpha_k) \cdot m_n(X_n) \geq \frac{7}{8}m_n(X_n)$, we have that

$$m_n(A \cap Y_{n,k}) \leq m_n(A) \leq \frac{1}{2}m_n(X_n) \leq \frac{4}{7}m_n(Y_{n,k}).$$

As $\{(Y_{n,k}, d_n, m_n)\}_{n\in\mathbb{N}}$ forms a sequence of measured asymptotic expanders for each fixed $k \in \mathbb{N}$, we can apply Lemma 4.12 to obtain two positive constants c' and R'_k only depending on c and R_k such that for any $k \in \mathbb{N}$ and $B \subseteq Y_{n,k}$ with $\frac{1}{2}m_n(Y_{n,k}) \leq m_n(B) \leq \frac{4}{7}m_n(Y_{n,k})$, we have $m_n(\partial^{Y_{n,k}}_{R'_k} B) > c' \cdot m_n(B)$. Hence, we obtain

$$m_n(\partial^{X_n}_{R'_k} A) \geq m_n\Big(\partial^{Y_{n,k}}_{R'_k}(A \cap Y_{n,k})\Big) > c' \cdot m_n(A \cap Y_{n,k}) \geq \frac{c'}{2} \cdot m_n(A).$$

Combining these two cases, we conclude that $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of measured asymptotic expanders, as desired. □

Remark 4.16. Note that the assumption of uniformly bounded geometry in Theorem 4.15 is only needed to guarantee the condition *(iii)*, which is automatically true when all m_n are counting measures. This is the reason why [24, Theorem 3.7] does not require the assumption of uniformly bounded geometry. Hence, Theorem 4.15 completely recovers [24, Theorem 3.7].

We end this subsection by showing that a sequence of measured asymptotic expanders always admits a "uniform exhaustion" by measured expanders with bounded ratios in measure as follows:

Let $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of measured asymptotic expanders with uniformly bounded geometry. For each fixed $k \in \mathbb{N}$, we consider a sequence of subspaces $\{Y_{n,k}\}_{n\in\mathbb{N}}$ satisfying the condition (2) in Theorem 4.15. We will make $\{Y_{n,k}\}_{n\in\mathbb{N}}$ into a sequence of finite measured graphs in the following sense:

Definition 4.17. A sequence of finite measured metric spaces $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ is called a sequence of *finite measured graphs* if each (V_n, E_n) is a finite (connected and undirected) graph equipped with the edge-path metric.

More concretely, we define the vertex set $V_{n,k} := Y_{n,k}$, the edge set $E_{n,k}$ by $u \sim_{E_{n,k}} v$ if and only if $u, v \in V_{n,k}$ satisfy $0 < d_n(u, v) \leq R_k$, and the measure $m_{n,k} := m_n|_{V_{n,k}}$. Then $Y_{n,k} \subseteq \mathrm{supp}(m_n)$ and condition (ii) in Theorem 4.15 implies that $m_{n,k}$ has full support and $(V_{n,k}, E_{n,k})$ is a finite (connected[14] and undirected) graph. Hence, $\{(V_{n,k}, E_{n,k}, m_{n,k})\}_{n\in\mathbb{N}}$ is a sequence of finite measured graphs. If we denote the edge-path metric on $V_{n,k}$ by $d_{n,k}$, then $d_n(u, v) \leq R_k \cdot d_{n,k}(u, v)$ for all $u, v \in V_{n,k}$ and $\partial^{V_{n,k}} A = \partial^{Y_{n,k}}_{R_k} A$ for $A \subseteq V_{n,k}$. In particular, $\{(V_{n,k}, E_{n,k})\}_{n\in\mathbb{N}}$ has uniformly bounded valency and the inclusion map $i_{n,k} : (V_{n,k}, d_{n,k}) \to (X_n, d_n)$ is R_k-Lipschitz.

Due to the above construction and Theorem 4.15 (2), we have the following:

(i) $m_{n,k}$ has full support and $m_{n,k}(V_{n,k}) \geq (1 - \alpha_k) \cdot m_n(X_n)$;
(ii) for each $A \subseteq V_{n,k}$ with $0 < m_{n,k}(A) \leq \frac{1}{2}m_{n,k}(V_{n,k})$, then $m_{n,k}(\partial^{V_{n,k}} A) > c \cdot m_{n,k}(A)$;
(iii) there exists $s_k \in (0, 1)$ such that for any adjacent pair of vertices $u \sim_{E_{n,k}} v$, we have $m_{n,k}(u) \sim_{s_k} m_{n,k}(v)$.

The following definition is derived from condition (ii) above and it is a measured version of the classical notion of expanders (see Definition 2.1 and we also refer the reader to [28] for more details about measured expanders):

[14] Just as for expanders, the expansion condition 4.15(2)(ii) forces any non-empty subset to have a non-empty boundary.

Definition 4.18. Let $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of finite measured graphs. We say that $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of *measured expanders*[15] if there exists $c > 0$ such that for any $n \in \mathbb{N}$ and $A \subseteq V_n$ with $0 < m_n(A) \le \frac{1}{2}m_n(V_n)$, we have $m_n(\partial^{V_n} A) > c \cdot m_n(A)$. In this case, we also say that $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of *c-measured expanders* (a.k.a. measured $(c, 1)$-asymptotic expanders, cf. Definition 4.9).

Remark 4.19. In Definition 4.18, observe that if all of m_n have full support then expansion in measure will force all graphs to be connected.

Before we state our structure theorem for measured asymptotic expanders by means of measured expanders, let us record the following lemma which is needed in Section 7.

Lemma 4.20. *Let $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of c-measured expanders for some $c > 0$. Then for any $\beta \in (0, 1)$, there exists some $c_\beta > 0$ depending only on c and β such that for any $n \in \mathbb{N}$ and $A \subseteq V_n$ with $0 < m_n(A) \le \beta \cdot m_n(V_n)$, we have $m_n(\partial^{V_n} A) > c_\beta \cdot m_n(A)$.*

Proof. If $0 < \beta \le \frac{1}{2}$, the conclusion follows from the definition of c-measured expanders.

If $\frac{1}{2} < \beta < 1$ and $m_n(A) \le \frac{1}{2}m_n(V_n)$, it again follows from the definition of c-measured expanders. If $\frac{1}{2} < \beta < 1$ and $m_n(A) > \frac{1}{2}m_n(V_n)$, it follows directly from Lemma 4.12 for $\underline{c} \equiv c$ and $\underline{R} \equiv 1$. □

We are ready to prove the following corollary:

Corollary 4.21. *Let $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of finite measured metric spaces with uniformly bounded geometry. Then the following are equivalent:*

(1) $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of measured asymptotic expanders;

(2) there exist $c > 0$, a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ in $(0, 1)$ with $\alpha_k \to 0$, a sequence $\{s_k\}_{k\in\mathbb{N}}$ in $(0, 1)$, and a positive sequence $\{R_k\}_{k\in\mathbb{N}}$ such that for any $n, k \in \mathbb{N}$ there exist a finite graph $(V_{n,k}, E_{n,k})$ and a R_k-Lipschitz injective map $i_{n,k} : V_{n,k} \to X_n$ satisfying the following:

   (i) the pullback measure[16] $m_{n,k} := i^*_{n,k}(m_n)$ on $V_{n,k}$ has full support and $m_{n,k}(V_{n,k}) \ge (1 - \alpha_k) \cdot m_n(X_n)$;

   (ii) for each $k \in \mathbb{N}$, $\{(V_{n,k}, E_{n,k}, m_{n,k})\}_{n\in\mathbb{N}}$ is a sequence of c-measured expanders with uniformly bounded valency;

   (iii) for any adjacent pair of vertices $u \sim_{E_{n,k}} v$, we have $m_{n,k}(u) \sim_{s_k} m_{n,k}(v)$.[17]

The hypothesis of (*iii*) *is* not *needed for the implication "*(2) ⇒ (1)*".*

Proof. "(1) ⇒ (2)": It follows exactly from our preceding construction.

[15]In the classical definition of expanders, the sequence of graphs is usually required to have $\lim_{n\to\infty} |V_n| = \infty$ and uniformly bounded valency. Here we take the liberty of excluding these restrictions for a greater generality.

[16]As $i_{n,k}$ is injective, the pullback measure is well-defined by the formula $i^*_{n,k}(m_n)(A) = m_n(i_{n,k}(A))$ for any $A \subseteq V_{n,k}$.

[17]Recall that given two positive numbers a and b, we denote $a \sim_s b$ for some $s \in (0, 1)$ if $sa \le b \le \frac{a}{s}$.

"(2) ⇒ (1)": For each fixed $n \in \mathbb{N}$, let $Y_{n,k} := i_{n,k}(V_{n,k})$ for every $k \in \mathbb{N}$. Since $i_{n,k}(\partial^{V_{n,k}} A) \subseteq \partial_{R_k}^{Y_{n,k}}(i_{n,k}(A))$ for any $A \subseteq V_{n,k}$, it follows that $\{Y_{n,k}\}_{k\in\mathbb{N}}$ is a sequence of subspaces in X_n satisfying (*i*) and (*ii*) in Theorem 4.15 as desired. □

As a direct consequence of Corollary 4.21, we obtain the following:

Corollary 4.22. *A metric space X does not measured weakly contain any measured expanders with uniformly bounded valency* if and only if *it does not measured weakly contain any measured asymptotic expanders with uniformly bounded geometry.*

5. The Poincaré inequality and spectral gaps

The aim of this section is to study the L^p-Poincaré inequality and spectral gaps for measured expanders with bounded measure ratios. To this end, we start by recalling the case of reversible random walks as established in [28].

5.1. Reversible random walks.

This subsection is devoted to recalling the L^p-Poincaré inequality and spectral gaps for reversible random walks. For more details about the theory of random walks, we refer to the textbooks [7, 43].

A *random walk* or a *Markov kernel* on a non-empty set V is a map $r : V \times V \longrightarrow [0, \infty)$ such that $\sum_{v\in V} r(u, v) = 1$ for any $u \in V$. A *stationary measure* μ for a random walk r is a function $\mu : V \longrightarrow (0, \infty)$ such that $\mu(u)r(u, v) = \mu(v)r(v, u)$ for any $u, v \in V$. A random walk is called *reversible* if it admits at least one stationary measure. In the reversible case, the map $a : V \times V \longrightarrow [0, \infty)$ defined by $a(u, v) := \mu(u)r(u, v)$ is called the *conductance* function. Clearly, a is *symmetric* in the sense that $a(u, v) = a(v, u)$ for all $u, v \in V$, and we also have $\mu(u) = \sum_{v\in V} a(u, v)$ for all $u \in V$. Conversely, let $a : V \times V \to [0, \infty)$ be a symmetric map such that $\mu(u) := \sum_{v\in V} a(u, v)$ is positive and finite for each $u \in V$. Then the formula $r(u, v) := \frac{a(u,v)}{\mu(u)}$ defines a reversible random walk on V with stationary measure μ.

Given a reversible random walk r on a non-empty set V with a stationary measure μ, we can endow V with a (not necessarily connected[18] but undirected) graph structure (V, E) by requiring that $u \sim_E v$ is an edge if and only if $r(u, v) > 0$ (which is also equivalent to that $r(v, u) > 0$). Since the corresponding conductance function a is symmetric, we define $a(e) := a(u, v) = a(v, u)$ for each edge $e \in E$ connecting vertices u and v. For $D \subseteq E$, its *area* is defined to be $a(D) := \sum_{e\in D} a(e)$ and $a(\emptyset) = 0$. Now let m be an arbitrary measure on V with full support such that (V, E, m) forms a finite measured graph. Then we define the (μ, a, m)-*Cheeger constant* of (V, E, m) to be

$$\min\Big\{\frac{a(\partial^E A)}{\mu(A)} : A \subseteq V \text{ with } 0 < m(A) \leq \frac{1}{2}m(V)\Big\},$$

where $\partial^E A$ is the edge boundary of A (*i.e.*, the set of edges with exactly one endpoint in A). Similarly to Remark 4.19, if the (μ, a, m)-Cheeger constant is positive, then the graph (V, E) is automatically connected.

[18]The constructed graph (V, E) is connected if and only if the random walk r is irreducible (see [7, Example 5.1.1] for details).

Consider the following Hilbert space:

$$\ell^2(V;\mu) := \Big\{ f : V \to \mathbb{C} \;\Big|\; \sum_{v\in V} |f(v)|^2 \mu(v) < \infty \Big\} \quad \text{with} \quad \langle f_1, f_2 \rangle_\mu := \sum_{v\in V} f_1(v)\overline{f_2(v)}\mu(v).$$

The *graph Laplacian* $\Delta \in \mathfrak{B}(\ell^2(V;\mu))$ associated to the reversible random walk r is defined as

$$(\Delta f)(v) := f(v) - \sum_{u\in V: u\sim_E v} f(u) r(v,u) \tag{5.1}$$

for $f \in \ell^2(V;\mu)$ and $u, v \in V$. In fact, the graph Laplacian Δ is a positive bounded operator with norm at most 2. When the constructed graph (V, E) is connected, $\Delta f = 0$ if and only if f is constant (see *e.g.*, [7, Proposition 5.2.2] for details).

We end this subsection by stating the L^p-Poincaré inequality and the spectral gap of the graph Laplacian for reversible random walks as established in [28] (we refer the reader to [1, 2, 18, 31, 37] for results of this type for classical expanders).

Proposition 5.1. [28, Proposition 3.2 and Proposition 3.8] *Let r be a reversible random walk on a non-empty and finite set V with a stationary measure μ such that a is the associated conductance function and (V, E) is the associated graph structure.*

If m is any non-trivial and finite measure on V of full support such that the (μ, a, m)-Cheeger constant c is positive, then the following hold:

(1) The spectrum of the graph Laplacian $\Delta \in \mathfrak{B}(\ell^2(V;\mu))$ is contained in $\{0\}\cup[c^2/2, 2]$;
*(2) For any $p \in [1,\infty)$, there exists a positive constant c_p only depending on c and p such that for any map $f : V \to \mathbb{C}$ we have the following L^p-*Poincaré inequality*:*

$$\sum_{u,v\in V: u\sim_E v} |f(u) - f(v)|^p a(u,v) \geq c_p \sum_{u,v\in V} |f(u) - f(v)|^p \frac{\mu(u)\mu(v)}{\mu(V)}. \tag{5.2}$$

5.2. **Spectral projections for measured expanders.** We now return to the general case of measured expanders and explore analogous L^p-Poincaré inequality and spectral gaps.

However, measures involved in the definition of measured expanders might not come from reversible random walks in general so that we cannot directly apply Proposition 5.1. To overcome this issue, we build auxiliary random walks whose stationary measures can uniformly control the original measures in measured expanders with bounded measure ratios on adjacent vertices. More precisely, we need the following key lemma:

Lemma 5.2. *Let $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of c-measured expanders for some $c > 0$ with valency uniformly bounded by $K \geq 1$. Assume that each m_n has full support and that there exists $s \in (0,1)$ such that $m_n(u) \sim_s m_n(v)$ for any edge $u \sim_{E_n} v$ and any $n \in \mathbb{N}$.*

Then for every $n \in \mathbb{N}$ there exists a reversible random walk $r_n : V_n \times V_n \to [0,\infty)$ such that the associated stationary measure $\mu_n : V_n \to (0,\infty)$ and the associated conductance function $a_n : V_n \times V_n \to [0,\infty)$ satisfy the following:

(1) $a_n(u,v) = m_n(u) + m_n(v)$ whenever $u, v \in V_n$ such that $u \sim_{E_n} v$;
(2) For $u, v \in V_n$, we have $r_n(u,v) > 0$ if and only if $u \sim_{E_n} v$;
(3) For every $n \in \mathbb{N}$ and $u \in V_n$, we have $\frac{s}{K(1+s)}\mu_n(u) \leq m_n(u) \leq \frac{1}{1+s}\mu_n(u)$;
(4) For every $n \in \mathbb{N}$, the (μ_n, a_n, m_n)-Cheeger constant is bounded below by $\frac{cs}{K}$.

Proof. For every $n \in \mathbb{N}$, we consider the symmetric function $a_n : V_n \times V_n \to [0, \infty)$ defined by

$$a_n(u,v) := \begin{cases} m_n(u) + m_n(v), & \text{if } u \sim_{E_n} v; \\ 0, & \text{otherwise,} \end{cases}$$

for $u, v \in V_n$. Since (V_n, E_n) is a connected finite graph by Remark 4.19, the associated stationary measure $\mu_n : V_n \to (0, \infty)$ is defined by

$$\mu_n(u) := \sum_{v \in V_n} a_n(u,v) \quad \text{for all } u \in V_n.$$

Then the formula $r_n(u,v) := a_n(u,v)/\mu_n(u)$ defines a reversible random walk on V_n with the stationary measure μ_n. Clearly, (1) and (2) hold by the above constructions. Moreover, (3) follows from the assumptions that $m_n(u) \sim_s m_n(v)$ for any edge $u \sim_{E_n} v$ and valencies are bounded by K together with the connectedness of the graph (V_n, E_n).

As for (4), the assumption of c-measured expanders implies that for any $n \in \mathbb{N}$ and $A \subseteq V_n$ with $0 < m_n(A) \leq \frac{1}{2} m_n(V_n)$ we have $m_n(\partial^{V_n} A) > c \cdot m_n(A)$. Hence,

$$\begin{aligned} a_n(\partial^{E_n} A) &= \sum_{e \in \partial^{E_n} A} a_n(e) = \sum_{u \in A, v \notin A, u \sim_{E_n} v} m_n(u) + m_n(v) \geq \sum_{v \in \partial^{V_n} A} (1+s) m_n(v) \\ &= (1+s) m_n(\partial^{V_n} A) > \frac{cs}{K} \mu_n(A), \end{aligned}$$

where we use (3) in the last inequality. Hence, we have verified (4) as desired. □

Now we are ready to prove the following L^p-Poincaré inequality for measured expanders with bounded measure ratios on adjacent vertices:

Corollary 5.3. *Let $\{(V_n, E_n, m_n)\}_{n \in \mathbb{N}}$ be a sequence of c-measured expanders for some $c > 0$ with valency uniformly bounded by $K \geq 1$. Assume that each m_n has full support and there exists $s \in (0,1)$ such that $m_n(u) \sim_s m_n(v)$ for any edge $u \sim_{E_n} v$ and any $n \in \mathbb{N}$.*

Then for any $p \in [1, \infty)$, there exists a positive constant c' only depending on c, s, p, K such that for any $n \in \mathbb{N}$ and any map $f : V_n \to \mathbb{C}$, we have the following L^p-Poincaré inequality:

$$\sum_{u,v \in V_n : u \sim_{E_n} v} |f(u) - f(v)|^p (m_n(u) + m_n(v)) \geq c' \sum_{u,v \in V_n} |f(u) - f(v)|^p \frac{m_n(u) m_n(v)}{m_n(V_n)}. \tag{5.3}$$

Proof. First of all, we notice that for every $n \in \mathbb{N}$ there exists a reversible random walk r_n on V_n such that the associated stationary measure μ_n and the associated conductance function a_n satisfy (1)-(4) in Lemma 5.2. Moreover, Lemma 5.2(2) tells us that (V_n, E_n) is exactly the associated graph structure coming from the reversible random walk r_n on V_n. Because of Lemma 5.2(4), Proposition 5.1 implies that for any $p \in [1, \infty)$ there exists a positive constant c_p only depending on c, s, p, K such that for any $n \in \mathbb{N}$ and any map $f : V_n \to \mathbb{C}$ we have that

$$\sum_{u,v \in V_n : u \sim_{E_n} v} |f(u) - f(v)|^p a_n(u,v) \geq c_p \sum_{u,v \in V_n} |f(u) - f(v)|^p \frac{\mu_n(u) \mu_n(v)}{\mu_n(V_n)}.$$

By Lemma 5.2(1) and (3), we deduce the inequality (5.3) for $c' := \frac{s(1+s)c_p}{K} > 0$. □

Finally, we show that the graph Laplacian (up to conjugacy) associated to measured expanders with bounded measure ratios has a spectral gap as expected.

Proposition 5.4. *Let $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of measured expanders with uniformly bounded valency. Assume that each m_n has full support and there exists $s \in (0,1)$ such that $m_n(u) \sim_s m_n(v)$ for any edge $u \sim_{E_n} v$ and any $n \in \mathbb{N}$. Let (V, d) be a coarse disjoint union of $\{V_n\}_{n\in\mathbb{N}}$ with respect to the edge-path metrics, and m be the direct sum measure of $\{m_n\}_{n\in\mathbb{N}}$ on V. Then for every $n \in \mathbb{N}$ there exists a reversible random walk on V_n with a stationary measure $\mu_n : V_n \to (0,\infty)$ such that the following hold:*

(1) If $\Delta_n \in \mathfrak{B}(\ell^2(V_n;\mu_n))$ is the graph Laplacian defined as in (5.1) and $\Lambda_n := W_n^*\Delta_n W_n \in \mathfrak{B}(\ell^2(V_n;m_n))$ where $W_n : \ell^2(V_n;m_n) \to \ell^2(V_n;\mu_n)$ is the "set-theoretic identity" operator, then the spectrum of $\Lambda := \bigoplus_{n\in\mathbb{N}} \Lambda_n \in \mathfrak{B}(\ell^2(V;m))$ is contained in $\{0\} \cup [\kappa,\infty)$ for some $\kappa > 0$;

(2) Let $\mathfrak{S}_n \in \mathfrak{B}(\ell^2(V_n;m_n))$ be the orthogonal projection onto the space of constant functions on V_n, and $\mathfrak{S} := \bigoplus_{n\in\mathbb{N}} \mathfrak{S}_n$. Then $\mathfrak{S} = \chi_{\{0\}}(\Lambda)$ is the spectral projection;

(3) $\sup\{d(u,v) : \langle \Lambda\delta_v, \delta_u\rangle_{\ell^2(V;m)} \neq 0\} \leq 1$.

Proof. From Lemma 5.2(3), we know that there exists a constant $M > 0$ such that $\|W_n\| \leq M$ for every $n \in \mathbb{N}$. In particular, we have $\Lambda = \bigoplus_{n\in\mathbb{N}} \Lambda_n \in \mathfrak{B}(\ell^2(V;m))$. To see the desired spectral gap of Λ, we apply Lemma 5.2(4) together with [28, Lemma 3.19 and Proposition 3.21], which establish bounds between the spectral gap of Λ and the best constant in a Poincaré inequality, and between the latter quantity and the Cheeger constant. This completes (1).

Since each Δ_n is positive, so are Λ_n and Λ. As $\chi_{\{0\}}(\Lambda) = \bigoplus_{n\in\mathbb{N}} \chi_{\{0\}}(\Lambda_n)$, it suffices to show $\mathfrak{S}_n = \chi_{\{0\}}(\Lambda_n)$ for every $n \in \mathbb{N}$. Clearly, $\chi_{\{0\}}(\Lambda_n)$ is the orthogonal projection onto the kernel of Λ_n. Hence, we need to check that the kernel of Λ_n exactly consists of constant functions on V_n. Since the identity operator W_n is invertible, it follows that if $\eta \in \ell^2(V_n;m_n)$ then $\Lambda_n\eta = 0$ if and only if $\Delta_n W_n \eta = 0$. Since each (V_n, E_n) is connected and W_n^{-1} is also the identity operator, this is also equivalent to η is a constant function on V_n. Hence, we have verified (2).

It is easily seen from (5.1) that $\sup\{d(u,v) : \langle \Delta_n\delta_v, \delta_u\rangle_{\ell^2(V_n;\mu_n)} \neq 0\} \leq 1$ for every $n \in \mathbb{N}$. This fact clearly ensures (3) as desired. □

6. Proofs of main Theorems

In previous sections, we introduced all necessary ingredients to prove our main theorems. Now we are in the position to prove Theorem B, which is the foundation of this paper. As its proof is rather technical, we split it into two parts: we first consider the case of uniform Roe algebras and then the case of Roe algebras.

6.1. **The case of uniform Roe algebras.**

In this subsection, we prove the following theorem for block-rank-one projections in uniform Roe algebras:

Theorem 6.1. *Let $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ be a sequence of finite metric spaces with uniformly bounded geometry and X be their coarse disjoint union. Let $P \in \mathfrak{B}(\ell^2(X))$ be a block-rank-one projection with respect to $\{(X_n, d_n)\}_{n\in\mathbb{N}}$. Then P is quasi-local* if and only if *P belongs to the uniform Roe algebra $C_u^*(X)$.*

Proof. The sufficiency holds trivially, so we only focus on the necessity. Suppose that $P = \bigoplus_{n\in\mathbb{N}} P_n$ is a quasi-local block-rank-one projection associated to the unit vectors $\{\xi_n\}_{n\in\mathbb{N}}$ in $\ell^2(X_n)$. Let m_n be the associated probability measure on X_n defined by $m_n(x) := |\xi_n(x)|^2$ for $x \in X_n$. By Lemma 4.5, Lemma 4.6 and Proposition 4.8, we may assume that each m_n has full support on X_n and $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of measured asymptotic expanders.

Then Corollary 4.21 guarantees that there exist $c > 0$, a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ in $(0, 1)$ with $\alpha_k \to 0$, a sequence $\{s_k\}_{k\in\mathbb{N}}$ in $(0, 1)$, and a positive sequence $\{R_k\}_{k\in\mathbb{N}}$ such that for any $n, k \in \mathbb{N}$ there exist a finite graph $(V_{n,k}, E_{n,k})$ with the edge-path metric $d_{n,k}$ and a R_k-Lipschitz injective map $i_{n,k} : V_{n,k} \to X_n$ satisfying the following:

(i) the pullback measure $m_{n,k} := i^*_{n,k}(m_n)$ on $V_{n,k}$ has full support and $m_{n,k}(V_{n,k}) \geq 1 - \alpha_k$;
(ii) for each $k \in \mathbb{N}$, $\{(V_{n,k}, E_{n,k}, m_{n,k})\}_{n\in\mathbb{N}}$ is a sequence of c-measured expanders with uniformly bounded valency;
(iii) for any adjacent vertices $u \sim_{E_{n,k}} v$, we have $m_{n,k}(u) \sim_{s_k} m_{n,k}(v)$.

Now we fix a $k \in \mathbb{N}$. Let (V_k, d_k) be a coarse disjoint union of $\{(V_{n,k}, d_{n,k})\}_{n\in\mathbb{N}}$ and m_k be the direct sum measure of $\{m_{n,k}\}_{n\in\mathbb{N}}$ on V_k. For each $n \in \mathbb{N}$, let $\mathfrak{S}_{n,k} \in \mathfrak{B}(\ell^2(V_{n,k}; m_{n,k}))$ be the orthogonal projection onto the space of constant functions on $V_{n,k}$ and $\mathfrak{S}_k := \bigoplus_{n\in\mathbb{N}} \mathfrak{S}_{n,k} \in \mathfrak{B}(\ell^2(V_k; m_k))$. Since $\{m_{n,k}(v)^{-\frac{1}{2}}\delta_v\}_{v\in V_{n,k}}$ forms an orthonormal basis in $\ell^2(V_{n,k}; m_{n,k})$ and $\{m_{n,k}(v)^{-\frac{1}{2}}\xi_n(i_{n,k}(v))\delta_{i_{n,k}(v)}\}_{v\in V_{n,k}}$ forms an orthonormal set in $\ell^2(X_n)$, we define a linear isometry $U_{n,k} : \ell^2(V_{n,k}; m_{n,k}) \to \ell^2(X_n)$ by the formula

$$m_{n,k}(v)^{-\frac{1}{2}}\delta_v \mapsto m_{n,k}(v)^{-\frac{1}{2}}\xi_n(i_{n,k}(v))\delta_{i_{n,k}(v)}, \text{ for } v \in V_{n,k}.$$

Thus, we have a linear isometry

$$U_k := \bigoplus_{n\in\mathbb{N}} U_{n,k} : \ell^2(V_k; m_k) \to \ell^2(X).$$

From Proposition 5.4, there exists an operator $\Lambda_k \in \mathfrak{B}(\ell^2(V_k; m_k))$ such that $\mathfrak{S}_k$ belongs to the unital C^*-subalgebra generated by Λ_k in $\mathfrak{B}(\ell^2(V_k; m_k))$ and

$$\sup\{d_k(u, v) : \langle \Lambda_k\delta_v, \delta_u\rangle_{\ell^2(V_k;m_k)} \neq 0\} \leq 1. \tag{6.1}$$

Hence, each $U_k\mathfrak{S}_kU_k^*$ belongs to the unital C^*-subalgebra generated by $U_k\Lambda_kU_k^*$ in $\mathfrak{B}(\ell^2(X))$. On the other hand, the bounded operator $U_k\Lambda_kU_k^*$ has propagation at most R_k as $i_{n,k}$ is R_k-Lipschitz together with (6.1). Consequently, both $U_k\Lambda_kU_k^*$ and $U_k\mathfrak{S}_kU_k^*$ belong to the uniform Roe algebra $C^*_u(X)$ for every $k \in \mathbb{N}$. In order to conclude $P \in C^*_u(X)$, it suffices to show that $U_k\mathfrak{S}_kU_k^* \to P$ as $k \to \infty$.

Recall that $\mathfrak{S}_{n,k}$ in $\mathfrak{B}(\ell^2(V_{n,k}; m_{n,k}))$ is the orthogonal projection onto the subspace spanned by the unit vector $\zeta_{n,k} \in \ell^2(V_{n,k}; m_{n,k})$, where $\zeta_{n,k}(v) = \frac{1}{\sqrt{m_{n,k}(V_{n,k})}} = \frac{1}{\sqrt{m_n(\mathrm{Im}\, i_{n,k})}}$ for every $v \in V_{n,k}$. It follows easily that $U_{n,k}\mathfrak{S}_{n,k}U_{n,k}^*$ is the orthogonal projection onto the one-dimensional subspace spanned by the unit vector $\eta_{n,k} := U_{n,k}\zeta_{n,k} \in \ell^2(X_n)$. A direct calculation gives us

$$\eta_{n,k}(x) = \begin{cases} \frac{\xi_n(x)}{\sqrt{m_n(\mathrm{Im}\, i_{n,k})}}, & x \in \mathrm{Im}\, i_{n,k}; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that ξ_n is the unit vector in $\ell^2(X_n)$ corresponding to the rank-one projection P_n in $\mathfrak{B}(\ell^2(X_n))$.

As $1 \geq m_n(\mathrm{Im} i_{n,k}) \geq 1 - \alpha_k$ for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}\|\eta_{n,k} - \xi_n\|^2 &= \sum_{x \in \mathrm{Im} i_{n,k}} \Big| \frac{\xi_n(x)}{\sqrt{m_n(\mathrm{Im} i_{n,k})}} - \xi_n(x)\Big|^2 + \sum_{x \in X_n \setminus \mathrm{Im} i_{n,k}} |\xi_n(x)|^2 \\ &= \Big(\frac{1}{\sqrt{m_n(\mathrm{Im} i_{n,k})}} - 1\Big)^2 \cdot m_n(\mathrm{Im} i_{n,k}) + m_n(X_n \setminus \mathrm{Im} i_{n,k}) \\ &\leq \Big(\frac{1}{\sqrt{1-\alpha_k}} - 1\Big)^2 + \alpha_k.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\|U_k \mathfrak{S}_k U_k^* - P\|^2 &= \sup_{n \in \mathbb{N}} \|U_{n,k} \mathfrak{S}_{n,k} U_{n,k}^* - P_n\|^2 \\ &\leq 4 \sup_{n \in \mathbb{N}} \|\eta_{n,k} - \xi_n\|^2 \\ &\leq 4 \left(\Big(\frac{1}{\sqrt{1-\alpha_k}} - 1\Big)^2 + \alpha_k \right) \to 0, \text{ as } k \to \infty.\end{aligned}$$

Consequently, we deduce that P belongs to $C_u^*(X)$ as desired. □

6.2. **The case of Roe algebras.** In this subsection, we complete Theorem B. In fact, we prove a stronger result (see Theorem 6.6). In order to state Theorem 6.6, we need the following notion:

Definition 6.2. Let (X,d) be a coarse disjoint union of a sequence of finite metric spaces $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ and $\mathcal{H}_0$ be a Hilbert space. Let $P = \bigoplus_{n \in \mathbb{N}} P_n \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ be a block-rank-one projection with respect to $\{X_n\}_{n \in \mathbb{N}}$. If ξ_n is a unit vector in $\ell^2(X_n;\mathcal{H}_0)$ associated to the rank-one projection P_n, then the unit vector $\widetilde{\xi_n} \in \ell^2(X_n)$, defined by $\widetilde{\xi_n}(x) := \|\xi_n(x)\|$ for $x \in X_n$, is called the *uniformization* of ξ_n. In this case, the rank-one projection $\widetilde{P_n} := \langle \cdot, \widetilde{\xi_n}\rangle \widetilde{\xi_n}$ in $\mathfrak{B}(\ell^2(X_n))$ is called the uniformization of P_n, and the block-rank-one projection $\widetilde{P} := \bigoplus_{n \in \mathbb{N}} \widetilde{P_n}$ in $\mathfrak{B}(\ell^2(X))$ is called the uniformization of P.

Remark 6.3. It is clear that P_n and $\widetilde{P_n}$ in Definition 6.2 have the same associated probability measure on X_n given by $X_n \ni x \mapsto \|\xi_n(x)\|^2 = |\widetilde{\xi_n}(x)|^2$. Hence, P is a ghost *if and only if* $\widetilde{P}$ is a ghost by Lemma 2.12.

The next lemma follows directly from Lemma 4.3 and the previous remark.

Lemma 6.4. *Let (X,d) be a coarse disjoint union of a sequence of finite metric spaces $\{(X_n, d_n)\}_{n \in \mathbb{N}}$, and $\mathcal{H}_0$ be a Hilbert space. Let $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ be a block-rank-one projection with respect to $\{X_n\}_{n \in \mathbb{N}}$, and let $\widetilde{P} \in \mathfrak{B}(\ell^2(X))$ be the uniformization of P. Then P is quasi-local* if and only if *$\widetilde{P}$ is quasi-local.*

As expected, we also have the following counterpart to Lemma 6.4:

Lemma 6.5. *Let (X,d) be a coarse disjoint union of a sequence of finite metric spaces $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ with uniformly bounded geometry, and $\mathcal{H}_0$ be an infinite-dimensional separable Hilbert space. Let $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ be a block-rank-one projection with respect to $\{X_n\}_{n \in \mathbb{N}}$. If $\widetilde{P} \in \mathfrak{B}(\ell^2(X))$ is the uniformization of P, then $P \in C^*(X)$* if and only if *$\widetilde{P} \in C_u^*(X)$.*

Proof. Let ξ_n be a unit vector in $\ell^2(X_n;\mathcal{H}_0)$ associated to the rank-one projection P_n for $n \in \mathbb{N}$, and we write $\xi_n = \sum_{x\in X_n} \delta_x \otimes \xi_n(x)$. Recall that its uniformization vector $\widetilde{\xi_n} \in \ell^2(X_n)$ is defined by $\widetilde{\xi_n}(x) = \|\xi_n(x)\|$ for $x \in X_n$. By Lemma 4.6, we can assume that both ξ_n and $\widetilde{\xi_n}$ have full support in X_n for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we consider the linear isometry $W_n : \ell^2(X_n) \longrightarrow \ell^2(X_n;\mathcal{H}_0)$ defined by

$$W_n(\delta_x) := \frac{\delta_x \otimes \xi_n(x)}{\|\xi_n(x)\|},\ x \in X_n.$$

Its adjoint operator $W_n^* : \ell^2(X_n;\mathcal{H}_0) \to \ell^2(X_n)$ is given by the formula

$$W_n^*(\delta_x \otimes \eta) = \frac{\langle \eta, \xi_n(x)\rangle}{\|\xi_n(x)\|} \cdot \delta_x,\ \text{for } x \in X_n \text{ and } \eta \in \mathcal{H}_0.$$

It follows that

$$\begin{aligned} W_n(\widetilde{\xi_n}) &= \sum_{x\in X_n} \|\xi_n(x)\| \cdot W_n(\delta_x) = \sum_{x\in X_n} \|\xi_n(x)\| \cdot \frac{\delta_x \otimes \xi_n(x)}{\|\xi_n(x)\|} = \sum_{x\in X_n} \delta_x \otimes \xi_n(x) = \xi_n; \\ W_n^*(\xi_n) &= \sum_{x\in X_n} W_n^*(\delta_x \otimes \xi_n(x)) = \sum_{x\in X_n} \frac{\langle \xi_n(x), \xi_n(x)\rangle}{\|\xi_n(x)\|} \cdot \delta_x = \sum_{x\in X_n} \|\xi_n(x)\| \cdot \delta_x = \widetilde{\xi_n}. \end{aligned}$$

As $P_n = \langle\, \cdot\, , \xi_n\rangle\xi_n$ and $\widetilde{P_n} = \langle\, \cdot\, , \widetilde{\xi_n}\rangle\widetilde{\xi_n}$, we deduce that $P_n = W_n \circ \widetilde{P_n} \circ W_n^*$ and $\widetilde{P_n} = W_n^* \circ P_n \circ W_n$ for every $n \in \mathbb{N}$. Thus, $P = W \circ \widetilde{P} \circ W^*$ and $\widetilde{P} = W^* \circ P \circ W$, where $W := \bigoplus_{n\in\mathbb{N}} W_n$ is the linear isometry from $\ell^2(X)$ to $\ell^2(X;\mathcal{H}_0)$.

As W and W^* have propagation zero, we obtain a bounded linear map $C^*(X) \to C_u^*(X)$ given by $T \mapsto W^* \circ T \circ W$ for $T \in C^*(X)$. In particular, if $P \in C^*(X)$ then $\widetilde{P} = W^* \circ P \circ W \in C_u^*(X)$. Conversely, we want to show that $P \in C^*(X)$ if $\widetilde{P} \in C_u^*(X)$. Let T be any bounded operator on $\ell^2(X)$ with finite propagation. If $x \in X_k$ and $y \in X_l$, then we have

$$(W \circ T \circ W^*)_{x,y}\xi = \frac{\langle \xi, \xi_l(y)\rangle \cdot T_{x,y}}{\|\xi_l(y)\| \cdot \|\xi_k(x)\|} \cdot \xi_k(x) \text{ for } \xi \in \mathcal{H}_0.$$

Thus, $W \circ T \circ W^*$ has finite propagation and is locally compact as $(W \circ T \circ W^*)_{x,y}$ is a rank-one operator on $\mathcal{H}_0$ for every $x, y \in X$. As a consequence, we obtain an injective $*$-homomorphism $C_u^*(X) \to C^*(X)$ given by $T \mapsto W \circ T \circ W^*$ so that $P = W \circ \widetilde{P} \circ W^* \in C^*(X)$. Hence, we complete the proof. □

Combining Theorem 6.1 with Lemma 6.4 and Lemma 6.5, we obtain our main theorem in this subsection and finish the proof of Theorem B:

Theorem 6.6. *Let (X, d) be a coarse disjoint union of a sequence of finite metric spaces $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ with uniformly bounded geometry, and $\mathcal{H}_0$ be an infinite-dimensional separable Hilbert space. Let $P \in \mathfrak{B}(\ell^2(X;\mathcal{H}_0))$ be a block-rank-one projection with respect to $\{X_n\}_{n\in\mathbb{N}}$. If $\widetilde{P} \in \mathfrak{B}(\ell^2(X))$ is the uniformization of P, then the following are equivalent:*

(1) $P \in C^*(X)$;
(2) P is quasi-local;
(3) $\widetilde{P} \in C_u^*(X)$;
(4) $\widetilde{P}$ is quasi-local.

Remark 6.7. If $P \in C_u^*(X)$ is a block-rank-one projection with respect to a coarse disjoint union $X = \bigsqcup_{n\in\mathbb{N}} X_n$, then so is $P \otimes e \in C^*(X)$ for any fixed rank-one projection $e \in \mathfrak{K}(\mathcal{H}_0)$. In this case, the uniformization of $P \otimes e$ is P itself. Consequently, Theorem 6.1 can be deduced from Theorem 6.6.

Let us finish this subsection with some consequences for the coarse Baum-Connes conjecture. First of all, we improve Proposition 3.5 by providing the following measured version of [24, Proposition 6.4]:

Proposition 6.8. *Let Y be a metric space of bounded geometry. Assume that there exists a sequence of measured asymptotic expanders $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ of uniformly bounded geometry which admits a measured weak embedding $\{f_n : (X_n, d_n, m_n) \to Y\}_{n\in\mathbb{N}}$ such that for every $y \in Y$ the preimage $f_n^{-1}(y)$ is empty for all but finitely many $n \in \mathbb{N}$. Then there exist a sparse subspace $Y' \subseteq Y$ and a block-rank-one ghost projection in $C_u^*(Y')$.*

If we additionally assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly *finite-to-one and each m_n is the counting measure, then the averaging projection of Y' is a ghost in $C_u^*(Y')$.*

Proof. Without loss of generality, we can assume that each m_n is a probability measure. Let X be a coarse disjoint union of $\{(X_n, d_n)\}_{n\in\mathbb{N}}$, and $f := \bigsqcup_{n\in\mathbb{N}} f_n : X \to Y$. As the sequence $\{f_n\}_{n\in\mathbb{N}}$ is coarse, we can assume that f is coarse as well by appropriately choosing the metric on X. From the assumption on the preimages of $\{f_n\}_{n\in\mathbb{N}}$, we know that f is finite-to-one.

By Proposition 4.8 and Theorem 6.1, the associated block-rank-one projection $P := \bigoplus_{n\in\mathbb{N}} P_n$ with respect to $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ belongs to $C_u^*(X)$, where each $P_n \in \mathfrak{B}(\ell^2(X_n))$ is the rank-one projection onto the subspace spanned by the unit vector $X_n \ni x \mapsto \sqrt{m_n(x)}$ in $\ell^2(X_n)$. We fix an infinite-dimensional separable Hilbert space $\mathcal{H}_0$ and a rank-one projection $e \in \mathfrak{K}(\mathcal{H}_0)$, then $P \otimes e \in C^*(X)$ and its uniformization is P itself by Remark 6.7. Moreover, Remark 6.3 tells us that each m_n is also the associated measure to $P_n \otimes e$. Hence, it follows from Proposition 3.5 that there exist a sparse subspace $Y' = \bigsqcup_{n\in\mathbb{N}} Y_n$ in Y where $Y_n = f_n(X_n)$, and a block-rank-one ghost projection $Q \in C^*(Y')$. By Theorem 6.6 and Remark 6.3, we conclude that its uniformization $\widetilde{Q}$ is a block-rank-one ghost projection in $C_u^*(Y')$ as desired.

Now we additionally assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly finite-to-one and each m_n is the probability counting measure on X_n. As $\widetilde{Q}$ is a block-rank-one ghost projection in $C_u^*(Y')$, it follows from Proposition 4.8 and Lemma 2.12 that $\{(Y_n, m'_n)\}_{n\in\mathbb{N}}$ forms a ghostly sequence of measured asymptotic expanders, where m'_n is the probability measure on Y_n associated to $\widetilde{Q}_n$. From the formula (3.1) in the proof of Proposition 3.5 we have that

$$m'_n(y) = \|\widetilde{Q}_n \delta_y\|^2 = |(\widetilde{Q}_n)_{y,y}| = m_n(f_n^{-1}(y)) = \frac{|f_n^{-1}(y)|}{|X_n|}, \text{ for } y \in Y_n.$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is uniformly finite-to-one and each f_n is surjective onto Y_n, there exists a constant $s \geq 1$ such that for any $n \in \mathbb{N}$ and any $y_1, y_2 \in Y_n$, we have

$$\frac{1}{s} m'_n(y_2) \leq m'_n(y_1) \leq s m'_n(y_2).$$

As a consequence of [27, Definition 6.4 and Lemma 6.5], we conclude that $\{Y_n\}_{n\in\mathbb{N}}$ is a sequence of asymptotic expanders with $|Y_n| \to \infty$ as $n \to \infty$. So the averaging projection of $\{Y_n\}_{n\in\mathbb{N}}$ is a ghost in $C_u^*(Y')$ by Theorem 6.1 or [24, Theorem C]. □

Note that in the previous proposition the condition "for every $y \in Y$ the preimage $f_n^{-1}(y)$ is empty for all but finitely many $n \in \mathbb{N}$" is redundant if Y is sparse. This follows from Corollary 4.22 and the following general fact:

Lemma 6.9. *Let $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence of finite measured graphs and $Y = \bigsqcup_{k\in\mathbb{N}} Y_k$ be a sparse space. If $\{f_n : (V_n, E_n, m_n) \to Y\}_{n\in\mathbb{N}}$ is a measured weak embedding, then for every $y \in Y$ the preimages $f_n^{-1}(y) \neq \emptyset$ for only finitely many n's.*

Proof. If this were not the case, then there would exist $y_0 \in Y$ such that $f_{n_i}^{-1}(y_0) \neq \emptyset$ for all $i \in \mathbb{N}$. As all (V_n, E_n) are connected graphs and all f_n are L-Lipschitz for some $L > 0$ by Remark 2.7, it follows that $f_{n_i}(V_{n_i}) \subseteq E := \bigsqcup_{k=1}^{M} Y_k$ for some natural number M (independent of i). Since E is a non-empty finite set and $f_{n_i}^{-1}(E) = V_{n_i}$ for all $i \in \mathbb{N}$, it follows that $m_{n_i}(V_{n_i}) = \sum_{y\in E} m_{n_i}(f_{n_i}^{-1}(y))$ for every $i \in \mathbb{N}$. In particular, for each $i \in \mathbb{N}$ there exists $y_i \in E$ such that $\frac{m_{n_i}(f_{n_i}^{-1}(y_i))}{m_{n_i}(V_{n_i})} \geq \frac{1}{|E|} > 0$, which contradicts with the assumption that $\{f_n\}_{n\in\mathbb{N}}$ is a measured weak embedding. □

The following theorem is a consequence of Corollary 4.22, Proposition 6.8 and Lemma 6.9 with an identical proof of [24, Theorem 6.7].

Theorem 6.10. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of finite metric spaces of uniformly bounded geometry and X be their coarse disjoint union. If X admits a fibred coarse embedding into a Hilbert space[19], and there exists a sequence of measured asymptotic expanders of uniformly bounded geometry which admits a measured weak embedding into X, then the following statements hold:*

(1) The coarse Baum–Connes assembly map for X is injective but non-surjective.
(2) The induced map $\iota_*\colon K_*(\mathfrak{K}) \to K_*(I_G)$ is injective but non-surjective, where $\iota\colon \mathfrak{K} \hookrightarrow I_G$ is the inclusion of the compact ideal $\mathfrak{K}$ into the ghost ideal I_G of the Roe algebra $C^*(X)$.
(3) The induced map $\pi_*\colon K_*(C^*_{max}(X)) \to K_*(C^*(X))$ is injective but non-surjective, where $\pi\colon C^*_{max}(X) \twoheadrightarrow C^*(X)$ is the canonical surjection from the maximal Roe algebra onto the Roe algebra. In particular, π is not injective.

In view of Lemma 2.8, we deduce the following corollary of Theorem 6.10:

Corollary 6.11. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of finite metric spaces of uniformly bounded geometry and X be their coarse disjoint union. If X admits a fibred coarse embedding into a Hilbert space, and there exists a sequence of ghostly measured asymptotic expanders of uniformly bounded geometry which admits a coarse embedding into X, then X violates the coarse Baum-Connes conjecture.*

6.3. **A geometric criterion for the rigidity problem.** Now we provide a geometric condition for which the rigidity problem holds. Let us start with the following characterisations for the analytic conditions introduced in Definition 3.1:

Corollary 6.12. *Let X be a metric space with bounded geometry. Then the following properties are equivalent:*

(1) all sparse subspaces of X contain no block-rank-one ghost projections in their Roe algebras;

[19]See [14, Definition 2.1] for the definition of fibred coarse embedding into Hilbert spaces.

(2) *all sparse subspaces of X contain no block-rank-one ghost projections in their uniform Roe algebras;*
(3) *X contains no sparse subspaces consisting of ghostly measured asymptotic expanders.*
(4) *X coarsely contains no sparse spaces consisting of ghostly measured asymptotic expanders with uniformly bounded geometry.*

In particular, all of these properties are coarsely invariant among metric spaces with bounded geometry.

Proof. "(1) ⇒ (2)" follows from Remark 6.3 and Remark 6.7. While "(2) ⇒ (1)" follows from Remark 6.3 and "(1) ⇒ (3)" in Theorem 6.6. The equivalence between (2) and (3) follows directly from Lemma 2.12, Proposition 4.8 and Theorem 6.1.

Since every isometric embedding is a coarse embedding, we clearly have "(4) ⇒ (3)". To see "(3) ⇒ (4)", we assume that X coarsely contains a sparse space X' consisting of ghostly measured asymptotic expanders with uniformly bounded geometry. If $f : X' \to X$ is the coarse embedding, then f must be finite-to-one because X' is sparse. As a consequence of Lemma 2.8 and Proposition 6.8, there exist a sparse subspace Z of X and a block-rank-one ghost projection in $C^*_u(Z)$. Hence, Z consists of ghostly measured asymptotic expanders by Lemma 2.12 and Proposition 4.8.

Finally, the last statement can be deduced from Proposition 3.7 together with the fact that every coarse embedding from a metric space with bounded geometry is uniformly finite-to-one. □

Condition (3) in Corollary 6.12 is the one we are most interested in, because it is formally the weakest geometric criterion which guarantees rigidity. More precisely, we obtain the following rigidity result (Theorem A) by combining Proposition 3.9 with Corollary 6.12.

Theorem 6.13. *Let X and Y be metric spaces with bounded geometry. Assume that either X or Y contains no sparse subspaces consisting of ghostly measured asymptotic expanders. Then the following are equivalent:*

(1) *X is coarsely equivalent to Y;*
(2) *$C^*_u(X)$ is Morita equivalent [20] to $C^*_u(Y)$;*
(3) *$C^*_s(X)$ is $*$-isomorphic to $C^*_s(Y)$;*
(4) *$UC^*(X)$ is $*$-isomorphic to $UC^*(Y)$;*
(5) *$C^*(X)$ is $*$-isomorphic to $C^*(Y)$.*

Remark 6.14. Very recently, it has been shown in [6] that (1) to (4) in Theorem 6.13 are all equivalent for general metric spaces with bounded geometry. At this moment, we believe that Theorem 6.13 is still the best-known result on the equivalence of (5) with the other items.

7. Rigid metric spaces

In this final section, we will apply Theorem 6.13 to obtain new examples of rigid spaces which are not obviously covered by previously existing results.

[20] By [13, Theorem 1.2], $C^*_u(X)$ and $C^*_u(Y)$ are Morita equivalent if and only if they are stably $*$-isomorphic.

Let us start with the following result, which asserts that measured asymptotic expanders are obstructions to measured weak embeddings into L^p-spaces.

Proposition 7.1. *Any sequence of measured asymptotic expanders with uniformly bounded geometry cannot be measured weakly embedded into any L^p-space for $p \in [1, \infty)$.*

Proof. We assume that there is a sequence of measured asymptotic expanders $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ with uniformly bounded geometry which could be measured weakly embedded into $L^p(Z; \nu)$ for some measure space (Z, ν) and $p \in [1, \infty)$. By Corollary 4.21, we can assume that $\{(X_n, d_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of measured expander graphs with valency uniformly bounded by $K \geq 1$ such that all m_n have full support with uniformly bounded measure ratio $s \in (0, 1)$ on adjacent vertices.

By Corollary 5.3, there exists a positive constant c such that for any $n \in \mathbb{N}$ and any map $g : X_n \to \mathbb{C}$, we have the following L^p-Poincaré inequality:

$$\sum_{u,v\in X_n:u\sim v} |g(u) - g(v)|^p(m_n(u) + m_n(v)) \geq c \sum_{u,v\in X_n} |g(u) - g(v)|^p \frac{m_n(u)m_n(v)}{m_n(X_n)}. \tag{7.1}$$

If $\{f_n : (X_n, d_n, m_n) \to L^p(Z, \nu)\}_{n\in\mathbb{N}}$ is a measured weak embedding, then all maps f_n can be chosen to be L-Lipschitz for some $L > 0$ by Remark 2.7. Integrating the inequality (7.1) over (Z, ν), we obtain that

$$\sum_{u,v\in X_n} \|f_n(u) - f_n(v)\|^p \frac{m_n(u)m_n(v)}{m_n(X_n)} \leq \frac{L^p}{c} \sum_{u,v\in X_n:u\sim v} (m_n(u) + m_n(v)) \leq \frac{(1 + s)KL^p}{sc} m_n(X_n).$$

As $\sum_{u\in X_n} \frac{m_n(u)}{m_n(X_n)} = 1$, using the pigeonhole principle we deduce that there exists at least one $u \in X_n$ such that

$$\sum_{v\in X_n} \|f_n(u) - f_n(v)\|^p \frac{m_n(v)}{m_n(X_n)} \leq \frac{(1 + s)KL^p}{sc}.$$

If we denote $M := \frac{(1+s)KL^p}{sc}$, then the set $\{v \in X_n : \|f_n(u) - f_n(v)\|^p \leq 2M\}$ has measure greater than $\frac{1}{2}m_n(X_n)$ for every $n \in \mathbb{N}$. Equivalently, we have

$$\frac{m_n(f_n^{-1}(B(f_n(u), (2M)^{1/p})))}{m_n(X_n)} > \frac{1}{2} \text{ for every } n \in \mathbb{N}.$$

This contradicts with the assumption that $\{f_n\}_{n\in\mathbb{N}}$ is a measured weak embedding. □

By Lemma 2.8 and Proposition 7.1, we obtain the following result which extends the L^p-case of [24, Theorem 4.2].

Corollary 7.2. *A coarse disjoint union of a sequence of ghostly measured asymptotic expanders with uniformly bounded geometry cannot coarsely embed into any L^p-space for $p \in [1, \infty)$.*

As a consequence of Theorem 6.13, rigidity holds for metric spaces which coarsely embed into some L^p-space for $p \in [1, \infty)$ as stated in Corollary E.

We finish this section by extending the main results in [4, 16] to the measured asymptotic case. More precisely, we show that there exist box spaces that do not coarsely embed into any L^p-space, but do not measured weakly contain any measured asymptotic expanders. Consequently, we obtain that the rigidity problem holds for such box spaces by Lemma 2.8 and Theorem 6.13.

To this end, we need the following key ingredient, which is a measured asymptotic analogue of [4, Proposition 2]:

Proposition 7.3. *Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of finitely generated groups with generating sets $\{S_n\}_{n\in\mathbb{N}}$ such that $|S_n| \leq k$ for all $n \in \mathbb{N}$. We assume that for each $n \in \mathbb{N}$, we have a short exact sequence*

$$1 \to N_n \to G_n \to Q_n \to 1$$

such that:

- *The sequence $(N_n)_{n\in\mathbb{N}}$ equipped with the induced metric coarsely embed into a Hilbert space;*
- *The sequence $(Q_n)_{n\in\mathbb{N}}$ equipped with the word metric associated to the projection T_n of S_n coarsely embeds into a Hilbert space.*

Then the coarse disjoint union $G := \bigsqcup_{n\in\mathbb{N}} G_n$ does not measured weakly contain any measured asymptotic expanders with uniformly bounded geometry.

Before we prove Proposition 7.3, we need the following lemma, which is the measured analogue of [4, Lemma 2.1 and Corollary 2.2]:

Lemma 7.4. *Let $p \in [1,\infty)$, $\alpha \in (0,1]$ and $c > 0$. If $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of c-measured expander graphs with valency uniformly bounded by $K \geq 1$ such that all m_n have full support with uniformly bounded measure ratio $s \in (0,1)$ on adjacent vertices, then there exists $D > 0$ depending only on p, α, c, K and s such that for any $n \in \mathbb{N}$, any $A \subseteq V_n$ with $m_n(A) \geq \alpha \cdot m_n(V_n)$ and any 1-Lipschitz map f from A to an L^p-space, we have*

$$\sum_{u,v\in A} \|f(u) - f(v)\|^p \frac{m_n(u)m_n(v)}{m_n(A)^2} \leq D. \tag{7.2}$$

In particular, there exists $x_0 \in A$ such that the set $\{v \in A : \|f(x_0) - f(v)\| \leq (2D)^{\frac{1}{p}}\}$ has measure greater than $m_n(A)/2$ for every $n \in \mathbb{N}$.

Proof. We will follow the same scheme as the proof of [4, Lemma 2.1] with minor modifications. There are two cases: $\alpha = 1$ and $\alpha \in (0,1)$.

If $\alpha = 1$, then $A = V_n$ because m_n has full support. Let f be any 1-Lipschitz map from V_n to an L^p-space. By Corollary 5.3, there exists a positive constant c' only depending on c, s, p, K such that

$$\sum_{u,v\in V_n} \|f(u) - f(v)\|^p \frac{m_n(u)m_n(v)}{m_n(V_n)^2} \leq \frac{1}{c' \cdot m_n(V_n)} \sum_{u,v\in V_n : u\sim_{E_n} v} (m_n(u) + m_n(v)) \leq \frac{(1+s)K}{sc'}.$$

In this case, we simply choose $D := \frac{(1+s)K}{sc'}$ so that (7.2) holds.

If $\alpha \in (0,1)$, then $1-\alpha \in (0,1)$. By Lemma 4.20, there exists a $c_{1-\alpha} > 0$ depending only on c and α such that

$$h^n_\alpha := \min\Big\{\frac{m_n(\partial^{V_n} B)}{m_n(B)} : B \subseteq V_n \text{ with } 0 < m_n(B) \leq (1-\alpha)\cdot m_n(V_n)\Big\} > c_{1-\alpha}$$

for all $n \in \mathbb{N}$.

Given $A \subseteq V_n$ with $m_n(A) \geq \alpha \cdot m_n(V_n)$. For every $i \in \mathbb{N}$, set $U_i := \mathcal{N}_i(A)^c$ and $W_i := U_i \setminus U_{i+1}$. As $(\partial^{V_n} U_{i+1}) \sqcup U_{i+1} \subseteq U_i$, and $m_n(U_{i+1}) \leq (1-\alpha)\cdot m_n(V_n)$ for every

$i \in \mathbb{N}$, it follows that

$$m_n(U_i) \geq m_n(\partial^{V_n} U_{i+1}) + m_n(U_{i+1}) \geq (1 + h_\alpha^n) \cdot m_n(U_{i+1}).$$

Thus, for every $i \in \mathbb{N}$

$$m_n(U_{i+1}) \leq \frac{m_n(U_1)}{(1 + h_\alpha^n)^i} < \frac{m_n(V_n)}{(1 + c_{1-\alpha})^i}.$$

Now we extend f from A to all of X as follows: for any $x \in A^c$, we choose a point $a_x \in A$ such that $d_n(x, A) = d_n(x, a_x)$ and we set $f(x) := f(a_x)$. If $x, y \in A^c$ with $x \sim_{E_n} y$, then $d_n(a_x, a_y) \leq d_n(x, A) + d_n(y, A) + 1$. Thus, we deduce that $d_n(a_x, a_y) \leq 2i + 4$ for every $x \in W_i$ and every $y \in V_n$ with $x \sim_{E_n} y$. As f is 1-Lipschitz on A, then $\|f(x) - f(y)\| \leq 2i + 4$ for every $x \in W_i$ with $x \sim_{E_n} y$. As $A^c = \bigsqcup_{i \in \mathbb{N}} W_i$, we have that

$$\begin{aligned}
&\sum_{u \in A^c} \sum_{v \in V_n:\ u \sim_{E_n} v} \|f(u) - f(v)\|^p (m_n(u) + m_n(v)) \\
&\leq \sum_{i \in \mathbb{N}} \sum_{u \in W_i} \sum_{v \in V_n:\ u \sim_{E_n} v} (2i + 4)^p (m_n(u) + m_n(v)) \\
&\leq \sum_{i \in \mathbb{N}} (2i + 4)^p \frac{(1 + s)K}{s} m_n(W_i) \\
&< \frac{(1 + s)K}{s} m_n(V_n) \sum_{i \in \mathbb{N}} \frac{(2i + 4)^p}{(1 + c_{1-\alpha})^{i-1}},
\end{aligned}$$

where $\sum_{i \in \mathbb{N}} \frac{(2i+4)^p}{(1+c_{1-\alpha})^{i-1}}$ is clearly convergent and its limit denotes by $\theta(p, c, \alpha)$. Moreover, we have $\|f(x) - f(y)\| \leq 2$ if $x \in A$ and $x \sim_{E_n} y$. Hence, it follows that

$$\sum_{u \in A} \sum_{v \in V_n:\ u \sim_{E_n} v} \|f(u) - f(v)\|^p (m_n(u) + m_n(v)) \leq 2^p \frac{(1 + s)K}{s} m_n(V_n).$$

Combining them together, we conclude that

$$\begin{aligned}
&\sum_{u, v \in V_n : u \sim_{E_n} v} \|f(u) - f(v)\|^p (m_n(u) + m_n(v)) \\
&< 2^p \frac{(1 + s)K}{s} m_n(V_n) + \frac{(1 + s)K}{s} \cdot \theta(p, c, \alpha) \cdot m_n(V_n) \\
&= \frac{(1 + s)K}{s} (2^p + \theta(p, c, \alpha)) \cdot m_n(V_n).
\end{aligned}$$

Now Corollary 5.3 provides us a positive constant c' depending only on p, K, s and c so that

$$\begin{aligned}
&\sum_{u, v \in V_n} \|f(u) - f(v)\|^p \frac{m_n(u) m_n(v)}{m_n(V_n)^2} \\
&\leq \frac{1}{c' \cdot m_n(V_n)} \sum_{u, v \in V_n : u \sim_{E_n} v} \|f(u) - f(v)\|^p (m_n(u) + m_n(v)) \\
&< \frac{(1 + s)K}{c' s} (2^p + \theta(p, c, \alpha)).
\end{aligned}$$

As $m_n(A) \geq \alpha \cdot m_n(V_n)$, we conclude that (7.2) holds for $D := \frac{(1+s)K}{\alpha^2 c' s}\Big(2^p + \theta(p, c, \alpha)\Big)$. As for the last statement, we simply apply the pigeonhole principle to (7.2) as in the proof of Proposition 7.1. □

Proof of Proposition 7.3. We assume that there is a sequence of measured asymptotic expanders with uniformly bounded geometry which could be measured weakly embedded into G. From Corollary 4.21, we obtain a sequence of c-measured expander graphs $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ with valency uniformly bounded by $K \geq 1$ such that all m_n have full support with uniformly bounded measure ratio $s \in (0,1)$ on adjacent vertices such that the sequence $\{(V_n, E_n, m_n)\}_{n\in\mathbb{N}}$ measured weakly embeds into G. If $\{g_n : (V_n, E_n, m_n) \to G\}_{n\in\mathbb{N}}$ is such a measured weak embedding, then for every $y \in G$ the preimages $g_n^{-1}(y) \neq \emptyset$ for only finitely many n's by Lemma 6.9. Since all (V_n, E_n) are connected graphs and all g_n are M-Lipschitz for some $M > 0$, we may remove finitely many V_n's if necessary so that every V_n in the remaining sequence is mapped into a single piece G_{k_n} for some $k_n \in \mathbb{N}$. Using Lemma 6.9 again, we may (after taking a subsequence if necessary) assume that each g_n maps V_n into G_n for all $n \in \mathbb{N}$.

We reach a contradiction with the fact that $\{g_n\}_{n\in\mathbb{N}}$ is a measured weak embedding by following the proof of [4, Proposition 2] step by step, except we use Lemma 7.4 instead of [4, Corollary 2.2], and replace the counting measure by m_n. □

Now we are able to strengthen [4, Theorem 1] as follows:

Theorem 7.5. *There exists a box space X of a finitely generated residually finite group such that X does not coarsely embed into any L^p-space for $1 \leq p < \infty$[21], yet does not measured weakly contain any measured asymptotic expanders with uniformly bounded geometry.*

Proof. The proof is identical to the proof of [4, Theorem 7.1] except that we use Proposition 7.3 instead of [4, Proposition 2] at the very end of the proof. For this reason, we shall not repeat the argument. □

In fact, the graphs constituting the box space X in Theorem 7.5 can be additionally required to have unbounded *girth* (*i.e.*, the length of the shortest cycle). For this, we need the following stronger version of [16, Proposition 2.4], whose proof is the same except that we use Proposition 7.3 instead of [4, Proposition 2] therein. To avoid too much word repetition, we omit its proof here.

Corollary 7.6. *Let G be a finitely generated, residually finite group and let $\{N_n\}_{n\in\mathbb{N}}$ be a sequence of nested finite index normal subgroups of G with trivial intersection. If $\{M_n\}_{n\in\mathbb{N}}$ is another sequence of finite index normal subgroups of G with $N_n > M_n$ for all $n \in \mathbb{N}$ and the box space $\bigsqcup_{n\in\mathbb{N}} G/N_n$ coarsely embeds into a Hilbert space, then the box space $\bigsqcup_{n\in\mathbb{N}} G/M_n$ does not measured weakly contain any measured asymptotic expanders with uniformly bounded geometry.*

Now we are ready to generalise [16, Theorem 4.5] in the following way:

Theorem 7.7. *There exists a box space X of the free group F_3 such that X does not coarsely embed into any L^p-space for $1 \leq p < \infty$, yet does not measured weakly contain any measured asymptotic expanders with uniformly bounded geometry.*

Proof. We refer to the proof of [16, Theorem 4.5] for details. In the proof, we have to apply our Corollary 7.6 instead of [16, Proposition 2.4] herein in order to obtain

[21]The box space X also does not coarsely embed into any *uniformly curved* Banach space (see [4, Definition 6.1] for the definition).

the second part of the statement, while the first part follows from the last remark in [16, Section 5][22]. □

References

[1] Noga Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.

[2] Noga Alon and Vitali D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B*, 38(1):73–88, 1985.

[3] Goulnara Arzhantseva, Erik Guentner, and Ján Špakula. Coarse non-amenability and coarse embeddings. *Geom. Funct. Anal.*, 22(1):22–36, 2012.

[4] Goulnara Arzhantseva and Romain Tessera. Relative expanders. *Geom. Funct. Anal.*, 25(2):317–341, 2015.

[5] Goulnara Arzhantseva and Romain Tessera. Admitting a coarse embedding is not preserved under group extensions. *Int. Math. Res. Not. IMRN*, (20):6480–6498, 2019.

[6] Florent P. Baudier, Bruno de Mendonça Braga, Ilijas Farah, Ana Khukhro, Alessandro Vignati, and Rufus Willett. Uniform Roe algebras of uniformly locally finite metric spaces are rigid. *arXiv:2106.11391*, 2021.

[7] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[8] Bruno M. Braga, Yeong Chyuan Chung, and Kang Li. Coarse Baum-Connes conjecture and rigidity for Roe algebras. *J. Funct. Anal.*, 279(9):108728, 2020.

[9] Bruno M. Braga and Ilijas Farah. On the rigidity of uniform Roe algebras over uniformly locally finite coarse spaces. *Trans. Amer. Math. Soc.*, 374(2):1007–1040, 2021.

[10] Bruno M. Braga, Ilijas Farah, and Alessandro Vignati. Embeddings of uniform Roe algebras. *Comm. Math. Phys.*, 377(3):1853–1882, 2020.

[11] Bruno M. Braga, Ilijas Farah, and Alessandro Vignati. Uniform Roe coronas. *Adv. Math.*, 389:Paper No. 107886, 2021.

[12] Bruno M. Braga and Alessandro Vignati. On the uniform Roe algebra as a Banach algebra and embeddings of ℓ_p uniform Roe algebras. *Bull. Lond. Math. Soc.*, 52(5):853–870, 2020.

[13] Lawrence G. Brown, Philip Green, and Marc A. Rieffel. Stable isomorphism and strong Morita equivalence of C^*-algebras. *Pacific J. Math.*, 71(2):349–363, 1977.

[14] Xiaoman Chen, Qin Wang, and Guoliang Yu. The maximal coarse Baum-Connes conjecture for spaces which admit a fibred coarse embedding into Hilbert space. *Adv. Math.*, 249:88–130, 2013.

[15] Yeong Chyuan Chung and Kang Li. Rigidity of ℓ^p Roe-type algebras. *Bull. Lond. Math. Soc.*, 50(6):1056–1070, 2018.

[16] Thiebout Delabie and Ana Khukhro. Box spaces of the free group that neither contain expanders nor embed into a Hilbert space. *Adv. Math.*, 336:70–96, 2018.

[17] Jintao Deng, Qin Wang, and Guoliang Yu. The coarse Baum-Connes conjecture for certain extensions and relative expanders. *arXiv preprint arXiv:2102.10617*, 2021.

[18] Jozef Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[19] Martin Finn-Sell. Fibred coarse embeddings, a-T-menability and the coarse analogue of the Novikov conjecture. *J. Funct. Anal.*, 267(10):3758–3782, 2014.

[20] Mikhael Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003.

[21] Erik Guentner and Jerome Kaminker. Geometric and analytic properties of groups. In *Noncommutative geometry*, volume 1831 of *Lecture Notes in Math.*, pages 253–262. Springer, Berlin, 2004.

[22] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[23] William B. Johnson and N. Lovasoa Randrianarivony. l_p $(p > 2)$ does not coarsely embed into a Hilbert space. *Proc. Amer. Math. Soc.*, 134(4):1045–1050, 2006.

[22] The authors claimed in this remark that their box space does not coarsely embed into ℓ^p, but it is straightforward to check that it actually cannot coarsely embed into any L^p-space for $1 \le p < \infty$ by the complex interpolation method as in the proof of [4, Proposition 4]).

[24] Ana Khukhro, Kang Li, Federico Vigolo, and Jiawen Zhang. On the structure of asymptotic expanders. *Adv. Math.*, 393:Paper No. 108073, 35, 2021.

[25] Kang Li, Piotr Nowak, Ján Špakula, and Jiawen Zhang. Quasi-local algebras and asymptotic expanders. *Groups Geom. Dyn.*, 15(2):655–682, 2021.

[26] Kang Li, Federico Vigolo, and Jiawen Zhang. A Markovian and Roe-algebraic approach to asymptotic expansion in measure. *arXiv:2008.12572, 2020*.

[27] Kang Li, Federico Vigolo, and Jiawen Zhang. Asymptotic expansion in measure and strong ergodicity. *arXiv:2005.05697, to appear in Journal of Topology and Analysis*, 2020.

[28] Kang Li, Ján Špakula, and Jiawen Zhang. Measured expanders. *arXiv:2104.06052, to appear in Journal of Topology and Analysis*, 2021.

[29] Kang Li, Zhijie Wang, and Jiawen Zhang. A quasi-local characterisation of L^p-Roe algebras. *J. Math. Anal. Appl.*, 474(2):1213–1237, 2019.

[30] Alexander Lubotzky. High dimensional expanders. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 705–730. World Sci. Publ., Hackensack, NJ, 2018.

[31] Jiří Matoušek. On embedding expanders into l_p spaces. *Israel J. Math.*, 102:189–197, 1997.

[32] Manor Mendel and Assaf Naor. Metric cotype. *Ann. of Math. (2)*, 168(1):247–298, 2008.

[33] Piotr W. Nowak. On coarse embeddability into l_p-spaces and a conjecture of Dranishnikov. *Fund. Math.*, 189(2):111–116, 2006.

[34] John Roe. An index theorem on open manifolds. I, II. *J. Differential Geom.*, 27(1):87–113, 115–136, 1988.

[35] John Roe. *Index theory, coarse geometry, and topology of manifolds*, volume 90 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996.

[36] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[37] Robert Michael Tanner. Explicit concentrators from generalized N-gons. *SIAM J. Algebraic Discrete Methods*, 5(3):287–293, 1984.

[38] Ján Špakula and Aaron Tikuisis. Relative commutant pictures of Roe algebras. *Comm. Math. Phys.*, 365(3):1019–1048, 2019.

[39] Ján Špakula and Rufus Willett. On rigidity of Roe algebras. *Adv. Math.*, 249:289–310, 2013.

[40] Ján Špakula and Jiawen Zhang. Quasi-locality and Property A. *J. Funct. Anal.*, 278(1):108299, 25, 2020.

[41] Stuart White and Rufus Willett. Cartan subalgebras in uniform Roe algebras. *Groups Geom. Dyn.*, 14(3):949–989, 2020.

[42] Rufus Willett and Guoliang Yu. *Higher Index Theory*. 189. Cambridge University Press, 2020.

[43] Wolfgang Woess. *Random walks on infinite graphs and groups*, volume 138 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.

[44] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.

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