UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING AND PHYSICAL SCIENCES
School of Physics and Astronomy

Holographic Correlators and Their (Hidden) Symmetries

by

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ABSTRACT

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In this thesis we discuss several aspects of correlators at strong coupling in three different theories: $\mathcal{N} = 4$ Super Yang-Mills (SYM) in four dimensions, the D3-D7 system and the D1-D5 CFT. In the first part we focus on $\mathcal{N} = 4$ SYM. After reviewing some background material, we outline a procedure to compute tree-level α' corrections to the four-point function of half-BPS operators, dual - via AdS/CFT - to the scattering of four Kaluza-Klein modes in $AdS_5 \times S^5$. The results enjoy remarkable features, ultimately ascribed to a hidden $10d$ conformal structure governing the dynamics of $\mathcal{N} = 4$ SYM in the supergravity regime. The method, that relies on the understanding of certain patterns in the CFT data at large N, also fixes uniquely a particular class of anomalous dimensions associated to long double-trace operators exchanged. These anomalous dimensions are the zeros of a characteristic polynomial which enjoys a lot of very intriguing properties which we describe in detail. In the second part of the thesis we explore two other conformal field theories: the D3-D7 system, described by a certain $4d$ $\mathcal{N} = 2$ superconformal field theory and the D1-D5 CFT, which is a $2d$ $\mathcal{N} = (4, 4)$ superconformal field theory. We show that the extreme simplicity of the tree-level dynamics in these theories - due to $8d$ and $6d$ hidden conformal symmetries - allows to define generalised Mellin transforms which make manifest many properties of the associated AdS amplitudes, such as the large p limit, and - in the case of the D3-D7 CFT - Bern-Carrasco-Johannson (BCJ) and double-copy relations. We then compute the anomalous dimensions of the long-double trace operators exchanged, and show that, like in $\mathcal{N} = 4$ SYM, they are simple rational function of their quantum numbers and manifest a residual degeneracy, as a consequence of the hidden symmetries. Lastly, we show that many of the formulae we recalled/derived in this thesis can be nicely assembled into compact expressions which interpolate between different theories.

Contents

List of Figures

List of Tables

Declaration of Authorship

I, Michele Santagata, declare that this thesis entitled "Holographic Correlators and Their (Hidden) Symmetries" and the work presented in it are my own and have been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as: [1–5]

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To my parents Marcello and Mirella, and my brother Giuseppe

Chapter 1

Introduction

Scattering amplitudes are among the most intriguing objects in Quantum Field Theories[1] (QFTs). They provide us with theoretical data which are then compared with experiments. Yet, they are a powerful playground for our understanding of QFTs; they very often reveal unexpected simplicity and hidden structures which are not manifest in the usual Feynman diagrammatic approach. This is ultimately due to the fact that scattering amplitudes, unlike some other quantities, are gauge-independent objects. A classic example is the celebrated Parke-Taylor formula [7–9], which provides the tree-level n-point Maximally Helicity Violating (MHV) amplitude of gluons in Yang-Mills (YM) theory. In the spinor-helicity formalism, it reads

$$A_n[1^+ \dots i^- j^- \dots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}. \tag{1.0.1}$$

There is no need to introduce the notation used in (1.0.1), because the point being made is as simple as this: the beauty and the compactness of the formula is surprising when compared to the huge number of Feynman diagrams - which grow factorially with the number of external states n - contributing to the process.

As another example, consider the four-point amplitude of gravitons in general relativity:

$$M_4[1^- 2^- 3^+ 4^+] = \frac{\langle 12 \rangle^7 [12]}{\langle 13 \rangle \langle 14 \rangle \langle 23 \rangle \langle 24 \rangle \langle 34 \rangle^2} = \frac{\langle 12 \rangle^4 [34]^4}{stu}. \tag{1.0.2}$$

where s, t, u are the Mandelstam variables. Again, despite a very bad-looking non-linear Lagrangian, the end result is just (1.0.2). These two significant examples capture the core idea of the *amplitude program*: one tries to focus on the full amplitude, rather than on objects which depend on field redefinitions or gauge choices such as Lagrangians and Feynman diagrams. In short, we can say that the modern amplitude perspective tries to address two (related) tasks:

[1]See [6] for a recent review on the subject.

1. find a framework where the simplicity is manifest. In the case of the MHV amplitude (1.0.1) this is the spinor-helicity formalism;

2. understand the reasons behind the simplicity, as the existence of symmetries. The compactness of the Parke-Taylor formula, for example, can be understood with the fact that it has support on the simplest curve (a straight line) in twistor space [10].

The study of scattering amplitudes has led to remarkable achievements in our understanding of gravity, gauge theories and their connection. In fact, while so different-looking - at least at the level of the Lagrangian - Einstein and Yang-Mills amplitudes are intimately related. Continuing with the two examples above, it is easy to show that

$$M_4[1^-2^-3^+4^+] = sA_4[1^-2^-3^+4^+]A_4[1^-2^-4^+3^+]. \tag{1.0.3}$$

Here, A_4 is the four-gluon MHV amplitude, i.e. (1.0.1) with $n = 4$:

$$A_4[1^-2^-3^+4^+] = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \frac{\langle 12\rangle^2[34]^2}{st}, \tag{1.0.4}$$

where in the second equality we made manifest the Mandelstam variable dependence. Equation (1.0.3) is the simplest of the so-called Kawai-Lewellen-Tye (KLT) relations, which were originally discovered in the context of string theory [11]. Together with the closely-related Bern-Carrasco-Johannson (BCJ), color-kinematic (CK) duality and double-copy constructions [12], they seem to support the beautiful idea that there is an underlying common framework for gravity and Yang-Mills theories, such as string theory. These relations are often summarised with the statement

$$\text{gravity} \sim \text{Yang-Mills} \times \text{Yang-Mills}, \tag{1.0.5}$$

i.e. gravity amplitudes are obtained by "squaring" Yang-Mills amplitudes. Despite - beyond tree-level [13–16] - a general proof for these relations is still lacking, there are by now explicit double-copy constructions for a wide class of theories (see e.g. [17] for an extensive review on the subject). Needless to say, the hope is that all of these properties will, one day, shed light on some of the most challenging and beautiful questions of theoretical physics:

> what are the UV completions of quantum gravity?
> And how are they related to string theory?

History has already taught us that amplitudes can certainly play a primary role in addressing these questions. In fact, the birth of string theory itself is exemplary from this point of view. In the mid-1960s, Veneziano constructed a four-point scattering amplitude to model certain features observed in hadron resonances. For various reasons, the model did not work. A decade later, in 1974, Schwarz and Scherk, and independently Yoneya, realised that the model was capable of describing spin-1 as well as spin-2 massless particles. They then proposed that string theory should be considered as a theory of all interactions, at a more microscopic level, rather than a model for strong interactions. Their idea was furthermore supported by the fact that the Veneziano (and its Virasoro-Shapiro cousin) amplitude enjoyed a number of properties, such as Regge trajectories and suppression of high-energy modes, which could in principle soften the bad ultraviolet behaviour of gravity.

It should be noted, however, that while many of these relations have been studied in flat space, much less is known in curved backgrounds, mainly because of the high complexity of performing such computations in curved spaces. This is nevertheless a very important subject, due to its connection to very active areas of modern physics like cosmology and black holes, where curvature effects are expected to become dominant. The simplest curved background where seeking and exploring new properties, is perhaps the Anti-De-Sitter (AdS) space. In fact, with the groundbreaking discovery of AdS/CFT correspondence [18–20], there is now a wealth of new tools to explore questions about quantum gravity in AdS.

The AdS/CFT correspondence states the equivalence between two *a priori* different theories:

- a theory of gravity in AdS in $d+1$ dimensions;
- a conformal field theory (CFT) in d dimensions living on the boundary, with no gravity.

The duality establishes an equivalence between the *dynamics* of these two theories and, quite surprisingly, reveals that gravity dynamics is captured by a quantum field theory in one dimension lower. Crucially, the correspondence maps scattering of states in AdS to CFT correlation functions, therefore the problem of computing scattering amplitudes in AdS can be turned into a computation of CFT correlators for which we have, in principle, more control. In fact, the rapid advancing of new CFT tools, and in particular of the bootstrap program, has led to stunning achievements in the computation of these so-called holographic correlators. The best studied example is undoubtedly the duality between IIB string theory in $AdS_5 \times S^5$ and $\mathcal{N}=4$ Super Yang-Mills (SYM), for which explicit results are now available at tree [21–32] and loop level [33–43] in supergravity as well as for tree [1, 3, 44–51] and loop [52, 53] string corrections[2].

In particular, all four-point functions of arbitrary (Kaluza-Klein) KK modes are now completely known at tree-level in supergravity. They were computed for the first time

[2]See [54, 55] for an up-to-date review on the state of the art of this program.

in [26, 27], with the help of a bootstrap approach in Mellin space. In the notation of [32], the amplitude reads

$$\mathcal{M}_{\mathrm{SUGRA}} = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}. \tag{1.0.6}$$

We will say more about notation and the many intriguing properties of this formula in the main body of the thesis. For now, let us just point out that tasks 1. and 2. we mentioned earlier have, in this case, been (perhaps partially?) addressed. In fact,

1. the amplitude admits a very compact representation in the large p formalism which has also the advantage of making explicit the so-called large p limit. This a refinement of the Mellin space formalism [56–59] which was already known to capture many features of the AdS dynamics;

2. the extreme simplicity of the correlator is due to the existence of an hidden $10d$ conformal symmetry [31], that controls the tree-level dynamics in supergravity. In particular, it allows the repackaging of all KK correlators into a single object (1.0.6).

Importantly, the hidden symmetry also provides an explanation for the observed simplicity in the spectrum of anomalous dimensions of the double-trace operators exchanged in the OPE at large N, which were computed in [33, 37].

Away from the supergravity limit, there is evidence that the hidden symmetry is broken [3, 50]. Nevertheless, its breaking still plays a crucial role in constraining the dynamics of four-point scattering beyond supergravity [3]. Supergravity and tree-level string corrections to four-point correlation functions of KK modes in $AdS_5 \times S^5$ are the main subject of the first part of the thesis.

The development of the bootstrap program has led to very important results also in other theories, such as the maximally supersymmetric backgrounds $AdS_4 \times S^7$ and $AdS_7 \times S^4$ [60, 61] as well as in theories with less supersymmetry such as $AdS_2 \times S^2$ [62], $AdS_3 \times S^3$ [4, 63–66] and $AdS_5 \times S^3$ [5, 67–70]. These last three theories, namely $AdS_2 \times S^2$, $AdS_5 \times S^3$ and $AdS_3 \times S^3$, enjoy hidden conformal symmetries at tree level analogous to that found in $\mathcal{N} = 4$. The second part of the thesis will be devoted to exploring the tree-level dynamics of the latter two theories.

The $AdS_5 \times S^3$ background arises as the low-energy limit of the so-called D3-D7 system and is dual to a certain $4d$ $\mathcal{N} = 2$ SCFT. The four-point function of half-BPS operators in the dual $4d$ $\mathcal{N} = 2$ theory computes the scattering of supergluons in $AdS_5 \times S^3$. Because of the hidden conformal symmetry, this system is an ideal, yet non-trivial, place for seeking AdS analogs of double-copy, colour-kinematic and BCJ relations. In fact, it turns out that such relations do hold in this background, at least for four-point functions, in a way strikingly similar to flat space [5, 68].

The $AdS_3 \times S^3$ background is dual to the so-called D1-D5 CFT, which is a $2d$ $\mathcal{N} = (4, 4)$ SCFT. The dual CFT, being two-dimensional, appears to be much more tractable than higher dimensional counterparts, therefore it is an ideal playground to test and, perhaps, derive the AdS/CFT correspondence [71]. Moreover, the theory has been the subject of much exploration at weak coupling in supergravity/strong coupling in the CFT [4, 63–66], particularly in the context of black hole physics and of the microstate program [72]. Interestingly, it has been found that there exists a family of pure CFT states dual to smooth horizonless supergravity solutions in the bulk [73–79], that suggest a possible resolution of the Hawking paradox [80, 81].

In the supergravity regime, the spectrum after KK reduction onto S^3, consists of two different multiplets. The tensor multiplet sector enjoys an hidden $6d$ conformal symmetry [64] and, as a consequence of this, the four-point function of half-BPS operators in the dual SCFT at strong coupling can be written, analogously to $\mathcal{N} = 4$, in terms of a single Mellin amplitude.

Outline of the thesis

We will give a slightly more detailed overview at the beginning of each chapter; here we present a very brief outline of the thesis.

The thesis is divided in two parts. In part I we discuss various aspects of four-point correlators of half-BPS operators in $\mathcal{N} = 4$ SYM at strong coupling, dual to the scattering of four KK modes in $AdS_5 \times S^5$. In chapter 2, we start by reviewing some well known facts about $\mathcal{N} = 4$ and their correlation functions. Then, in chapter 3, we review some basics of AdS/CFT, with a particular emphasis on the archetypal example of duality between $\mathcal{N} = 4$ and IIB string theory on $AdS_5 \times S^5$. In chapter 4 we discuss the supergravity correlator in $AdS_5 \times S^5$ in more detail, and introduce the so-called large p formalism [32]. The large p formalism turns out to be very useful also for tree-level string corrections. These are the subjects of chapter 5. There, we present the results for tree-level string correlators obtained via a bootstrap approach developed in [1, 3], which we will explain in great detail in chapter 7. In order to better understand that, we first need to review (chapter 6) the supergravity spectrum and in particular the computation of anomalous dimensions, first done in [33, 37].

Part II is devoted to the study of four-point functions of half-BPS operators in $AdS_5 \times S^3$ and $AdS_3 \times S^3$ backgrounds. In chapter 8, we recall some facts about the $AdS_5 \times S^3$ setup. We adapt the large p formalism of [32] to the $AdS_5 \times S^3$ case, and define a suitable Mellin transform which has the advantage of making manifest many properties of the amplitude. In chapter 9 we review BCJ and double copy constructions in flat space amplitudes and then, after presenting the tree-level correlators in the large p formalism, we show that analogous relations hold in $AdS_5 \times S^3$. In chapter 10 we study the double-trace spectrum of the theory and compute all leading order anomalous dimensions. The results are reminiscent of the ones found in $\mathcal{N} = 4$, confirming once again that the hidden symmetry plays a prominent role in constraining these CFT data.

Next, we move to the D1-D5 CFT, which is a $2d$ $\mathcal{N} = (4, 4)$ SCFT dual to string theory on $AdS_3 \times S^3$. By following the same structure as the other setups, we first recall (chapter 11) some basics of the theory. Then, following [4], we define an $AdS_3 \times S^3$ Mellin transform which makes manifest the hidden $6d$ conformal symmetry enjoyed by these correlators. We then investigate the double-trace spectrum of the theory at large N. To do so, we first compute the relevant long superconformal blocks and then find the anomalous dimensions. This is explained in chapter 12.

In chapter 13 we present some general (and intriguing) formulae which interpolate between these three cases. In the last section we discuss some possible string corrections to $AdS_3 \times S^3$ and $AdS_5 \times S^3$ amplitudes and the associated string-corrected spectrum.

Finally, we draw some conclusions, by giving an overview of the thesis and discussing some possible future works.

Part I

$\mathcal{N} = 4$ SYM at strong coupling

Prologue I

We begin the journey with $\mathcal{N} = 4$ SYM, the most supersymmetric gauge theory in four dimensions. The theory has been subject of a considerable amount of work in the past several years, both at weak [82–100] as well as strong coupling [1, 3, 21–34, 37–50, 52, 53] in the gauge parameter.

Surprisingly, $\mathcal{N} = 4$ SYM appears to be, in the so-called planar limit, integrable - a feature which is usually confined to two-dimensional models[3]. The many hints of simplicity make the theory an ideal model for understanding properties shared by some more phenomenologically relevant cousins, such as QCD.

$\mathcal{N} = 4$ SYM is a very useful theoretical laboratory not only for testing our understanding of quantum field theories, but also for quantum gravity. In fact, as mentioned in the introduction, it is dual to IIB superstring theory on $AdS_5 \times S^5$ [18–20]. In particular, the strong coupling regime of $\mathcal{N} = 4$ SYM maps to the weakly-coupled regime of superstring theory, therefore, together with AdS/CFT, it provides us a very concrete way to explore various aspects of quantum gravity.

It is the purpose of this first part to go through some recent advances in the computation of four-point functions of half-BPS operators in $\mathcal{N} = 4$ at strong coupling. We will discuss a recently proposed bootstrap method [1, 3] to compute tree-level correlators at any order in α', dual to the (low-energy expansion of) Virasoro-Shapiro amplitude in $AdS_5 \times S^5$. The method relies on the understanding of a pattern on the structure of the anomalous dimensions of the operators exchanged in the OPE at leading order. Reassuringly, the amplitudes obtained in this way precisely match with those of [50] where a different method is used. The results indicate that a magic hidden structure, yet to be fully understood, controls these observables.

[3]See [101] for a review on AdS/CFT integrability.

Chapter 2

Basic facts about $\mathcal{N} = 4$ SYM

In this chapter we review some well known facts about $\mathcal{N} = 4$ SYM. We begin by recalling a few properties of the theory; we then introduce the notion of of superconformal primary operators and, in particular, the half-BPS primary operators (section 2.1). In section 2.2 we discuss two-point, three-point and four-point functions of these operators. Finally, in section 2.3, we introduce the important concept of the operator product expansion (OPE) and the related superconformal blocks.

$\mathcal{N} = 4$ SYM has 16 real supercharges, which is the maximum allowed number in four dimensions if we do not want to include gravity. On top of the gauge group, which we will take to be $SU(N)$, the theory is invariant under the superconformal group[1] $PSU(2, 2|4)$. The statement holds not only at classical level, but also at quantum level - due to the vanishing of the β function. The superconformal algebra consists of bosonic generators, that span the $so(2, 4) \times su(4)_{\mathrm{R}}$ algebra, where the first factor is the conformal algebra and the second is the R-symmetry algebra, as well as fermionic generators. The conformal algebra is spanned by the usual Poincarè generators $P_\mu, M_{\mu\nu}$, the dilation generator Δ and the special conformal generators K_μ. The fermionic generators on the other hand, consist of the supercharges $Q^i_\alpha, \bar{Q}_{\dot{\alpha}i}$ and the special fermionic generators $S^i_\alpha, \bar{S}_{\dot{\alpha}i}$. Both Q and S transform in the fundamental representation of $SU(4)_{\mathrm{R}}$, $i = 1, \ldots 4$ and in the $(\frac{1}{2}, 0)$ (or $(0, \frac{1}{2})$) representation of the Lorentz group. We will refrain from writing all commutation relations explicitly. Let us just recall a few of them, involving the dilation operator Δ. We have

$$[\hat{\Delta}, P_\mu] = P_\mu, \qquad [\hat{\Delta}, K_\mu] = -K_\mu, \tag{2.0.1}$$

as well as

$$[\hat{\Delta}, Q^i_\alpha] = \frac{1}{2} Q^i_\alpha, \qquad [\hat{\Delta}, S^i_\alpha] = -\frac{1}{2} S^i_\alpha. \tag{2.0.2}$$

[1] These are the "obvious" symmetries of $\mathcal{N} = 4$ SYM; it turns out that $\mathcal{N} = 4$ SYM also enjoys *dual* superconformal symmetry [102] which lifts the superconformal group to a Yangian [103].

These commutation relations will come back in the next section when we will introduce the concept of (super)conformal primary operator.

Maximal supersymmetry repackages all fields in one self-CPT conjugate multiplet, transforming in the adjoint of the $SU(N)$ gauge group, with 16 on-shell degrees of freedom so divided:

- a spin 1 gauge field A_μ, singlet under $SU(4)_R$;
- 4 fermion fields ψ^i, living in the fundamental of $SU(4)_R$;
- 6 scalars ϕ^a transforming in the adjoint of $SU(4)_R$.

2.1 Superconformal primary operators

It is helpful to work in a basis in which the dilation operator acting on operators at the origin is diagonalised, i.e.

$$[\hat{\Delta}, \mathcal{O}(0)] = \Delta\, \mathcal{O}(0) \tag{2.1.3}$$

where $\mathcal{O}(0)$ is a generic operator. Now, note that by virtue of (2.0.2) and (2.1.3), we can write

$$[\hat{\Delta}\,, [S^i_\alpha, \mathcal{O}(0)]] = [[\hat{\Delta}, S^i_\alpha], \mathcal{O}(0)] + [S^i_\alpha, [\hat{\Delta}, \mathcal{O}(0)]] = \left(\Delta - \frac{1}{2}\right)[S^i_\alpha, \mathcal{O}(0)], \tag{2.1.4}$$

therefore, the superconformal generator S^i_α lowers the dimension by 1/2. An analogous statement obviously also holds for $\bar{S}_{\dot{\alpha}i}$. Since in physically sensible theories the dimension of operators is bounded from below, there must exist an operator such that

$$[S^i_\alpha, \mathcal{O}_{\text{prim}}] = 0, \qquad [\bar{S}_{\dot{\alpha}i}, \mathcal{O}_{\text{prim}}] = 0. \tag{2.1.5}$$

$\mathcal{O}_{\text{prim}}$ is called *superconformal primary operator* (SCPO) and its superconformal descendants are obtained by acting with the operators $Q^i_\alpha, \bar{Q}_{\dot{\alpha}i}$. Note that $Q, \bar{Q}$ increases the dimension by $\frac{1}{2}$.[2] The *superconformal multiplet* is the multiplet obtained starting from the lowest operator and acting with all possible $Q, \bar{Q}$'s.

Analogously, one can define a conformal primary operator (CPO) by requiring that is annihilated by the conformal generator K_μ[3]. A SCPO is also a CPO but the viceversa is of course not true. The full conformal multiplet is then obtained by acting on the SCPOs with the translation generator P_μ.

SCPOs can satisfy shortening conditions, i.e. they can commute with some of the supercharges so that the resulting multiplet is shorter. The classification of superconformal

[2] In the case of an operator transforming non trivially under R-symmetry we refer, as is common in this context, to the SCPO as the whole set of primaries forming the given R-symmetry representation.

[3] Note that, by virtue of (2.0.1), K_μ lowers the dimension by 1.

Operator	$\#Q$	$SU(4)_{\mathrm{R}}$	Δ
1/2 BPS	8	$[0,p,0]$	p
1/4 BPS	4	$[q,p,q]$	$p+2q$
1/8 BPS	2	$[q,p,q+2r]$	$p+2q+3r$
long	0	any	unprotected

Table 2.1: Different supermultiplets in $\mathcal{N}=4$.

multiplets in $\mathcal{N}=4$ SYM has been extensively studied in the past two decades, see e.g. [104–107]. There are four different types of multiplets classified according to their length[4]. In table (2.1) we briefly summarise the different types of multiplets, highligthing the number of charges that leave the primary invariant and their $SU(4)_{\mathrm{R}}$ representation.

As usual, we specify the R-symmetry representation via its Dynkin labels, which in the case of $SU(4)$ are three and denoted by $[a_1,a_2,a_3]$.

BPS operators are very special: their dimension is unrenormalised or, in other words, they are *protected* from quantum corrections. This is not true for long operators, which indeed acquire an anomalous dimension.

A primary role in AdS/CFT (and in this thesis) is played by half-BPS operators. The simplest half-BPS are the so-called *single-trace* operators and can be written in terms of the scalar field ϕ as

$$\mathcal{O}_p^{i_1,\dots i_n}=\mathrm{sTr}(\phi^{\{i_1}\dots\phi^{i_p\}}) \tag{2.1.6}$$

where sTr denotes the symmetrised trace over the gauge algebra and the curly brackets stand for the traceless part of the tensor. They have (protected) scaling dimension p and live in the $[0,p,0]$ representation of $SU(4)_{\mathrm{R}}$. An equivalent but very useful way to keep track of the R-symmetry is to introduce null vectors $\vec{y}$,

$$\mathcal{O}_p(x,y)=y_{i_1}\cdots y_{i_p}\,\mathrm{Tr}(\phi^{i_1}\dots\phi^{i_p}),\qquad \vec{y}.\vec{y}=0. \tag{2.1.7}$$

In this way, an half-BPS operator is generically specified by the spacetime coordinate x and the $SU(4)$ coordinate y. From now on, we will suppress the dependence on (x,y) when this does not create confusion and write $\mathcal{O}_p(x,y)\equiv\mathcal{O}_p$.

We can also build half-BPS multi-trace operators by taking product of single-trace operators of the form

$$\mathcal{O}_{p_1,\dots,p_n}=\mathcal{O}_{p_1}\cdots\mathcal{O}_{p_n}\Big|_{[0,p,0]},\qquad p=p_1+\cdots p_n, \tag{2.1.8}$$

and projecting onto the $[0,p,0]$ representation.

[4]Strictly speaking the multiplets are 5 because there is the identity, which is annihilated by all supercharges and is obviously the only operator in its multiplet.

Certainly not all multi-trace operators are half-BPS. One famous example are the long double-trace operators of the schematic form

$$O_p\Box^n\partial_{\{\mu_1}\cdots\partial_{\mu_l\}}\mathcal{O}_q\big|_{[a,b,a]}, \tag{2.1.9}$$

where the curly brackets mean that the operator is projected onto a symmetric traceless representation with spin l and classical dimension $\Delta_{\text{free}}=p+q+2n+l$. These operators are SCPOs of long multiplets; they are particularly relevant in this context since it can be shown that, in CFTs which admit a large N expansion[5], these are actually the only long operators exchanged in the OPE of two half-BPS operators at large N [58]. Although these operators are unprotected, they only acquire an anomalous dimension at order $1/N^2$, because interactions are suppressed at large N. These anomalous dimensions are an important piece of the bootstrap program, as we will see more in detail later on. Let us now discuss some properties of the correlation function of half-BPS operators.

2.2 Kinematics of four-Point functions

From now on, we will focus on correlation functions of scalar operators; analogous statements are available for operators with non-trivial spin structure. (Super)conformal symmetry heavily constrains correlation functions. In particular two-point functions are completely fixed, up to a normalisation which can be reabsorbed into the redefinition of the field. We have,

$$\langle\mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\rangle=C\delta_{p_1,p_2}g_{12}^{p_1},\qquad g_{12}=\frac{y_{12}^2}{x_{12}^2} \tag{2.2.10}$$

where $x_{12}^2=(x_1-x_2)^2$, $y_{12}^2=(y_1-y_2)^2=y_1\cdot y_2$. For future convenience, in $\mathcal{N}=4$ SYM we will choose the normalisation[6] to be $C=\frac{1}{p_1}$. Note that the power of x_{12} is fixed by requiring that (2.2.10) satisfies the Ward identity associated to dilation symmetry:

$$\langle\mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\rangle=\lambda^{-2p_1}\langle\mathcal{O}_{p_1}(x_1',y_1)\mathcal{O}_{p_2}(x_2',y_2)\rangle \tag{2.2.11}$$

with $x_1'=\lambda x_1$. An analogous statement also holds for the internal coordinates y. Similarly, for three-point functions we have

$$\langle\mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\mathcal{O}_{p_3}(x_3,y_3)\rangle=C_{123}\,g_{12}^{p_1+p_2-p_3}g_{23}^{p_2+p_3-p_1}g_{13}^{p_1+p_3-p_2}, \tag{2.2.12}$$

where C_{123} is an overall constant. Two-point and three-point functions of BPS operators are special. In fact, not only they are completely fixed by conformal symmetry, but they

[5]We refer to chapter 3 for the definition of large N expansion.

[6]In fact, a more natural normalisation for operators dual to single particle states would be $C=p_1N$ where N is the rank of the gauge group. We will come back to this in the next chapter, when we will discuss the single particle operators.

take the same form as their free theory value, i.e. they are coupling-independent.

The situation starts to be different for four-point functions, which have non-trivial dynamics. As is common in this context, let us define spacetime and internal cross-ratios via

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = U = x\bar{x}\,, \qquad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = V = (1-x)(1-\bar{x})\,,$$
$$\frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} = \tilde{U} = y\bar{y}\,, \qquad \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2} = \tilde{V} = (1-y)(1-\bar{y})\,.$$

For future convenience, we also defined the variables $x, \bar{x}, y, \bar{y}$ which can be viewed as of square roots of cross ratios and should not be confused with spacetime and internal coordinates x_i, y_i. It is easy to show that the Ward identities for bosonic generators imply that we can always extract a prefactor from four-point functions such that the remaining function only depends on $U, V, \tilde{U}, \tilde{V}$. In fact, we can write:

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle = \mathcal{P}[\{g_{ij}\}]\mathcal{G}_{\vec{p}}(U,V,\tilde{U},\tilde{V}) \tag{2.2.13}$$

where in our conventions the prefactor $\mathcal{P}$ is

$$\mathcal{P}[\{g_{ij}\}] = g_{12}^{k_s} g_{14}^{k_t} g_{24}^{k_u} \left(g_{13}g_{24}\right)^{p_3}, \qquad g_{ij} = \frac{y_{ij}^2}{x_{ij}^2}. \tag{2.2.14}$$

where

$$k_s = \frac{1}{2}\left(p_1+p_2-p_3-p_4\right), \quad k_t = \frac{1}{2}\left(p_1+p_4-p_2-p_3\right), \quad k_u = \frac{1}{2}\left(p_2+p_4-p_1-p_3\right), \tag{2.2.15}$$

and we have introduced the shorthand $\vec{p} = (p_1, p_2, p_3, p_4)$ to specify the charge dependence. Without loss of generality, from now on we will assume $p_4 - p_3 \geq p_2 - p_1 \geq 0$. Note that $\mathcal{G}_{\vec{p}}$ is now invariant under dilations with the conformal weight entirely carried by $\mathcal{P}[\{g_{ij}\}]$.

We can further constrain the function $\mathcal{G}_{\vec{p}}$, by solving the Ward identities for the fermionic generators[7] [108]. The solution takes the following form:

$$\mathcal{G}_{\vec{p}} = \mathcal{G}_{\vec{p},\text{free}} + \mathcal{G}_{\vec{p},\text{dynamical}} \tag{2.2.16}$$

where $\mathcal{G}_{\vec{p},\texttt{free}}$ is independent of the coupling and $\mathcal{G}_{\vec{p},\texttt{dynamical}}$, which encodes all the non-trivial dynamics of the theory, factorises into

$$\mathcal{G}_{\vec{p},\texttt{dynamical}} = \mathcal{I}\,\mathcal{A}_{\vec{p}} \tag{2.2.17}$$

[7] Note that similar statements also hold for other SCFTs, as we will see in the second part of the thesis.

where

$$\mathcal{I}=(x-y)(\bar{x}-y)(x-\bar{y})(\bar{x}-\bar{y}) \tag{2.2.18}$$

is the so-called Intriligator factor and we have temporarily suppressed the dependence on $U,V,\tilde{U},\tilde{V}$. Equation (2.2.16) is often referred to as *partial non-renormalisation*[8] [109]. The function $\mathcal{A}_{\vec{p}}(U,V,\tilde{U},\tilde{V})$ - and its Mellin transform which will be defined later on - is the main protagonist of the next chapters. It is a function of cross-ratios as well as external charges. Moreover, note that $\mathcal{A}_{\vec{p}}$ has reduced degree in $y,\bar{y}$ compared to the original correlator.

2.3 OPE and superconformal blocks

An important tool of (S)CFTs is the so-called operator product expansion (OPE). This allows to write the product of two operators as a sum over primaries. In the case of $\mathcal{N}=4$ the OPE of two half-BPS operators reads,

$$\mathcal{O}_{p_1}(x_1)\mathcal{O}_{p_2}(x_2)=\sum_{\mathcal{O}_\Delta^{(l)}} g_{12}^{\frac{p_1+p_2-\Delta}{2}} C_{12\mathcal{O}}\mathcal{K}(x_{12},\partial_2)\,\mathcal{O}_\Delta^{(l)}(x_2) \tag{2.3.19}$$

where the sum runs over *all* superconformal primary operators. Here, $\mathcal{K}$ is a differential operator that captures the contribution of all descendants of the super primary $\mathcal{O}$ and C_{12O} is the coefficient of the three-point function $\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_\Delta\rangle$, defined by (2.2.12). To see this, just multiply both sides of (2.3.19) by $\mathcal{O}_\Delta$ and then take the correlation function in both sides.

An important consequence of the OPE is that it allows to reduce an n-point function to a sum of products of three-point functions. For example, in the case of a four-point function, we can write

$$\langle\mathcal{O}_{p_1}(x_1)\mathcal{O}_{p_2}(x_2)\mathcal{O}_{p_3}(x_3)\mathcal{O}_{p_4}(x_4)\rangle=\sum_{\vec{\tau}}\mathcal{C}_{\vec{p},\vec{\tau}}\mathbb{S}_{\vec{p},\vec{\tau}}, \qquad \mathcal{C}_{\vec{p},\vec{\tau}}=\sum_i C_{12\mathcal{O}_i}C_{34\mathcal{O}_i}, \tag{2.3.20}$$

where we have used the shorthand $\vec{\tau}=(\tau,l,\mathcal{R})$ to specify the quantum labels of the operator. Here, $\tau\equiv\Delta-l$ is the so-called twist and $\mathcal{R}$ stands for any R-symmetry representation belonging to $([0,p_1,0]\otimes[0,p_2,0])\cap([0,p_3,0]\otimes[0,p_4,0])$. The possible R-symmetry representations exchanged in $([0,p_1,0]\otimes[0,p_2,0])$ are always of the form $[a,b,a]$ and they run over a set which depends on the charges as well as the type of multiplet exchanged. We also introduced a sum over i because in principle there can be different operators with the same scaling dimension Δ (or, equivalently, same twist τ). This is in fact exactly what happens to the spectrum of long double-trace operators in

[8] Note that this only states that the interacting piece necessarily factors out I, but it does not prevent (some part of) free theory to factor out an I as well.

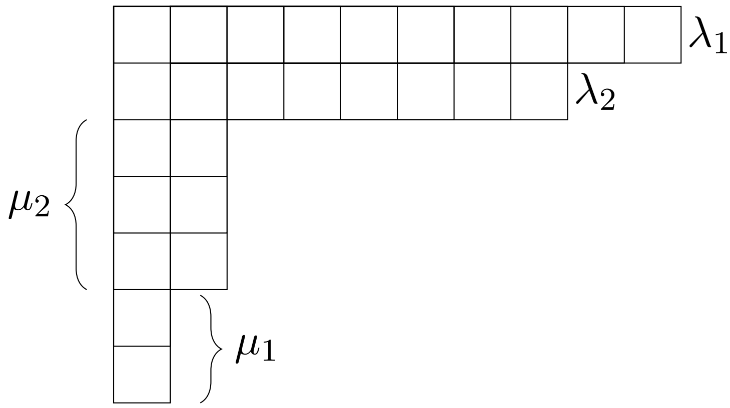


Figure 2.1: A permitted Young diagram for $\mathcal{N} = 4$ reps. The length of the rows satisfies $\lambda_i \leq 2$ with $i \geq 3$. In this example the diagram is associated to a long operator with $\lambda_1 = 10, \lambda_2 = 8, \mu_1 = 2, \mu_2 = 3$.

$\mathcal{N} = 4$ SYM at large N. We will come back to this in the next chapters when we will discuss the operator mixing.

The contribution from all superconformal descendants is captured by the functions $\mathbb{S}_{\vec{p},\vec{\tau}}$ which are known as *superconformal blocks*. There has been a lot of extensive work in the computation of superconformal blocks in different dimensions with different number of supercharges, by using various approaches [104, 105, 107, 110–115]. In this thesis we will need the superconformal blocks of four point of half-BPS operators in three theories, namely $\mathcal{N} = 4$ in four dimensions, $\mathcal{N} = 2$ in four dimensions and $\mathcal{N} = (4, 4)$ in two dimensions. We will borrow them from [107], whose formalism is valid for all theories with $SU(m, m|2n)$ symmetry group and makes use of a group-theoretic approach and has the advantage of dealing with all different representations (short, semi-short, long) in a uniform way[9]. Let us focus on the $\mathcal{N} = 4$ blocks for now, leaving the discussion for the other two theories to the second part of the thesis. In this formalism, an $\mathcal{N} = 4$ superconformal primary operator $\mathcal{O}_{\gamma,\underline{\lambda}}$ is labelled by a number γ and a finite dimensional representation of $GL(2|2)$, where the latter is defined by a Young diagram $\underline{\lambda} = (\lambda_1 \ldots, \lambda_n)$ where λ_i represents the length of the row i. These Young diagrams cannot have an arbitrary shape; they are restricted to fit into hook shapes, i.e. they can have at most two rows with length greater than 2 and at most two columns with height greater than 2. In particular, there are two types of representations: atypical, which correspond to multiplets satisfying shortening conditions, and typical, which are associated to long multiplets. We will mainly be interested in the latter. A typical Young diagram is shown in figure 2.1. The associated long superblocks are specified by four quantum labels; the length of the first two rows, λ_1, λ_2, which identify the conformal representation, and the number of rows with length 1 and 2, labelled by μ_1 and μ_2, respectively, which instead specify the R-symmetry representation. The translation

[9] Recently, Aprile and Heslop have generalised this method and have computed the superconformal blocks for scalar correlators in many dimensions, even and odd, with different number of supercharges [115].

multiplet	$GL(2,2)$ rep	τ	l	$SU(4)_{\rm R}$ rep
1/2 BPS	$[0]$	γ	0	$[0,\gamma,0]$
1/4 BPS	$[1^\mu]$	γ	0	$[\mu,\gamma-2\mu,\mu]$
1/8 BPS	$[\lambda,1^\mu]$, $\lambda\geq 2$	γ	$\lambda-2$	$[\mu,\gamma-2\mu-2,\mu]$
long	$[\lambda_1,\lambda_2,1^{\mu_1},2^{\mu_2}]$, $\lambda_2\geq 2$	$\gamma+2\lambda_2-4$	$\lambda_1-\lambda_2$	$[\mu_1,\gamma-2\mu_1-2\mu_2-4,\mu_1]$

Table 2.2: Translation between superconformal reps and superfields [107]. Here, μ_1 and μ_2 label the number of columns with length 1 and 2, respectively.

between these and the more common $\mathcal{N}=4$ quantum labels τ,l,b,a is given in table 2.2. The long blocks are the simplest among the superconformal blocks, they do not depend on γ and take the following fully factorised form:

$$\mathbb{L}_{\vec{\tau}}=\mathcal{P}[\{g_{ij}\}]\left(\frac{\tilde{U}}{U}\right)^{p_3}\mathcal{I}\,\mathcal{G}_{\tau,l}(x,\bar{x})\,\mathcal{H}_{b,a}(y,\bar{y}) \tag{2.3.21}$$

where $\mathcal{P}[\{g_{ij}\}]$ and $\mathcal{I}$ were defined around (2.2.16) and

$$\begin{aligned}\mathcal{G}_{\tau,l}(x,\bar{x})&=\frac{(-1)^l}{(x-\bar{x})U^{1+\frac{p_{43}}{2}}}\left[\mathcal{F}^+_{\frac{\tau}{2}+2+l}(x)\mathcal{F}^+_{\frac{\tau}{2}+1}(\bar{x})-\mathcal{F}^+_{\frac{\tau}{2}+1}(x)\mathcal{F}^+_{\frac{\tau}{2}+2+l}(\bar{x})\right],\\ \mathcal{H}_{b,a}(y,\bar{y})&=\frac{1}{(y-\bar{y})\tilde{U}^{1-\frac{p_{43}}{2}}}\left[\mathcal{F}^-_{-\frac{b}{2}-a-1}(y)\mathcal{F}^-_{-\frac{b}{2}-1}(\bar{y})-\mathcal{F}^-_{-\frac{b}{2}-a-1}(y)\mathcal{F}^-_{-\frac{b}{2}-1}(\bar{y})\right].\end{aligned} \tag{2.3.22}$$

Here,

$$\mathcal{F}^\pm_h(x)=x^h{}_2F_1\left[h\mp\tfrac{p_{12}}{2},h\mp\tfrac{p_{43}}{2},2h\right](x) \tag{2.3.23}$$

and ${}_2F_1$ is the standard hypergeometric function. We have used the symbol $\mathbb{L}_{\vec{\tau}}$ to highlight that they capture long (unprotected) representations. For future convenience, let us also define another set of labels $\vec{h}$ related to $\vec{\tau}$ via

$$h=\frac{\tau}{2}+l+2,\quad \bar{h}=\frac{\tau}{2}+1,\qquad j=-\frac{b}{2}-a-1,\qquad \bar{j}=-\frac{b}{2}. \tag{2.3.24}$$

This set of labels will be very helpful when we will discuss the other two theories.

Note that the $\mathcal{H}_{b,a}$ are nothing but $SO(6)\sim SU(4)$ spherical harmonics; they can also be written in terms of Jacobi polynomials. We can think of the decomposition of the correlator under the R-symmetry (or "internal") blocks as a linear change of basis from the "monomial" basis (i.e. $\tilde{U},\tilde{V}$) to the spherical harmonics basis. In fact, (2.3.22), for any given choice of charges, is just a polynomial in $\tilde{U},\tilde{V}$. For long multiplets, the sum in (2.3.20) runs over a certain set of representations $[a,b,a]$, which depends on the charges $\vec{p}$. In particular, the values of a run over the following set:

$$0\leq a\leq\kappa_{\vec{p}}$$

where

$$\kappa_{\vec{p}} = \frac{\min(p_1+p_2, p_3+p_4) - p_{43} - 4}{2} \tag{2.3.25}$$

is the 'degree of extremality' and $p_{43} = p_4 - p_3$. For each value of a, b runs over the set

$$0 \leq \frac{b - p_{43}}{2} \leq (\kappa_{\vec{p}} - a). \tag{2.3.26}$$

Note that the spherical harmonics $\mathtt{Y}_{b,a}$ automatically vanish when a, b are not in the set (2.3.25),(2.3.26).

Chapter 3

$AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ SYM

The AdS/CFT correspondence, originally proposed by Maldacena in 1998 [18], is a duality between two *a priori* unrelated theories. On one side, we have a theory of gravity in $d+1$ dimensions in a space with an AdS factor; on the other side we have a CFT in d dimensions living on the (conformal) boundary of AdS with *no* spin-2 states. The correspondence states that the *full* dynamics - correlation functions, states, etc. - of these two theories are equivalent; thus - amazingly - gravity dynamics is captured from a theory in one dimension lower. In fact, the AdS/CFT provides a concrete realisation of the so-called holographic principle, which made its first appearance in the context of black hole physics. Bekenstein [116, 117] observed that the entropy of a black-hole scales with its area rather than the volume, as is usual in thermodynamics. This led to the idea that in gravity systems the information stored in a certain volume V is encoded on its boundary surface A. Then, in the nineties, inspired by 't Hooft [118] and Thorn [119], Susskind introduced the concept for the first time in string theory [120].

AdS/CFT, besides its intrinsic relevance in theoretical physics, is by now a firmly established tool used to investigate corners of theories particularly challenging, both from gravity and gauge theory side. In this first part of the thesis we focus on the archetypal example of AdS/CFT, i.e. the duality between

- type IIB string theory on $AdS_5 \times S^5$ with both factors having the same radius R and N units of F_5 flux on S^5;
- $\mathcal{N} = 4$ SYM in 4 dimensions with gauge group $SU(N)$ in its unbroken phase,

with the parameters identified in the following way:

$$g_s = \frac{g_{\rm YM}^2}{4\pi}, \qquad R^4 = 4\pi g_s N \alpha'^2. \tag{3.0.1}$$

Here, g_s is the string coupling and α' is the Regge slope. Note also the symmetry matching on both sides: the $SU(4) \sim SO(6)$ R-symmetry group is realised in the gravity side as isometry group of the sphere while the conformal group $SO(2,4)$ is the isometry group of the AdS factor.

In the second part of the thesis we will discuss two other examples, namely the D3-D7 and the D1-D5 systems which are dual to certain $4d$ $\mathcal{N}=2$ and $2d$ $\mathcal{N}=(4,4)$ SCFTs, respectively.

The remainder of this chapter is organised as follows. In section 3.1 we introduce a very important and useful limit to explore the correspondence, namely the 't Hooft limit. In section 3.2, we briefly sketch how the duality arises naturally from a string theory perspective, loosely following refs [121, 122]. We end the chapter by providing the definition of *single particle operators*, i.e. operators dual to single particle states in the bulk, and highlighting some of their properties.

3.1 The 't Hooft limit

The AdS/CFT correspondence is a strong-weak coupling duality, in the sense that when one side is at weak-coupling, the other is at strong-coupling. For this reason, it is hard to work with it in its full form. Despite the conjecture is supposed to hold for all values of its parameters, in practice, it is very convenient and, very often, necessary, to take some limits. One very important limit which we will use throughout this thesis is the so-called 't Hooft limit, originally proposed by 't Hooft to understand some features of the strong interactions [123]. First, let us define the 't Hooft parameter via

$$\lambda = g_{\rm YM}^2 N. \tag{3.1.2}$$

Now, the 't Hooft limit is defined as the limit

$$\boxed{N \to \infty, \qquad \lambda \text{ fixed.}}$$

In the field theory side this limit is well defined and corresponds to a topological expansion of Feynman diagrams. It is also known as planar limit, the reason being that in this regime only Feynman diagrams which can be drawn on a plane without crossing lines survive. In the gravity side, upon taking $\lambda \to \infty$, we approach the supergravity limit. The $1/N$ expansion then becomes a loop expansion in supergravity. On the other hand, the expansion in $1/\lambda$ corresponds to consider string corrections. In table 3.1 we summarise the different regimes for $\mathcal{N}=4$ and IIB string theory on $AdS_5 \times S^5$.

All observables - including correlation functions - inherit double-expansion in $1/N$ and

$\mathcal{N} = 4$ SYM	type IIB string theory on $AdS_5 \times S^5$
all N, λ	full quantum type IIB string theory
$N \to \infty$, λ fixed	classical IIB string theory
$N \to \infty$, $\lambda \to \infty$	type IIB supergravity on $AdS_5 \times S^5$
$N \to \infty$, $1/\lambda$ expansion	type IIB supergravity, α' corrections
$N \to \infty$, $1/N$ expansion	type IIB supergravity, loop (g_s) correction

Table 3.1: Different regimes in $\mathcal{N} = 4$ and their supergravity/string theory interpretation.

α'. In particular, in the case of the four-point function of half-BPS operators we have

$$\begin{aligned}\mathcal{G} =& \mathcal{G}_{\text{free,disc}} + \frac{1}{N^2}\mathcal{G}_{\text{free,conn}} + \\ &+ \frac{1}{N^2}\left(\mathcal{G}_{\text{SUGRA}} + \mathcal{G}_0\alpha'^3 + \mathcal{G}_2\alpha'^5 + \mathcal{G}_3\alpha'^6\right) + \frac{1}{N^4}\left(\mathcal{G}_{\text{SUGRA-1}} + \alpha'^3\mathcal{G}_{\text{0-1}} + \alpha'^5\mathcal{G}_{\text{2-1}}\right) + \cdots\end{aligned} \tag{3.1.3}$$

where the leading N^0 contribution is the disconnected piece, $1/N^2$ is the tree-level contribution, $1/N^4$ the one-loop, and so on. Note that from a CFT perspective we have two $1/N^2$ contributions at $\alpha' = 0$, namely a connected free theory and a genuine interacting term. On the other hand, in supergravity there is no such a distinction and the tree-level SUGRA amplitude is more properly the sum of both $1/N^2$ contributions. In fact, as we will see, $\mathcal{G}_{\text{SUGRA}}$ contains poles corresponding to heavy string states which decouple in the supergravity limit; these poles correctly cancel against those present in $\mathcal{G}_{\text{free,conn}}$ [37].

The tree-level contribution corresponds to the $AdS_5 \times S^5$ completion of the well known (α' expansion of the) Virasoro-Shapiro (VS) amplitude, i.e. the tree-level scattering of four closed strings. We will come back to this later on.

3.2 An open-closed string duality

As we mentioned already, the AdS/CFT correspondence arises quite naturally in the context of string theory, and in particular in D-brane physics. D-branes can be viewed either as higher-dimensional objects on which open strings end (open string perspective) or solitonic solutions of (the low energy limit of) superstring theory (closed string perspective). The validity of both perspectives depends on the coupling: the open string perspective can be trusted at small values of the coupling, viceversa for the closed string perspective.

3.2.1 Open string perspective

Consider type IIB string theory in flat $10d$ Minkowski background and add a stack of N D3-branes, say along the 0123 directions.

	0	1	2	3	4	5	6	7	8	9
D3	•	•	•	•	-	-	-	-	-	-

In the perturbative limit, $g_s N \ll 1$, there are two types of excitations, open string excitations, which are excitations of the D3-branes, and closed string excitations which are instead excitations of the full $10d$ spacetime. In the low energy limit $\alpha' \to 0$, open and closed strings decouple. Closed strings are insensitive to the D-branes and propagate in the full $10d$ spacetime. On the other hand, open strings give rise to a $4d$ $\mathcal{N} = 4$ supermultiplet built out of the 16 supercharges preserved by this configuration of branes. Moreover, the $\mathcal{N} = 4$ multiplet is valued in the adjoint of an $U(N)$ gauge group. The dynamics is therefore described by

IIB supergravity in flat $10d \oplus \mathcal{N} = 4$ SYM with gauge group $U(N)$

provided that we identify $g^2_{\text{YM}} = 4\pi g_s$.

3.2.2 Type IIB supergravity and Kaluza-Klein reduction

Let us now take the closed string perspective - that can be trusted at strong coupling, i.e. $g_s N \gg 1$ - and sketch how the $AdS_5 \times S^5$ solution emerges from type IIB supergravity. Type IIB supergravity has a solution in which the metric takes the following form [124]

$$ds^2_{10} = \frac{1}{\sqrt{H(y)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(y)}\delta_{ab}dy^a dy^b, \qquad H(y) = 1 + \frac{R^4}{r^4}, \tag{3.2.4}$$

with $r = \sqrt{y^a y^a}$. Here $\mu = 0, \ldots, 3$ are Minkowski coordinates and $a = 6, \ldots, 10$ label the transverse coordinates. String theory tells us that the solution describes a stack of N D3-branes located at the origin of the spacetime; this fixes the constant R - which remains undetermined in supergravity - to be

$$R^4 = 4\pi g_s N \alpha'^2. \tag{3.2.5}$$

We can see that there are two different regions, depending on r. For large r, (3.2.4) asymptotes to a flat 10d metric. On the other hand, near the so-called throat, i.e. the region $r \to 0$, the metric asymptotes to an $AdS_5 \times S^5$ metric

$$ds^2_{10}\big|_{r\to 0} = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 + R^2 d\Omega^2_5. \tag{3.2.6}$$

Now, in the limit $\alpha' \to 0$, these two regions decouple and we have that the dynamics of closed strings is described by:

$$\text{IIB supergravity in } 10d \text{ flat} \oplus \text{IIB supergravity on } AdS_5 \times S^5$$

Now, since the two perspectives - open and closed - should be equivalent descriptions of the same physics, Maldacena conjectured that IIB supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in $4d$ with gauge group $SU(N)$ are duals of each other[1].

A useful approach to study fluctuations of the supergravity fields about the $AdS_5 \times S^5$ solution is the Kaluza-Klein reduction. In short, one expands the supergravity fields in $SO(6)$ spherical harmonics such that each component is a field on AdS_5. Schematically,

$$\phi \sim \sum_k \phi_k Y_k \tag{3.2.7}$$

where ϕ_k lives on AdS_5. By looking at the linearised equations of motion, it turns out that - amazingly - there is a one-to-one correspondence between the KK modes ϕ_k and the half-BPS spectrum of $\mathcal{N} = 4$ with mass of the field and dimension of the dual operators related by

$$\Delta = 2 + \sqrt{4 + m^2 R^2}. \tag{3.2.8}$$

In particular, the scalar contained in the multiplet of the graviton corresponds to the half-BPS operator $\mathcal{O}_2$ we introduced in the previous chapter, that, among its descendants, contains the stress-energy tensor of $\mathcal{N} = 4$.

The precise form of the field-operator map requires some care because it involves a procedure of renormalisation. The upshot of this construction is that the supergravity action is (related to) the generating functional for connected Green's function of gauge invariant operators via

$$\left\langle \exp\left(\int d^d x \phi_0(x) \mathcal{O}(x) \right) \right\rangle_{\text{CFT}} = e^{-S_{\text{SUGRA}}[\phi_0]} \tag{3.2.9}$$

where $S_{\text{SUGRA}}[\phi_0]$ is the supergravity action, and the boundary field ϕ_0 is obtained by "pushing" the supergravity field ϕ on the boundary, acting as a source for the operator $\mathcal{O}$. This is the "weak form" of the correspondence. In the "strong form" $e^{-W[\phi_0]}$ gets replaced by the string partition function, Z_{string}. In the saddle point approximation, $Z_{\text{string}} \sim e^{-S_{\text{SUGRA}}[\phi_0]}$.

[1]Note that in the open string perspective the gauge group is actually $U(N)$; however, the overall $U(1)$ does not propagate into the bulk.

3.3 Single-particle operators

As we recalled, the AdS/CFT correspondence maps supergravity states to gauge invariant operators in the CFT. However, there is a subtlety to consider in the mapping between *single-particle* supergravity states and their dual gauge theory operators. In fact, while it is true that single-particle operators are protected half-BPS operators, the space of half-BPS operators of given charge is degenerate, and only in the planar limit can the single-particle operators be identified with the single-trace half-BPS operators. This identification was indeed known to receive $1/N$ suppressed contributions from multi-trace admixtures[2] already several years ago [125, 126] and the first order double-trace corrections have been worked out recently directly from supergravity [29, 30]. A non-perturbative, deceptively simple, definition of "single-particle operators", i.e. operators dual to single particle states in AdS, has been given in [37]:

> single-particle operators (SPOs) are half-BPS operators which have vanishing two-point functions with all multi-trace operators.

In fact, in a natural basis of scattering states, multi-particle states should be orthogonal to single-particle ones. For the purpose of this thesis, the difference between single-particle and single-trace operators is mostly irrelevant, since we will work in the planar limit. It is interesting however to recall a few properties of the SPOs and we will do so in the remainder of the section. We will follow [2], where we obtained general formulae for the SPOs as well as some of their correlation functions in free theory[3]. The interested reader might want to look at that paper for more details.

Let us thus define the trace basis as the basis built out of the single-trace operators, i.e.

$$\mathcal{T}_p(x, y) = y_{i_1} \cdots y_{i_p} \operatorname{Tr}(\phi^{i_1} \dots \phi^{i_p}), \qquad \vec{y}.\vec{y} = 0, \tag{3.3.10}$$

and their multi-trace admixtures

$$\mathcal{T}_{p_1,\dots,p_n} = \mathcal{T}_{p_1} \cdots \mathcal{T}_{p_n}\big|_{[0,p,0]}, \qquad p = p_1 + \cdots p_n. \tag{3.3.11}$$

Note that we are here denoting the half-BPS operators in the trace basis by $\mathcal{T}$, while in the previous chapter they were indicated by $\mathcal{O}$. Instead, we will now use the symbol $\mathcal{O}$ to refer to a single-particle operator, defined below. In the rest of the thesis the distinction between the two will not really matter and we will identify $\mathcal{T} \sim \mathcal{O}$, unless stated otherwise.

It is easy to check, by just performing Wick contractions, that

$$\langle \mathcal{T}_p(x_1, y_1) \mathcal{T}_{p_1,\dots,p_n}(x_2, y_2) \rangle \neq 0. \tag{3.3.12}$$

[2]These admixtures are half-BPS and are of the form (2.1.8).

[3]A similar analysis of mixing in $AdS_3 \times S^3$ has been carried out recently in [127].

On the other hand, a single-particle operator is an operator $\mathcal{O}_p \equiv \mathcal{T}_p + \cdots$, where $\cdots$ refers to multi-trace admixtures, such that

$$\langle \mathcal{O}_p(x_1, y_1) \mathcal{T}_{p_1,\ldots,p_n}(x_2, y_2) \rangle = 0. \tag{3.3.13}$$

This immediately implies

$$\langle \mathcal{O}_p(x_1, y_1) \mathcal{O}_{p_1,\ldots,p_n}(x_2, y_2) \rangle = 0. \tag{3.3.14}$$

Let us now show a couple of explicit examples. With the gauge group taken to be $SU(N)$, we have $\mathcal{T}_1 = 0$, therefore

$$\mathcal{O}_2 = \mathcal{T}_2, \qquad \mathcal{O}_3 = \mathcal{T}_3. \tag{3.3.15}$$

The first non-trivial cases arise when $p = 4, 5$. By performing Wick contractions and using the orthogonality condition (3.3.13), we get:

$$\mathcal{O}_4 = \mathcal{T}_4 - \frac{2N^2 - 3}{N(N^2 + 1)} \mathcal{T}_{2,2}\,, \tag{3.3.16}$$

$$\mathcal{O}_5 = \mathcal{T}_5 - 5 \frac{N^2 - 2}{N(N^2 + 5)} \mathcal{T}_{2,3}. \tag{3.3.17}$$

In [2] we were able to prove, through group theory techniques, a general formula which resolves the expansion of the SPOs in terms of multi-traces. Here we just quote the result:

$$\mathcal{O}_p(x) = \sum_{\{q_1..q_m\} \vdash p} C_{q_1,\ldots,q_m} T_{q_1,\ldots,q_m}(x) \tag{3.3.18}$$

$$C_{q_1,..q_m} = \frac{|[\sigma_{q_1..q_m}]|}{(p-1)!} \sum_{s \in \mathcal{P}(\{q_1,..,q_m\})} \frac{(-1)^{|s|+1}(N+1-p)_{p-\Sigma(s)}(N+p-\Sigma(s))_{\Sigma(s)}}{(N)_p - (N+1-p)_p}.$$

The group theory data consists of $\mathcal{P}(\{q_1, \ldots, q_m\})$, the powerset[4] of the traces $T_{q_1,\ldots,q_m}$, then $|s|$ is the cardinality of s and $\Sigma(s) = \sum_{s_i \in s} s_i$. Finally, $|[\sigma_{q_1..q_m}]|$ is the size of the conjugacy classes of a permutation $\sigma \in S_n$ where S_n is the symmetric group, with length cycles $q_1 \ldots q_m$. For example, in the case of $\mathcal{O}_4$, we only need to consider the partition 22. The powerset is $\mathcal{P}(\{2, 2\}) = \{\{\}, \{2\}, \{2\}, \{2, 2\}\}$, the size of the conjugacy class is $|[\sigma_{22}]| = 3$. The sum over the four partitions precisely gives[5] (3.3.16).

Let us now briefly discuss some properties of correlators of SPOs in free theory. We

[4] We remind that the powerset of a set S is the set of all subsets of S, including the empty set and the set S itself.

[5] For each of the four partitions, $\Sigma(s)$ takes the values $0, 2, 2, 4$ respectively.

start with two-point functions. It is possible to show that[6]:

$$\langle \mathcal{O}_p(x_1)\mathcal{O}_p(x_2)\rangle = R_p\, g_{12}^p \tag{3.3.19}$$

where R_p takes the form

$$R_p = p^2(p-1)\left[\frac{1}{(N-p+1)_{p-1}} - \frac{1}{(N+1)_{p-1}}\right]^{-1}. \tag{3.3.20}$$

Note that in the large N limit we have $R_p \sim pN^p$; however, for convenience, from next chapter on, we will renormalise the operators such that $\langle \mathcal{O}_p(x_1)\mathcal{O}_p(x_2)\rangle = \frac{1}{p} g_{12}^p$, as mentioned already around (2.2.10).

An important point about two-point functions of single particle operators is that they automatically vanish when the charge of the operators exceed the number of colours, i.e. $p > N$. This should be compared with the trace basis where instead the single-trace operators, for $p > N$, become linear combinations of multi-trace operators. This also provides an intuitive reason on why single particle operators vanish: for $p > N$ all operators are multi-trace therefore, since by definition the single-particle operators are orthogonal to all operators, they must vanish. This property of the operators dual to single-particle states has long been expected from AdS/CFT and follows from the string exclusion principle [129]. As the angular momentum of the gravitons increases they become less and less pointlike, eventually growing into giant gravitons, D3-branes wrapping an $S^3 \subset S^5$ [130] which can not grow bigger than the size of the S^5 sphere. In [131] (sub)-determinant half-BPS operators were defined as duals to these predicted sphere giants and shortly later the Schur polynomial basis of half-BPS operators was defined and the sphere giant gravitons associated with the completely antisymmetric (single column Young tableau) Schur polynomials [132]. We find that at large N, the single-particle operators with charge close to N indeed approach these (sub)-determinant operators.

Let us now discuss higher-point functions. Consider the free theory correlator

$$\langle \mathcal{O}_p(x)\mathcal{O}_{q_2}(x_2)\cdots\mathcal{O}_{q_n}(x_n)\rangle, \tag{3.3.21}$$

and, without loss of generality, let p be the largest charge. In [2] we prove that[7]

$$\langle \mathcal{O}_p(x)\mathcal{O}_{q_2}(x_2)\cdots\mathcal{O}_{q_n}(x_n)\rangle = 0, \qquad k \leq n-3, \qquad k = \frac{1}{2}\left(\sum_{i=1}^{n-1} q_i - p\right). \tag{3.3.22}$$

For example, we have

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_4\rangle = \langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_6\rangle = 0. \tag{3.3.23}$$

[6] This formula was already given in [128], albeit without explicit physical description as single particle operators.

[7] Note that when $k < 0$ the correlator vanishes for R-symmetry rules.

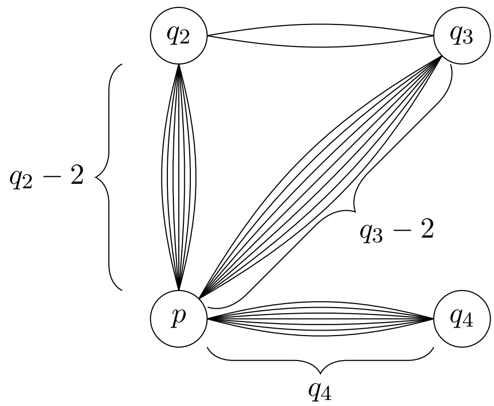


Figure 3.1: An example of vanishing "π-shape" diagram with a propagator structure $g_{12}^{q_2-2} g_{13}^{q_3-2} g_{14}^{q_4} g_{23}^{2}$. Here $p = q_2 + q_3 + q_4 - 4$; note that there are no propagators between q_3 and q_4.

The first non-zero correlators are therefore those with $k = n - 2$. In [2], we found a general formula for this class of correlators for any n. We will just quote a couple of examples. Three-point functions read

$$\langle \mathcal{O}_p(x_1) \mathcal{O}_{q_2}(x_2) \mathcal{O}_{q_3}(x_3) \rangle = q_2 q_3 R_p \times g_{12}^{q_1-1} g_{13}^{q_2-1} g_{23}, \qquad p = q_2 + q_3 - 2. \quad (3.3.24)$$

As last example, let us consider four-point functions with $k = 2$. An example of a correlator belonging to this class is $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$. In the next chapters we will see these correlators have the feature that the SCPOs in the long sector have a single $SU(4)_{\mathrm{R}}$ representation. We have[8]

$$\langle \mathcal{O}_p \mathcal{O}_{q_2} \mathcal{O}_{q_3} \mathcal{O}_{q_4} \rangle = q_2 q_3 q_4 R_p \times \Big((q_2 - 1) g_{12}^{q_1-2} g_{13}^{q_3-1} g_{14}^{q_4-1} g_{23} g_{24} + \\ + (q_3 - 1) g_{12}^{q_1-1} g_{13}^{q_3-2} g_{14}^{q_4-1} g_{23} g_{34} + (q_4 - 1) g_{12}^{q_1-1} g_{13}^{q_3-1} g_{14}^{q_4-2} g_{24} g_{34} \Big), \quad (3.3.25)$$

where here $p = q_1 + q_2 + q_3 - 4$. In particular, note that propagator structures like $g_{12}^{q_2-2} g_{13}^{q_3-2} g_{14}^{q_4} g_{23}^{2}$ are absent. In fact, this is the original motivation why single-particle operators were introduced; these "π-shape diagrams" are absent in supergravity while they do *not* vanish in CFT if one considers single-trace rather than single-particle operators. An example of this diagram is given in figure 3.1.

[8] To be precise, this is just the connected part of the correlator. The disconnected part, when exists, is given by products of two-point functions.

Chapter 4

Large p formalism and the supergravity correlator

Now that we have reviewed the necessary material, we are ready to discuss the four-point function of half-BPS operators at strong coupling in the CFT. Let us remind again that the correlator can be split in two pieces: a free-theory contribution, independent of the coupling, and an interacting term, which carries all the non-trivial dynamical information[1]:

$$\mathcal{G} = \mathcal{G}_{\text{free}} + \mathcal{G}_{\text{dynamical}}. \tag{4.0.1}$$

In particular, the free-theory term contains a disconnected and a (suppressed) connected term:

$$\mathcal{G}_{\text{free}} = \mathcal{G}_{\text{free,disc}} + \frac{1}{N^2}\mathcal{G}_{\text{free,conn}}. \tag{4.0.2}$$

Superconformal symmetry forces $\mathcal{G}_{\text{dynamical}}$ to take the form

$$\mathcal{G}_{\text{dynamical}} = \mathcal{I}\mathcal{A} \tag{4.0.3}$$

with

$$\mathcal{I} = (x-y)(\bar{x}-y)(x-\bar{y})(\bar{x}-\bar{y}). \tag{4.0.4}$$

The ”reduced” correlator $\mathcal{A}$ admits a double expansion: a (loop) $1/N$ expansion

$$\mathcal{A} = \frac{1}{N^2}\mathcal{A}_{\text{tree}} + \frac{1}{N^4}\mathcal{A}_{\text{1-l}}, \tag{4.0.5}$$

[1]We will omit the subscript $\vec{p}$ when possible, to avoid cluttering the notation.

as well as an α' expansion.[2] In this thesis we will be interested in the tree-level amplitude, whose expansion we parametrise in the following way[3]:

$$\mathcal{A}_{\text{tree}} = \big(\mathcal{A}_{\text{SUGRA}} + \mathcal{A}_0\alpha'^3 + \mathcal{A}_2\alpha'^5 + \mathcal{A}_3\alpha'^6\big) + \cdots \,. \tag{4.0.6}$$

In this chapter we will focus on the supergravity term, namely $\mathcal{A}_{\text{SUGRA}}$. The problem of computing supergravity correlators has been attacked in various ways in the last two decades. The first computations of this sort were performed directly in supergravity. This approach involves the evaluation of Witten diagrams whose vertices are encoded in the AdS_5 effective action obtained by KK reducting IIB supergravity on S^5 and it was carried out for a number of different cases [21–25]. The procedure, however, becomes cumbersome and, in practice, is very difficult to go beyond low-charge cases. Despite the complexity of the computation, in all cases it was found that the amplitude could be written in terms of a restricted set of functions, the so-called $\bar{D}$ functions, hinting that a general formula for all charges was possible. In fact, several years later, Rastelli and Zhou, by using a bootstrap approach in Mellin space, conjecture a very elegant formula valid for arbitrary KK modes [26, 27], which agrees with all previous computations and has also been checked in new cases [28–30].

We now understand that the extreme simplicity of the result is a consequence of an hidden $10d$ conformal symmetry [31], that allows to repackage all correlators in a single $10d$ object. This becomes evident in the large p formalism of Aprile and Vieira [32], which has also the advantage of making the so-called large p limit manifest, as we will see.

The rest of the chapter is organised as follows. In section 4.1 we introduce the $AdS_5 \times S^5$ Mellin transform, which, as we will see, turns out to be very useful also in the construction of tree-level string amplitudes. Then, in section 4.2 we present the supergravity correlator and discuss the consequences of the hidden conformal symmetry.

4.1 $AdS_5 \times S^5$ Mellin transform and large p limit

A very natural language to represent holographic correlators is the Mellin formalism, initiated in [56, 57] and further developed in [58, 59]. The Mellin transform plays, in the context of holographic correlators, the same role the Fourier transform plays in flat space scattering amplitudes. For example, contact interactions map to polynomial Mellin amplitudes. In particular, a contact Witten diagram with $2l$ number of derivatives will give rise to a polynomial of degree l in the Mellin variables. In position space, these

[2]We remind that $\alpha' \sim \lambda^{-\frac{1}{2}}$, therefore the low-energy α' expansion corresponds to the strong coupling expansion in the 't Hooft coupling.

[3]The absence of α'^1, α'^2 terms will become clear in the next chapter when we will discuss the flat space limit.

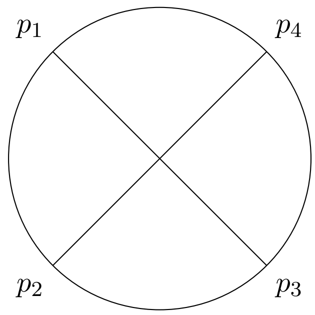


Figure 4.1: A contact AdS Witten diagram which gives rise to a polynomial Mellin amplitude. The degree of the polynomial depends on the number of derivatives hitting the vertex.

correspond to a finite sum of $\bar{D}$ function[4]. The latter are derivatives of the box integral $\bar{D}_{1111}$ which admits the following representation in position space:

$$\bar{D}_{1111}(U,V) = \frac{2\text{Li}_2(x) - 2\text{Li}_2(\bar{x}) + \log(x\bar{x})(\log(1-x) - \log(1-\bar{x}))}{x - \bar{x}}. \tag{4.1.7}$$

While the many benefits of employing the Mellin transform in AdS correlators were already known since the pioneering work by Mack and Penedones, a recent paper [32] pointed out that, at least for $\mathcal{N}=4$, the Mellin transform can be extended to the sphere, such that the large p limit - which we are going to define in a moment - is made manifest.

Following [32], let us thus define the $AdS_5 \times S^5$ Mellin trasform of the reduced correlator $\mathcal{A}$ via

$$\mathcal{A}_{\vec{p}}(U,V,\tilde{U},\tilde{V}) = \oint dsdt \sum_{\tilde{s},\tilde{t},\tilde{u}} \; U^s V^t \tilde{U}^{\tilde{s}} \tilde{V}^{\tilde{t}} \times \Gamma_{\otimes} \times \mathcal{M}_{\vec{p}} \tag{4.1.8}$$

where the kernel $\Gamma_{\otimes}$ is factorised into $AdS_5 \times S^5$, and reads

$$\Gamma_{\otimes} = \frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]\Gamma[-s+k_s]\Gamma[-t+k_t]\Gamma[-u+k_u]}{\Gamma[1+\tilde{s}]\Gamma[1+\tilde{t}]\Gamma[1+\tilde{u}]\Gamma[1+\tilde{s}+k_s]\Gamma[1+\tilde{t}+k_t]\Gamma[1+\tilde{u}+k_u]} \tag{4.1.9}$$

where

$$s+t+u = -p_3 - 2, \qquad \tilde{s}+\tilde{t}+\tilde{u} = p_3 - 2, \tag{4.1.10}$$

and we recall that

$$k_s = \frac{p_1+p_2-p_3-p_4}{2}, \qquad k_t = \frac{p_1+p_4-p_2-p_3}{2}, \qquad k_u = \frac{p_2+p_4-p_1-p_3}{2}. \tag{4.1.11}$$

A number of comments are in order.

- Firstly, the Mellin amplitude $\mathcal{M}_{\vec{p}}$ is, in principle, a function of AdS_5 variables $s, t,$

[4]To be precise, they are proportional to $\bar{D}$ functions, where the proportionality coefficient carries the conformal weight, see e.g. [21] for more details on $\bar{D}$ functions.

S^5 variables $\tilde{s}, \tilde{t}$ as well as charges $\vec{p}$;

- the contour integral in s and t is a standard Mellin-Barnes contour, separating left from right poles in the complex planes s and t. This is the "original" Mellin transform;
- note that the sum (11.3.11) is restricted to the triangle $T = \{\ \tilde{s} \geq -\min(0, k_s), \tilde{t} \geq -\min(0, k_t), \tilde{u} \geq -\min(0, k_u)\}$ due to the Γ function in the denominator of $\Gamma_\otimes$. In fact, since in our conventions $p_4 - p_3 \geq p_2 - p_1 \geq 0$, the domain is really the triangle $T = \{\ \tilde{s} \geq -\min(0, -k_s)\ ;\ \tilde{t}, \tilde{u} \geq 0\}$;
- in [32], the authors show that the sum can also be written as a double integral. This is useful because, in the limit of large charges p, the integral localises on a classical saddle point. The computation matches that of four geodesics shooting from the boundary and meeting in a common bulk point at which the particles scatter as they were in flat space. At the saddle point, the "bold-face" variables

$$\mathbf{s} = s + \tilde{s}, \qquad \mathbf{t} = t + \tilde{t}, \qquad \mathbf{u} = u + \tilde{u}, \qquad \mathbf{s} + \mathbf{t} + \mathbf{u} = -4 \tag{4.1.12}$$

become proportional to the flat space Mandelstam variables. In this large p limit, as we are going to see, $\mathcal{M}_{\vec{p}}$ approaches the flat S-matrix as a function of bold-face variables.

4.2 Four-graviton scattering in supergravity

We can now present the supergravity correlator, first computed by Rastelli and Zhou [26, 27]. In the large p formalism, it takes a surprisingly simple form [32],

$$\mathcal{M}_{\mathrm{SUGRA},\vec{p}} = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)} \tag{4.2.13}$$

where the bold-face variables are defined in (4.1.12). The compactness of the result demands and deserves some more explanation, which we are going to give in a moment. First, note the presence of single poles at the location $\mathbf{s} = -1$, etc. These have a very simple interpretation. As we mentioned already, they are needed to cancel unwanted string states which decouple in the SUGRA limit. In fact, the same poles are present in connected free theory which is of order $1/N^2$; the sum of both terms, which is the actual tree-level supergravity amplitude, is free of such poles. Historically, this has been used to fix the overall normalisation of $\mathcal{M}_{\mathrm{SUGRA},\vec{p}}$ [37], which we now understand as being part of a generalised $AdS_5 \times S^5$ Mellin kernel. Crucially, in order for these cancellation to occur, the external operators have to be the single-particle operators and not the single-trace operators [37].

Let us now consider the limit when all variables are taken to be large. This is formally achieved by rescaling all variables by a common factor, say p, and then taking the limit as p goes to infinity:

$$\lim_{p\to\infty}\left(p^3\mathcal{M}_{\vec{p}}(ps,pt,p\tilde{s},p\tilde{t},pt,pp_i)\right)=\frac{1}{\mathbf{s}\,\mathbf{t}\,\mathbf{u}}, \tag{4.2.14}$$

i.e. as we anticipated before, the correlator approaches the well known supergravity amplitude[5] in flat $10d$, as a function of bold-face variables. The large p limit is a generalisation of the flat space limit proposed by Penedones [58], in the sense that now both AdS and S variables are taken to be large. We will come back to this in the next chapter, when we will discuss the flat space limit in more detail.

Let us have a closer look to $\mathcal{M}_{\vec{p}}$. Away from large p, nothing would prevent the correlator from depending on all variables. In fact, notice that it only depends $\mathbf{s},\mathbf{t}$ - rather than $s,\tilde{s}$ separately - not just at large p. This is nothing but a consequence of a surprising hidden $10d$ conformal symmetry which was discovered by Caron-Huot and Trinh [31]. As explained in that paper, all $AdS_5\times S^5$ tree level correlators $\mathcal{A}_{\vec{p}}(U,V,\tilde{U},\tilde{V})$ in position space can be obtained by Taylor expanding a generating function $\mathcal{G}$, which corresponds to a $10d$ version of the 2222 correlator, namely $\mathcal{G}(U_{10},V_{10})=U_{10}^4\mathcal{A}_{2222}(U_{10},V_{10})$, where U_{10} and V_{10} are now $10d$ cross ratios, rather than $AdS_5\times S^5$ cross ratios. A nice way to represent this expansion is to use operators $\widehat{\mathcal{D}}_{\vec{p}}$ such that, directly on $AdS_5\times S^5$, we have

$$\mathcal{A}_{\vec{p}}(U,V,\tilde{U},\tilde{V})=\widehat{\mathcal{D}}_{\vec{p}}\Big[U^4\mathcal{A}_{2222}(U,V)\Big]. \tag{4.2.15}$$

These operators were found explicitly in [32],

$$\widehat{\mathcal{D}}_{\vec{p}}=\frac{1}{U^4}\sum_{\tilde{s},\tilde{t}}\left(\frac{\tilde{U}}{U}\right)^{\tilde{s}}\left(\frac{\tilde{V}}{V}\right)^{\tilde{t}}\widehat{\mathcal{D}}^{(0,0,0)}_{\vec{p},(\tilde{s},\tilde{t})}\widehat{\mathcal{D}}^{(k_s,k_t,k_u)}_{\vec{p},(\tilde{s},\tilde{t})} \tag{4.2.16}$$

where

$$\widehat{\mathcal{D}}^{(a,b,c)}_{\vec{p},(\tilde{s},\tilde{t})}=\frac{(U\partial_U-3-\tilde{s}-a)_{\tilde{s}+a}}{(-)^a(\tilde{s}+a)!}\frac{(V\partial_V+1-\tilde{t}-b)_{\tilde{t}+b}}{(-)^b(\tilde{t}+b)!}\frac{(U\partial_U+V\partial_V)_{\tilde{u}+c}}{(\tilde{u}+c)!} \tag{4.2.17}$$

and $(\cdots)_i$ is the Pochhammer symbol. Let us prove that (4.2.15) gives indeed the right Mellin amplitude (4.2.13). To see this, first note that

$$\mathcal{A}_{2222}=\oint dsdt\,U^sV^t\tilde{U}^{\tilde{s}}\tilde{V}^{\tilde{t}}\times\Gamma[-s]^2\Gamma[-t]^2\Gamma[-u]^2\times\frac{1}{(s+1)(t+1)(u+1)}. \tag{4.2.18}$$

In particular, $\tilde{s}=\tilde{t}=\tilde{u}=0$ and the S^5 kernel in this case is just 1. Let us now consider the action of the operator on the spacetime dependent part, $\widehat{\mathcal{D}}_{\vec{p}}[U^{s+4}V^t]$, and focus on

[5] More precisely, it only captures the Mandelstam dependence of the IIB supergravity amplitude. The IIB supergravity amplitude is $\frac{1}{stu}\times\delta^{(16)}$ where $\delta^{(16)}$ is a "fermionic delta function" due to supersymmetry, that captures all the different components in the multiplet.

a given propagator structure (i.e. fixed $\tilde{s}, \tilde{t}$). We have

$$\begin{aligned}\hat{D}_{\tilde{s},\tilde{t}}(\vec{p})[U^{s+4}V^t] = \frac{(s+1-\tilde{s})_{\tilde{s}}}{\tilde{s}!}\frac{(s+1-\tilde{s}-k_s)_{\tilde{s}+k_s}}{(-1)^{k_s}(\tilde{s}+k_s)!}\times \\ \times\frac{(t+1-\tilde{t})_{\tilde{t}}}{\tilde{t}!}\frac{(t+1-\tilde{t}-k_t)_{\tilde{t}+k_t}}{(-1)^{k_t}(\tilde{t}+k_t)!}\times\frac{(s+t+4)_{\tilde{u}}}{\tilde{u}!}\frac{(s+t+4)_{\tilde{u}+k_u}}{(\tilde{u}+k_u)!}U^{s-\tilde{s}}V^{t-\tilde{t}}.\end{aligned} \tag{4.2.19}$$

The denominator can be immediately recognised as the S^5 kernel in $\Gamma_\otimes$ for $\mathcal{A}_{\vec{p}}$. We therefore just need to deal with the numerator. For the part dependent on s-type variables (i.e. $s, \tilde{s}, k_s$) we have:

$$\begin{aligned}(s+1-\tilde{s})_{\tilde{s}}(s+1-\tilde{s}-k_s)_{\tilde{s}+k_s} = \frac{\Gamma(s+1)}{\Gamma(s+1-\tilde{s})}\frac{\Gamma(s+1)}{\Gamma(s+1-\tilde{s}-k_s)} = \\ = \frac{1}{\Gamma(-s)^2}\Gamma(-s+\tilde{s})\Gamma(-s+\tilde{s}+k_s)\end{aligned} \tag{4.2.20}$$

where in the last line we have used a consequence of Euler's reflection identity. The t-type dependent part works exactly in the same way. Finally, for the u-type part (remember for $\mathcal{A}_{2222}$ we have $s+t+u=-4$):

$$\begin{aligned}(-u)_{\tilde{u}}(-u)_{\tilde{u}+k_u} = \frac{\Gamma(-u+\tilde{u})}{\Gamma(-u)}\frac{\Gamma(-u+\tilde{u}+k_u)}{\Gamma(-u)} = \\ = \frac{1}{\Gamma(-u)^2}\Gamma(-u+\tilde{u})\Gamma(-u+\tilde{u}+k_u).\end{aligned} \tag{4.2.21}$$

Now, notice that

$$-u+\tilde{u} = s+t+p_3+2-\tilde{s}-\tilde{t}, \tag{4.2.22}$$

therefore the p_3 in $\tilde{u}$ uplifts the u of $\mathcal{A}_{2222}$ to the u of $\mathcal{A}_{\vec{p}}$ which satisfies the on-shell constraint $s+t+u=-p_3-2$. Restoring back $\Gamma_\otimes$ and shifting $s\to s+\tilde{s}$, $t\to t+\tilde{t}$ we finally get the amplitude (4.2.13) accompanied by the correct kernel (4.1.9).

To summarise, we have shown that the action of (4.2.16) on the Mellin transform provides the "covariantisation" of $\mathcal{M}_{2222}$, in the sense that the correlator with generic charges is obtained via

$$\mathcal{M}_{\vec{p}} = \mathcal{M}_{2222}(\mathbf{s},\mathbf{t},\mathbf{u}), \tag{4.2.23}$$

with the kernel of gamma functions $\Gamma_\otimes$ in the Mellin transform given by (4.1.9). In other words, the amplitude for general charges is obtained from $\mathcal{M}_{2222}$ by replacing[6] s, t with $\mathbf{s}, \mathbf{t}$. In the next chapter we will see that this is no longer true when considering α' corrections to the SUGRA amplitude. In this sense, α' corrections explicitly *break* the hidden conformal symmetry.

The simplicity of (4.2.13) allows us, once again, to appreciate the spirit behind the amplitude program. The observables seem to be much more natural objects, and unveil structures which are often hidden in lagrangians, especially when the latter are, at first

[6]Note that, for $\mathcal{M}_{2222}$, $\tilde{s}=\tilde{t}=0$, therefore $s,t=\mathbf{s},\mathbf{t}$

glance, very complicated.

Chapter 5

The Virasoro-Shapiro amplitude in $AdS_5 \times S^5$

So far, we have discussed the SUGRA term, which arises in the strict 't Hooft limit with the coupling λ taken to be infinity. As we reviewed, the AdS/CFT correspondence tells us that λ corrections correspond to add α' corrections to the SUGRA amplitude. In particular, in the case of $\mathcal{N} = 4$, this expansion reproduces the $AdS_5 \times S^5$ analogue of the well-known Virasoro-Shapiro (VS) amplitude in flat space, i.e. the scattering of four-closed strings in IIB string theory. We will refer to it as the Virasoro-Shapiro amplitude in $AdS_5 \times S^5$.

On general grounds, we expect α' corrections to be polynomial of a certain degree at each order in all its variables. In fact, all poles corresponding to massive string states disappear since massive modes get an infinite mass and decouple in the low-energy limit. From an effective field theory perspective, we can imagine these polynomial as arising from higher curvature corrections $D^{2n}\mathcal{R}^4$ to the type IIB supergravity action with vertices containing an increasing number of derivatives. In complete analogy with the momentum transform in flat space, we expect the term $D^{2n}\mathcal{R}^4$ to give rise to a polynomial of *maximum* degree n in the $AdS_5 \times S^5$ Mellin variables. In fact, one extra layer of complication of AdS string amplitudes is that, unlike their flat space counterparts, they are *not* homogeneous polynomial of fixed degree at each order in the expansion, rather they come with a whole tower of lower degree polynomials, which come from terms in 10 dimensions with legs on S^5.

The problem of computing string corrections has been attacked in various ways, with the help of flat space limit techniques [44, 45], localisation [46–49, 51, 133], bootstrap [1, 3] and effective field theory approaches [50]. However, until [1], no results were known for general charges, except for the α'^3 amplitude which, as we will review, is completely fixed by flat space limit [45]. In that paper we addressed the problem for the first time at the order α'^5. We made an ansatz for the amplitude which we were then able to fix with a bootstrap approach which relied on some observed patterns in the anomalous dimensions. We will explain this in detail in the next chapters. Quite nicely, Aprile and Vieira showed that the result - originally written in the spherical harmonics basis - admits a very compact representation in the large p formalism [32]. We then took advantage of the formalism to upgrade our method and bootstrap higher α' corrections [3]. Reassuringly, we found that the results are consistent with those of [50], where a different approach, based on some generalised Witten diagrams, was used.

Before presenting the method and the results, it is useful to recall some properties of the VS amplitude in flat space and we do so in section 5.1. This is a natural starting point since, as we mentioned already, there is a close relation between scattering amplitudes in flat space and AdS. Then, in section 5.2 we describe the general ansatz dictated by the large p limit; in section 5.4 we present the bootstrap results, postponing the discussion on the method we used to fix the ansatz to chapter 7. We conclude the chapter by sketching the main idea of the effective field theory approach of [50], with which we find perfect agreement. This will help us to gain some new intuition on our results.

5.1 VS in flat space and the flat space limit

The Virasoro-Shapiro amplitude is the tree-level four-point amplitude of type IIB string theory in a flat background. As mentioned in the introduction, the formula encodes a lot of features which, since the dawn of string theory, suggested string theory could be a good candidate for a theory of quantum gravity. The amplitude is the product of a kinematical factor, which takes into account polarisation information, and a dynamical factor, function of Mandelstam variables. With a slight abuse of language, we will refer to the latter as the Virasoro-Shapiro amplitude. This takes a very simple form[1]

$$\mathcal{V}_{\text{flat}} = -\alpha'^3 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)\Gamma(-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}. \tag{5.1.1}$$

where the Mandelstam variables satisfy $s+t+u=0$. There are a number of things worth noticing. First, the amplitude contains a (infinite) sequence of poles that correspond to the (infinite) tower of massive modes going on shell. These massive particles decouple in the low-energy limit, and, as a consequence of this, the poles disappear when we perform an α' expansion about zero:

$$\begin{aligned} &\mathcal{V}_{\text{flat}} = \mathcal{V}_{\text{SUGRA}} + \sum_n \mathcal{V}_n \alpha'^{n+3} = \\ &\frac{1}{s\,t\,u} + \alpha'^3\, 2\zeta_3 + \alpha'^5\, \zeta_5(s^2+t^2+u^2) + \alpha'^6\, 2\zeta_3^2 s\,t\,u + \alpha'^7\, \zeta_7(s^4+t^4+u^4) + \cdots \end{aligned} \tag{5.1.2}$$

[1] For convenience, we have also rescaled α' by a factor of 4 with respect to the actual amplitude.

Note that the amplitude only contains odd zetas. In fact, (5.1.1) can be written - within a certain radius of convergence - as an exponential:

$$\mathcal{V}_{\text{flat}} = \frac{1}{s\,t\,u} \exp\left[\sum \frac{2\zeta_{2n+1}}{2n+1} \alpha'^{2n+1}(s^{2n+1} + t^{2n+1} + u^{2n+1})\right]. \tag{5.1.3}$$

Note also that the amplitude is, at given order in α', an homogenous polynomial in the Mandelstam variables.

In AdS, we expect the above flat space VS to be related to the *reduced* Mellin amplitude defined via the integral transform (4.1.8). This reduced amplitude inherits from the correlator an α' expansion which we parametrise in the following way

$$\mathcal{M} = \mathcal{M}_{\text{SUGRA}} + \sum_n \mathcal{M}_n \alpha'^{n+3} \tag{5.1.4}$$

where $\mathcal{M}_{\text{SUGRA}}$ is the supergravity amplitude (4.2.13).

$\mathcal{M}_n$ is a non-homogeneous polynomial of all its variables, and we expect it to be captured by the following ansatz

$$\mathcal{M}_n = \sum_{\ell=0}^{n-1} (\Sigma - 1)_{\ell+3} \mathcal{M}_{n,l} + (\Sigma - 1)_{n+3}\, \mathcal{M}_{n,n}(\mathbf{s}, \mathbf{t}, \mathbf{u}) \tag{5.1.5}$$

where $\Sigma = \frac{1}{2}(p_1+p_2+p_3+p_4)$ and we remind that $(\cdots)_n$ denotes the Pochhammer symbol (or rising factorial). Here, $\mathcal{M}_{n,l}$ are polynomials of degree $l < n$ in $\mathbf{s}, \mathbf{t}$, subleading with respect to $\mathcal{M}_{n,n}$, and we will deal with them in the next section. For now, let us notice that the leading power in $\mathcal{M}_n$, denoted by $\mathcal{M}_{n,n}$, is fixed by the flat space limit, i.e. $\mathcal{M}_{n,n} = \mathcal{V}_n$. In fact, by incorporating the large p limit with the Penedones flat space limit [58], we get the following relation between flat and $AdS_5 \times S^5$ Virasoro-Shapiro amplitude:

$$\mathcal{M}_n\big|_{\text{leading}} = \frac{1}{\Gamma(\Sigma - 1)} \int_0^\infty d\alpha\; e^{-\alpha} \alpha^{1+\Sigma}\, \mathcal{V}_n(\alpha \mathbf{s}, \alpha \mathbf{t}) = (\Sigma - 1)_{n+3}\, \mathcal{V}_n(\mathbf{s}, \mathbf{t}, \mathbf{u}) \tag{5.1.6}$$

where $\mathcal{V}_n$ is defined by (5.1.2). This also justifies the presence of $(\Sigma - 1)_{n+3}$ in the ansatz[2]. For example, at α'^3 we have [45],

$$\mathcal{M}_0 = (\Sigma - 1)_3 \mathcal{M}_{0,0} = (\Sigma - 1)_3 \mathcal{V}_0 = 2(\Sigma - 1)_3 \zeta_3. \tag{5.1.7}$$

Note that in this case the flat space limit reproduces the full answer because there is no room left for lower degree polynomials. At α'^5 we have

$$\mathcal{M}_2 = (\Sigma - 1)_5 \mathcal{M}_{2,2} + \cdots = (\Sigma - 1)_5 \zeta_5 (\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2) + \cdots \tag{5.1.8}$$

[2] *A priori*, there is no obvious reason why lower degree coefficients should be accompanied by similar Pochhammers. However, their appearance is in perfect agreement with bootstrap and localisation results.

where $\cdots$ refers to terms of lower degree in $\mathbf{s}, \mathbf{t}, \tilde{s}, \tilde{t}, \vec{p}$ which are not fixed by the flat space limit.

5.2 A novel large p stratification

The general idea is to start with an ansatz dictated by the large p limit and bootstrapping it by imposing crossing symmetry and a special requirement on the number of operators exchanged in the OPE, based on a certain $10d$ spin, which we will define later on. In this chapter we will focus on the construction of the ansatz and the explicit results, by postponing the discussion on the procedure used to fix the ansatz to the next chapters because this will require introducing various details of the double-trace spectrum of $\mathcal{N}=4$ at strong coupling.

As mentioned in the previous section, we expect to accommodate the $AdS_5 \times S^5$ version of the VS amplitude in the polynomial ansatz,

$$\mathcal{M}_n = \sum_{\ell=0}^{n-1} (\Sigma-1)_{\ell+3} \mathcal{M}_{n,l} + (\Sigma-1)_{n+3}\, \mathcal{M}_{n,n}(\mathbf{s}, \mathbf{t}, \mathbf{u}) \tag{5.2.9}$$

where $\mathcal{M}_{n,l}$ are polynomial coefficient functions to be determined and the subscript n stands for the amplitude at the order α'^{n+3}. The ansatz starts with $\mathcal{M}_{n,n}$, that, as we explained before, is fixed by the flat space VS amplitude.

Now, in the large p limit both Mellin variables and charges scale in the same way, say with p. Thus the large p limit of $\mathcal{M}_n$ is $(\Sigma-1)_{n+3}\mathcal{M}_{n,n}$ by construction, and enjoys a $10d$ symmetry, precisely because it depends on bold-face variables only. The completion of it in $AdS_5 \times S^5$ has more structures, which are parametrised by the strata $\mathcal{M}_{n,l}$ and are in general different from zero. As we will see, these in general do not just depend on bold-variables. Thus we can already anticipate that

α' corrections *break* the hidden $10d$ conformal symmetry.

Let us now deal with $\mathcal{M}_{n,l}$. A first bound on this polynomial comes from the large p limit. Note that the leading term in $(\Sigma-1)_{n+3}\mathcal{M}_{n,n}$ scales like p^{2n+3}, thus the next-to-leading term should scale at most as p^{2n+3-1} in order not to conflict with the large p limit. However, *a posteriori*, we observe that the various $\mathcal{M}_{n,l}$ satisfy a more strict limit. The scaling behaviour of each term is given in the table below.

$$\begin{array}{c|ccccc} \ell & 0 & 1 & 2 & 3 & \dots \\ \hline \mathcal{M}_{0,0} & p^3 & & & & \\ \mathcal{M}_{1,l} & p^4 & p^5 & & & \\ \mathcal{M}_{2,l} & p^5 & p^6 & p^7 & & \\ \mathcal{M}_{3,l} & p^6 & p^7 & p^8 & p^9 & \\ \vdots & & & & & \end{array} \tag{5.2.10}$$

We will call this feature *large p stratification.*

Given the above scaling properties, we can then define $\mathcal{M}_{n,\ell}$ in the following way:

> $\mathcal{M}_{n,\ell}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{s},\tilde{t},\vec{p})$ is a crossing symmetric polynomial in all its variables, of maximum degree n, such that only monomials of degree ℓ in **s**, **t** and **u** appear.

Therefore $\mathcal{M}_{n,\ell}$ is not an homogeneous polynomial, but of course can be written recursively,

$$span\big(\mathcal{M}_{n,\ell}\big) = span\big(\mathcal{H}_{n,(\ell,n-\ell)}\,,\,\mathcal{M}_{n-1,\ell}\big) \tag{5.2.11}$$

by isolating each time a new homogeneous polynomial. In fact,

> $\mathcal{H}_{n,(\ell,n-\ell)}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{s},\tilde{t},\vec{p})$ is a crossing symmetric polynomial in all its variables, of fixed degree n, such that only monomials of degree ℓ in **s**, **t** and **u** appear.

Consequently, $\mathcal{H}_{n,(\ell,n-\ell)}$ has degree $n-\ell$ in all other variables $\tilde{s},\tilde{t}$, and $p_1p_2p_3p_4$.

For what we just said, $\mathcal{M}_{n,\ell}$ can be parametrised as following

$$\mathcal{M}_{n,\ell} = \sum_{0\leq d_1+d_2\leq \ell} C^{(n)}_{\ell;\,d_1d_2}(\tilde{s},\tilde{t},\vec{p})\,\mathbf{s}^{d_1}\mathbf{t}^{d_2} \tag{5.2.12}$$

where $C^{(n)}_{\ell;\,d_1d_2}$ are polynomials in the remaining variables. Note that, even though $\mathcal{M}_{n,\ell}$ is by definition a polynomial of fixed degree in $\mathbf{s},\mathbf{t},\mathbf{u}$, in the sum we do need to include all lower powers $d_1+d_2\leq\ell$. The reason is that a crossing symmetric polynomial will depend on $\mathbf{s},\mathbf{t}$, and $\mathbf{u}=-\mathbf{s}-\mathbf{t}-4$, thus any power of $\mathbf{u}$ brings down lower powers of $\mathbf{s}$ and $\mathbf{t}$ in the stratum. This introduces a subtlety in the construction of an ansatz for $\mathcal{M}_{n,\ell}$. In fact, while $\mathcal{M}_{n,\ell}$ can always be decomposed as in (5.2.12), the converse is not true, in the sense that given a generic polynomial as in the r.h.s. of (5.2.12), this in general does not contain *only* $\mathcal{M}_{n,\ell}$ but also polynomials of lower degree in $\mathbf{s},\mathbf{t},\mathbf{u}$. In appendix A we give some more detail on how to read off $\mathcal{M}_{n,\ell}$ starting from a generic polynomial ansatz.

Let us now parametrise the polynomial $C^{(n)}_{\ell;\, d_1 d_2}(\tilde{s},\tilde{t},\vec{p})$. Large p stratification and the fact that $\mathbf{s}^{d_1}\mathbf{t}^{d_2}$ has maximum degree l imply

$$\lim_{p\to\infty} C^{(n)}_{\ell;\, d_1 d_2}(\tilde{s},\tilde{t},\vec{p}) \;\sim\; p^{(n-\ell)}, \tag{5.2.13}$$

and we can therefore write

$$C^{(n)}_{\ell;\, d_1 d_2} = \sum_{0\leq \delta_1+\delta_2 \leq (n-\ell)} c^{(n)}_{\ell;\, d_1 d_2, \delta_1 \delta_2}(\vec{p})\, \tilde{s}^{\delta_1} \tilde{t}^{\delta_2} \tag{5.2.14}$$

where finally $c^{(n)}_{\ell;\, d_1 d_2,\delta_1\delta_2}$ is a polynomial in p_1,p_2,p_3,p_4 of max degree $(n-\ell)-\delta_1-\delta_2$.

With the information about $C^{(n)}_{\ell}$ at hand, we can now bootstrap the correlator. We will directly start from the sharper ansatz dictated by the large p stratification. Let us however stress that is *not* necessary. Had we started from a wider ansatz, for example without imposing large p stratification, we would have arrived to the same conclusions after applying all the bootstrap constraints, see also section 5.3.

Let us first deal with crossing symmetry. This is a statement about the full correlator and in particular about the equality

$$\langle \mathcal{O}_{p_1}(\mathrm{x}_{\sigma_1})\mathcal{O}_{p_2}(\mathrm{x}_{\sigma_2})\mathcal{O}_{p_3}(\mathrm{x}_{\sigma_3})\mathcal{O}_{p_4}(\mathrm{x}_{\sigma_4})\rangle = \langle \mathcal{O}_{p_{\sigma_1}}(\mathrm{x}_1)\mathcal{O}_{p_{\sigma_2}}(\mathrm{x}_2)\mathcal{O}_{p_{\sigma_3}}(\mathrm{x}_3)\mathcal{O}_{p_{\sigma_4}}(\mathrm{x}_4)\rangle, \tag{5.2.15}$$

where $\mathtt{x}$ is a shorthand for the pair (x,y), with x being the spacetime coordinate and y the internal. The possible permutations σ are six. Considering the Mellin transform of the correlator, we then deduce what relations the Mellin amplitude satisfies

$$\begin{aligned}
&\mathcal{M}(s,u,\tilde{s},\tilde{u};p_2,p_1,p_3,p_4) = \mathcal{M}(s,t,\tilde{s},\tilde{t};p_1,p_2,p_3,p_4),\\
&\mathcal{M}(t,s,\tilde{t},\tilde{s};p_1,p_4,p_3,p_2) = \mathcal{M}(s,t,\tilde{s},\tilde{t};p_1,p_2,p_3,p_4),\\
&\mathcal{M}(u,t,\tilde{u},\tilde{t};p_4,p_2,p_3,p_1) = \mathcal{M}(s,t,\tilde{s},\tilde{t};p_1,p_2,p_3,p_4),\\
&\mathcal{M}(s,u-k_u,\tilde{s},\tilde{u}+k_u;p_1,p_2,p_4,p_3) = \mathcal{M}(s,t,\tilde{s},\tilde{t};p_1,p_2,p_3,p_4),\\
&\mathcal{M}(t-k_t,s-k_s,\tilde{t}+k_t,\tilde{s}+k_s;p_3,p_2,p_1,p_4) = \mathcal{M}(s,t,\tilde{s},\tilde{t};p_1,p_2,p_3,p_4),\\
&\mathcal{M}(u-k_u,t,\tilde{u}+k_u,t;p_1,p_3,p_2,p_4) = \mathcal{M}(s,t,\tilde{s},\tilde{t};p_1,p_2,p_3,p_4).
\end{aligned} \tag{5.2.16}$$

The best we can do at this point is to make crossing symmetry manifest by identifying variables such that the transformations above act in a ‘block diagonal’ form. The large p limit suggests first to pick $\mathbf{s},\mathbf{t},\mathbf{u}$ and we will accompany this with another set. In total

$$\begin{aligned}
&\mathbf{s} = s+\tilde{s}, \qquad \mathbf{t} = t+\tilde{t}, \qquad\qquad \mathbf{s}+\mathbf{t}+\mathbf{u} = -4,\\
&\tilde{\mathbf{s}} = k_s+2\tilde{s}, \qquad \tilde{\mathbf{t}} = k_t+2\tilde{t}, \qquad \tilde{\mathbf{s}}+\tilde{\mathbf{t}}+\tilde{\mathbf{u}} = \Sigma-4,
\end{aligned} \tag{5.2.17}$$

$$k_s = \tfrac{p_1+p_2-p_3-p_4}{2}, \quad k_t = \tfrac{p_1+p_4-p_2-p_3}{2}, \quad k_u = \tfrac{p_2+p_4-p_3-p_1}{2}, \quad \Sigma = \tfrac{p_1+p_2+p_3+p_4}{2}.$$

In these variables, crossing becomes

$$\begin{aligned}
\mathcal{M}(\mathbf{s},\mathbf{u},\mathbf{t},\tilde{\mathbf{s}},\tilde{\mathbf{u}},\tilde{\mathbf{t}},+k_s,+k_u,+k_t,\Sigma) &= \mathcal{M}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)\\
\mathcal{M}(\mathbf{t},\mathbf{s},\mathbf{u},\tilde{\mathbf{t}},\tilde{\mathbf{s}},\tilde{\mathbf{u}},+k_t,+k_s,+k_u,\Sigma) &= \mathcal{M}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)\\
\mathcal{M}(\mathbf{u},\mathbf{t},\mathbf{s},\tilde{\mathbf{u}},\tilde{\mathbf{t}},\tilde{\mathbf{s}},+k_u,+k_t,+k_s,\Sigma) &= \mathcal{M}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)\\
\mathcal{M}(\mathbf{s},\mathbf{u},\mathbf{t},\tilde{\mathbf{s}},\tilde{\mathbf{u}},\tilde{\mathbf{t}},+k_s,-k_u,-k_t,\Sigma) &= \mathcal{M}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)\\
\mathcal{M}(\mathbf{t},\mathbf{s},\mathbf{u},\tilde{\mathbf{t}},\tilde{\mathbf{s}},\tilde{\mathbf{u}},-k_t,-k_s,+k_u,\Sigma) &= \mathcal{M}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)\\
\mathcal{M}(\mathbf{u},\mathbf{t},\mathbf{s},\tilde{\mathbf{u}},\tilde{\mathbf{t}},\tilde{\mathbf{s}},-k_u,+k_t,-k_s,\Sigma) &= \mathcal{M}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)
\end{aligned} \tag{5.2.18}$$

Each set of three transforms in the same way, modulo ± 1 signs. Σ is obviously singlet. To summarise, the combination of crossing symmetry and large p stratification provides us with the initial ansatz for the VS amplitude in $AdS_5 \times S^5$. The results are summarised by the formula

$$\mathcal{M}_n = \sum_{\ell=0}^{n-1}(\Sigma-1)_{\ell+3}\,\mathcal{M}_{n,\ell}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)+(\Sigma-1)_{n+3}\,\mathcal{M}^{flat}_{n,n}(\mathbf{s},\mathbf{t},\mathbf{u}) \tag{5.2.19}$$

where recursively we get

$$span\big(\,\mathcal{M}_{n,\ell}\,\big) = span\big(\,\mathcal{H}_{n,(\ell,n-\ell)}\,,\,\mathcal{M}_{n-1,\ell}\big). \tag{5.2.20}$$

Here, $\mathcal{H}_{n,(\ell,n-\ell)}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)$ is an homogeneous polynomial of degree n such that only monomials of degree ℓ in $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{u}$ appear. Note that

$$\mathcal{H}_{n,(\ell,n-\ell)}(\mathbf{s},\mathbf{t},\mathbf{u},\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}},k_s,k_t,k_u,\Sigma)$$

with $0 \leq l \leq n$ is the new genuine contribution at the order α'^{n+3}. The construction of all possible terms which can contribute to the amplitude is quite interesting. In appendix A we describe the method we used. A counting of initial parameters is given in the table below. The notation $|\mathcal{H}|$ stands for the number of crossing invariant terms.

α'^3	α'^4	α'^5	α'^6	α'^7	α'^8	α'^9
$\lvert\mathcal{H}_{0,(0,0)}\rvert = 1$	$\lvert\mathcal{H}_{1,(0,1)}\rvert = 1$	$\lvert\mathcal{H}_{2,(0,2)}\rvert = 3$	$\lvert\mathcal{H}_{3,(0,3)}\rvert = 6$	$\lvert\mathcal{H}_{4,(0,4)}\rvert = 11$	$\lvert\mathcal{H}_{5,(0,5)}\rvert = 18$	$\lvert\mathcal{H}_{6,(0,6)}\rvert = 32$
	$\lvert\mathcal{H}_{1,(1,0)}\rvert = 0$	$\lvert\mathcal{H}_{2,(1,1)}\rvert = 1$	$\lvert\mathcal{H}_{3,(1,2)}\rvert = 3$	$\lvert\mathcal{H}_{4,(1,3)}\rvert = 6$	$\lvert\mathcal{H}_{5,(1,4)}\rvert = 14$	$\lvert\mathcal{H}_{6,(1,5)}\rvert = 26$
		$\lvert\mathcal{H}_{2,(2,0)}\rvert = 1$	$\lvert\mathcal{H}_{3,(2,1)}\rvert = 2$	$\lvert\mathcal{H}_{4,(2,2)}\rvert = 6$	$\lvert\mathcal{H}_{5,(2,3)}\rvert = 12$	$\lvert\mathcal{H}_{6,(2,4)}\rvert = 25$
			$\lvert\mathcal{H}_{3,(3,0)}\rvert = 1$	$\lvert\mathcal{H}_{4,(3,1)}\rvert = 2$	$\lvert\mathcal{H}_{5,(3,2)}\rvert = 6$	$\lvert\mathcal{H}_{6,(3,3)}\rvert = 14$
				$\lvert\mathcal{H}_{4,(4,0)}\rvert = 1$	$\lvert\mathcal{H}_{5,(4,1)}\rvert = 3$	$\lvert\mathcal{H}_{6,(4,2)}\rvert = 9$
					$\lvert\mathcal{H}_{5,(5,0)}\rvert = 1$	$\lvert\mathcal{H}_{6,(5,1)}\rvert = 3$
						$\lvert\mathcal{H}_{6,(6,0)}\rvert = 2$

(5.2.21)

As an example, $\mathcal{H}_{2,(0,2)}$ is spanned by all crossing symmetric terms of degree 0 in $\mathbf{s}, \mathbf{t}, \mathbf{u}$ and degree 2 in all other variables:

$$span(\mathcal{H}_{2,(0,2)}) = \left\{ \tilde{\mathbf{s}}^2 + \tilde{\mathbf{t}}^2 + \tilde{\mathbf{u}}^2,\, k_s^2 + k_t^2 + k_u^2,\, \Sigma^2 \right\}. \tag{5.2.22}$$

Note that crossing symmetry forbids terms like $k_s \tilde{\mathbf{s}} + \texttt{crossing}$. The ansatz for $\mathcal{M}_{n,\ell}$ is obtained from the recursion in (5.2.20), so that the total number of terms is given by summing along the rows, from right to left.

Moreover, notice that inside a given $\mathcal{H}_{n,(\ell,n-\ell)}$ we can add another level, which is the one given by terms of the form ($\Sigma^{\#} \times$ crossing invariants), where usually the latter already appeared at previous orders. For example,

$$span(\,\mathcal{H}_{3,(2,1)}\,) = \left\{ \,\mathbf{s}^2\tilde{\mathbf{s}} + \mathbf{t}^2\tilde{\mathbf{t}} + \mathbf{u}^2\tilde{\mathbf{u}}\ ,\ \Sigma \times \left(\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2\right) \right\}. \tag{5.2.23}$$

Terms of the form ($\Sigma^{\#} \times$ crossing invariants) are the first instance of the more general class of terms of the form (crossing invariant) $\times$ (crossing invariant). In the case above one of the two is simply a power of Σ.

The next step is to impose constraints on the free parameters in our initial ansatz, at each order in the α' expansion. This is done essentially by imposing a bound on the spectrum of two-particle operators visible by $\mathcal{M}_n$. We will explain what we mean by this in chapter 7. For the moment, let us just point out that there will be an infinite number of constraints - which we will refer to as *rank constraints*, for reasons we will explain later on - but finitely many parameters in our ansatz. The outcome will be our proposal for the VS amplitude in $AdS_5 \times S^5$ up to certain ambiguities, at its first stage. For example, we will see that we will not be able to fix the ambiguity of adding previous amplitudes $\mathcal{M}_{k\leq n-1}$ to our result for $\mathcal{M}_n$, within the bootstrap. Nevertheless, the problem of finding certain CFT data in what we call the "edge" *is fully determined* at each order in α', therefore for each new amplitude that we bootstrap, we can extract novel CFT data out of it, and feed these new data into the OPE relations governing the amplitudes at higher orders, thus reducing the number of free parameters at the first stage.

5.3 *Intermezzo*: from the spherical harmonics basis to the large p formalism

Before presenting the results in the large p formalism, we open here a small digression on the way we computed the α'^5 amplitude in [1]. As we mentioned at the beginning of the chapter, we originally bootstrapped an ansatz in the spherical harmonics basis. We remind that this is related to $\tilde{U}, \tilde{V}$ by a linear change of basis, see (2.3.22) where we write $\mathtt{Y}_{b,a}$ in terms of $y, \bar{y}$ or, equivalently, $\tilde{U}, \tilde{V}$. In the spherical harmonics basis, the amplitude becomes a function of $(s, t, b, a, \vec{p})$, rather than $(s, t, \tilde{s}, \tilde{t}, \vec{p})$. The method we used to bootstrap the ansatz is based on the same idea we used in [3] which we are going to describe in the next chapters. The main difference is that the ansatz we proposed in the spherical harmonics basis was much wider, because we did not input any information about large p limit and stratification. Reassuringly, the solution obtained in this way was then shown to respect the large p limit [32]. The understanding of the large p limit then allowed us to bootstrap higher order corrections starting from a much more strict ansatz, as explained in the previous section. In the next section we will present all the results directly in the large p formalism.

5.4 Explicit results and remarkable simplifications

We are now ready to present the results for the first few orders in α'. Let us stress again that there will be ambiguities showing up in the results, i.e. coefficients not fixed by the bootstrap. To fix the ambiguities we will need additional input. One source of such information is the relation between the integrated correlators and derivatives of the partition function w.r.t. deformations of $\mathcal{N} = 4$ SYM on the sphere, computed by supersymmetric localisation [46, 47]. Some ambiguities can be fixed with the currently available data, and our formalism will make more transparent how these contribute.

Let us start with the α'^3 amplitude. As we mentioned already, at this order, the flat space limit fully fixes the amplitude which reads

$$\mathcal{M}_0 = 2(\Sigma - 1)_3 \zeta_3. \tag{5.4.24}$$

At α'^4 the flat space contribution vanishes but we do find a non zero ansatz in $AdS_5 \times S^5$, i.e.

$$\mathcal{M}_1 = (\Sigma - 1)_3 (z_1 + z_2 \Sigma) \tag{5.4.25}$$

These constants are set to zero by localisation [46]. Independently, the rank constraints [3] will also set to zero the term $\Sigma \times z_2$.

The first non-trivial amplitude is at α'^5. We have

$$\mathcal{M}_2 = (\Sigma - 1)_3 \mathcal{M}_{2,0} + (\Sigma - 1)_4 \mathcal{M}_{2,1} + (\Sigma - 1)_5 \zeta_5 \times (\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2) \tag{5.4.26}$$

with the strata given by

$$\begin{aligned} \mathcal{M}_{2,0} &= z_{3,1}\Sigma^2 + z_{3,2}(k_s^2 + k_t^2 + k_u^2) + z_{3,3}\left(\tilde{\mathbf{s}}^2 + \tilde{\mathbf{t}}^2 + \tilde{\mathbf{u}}^2\right) + z_{3,4}\Sigma + z_{3,5} \\ \mathcal{M}_{2,1} &= z_{4,1}\left(\mathbf{s}\tilde{\mathbf{s}} + \mathbf{t}\tilde{\mathbf{t}} + \mathbf{u}\tilde{\mathbf{u}}\right). \end{aligned} \tag{5.4.27}$$

[3] We refer to chapters 6 and 7 for what we precisely mean by rank constraints.

The rank constraints imply

$$z_{4,1} = -5, \qquad z_{3,3} = 5, \qquad z_{3,2} - z_{3,1} = 11, \qquad z_{3,4} = 0. \tag{5.4.28}$$

Note that there are two free parameters, $z_{3,5}$, which is a constant as the amplitude at α'^3, and the other, say $z_{3,1}$, that goes with the combination $\Sigma^2 + k_s^2 + k_t^2 + k_u^2$. They can be fixed with the help of localisation [46–49, 51, 133], and we get

$$z_{3,1} = -\tfrac{27}{2}, \qquad z_{3,5} = \tfrac{33}{2}\,. \tag{5.4.29}$$

The parametrisation of the VS amplitude at α'^6 is

$$\mathcal{M}_3 = (\Sigma-1)_3\mathcal{M}_{3,0} + (\Sigma-1)_4\mathcal{M}_{3,1} + (\Sigma-1)_5\mathcal{M}_{3,2} + (\Sigma-1)_6 \times \tfrac{2}{3}\zeta_3^2\big(\mathbf{s}^3+\mathbf{t}^3+\mathbf{u}^3\big) \tag{5.4.30}$$

with the lower strata given by

$$\mathcal{M}_{3,2} = z_{5,1}\left(\mathbf{s}^2\tilde{\mathbf{s}} + \mathbf{t}^2\tilde{\mathbf{t}} + \mathbf{u}^2\tilde{\mathbf{u}}\right) + \left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)(\Sigma z_{5,2} + z_{5,3}) \tag{5.4.31}$$

$$\mathcal{M}_{3,1} = z_{4,1}\left(\mathbf{s}\tilde{\mathbf{s}}^2 + \mathbf{t}\tilde{\mathbf{t}}^2 + \mathbf{u}\tilde{\mathbf{u}}^2\right) + z_{4,2}\left(k_s^2\mathbf{s} + k_t^2\mathbf{t} + k_u^2\mathbf{u}\right) + (\Sigma z_{4,3} + z_{4,4})\left(\mathbf{s}\tilde{\mathbf{s}} + \mathbf{t}\tilde{\mathbf{t}} + \mathbf{u}\tilde{\mathbf{u}}\right)$$

$$\begin{aligned}\mathcal{M}_{3,0} = {} & z_{3,1}(\tilde{\mathbf{s}}^3 + \tilde{\mathbf{t}}^3 + \tilde{\mathbf{u}}^3) + z_{3,2}\left(k_s^2\tilde{\mathbf{s}} + k_t^2\tilde{\mathbf{t}} + k_u^2\tilde{\mathbf{u}}\right) + z_{3,3}\Sigma^3 + z_{3,4}(k_s^2+k_t^2+k_u^2)\Sigma \\ & + z^{(3)}_{3,6}k_s\,k_t\,k_u + (z_{3,5}\Sigma + z_{3,9})\left(\tilde{\mathbf{s}}^2 + \tilde{\mathbf{t}}^2 + \tilde{\mathbf{u}}^2\right) + z_{3,7}\Sigma^2 + z_{3,8}(k_s^2+k_t^2+k_u^2) + \Sigma z_{3,10} + z_{3,11}.\end{aligned} \tag{5.4.32}$$

From left to right we first wrote the terms corresponding to the homogeneous polynomial $\mathcal{H}_{3,(\ell,3-\ell)}$ and then the terms coming from previous orders, which in this case are simple to recognise.

The rank constraints impose

$$\begin{aligned}
& z_{5,1} = -6, && z_{5,2} = 4, \\
& z_{4,1} = +15, && z_{4,2} = -\tfrac{7}{3} - \tfrac{1}{32}z_{3,10}, \quad z_{4,3} = -\tfrac{58}{3} + \tfrac{1}{16}z_{3,10}, \quad z_{4,4} = -\tfrac{4}{3} - 5z_{5,3} - \tfrac{1}{8}z_{3,10}, \\
& z_{3,1} = -10, && z_{3,2} = \tfrac{14}{3} + \tfrac{1}{16}z_{3,10}, \quad z_{3,3} = -32 + \tfrac{1}{8}z_{3,10}, \quad z_{3,4} = -\tfrac{7}{3} - \tfrac{1}{32}z_{3,10}, \\
& && z_{3,5} = \tfrac{55}{3} - \tfrac{5}{32}z_{3,10}, \quad z_{3,7} = -\tfrac{22}{3} - \tfrac{11}{16}z_{3,10} + z_{3,8} - 11z_{5,3}, \\
& && z_{3,9} = \tfrac{10}{3} + \tfrac{5}{16}z_{3,10} + 5z_{5,3}, \quad z_{3,6} = 0.
\end{aligned} \tag{5.4.33}$$

The four free parameters are: $z_{3,11}$ and $z_{3,8}$, i.e. the ambiguities we also found at order α'^5, then $z_{5,3}$, i.e. the ambiguity corresponding to a shift by the same amplitude as $\mathcal{V}_2$, and finally $z_{3,10}$.

At this point we can use the OPE once more by considering what information at α'^6 comes from the amplitude at α'^5, in particular from the solution of the partial degeneracy of operators at $m^* = 2$. We explain the details of this procedure in section 7.2.2. Remarkably, the new constraints we obtain in this way are automatically satisfied, therefore we are left with four genuine bootstrap ambiguities.

By imposing on our bootstrapped amplitude consistency with the results from supersymmetric localisation for $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle$ and $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$, see e.g. [46], we obtain three additional equations [4]

$$z_{3,8} = \tfrac{4}{3} - \tfrac{1}{16} z_{3,10}, \qquad z_{3,11} = 0, \qquad z_{5,3} = -2, \tag{5.4.34}$$

leaving us finally with only one free parameter, $z_{3,10}$. In contrast to order α'^5, here localisation is not yet sufficient to fix the full amplitude.

Remarkable simplicity: a generalised flat space limit

The results for $(\alpha')^{7,8,9}$ can be presented as above, and we do so in an ancillary file. Here, we will observe a further structure in the pattern of the coefficients leading to remarkable simplicity. For example, returning to $\mathcal{M}_2$ given in (5.4.26)-(5.4.29), if we expand in terms of the original $AdS_5 \times S^5$ Mellin variables s, t, u and $\tilde{s}, \tilde{t}, \tilde{u}$ we observe the terms of the form $s^l \tilde{s}^a$ with $a + l = 2$ have coefficients

$$(\Sigma - 1 - a)_{a+l+3} \frac{(a+l)!}{a!l!} s^l \tilde{s}^a \,. \tag{5.4.35}$$

Note that these coefficients arise from different strata in $\mathcal{M}_2$ thus they are non trivial. A similar pattern is observed at higher orders for the terms with $a+l = n$. This observation suggests a rescaling of the variables according to an integral transform which generalises the one used by Penedones in [58]. The integral transform we have in mind is

$$\mathcal{M}_n = \frac{i}{2\pi} \int_0^\infty d\alpha \int_{\mathcal{C}} d\beta \; e^{-\alpha-\beta} \alpha^{1+\Sigma} (-\beta)^{1-\Sigma} \tilde{\mathcal{M}}_n(\alpha, \beta) \tag{5.4.36}$$

where $\mathcal{C}$ is the Hankel contour. Here $\mathcal{M}_n$ is given by our bootstrap results, and $\tilde{\mathcal{M}}_n$ is a simplified amplitude, defined in terms of the following variables,

$$S = \alpha\hat{s} - \beta\check{s}, \quad \tilde{S} = \alpha\hat{s} + \beta\check{s}, \qquad \begin{cases} \hat{s} = s - \frac{1}{2}k_s + 1\,, \\ \check{s} = \tilde{s} + \frac{1}{2}k_s + 1\,, \end{cases} \tag{5.4.37}$$

and similarly for t-type and u-type variables. The integral transform (5.4.36) provides Γ functions, direct and inverse, and produces the Pochhammer in (5.4.35) for the relevant terms. Quite remarkably, all terms $s^l \tilde{s}^a$ in $\tilde{\mathcal{M}}$ with $a + l = n$ then recombine into the binomial expansion of powers of the combinations S, T, U, while the combinations $\tilde{S}, \tilde{T}, \tilde{U}$ only arise from terms with $a + l < n$.

[4] We thank Shai Chester for sharing these results at orders α'^6, obtained using the methods described in [46].

Let us quote the results for the $\tilde{\mathcal{M}}_n$. For completeness, $\tilde{\mathcal{M}}_0 = 2\zeta_3$ and $\tilde{\mathcal{M}}_1 = 0$. We will split the amplitude as a *particular* contributions plus a choice of ambiguities. At order α'^5 we have

$$\tilde{\mathcal{M}}_2 = \zeta_5 \big[\tilde{\mathcal{M}}_2^{\text{ptic}} + \tilde{\mathcal{M}}_2^{\text{amb}}\big] \tag{5.4.38}$$

with the remaining free parameters (after the rank constraints have been imposed) in the second term. The two terms are given explicitly by

$$\tilde{\mathcal{M}}_2^{\text{ptic}} = S^2 + T^2 + U^2 + 3\Sigma^2, \qquad \tilde{\mathcal{M}}_2^{\text{amb}} = b_1 I_2 + b_2 \tag{5.4.39}$$

where $I_2 \equiv k_s^2 + k_t^2 + k_u^2 + \Sigma^2 = \sum_i p_i^2$. Localisation fixes $b_1 = -\frac{5}{2}$ and $b_2 = \frac{41}{2}$. As we mentioned above, constraints from localisation at this order fully fix the amplitude.

At order α'^6 we have

$$\tilde{\mathcal{M}}_3 = \zeta_3^2 \big[\tilde{\mathcal{M}}_3^{\text{ptic}} + \tilde{\mathcal{M}}_3^{\text{amb}}\big], \qquad \begin{cases} \tilde{\mathcal{M}}_3^{\text{ptic}} &= \frac{2}{3}\big(S^3 + T^3 + U^3 - 2\Sigma(\Sigma^2 - 4)\big) \\ \tilde{\mathcal{M}}_3^{\text{amb}} &= b_1 \tilde{\mathcal{M}}_3^{\text{amb},1} + b_2 \tilde{\mathcal{M}}_2^{\text{ptic}} + b_3 I_2 + b_4 \end{cases} \tag{5.4.40}$$

where the new ambiguous contribution is

$$\tilde{\mathcal{M}}_3^{\text{amb},1} = S(2\tilde{S} + k_s^2) + T(2\tilde{T} + k_t^2) + U(2\tilde{U} + k_u^2) + \Sigma(12 - k_s^2 - k_t^2 - k_u^2)\,. \tag{5.4.41}$$

In this case the constraints from localisation quoted in (5.4.34) become

$$b_1 = -3 - \overline{k}\,, \quad b_2 = 2\overline{k}\,, \quad b_3 = -2\overline{k}\,, \quad b_4 = 8\overline{k}\,, \tag{5.4.42}$$

for some free parameter $\overline{k}$.

At order α'^7 we have $\tilde{\mathcal{M}}_4 = \zeta_7 \big[\tilde{\mathcal{M}}_4^{\text{ptic}} + \tilde{\mathcal{M}}_4^{\text{amb}}\big]$ with

$$\begin{aligned} \tilde{\mathcal{M}}_4^{\text{ptic}} \;=\; & S^4 + T^4 + U^4 + 8(S^2 + T^2 + U^2)\Sigma^2 + 9(S\tilde{S} + T\tilde{T} + U\tilde{U})\Sigma \\ & -\tfrac{1}{2}(\tilde{S}k_s^2 + \tilde{T}k_t^2 + \tilde{U}k_u^2) - \tfrac{1}{4}\Sigma\big[\Sigma(I_2 - 16) - 6k_s k_t k_u - 56\Sigma^3\big]\,. \end{aligned} \tag{5.4.43}$$

and ten ambiguities in total,

$$\begin{aligned} \tilde{\mathcal{M}}_4^{\text{amb}} \;=\; & b_1 \tilde{\mathcal{M}}_4^{\text{amb},1} + b_2 \tilde{\mathcal{M}}_4^{\text{amb},2} + b_3 \tilde{\mathcal{M}}_4^{\text{amb},3} + b_4 I_2 \tilde{\mathcal{M}}_2^{\text{ptic}} + b_5 (I_2)^2 \\ & + b_6 \tilde{\mathcal{M}}_3^{\text{ptic}} + b_7 \tilde{\mathcal{M}}_3^{\text{amb},1} + b_8 \tilde{\mathcal{M}}_2^{\text{ptic}} + b_9 I_2 + b_{10}\,. \end{aligned} \tag{5.4.44}$$

Those in the first line above are either products of terms from previous orders or given by

$$
\begin{aligned}
\tilde{\mathcal{M}}_4^{\text{amb},1} &= S^2(2\tilde{S}+k_s^2+\Sigma^2)+T^2(2\tilde{T}+k_t^2+\Sigma^2)+U^2(2\tilde{U}+k_u^2+\Sigma^2)\\
&\qquad -\Sigma\big(2(Sk_s^2+Tk_t^2+Uk_u^2)+3k_sk_tk_u\big)+\Sigma^2\big(\tfrac{5}{2}I_2-2\Sigma^2+8\big)\,,\\
\tilde{\mathcal{M}}_4^{\text{amb},2} &= k_s^4+k_t^4+k_u^4+12k_sk_tk_u\Sigma+\Sigma^4\,,\\
\tilde{\mathcal{M}}_4^{\text{amb},3} &= (k_s^2+2\tilde{S})^2+(k_t^2+2\tilde{T})^2+(k_u^2+2\tilde{U})^2+28\Sigma^2+2(k_s^2+k_t^2+k_u^2)\Sigma^2-\Sigma^4\,.
\end{aligned}
\tag{5.4.45}
$$

At order α'^8 we just quote the full result in the form,

$$
\begin{aligned}
\tilde{\mathcal{M}}_5 = \tfrac{4}{5}\zeta_3\zeta_5\Big[&\big[S^5+15S^3\Sigma^2+25S^2\tilde{S}\Sigma-\tfrac{5}{2}S\tilde{S}(k_s^2+\Sigma^2)-\tfrac{5}{8}Sk_s^4-\tfrac{5}{4}Sk_s^2\Sigma^2+\tfrac{5}{2}\tilde{S}k_s^2\Sigma+\tfrac{5}{4}k_s^4\Sigma\\
&+\tfrac{5}{4}k_s^2\Sigma^3-5k_s^2\Sigma+(t\text{-type })+(u\text{-type})\big]-\Sigma(32\Sigma^4-135\Sigma^2+88)\\
&+16\text{ ambiguities}\Big]\,.
\end{aligned}
\tag{5.4.46}
$$

The 16 ambiguities for this case can be found in the ancillary file.

The ambiguities at $(\alpha')^{7,8}$ can be further constrained by using the OPE and the data extracted from the amplitude at α'^5, as we tried to do with α'^6 where it turned out the extra constraints were automatically satisfied. In section 7.2.2 we will find a new constraint at α'^7 and two new constraints at α'^8.

The simplicity of the rescaled amplitudes is quite remarkable, with $\tilde{\mathcal{M}}_n$ simply given by the corresponding term in the Virasoro-Shapiro amplitude in terms of S, T, U plus terms of lower order in $S, T, U, \tilde{S}, \tilde{T}, \tilde{U}$. Nicely, this continues to hold even at order α'^9 where there are two distinct contributions coming with different combinations of zeta values,

$$
\tilde{\mathcal{M}}_6 = \zeta_9(S^6+T^6+U^6)-\tfrac{1}{27}(7\zeta_9-4\zeta_3^3)(S^3+T^3+U^3)^2+\dots \tag{5.4.47}
$$

where the dots refer to terms of lower degree. These relations are strongly suggestive of an even more restrictive relation of the Mellin amplitudes to the flat space amplitudes, enhancing that of [32] which itself enhanced that of [58] in the case of $AdS_5 \times S^5$. In the next section we summarise the different types of flat space limits and their connections.

Finally, let us observe that the rescaled amplitudes exhibit properties under an interesting Z_2 transformation which exchanges AdS_5 and S^5 quantities,

$$
\{S,T,U\} \leftrightarrow \{-S,-T,-U\}\,, \qquad \{\tilde{S},\tilde{T},\tilde{U}\} \leftrightarrow \{\tilde{S},\tilde{T},\tilde{U}\}\,, \qquad p_i \leftrightarrow -p_i\,. \tag{5.4.48}
$$

At each order the term $\tilde{\mathcal{M}}_n^{\text{ptic}}$ is even/odd under the transformation depending on whether n is even or odd. Each ambiguity also has a definite parity under the transformation. If one insists that at each order ambiguities of the opposite parity compared to $\tilde{\mathcal{M}}_n^{\text{ptic}}$ are ruled out, then we find that the remaining parameter $\overline{k}$ in eq. (5.4.42) vanishes and that imposing the possible symmetry is simultaneously consistent with the three conditions from localisation. A similar statement also holds at order α'^4 where the constant contribution removed by localisation is also of odd parity.[5] In a similar way the symmetry would also imply $b_6 = b_7 = 0$ in (5.4.44).

5.5 Towards a more general flat space limit

We conclude this chapter by sketching an alternative method, based on an effective field theory approach [50], which leads to the exact same solution we found with our bootstrap method. This approach will help us to gain intuition on the nature of the ambiguities. The idea is to use generalised $AdS_5 \times S^5$ Witten diagrams, where the vertices are obtained by a certain action built in the following way. Note that, excluding the supergravity term, the polynomials in the VS amplitude (5.1.2) can be seen as arising from an effective potential with an infinite number of contact terms with increasing number of derivatives:

$$V = \frac{1}{4!} 2\zeta_3 \alpha'^3 \phi^4 + \frac{1}{2}\zeta_5 \alpha'^5 (\partial_\mu \phi \partial^\mu \phi)^2 + \frac{1}{3} 2\zeta_3^2 \alpha'^6 (\partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi)(\partial_\rho \phi \partial^\rho \phi) + \cdots \tag{5.5.49}$$

The idea of [50] is to uplift this potential to an $AdS_5 \times S^5$ background with the partial derivatives replaced by suitable $AdS_5 \times S^5$ ones. Now, the point is that the uplift is not unique for two reasons: first, the derivatives no longer commute and moreover there are terms involving lower number of derivatives, compensated by the AdS radius which no longer vanish. At the first few orders we have:

$$\begin{aligned} V =& \frac{1}{4!} A \alpha'^3 \phi^4 + \frac{1}{2}\alpha'^5 \Big(B(\nabla_\mu \phi \nabla^\mu \phi)^2 + C(\nabla^2 \nabla_\mu \phi \nabla^\mu \phi)\phi^2 \Big) + \\ &+ \frac{1}{3}\alpha'^6 \Big(D(\nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi)(\nabla_\rho \phi \nabla^\rho \phi) + E(\nabla^2 \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi)\phi^2 \Big) + \cdots \end{aligned} \tag{5.5.50}$$

[5] A similar conclusion has been reached by the author of [50].

where

$$\begin{aligned} A &= 2\zeta_3 + A_1 \frac{\alpha'}{R^2} + A_2 \left(\frac{\alpha'}{R^2}\right)^2 + \cdots \\ B &= \zeta_5 + B_1 \frac{\alpha'}{R^2} + B_2 \left(\frac{\alpha'}{R^2}\right)^2 + \cdots \\ C &= C_0 + C_1 \frac{\alpha'}{R^2} + C_2 \left(\frac{\alpha'}{R^2}\right)^2 + \cdots \\ D &= 2\zeta_3^2 + D_1 \frac{\alpha'}{R^2} + D_2 \left(\frac{\alpha'}{R^2}\right)^2 + \cdots \\ E &= E_0 + E_1 \frac{\alpha'}{R^2} + E_2 \left(\frac{\alpha'}{R^2}\right)^2 + \cdots \end{aligned} \tag{5.5.51}$$

Note these are precisely the two different types of ambiguities we also encountered in our ansatz. For example, A_2 and C_0 are the ambiguities of the α'^5 amplitude and correspond to (linear combination of) $z_{3,1}$ and $z_{3,5}$ in (5.4.26).

Let us end the chapter by commenting on relation between the different types of flat space limits we encountered in this thesis. These are shown pictorially in figure 5.1. The innermost circle is the "improved" Penedones flat space limit, where the Mellin variables have been replaced by bold-face variables $\mathbf{s}, \mathbf{t}, \mathbf{u}$, see (5.1.6). There is evidence that this limit arises as a particular case of the double integral (5.4.36), or, in other words, the *AdS* amplitude always contains a sub-amplitude given by the flat space VS written as a function of the S, T, U variables (5.4.37). In [50], the authors conjecture a more general notion of flat space limit, which includes as a particular case all the flat space limits we discussed, and corresponds to replacing partial derivatives with covariant ones. As we mentioned before, this precisely matches the full sub-amplitude fixed by the bootstrap. Unfortunately, a closed formula for this sub-amplitude is still missing. In the outermost circle we have the full AdS Virasoro-Shapiro amplitude, which is the sum of the sub-amplitude and all the ambiguities. The latter are genuine AdS terms which do not have a counterpart in flat space, because they vanish in the flat space limit.

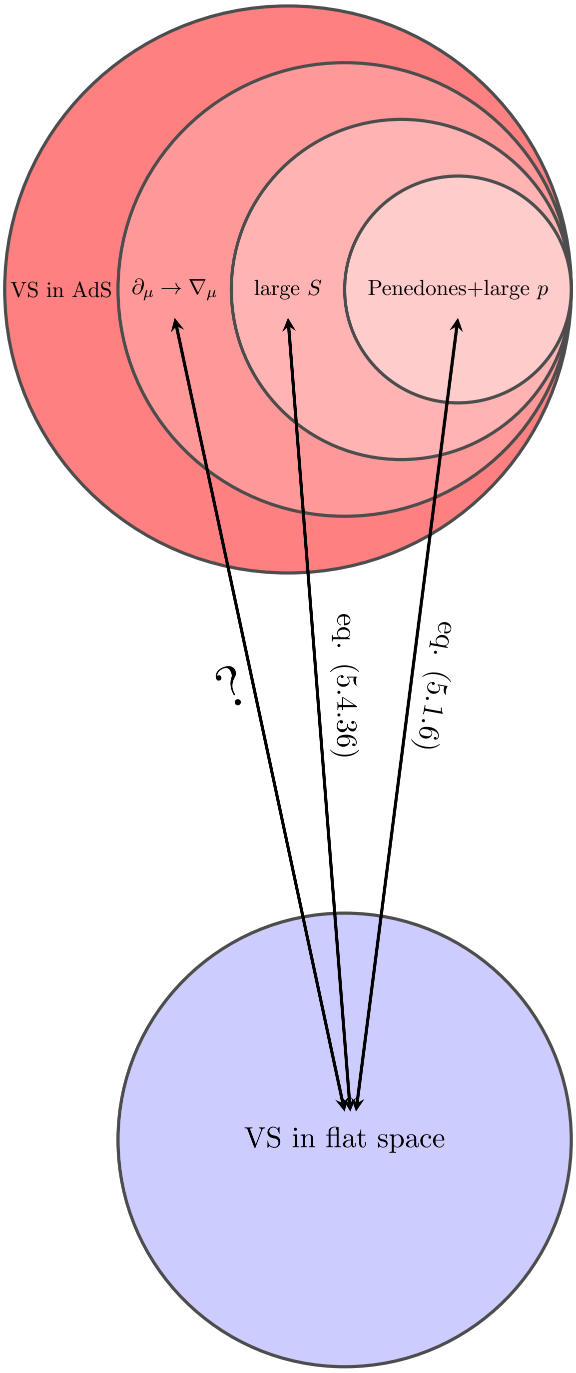


Figure 5.1: The relation between different flat space limits. Note that the flat space VS - represented by a blue circle - is of the same size as the $\partial_\mu \to \nabla_\mu$ circle, meaning that they should contain the same information. On the other hand, the full VS in AdS contains some information which is not inherited from flat space; in the bootstrap this manifests in the form of ambiguities, which can only by fixed with other methods, such as supersymmetric localisation.

Chapter 6

The double-trace spectrum in supergravity

As mentioned in previous chapters, our bootstrap algorithm exploits certain properties of the OPE and, in particular, of the spectrum of double-trace operators. In order to explain the method, we therefore need to recall a few properties of the spectrum. In this chapter we begin with the double-trace spectrum in the supergravity limit, first analysed in [33, 37]. In section 6.1 we derive the unmixing equations, valid at any order in α', and cast them in a form that allows to turn the unmixing problem in an eigenvalue problem [33, 37]. In section 6.2 we discuss the formula for the block coefficients of the unprotected part in disconnected free theory [31, 37], and rewrite it in a form which makes manifest the connection between different theories, as we will see in the second part of the thesis. In section 6.3 we show the solution of the "unmixing problem", i.e. the splitting of the dimensions of certain double-trace operators which are degenerate in free theory. Remarkably, the anomalous dimensions responsible for the splitting turn out to be very simple. We conclude the chapter by commenting on the relation between anomalous dimensions and the hidden conformal symmetry, which was, in fact, one of the first hints of the existence of the symmetry [31]. The discussion will also provide an heuristic argument for the assumption we are going to make on the string-corrected spectrum in the next chapter.

At leading order in the large N expansion only long two-particle multiplets receive an anomalous dimension in the interacting theory; these are precisely the operators responsible for the sequence of poles in s, t, u captured by $\Gamma_{\otimes}$. The corresponding primaries in the free theory have the schematic form

$$\mathcal{O}_{pq} = \mathcal{O}_p \partial^l \Box^{\frac{1}{2}(\tau-p-q)} \mathcal{O}_q \,, \qquad (p < q) \tag{6.0.1}$$

For given quantum numbers $\vec{\tau} = (\tau, l, [aba])$, many of these operators are degenerate.

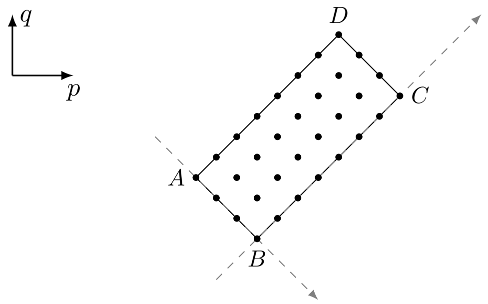
Their number is equal to the pairs (pq) filling in a rectangle [37]

$$R_{\vec{\tau}} := \left\{ (p,q) : \begin{array}{c} p = i+a+2+r \\ q = i+a+2+b-r \end{array} \quad \text{for} \quad \begin{array}{c} i = 0,\dots,(t-2) \\ r = 0,\dots,(\mu-1) \end{array} \right\} \tag{6.0.2}$$

where

$$t \equiv \frac{(\tau-b)}{2} - a \qquad ; \qquad \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a+l \text{ even}, \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a+l \text{ odd}. \end{cases} \tag{6.0.3}$$

This rectangle $R_{\vec{\tau}}$ consists of $d = \mu(t-1)$ allowed lattice points and, for reasons which will be clear when we discuss the unmixing, is depicted with 45° orientation in the (p,q) plane, as shown in the figure below.



$A = (a+2, a+b+2)$;
$B = (a+1+\mu, a+b+3-\mu)$;
$C = (a+\mu+t-1, a+b+1+t-\mu)$;
$D = (a+t, a+b+t)$;

(6.0.4)

Note that, for some values of the quantum numbers, the rectangle $R_{\vec{\tau}}$ can degenerate to a line. When $\mu = 1$ the rectangle collapses to a line with +45° orientation; when $\tau = 2a+b+4$, with $\mu > 1$, which corresponds to the first available twist for the rep $[aba]$, the rectangle also collapses to a line, this time with −45° orientation. Then, as the twist increases the rectangle opens up in the plane.

6.1 Unmixing equations in supergravity

Free theory long operators $\mathcal{O}_{pq}$ mix when interactions are turned on. Let us denote the true two-particle operators in the interacting theory, i.e. the eigenstates with well-defined scaling dimensions, by $\mathcal{K}_{pq}$. As shown in [33, 37], the mixing problem can be turned into an eigenvalue problem in the following way. First arrange a $(d \times d)$ matrix of correlators $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}|$ and $|\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$ with both (p_1p_2) and (p_3p_4) ranging over the same $R_{\vec{\tau}}$. Then, define the matrices $\mathbf{L}_{\vec{\tau}}$ from the long sector of disconnected free theory and $\mathbf{M}_{\vec{\tau}}(\alpha')$ from the leading $\log U$ discontinuity at tree level (including all α' corrections),

$$\begin{aligned} O(N^0): \quad & \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle\Big|_{\text{disc, long}} = \textstyle\sum_{\vec{\tau}} \mathbf{L}_{\vec{\tau}} \mathbb{L}_{\vec{\tau}}, \\ O(N^{-2}): \quad & \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle\Big|_{\log U} = \textstyle\sum_{\vec{\tau}} \mathbf{M}_{\vec{\tau}}(\alpha')\mathbb{L}_{\vec{\tau}} \end{aligned} \tag{6.1.5}$$

where $\mathbb{L}_{\vec{\tau}}$ are the long blocks defined in (2.3.21). Notice that $\mathbf{L}_{\vec{\tau}}$ is always diagonal, because disconnected free theory only exists when the charges of the external operators are pairwise equal. We stress once again that $\mathbf{L}_{\vec{\tau}}$ and $\mathbf{M}_{\vec{\tau}}$ are known coefficients once disconnected and the tree-level correlators are known. Now, it is not difficult to see that we can relate, via the OPE, $\mathbf{L}_{\vec{\tau}}, \mathbf{M}_{\vec{\tau}}$ to the CFT data, i.e. three-point functions and anomalous dimensions. In fact, the following equations hold:

$$\mathbf{C}_{\vec{\tau}}(\alpha')\mathbf{C}^T_{\vec{\tau}}(\alpha') = \mathbf{L}_{\vec{\tau}}, \qquad \mathbf{C}_{\vec{\tau}}(\alpha')\boldsymbol{\eta}_{\vec{\tau}}(\alpha')\mathbf{C}^T_{\vec{\tau}}(\alpha') = \mathbf{M}_{\vec{\tau}}(\alpha') \tag{6.1.6}$$

where $\mathbf{C}_{(pq),(\tilde{p}\tilde{q})}$ is a $(d\times d)$ matrix of three-point functions $\langle \mathcal{O}_p\mathcal{O}_q\mathcal{K}_{\tilde{p}\tilde{q}}\rangle$ and $\boldsymbol{\eta}$ is a diagonal matrix encoding the anomalous dimensions of the eigenstates $\mathcal{K}_{pq}$,

$$\Delta_{pq} = \tau + l + \frac{2}{N^2}\eta_{pq}(\alpha') + O\left(\frac{1}{N^4}\right) . \tag{6.1.7}$$

Note that if there was no mixing, there would have been one-to-one correspondence between coefficients and OPE data. Instead, because there is mixing, for any given set of quantum numbers $\vec{\tau}$ we have two sets of *matrix* equations[1].

Our notation for the α' expansion will be

$$\boldsymbol{\eta} = \boldsymbol{\eta}^{(0)} + \alpha'^3\boldsymbol{\eta}^{(3)} \ + \alpha'^5\boldsymbol{\eta}^{(5)} + \dots, \qquad \mathbf{C} = \mathbf{C}^{(0)} + \alpha'^3\mathbf{C}^{(3)} + \alpha'^5\mathbf{C}^{(5)} + \dots . \tag{6.1.8}$$

In this chapter we deal with the supergravity CFT data, for which the equations (6.1.6) reduce to [37],

$$\mathbf{c}^{(0)}_{\vec{\tau}}\mathbf{c}^{(0)T}_{\vec{\tau}} = \mathbf{I}_{\vec{\tau}}, \qquad \mathbf{c}^{(0)}_{\vec{\tau}}\boldsymbol{\eta}_{\vec{\tau}}{}^{(0)}\mathbf{c}^{(0)T}_{\vec{\tau}} = \mathbf{N}^{(0)}_{\vec{\tau}} \tag{6.1.9}$$

where we have defined

$$\mathbf{c}^{(0)}_{\vec{\tau}} = \mathbf{L}^{-\frac{1}{2}}_{\vec{\tau}}\mathbf{C}^{(0)}_{\vec{\tau}}, \qquad \mathbf{N}^{(0)}_{\vec{\tau}} = \mathbf{L}^{-\frac{1}{2}}_{\vec{\tau}}\mathbf{M}^{(0)}_{\vec{\tau}}\mathbf{L}^{-\frac{1}{2}}_{\vec{\tau}}. \tag{6.1.10}$$

In the case of $\mathbf{L}_{\vec{\tau}}$ a general formula can be found, and we will show it in the next section. A closed formula for $\mathbf{N}^{(0)}_{\vec{\tau}}$ appears instead to be more challenging. This is however not necessary: one can directly focus on the eigenvalues of the matrix $\mathbf{N}^{(0)}_{\vec{\tau}}$, which are the supergravity anomalous dimensions and compute them the for various quantum numbers. These can fitted quite easily and they have a remarkably simple form, as we will see.

[1] In the case of no mixing, the matrix is 1×1.

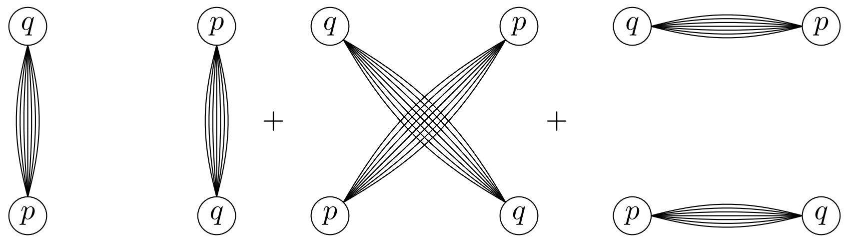


Figure 6.1: The three disconnected diagrams. Note that the identity (first diagram) and the t-channel (third diagram) only exist when $p = q$.

6.2 Long disconnected free theory in $\mathcal{N} = 4$

Let us start with SCPW decomposition of the long sector of disconnected free theory. It is easy to see that the only correlators with non-zero disconnected diagrams are those with pairwise equal charges[2] and their spacetime dependence can be computed by performing simple Wick contractions. We have

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle\Big|_{\text{disc}} = \frac{1}{pq}\left(\delta_{pq}\; g_{12}^{p}g_{34}^{p} + \underbrace{g_{13}^{p}g_{24}^{q}}_{\texttt{u-channel}} + \delta_{pq}\; \underbrace{g_{14}^{p}g_{23}^{p}}_{\texttt{t-channel}}\right) \tag{6.2.11}$$

Now, following [108], one first extracts the unprotected contribution and then decompose it in long superblocks. Obviously, no long operators are exchanged in the identity, therefore the identity does not contribute to the long decomposition. The block decomposition reads

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle\Big|_{\text{disc, long}} = \sum_{\vec{\tau}} L_{\vec{\tau}}\mathbb{L}_{\vec{\tau}}, \tag{6.2.12}$$

where $\mathbb{L}_{\vec{\tau}}$ are the long superblocks defined in (2.3.21). By performing the expansion for various cases, it is not difficult to find an explicit formula for the block coefficients. [31, 37]. In our notation it reads

$$L_{\vec{\tau}} = -\frac{1 + (-1)^{a+l}\delta_{pq}}{pq} A_h A_{\bar{h}} B_j B_{\bar{j}} \boldsymbol{\delta}\,. \tag{6.2.13}$$

For future convenience, we have defined the functions A and B,

$$\begin{aligned} A_h =& \frac{\Gamma(h + \frac{p-q}{2})\Gamma(h - \frac{p-q}{2})\Gamma(h + \frac{p+q}{2})}{\Gamma(2h-1)\Gamma(h - \frac{p+q}{2})}, \\ B_j =& \frac{\Gamma(2-2j)}{\Gamma(1-j+\frac{p-q}{2})\Gamma(1-j-\frac{p-q}{2})} \frac{1}{\Gamma(\frac{p+q}{2}+j-1)\Gamma(\frac{p+q}{2}-j)}\,, \end{aligned} \tag{6.2.14}$$

[2]We remind that with our conventions $p_4 - p_3 \geq p_2 - p_1 \geq 0$, therefore the only allowed correlators are those with $q > p$.

as well as $\boldsymbol{\delta}$:

$$\boldsymbol{\delta} = \frac{\delta^{(4)}_{h,\bar{h},j,\bar{j}} - \delta^{(4)}_{h,\bar{h},\bar{j},j}}{\delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j}}, \qquad \delta^{(4)}_{h,\bar{h},j,\bar{j}} \equiv \delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},\bar{j}}, \qquad \delta^{(2)}_{h,j} = (h-j)(h+j-1), \tag{6.2.15}$$

where we remind that

$$h = \frac{\tau}{2} + l + 2, \quad \bar{h} = \frac{\tau}{2} + 1, \qquad j = -\frac{b}{2} - a - 1, \qquad \bar{j} = -\frac{b}{2}. \tag{6.2.16}$$

To simplify the notation, from now on we will write $\delta^{(4)}_{h,\bar{h},j,\bar{j}} \equiv \delta^{(4)}$ and $\delta^{(4)}_{h,\bar{h},\bar{j},j} \equiv \bar{\delta}^{(4)}$. We will later see that these polynomials are very natural and will make an appearance also in other theories.

A fundamental observation came from [31]. There, it was noticed that $\delta^{(8)} \equiv \delta^{(4)}\bar{\delta}^{(4)}$ is the eigenvalue of a certain Casimir operator acting on the blocks. In fact, note that the functions $\mathcal{F}^{\pm}_h$ defined in (2.3.23) satisfy the following equality,

$$D^{\pm}_x \mathcal{F}^{\pm}_h(x) = h(h-1)\mathcal{F}^{\pm}_h(x) \tag{6.2.17}$$

where $D^{\pm}_x$ is [134]

$$D^{\pm}_x = x^2\partial_x(1-x)\partial_x \pm (p_{12}+p_{34})x^2\partial_x - p_{12}p_{34}\, x. \tag{6.2.18}$$

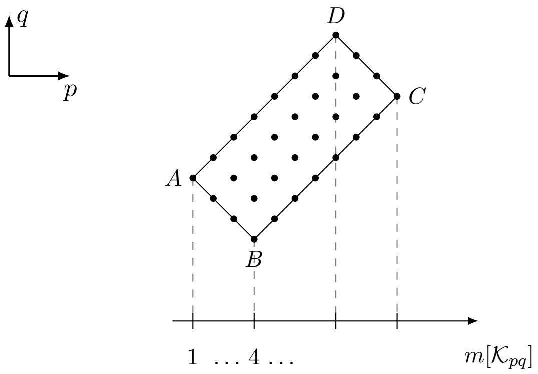
With the help of the above eigenvalue equation, it is immediate to check that

$$\begin{aligned} &\mathcal{D}_8\big((x-\bar{x})U^{1+\frac{p_{43}}{2}}(y-\bar{y})\tilde{U}^{1-\frac{p_{43}}{2}}\mathcal{G}_{\tau,l}\mathcal{H}_{b,a}\big) \\ &= \delta^{(4)}\bar{\delta}^{(4)}\ \big((x-\bar{x})U^{1+\frac{p_{43}}{2}}(y-\bar{y})\tilde{U}^{1-\frac{p_{43}}{2}}\mathcal{G}_{\tau,l}\mathcal{H}_{b,a}\big)\,, \end{aligned} \tag{6.2.19}$$

where $\mathcal{D}_8$ is given by

$$\mathcal{D}_8 \equiv \mathcal{D}_4\bar{\mathcal{D}}_4, \qquad \mathcal{D}_4 = (D^{+}_x - D^{-}_y)(D^{+}_{\bar{x}} - D^{-}_y), \qquad \bar{\mathcal{D}}_4 = (D^{+}_x - D^{-}_{\bar{y}})(D^{+}_{\bar{x}} - D^{-}_{\bar{y}}). \tag{6.2.20}$$

In the next section we will see that $\delta^{(8)}$ is nothing but the numerator of the anomalous dimensions. The presence of $\delta^{(8)}$ suggests that the hidden symmetry in free theory is realised not on the correlator of the $\mathcal{O}_p$ but on a correlator of superconformal descendants of $\mathcal{O}_p$, obtained by action of the Casimir. A more detailed discussion can be found in [31] and in [62] for the $AdS_2 \times S^2$ background, where the logic is exactly the same.



$$m_{\mathcal{K}_{pq}} = p - a - 1 \tag{6.3.23}$$

Figure 6.2: The level-splitting label m, counting the distance on the p axis from $a+2$. It will acquire a particular meaning in the next chapter when we will discuss the string-corrected spectrum.

6.3 Anomalous dimension and residual degeneracy

We will refrain from writing down explicit examples, which can be found in [33], and just present the general formulae[3]. As we mentioned, the idea is to compute the eigenvalues of the matrix $\mathbf{N}^{(0)}_{\vec{\tau}}$ for different values of the quantum numbers τ. In [37], the authors find that the anomalous dimensions are given by a very simple formula:

$$\eta^{(0)}_{pq} = -2 \frac{\delta^{(8)}}{(l - a + 2p - 2 - \frac{1+(-1)^{a+l}}{2})_6} \tag{6.3.21}$$

where we repeat here for convenience the definition of $\delta^{(8)}$,

$$\delta^{(8)} \equiv \delta^{(4)}_{h,\bar{h},j,\bar{j}} \delta^{(4)}_{h,\bar{h},\bar{j},j} = (h-j)(h-j+1)(\bar{h}-j)(\bar{h}+j-1)(\bar{h}-\bar{j})(\bar{h}+\bar{j}-1)(h-\bar{j})(h+\bar{j}-1). \tag{6.3.22}$$

The first thing to note is that the anomalous dimensions are all rationals. Let us stress again that these are eigenvalues of $d \times d$ matrices, therefore the result is highly non-trivial and strongly suggesting of the existence of an hidden structure, which is, in fact, the $10d$ conformal symmetry [31]. Moreover, they only depend on p, rather than the pair (pq), so all operators in $R_{\vec{\tau}}$ with the same value of p but varying q have the same anomalous dimension. In other words, operators on the same vertical line in the rectangle remain degenerate in supergravity. This brings to the conclusion that

the resolution of the operator mixing in tree-level supergravity is only partial!

To help visualising the partial degeneracy, let us introduce the level-splitting label m of an operator $\mathcal{K}_{pq}$ in $R_{\vec{\tau}}$, which measures the distance on the p axis from the value $p_A = a + 2$, as shown in figure 6.2. For each anomalous dimension labelled by m, the partial degeneracy is counted by the number of points on the q axis. The left most corner

[3]We will go through some examples for specific quantum numbers in the second part of the thesis for the other two theories we consider, where the logic is very similar.

of the rectangle $A = (p_A, q_A)$ corresponds to the most negative anomalous dimension. The partial degeneracy is bounded by the parameter μ introduced in (6.0.3), but notice that the level-splitting label m and the parameter μ are not the same.

Because of the residual degeneracy, the eigenvalue problem on $R_{\vec{\tau}} \otimes R_{\vec{\tau}}$ is well-posed, but the leading order three-point functions are not uniquely fixed, since these are determined by the columns of $\mathbf{c}^{(0)}_{\vec{\tau}}$ which, we recall, are the eigenvectors of $\mathbf{N}^{(0)}_{\vec{\tau}}$. If the anomalous dimension is degenerate, only a certain hyperplane is singled out, whose dimension is given by the partial degeneracy of $\eta(m)$. If the anomalous dimension is non degenerate, this dimension is unity and a unique vector is singled out. In any case we can fix a basis of eigenvectors and provide an orthogonal decomposition of $\mathbb{R}^d$,

$$\mathbb{V}_{\vec{\tau},1} \oplus \mathbb{V}_{\vec{\tau},2} \oplus \ldots \simeq \mathbb{R}^d \tag{6.3.24}$$

where $\mathbb{V}_{\vec{\tau},m}$ span the hyperplane labelled by $\eta^{(0)}(m)$. Obviously $d = \mu(t-1)$ counts the total number of operators, as explained around (6.0.3).

We end the chapter by commenting on the relation between the form of the CFT data and the hidden $10d$ conformal symmetry. In [31], it was recognised that the denominator of the anomalous dimensions resembles a quantity which can be computed in flat space. In particular, they noticed that the coefficients of the partial wave expansion[4] of the $2 \to 2$ scattering of axi-dilatons in IIB supergravity goes like $\sim 1/(l+1)_6$ where l is the flat $10d$ spin. This led the authors to conjecture that the quantity

$$l_{10} \equiv l + a + 2\,m - \tfrac{1+(-1)^{a+l}}{2} - 1 \tag{6.3.25}$$

present in the denominator of (6.3.21), behaves as an effective $10d$ spin. Here, $m = p - a - 1$ is the level-splitting label we introduced earlier. Then, they ascribed the reason of this similarity to the existence of a $10d$ conformal symmetry governing the supergravity dynamics. The similarity between these two quantities can be intuitively explained as follows: in the flat space limit, the correlator reproduces the flat amplitude and the conformal blocks reduce to Gegenbauer polynomials. As a consequence, the coefficient of the block expansion - which are the anomalous dimensions - should also be related to the coefficients of partial wave expansion.

In the next chapter we will see that l_{10} also plays a crucial role in string theory, since it dictates which operators are turned on at a given order.

[4]The partial waves can be written in terms of Gegenbauer polynomials, see e.g. [135].

Chapter 7

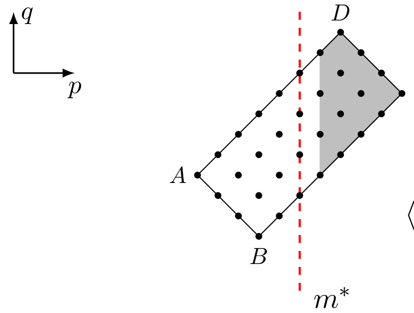
The double-trace spectrum in string theory

We have seen that the unmixing in supergravity does not completely lift the free-theory degeneracy, rather it leaves a residual degeneracy which is nicely depicted with a 45° rotated rectangle in the (p, q) plane. A natural question to ask is whether the residual degeneracy is lifted when adding α' corrections. In this chapter we show that an affirmative answer is indeed consistent with the (infinite) bootstrap equations given by the OPE. This, on one side, allows to fix the coefficients in the ansatz we described in chapter 5 and, on the other side completely fixes a certain class of anomalous dimensions. These are specified by a certain characteristic polynomial, which, as we will see, has a lot of very peculiar features.

The chapter is organised as follows. In section 7.1 we present the main conjecture; then, in section 7.2, we discuss the relevant α'-corrected OPE equations α'. In section 7.3 we present the simplest anomalous dimensions, i.e. those associated to the so-called rank=1 problem. This will be a useful starting point to discuss the more general eigenvalue problem, detailed in section 7.4. We will find that the solution, unlike supergravity, cannot be written in terms of radicals, therefore we will necessarily need to focus on the associated characteristic polynomial, rather than its zeros, i.e. the anomalous dimensions. Finally, in section 7.5, we list a number of properties enjoyed by the characteristic polynomial.

7.1 A bound on l_{10}

The general idea behind our bootstrap approach is that, by imposing a bound on the effective $10d$ spin, we can fix the coefficients of the ansatz, up to a certain number of ambiguities. As we anticipated already, this requirement will give an infinite number of



$$\eta_{(m)}\Big|_{\alpha'^{n+3}} = 0 \;\; \forall m > m^*$$

$$\langle \mathcal{O}_{p_i}\mathcal{O}_{p_j}|\mathcal{K}_{(m)}\rangle\Big|_{\alpha'^{n+3}} = 0 \;\; \forall m > m^*$$

Figure 7.1: The rectangle $R_{\vec{\tau}}$ of operators $\mathcal{K}_{pq}$ which are degenerate at leading order. The lifting in supergravity is only partial with the anomalous dimension depending only on the column. At the order α'^{n+3} the operators in the grey area turn out to be uncorrected.

constraints[1] for a finite number of free parameters, therefore the fact that the conjecture is not ruled out is quite reassuring. The main conjecture is that operators turned on at the order $\alpha'^{(n+3)}$ are those for which their labels satisfy the following inequality

$$l_{10} \leq n, \qquad @\,\alpha'^{n+3}, \qquad n \in 2\mathbb{N} \tag{7.1.1}$$

where, for convenience of the reader, we repeat here the definition of l_{10}:

$$l_{10} \equiv l + a + 2\,m - \tfrac{1+(-1)^{a+l}}{2} - 1. \tag{7.1.2}$$

We can also turn the inequality (7.1.2) into an inequality for level splitting label $m = p - a - 1$:

$$m \leq m^*, \qquad m^* = \frac{n - (a+l) - \frac{1-(-1)^{a+l}}{2}}{2} + 1, \qquad n \in \mathbb{N} \text{ even}. \tag{7.1.3}$$

The conjecture essentially states that

1. $\eta_{(m)}\Big|_{\alpha'^{n+3}} = 0, \qquad \forall m > m^*;$
2. $\langle \mathcal{O}_{p_i}\mathcal{O}_{p_j}|\mathcal{K}_{(m)}\rangle\Big|_{\alpha'^{n+3}} = 0, \qquad \forall m > m^*$

i.e. that there is a class of anomalous dimensions and three point functions which is zero. This can be translated into concrete equations for OPE coefficients, and we are going to do so in the next section.

The inequality on l_{10} can be somewhat justified following the logic of [31] for the supergravity case which we recalled at the end of the previous chapter. In fact, in flat space,

[1]This is because the number of equations vary with τ, b which are unbounded.

the partial wave decomposition of any given term in the α' expansion of the Virasoro-Shapiro amplitude (and of polynomials in general) is bounded in spin. In particular, a polynomial of degree n in the Mandelstam variable t contributes up to spin n in the partial wave decomposition. Now, since conformal blocks reduce to partial waves in the flat space limit, then (7.1.1) is, in a sense, the statement that l_{10} cannot exceed the highest possible spin in flat space, i.e. $l_{10}^{\text{flat}} = n$.

In figure (7.1) we describe the situation pictorially. The anomalous dimensions of the operators in the grey area are zero. Moreover we will see that the ambiguities we described in chapter 5, only affect the anomalous dimensions with labels satisfying $l_{10} < n$, while operators for which $l_{10} = n$ do not suffer ambiguities. This is in agreement with the intuitive picture offered by the effective field theory approach of [50]: there is a sub-amplitude in the $AdS_5 \times S^5$ Virasoro-Shapiro amplitude which directly descends from flat space. On the other hand, there are some other terms which instead capture curvature effects and are insensitive to the bootstrap. Note that we have specified $n \in 2\mathbb{N}$, the reason being that for odd n, a crossing symmetric polynomial goes like

$$s^n + t^n + u^n \sim st^{n-1}, \tag{7.1.4}$$

therefore for odd n the flat spin is $n-1$. A more precise argument is given in appendix C. Thus, for general n the inequality updates to

$$l_{10} \leq n - \frac{1-(-1)^n}{2} \qquad @\alpha'^{n+3}, \qquad n \in 2\mathbb{N}. \tag{7.1.5}$$

At this point, we should note that the spin appearing at odd n already appeared at the previous order. As a consequence of this, we expect that the spectrum of double-trace operators at odd n will contain ambiguities for *all* values of l_{10}.

The inequality above implies that we will get average CFT data stratified as

$$a + l = n, n-1, n-2, \dots 0 \tag{7.1.6}$$

For each value of $a + l$ we then read off the bound $m \leq m^*$ where

$a+l=$	n	$n-1$	$n-2$	$n-3$	$\dots$
$m^* =$	1	1	2	2	$\dots$

(7.1.7)

7.2 Unmixing equations at stringy level

Let us now translate the discussion we had so far into formulae by considering the OPE. The OPE will be referred to an orthonormal basis in supergravity, as in (6.3.24), for which we have the relation

$$span\big(\,\texttt{columns of }\mathbf{c}^{(0)}_{\vec{\tau}}\,\big) \simeq \Big[\mathbb{V}_{\vec{\tau},1}, \mathbb{V}_{\vec{\tau},2}, \ldots\Big] \tag{7.2.8}$$

The most general constraints from the OPE are discussed in appendix B. The ones we need in this section can be written as the following matrix equation,

$$\mathbb{V}^T_{\vec{\tau},m}\Big(\mathbf{c}^{(0)}\boldsymbol{\eta}^{(n+3)}\mathbf{c}^{(0)T} + \mathbf{D}^{(n+3)}\mathbf{N}^{(0)} \;+\; \mathbf{N}^{(0)}\mathbf{D}^{(n+3)T}\Big)\mathbb{V}_{\vec{\tau},m'} = \mathbb{V}^T_{\vec{\tau},m}\mathbf{N}^{(n+3)}\mathbb{V}_{\vec{\tau},m'}, \qquad \forall m \geq 1, \qquad \forall m' \geq m^*, \tag{7.2.9}$$

where

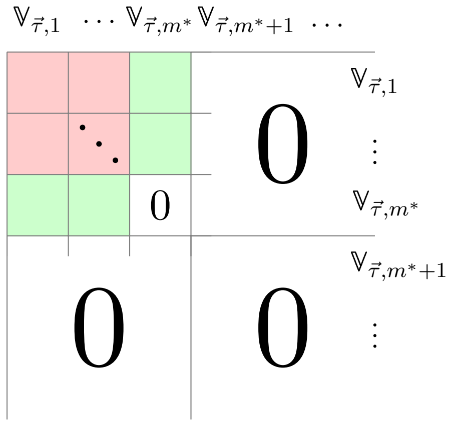
$$\mathbf{D}^{(k)} = \mathbf{L}^{-\frac{1}{2}}\left(\mathbf{C}^{(k)}\mathbf{C}^{(0)T}\right)\mathbf{L}^{-\frac{1}{2}} = \mathbf{L}^{-\frac{1}{2}}\,\mathbf{C}^{(k)}\,\mathbf{c}^{(0)T} \tag{7.2.10}$$

and

$$\mathbf{c}^{(0)}_{\vec{\tau}} = \mathbf{L}^{-\frac{1}{2}}_{\vec{\tau}}\mathbf{C}^{(0)}_{\vec{\tau}}, \qquad \mathbf{N}^{(n+3)}_{\vec{\tau}} = \mathbf{L}^{-\frac{1}{2}}_{\vec{\tau}}\mathbf{M}^{(n+3)}_{\vec{\tau}}\mathbf{L}^{-\frac{1}{2}}_{\vec{\tau}}. \tag{7.2.11}$$

The matrix of rotated three-point couplings $\mathbf{D}^{(n+3)}$ has a block structure depicted below,

$$\left(\mathbf{D}^{(n+3)}_{\vec{\tau}}\right)_{mm'} =$$



(7.2.12)

The block structure of $\mathbf{D}$ goes together with the obvious diagonal structure of the matrix of anomalous dimensions $\boldsymbol{\eta}$. Moreover

1. in the subspaces $\mathbb{V}_{\vec{\tau},m\geq 1} \otimes \mathbb{V}_{\vec{\tau},m>m^*}$ both $\mathbf{D}^{(n+3)}$ and $\boldsymbol{\eta}^{(n+3)}$ vanish under the assumption that the operators $\mathcal{K}_{(pq),\vec{\tau}}$ with $m > m^*$ are decoupled at that order;
2. in the subspace $\mathbb{V}_{\vec{\tau},m\leq m^*} \otimes \mathbb{V}_{\vec{\tau},m^*}$, $\mathbf{D}^{(n+3)}$ is anti-symmetric (green part).

We prove this latter statement in appendix B, where we also address the content of the red part, which is not important in the discussion.

Combining the information from the OPE on the l.h.s. of (7.2.9) with the r.h.s. determined from the superblock decomposition of $\mathcal{M}_n$, we find, in correspondence of the previous items,

1. the rank constraints,

$$\mathbb{V}^T_{\vec{\tau},m}\,\mathbf{N}^{(n+3)}_{\vec{\tau}}\,\mathbb{V}_{\vec{\tau},m'}=0,\qquad \forall m'>m^*,\quad \forall m\geq 1; \tag{7.2.13}$$

2. the level splitting problem, written as the eigenvalue problem of

$$\mathbf{E}^{(n+3)}_{\vec{\tau},m^*}=\left(\mathbf{v}^T_{\vec{\tau},I}\,\mathbf{N}^{(n+3)}_{\vec{\tau}}\,\mathbf{v}_{\vec{\tau},J}\right)_{I,J},\qquad \mathbf{v}_{\vec{\tau},I}\in\mathbb{V}_{\vec{\tau},m^*}. \tag{7.2.14}$$

The matrix $\mathbf{E}_{\vec{\tau},m^*}$ is a square matrix of dimension $dim(\mathbb{V}_{\vec{\tau},m^*})$.[2] As explained in more detail in appendix B, its eigenvalues are the new corrections to the tree level anomalous dimensions of the operators $\mathcal{K}_{(pq),\vec{\tau}}$, with level splitting label m^*,

$$\boldsymbol{\eta}_{\vec{\tau}}\Big|_{\mathbb{V}_{\vec{\tau},m^*}}=\texttt{eigenvalues}[\,\mathbf{E}_{m^*,\vec{\tau}}\,] \tag{7.2.15}$$

and will provide the lift of the partial degeneracy of tree level supergravity. This is always a zeta-odd valued function, i.e. $\sim\zeta_{n+3}\alpha'^{n+3}$. The eigenvectors of $\mathbf{E}$ single out particular directions on the hyperplane $\mathbb{V}_{\vec{\tau},m^*}$, and the full three point functions are given by

$$\mathbf{c}^{(0)}_{\vec{\tau}}\Big|_{\mathbb{V}_{\vec{\tau},m^*}}=\mathbb{V}_{\vec{\tau},m^*}\cdot\texttt{eigenvectors}[\,\mathbf{E}_{m^*,\vec{\tau}}\,] \tag{7.2.16}$$

where the `eigenvectors` are taken to be orthonormal. In this way the computation of the matrix $\mathbf{c}^{(0)}$ is complete, and the spectrum of operators at genus zero is fully unmixed.

Finally, let us point out a consequence of the relation $a+l\sim n-m^*$. The value of n here sets the order of the α' expansion, therefore, an operator with fixed level-splitting label m^*, in a given $SU(4)_{\rm R}$ channel $[aba]$, but varying spin l, receive for the first time a correction to its supergravity anomalous dimension at order $\sim\alpha'^{m^*+a+l}$. To study operators with large spin l in the same $R_{\vec{\tau}}$ we then have to look at high orders in perturbation theory. We therefore conclude that

the level splitting problem is *not* a problem of fixed order in the α' expansion.

Before entering the details of how we impose the constraints (7.2.13), let us discuss some simple cases to let the reader familiarise with our various statements.

7.2.1 Rank formula

It is nice to understand the rank constraints in (7.2.13) as a sort of exclusion plot, i.e. we know how many eigenvectors of tree level supergravity are in the kernel of $\mathbf{N}^{(n+3)}$,

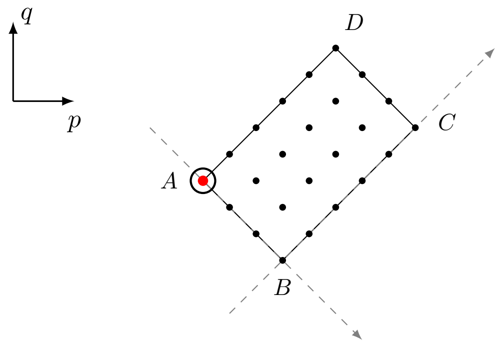
[2] Given m^* and $a+l$ we will avoid the label $(n+3)$ from now on, since n follows from (7.1.3).

at given order in α', therefore we know that the ones not in the kernel give us the rank. To fix ideas consider a *generic* rectangle $R_{\vec{r}}$. The table below shows for the first few orders in the α' expansion, and varying values of $a+l$, the expected rank of $\mathbf{N}^{(n+3)}$ and the position of the edge m^*,

α'^3	$a+l=0$	rank $=1$	$m^*=1$
α'^5	$a+l=2,1$	rank $=1$	$m^*=1$
	$a+l=0$	rank $=3=1+2$	$m^*=2$
α'^7	$a+l=4,3$	rank $=1$	$m^*=1$
	$a+l=2,1$	rank $=3=1+2$	$m^*=2$
	$a+l=0$	rank $=6=1+2+3$	$m^*=3$

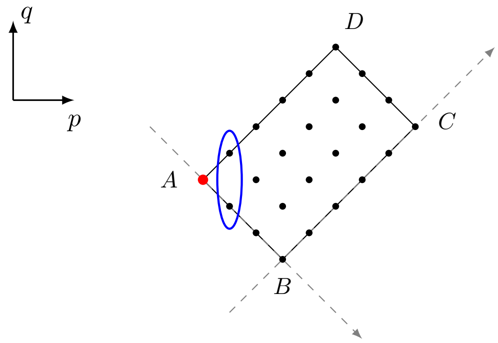
(7.2.17)

Let us start by discussing the rank$=1$ problem, that is when $a+l=n,n-1$. This is a special case since we are not actually unmixing any residual degeneracy. In fact, there is only one operator for any $R_{\vec{r}}$ corresponding to the operator labelled by the left most corner, which we defined by A, i.e. the one highlighted in red in figure (7.2.18).



(7.2.18)

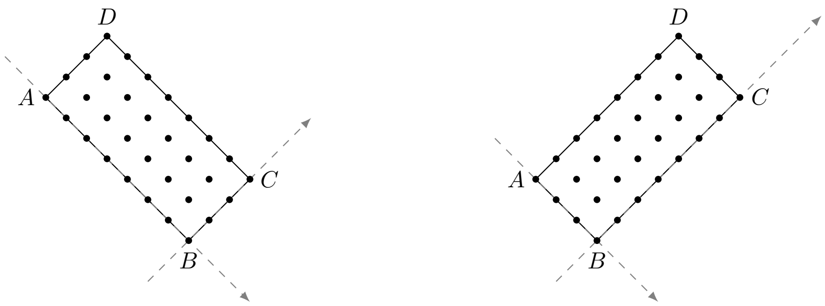
When $a+l=n-2,n-3$ operators with level splitting label $m=2$ are visible and $\mathbf{N}_{\vec{r}}$ has rank$=3$. The three operators in question are simply the ones labelled by A and the pair $A+(1,1)$ and $A+(1,-1)$. In figure (7.2.19), the pair is encircled in blue.



(7.2.19)

Note that A receives the first correction to its anomalous dimension from the rank $=1$ problem, therefore, the rank$=3$ problem will add a second correction to A, which we will not study. As we mentioned already, this correction depends on ambiguities which are not fixed within the bootstrap. Instead, operators labelled by $A+(1,1)$ and $A+(1,-1)$, receive a string correction for the first time and are fully fixed by the bootstrap. The reasoning for the rank$=6$ example at $a+l=n-4, n-5$ is very similar, and we conclude that this is responsible for lifting the SUGRA degeneracy among $A+(2,2)$, $A+(2,0)$ and $A+(2,-2)$, i.e. the ones with level splitting label $m=3$. Analogously to the previous case, the rank$=6$ problem will add another correction to A, $A+(1,1)$ and $A+(1,-1)$ but these will suffer ambiguities.

In our table above we assumed a large rectangle $R_{\vec{\tau}}$ to start with, therefore for small values of m we only explored operators labelled by points in between A and B. The level splitting for operators lying on the right of B takes place for values of n and $a+l$ as in (7.1.3), but one has to pay attention to the actual numerical value of the rank. Graphically there are two situations,



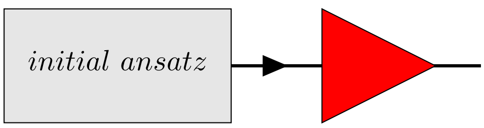
and the general formula is

$$rank\ \mathbf{N}_{\vec{\tau}}\Big|_{m^*} = \#\{(p,q) \text{ with } \ 2+a \le p \le a+m^*+1\} \tag{7.2.20}$$

where the r.h.s. is simply counting the points in $R_{\vec{\tau}}$ of the form (p,q) with $p \le a+m^*+1$.

7.2.2 Tailoring the bootstrap program

Our bootstrap algorithm begins by taking a crossing invariant ansatz for $\mathcal{M}_n$, i.e. the one we built in section 5, and computing the matrices $\mathbf{N}^{(n+3)}_{\vec{\tau}}$, as function of the parameters in the ansatz. Then, we impose the rank constraints



$$\mathbb{V}^T_{\vec{\tau},m}\,\mathbf{N}^{(n+3)}_{\vec{\tau}}\,\mathbb{V}_{\vec{\tau},m'} = 0, \qquad \forall m' > m^*, \quad \forall m \ge 1. \tag{7.2.21}$$

Notice that a necessary intermediate step here is to compute a basis of orthonormal eigenvectors of the supergravity matrix $\mathbf{N}^{(0)}_{\vec{\tau}}$, which we borrow from [37]. Equations

(7.2.21) are linear in the parameters $\vec{z}$ of the ansatz and can be rearranged as a linear system of the form $\mathcal{L} \cdot \vec{z} = \vec{f}$, where on the r.h.s. we have put the covariantised flat space contribution, which is known. We repeat our procedure for many rectangles $R_{\vec{\tau}}$ until the solution of the linear system saturates. A convenient way to do so is to consider first a selection of quantum numbers τ and $[aba]$ with $a + l = n$, then add the results from another selection of quantum numbers τ and $[aba]$ with $a + l = n - 1$, and keep going until $a + l = 0$. In principle we can take infinite values of τ and b. In practise we have taken finitely many for each $[aba]$, and we have seen the system saturates after a few values.

In the table below we summarise how many independent conditions are imposed from the rank constraints,

	initial ansatz	*rank constraints*
$\mathcal{M}_2$	6	4
$\mathcal{M}_3$	18	14
$\mathcal{M}_4$	44	34
$\mathcal{M}_5$	98	82

(7.2.22)

The number of initial parameters is the one counted by using the table in (5.2.21), where $\mathcal{M}_{n,n}$ is assumed. As n increases the number of new crossing invariants $\mathcal{H}_{n,(\ell,n-\ell)}$ grows as well, and moreover, more spin structures are turned on, as it is the case in flat space (see discussion in appendix C). The first case in which more than one spin structure is turned on in flat space is at α'^9, i.e. the spin six contribution $(\mathbf{s}^6 + \mathbf{t}^6 + \mathbf{u}^6)$ and the spin four $(\mathbf{s}^3 + \mathbf{t}^3 + \mathbf{u}^3)^2$ contribution. In this case there are two different problems,

	initial ansatz	*rank constraints*
$\mathcal{M}_{6,spin=6}$	208(−17)	176
$\mathcal{M}_{6,spin=4}$	208(−17)	176

(7.2.23)

where for computational simplicity we also fixed a particular gauge.[3] In both cases we have found the same number of constraints, as shown in the ancillary file.

Notice that since $\mathcal{V}_6$ is the completion of two spin structures, rather than one, the number of new crossing invariants in $\mathcal{V}_6$ essentially doubles compared to $\mathcal{V}_5$, see the counting in table (5.2.21).

In some cases we can exploit the OPE even further, especially if we can set something to

[3]With the word gauge here we mean the freedom to set some parameters to zero without changing the anomalous dimensions on the edge of the rectangle. In particular, we set $span(\mathcal{H}_{2,1,0})$ to zero and we only kept the terms with no Σ in $span(\mathcal{H}_3)$ and set to zero the others. In total we used a gauge with 17 parameters set to zero, as given in the ancillary file.

zero. For example, at $(\alpha')^{6,7}$, we can look at the subspace of operators with $a+l=0$ and $m=2$, consisting of two degenerate operators at tree level in SUGRA. These operators were at the edge of the α'^5 contribution, thus the amplitude $\mathcal{M}_2$ unmixes them and returns well defined three-point coupling in the $\mathbf{c}^{(0)}_{\vec{\tau}}$ matrix, let's say $[\mathbf{c}^{(0)}_{i=1,2}]_{\tau,0,[0b0],m=2}$ orthonormal. From the OPE follows that

$$0=\left[[\mathbf{c}^{(0)}_1]^T\Big(\mathbf{c}^{(0)}\boldsymbol{\eta}^{(k)}\mathbf{c}^{(0)T}+\mathbf{D}^{(k)}\mathbf{N}^{(0)}\ +\ \mathbf{N}^{(0)}\mathbf{D}^{(k)T}\Big)[\mathbf{c}^{(0)}_2]\right]_{\tau,0,[0b0],m=2}\,,\qquad k=6,7 \tag{7.2.24}$$

because both $\mathbf{D}^{(6)}$ and $\mathbf{D}^{(7)}$ are just anti-symmetric at this order. Therefore,

$$0=\left[[\mathbf{c}^{(0)}_1]^T\,\mathbf{N}^{(k)}\,[\mathbf{c}^{(0)}_2]\right]_{\tau,0,[0b0],m=2}\,,\qquad \forall\tau,b,\quad k=6,7. \tag{7.2.25}$$

This is a new condition in addition to the rank constraints, which we expect to saturate for all values of b and τ. For α'^6 we find that no independent constraint is added, while at α'^7 we find a new relation among free parameters. This is reasonable because the operators we are using here are strictly below the edge of α'^7, but not for α'^6. At order α'^8 we find two more constraints, and we checked instead that some of the new data from unmixing operators at the edge of α'^7 is automatically implemented, similarly to the behaviour between α'^6 and α'^5. We have attached the results in the ancillary file.

7.2.3 Ambiguity-free CFT data at the edge

We already anticipated that the CFT data at the edge are uniquely fixed. We conclude this section by commenting on this very important feature of this bootstrap program.

The reformulation of the rank constraints as a linear system $\mathcal{L}\cdot\vec{z}=\vec{f}$, where f comes solely from the flat space contribution, explains how $\mathcal{M}_{n,n}$ propagates into $\mathcal{M}_n$. If f is determined uniquely, the solution consists of a particular one, supplemented by $\texttt{ker}\mathcal{L}$. The particular solution, which depends both on $\mathcal{L}$ and f, is the most interesting part for the CFT data, because it uniquely identifies the level splitting matrix

$$\mathbf{E}^{(n+3)}_{\vec{\tau},m^*}=\Big(\mathbf{v}^T_{\vec{\tau},I}\,\mathbf{N}^{(n+3)}_{\vec{\tau}}\,\mathbf{v}_{\vec{\tau},J}\Big)_{I,J}\,,\qquad \mathbf{v}_{\vec{\tau},I}\in\mathbb{V}_{\vec{\tau},m^*}. \tag{7.2.26}$$

For concreteness, consider again the amplitude at α'^5,

$$\mathcal{M}_2=\ (\Sigma-1)_3\mathcal{M}_{2,0}+(\Sigma-1)_4z_{4,1}\left(\mathbf{s}\tilde{\mathbf{s}}+\mathbf{t}\tilde{\mathbf{t}}+\mathbf{u}\tilde{\mathbf{u}}\right)+(\Sigma-1)_5(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2), \tag{7.2.27}$$

$$\mathcal{M}_{2,0}=\ z_{3,1}\Sigma^2+z_{3,2}(k_s^2+k_t^2+k_u^2)+z_{3,3}\left(\tilde{\mathbf{s}}^2+\tilde{\mathbf{t}}^2+\tilde{\mathbf{u}}^2\right)+z_{3,4}\Sigma+z_{3,5}.$$

The rank constraints give four relations for six coefficients, namely

$$z_{4,1} = -5, \qquad z_{3,3} = 5, \qquad z_{3,2} - z_{3,1} = 11, \qquad z_{3,4} = 0. \tag{7.2.28}$$

It turns out that, even though $\mathbf{N}_{\vec{\tau}}$ still depends on free parameters, when we project on $\mathbb{V}_{\vec{\tau},m^*}$ they cancel out. It is simple to confirm with computer algebra that both $\mathbf{E}^{(5)}_{\vec{\tau},2}$ at $a+l=0$, and the CFT data at $m^*=1$ with $a+l=2,1$, do not depend on the two remaining free parameters, despite the fact that the amplitude at this point still does.

In general, $\vec{f}$ can be ambiguously or unambiguously determined depending on the $10d$ spin of the flat term. We have two cases:

- n even with leading $10d$ spin;
- n odd or more generally sub-leading $10d$ spins, that is any contribution in the flat space amplitude given by products of amplitudes at previous orders.

In the first case, $\vec{f}$ is unambiguous and the CFT data at the edge $m=m^*$ are completely determined; in the second case $\vec{f}$ does depend on ambiguities contributing to the same $10d$ spin, meaning that f is ambiguous. A nice example is α'^9 which contains both $\mathbf{s}^6+\mathbf{t}^6+\mathbf{u}^6$ and $(\mathbf{stu})^2$. The first one has $l_{10}=6$, and it is the first time that this value appears in the α' expansion, while the second one has $l_{10}=4$, so it will mix with $\mathbf{s}^4+\mathbf{t}^4+\mathbf{u}^4$ present in the ansatz. The CFT data at the edge of α'^9 is the one corresponding to $l_{10}=6$, for which the term $\vec{f}$ is uniquely determined by $\mathbf{s}^6+\mathbf{t}^6+\mathbf{u}^6$. In fact, we experimentally checked that if we introduce a parameter q to deform the spin six problem as $\mathbf{s}^6+\mathbf{t}^6+\mathbf{u}^6+q(\mathbf{stu})^2$, the CFT data at the edge is independent of q.

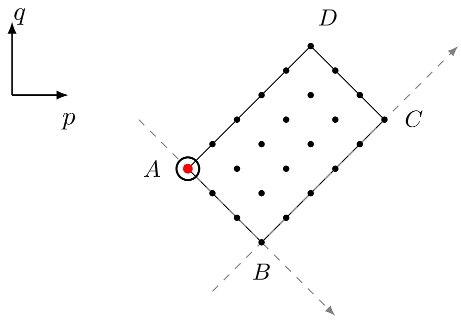
Summarising, we believe that the uniqueness of the CFT data at the edge at $m=m^*$ strongly suggests that there is a preferred sub-amplitude that corresponds to the covariantisation of the flat effective action [50].

7.3 All rank= 1 anomalous dimensions

Before exploring the general level-splitting problem, we study in isolation the case of rank= 1 at $a+l=n,n-1$, because it can be solved *independently* at all orders in α'. This will help us to explore various properties of the $m^*=1$ anomalous dimensions w.r.t. the quantum numbers $\vec{\tau}$ which we will then use to analyse the more general characteristic polynomial when $m^*>1$ in the next section.

We remind that when the matrix $\mathbf{N}^{(n+3)}_{\vec{\tau}}$ has rank= 1 only the operator on the left most corner of $R_{\vec{\tau}}$, encircled in red in the figure below, gets a correction to its CFT data, i.e.

$m = m^* = 1$.



(7.3.29)

In this case the only quantity to compute is the anomalous dimension, since the correction to the three-point function is a vanishing one-by-one anti-symmetric matrix. The first set of $m^* = 1$ anomalous dimensions are found for $a + l = n$ even, in the following channels

$$\alpha'^{n+3}, \quad n = 0, 2, 4 \dots, \qquad [aba], \qquad \begin{array}{lllll} a = & n, & n-1, & \dots \\ l = & 0, & 1, & \dots \end{array} \tag{7.3.30}$$

Now, we just look at the eigenvalues of the level splitting matrix $\mathbf{E}^{(n+3)}_{\vec{\tau},m^*}$ defined in (7.2.26) for various labels τ, l, a, b with $a + l = n$ and fit the data. The eigenvalues are all zero, except for one, that corresponds to the anomalous dimension in the left-most corner. Remarkably, the resulting anomalous dimensions have a very simple form, given by

$$\eta^*_{\vec{\tau}} = -2 \times \zeta_{n+3} \frac{n!(n+4)!}{(2n+8)!} \, \delta^{(8)} \left(\frac{\tau}{2} - \frac{b+2a+2}{2} \right)_{n+3} \left(\frac{\tau}{2} + \frac{b+2}{2} \right)_{n+3}. \tag{7.3.31}$$

The second set of $m^* = 1$ anomalous dimensions are found for $a + l = n - 1$ odd in the following channels,

$$\alpha'^{n+3}, \qquad n = 2, 4 \dots, \qquad [aba] \; ; \qquad \begin{array}{lllll} a = & n-1, & n-2, & \dots \\ l = & 0, & 1, & \dots \end{array} \tag{7.3.32}$$

In this case, we find that the anomalous dimensions fit with the following polynomial

$$\eta^*_{\vec{\tau}} = -\mathcal{F}_{\vec{\tau},n} \times \frac{\tau(\tau + 2l + 4) - b(b + 2a + 4)}{4} \tag{7.3.33}$$

where we defined

$$\mathcal{F}_{\vec{\tau},n} \equiv +2 \times \zeta_{n+3} \frac{n!(n+4)!}{(2n+8)!} \, \delta^{(8)} \left(\frac{\tau}{2} - \frac{b+2a+2}{2} \right)_{a+l+3} \left(\frac{\tau}{2} + \frac{b+2}{2} \right)_{a+l+3}. \tag{7.3.34}$$

We conclude the section with some observations which will be useful for the more general case $m^* \neq 1$:

- notice that for both $a+l = n$ and $a+l = n-1$, the total degree in twist of η^* is unchanged. The -1 lost in $\mathcal{F}_{\vec{\tau},n}$ in the odd case is regained by the $\frac{\tau(\tau+2l+4)-b(b+2a+4)}{4}$ contribution. This polynomial is more compactly $T-B$, where

$$T \equiv \tfrac{1}{4}\tau(\tau+2l+4), \qquad B \equiv \tfrac{1}{4}b(b+2a+4); \tag{7.3.35}$$

- the anomalous dimension are *odd* under the symmetry $T \leftrightarrow B$ and $l \leftrightarrow a$;

- both anomalous dimensions in (7.3.31) and (7.3.33) are negative definite for physical values of $\vec{\tau}$, and for given value of n can be written solely in terms of T and B, a and l;

- Upon factoring out $\mathcal{F}$, we find unity when $a+l=n$ even, and $T-B$ when $a+l=n-1$ odd. Notice that if we assume T is present, then we know that B is also present because the flat space amplitude cannot distinguish τ from b, and a from l, thus they have to appear on equal footing at leading order. This is equivalent to saying that the flat space Mellin amplitude only depends on one set of variables (namely, the S,T,U variables defined in (5.4.37)). We infer in this way that the flat space limit is implemented at the level of CFT data as the limit in which T and B scale in the same way and are large.

After this warm-up, we can now study the general splitting of degenerate long two-particle operators at tree level in supergravity.

7.4 Level splitting and the characteristic polynomial

Following the discussion for the case $m^*=1$ of the previous section, it will be convenient to define a rescaled anomalous dimension:

$$\eta^*_{\vec{\tau},m} \;=\; \mathcal{F}_{\vec{\tau},n}\,\tilde{\eta}_{\vec{\tau},m}. \tag{7.4.36}$$

The factor $\mathcal{F}$ is precisely the one in (7.3.34),

$$\mathcal{F}_{\vec{\tau},n} \equiv +2\times\zeta_{n+3}\frac{n!(n+4)!}{(2n+8)!}\,\delta^{(8)}_{[aba],\tau,l}\left(\frac{\tau}{2}-\frac{b+2a+2}{2}\right)_{a+l+3}\left(\frac{\tau}{2}+\frac{b+2}{2}\right)_{a+l+3}. \tag{7.4.37}$$

The information about the new anomalous dimensions is carried by the characteristic polynomial

$$\mathcal{P}^*_{\vec{\tau},m} \;=\; \frac{(-)^m}{(\mathcal{F}_{\vec{\tau},n})^m}\det\left[\mathbf{E}_{\vec{\tau},m}-\eta^*_{\vec{\tau},m}\,\mathbf{1}\right]. \tag{7.4.38}$$

The simplest observation we can make about η^* has to do with the flat space contribution in the capital variables S,T,U, which is blind to the level splitting. We can access this limit by taking the twist τ to be large in the anomalous dimensions, then we expect the polynomial to covariantise[4], and collapse in such a way that all roots are equal,

$$\mathcal{P}^*_{\vec{\tau},m} \underset{\tau\gg 1}{\longrightarrow} \Big(\tilde{\eta} + (T-B)^{n-a-l}\Big)^m + \ldots \tag{7.4.39}$$

with the variables T and B as in (7.3.35).

We know the exponent of the term $(T-B)$ after comparing $\tilde{\eta}$ with the anomalous dimension for the rank= 1 problem. This is simply $\mathcal{F}_{\vec{\tau},n}$ with $a+l=n$, i.e. the formula we gave in (7.3.31). When we increase the values of $m^* > 1$, equivalently we decrease the value of $a+l$ w.r.t. n, the mismatch in powers of T is precisely $n-a-l$. The flat space limit (7.4.39) tells us what is the maximum degree in T and B of the coefficients in $\tilde{\eta}$ of the characteristic polynomial

$$\begin{aligned} \mathcal{P}^*_{\vec{\tau},m} &= \tilde{\eta}^m + K_{m,1}(T,B,a,l)\,\tilde{\eta}^{m-1} + \ldots + K_{m,m}(T,B,a,l) \\ deg[K_{m,j}] &\leq j \times \Big(2m - 2 + \tfrac{1}{2}(1-(-1)^{a+l})\Big)\,. \end{aligned} \tag{7.4.40}$$

In the next section we are going to study the case $m^* = 2$, for which there are only two coefficients, namely $K_{2,j}$ with $j = 1, 2$. Since this case is associated to a degree 2 polynomial, we can actually focus on its roots, and look for some more properties. This will help us to deal with the more general $m^* \geq 3$ case where we can only investigate properties of the coefficients $K_j(T,B,a,l)$ w.r.t. the quantum numbers. We will do so in section 7.5.

7.4.1 $m^* = 2$ operators at all orders in α'

We now discuss the level splitting of $m^* = 2$ operators with $a+l = n-2$ even first, and then $a+l = n-3$ odd. Given the simplicity of the degree two characteristic polynomial in these cases, we will be able to include explicitly all orders in α'.

We start with $a+l = n-2$. By using our result at $\alpha'^{5,7,9}$ we gathered data for $a+l = 0, 2, 4$, respectively. Let us quote an example for concreteness,

$$\mathbf{E}_{\tau=12,l=0,[040]} = \begin{pmatrix} -\frac{8070480000}{7} & \frac{118800000\sqrt{187}}{7} \\ \frac{118800000\sqrt{187}}{7} & -\frac{8624880000}{7} \end{pmatrix}. \tag{7.4.41}$$

Even though the numbers look (very) nasty, we will see that the function is in fact very simple. The quantum numbers b and τ are arbitrary in principle, subject only to the bound $\tau \geq b + 2a + 4$, thus as done for the case $m^* = 1$, we first fitted the characteristic polynomial as functions of T and B, keeping $a+l$ fixed. Collecting all pairs (a,l) we then

[4]With covariantisation here we mean the observed property that the T dependence upgrades to a T, B dependence.

looked at the dependence on a and l. For $m^* = 2$ we had the bonus of looking directly to the roots, rather than the individual coefficients of the characteristic polynomial. This was fruitful because suggested the following representation of the characteristic polynomial,

$$\mathcal{P}^*_{\vec{\tau},2} = (\tilde{\eta} + r)^2 + (\tilde{\eta} + r)\gamma_{2,1} + \gamma_{2,0} \tag{7.4.42}$$

where

$$\gamma_{2,1} = -\frac{(n+2)(n+3)}{2n+5}\Big(B(2l+5) + (2a+5)T - (a+2)(l+2)\Big), \tag{7.4.43}$$

$$\gamma_{2,0} = +\frac{(n+2)^2(n+3)^2}{2n+5} B\,T \tag{7.4.44}$$

and the shift is

$$r = (T-B)^2 + B(2+l) + (2+a)T. \tag{7.4.45}$$

By construction, the square root responsible for splitting the anomalous dimensions does not depend on r, and is quite simple $\pm(\gamma^2_{2,1} - 4\gamma_{2,0})^{\frac{1}{2}}$.

Let us now switch to the explicit form,

$$\mathcal{P}^*_{\vec{\tau},2} = \tilde{\eta}^2 + K_{2,1}(T,B,a,l)\tilde{\eta} + K_{2,2}(T,B,a,l) \tag{7.4.46}$$

and look for additional properties. The first observation is that

$$K_{2,j}(T,B,a,l) = K_{2,j}(B,T,l,a), \qquad j = 1,2 \tag{7.4.47}$$

and in fact the rescaled anomalous dimension $\tilde{\eta}$ is even under the symmetry.

The second observation is about the covariantised flat space limit in T and B. This is manifest in the parametrisation (7.4.42), and to see it scale $\eta \to \epsilon^2\eta$ and $(B,T) \to \epsilon(B,T)$, and take the limit ϵ large. At leading order,

$$\mathcal{P}^*_{\vec{\tau},2}(\epsilon^2\tilde{\eta}, \epsilon B, \epsilon T)\Big|_{\epsilon^4} = (\tilde{\eta} + (T-B)^2)^2 \tag{7.4.48}$$

where the term $(T-B)$ comes just from the shift by r. This collapsed polynomial has indeed two equal roots, as we anticipated already in (7.4.39). A nice experiment is to go beyond the leading term, and see how the anomalous dimensions split, since we know they will split. The ϵ expansion reads,

$$\mathcal{P}^*_{\vec{\tau},2}(\epsilon^2\tilde{\eta}, \epsilon B, \epsilon T) = \epsilon^4\left[\tilde{\eta}^2_{flat} - \frac{1}{\epsilon}\tilde{\eta}_{flat}\left[T + B + \frac{n^2+3n+1}{2n+5}(T(2a+5) + (2l+5)B)\right] + \ldots\right] \tag{7.4.49}$$

where $\tilde{\eta}_{flat} = \tilde{\eta} + (T-B)^2$. Remarkably, keeping the first correction we find the solutions

$$O(\epsilon^{-1}), \qquad \tilde{\eta}_{flat} = 0 \qquad ; \qquad \tilde{\eta}_{flat} = \frac{1}{\epsilon}\Big[T + B + \dots\Big] \tag{7.4.50}$$

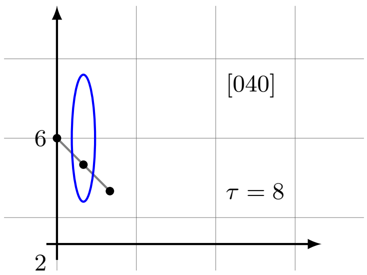
We learn from this formula that as we move away from flat space, the degeneracy is lifted sequentially, one of the two roots is still at the flat space locus, while the other is shifted.

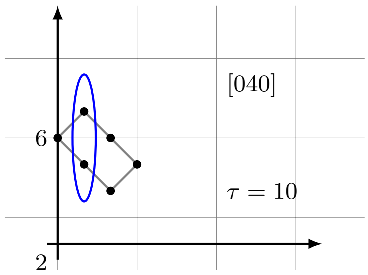
Our next observation has to do with a factorisation in the coefficient $K_{2,2}$, which is not manifest in (7.4.42), but it becomes apparent in (7.4.46) upon replacing $n = a + l + 2$. Very nicely we find

$$K_{2,2}(B,T,a,l) = \\ \left(\frac{\tau+b}{2}\right)\left(\frac{\tau+b}{2} + a + l + 4\right)\left(\frac{\tau-b}{2} - a - 2\right)\left(\frac{\tau-b}{2} + l + 2\right)\tilde{K}_{2,2}(T,B,a,l) \tag{7.4.51}$$

for a non factorisable $\tilde{K}_{2,2}$ such that $\deg[\tilde{K}_{2,2}] \leq 2$, as expected from (7.4.40).

This factorisation can be interpreted in the following way. Note that $K_{2,2}$ above vanishes precisely at $\tau = b + 2a + 4$, which is the minimum value of τ for a two-particle operator. Let us recall now that at the minimum twist, the rectangle collapses to a single line with $-45°$ orientation and there is no residual degeneracy in this case (see chapter 6). The partial degeneracy for $m^* = 2$ will start showing up at $\tau = b + 2a + 6$. For example,





(7.4.52)

When there is no degeneracy, the two particle operator with label $m^* = 2$ is already identified by the SUGRA eigenvalue problem, therefore the α' correction is linear and is obtained by the following direct computation,

$$\eta^*_{\vec{\tau}}\Big|_{\tau=b+2a+4} = \mathbb{V}^T_{\vec{\tau},2} \cdot \mathbf{N}^{(n+3)}_{\vec{\tau}} \cdot \mathbb{V}_{\vec{\tau},2}\Big|_{\tau=b+2a+4}, \tag{7.4.53}$$

where $\mathbb{V}_{b+2a+4,2}$ consists of a single eigenvector. Let us emphasise that there is *no* 2×2 level splitting matrix corresponding to this case. Remarkably what we find by looking at the characteristic polynomial, and forcing $\tau = b + 2a + 4$ is

$$\mathcal{P}^*_{\vec{\tau},2}\Big|_{\tau=b+2a+4} = \tilde{\eta}\,(\tilde{\eta} + \gamma_{2,1} + 2r)\Big|_{\tau=b+2a+4}. \tag{7.4.54}$$

Thus, one root of the polynomial goes to zero, and upon inspection the other root precisely coincides with the rescaled anomalous dimension from (7.4.53)! We interpret the above phenomenon as follows. Because the characteristic polynomial is analytic in the quantum numbers, we can think of its roots as the anomalous dimensions of two analytically continued operators. The reduction in (7.4.54) shows the decoupling of one of the two operators, when physically only one operator exists in the theory. A priori there would be no reason to expect the non zero root to correctly reproduce the rescaled anomalous dimension of the physical operator, since there is really no 2×2 level splitting matrix at $\tau = b + 2a + 4$. Quite surprisingly we find that it does, here and in all other examples that we will check.

The case $a + l = n - 3$, generalises in a simple way the previous case. We will focus mainly on the characteristic polynomial, which we can write as

$$\mathcal{P}^*_{\vec{\tau},2} = (\tilde{\eta} + r)^2 + (\tilde{\eta} + r)\gamma_{2,1} + \gamma_{2,0} \tag{7.4.55}$$

in terms of a new shift

$$r = (T - B)^3 + (T - B)(B(3l + 7) + (3a + 7)T) + (a - l)(B(l + 2) + (a + 2)T) \tag{7.4.56}$$

and new coefficients

$$\gamma_{2,1} = -\frac{(n+2)(n+3)}{2n+5}(T - B)\Big(B(2l + 7) + (2a + 7)T - 3al - 7(a + l) - 16\Big) \tag{7.4.57}$$

$$\gamma_{2,0} = +\frac{(n+2)^2(n+3)^2}{2n+5}(T - B)^2\Big(3BT - B(l + 2) - (a + 2)T\Big). \tag{7.4.58}$$

The shift by r makes manifest the flat space limit, which this times goes with

$$\mathcal{P}^*(\epsilon^3\tilde{\eta}, \epsilon T, \epsilon B)\Big|_{\epsilon^6} = \big(\tilde{\eta}_{flat}\big)^2 \qquad ; \qquad \tilde{\eta}_{flat} = \tilde{\eta} + (T - B)^3 \tag{7.4.59}$$

The power of $(T - B)$ is one more compared to $a + l = n - 2$ even. In general $T > B$ therefore there is no ambiguity with odd powers. This odd power remind us that in this case the rescaled anomalous dimension $\tilde{\eta}$ is odd under symmetry $T \leftrightarrow B$ and $a \leftrightarrow l$, which implies on the polynomial

$$K_{2,j}(T, B, a, l) = (-)^j K_{2,j}(B, T, l, a), \qquad j = 1, 2 \tag{7.4.60}$$

As in the previous case, the splitting of the anomalous dimensions away from $\tilde{\eta}_{flat} = 0$ is sequential, and the rank reduction at the minimum twist decouples one of the two analytically continued operators, and reproduces the anomalous dimension of the physical operator.

7.4.2 Unmixed three-point couplings

The three point couplings of the newly identified two-particle operators are given by the columns of the $\mathbf{c}^0_{\vec{\tau}}$ matrix, as explained around (7.2.16), namely

$$\mathbf{c}^{(0)}_{\vec{\tau}}\Big|_{\mathbb{V}_{\vec{\tau},2}} = \mathbb{V}_{\vec{\tau},2} \cdot \texttt{eigenvectors}[\,\mathbf{E}_{2,\vec{\tau}}\,] \tag{7.4.61}$$

where the **eigenvectors** are taken to be orthonormal. This formula simply means that the three point couplings are given by taking an orthonormal basis for $\mathbb{V}_{\vec{\tau},2}$, from the SUGRA eigenvalue problem, then solve the stringy eigenvalue problem in that basis, and use the stringy eigenvectors to fix the residual freedom on $\mathbb{V}_{\vec{\tau},2}$.[5]

The general form of the three-point couplings is

$$\texttt{cln}\Big(\mathbf{c}^{(0)}_{\vec{\tau}}\Big) = \begin{bmatrix} \boxed{\mathcal{T}_1} \\ \boxed{\mathcal{T}_2} \\ \vdots \\ \boxed{\mathcal{T}_\mu} \end{bmatrix} \qquad\qquad \mathcal{T}_{\beta,\vec{\tau}} = \texttt{Table}\Big[\ \ldots\,, \{i,1,t-1\}\Big] \tag{7.4.62}$$

We will now label the new three-point couplings at $m^* = 2$ with $\pm$ signs,

$$\mathcal{T}^{\pm}_{\beta,\vec{\tau}} = \sqrt{\mathcal{N}_{\vec{\tau},\beta} \times \frac{(\frac{\tau+b}{2}-\beta+2)_{\beta-2}(\frac{\tau-b}{2}+l+3)_{\beta-2}}{(\frac{\tau-b}{2}+\mu-2)_{\beta+2-\mu}(\frac{\tau+b}{2}+l+3-\beta)_{\beta+2-\mu}}} \times \widetilde{\mathcal{T}}^{\pm}_{\beta,\vec{\tau}} \tag{7.4.63}$$

$$\mathcal{N}_{\vec{\tau},\beta} = \frac{1}{(\tau-1)_{2l+7}}\frac{(\frac{\tau+b}{2}+a+l+5)_{-a-\mu}}{(\frac{\tau-b}{2}-a-2)_{+a+\mu}}\begin{cases} (\frac{\tau+b}{2})(\frac{\tau-b}{2}+l+2); & \beta=1 \\ 1 & 2\le\beta\le\mu-1 \\ (\frac{\tau-b}{2}+\mu-1)(\frac{\tau+b}{2}-\mu+l+3); & \beta=\mu \end{cases}$$

where

$$\widetilde{\mathcal{T}}^{\pm}_{\beta,\vec{\tau}} = \texttt{Table}\Bigg[\sigma_{\beta,i}\left[(\tfrac{\tau+b}{2}+a+i+2)_{l+1}(\tfrac{\tau-b}{2}-i-a)_{l+1}\left(\widetilde{\mathcal{P}}_{\beta,1}(T,I) \pm \frac{\widetilde{\mathcal{P}}_{\beta,2}(T,I)}{\sqrt{\gamma^2_{2,1}-4\gamma_{2,0}}}\right)\right]^{\frac{1}{2}}, \{i,1,t-1\}\Bigg] \tag{7.4.64}$$

with $\sigma^2 = 1$ and $\mathcal{P}_\beta$ polynomials in the variables T and $I \equiv i(i+b+2a+2)$, containing

[5] Differently from the characteristic polynomial, which is computable for any m^*, the computation of (7.4.61) requires knowledge of the roots. Thus, the three-point couplings will remain somewhat implicit/numerical in the general case $m^* \geq 3$.

non factorisable pieces in most of the cases.

The general form of the polynomials $\widetilde{\mathcal{P}}_{\beta,1}$ and $\widetilde{\mathcal{P}}_{\beta,2}$ is of course complicated. Ultimately, they come from combining two eigenvalue problems, because of the very definition of $\mathbf{c}^{(0)}$ in (7.4.61). For example, in the $SU(4)_{\rm R}$ channel [040] and $l=0$ we find

$$\widetilde{\mathcal{P}}_{\beta,1}=\begin{bmatrix} 440(-270(9+I)^2+3(9+I)(103+7I)T-2(89+9I)T^2+5T^3) \\ 88T(-27(505+I(130+9I))+12(609+I(158+11I))T-16(62+9I)T^2+40T^3) \\ 132(-72(5+I)^2+(5+I)(323+55I)T-6(53+9I)T^2+15T^3) \end{bmatrix}, \tag{7.4.65}$$

$$\widetilde{\mathcal{P}}_{\beta,2}=\begin{bmatrix} 1760(-9720(9+I)^2+54(9+I)(161+9I)T-3(3539+I(550+19I))T^2+2(427+45I)T^3-25T^4) \\ -352T(972(305+I(50+I))+27(-3379+I(-246+29I))T+12(2755+I(674+53I))T^2-16(292+45I)T^3+200T^4) \\ 528(2592(5+I)^2-36(5+I)(313+53I)T-(4147+I(1654+211I))T^2+6(247+45I)T^3-75T^4) \end{bmatrix}.$$

Rather than looking for a general formula, in the following it will be more illuminating to discuss features of the three-point couplings related to the flat space limit and the rank reduction, by making a parallel with the discussion about the characteristic polynomial.

In (7.4.65), the degree in T of $\widetilde{\mathcal{P}}_{\beta,2}$ is one power higher than $\widetilde{\mathcal{P}}_{\beta,1}$, but what enters the three-point couplings is the combination $\widetilde{\mathcal{P}}_{\beta,2}\times(\gamma_{2,1}^2-4\gamma_{2,0})^{\frac{1}{2}}$. The square root precisely lowers the degree by one in the regime of large T. In fact, [6]

$$\lim_{T\gg 1}\ \widetilde{\mathcal{P}}_{\beta,1}\to\begin{bmatrix} +2200T^3-(78320+7920I)T^2+O(T) \\ +3520T^4-(87296+12672I)T^3+O(T^2) \\ +1980T^3-(41976+7128I)T^2+O(T) \end{bmatrix}, \tag{7.4.66}$$

$$\lim_{T\gg 1}\ \frac{\widetilde{\mathcal{P}}_{\beta,2}}{4\sqrt{(36+5T)^2-288T}}\to\begin{bmatrix} -2200T^3+(78320+7920I)T^2+O(T) \\ -3520T^4+(87296+12672I)T^3+O(T^2) \\ -1980T^3+(41976+7128I)T^2+O(T) \end{bmatrix}.$$

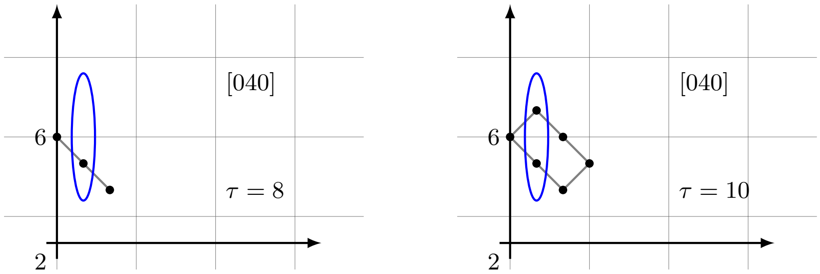
When we add/subtract (7.4.66) to build $\mathcal{T}^{\pm}$ we find that in the flat space limit $\mathcal{T}^{+}_{\beta,\vec{\tau}}$ vanishes at leading and subleading order, while $\mathcal{T}^{-}_{\beta,\vec{\tau}}$ survives.

Next we would like to see what happens when we go to the minimum twist.[7] Reconsider

[6] Notice that our normalisation $\mathcal{N}$ extracts a factor that we understood to be present always, for the first and the last block, i.e. $\beta=1,\mu$, otherwise all components of $\widetilde{\mathcal{P}}$ will scale the same.

[7] We thank Pedro Vieira for motivating this investigation.

our previous picture, which was suited for the example we are illustrating here.



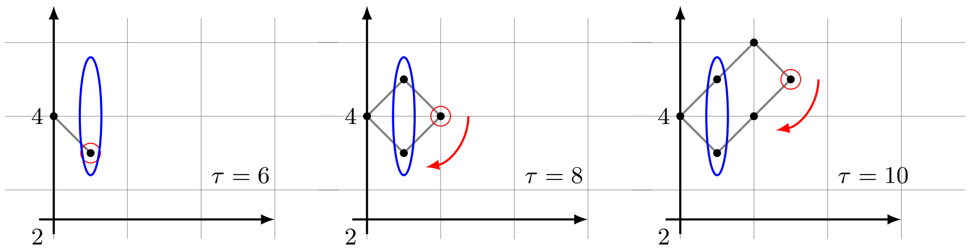
(7.4.67)

For the characteristic polynomial at the minimum twist we understood the appearance of a vanishing root (out of two) as a form of decoupling of one of the two analytically continued operators. For the three-point couplings we expect something different to happen. Continuing with our example (7.4.65), we find

$$\widetilde{\mathcal{P}}_{\beta,1}\bigg|_{\tau=8} = \begin{bmatrix} 102960(-1+i)^2(7+i)^2 \\ 6177600(-1+i)^2(7+i)^2 \\ 164736(-1+i)^2(7+i)^2 \end{bmatrix}, \quad \frac{\widetilde{\mathcal{P}}_{\beta,2}}{4\sqrt{(36+5T)^2-288T}}\bigg|_{\tau=8} = \begin{bmatrix} -102960(-1+i)^2(7+i)^2 \\ -6177600(-1+i)^2(7+i)^2 \\ -164736(-1+i)^2(7+i)^2 \end{bmatrix}. \tag{7.4.68}$$

Both polynomials vanish independently and we conclude that the two analytically continued operators decouple[8].

To understand the physics of the three-point decoupling, let us start again, this time from a simpler case, i.e. [020] even spin, $\mu = 2$. Varying the twist we would find the following picture



(7.4.69)

The red circle is pinning the operator which together with $\mathcal{K}_{m^*=1}$ is a singlet eigenvector of the SUGRA eigenvalue problem. This operator has its own analytic trajectory and the arrow indicates that the three-point coupling goes analytically in twist, from right to left. When we move from $\tau = 8$ to $\tau = 6$ the red colored operator takes the place of an $m^* = 2$ operator, but already in SUGRA it is not the analytic continuation of the pair of degenerate operators, which therefore has to decouple. Again, when $i = 1$ the three-point couplings vanish at the minimum twist. The example in (7.4.67) is more complicated, since it comes with $\mu = 3$ to begin with, but the fate of the pair at $m^* = 2$ pair at the minimum twist is the same.[9]

[8]Remember that only the $i = 1$ component exists at the minimum twist.

[9]In (7.4.67) the operator in the middle at $\tau = 8$ will come from the reduction of the $m^* = 3$ operators.

The three-point decoupling is thus stronger compared to the decoupling in the characteristic polynomial, which in this sense is quite smart because it retains information about all physical operators.

7.5 General properties of the characteristic polynomial

The characteristic polynomial $\mathcal{P}^*_{\vec{\tau},m}$ associated to the level splitting problem is a novel and very intriguing object. This is defined by

$$\mathcal{P}^*_{\vec{\tau},m} \;=\; \frac{(-)^m}{(\mathcal{F}_{\vec{\tau},n})^m}\det\left[\,\mathbf{E}_{\vec{\tau},m}-\eta^*_{\vec{\tau},m}\,\mathbf{1}\,\right], \qquad \eta^*_{\vec{\tau},m}=\mathcal{F}_{\vec{\tau},n}\,\tilde{\eta}_{\vec{\tau},m} \tag{7.5.70}$$

with the level splitting matrix defined in (7.2.14) and the normalisation $\mathcal{F}$ introduced in (7.3.34). This object nicely packages the CFT data from the $AdS_5\times S^5$ VS amplitude which lifts the partial degeneracy of the SUGRA anomalous dimensions.

Analyticity of $\mathcal{P}^*_{\vec{\tau},m}$ w.r.t. T,B,a,l might be obvious for $m^*=2$, since the level splitting matrix is just 2×2. However, this is not so intuitive in more complicated cases as

$$\frac{\mathbf{E}_{\vec{\tau}}}{\mathcal{F}_{\vec{\tau},6}}\Bigg|_{\substack{\tau=14\\ l=2,[040]}} = \begin{pmatrix} -\frac{6945359904}{499} & \frac{48965850432}{499}\sqrt{\frac{69}{41735}} & -1524096\sqrt{\frac{690690}{4165153}} \\ \frac{48965850432}{499}\sqrt{\frac{69}{41735}} & -\frac{337620067080624}{20825765} & \frac{33255693072}{8347}\sqrt{\frac{2002}{499}} \\ -1524096\sqrt{\frac{690690}{4165153}} & \frac{33255693072}{8347}\sqrt{\frac{2002}{499}} & -\frac{183139846560}{8347} \end{pmatrix} \tag{7.5.71}$$

and the characteristic polynomial will certainly not be analytic if the square roots remain in the final result. For general m^* there is a short computation we can do to actually see what determines the analytic properties of $\mathcal{P}^*$ and it uses the known formula,[10]

$$K_j=\frac{(-)^j}{j!}\det\begin{bmatrix} \mathtt{tr}\mathbf{E} & j-1 & 0 & \dots & \dots \\ \mathtt{tr}\mathbf{E}^2 & \mathtt{tr}\mathbf{E} & j-2 & 0 & \dots \\ \vdots & & & & \\ \vdots & & & & \vdots \\ & & & & 1 \\ \mathtt{tr}\mathbf{E}^j & \mathtt{tr}\mathbf{E}^{j-1} & \dots & \dots & \mathtt{tr}\mathbf{E} \end{bmatrix}, \qquad \mathcal{P}^*_{\vec{\tau},m}=\tilde{\eta}^{\,m}+\sum_{j=1}^{m}K_{m,j}\,\tilde{\eta}^{m-j}. \tag{7.5.72}$$

The analytic properties of $K_{m,j}(T,B,a,l)$ then follow from those of $\mathtt{tr}\,\mathbf{E}^k$. Let us consider $k=1$, since the general case will be analogous. From the definition of the level splitting matrix in (7.2.14), we find

$$\mathtt{tr}\,\mathbf{E}_{\vec{\tau},m}=\mathtt{tr}\left[\mathbf{M}_{\vec{\tau}}\mathbf{L}_{\vec{\tau}}^{-\frac{1}{2}}\mathbf{P}_{\vec{\tau},m}\mathbf{L}_{\vec{\tau}}^{-\frac{1}{2}}\right], \qquad \mathbf{P}_{\vec{\tau},m}=\left(\sum_{I=1}^{m}\mathbf{v}_I\mathbf{v}_I^T\right)_{1\le i,j\le\mu(t-1)} \tag{7.5.73}$$

[10] For example, see `https://en.wikipedia.org/wiki/Characteristic_polynomial`.

where $\mathbf{P}_m$ is the projector onto the hyperplane $\mathbb{V}_{\vec{\tau},m}$ spanned by the vectors $\mathbf{v}_I$. Analyticity of $\mathtt{tr}\,\mathbf{E}_{\vec{\tau},m}$ will hold if both $\mathbf{M}_{\vec{\tau}}$ and the combination involving $\mathbf{P}_{\vec{\tau},m}$ are analytic.[11] By definition $\mathbf{M}_{\vec{\tau}}$ collects the superblock decomposition of the VS amplitude on $R_{\vec{\tau}} \otimes R_{\vec{\tau}}$, thus is analytic in $\vec{\tau}$ when the superblock decomposition is analytic. Now, notice that the combination involving $\mathbf{P}_{\vec{\tau},m}$ is also analytic if the 'square' of three-point couplings is. Indeed we can rewrite it as

$$\left(\mathbf{L}_{\vec{\tau}}^{-\frac{1}{2}}\mathbf{P}_{\vec{\tau},m}\mathbf{L}_{\vec{\tau}}^{-\frac{1}{2}}\right)_{ij} = \left(\mathbf{L}_{\vec{\tau}}^{-1}\right)_{ii}\left[\sum_{I=1}^{m}\left(\mathbf{C}_{\vec{\tau}}^{(0)}\right)_{iI}\left(\mathbf{C}_{\vec{\tau}}^{(0)T}\right)_{Ij}\right]\left(\mathbf{L}_{\vec{\tau}}^{-1}\right)_{jj} \tag{7.5.74}$$

since $span\big(\mathbf{v}_I\big) \simeq span\Big(\big(\mathbf{L}_{\vec{\tau}}^{-\frac{1}{2}}\mathbf{C}_{\vec{\tau}}^{(0)}\big)_{iI}\Big)$, up to unitary transformations on the hyperplane.

The domain of definition of $\mathcal{P}^*_m(T,B,a,l)$ is the physical domain of existence of the level splitting matrix $\mathbf{E}_{m,\vec{\tau}}$,

$$\tau \geq b+2a+4+2(m-1), \qquad \begin{cases} b \geq 2m-2 & \text{if} \quad a+l \ \textit{even} \\ b \geq 2m-1 & \text{if} \quad a+l \ \textit{odd} \end{cases} \tag{7.5.75}$$

In relation to this, analyticity in $\vec{\tau}$ is now quite important because allows us to think about the roots of the characteristic polynomial as the stringy anomalous dimensions of analytically continued two-particle operators, outside the physical domain of definition. In this sense our experiments on the $m^* = 2$ problem had two amazing outcomes. Firstly, we learned that the new anomalous dimensions start splitting sequentially as we move away from the flat space limit. Secondly, we learned that as the space of physical operators reduces, the characteristic polynomial reduces as well, factorising a zero root each time. Already for $m^* = 2$ we saw that the non vanishing root carries the correct information about the physical spectrum of operators, which is not an obvious feature. We will refer to this process as 'rank reduction'. This phenomenon is quite beautiful and yet to be fully understood.

We will now generalise both the sequential splitting and the rank reduction for arbitrary $m \geq 2$, and we will demonstrate that they hold for the case $m^* = 3, 4$, i.e. the first cases for which the roots of $\mathcal{P}^*$ are not explicit. It will be convenient to use the notation

$$K_{m,j} = \sum_{0\leq x,y\leq deg} T^x\left(\mathbf{K}_{m,j}(a,l)\right)_{xy} B^y \tag{7.5.76}$$

where $\mathbf{K}_{m,j}$ is a $deg \times deg$ matrix. Recall $deg[K_{m,j}] \leq j \times \big(2m-2+\frac{1}{2}(1-(-1)^{a+l})\big)$.

We computed $\mathcal{P}^*_{m=3,4}$ as function of T and B without imposing any constraint to start with. The results are attached in an ancillary file. Then, we checked that both the sequential splitting and the rank reduction hold. Thus, we repeated the computation

[11]The difference w.r.t. the SUGRA eigenvalue problem is the projector $\mathbf{P}_{\vec{\tau},m}$, i.e. in SUGRA we would find the resolution of the identity, rather than the sum from 1 to m. In fact, the SUGRA anomalous dimensions at tree level are also the eigenvalues of $\mathbf{ML}^{-1}$, as shown in [38].

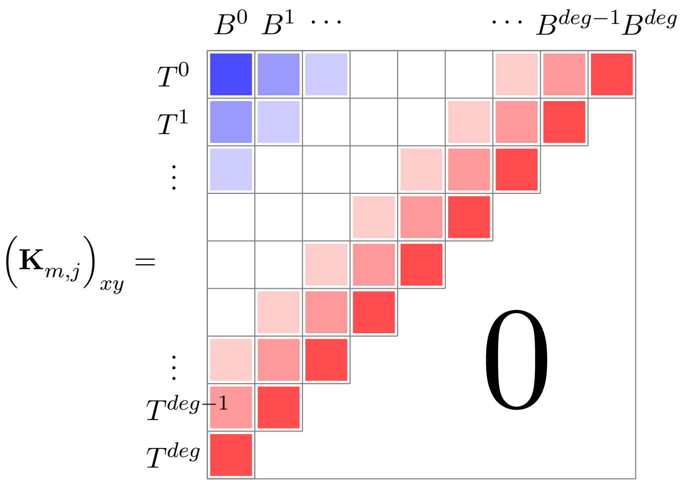


Figure 7.2: The different constraints on the coefficients of the characteristic polynomial.

the other way round. What happens on $\mathbf{K}_{m,j}$ is quite instructive, and is depicted in figure 7.2. The flat space limit determines the entries on the diagonal. Then, constraints from the sequential splitting impose relations spreading on the other diagonals in red. Constraints from rank reduction instead impose relations spreading from the left top corner in blue. The rest of $\mathbf{K}_{m,j}$ can only be determined by looking at the characteristic polynomial from the actual computation of the VS amplitude.

We find that the symmetry $T \leftrightarrow B$ and $a \leftrightarrow l$ holds beyond the flat space approximation, and in fact, it relates $\mathbf{K}$ to its transposed with swapped parameters,

$$\mathbf{K}_{m,j}(a,l) \quad = \quad \Big[\mathbf{K}_{m,j}(l,a)\Big]^{\mathrm{T}} \qquad a+l \text{ even}, \tag{7.5.77}$$

$$\mathbf{K}_{m,j}(a,l) \quad = \quad \Big[(-)^j\mathbf{K}_{m,j}(l,a)\Big]^{\mathrm{T}} \qquad a+l \text{ odd}. \tag{7.5.78}$$

Considering that the level splitting problem is uniquely determined within our bootstrap program, and assuming that we have been able to isolate a sub-amplitude inside the full VS amplitude in $AdS_5 \times S^5$, responsible just for the level splitting problem, then we infer from (7.5.77) that this subamplitude will have a non trivial duality in its Mellin representation reflecting the symmetry above. Moreover this duality will be *non perturbative*, since the level splitting problem is *not* a problem at fixed order in the α' expansion. In fact, for fixed α', a and l are fixed, therefore the duality, which exchanges a with l, becomes invisible.

7.5.1 Sequential splitting away from flat space

The flat space limit formalises as follows

$$\mathcal{P}^*_{\vec{\tau},m}(\epsilon^{n-a-l}\tilde{\eta}, \epsilon T, \epsilon B)\Big|_{\epsilon^{m(n-a-l)}} = \Big(\tilde{\eta} + (T-B)^{n-a-l}\Big)^m . \tag{7.5.79}$$

Then, the expansion away from the flat space limit is an expansion in the shifted anomalous dimension

$$\eta_{flat} = \tilde{\eta} + (T-B)^{n-a-l}. \tag{7.5.80}$$

The sequential splitting is now the statement that the roots of the characteristic polynomial move away from the degenerate locus $\eta_{flat} = 0$ one by one, sequentially for each extra term we keep in the ϵ expansion. This means that $\mathcal{P}^*_{\vec{\tau},m}$ is such that

$$\mathcal{P}^*_{\vec{\tau},m}(\epsilon^{n-a-l}\tilde{\eta}, \epsilon T, \epsilon B) = \epsilon^{m(n-a-l)} \Bigg[(\eta_{flat})^m + \frac{1}{\epsilon}(\eta_{flat})^{m-1} C_1(\tilde{\eta}, B, T) + \frac{1}{\epsilon^2}(\eta_{flat})^{m-2} C_2(\tilde{\eta}, B, T) + \dots \Bigg] \tag{7.5.81}$$

with generic C_i. The dependence on a and l is understood.

If we consider an ansatz for the coefficients $K_{m,j}$, we know the degree w.r.t. to B and T, given in (7.5.76), and then we know the diagonal entries of the $\mathbf{K}_{m,j\geq 1}$, since these are determined by the flat space limit. The flat space limit is a universal constraint for any $\mathbf{K}_{m,j\geq 1}$, but is the only constraint for $\mathbf{K}_{m,1}$. The sequential splitting takes $\mathbf{K}_{m,1}$ as an input and moves forward. At the first step we find that the diagonal of $\mathbf{K}_{m,2}$, which is next-to-the-flat space limit, is determined by $\mathbf{K}_{m,1}$. Then, we find that the next-to- and next-to-next-to-the-flat space diagonals of $\mathbf{K}_{m,3}$ depend on $\mathbf{K}_{m,1}$, and $\mathbf{K}_{m,j=1,2}$, respectively, and so on so forth. The flow according to which the constraints move sequentially away from flat space, as we look to coefficients $K_{j>1}$, is represented by the shadowing in red in figure 7.2.

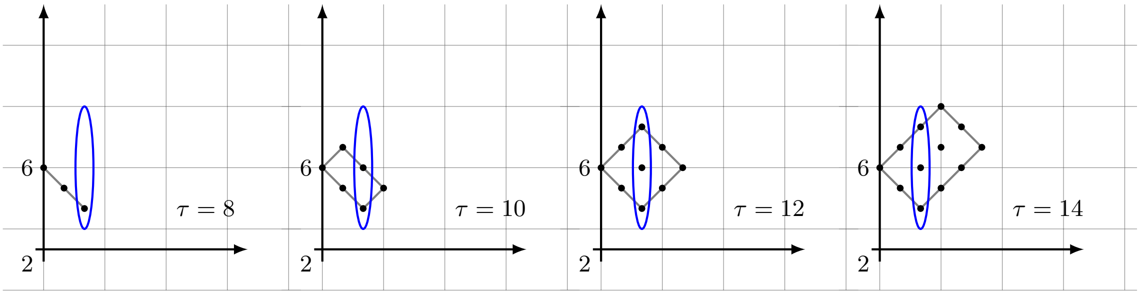
7.5.2 Rank reduction and multiple zeros

The rank reduction is better phrased using the concept of 'filtration'.[12] Consider a rectangle $R_{\tau,l,[aba]}$ for fixed values of l and $[aba]$ and varying twist. The minimum available twist is $\tau_{min} = b + 2a + 4$, but otherwise τ can grow unbounded. Then, the sequence

$$R_{b+2a+4,l,[aba]} \subset R_{b+2a+6,l,[aba]} \subset R_{b+2a+8,l,[aba]} \subset \dots \tag{7.5.82}$$

is a filtration. Graphically,

[12] Filtration is the name for a sequence of sets $\{S_i\}_{i\in\mathbb{N}}$ labelled by an integer, such that $S_i \subset S_{i+1}$.



Consider now a value of the level splitting label in the filtration, say m. The figure above has $m = 3$. By varying the twist, the number of physical operators with that m varies: it goes from m operators in the domain (7.5.75) for generic twist, down to a single physical operator at the minimum twist $\tau = b + 2a + 4$. In particular, we are always outside the domain of definition of the characteristic polynomial as long as $b + 2a + 4 \leq \tau < b + 2a + 4 + 2(m-1)$.

Following on what happens at $m^* = 2$, we should find that as decrease the twist below the domain of definition the coefficients of the characteristic polynomial vanish in such a way to factor a zero root each time. The pattern is[13]

$$\begin{array}{lcl} K_{m,m} \quad = 0 & @ & t = 2, \ldots \ldots \ldots m \\ K_{m,m-1} = 0 & @ & t = 2, \ldots \quad m-1 \\ K_{m,m-2} = 0 & @ & t = 2, \ldots m-2 \\ \quad \vdots & & \quad \vdots \\ \quad K_{m,2} = 0 & @ & t = 2 \end{array} \tag{7.5.83}$$

which is solved by

$$K_{m,j}(T,B,a,l) = \mathtt{R}_{j,\vec{\tau}} \times \tilde{K}_{m,j}(T,B,a,l) \tag{7.5.84}$$

where

$$\mathtt{R}_{j,\vec{\tau}} \equiv (\tfrac{\tau+b}{2} - j + 2)_{j-1}(\tfrac{\tau+b}{2} + a + l + 4)_{j-1}(\tfrac{\tau-b}{2} - a - j)_{j-1}(\tfrac{\tau-b}{2} + l + 2)_{j-1} \tag{7.5.85}$$

and $\tilde{K}$ is a polynomial of reduced degree. The prefactor is the unique polynomial in B and T of minimal degree which vanishes at the locus (7.5.83). In practise, the characteristic polynomial $\mathcal{P}^*_{\vec{\tau},m}$ always has m roots, and since it is analytic we think of these roots as describing the anomalous dimensions of m analytically continued operators. But as the rank of $\mathbf{N}_{\vec{\tau}}$ reduces, the number of physical operators changes. This is the picture given by the filtration. By experiment, analytically continued operators which do not correspond to physical operators localises on the vanishing roots.[14] Considering that the bulk interpretation of the level splitting is the formation of energetically favourable bound states, exchanged in the S-matrix, we infer the bound $\tilde{\eta} \leq 0$. This would explain

[13] We remind that $t = \frac{\tau-b}{2} - a$.

[14] This actually suggests that an alternative definition of this characteristic polynomial might exists such that it always lives on the space of $m \times m$ matrices.

the degeneration of the roots onto $\eta = 0$, that we see experimentally.

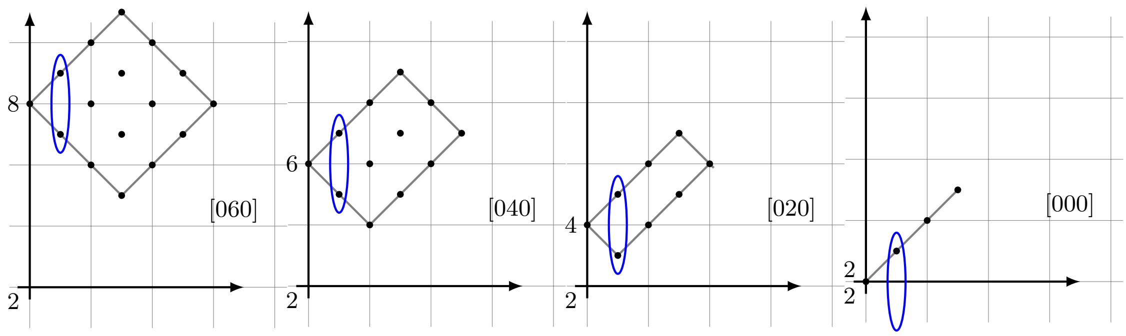
We have attached in an ancillary file the characteristic polynomials for $m^* = 3$ and $a + l = 0, 1, 2$, and the characteristic polynomial for $m^* = 4$ and $a + l = 0$. Looking at the results we can see that there are in fact multiple factorisations compared to the ones justified by the rank reduction, summarised by equation (7.5.84). In practise, we find the pattern

$$\begin{aligned} &K_{m=2,2} = \mathtt{R}_{2,\vec{\tau}} \times \tilde{K}_{2,2} \\ &K_{m=3,2} = \mathtt{R}_{2,\vec{\tau}} \times \tilde{K}_{3,2} \quad ; \quad K_{m=3,3} = \mathtt{R}_{3,\vec{\tau}}\mathtt{R}_{2,\vec{\tau}} \times \tilde{K}_{3,3} \\ &K_{m=4,2} = \mathtt{R}_{2,\vec{\tau}} \times \tilde{K}_{4,2} \quad ; \quad K_{m=4,3} = \mathtt{R}_{3,\vec{\tau}}\mathtt{R}_{2,\vec{\tau}} \times \tilde{K}_{4,3} \quad ; \quad K_{m=4,4} = \mathtt{R}_{4,\vec{\tau}}\mathtt{R}_{3,\vec{\tau}}\mathtt{R}_{2,\vec{\tau}} \times \tilde{K}_{4,4} \end{aligned} \tag{7.5.86}$$

In other words, the various $\mathtt{R}_{m,j}$ appear multiple times, giving extra multiplicity to the individual factors in (7.5.84), and the $\tilde{K}_{m,j}(B, T, a, l)$ have reduced degree compared to $K_{m,j}$.

7.5.3 Low b factorisation

Let us now investigate how the characteristic polynomial behaves when we vary the label b of the $[aba]$ rep, keeping the level splitting label fixed. Consider for example the fate of the operators at $m^* = 2$ across various $SU(4)_{\rm R}$ channels, depicted in the figure below.



This figure wants to show that a physical pair of operators with level splitting label $m^* = 2$ exists for any $[0b0]$ with $b \geq 2$ and $\tau \geq b + 2a + 6$, but when we go to [000] or [010] we find only one physical operator. For clarity, let us remark that $\mathcal{P}^*(T, B, a, l)$ is defined in the domain (7.5.75) and here we are discussing what happens outside that domain, as it was the case for the rank reduction in twist, but this time in the b direction.

Consider then the characteristic polynomial given in (7.4.42), and check what happens in the case of the figure. From right to left the significative cases are [000] and [020].

The result is included in the general formula

$$[a0a] \qquad n = a + l + 2 \tag{7.5.87}$$

$$\mathcal{P}^* = \Big(\tilde\eta + \tfrac{\tau(\tau+2l+4)(8+4a+4\tau+2l\tau+\tau^2)}{16}\Big)\Big(\tilde\eta + \tfrac{(\tau-2a-4)(2n+4+\tau)}{4}\big(T + \tfrac{3a-3n+an-n^2}{2n+5}\big)\Big)\,.$$

Thus, amazingly, the characteristic polynomial factorises and, upon inspection, the n-dependent root coincides with the anomalous dimension of the physical operator at $\tau = b + 2a + 6$.[15] Similarly,

$$[a1a] \qquad n = a + l + 2 \tag{7.5.88}$$

$$\mathcal{P}^* = \Big(\tilde\eta + \tfrac{(\tau-1)(\tau+1)(\tau+2l+3)(\tau+2l+5)}{16}\Big)\Big(\tilde\eta + \tfrac{(\tau-2a-5)(2n+5+\tau)}{16}\big(T + \tfrac{12+2a+4l}{2n+5}\big)\Big)\,.$$

Notice the appearance of a fully factorised root, and recall now that it is $b = 1$, rather than $b = 0$, the first value of b to lie outside the definition domain of the characteristic polynomial. From [020] upward we will find two physical roots with square root splitting, as shown in section 7.4.1.

Consider also the characteristic polynomial given in (7.4.55). Again we find

$$[a2a] \qquad n = a + l + 3 \tag{7.5.89}$$

$$\mathcal{P}^* = \Big(\tilde\eta + \tfrac{(\tau-2)\tau(\tau+2)(\tau+2l+2)(\tau+2l+4)(\tau+2l+6)}{64}\Big)\Big(\tilde\eta + \tfrac{(\tau-2a-6)(2n+4+\tau)}{4}\Big(T^2 - \tfrac{22+2a+12n+an+n^2}{2n+5} + \tfrac{(n+3)(2-4a-a^2+5n+an)}{2n+5}\Big)\Big)\,.$$

Again the n-dependent root coincides with the physical anomalous dimensions of the operator at the minimum twist. Even more interestingly, the value $b = 2$ is the first value of b to lie outside the domain of definition of the characteristic polynomial, and again we find a fully factorise root.

The pattern of factorised non physical roots continues for $m^* = 3, 4$, in particular,

$$\mathcal{P}^*_{m,[aba]}\Big|_{b=2m-3} = \Big(\tilde\eta + (\tfrac{\tau+1}{2} - m + 1)_{2m-2}(\tfrac{\tau+1}{2} + l - m + 3)_{2m-2}\Big)\Big(\ldots\Big) \qquad a+l \text{ even},$$
$$\tag{7.5.90}$$
$$\mathcal{P}^*_{m,[aba]}\Big|_{b=2m-2} = \Big(\tilde\eta + (\tfrac{\tau}{2} - m + 1)_{2m-1}(\tfrac{\tau}{2} + l - m + 3)_{2m-1}\Big)\Big(\ldots\Big) \qquad a+l \text{ odd}.$$

The transition in b is thus different compared to the rank reduction in the twist τ. It involves remarkable factorisations of the characteristic polynomial, but brings some of the analytically continued operators to a non physical sheet. Nevertheless, all physical roots are correctly captured, and we verified this statement for all characteristic polynomials in the ancillary file attached.

Finally, consider combining the transition in b and the reduction in τ. Take first a

[15]This is the first value of the twist for an $m^* = 2$ two-particle operator with $a + l = n - 2$.

$\mathcal{P}_m^*(T, B, a, l)$ and vary b to a value such that m^c analytically continued operators do not belong to the physical sheet, while $m - m^c$ instead remain. At this point the characteristic polynomial factorises a polynomial of degree m^c, which we discard, and an $m - m^c$ polynomial on the physical sheet remains. We can now reduce the latter in τ, and ask whether the rank reduction works in the same way as for the generic case. For all the examples we have, it works!

Epilogue I

In this first part of the thesis we have analysed some aspects of $\mathcal{N} = 4$ SYM at strong coupling. We saw that the dynamics in the supergravity limit is very simple as a consequence of a surprising accidental $10d$ conformal symmetry [31], which allows to to write *all* four-point KK correlators in terms of a single $10d$ object

$$\mathcal{M}_{\text{SUGRA}} = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}, \qquad \mathbf{s}+\mathbf{t}+\mathbf{u} = -4 \tag{7.5.91}$$

where $\mathbf{s}, \mathbf{t}, \mathbf{u}$ are the bold-face variables introduced in [32]. The hidden symmetry is nicely reflected in the structure of the anomalous dimensions of the long-double trace operators exchanged at large N, which are all rationals and manifest a residual degeneracy [37]:

$$\eta^{(0)}_{pq} = -2\frac{\delta^{(8)}}{(l_{10}+1)_6}, \qquad l_{10} \equiv l + a + 2\,m - \frac{1+(-1)^{a+l}}{2} - 1 \tag{7.5.92}$$

where $m = p - a - 1$ is the level splitting label.

We then moved away from the supergravity limit by considering the α' deformation of the theory and outlined a method, based on [1, 3], to compute the amplitude (up to a certain number of ambiguities) at any order in α'. The procedure allowed us to fix, on one side, the amplitude, and on the other side, a certain class of anomalous dimensions, which live at the edge of the rectangle $R_{\vec{\tau}}$. From explicit results, we observed that the correlators manifest additional simplicity. Interestingly, we found that the amplitudes are best written in terms of a pre-amplitude, defined via (5.4.36). As an example, the α'^5 amplitude reads

$$\tilde{\mathcal{M}}_2 = S^2 + T^2 + U^2 + 3\Sigma^2 + b_1(\textstyle\sum_i p_i^2) + b_2 \tag{7.5.93}$$

where b_1 and b_2 are fixed by localisation. Let us emphasize once again that the result is valid for *arbitrary* KK modes. Remarkably, in the S, T, U variables, the amplitude closely resembles its flat space cousin, suggesting the existence of a more general version of flat space limit, which generalises the one introduced by Penedones [58]. Ultimately, we expect this flat space limit to be a simplified version of a more general flat space limit, obtained by replacing partial with covariant derivatives in a $10d$ effective action [50]. We have summarised the different notions of flat space limit and their relations figure 5.1.

We then studied the properties of the anomalous dimensions of operators that saturate the $10d$ bound - which are the ones that are completely fixed by the bootstrap - and their associated characteristic polynomial. To quote an example, the anomalous dimensions of operators with $m = 2$ are the zeros of the following characteristic polynomial

$$\mathcal{P}^*_{\vec{\tau},2} = (\tilde{\eta} + r)^2 + (\tilde{\eta} + r)\gamma_{2,1} + \gamma_{2,0}, \tag{7.5.94}$$

with

$$\gamma_{2,1} = -\frac{(n+2)(n+3)}{2n+5}\Big(B(2l+5) + (2a+5)T - (a+2)(l+2)\Big), \tag{7.5.95}$$

$$\gamma_{2,0} = +\frac{(n+2)^2(n+3)^2}{2n+5}B\,T, \tag{7.5.96}$$

and

$$r = (T-B)^2 + B(2+l) + (2+a)T. \tag{7.5.97}$$

Interestingly, these results interpolate between different orders[16] in α'. This means that the characteristic polynomial is intrinsically a *non*-perturbative object.

The structure of the characteristic polynomial is quite fascinating and satisfies different properties, such as flat space limit, a Z_2 duality which swaps B, a with T, l, rank-reduction and low b factorisation, strongly suggesting of the existence of an integrable structure behind it. Crucially, the characteristic polynomial seems to be 1-to-1 with the (complete) generalised flat space limit of the Virasoro-Shapiro amplitude. As an example, we saw that the top powers in $s, \tilde{s}$, etc rearrange themselves and combine into S, T, U variables. In the characteristic polynomial this property reflects into the fact that the variables b, a, τ, l reorganise in such a way that the top power only depends on the combination $T - B$. We believe that unveiling all of its properties will help to shed light not only on the VS amplitude in $AdS_5 \times S^5$, but, perhaps, also on the relation between AdS and flat space scattering amplitudes. We hope to report on this in the future.

[16]This is because, in order to access different values of a and l, we necessarily need to go cross different orders in α', as we can see from formula (7.1.3).

Part II

D1-D5 and D3-D7 systems

Prologue II

In this second part of the thesis we will focus on two other theories, namely $AdS_3 \times S^3$ and $AdS_5 \times S^3$.

As mentioned in the introduction, the $AdS_5 \times S^3$ background arises in two different string theory setups. In both setups, the boundary theory is a $4d$ $\mathcal{N} = 2$ SCFT with flavour group G_F. On the other hand, the $AdS_3 \times S^3$ background arises as the low-energy limit of the D1-D5 system. This is described by a $2d$ $\mathcal{N} = (4, 4)$ SCFT dual to string theory on $AdS_3 \times S^3 \times M$, where M is either $K3$ or T^4.

The strong-coupling tree-level dynamics of certain subsectors of these theories appears to be quite simple, as a consequence of hidden conformal symmetries, analogous to that of $\mathcal{N} = 4$ SYM. In particular, the gluon sector in $AdS_5 \times S^3$ enjoys an $8d$ conformal symmetry [67]. Similarly, $AdS_3 \times S^3$ correlators enjoy an $6d$ hidden conformal symmetry; this feature was first observed in the tensor multiplet subsector [64] - which is the one we will focus on in this thesis - and it was then generalised to all sectors [66].

In this second part of the thesis we proceed in direct parallel with $\mathcal{N} = 4$ SYM. Following [3, 5], we first introduce $AdS_5 \times S^3$ and $AdS_3 \times S^3$ generalised Mellin transforms, that make manifest the large p limit. This is a quite useful representation, because it makes manifest many properties of these amplitudes. Then, we study the double-trace spectrum of both theories, giving explicit formulas for the coefficients of the disconnected part of free theory and the leading order anomalous dimensions. These very much resemble the analogous quantities in $\mathcal{N} = 4$ SYM and, in particular, manifest a residual degeneracy, as a consequence of the hidden conformal symmetries.

In the last chapter of the thesis we present some remarkable formulae which interpolate between the different backgrounds. In particular, we present a novel formula for the block decomposition of *all* free theory diagrams in *all* theories with $SU(m, m|2n)$ symmetry. Finally, we close the stage with a more speculative section on possible α' corrections, along the lines of what was done in $\mathcal{N} = 4$ SYM.

Chapter 8

Supergluons in $AdS_5 \times S^3$

We start this second part with $AdS_5 \times S^3$ and, in particular, we analyse the four-point function of half-BPS operators at large N in a certain $4d\ \mathcal{N}=2$ SCFT, dual to the tree-level scattering of four supergluons in $AdS_5 \times S^3$. In this chapter we briefly describe the set-up (section 8.1). Then, inspired by the $\mathcal{N}=4$ SYM case, we propose an $AdS_5 \times S^3$ Mellin transform, which, as we will see, it has the advantage of making manifest many properties of the amplitude (section 8.2).

8.1 The set-up

The $AdS_5 \times S^3$ background arises in two basic stringy setups. One can either consider a stack of N D3-branes probing F-theory 7-brane singularities or a stack of N_F D7-branes wrapping an $AdS_5 \times S^3$ subspace in the $AdS_5 \times S^5$ geometry of a stack of N D3-branes. For concreteness, we will focus on the latter. We consider a stack of N D3-branes embedded into the world volume of N_F D7-branes, as shown in the table below.

	0	1	2	3	4	5	6	7	8	9
D3	•	•	•	•	-	-	-	-	-	-
D7	•	•	•	•	•	•	•	•	-	-

The configuration breaks 8 out of the 16 supercharges of $\mathcal{N}=4$ SYM. Moreover, the $SO(6)$ isometry group is broken to $SO(4) \times SO(2) \sim SU(2)_R \times SU(2)_L \times SO(2)$. Here, $SO(4)$ rotates the 5678 directions along the D7-branes, $SO(2)$ rotates the 89 directions orthogonal to the D7. In the large N limit, the dynamics of the strings stretching between D7-branes decouples from the rest, therefore the corresponding low-energy theory has the usual $\mathcal{N}=4$ supermultiplet - generated by strings with both ends on the D3-branes - and $\mathcal{N}=2$ supermultiplets - which are generated by strings stretching between D3- and D7-branes [136]. The β function of the theory, $\beta \sim \lambda^2 \frac{N_F}{N}$, goes to zero in the large

N limit, for fixed 't Hooft coupling and N_F small, therefore in this limit, the theory is conformal. To summarise, the low energy theory of this brane set-up is a $4d$ $\mathcal{N}=2$ CFT with flavour group G_F[1]. Note that, in the CFT, $SU(2)_R$ is realised as R-symmetry group, $SU(2)_L$ becomes part of the flavour group[2].

We are interested in the dynamics of supergluons in AdS. These correspond to an $\mathcal{N}=1$ vector multiplet which transforms in the adjoint of G_F. Upon reducing on the sphere, it provides an infinite tower of Kaluza-Klein modes organised in different multiplets. In the dual CFT, the super primaries of these multiplets are half-BPS scalar operators of the form $\mathcal{O}_p^{Ia_1a_2\ldots a_p;\bar{a}_1\bar{a}_2\ldots\bar{a}_{p-2}}$. Here I is the colour index, p is the scaling dimension of the operator, $a_1,\ldots,a_p$ are symmetrised $SU(2)_R$ R-symmetry indices and similarly $\bar{a}_i$ are indices of the additional $SU(2)_L$ flavour group. Note that these operators have spin $\frac{p}{2}$ under $SU(2)_R$ and spin $\frac{p}{2}-1$ under $SU(2)_L$. Similarly to $\mathcal{N}=4$, it is convenient to contract the indices with auxiliary bosonic two-component vectors η and $\bar{\eta}$ to keep track of the $SU(2)_R \times SU(2)_L$ indices:

$$\mathcal{O}_p^I \equiv \mathcal{O}_p^{I;a_1a_2\ldots a_p;\bar{a}_1\bar{a}_2\ldots\bar{a}_{p-2}}\eta_{a_1}\ldots\eta_{a_p}\bar{\eta}_{\bar{a}_1}\ldots\bar{\eta}_{\bar{a}_{p-2}}\,. \tag{8.1.1}$$

We will denote the amplitude of supergluons by

$$G_{\vec{p}}^{I_1I_2I_3I_4}(x_i,\eta_i,\bar{\eta}_i) \equiv \langle \mathcal{O}_{p_1}^{I_1}\mathcal{O}_{p_2}^{I_2}\mathcal{O}_{p_3}^{I_3}\mathcal{O}_{p_4}^{I_4}\rangle. \tag{8.1.2}$$

A crucial point is that, the strength of the self-gluon coupling is larger than the coupling of gluons to gravitons [67]. In light of this, one can perform an expansion in $1/N$ in which the graviton-exchange is $1/N$ suppressed. Schematically, we have

$$G_{\vec{p}}^{I_1I_2I_3I_4} = G_{\text{disc},\vec{p}}^{I_1I_2I_3I_4} + \frac{1}{N}G_{\text{tree-gluon},\vec{p}}^{I_1I_2I_3I_4} + \frac{1}{N}G_{\text{tree-graviton},\vec{p}}^{I_1I_2I_3I_4} + \cdots \tag{8.1.3}$$

Much like the $\mathcal{N}=4$ case we discussed in the first part, the 'disconnected' term is a sum over products of two-point functions and takes the form of (generalised) free theory. In terms of OPE data it contains the leading order contributions to the three-point functions of the external operators with exchanged two-particle operators. We will refer to $G_{\text{tree-gluon},\vec{p}}^{I_1I_2I_3I_4}$ as the 'tree-level' amplitude.

8.2 $AdS_5 \times S^3$ Mellin transform

Let us deal with superconformal symmetry. We will skip the details, since the logic is very similar to the $\mathcal{N}=4$ case discussed in the first part. Following the definitions given

[1]In this set-up the group is actually $U(N_F)$ but we will keep it generic because it is irrelevant for the details considered in this thesis.

[2]The remaining $SO(2)$ factor does not play any role here because the half-BPS operators we will consider in this thesis are chargeless under this symmetry.

there, we analogously define[3]

$$g_{ij} = \frac{y_{ij}^2}{x_{ij}^2} \tag{8.2.4}$$

where $y_{ij}^2 = \langle \eta_i \eta_j \rangle \langle \bar{\eta}_i \bar{\eta}_j \rangle$ with $\langle \eta_i \eta_j \rangle = \eta_{ia} \eta_{jb} \epsilon^{ab}$ and similarly $\langle \bar{\eta}_i \bar{\eta}_j \rangle = \bar{\eta}_{i\bar{a}} \bar{\eta}_{j\bar{b}} \epsilon^{\bar{a}\bar{b}}$. As usual, the cross cross-ratios are defined via

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = U = x\bar{x}\,, \qquad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = V = (1-x)(1-\bar{x})\,,$$
$$\frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} = \tilde{U} = y\bar{y}\,, \qquad \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2} = \tilde{V} = (1-y)(1-\bar{y})\,.$$

Note that in this case we can write the $y, \bar{y}$ variables in terms of the η and $\bar{\eta}$ variables as

$$y = \frac{\langle \eta_1 \eta_2 \rangle \langle \eta_3 \eta_4 \rangle}{\langle \eta_1 \eta_3 \rangle \langle \eta_2 \eta_4 \rangle}, \qquad \bar{y} = \frac{\langle \bar{\eta}_1 \bar{\eta}_2 \rangle \langle \bar{\eta}_3 \bar{\eta}_4 \rangle}{\langle \bar{\eta}_1 \bar{\eta}_3 \rangle \langle \bar{\eta}_2 \bar{\eta}_4 \rangle}\,. \tag{8.2.5}$$

The correlator can be split into a protected and an unprotected sector, each separately respecting crossing symmetry,

$$G_{\text{tree-gluon},\vec{p}} = G_{0,\vec{p}} + \mathcal{P}\,\mathcal{I}\,\mathcal{A}_{\vec{p}}\,. \tag{8.2.6}$$

The term $G_{0,\vec{p}}$ contains all contributions due to protected multiplets[4] at this order in $1/N$. The second term contains all the logarithmic terms which arise due to two-particle operators receiving anomalous dimensions. Thanks to bosonic and fermionic symmetries, we can factor out $\mathcal{P}$ and $\mathcal{I}$ which in this case are given by

$$\mathcal{P} \equiv \frac{g_{12}^{k_s} g_{14}^{k_t} g_{24}^{k_u} \left(g_{13} g_{24}\right)^{p_3}}{\langle \bar{\eta}_1 \bar{\eta}_3 \rangle^2 \langle \bar{\eta}_2 \bar{\eta}_4 \rangle^2}\,, \quad \mathcal{I} = (x-y)(\bar{x}-y)\,, \tag{8.2.7}$$

where we remind that

$$k_s = \frac{p_1 + p_2 - p_3 - p_4}{2}, \qquad k_t = \frac{p_1 + p_4 - p_2 - p_3}{2}, \qquad k_u = \frac{p_2 + p_4 - p_3 - p_1}{2}. \tag{8.2.8}$$

A few comments are in order:

- note that $\mathcal{I}$ has degree two in y but no dependence on $\bar{y}$. In fact, $\mathcal{I}$ is fixed by superconformal symmetry, therefore it can only depend on R-symmetry variables (y) as well as spacetime variables $x, \bar{x}$, while $\bar{y}$ is a flavour variable; in $\mathcal{N} = 4$ the analogous factor[5] is instead a function of both $y, \bar{y}$, cf. (2.2.18);
- since $G_{\text{tree-gluon},\vec{p}}$ is a polynomial whose degree in $\bar{y}$ is two units lower than the

[3] To make direct parallel with the $\mathcal{N} = 4$ case, we will use the same symbols we used for $\mathcal{N} = 4$ quantities, even though their definition is now different.

[4] We stress again that the theorem only says that *all* contributions due to protected multiplets are contained in $G_{0,\vec{p}}$ but this does *not* mean that $G_{0,\vec{p}}$ cannot contain contributions coming from unprotected multiplets. In fact, we already know that generalised free theory does contain such contributions.

[5] We stress again that the definition of $y, \bar{y}$ variables is different in both theories.

degree in y, once we extract the factor $\mathcal{I}$, the remaining function $\mathcal{A}_{\vec{p}}^{I_1I_2I_3I_4}$ has the same degree in $y, \bar{y}$;

- this last observation is very important because it guarantees that, if $\mathcal{A}_{\vec{p}}^{I_1I_2I_3I_4}$ is symmetric under $y, \bar{y}$ exchange, which is the case for the amplitude we consider in this paper, it can be written as a function of $\tilde{U}, \tilde{V}$ as well as U and V and the charges $\vec{p}$. This is crucial because it allows us to define an $AdS_5 \times S^3$ Mellin transform, by adapting the large p formalism to this set up.

We are now ready to define the $AdS_5 \times S^3$ Mellin transform $\mathcal{M}$ via

$$\mathcal{A}_{\vec{p}}^{I_1I_2I_3I_4} = -\oint dsdt \sum_{\tilde{s},\tilde{t}} U^s V^t \tilde{U}^{\tilde{s}} \tilde{V}^{\tilde{t}}\, \Gamma_\otimes\, \mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4} \tag{8.2.9}$$

where $\mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4} \equiv \mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4}(s,t,\tilde{s},\tilde{t})$. As in $\mathcal{N}=4$ SYM, the kernel $\Gamma_\otimes$ is factorised into AdS_5 and S^3 contributions and takes the form $\Gamma_\otimes = \Gamma_s\Gamma_t\Gamma_u$ with

$$\Gamma_s = \frac{\Gamma[-s]\Gamma[-s+k_s]}{\Gamma[1+\tilde{s}]\Gamma[1+\tilde{s}+k_s]} \tag{8.2.10}$$

and Γ_t, Γ_u defined similarly. This time the Mellin variables obey the relations

$$s+t+u = -p_3 - 1, \quad \tilde{s}+\tilde{t}+\tilde{u} = p_3 - 2\,, \tag{8.2.11}$$

which may be used to eliminate u and $\tilde{u}$. Note also that, like in $\mathcal{N}=4$, the gamma functions in the denominator automatically restrict the sum over $\tilde{s}$, $\tilde{t}$ to the triangle

$$T = \{\tilde{s} \geq max(0, -k_s),\ \tilde{t}, \tilde{u} \geq 0\}\,. \tag{8.2.12}$$

The contour integral in s and t requires a little care and we will return to this point in the next chapter when we will introduce the colour-ordered amplitude. The double integral (8.2.9), when combined with the amplitude $\mathcal{M}_{\vec{p}}$ given in the next section, precisely coincides with the result given in [67].

In the next chapter we will see that, at large p, the $AdS_5 \times S^3$ Mellin amplitude, approaches the flat space S-matrix with the Mandelstam variables replaced by the bold-face variables

$$\mathbf{s} = s+\tilde{s}, \quad \mathbf{t} = t+\tilde{t}, \quad \mathbf{u} = u+\tilde{u}, \quad \mathbf{s}+\mathbf{t}+\mathbf{u} = -3. \tag{8.2.13}$$

Note that, unlike $\mathcal{N}=4$ SYM, the bold-face variables sum to -3 rather than -4. This will be crucial in order for the integrand $\mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4}$ to satisfy BCJ and double-copy relations incorporating all Kaluza-Klein modes.

Chapter 9

BCJ and CK duality at tree-level

As we mentioned in the introduction, flat space scattering amplitudes of gluons and gravitons are known to satisfy a number of relations, such as colour-kinematic and the associated double-copy, suggesting an underlying common structure shared by Yang-Mills and gravity theories. Intuitively, one would expect that these features are intrinsic properties of these interactions and therefore should persist also in curved backgrounds. In this chapter we show that, at least in $AdS_5 \times S^3$, four-point functions of supergluons do satisfy these properties in a way surprisingly similar to flat space. Before doing that, it is useful to quickly review the story in flat space.

9.1 BCJ and CK in flat space

This section is devoted to review some properties of four-gluon amplitudes in flat space. The full tree-level n-point amplitude can be written in terms of the so-called colour-ordered amplitudes as follows

$$A^{I_1 I_2 \ldots I_n} = \sum_{\mathcal{P}(2,\ldots,n)} \mathrm{Tr}\left(T^{I_1} T^{I_2} \ldots T^{I_n}\right) A(1,2,\ldots,n) \tag{9.1.1}$$

where the sum is over all permutations of points $2,\ldots,n$ and $A(1,2\ldots,n)$ are the colour-ordered amplitudes[1]. This is called "trace basis". The trace basis is overcomplete: the $(n-1)!$ colour-ordered amplitudes in (9.1.1) are not all linearly independent and satisfy various relations. In particular, an n-point function satisfies cyclicity, which in (9.1.1) has already been used to fix the first entry, reflection, Kleiss-Kuijf relations and the $U(1)$ decoupling identity that reduce[2] the number of independent colour-ordered amplitudes

[1]Note that one can use the ciclicity of the trace to fix the first entry so that the permutation is over $n-1$ points.

[2]In fact, it is possible to write down the amplitude in a non-redundant basis, known as DDM basis, in which only the independent $(n-2)!$ colour-ordered amplitudes show up [137, 138].

to $(n-2)!$. These last two relations coincide for a four-point function and take the form

$$A(1,2,3,4)+A(1,2,4,3)+A(1,3,2,4)=0. \tag{9.1.2}$$

Bern, Carrasco and Johannson [12] noticed that colour-ordered amplitudes satisfy further relations, known as BCJ relations, that reduce the number of independent colour-ordered amplitudes to $(n-3)!$. In the case of four-point functions the BCJ relations read

$$\begin{aligned} tA(1,2,3,4) &= uA(1,3,4,2)\,, \\ sA(1,2,3,4) &= uA(1,4,2,3)\,, \\ tA(1,4,2,3) &= s\,A(1,3,4,2)\,, \end{aligned} \tag{9.1.3}$$

where here s,t,u are the usual Mandelstam variables. Importantly, the existence of these relations allows to rewrite the full colour-dressed amplitude in another representation, often dubbed "colour basis":

$$A^{I_1I_2I_3I_4}=\frac{n_sc_s}{s}+\frac{n_tc_t}{t}+\frac{n_uc_u}{u} \tag{9.1.4}$$

where $c_s=f^{I_1I_2I_5}f^{I_5I_3I_4}$, $f^{I_1I_2I_3}$ are the structure constants, and c_t,c_u defined similarly. On the other hand, n_s,n_t,n_u are kinematic factors and are functions of polarisations and Mandelstam variables. Note that, by virtue of the Jacobi identity, the c_i satisfy:

$$c_s+c_t+c_u=0. \tag{9.1.5}$$

Now, it can be shown that n_i satisfy an analogous relation

$$n_s+n_t+n_u=0. \tag{9.1.6}$$

The pair of equations (9.1.5), (9.1.6) is a manifestation of the so-called *colour-kinematic duality*. Let us now prove that (9.1.6) does imply BCJ relations among colour-ordered amplitudes. To see this, we need to switch to the trace basis via

$$\begin{aligned} c_s &= \mathrm{Tr}\left(T^{I_1}T^{I_2}T^{I_3}T^{I_4}\right)+\mathrm{Tr}\left(T^{I_1}T^{I_4}T^{I_3}T^{I_2}\right)-\mathrm{Tr}\left(T^{I_1}T^{I_2}T^{I_4}T^{I_3}\right)-\mathrm{Tr}\left(T^{I_1}T^{I_3}T^{I_4}T^{I_2}\right), \\ c_t &= \mathrm{Tr}\left(T^{I_1}T^{I_4}T^{I_2}T^{I_3}\right)+\mathrm{Tr}\left(T^{I_1}T^{I_3}T^{I_2}T^{I_4}\right)-\mathrm{Tr}\left(T^{I_1}T^{I_4}T^{I_3}T^{I_2}\right)-\mathrm{Tr}\left(T^{I_1}T^{I_2}T^{I_3}T^{I_4}\right), \\ c_u &= \mathrm{Tr}\left(T^{I_1}T^{I_3}T^{I_4}T^{I_2}\right)+\mathrm{Tr}\left(T^{I_1}T^{I_2}T^{I_4}T^{I_3}\right)-\mathrm{Tr}\left(T^{I_1}T^{I_3}T^{I_2}T^{I_4}\right)-\mathrm{Tr}\left(T^{I_1}T^{I_4}T^{I_2}T^{I_3}\right). \end{aligned} \tag{9.1.7}$$

Now, plugging (9.1.7) into (9.1.4), we can write a relation between colour factors and colour ordered amplitudes:

$$\begin{pmatrix} A(1,2,3,4) \\ A(1,3,4,2) \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & -\frac{1}{t} \\ -\frac{1}{u}-\frac{1}{s} & -\frac{1}{u} \end{pmatrix} \begin{pmatrix} n_s \\ n_t \end{pmatrix} \tag{9.1.8}$$

where we have used $n_u = -n_t - n_s$. The determinant of this matrix[3] is proportional to $s+t+u=0$, therefore the matrix is rank one and cannot be inverted. This means that the colour-ordered amplitudes must satisfy some linear relations, which in fact are just the BCJ relations.

Surprises do not finish here. Bern, Carrasco and Johannsson also noticed that if we replace colour with kinematic factors in (9.1.4), i.e.

$$M = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}, \tag{9.1.9}$$

we get the scattering of four gravitons! Thus, in this sense, gravity=(Yang-Mills)^2.

As an example, let us check the above properties with the simple MHV amplitude mentioned in the introduction. We have

$$A[1^-2^-3^+4^+] = \frac{\langle 12\rangle^2[34]^2}{st}, \tag{9.1.10}$$

and the other colour-ordered amplitudes related to it by crossing. The spinor brackets in the numerator carry helicity information and can be though of "square roots" of the Mandelstam variables, and s, t are the Mandelstam variables. In the trace-basis the kinematic factors are

$$\begin{aligned} n_s &= \langle 12\rangle^2[34]^2\frac{1}{3}\left(\frac{1}{t} - \frac{1}{u}\right), \\ n_t &= \langle 12\rangle^2[34]^2\frac{1}{3}\left(\frac{1}{u} - \frac{1}{s}\right), \\ n_u &= \langle 12\rangle^2[34]^2\frac{1}{3}\left(\frac{1}{s} - \frac{1}{t}\right) \end{aligned} \tag{9.1.11}$$

and one can easily check that they satisfy (9.1.6). Finally, replacing colour with kinematic, we get

$$M[1^-2^-3^+4^+] = \frac{\langle 12\rangle^4[34]^4}{s\,t\,u} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u} \tag{9.1.12}$$

where $M[1^-2^-3^+4^+]$ is the four-graviton MHV amplitude in Einstein theory.

In the introduction we mentioned another double-copy procedure, namely the one realised through KLT relations. By switching to the trace basis and using the BCJ relations above we can easily verify that at tree-level KLT and the BCJ relations are equivalent.

The double-copy is by now a well-established method to construct amplitudes in theories using as input other, simpler, theories. It has been shown to hold in a wide class of theories, both at tree and loop level, see e.g. [17, 140] for the state of the art of this program.

Having recalled the main features of double-copy and BCJ relations in flat space, we are

[3]This matrix was introduced in [139] and dubbed propagator matrix.

now ready to switch to $AdS_5 \times S^3$, where we are going to show that analogous relations hold. Perhaps with no much surprise, the space in which they manifest their simplicity is the generalised Mellin space we introduced in the previous chapter.

9.2 BCJ and CK in $AdS_5 \times S^3$

Let us consider the field theory amplitude computed in [67]. In the large p formalism we introduced in chapter 8 takes the following very simple form

$$\mathcal{M}_{\vec{p}}^{I_1 I_2 I_3 I_4} = \frac{\mathbf{n_s} c_s}{\mathbf{s}+1} + \frac{\mathbf{n_t} c_t}{\mathbf{t}+1} + \frac{\mathbf{n_u} c_u}{\mathbf{u}+1}\,, \tag{9.2.13}$$

where we remind that $\mathbf{s}+\mathbf{t}+\mathbf{u}=-3$. Here we have

$$\begin{aligned}
\mathbf{n_s} &= \frac{1}{3}\left(\frac{1}{\mathbf{t}+1} - \frac{1}{\mathbf{u}+1}\right), \qquad & c_s &= f^{I_1 I_2 J} f^{I_3 I_4 J},\\
\mathbf{n_t} &= \frac{1}{3}\left(\frac{1}{\mathbf{u}+1} - \frac{1}{\mathbf{s}+1}\right), \qquad & c_t &= f^{I_1 I_4 J} f^{I_2 I_3 J},\\
\mathbf{n_u} &= \frac{1}{3}\left(\frac{1}{\mathbf{s}+1} - \frac{1}{\mathbf{t}+1}\right), \qquad & c_u &= f^{I_1 I_3 J} f^{I_2 I_4 J}.
\end{aligned} \tag{9.2.14}$$

The large p limit ensures that the amplitude reduces to the associated flat amplitude, which in this case is the $\mathcal{N}=1$ four-point amplitude in $8d$, which is roughly given by equation (9.1.10) with the helicity factors $\langle 12\rangle^2\langle 34\rangle^2$ replaced by some fermionic delta function that carries the helicity dependence of the full $\mathcal{N}=1$ supermultiplet. It is immediate to see that in the limit of large p, (9.2.14) reduces to (the kinematic part of) (9.1.11), with the Mandelstam replaced by the bold-font variables.

In the first part of the thesis we saw that the reason why the $\mathcal{N}=4$ supergravity correlator does not depend on $s, \tilde{s}, \ldots$ separately is due to the an accidental hidden symmetry. The same mechanism holds here: in fact, we can promote the correlator $\mathcal{M}_{2222}^{I_1 I_2 I_3 I_4}$ to a generating function for correlators with arbitrary charges $\vec{p}$. Then, $\mathcal{M}_{\vec{p}}^{I_1 I_2 I_3 I_4}$ follows from 'covariantising' $\mathcal{M}_{2222}^{I_1 I_2 I_3 I_4}$:

$$\mathcal{M}_{2222}^{I_1 I_2 I_3 I_4}(s,t) \xrightarrow{\texttt{8d-symm}} \mathcal{M}_{\vec{p}}^{I_1 I_2 I_3 I_4} = \mathcal{M}_{2222}^{I_1 I_2 I_3 I_4}(\mathbf{s},\mathbf{t}).$$

In this case, the differential operator reads

$$\mathcal{A}_{\vec{p}}^{I_1 I_2 I_3 I_4} = \widehat{\mathcal{D}}_{\vec{p}}\Big[U^3 \mathcal{A}_{2222}^{I_1 I_2 I_3 I_4}\Big] \tag{9.2.15}$$

where

$$\widehat{\mathcal{D}}_{\vec{p}} = U^{-3} \sum_{\tilde{s},\tilde{t}} \left(\frac{\tilde{U}}{U}\right)^{\tilde{s}} \left(\frac{\tilde{V}}{V}\right)^{\tilde{t}} \widehat{\mathcal{D}}^{(0,0,0)}_{\vec{p},\tilde{s},\tilde{t}} \widehat{\mathcal{D}}^{(k_s,k_t,k_u)}_{\vec{p},\tilde{s},\tilde{t}} \tag{9.2.16}$$

and

$$\widehat{\mathcal{D}}^{(a,b,c)}_{\vec{p},\tilde{s},\tilde{t}} = \frac{(U\partial_U - 2 - \tilde{s} - a)_{\tilde{s}+a}}{(-)^a(\tilde{s}+a)!}\frac{(V\partial_V + 1 - \tilde{t} - b)_{\tilde{t}+b}}{(-)^b(\tilde{t}+b)!}\frac{(U\partial_U + V\partial_V)_{\tilde{u}+c}}{(\tilde{u}+c)!} \tag{9.2.17}$$

We will skip the proof that the operator generates the correct amplitude for *all* Kaluza-Klein modes, because is essentially the same as the $\mathcal{N}=4$ case we recalled in the first part.

Let us now have a closer look to the amplitude. From its form, it is clear that all relations obeyed by the flat amplitude will obviously hold here. In fact our variables obey $\mathbf{s}+\mathbf{t}+\mathbf{u}=-3$. Therefore the Mellin amplitude $\mathcal{M}$ is literally the same function as the flat space amplitude with the Mandelstam variables s,t,u replaced by the shifted bold face variables $(\mathbf{s}+1)$, $(\mathbf{t}+1)$, $(\mathbf{u}+1)$. We stress again that it is not trivial that this holds; for example, as we saw, the on-shell relation in $AdS_5\times S^5$ is $\mathbf{s}+\mathbf{t}+\mathbf{u}=-4$. As an example of the properties obeyed by $\mathcal{M}$ we have that

$$\begin{aligned} \mathbf{n_s}+\mathbf{n_t}+\mathbf{n_u}&=0\,,\\ c_s+c_t+c_u&=0\,, \end{aligned} \tag{9.2.18}$$

which gives an AdS version of the colour-kinematic duality, which was already observed in [68]. Note that (9.2.18) captures this duality for *all* Kaluza-Klein modes. We saw in the previous section that this duality is intimately connected with the BCJ relations between colour-ordered amplitudes. Using the change of basis (9.1.7), we can read off the colour-ordered amplitudes

$$\begin{aligned} \mathcal{M}_{\vec{p}}(1,2,3,4)=\mathcal{M}_{\vec{p}}(1,4,3,2)&=\frac{\mathbf{n_s}}{\mathbf{s}+1}-\frac{\mathbf{n_t}}{\mathbf{t}+1}\,,\\ \mathcal{M}_{\vec{p}}(1,2,4,3)=\mathcal{M}_{\vec{p}}(1,3,4,2)&=\frac{\mathbf{n_u}}{\mathbf{u}+1}-\frac{\mathbf{n_s}}{\mathbf{s}+1}\,,\\ \mathcal{M}_{\vec{p}}(1,3,2,4)=\mathcal{M}_{\vec{p}}(1,4,2,3)&=\frac{\mathbf{n_t}}{\mathbf{t}+1}-\frac{\mathbf{n_u}}{\mathbf{u}+1}\,. \end{aligned} \tag{9.2.19}$$

which in fact satisfy a $U(1)$ decoupling identity

$$\mathcal{M}_{\vec{p}}(1,2,3,4)+\mathcal{M}_{\vec{p}}(1,2,4,3)+\mathcal{M}_{\vec{p}}(1,3,2,4)=0 \tag{9.2.20}$$

as well as BCJ relations

$$\begin{aligned} (\mathbf{t}+1)\mathcal{M}_{\vec{p}}(1,2,3,4)&=(\mathbf{u}+1)\mathcal{M}_{\vec{p}}(1,3,4,2)\,,\\ (\mathbf{s}+1)\mathcal{M}_{\vec{p}}(1,2,3,4)&=(\mathbf{u}+1)\mathcal{M}_{\vec{p}}(1,4,2,3)\,,\\ (\mathbf{t}+1)\mathcal{M}_{\vec{p}}(1,4,2,3)&=(\mathbf{s}+1)\,\mathcal{M}_{\vec{p}}(1,3,4,2)\,, \end{aligned} \tag{9.2.21}$$

where we used the on-shell relation $\mathbf{s}+\mathbf{t}+\mathbf{u}=-3$. We stress again that the relations (9.2.21) capture the appearance of BCJ relations in AdS for *all* Kaluza-Klein modes. Such relations are manifest at level of the reduced Mellin amplitude while they do not hold, at least directly, for the full Mellin amplitude [70]. It is an interesting open question how such relations might extend to higher point amplitudes in AdS and what the role of a reduced Mellin amplitude might be in this regard.

Having introduced the colour-ordered amplitudes, let us return to the issue of the contour in the Mellin integral (8.2.9). It should be noted that the presence of poles at $\mathbf{s} = -1$, $\mathbf{t} = -1$ and $\mathbf{u} = -1$ is potentially a problem for the contour of integration. In fact, since $\mathbf{s} + \mathbf{t} + \mathbf{u} = -3$, the simultaneous presence of these poles leaves no region in the real s, t plane for the contour to pass through, while separating left moving and right moving sequences of poles in the Mellin integrand. Thus the same property which leads to the direct analogy with the flat space amplitudes also leads to a subtlety in returning to position space from Mellin space. For the colour ordered amplitudes, one does not have all three poles present simultaneously. Thus we propose that the correct definition for the contour is tied to the colour-ordering and we define analogously a colour-ordered correlator,

$$\mathcal{A}(1,2,3,4) = -\oint ds dt \oint d\tilde{s} d\tilde{t}\, U^s V^t \tilde{U}^{\tilde{s}} \tilde{V}^{\tilde{t}}\, \Gamma\, \mathcal{M}(1,2,3,4)\,,$$

The contour can now be taken to lie slightly below $\mathbf{s} = -1$ and $\mathbf{t} = -1$. Note then that this introduces a subtlety in interpreting the BCJ relations (9.2.21) back in position space, since the left and right hand sides of these equations are to be integrated over slightly different contours.

To conclude, note that an AdS version of the double-copy prescription also holds [68]. In the large p formalism this becomes completely manifest. In fact, by replacing colour with kinematic factors we get

$$\begin{aligned} \mathcal{M}_{\vec{p}}^{I_1 I_2 I_3 I_4} &\xrightarrow[c_i \to \mathbf{n}_i]{} \frac{\mathbf{n}_{\mathbf{s}}^2}{\mathbf{s}+1} + \frac{\mathbf{n}_{\mathbf{t}}^2}{\mathbf{t}+1} + \frac{\mathbf{n}_{\mathbf{u}}^2}{\mathbf{u}+1} \\ &= \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)} \propto \mathcal{M}_{\vec{p}}^{\text{SUGRA}}. \end{aligned} \tag{9.2.22}$$

This is nothing but the SUGRA amplitude (4.2.13) upon reinterpreting $\mathbf{s}, \mathbf{t}, \mathbf{u}$ as the $\mathcal{N} = 4$ variables, i.e. subject to the constraint $\mathbf{u} = -\mathbf{s} - \mathbf{t} - 4$. We should however point out that the Mellin variables satisfy a different on-shell constraints, thus the double copy construction seems to be, strictly speaking, slightly different from its flat space counterpart.

Chapter 10

The double-trace spectrum of super gluons

With the amplitude in our hand, we are now ready to study the double-trace spectrum in this theory. Similarly to $\mathcal{N} = 4$, we expect the anomalous dimensions to be rational functions of the quantum numbers, and in particular of the form

$$\eta \sim \frac{\delta}{l_{8d} + 1} \tag{10.0.1}$$

where δ is a certain rational function of twist and other quantum numbers, and l_{8d} is an "effective" $8d$ spin. An explicit computation will confirm the guess. To see this, all we need is the superconformal block decomposition of disconnected generalised free theory and that of the $\log U$ discontinuity of the tree-level correlator.

The chapter is organised as follows. In section 10.1 we give an explicit form for the long superconformal blocks, following the approach of [107]. Then, in section 10.2 we compute the long disconnected free theory matrix. Finally, in section 10.3 we unmix the spectrum of long double trace operators by computing *all* the anomalous dimensions at order $1/N$. One important difference between this theory and $\mathcal{N} = 4$ SYM is that, on top of the above mentioned usual technology, we also have to deal with the non-trivial flavour structure of the amplitude. However, since this just amounts to considering certain symmetric or antisymmetric combinations built out of the correlator, we postpone the discussion on flavour structures to the end of the chapter. A more detailed discussion can be found in [69].

10.1 Superconformal blocks in $AdS_5 \times S^3$

Let us briefly review the superconformal block technology needed in this chapter. We will only be interested in the long superconformal blocks which can be found in [107]. A

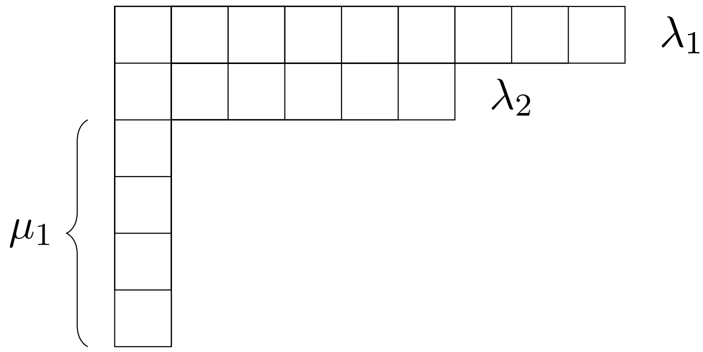


Figure 10.1: An allowed Young diagram for $\mathcal{N}=2$ reps. In this case we have $\lambda_i \leq 1$ with $i \geq 3$. In this example the diagram is associated to a long operator with $\lambda_1 = 9, \lambda_2 = 6$, $\mu_1 = 4$.

typical Young diagram associated to an $\mathcal{N}=2$ long rep has two rows and one column, as shown in figure 10.1. We should however note that the column only captures the $SU(2)_R$ dependence, that is the R-symmetry group of the $4d$ $\mathcal{N}=2$ theory. To deal with the $SU(2)_L$ flavour factor, all we need to do is to multiply the $\mathcal{N}=2$ long blocks by another $SU(2)$ spherical harmonics, or in the language of Young diagrams, long representations will be accompanied by a further one-column Young diagram. To summarise, a generic rep is specified by four quantum numbers which are in correspondence with the two Young diagrams quantum number[1]. All in all

$$\mathbb{L}_{\vec{\tau}} = \mathcal{P}(x-y)(\bar{x}-y)\left(\frac{\tilde{U}}{U}\right)^{p_3} \mathcal{G}_{\tau,l}(x,\bar{x})\mathcal{H}_{b,a}(y,\bar{y})\,, \tag{10.1.2}$$

where

$$\begin{aligned} \mathcal{G}_{\tau,l}(x,\bar{x}) &= \frac{(-1)^l}{(x-\bar{x})U^{\frac{p_{43}}{2}}}\Big(\mathcal{F}^+_{\frac{\tau}{2}+1+l}(x)\mathcal{F}^+_{\frac{\tau}{2}}(\bar{x}) - \mathcal{F}^+_{\frac{\tau}{2}}(x)\mathcal{F}^+_{\frac{\tau}{2}+1+l}(\bar{x})\Big), \\ \mathcal{H}_{b,a}(y,\bar{y}) &= \frac{1}{\tilde{U}^{2-\frac{p_{43}}{2}}}\mathcal{F}^-_{-\frac{b}{2}-a}(y)\mathcal{F}^-_{-\frac{b}{2}}(\bar{y}), \end{aligned} \tag{10.1.3}$$

with

$$\mathcal{F}^{\pm}_h(x) = x^h {}_2F_1\left[h \mp \tfrac{p_{12}}{2}, h \mp \tfrac{p_{43}}{2}, 2h\right](x). \tag{10.1.4}$$

Here, $\mathcal{G}_{\tau,l}(x,\bar{x})$ are the standard $4d$ conformal blocks (up to a shift by 2 in the twist τ) and $\mathcal{H}_{b,a}(y,\bar{y})$ are the internal blocks. Note that the latter are the product of two $SU(2)$ spherical harmonics, one corresponding to the R-symmetry group $SU(2)_R$ and the other corresponding to the flavour group $SU(2)_L$. Finally, τ, l are, respectively, twist and spin, and b, a label the different representation of $SO(4) \sim SU(2)_R \times SU(2)_L$. This way of labelling the internal representations is useful because, as we saw in the first part of the thesis, b, a can be viewed as the analogues of twist and spin on the sphere, respectively.

[1] In principle it could depend on another number γ, which however drops out from the long blocks, as it was the case also for $\mathcal{N}=4$

As in $\mathcal{N}=4$ SYM, it is useful to introduce $\vec{h}$ labels through

$$h=\frac{\tau}{2}+1+l, \quad \bar{h}=\frac{\tau}{2}, \quad j=-\frac{b}{2}-a, \quad \bar{j}=-\frac{b}{2}. \tag{10.1.5}$$

Note that the dictionary between $\vec{h}$ and $\vec{\tau}$ labels differs from the one we introduced in $\mathcal{N}=4$ SYM, c.f. (6.2.16). Moreover, note that $j,\bar{j}$ are nothing but the two spins labelling the $SU(2)$ representation [2].

It is also important to remark that the internal blocks are not invariant under $y \leftrightarrow \bar{y}$ exchange. This is a consequence of the fact that the two $SU(2)$ have a different nature: one is the R-symmetry group, the other is a flavour group. This means that, unlike[3] $\mathcal{N}=4$ SYM, the decomposition is extended to spherical harmonics with label $a<0$ or, in other words, in the OPE of two half BPS operators there are more representation exchanged. More specifically, for given charges p_i, we decompose a function in spherical harmonics labelled by $[ab]$ with values of a run over the following set:

$$-\kappa_{\vec{p}} \leq a \leq \kappa_{\vec{p}}$$

where, as in $\mathcal{N}=4$,

$$\kappa_{\vec{p}}=\frac{\min(p_1+p_2,p_3+p_4)-p_{43}-4}{2} \tag{10.1.6}$$

is the 'degree of extremality' and $p_{43}=p_4-p_3$. For each value of a, b runs over

$$-\min(a,0) \leq \frac{b-p_{43}}{2} \leq (\kappa_{\vec{p}}-a+\min(a,0)). \tag{10.1.7}$$

10.2 Long disconnected free theory

We are now ready to study the superblock decomposition of disconnected free theory. Wick contractions give

$$\begin{aligned} G^{I_1I_2I_3I_4}_{\text{disc},pqpq} =& \delta^{I_1I_2}\delta^{I_3I_4}\delta_{pq}\frac{g^p_{12}g^p_{34}}{\langle\bar{\eta}_1\bar{\eta}_2\rangle^2\langle\bar{\eta}_3\bar{\eta}_4\rangle^2} \\ &+\underbrace{\delta^{I_1I_3}\delta^{I_2I_4}\frac{g^p_{13}g^p_{24}}{\langle\bar{\eta}_1\bar{\eta}_3\rangle^2\langle\bar{\eta}_2\bar{\eta}_4\rangle^2}}_{\texttt{u-channel}} \\ &+\underbrace{\delta^{I_1I_4}\delta^{I_2I_3}\delta_{pq}\frac{g^p_{14}g^p_{23}}{\langle\bar{\eta}_1\bar{\eta}_4\rangle^2\langle\bar{\eta}_2\bar{\eta}_3\rangle^2}}_{\texttt{t-channel}}. \end{aligned} \tag{10.2.8}$$

Now, due to the non-trivial colour structure of the amplitude, only representations with a definite parity under $t\leftrightarrow u$ exchange enter the OPE. In practice, we need to decompose

[2] In fact, strictly speaking, the spins are really $-j,-\bar{j}$ because $j,\bar{j}$ are negative definite.
[3] In fact, we can anticipate that also in $AdS_3\times S^3$ the sum is restricted to reps with $a>0$.

the following combinations of diagrams

$$G^{\pm}_{\text{disc},pqpq} = \delta_{pq}\frac{g^p_{14}g^p_{23}}{\langle\bar\eta_1\bar\eta_4\rangle^2\langle\bar\eta_2\bar\eta_3\rangle^2} \pm \frac{g^p_{13}g^p_{24}}{\langle\bar\eta_1\bar\eta_3\rangle^2\langle\bar\eta_2\bar\eta_4\rangle^2}. \tag{10.2.9}$$

The block decomposition of the unprotected part reads

$$G^{\pm}_{\text{disc},pqpq}\big|_{\text{long}} = \sum_{\vec\tau} L^{\pm}_{\vec\tau}\mathbb{L}_{\vec\tau}, \tag{10.2.10}$$

where $\mathbb{L}_{\vec\tau}$ are the long superblocks. The coefficients take a form very similar to the $\mathcal{N}=4$ ones,

$$L^{\pm}_{\vec\tau} = -\frac{\pm 1 + (-1)^{a+l}\delta_{pq}}{(p-1)(q-1)} A_h A_{\bar h} B_j B_{\bar j}\boldsymbol{\delta}\,. \tag{10.2.11}$$

In this case the A and B factors are given by

$$\begin{aligned} A_h =& \frac{\Gamma(h+\frac{p-q}{2})\Gamma(h-\frac{p-q}{2})\Gamma(h+\frac{p+q}{2}-1)}{\Gamma(2h-1)\Gamma(h-\frac{p+q}{2}+1)}, \\ B_j =& \frac{\Gamma(2-2j)}{\Gamma(1-j+\frac{p-q}{2})\Gamma(1-j-\frac{p-q}{2})}\frac{1}{\Gamma(\frac{p+q}{2}+j-1)\Gamma(\frac{p+q}{2}-j)}, \end{aligned} \tag{10.2.12}$$

while $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta} = \frac{\delta^{(2)}_{h,j} - \delta^{(2)}_{\bar h,\bar j}}{\delta^{(2)}_{h,j}\delta^{(2)}_{\bar h,\bar j}}, \qquad \delta^{(2)}_{h,j} = (h-j)(h+j-1) \tag{10.2.13}$$

and we remind that

$$h = \frac{\tau}{2}+1+l, \quad \bar h = \frac{\tau}{2}, \quad j = -\frac{b}{2}-a, \quad \bar j = -\frac{b}{2}. \tag{10.2.14}$$

It is worth pointing out the different ways the two internal $SU(2)$ factors enter the coefficients. On the one hand, $SU(2)_L$ only comes in through the function $B_{\bar j}$. On the other hand, the decomposition under the R-symmetry group $SU(2)_R$ produces also the function $\boldsymbol{\delta}$ and, in particular the combination $\delta^{(2)}_{h,j}\delta^{(2)}_{\bar h,\bar j}$. The strong similarity with $\mathcal{N}=4$ SYM suggests that this object is the eigenvalue of a Casimir operator operator acting on the blocks, and therefore it cannot depend on flavour labels. In fact,

$$\begin{aligned} &\mathcal{D}_4\big(U^{\frac{p_{43}}{2}}\tilde U^{2-\frac{p_{43}}{2}}(x-\bar x)\mathcal{G}_{\tau,l}\mathcal{H}_{b,a}\big) \\ &\quad = \delta^{(2)}_{h,j}\delta^{(2)}_{\bar h,\bar j}\,\big(U^{\frac{p_{43}}{2}}\tilde U^{2-\frac{p_{43}}{2}}(x-\bar x)\mathcal{G}_{\tau,l}\mathcal{H}_{b,a}\big)\,. \end{aligned} \tag{10.2.15}$$

Here the differential operator $\mathcal{D}_4$ is given by

$$\mathcal{D}_4 = (D^+_x - D^-_y)(D^+_{\bar x} - D^-_y), \;\; D^{\pm}_x\mathcal{F}^{\pm}_h(x) = h(h-1)\mathcal{F}^{\pm}_h(x),$$

where $D_x^{\pm}$ is [134],

$$D_x^{\pm} = x^2\partial_x(1-x)\partial_x \pm (p_{12}+p_{34})x^2\partial_x - p_{12}p_{34}\,x. \tag{10.2.16}$$

Note that $\mathcal{F}^{-}_{\bar{j}}(\bar{y})$ is a spectator in (10.2.15). Once again, this is because $SU(2)_L$ is not part of the superconformal algebra.

10.3 Anomalous dimensions and residual degeneracy

The main difference with the $\mathcal{N}=4$ case is that here double-trace operators have a flavour structure. Because of this, there will be two types of anomalous dimensions, those of operators exchanged in symmetric or antisymmetric channels.

At large N, the operators acquiring anomalous dimensions are of the schematic form,

$$\mathcal{O}^{\pm}_{pq} = \mathbb{P}^{\pm}_{I_1 I_2}\mathcal{O}^{I_1}_p \partial^l \Box^{\frac{1}{2}(\tau-p-q)}\mathcal{O}^{I_2}_q \tag{10.3.17}$$

where $\mathbb{P}_{ij}$ is an appropriate projector that projects onto symmetric or antisymmetric representations of the gauge group exchanged in the OPE. For any given quantum numbers $\vec{\tau}=(\tau,b,l,a)$, the number of operators exchanged in the OPE can be represented with the number of pairs (pq) filling our favourite rectangle. In this case we have

$$R_{\vec{\tau}} := \left\{ (p,q) : \begin{array}{l} p = i+|a|+1+r \\ q = i+a+1+b-r \end{array}, \quad \begin{array}{l} i=1,\ldots,(t-1) \\ r=0,\ldots,(\mu-1) \end{array} \right\} \tag{10.3.18}$$

and the rectangle $R_{\vec{\tau}}$ is spanned by the $d=\mu(t-1)$ allowed lattice points where

$$t \equiv \frac{(\tau-b)}{2} - \frac{(a+|a|)}{2}, \quad \mu \equiv \begin{cases} \left\lfloor \frac{b+a-|a|+2}{2} \right\rfloor & a+l \text{ even}, \\ \left\lfloor \frac{b+a-|a|+1}{2} \right\rfloor & a+l \text{ odd}. \end{cases}$$

Figure 10.2 shows an example with $\mu=4, t=9$. Note the appearance of absolute values for a. This is a consequence of the fact the the theory is not symmetric under $y \leftrightarrow \bar{y}$ exchange, as it was the case for $\mathcal{N}=4$ SYM.

Let us now consider the OPE at genus zero. Analogously to what we did in the first part, we need to consider the $d \times d$ matrix of correlators [33],

$$\delta_{p_1p_3}\delta_{p_2p_4} G^{\pm}_{\text{disc},\vec{p}}\big|_{\text{long}} + \frac{1}{N}\mathcal{P}(x-y)(\bar{x}-y)\mathcal{A}^{\pm}_{\vec{p}} \tag{10.3.19}$$

with the pairs (p_1,p_2) and (p_3,p_4) running over the same $R_{\vec{\tau}}$. Here, we denote by $\mathcal{A}^{\pm}_{\vec{p}}$

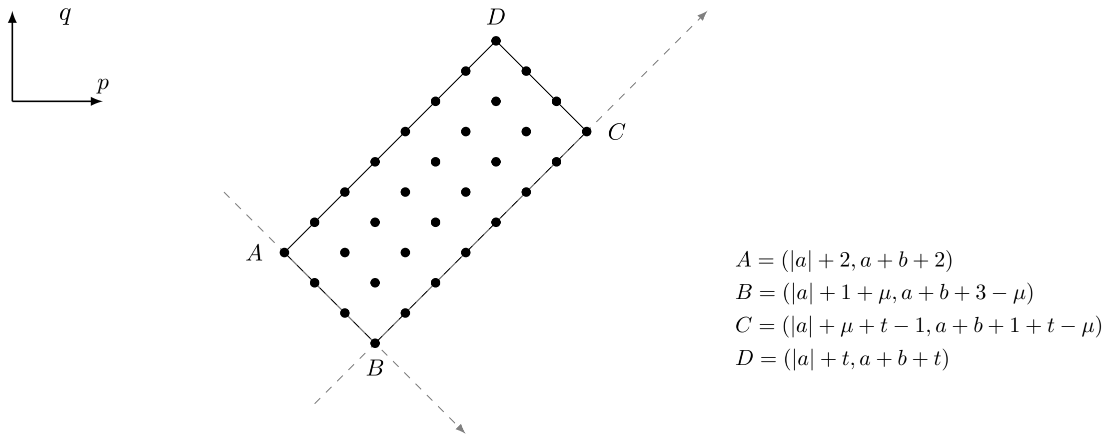


Figure 10.2: A rectangle of degenerate operators for $AdS_5 \times S^3$ with $\mu = 4, t = 9$.

the inverse Mellin transform of the following Mellin amplitudes[4],

$$\begin{aligned}\mathcal{M}^{\pm}_{\vec{p}} &= 2\big(\mathcal{M}_{\vec{p}}(1,2,3,4) \pm \mathcal{M}_{\vec{p}}(1,3,4,2)\big) \\ &= 2\frac{1}{\mathbf{s}+1}\left(\frac{1}{\mathbf{t}+1} \pm \frac{1}{\mathbf{u}+1}\right).\end{aligned} \tag{10.3.20}$$

The OPE equations then read

$$\begin{aligned}\mathbf{C}^{\pm}_{\vec{\tau}}\mathbf{C}^{\pm^T}_{\vec{\tau}} &= \mathbf{L}^{\pm}_{\vec{\tau}}\,, \\ \mathbf{C}^{\pm}_{\vec{\tau}}\boldsymbol{\eta}^{\pm}_{\vec{\tau}}\mathbf{C}^{\pm^T}_{\vec{\tau}} &= \mathbf{M}^{\pm}_{\vec{\tau}}\,.\end{aligned} \tag{10.3.21}$$

Here, $\mathbf{L}^{\pm}_{\vec{\tau}}$ is a (diagonal) matrix of CPW coefficients of disconnected free theory defined by (10.2.10), while $\mathbf{M}^{\pm}_{\vec{\tau}}$ is a matrix of CPW coefficients of the $\log U$ discontinuity of $\mathcal{A}^{\pm}_{\vec{p}}$

$$\mathcal{P}(x-y)(\bar{x}-y)\mathcal{A}^{\pm}_{\vec{p}}\big|_{\log\,U} = \sum_{\vec{\tau}} M^{\pm}_{\vec{\tau}}\mathbb{L}_{\vec{\tau}}. \tag{10.3.22}$$

Note that there here we have two sets of equations, in correspondence with symmetric and antisymmetric channels. Finally, $\boldsymbol{\eta}^{\pm}_{\vec{\tau}}$ is a diagonal matrix of anomalous dimensions and $\mathbf{C}^{\pm}_{\vec{\tau}} = \langle \mathcal{O}_p\mathcal{O}_q\mathcal{K}^{\pm}_{rs}\rangle$ is a matrix of three-point functions with two half-BPS and one double-trace operator. Here, we denote with $\mathcal{K}^{\pm}_{rs}$ the true two-particle operator in interacting theory, that differs by $\mathcal{O}^{\pm}_{pq}$, precisely because there is mixing. Note that, since $\mathcal{A}^{\pm}_{\vec{p}}$ can be written as a function of $\tilde{U}$ and $\tilde{V}$, the $SU(2)_L \times SU(2)_R$ representations contributing to $\mathbf{M}^{\pm}_{\vec{\tau}}$ can be reorganised into $SO(4)$ representations, while this is not possible for the disconnected contribution $\mathbf{L}^{\pm}_{\vec{\tau}}$ that is not symmetric under $y \leftrightarrow \bar{y}$ exchange. This will have precise implication on the symmetry properties of the corresponding CFT data, as we will see.

[4]The factor of 2 is a convention but the choice is really arbitrary because the overall constant depends on the gauge group G_F and the representation exchanged.

10.3.1 Some unmixing examples

The anomalous dimensions are the eigenvalues of the matrix $\mathbf{M}_{\vec{\tau}}\left(\mathbf{L}^{\pm}_{\vec{\tau}}\right)^{-1}$. Following the procedure outlined in the first part for $\mathcal{N}=4$, we will first compute the matrix for many quantum numbers and then try to spot a pattern. Before giving the general formula, it is instructive to go through some explicit examples. For concreteness, we will focus on the symmetric representations only, the computation for the antisymmetric being equivalent. We begin with the easier representation, [00], i.e. the singlet. At twist 4, there is only one long operator for any l, therefore there is no actual unmixing to do. For the first few spins[5] we get

$$\begin{aligned}
\mathbf{L}^{+}_{\tau=4,l=0,[00]} &= \frac{2}{3}, & \mathbf{M}^{+}_{\tau=4,l=0,[00]} &= -\frac{2}{3},\\
\mathbf{L}^{+}_{\tau=4,l=2,[00]} &= \frac{9}{35}, & \mathbf{M}^{+}_{\tau=4,l=2,[00]} &= -\frac{2}{35},\\
\mathbf{L}^{+}_{\tau=4,l=4,[00]} &= \frac{10}{231}, & \mathbf{M}^{+}_{\tau=4,l=4,[00]} &= -\frac{1}{231}.
\end{aligned} \tag{10.3.23}$$

By computing the matrices for many spins we find that the corresponding anomalous dimensions follow the pattern

$$\eta^{+}_{22}\big|_{\tau=4,l,[00]} = \texttt{Eigenvalues}[\mathbf{M}^{+}_{\vec{\tau}}(\mathbf{L}^{+}_{\vec{\tau}})^{-1}] = -\frac{4}{(l+1)(l+4)} \tag{10.3.24}$$

where the subscript in η^{+}_{22} refers to the pair pq which labels the double-trace operator exchanged. Note that in this case the rectangle degenerates to a segment whose length depends on the twist.

Let us now show a less trivial case, where the unmixing matrix is not 1×1. At $\tau=6, l=0$ we have

$$\mathbf{M}^{+}_{\tau=6,l=0,[00]} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{11}{10} \end{pmatrix}, \qquad \mathbf{L}^{+}_{\tau=6,l=0,[00]} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \tag{10.3.25}$$

which gives the following anomalous dimensions

$$(\eta^{+}_{22},\eta^{+}_{33})\big|_{\tau=6,0,[00]} = \texttt{Eigenvalues}[\mathbf{M}^{+}_{\vec{\tau}}(\mathbf{L}^{+}_{\vec{\tau}})^{-1}] == \left(-6,-\frac{2}{5}\right). \tag{10.3.26}$$

As another example, consider the channel [02]. In this case there is already mixing at lowest available twist, i.e. $\tau=6$. The correlators we need to consider are $\mathcal{A}_{2424}$, $\mathcal{A}_{3333}$, $\mathcal{A}_{3324}$. We get

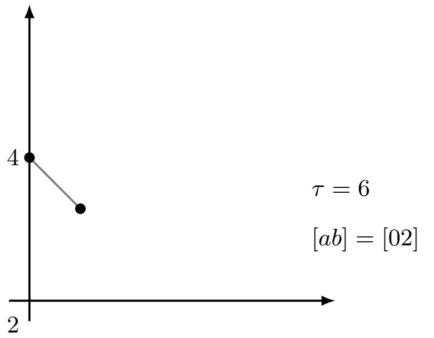
$$\mathbf{M}^{+}_{\tau=6,l=0,[02]} = \begin{pmatrix} -\frac{6}{5} & \frac{8}{5} \\ \frac{8}{5} & -\frac{14}{5} \end{pmatrix}, \qquad \mathbf{L}^{+}_{\tau=6,l=0,[02]} = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & \frac{9}{5} \end{pmatrix} \tag{10.3.27}$$

[5]Note that, by crossing symmetry, in the singlet only even spins can be exchanged. For the antisymmetric amplitude we have the opposite, i.e. only odd spins can be exchanged.

which gives

$$(\eta^+_{24}, \eta^+_{33})\big|_{\tau=6,0,[02]} = \texttt{Eigenvalues}[\mathbf{M}^+_{\vec{\tau}}(\mathbf{L}^+_{\vec{\tau}})^{-1}] = \left(-\frac{10}{3}, -\frac{2}{9}\right). \tag{10.3.28}$$

Note that also in this case the rectangle collapses to a line, like in the figure below.



(10.3.29)

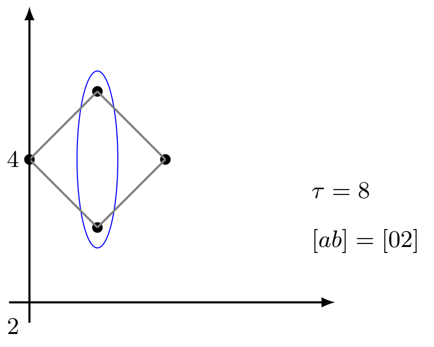
As a last example, let us show a case when there is degeneracy, for example in the [02] channel at $\tau = 8$. This time the matrices are 4×4,

$$\mathbf{M}^+_{\tau=8,l=0,[02]} = \begin{pmatrix} -\frac{16}{21} & \frac{8}{7} & \frac{8}{7} & \frac{12}{7} \\ -\frac{8}{7} & -\frac{129}{56} & \frac{12}{7} & -\frac{22}{7} \\ -\frac{8}{7} & \frac{12}{7} & -\frac{72}{35} & \frac{104}{35} \\ -\frac{12}{7} & \frac{-22}{7} & \frac{104}{35} & -\frac{228}{35} \end{pmatrix}, \qquad \mathbf{L}^+_{\tau=8,l=0,[02]} = \begin{pmatrix} \frac{4}{21} & 0 & 0 & 0 \\ 0 & \frac{3}{8} & 0 & 0 \\ 0 & 0 & \frac{24}{35} & 0 \\ 0 & 0 & 0 & \frac{8}{5} \end{pmatrix}, \tag{10.3.30}$$

and we get

$$(\eta^+_{24}, \eta^+_{33}, \eta^+_{35}, \eta^+_{44})\big|_{\tau=8,0,[02]} = \texttt{Eigenvalues}[\mathbf{M}^+_{\vec{\tau}}(\mathbf{L}^+_{\vec{\tau}})^{-1}] == \left(-15, -1, -1, -\frac{3}{14}\right). \tag{10.3.31}$$

Note that two anomalous dimensions are equal, i.e. operators with the same p remain degenerate at this order in $1/N$. The figure below shows the rectangle for this case; the operators that remain degenerate are circled in blue.



(10.3.32)

10.3.2 All anomalous dimensions in $AdS_5 \times S^3$

By computing the matrices for various quantum numbers, we find that the corresponding anomalous dimensions follow the pattern

$$\eta^{\pm}_{\vec{\tau}} = -\frac{2}{N}\frac{\delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},j}}{(l^{\pm}_{8d}+1)_4} \tag{10.3.33}$$

where l_{8d} is

$$l^{\pm}_{8d} = l + 2(p-2) + \frac{1 \mp (-1)^{a+l}}{2} - |a|\,, \tag{10.3.34}$$

and can be interpreted as a sort of effective $8d$ spin. As in $\mathcal{N}=4$ SYM, the definition is suggested by the partial wave decomposition of the flat amplitude in $8d$ [31].

Note that, as we anticipated, (10.3.33) only depends on p, not q, or in other words, operators on the same vertical line in the rectangle will acquire the same anomalous dimensions. We stress once again that these are the anomalous dimensions associated to the double-trace operators exchanged in the amplitudes $\mathcal{M}^{\pm}_{\vec{p}}$: the gauge group enters the anomalous dimensions only through an overall constant which does not play any significant role in the computation.

Another peculiar feature of these anomalous dimensions is that the denominator only depends on the absolute value of $|j-\bar{j}| = |a|$. This is because, in a sense, the denominator directly descends from the amplitude, which is symmetric under $y \leftrightarrow \bar{y}$ exchange. On the other hand, the numerator only depends on one label, j, and breaks the $j, \bar{j}$ symmetry. This was not case for $\mathcal{N}=4$ SYM, where the theory is invariant under $y \leftrightarrow \bar{y}$ at all orders.

As a comparison, in $\mathcal{N}=4$ SYM the anomalous dimensions are given by (6.3.21) which we recall here for convenience,

$$\eta_{\vec{\tau}} = -\frac{2}{N^2}\frac{\delta^{(4)}\delta^{(4)}}{(l_{10d}+1)_6}\,, \tag{10.3.35}$$

where

$$\delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j} \equiv \delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},\bar{j}}\delta^{(2)}_{h,\bar{j}}\delta^{(2)}_{\bar{h},j}. \tag{10.3.36}$$

We can see that the numerator is doubled with respect to the $AdS_5 \times S^3$ case, as a consequence of the fact that supersymmetry is also doubled. Finally, it is worth pointing out that the object $\delta^{(2)}_{h,\bar{j}}$ appearing ubiquitously is, perhaps with no much surprise, nothing but the anomalous dimension of the two-derivative sector in $AdS_2 \times S^2$ [62].

We conclude the chapter by commenting on the flavour structure of the correlator. One way to deal with it is to decompose t, u channel flavour structures (of both disconnected and tree-level correlators) in a basis of representations appearing in the tensor product of two adjoint representations in the s channel. We then read off the coefficients associated to each flavour structure which are of the form

$$
\begin{aligned}
G_a^{I_1I_2I_3I_4} &\propto G_t^{I_1I_2I_3I_4} + G_u^{I_1I_2I_3I_4} \qquad && a \in \text{symm} \\
G_a^{I_1I_2I_3I_4} &\propto G_t^{I_1I_2I_3I_4} - G_u^{I_1I_2I_3I_4} \qquad && a \in \text{anti}
\end{aligned}
$$

where a runs over all symmetric (antisymmetric) representations in $\texttt{adj} \otimes \texttt{adj}$ with the proportionality coefficient depending on the specific group as well as the exchanged representation. Examples of such coefficients are given in [69]. The unmixing procedure can then be consistently carried for each a separately. For the symmetric (antisymmetric) representations the relevant double-trace operators exchanged are of the type $\mathcal{O}^+_{pq}$ ($\mathcal{O}^-_{pq}$) with the respective anomalous dimensions proportional to $\eta^+_{\vec{\tau}}$ ($\eta^-_{\vec{\tau}}$). Actually, it turns out that the only antisymmetric representation exchanged is the adjoint itself.

Chapter 11

The D1-D5 system

The last theory we examine in this thesis is the tensor multiplet sector in the so-called D1-D5 CFT. This is a $2d$ $\mathcal{N} = (4,4)$ SCFT dual to string theory on $AdS_3 \times S^3 \times M$, where M is either $K3$ or T^4. In [64], it was conjectured that the four-point function of the tensor multiplet sector of the theory enjoys an hidden $6d$ conformal symmetry. Despite a formal proof is lacking, the results are consistent with computations obtained with other independent methods [63–65]. From the lesson learned in $\mathcal{N} = 4$ SYM, we expect the correlator to depend on suitable bold-face variables only. This will in fact be the case.

The rest of the chapter is organised as follows. We start by reviewing the $AdS_3 \times S^3$ solution of IIB supergravity; in section 11.2 we discuss kinematics of four-point functions in the D1-D5 CFT. Then, in section 11.3 we present the four-point function of tensor multiplets written in the large p formalism.

11.1 Generalities

Consider type IIB string theory on $\mathbb{R}^{1,4} \times S^1 \times M$, where we take the directions 1234 to be flat, and M is either $K3$ or T^4. Consider now a stack of N_1 $D1$-branes and N_5 $D5$-branes, with the $D1$ wrapping S^1 and the $D5$ wrapping $S^1 \times M$ as shown in the table below.

	0	1	2	3	4	5	6	7	8	9
D1	•	-	-	-	-	•	-	-	-	-
D5	•	-	-	-	-	•	•	•	•	•

At low energies, and when the size of M is small compared to S^1, the system is described by a $2d$ $\mathcal{N} = (4,4)$ SCFT. On the other hand, when $N \equiv N_1 N_5 \gg 1$, the near horizon limit of the supergravity solution corresponding to this system is $AdS_3 \times S^3 \times M$. A compactification on M gives rise to $6d$ $(2,0)$ SUGRA on $AdS_3 \times S^3$ coupled to n tensor multiplets[1]. This led Maldacena [18] to conjecture that $6d$ $(2,0)$ SUGRA on $AdS_3 \times S^3$ coupled to n tensor multiplets is dual to a $2d$ $\mathcal{N} = (4,4)$ SCFT.

The spectrum after Kaluza-Klein reduction on S^3 consists of two infinite towers of Kaluza Klein modes, one associated to the graviton and one to the tensor multiplet. We will focus on the latter. The dual operators of the tensor multiplet are half-BPS operators which carry a flavour index as well as $SO(4) \sim SU(2)_{\rm L} \times SU(2)_{\rm R}$ R-symmetry indices. As done previously, let us deal with the R-symmetry indices by contracting them with null vectors, by defining

$$\mathcal{O}^I_p(z,\bar{z},\eta,\bar{\eta}) \equiv \mathcal{O}_p^{I;a_1a_2\ldots a_p;\bar{a}_1\bar{a}_2\ldots\bar{a}_p}\eta_{a_1}\ldots\eta_{a_p}\bar{\eta}_{\bar{a}_1}\ldots\bar{\eta}_{\bar{a}_p}\,. \tag{11.1.1}$$

These operators transform under the $(\frac{p}{2},\frac{p}{2})$ representation of the R-symmetry group and have (protected) conformal dimension $(\frac{p}{2},\frac{p}{2})$.

Notice that, since the CFT is 2-dimensional, coordinates split into holomorphic (z,η) and antiholomorphic $(\bar{z},\bar{\eta})$ components.

We will denote the four point function of these half-BPS operators by

$$\mathcal{G}^{I_1I_2I_3I_4}_{\vec{p}} = \langle \mathcal{O}^{I_1}_{p_1}\mathcal{O}^{I_2}_{p_2}\mathcal{O}^{I_3}_{p_3}\mathcal{O}^{I_4}_{p_4}\rangle. \tag{11.1.2}$$

This four-point function contains a disconnected as well as a tree-level contribution

$$G^{I_1I_2I_3I_4}_{\vec{p}} = G^{I_1I_2I_3I_4}_{\text{disc},\vec{p}} + \frac{1}{N}G^{I_1I_2I_3I_4}_{\text{tree-tensor},\vec{p}} + \cdots \tag{11.1.3}$$

In the next section we focus on the tree-level contribution.

11.2 Kinematics in $AdS_3 \times S^3$

Let us first deal with the kinematics. We define the propagator via

$$g_{ij} = \frac{y^2_{ij}}{z^2_{ij}} \tag{11.2.4}$$

[1]To be precise, cancellation of anomalies implies $n = 5$ for $M = T^4$ and $n = 21$ for $M = K3$, but we will keep it generic because it does not play any role in the computations we consider in this thesis.

where $z_{ij}^2 = (z_i - z_j)(\bar{z}_i - \bar{z}_j)$, $y_{ij}^2 = \langle \eta_i \eta_j \rangle \langle \bar{\eta}_i \bar{\eta}_j \rangle$ with $\langle \eta_i \eta_j \rangle = \eta_{ia} \eta_{jb} \epsilon^{ab}$ and similarly $\langle \bar{\eta}_i \bar{\eta}_j \rangle = \bar{\eta}_{i\bar{a}} \bar{\eta}_{j\bar{b}} \epsilon^{\bar{a}\bar{b}}$. Then, the cross-ratios are defined via

$$\frac{z_{12}^2 z_{34}^2}{x_{13}^2 z_{24}^2} = U = x\bar{x}\,, \qquad \frac{z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2} = V = (1-x)(1-\bar{x})\,,$$
$$\frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} = \tilde{U} = y\bar{y}\,, \qquad \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2} = \tilde{V} = (1-y)(1-\bar{y})\,.$$

Note that both spacetime and internal cross ratios naturally split into an holomorphic and an antiholomorphic part. In fact, we have

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\,, \qquad \bar{x} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_4 - \bar{z}_1)(\bar{z}_2 - \bar{z}_3)}\,, \tag{11.2.5}$$

$$y = \frac{\langle \eta_1 \eta_2 \rangle \langle \eta_3 \eta_4 \rangle}{\langle \eta_1 \eta_3 \rangle \langle \eta_2 \eta_4 \rangle}\,, \qquad \bar{y} = \frac{\langle \bar{\eta}_1 \bar{\eta}_2 \rangle \langle \bar{\eta}_3 \bar{\eta}_4 \rangle}{\langle \bar{\eta}_1 \bar{\eta}_3 \rangle \langle \bar{\eta}_2 \bar{\eta}_4 \rangle}\,. \tag{11.2.6}$$

The Ward identities imply that we can break the correlator into a protected and an unprotected sector, each separately respecting crossing symmetry[2],

$$G_{\text{tree-tensor},\vec{p}} = G_{0,\vec{p}} + \mathcal{P}\,\mathcal{I}\,\mathcal{A}_{\vec{p}} \tag{11.2.7}$$

where we remind that the term $G_{0,\vec{p}}$ contains all contributions due to protected multiplets at this order in $1/N$, while the second term contains the logarithmic discontinuities which arise due to two-particle operators receiving anomalous dimensions. The factors $\mathcal{P}$ and $\mathcal{I}$ in the case of $AdS_3 \times S^3$ are given by

$$\mathcal{P} \equiv g_{12}^{k_s} g_{14}^{k_t} g_{24}^{k_u} \left(g_{13} g_{24}\right)^{p_3}, \qquad \mathcal{I} = (x-y)(\bar{x}-\bar{y})\,, \tag{11.2.8}$$

where, as usual,

$$k_s = \frac{p_1 + p_2 - p_3 - p_4}{2}, \qquad k_t = \frac{p_1 + p_4 - p_2 - p_3}{2}, \qquad k_u = \frac{p_2 + p_4 - p_3 - p_1}{2}. \tag{11.2.9}$$

At this point, we should note one important difference with respect to the theories we discussed previously. In this $2d$ CFT, the correlator is only invariant under the *simultaneous* exchange $(x, y) \leftrightarrow (\bar{x}, \bar{y})$. Therefore, expressing the correlator in terms of cross ratios is in general ambiguous, due to the presence of square roots. We can however define a symmetric and an antisymmetric part in the following way,

$$\mathcal{A}_{\vec{p}} = \mathcal{A}_{\vec{p}}^{\mathbb{S}}(U, V, \tilde{U}, \tilde{V}) + (x-\bar{x})(y-\bar{y})\mathcal{A}_{\vec{p}}^{\mathbb{A}}(U, V, \tilde{U}, \tilde{V}) \tag{11.2.10}$$

where the factor $(x-\bar{x})(y-\bar{y})$ should not be confused with $\mathcal{I}$ and is necessary in order for $\mathcal{A}_{\vec{p}}^{\mathbb{A}}$ to be antisymmetric under $x \leftrightarrow \bar{x}$ (and $y \leftrightarrow \bar{y}$) exchange.

[2]We will sometimes drop the flavour indices I_i to simplify the notation.

Explicit computations [64, 65] have shown that $\mathcal{A}_{\vec{p}}^{\mathbb{A}}$ is zero for different values of the charges. However, a proof that this is true for all charges is still lacking. Importantly, in the next section we will see that *if* the correlators enjoy an hidden conformal symmetry, *then* $\mathcal{A}_{\vec{p}}^{\mathbb{A}}(U, V, \tilde{U}, \tilde{V})$ is necessarily zero.

It is now time to define an $AdS_3 \times S^3$ Mellin representation for $\mathcal{A}_{\vec{p}}^{\mathbb{S}}$.

11.3 Four-point function of tensor multiplets

In [64], the authors conjecture that the tree-level dynamics of these tensor multiplets is controlled by a $6d$ hidden symmetry. Therefore, we expect the correlator to admit a generalised Mellin representation, such that the associated Mellin amplitude depends on bold-face variables only. In fact, it is easy to show that the following representation for $\mathcal{A}_{\vec{p}}^{\mathbb{S}}$ is consistent with the results of [64, 65]:

$$\mathcal{A}_{\vec{p}}^{\mathbb{S}} = -\oint dsdt \sum_{\tilde{s},\tilde{t},\tilde{u}} U^s V^t \tilde{U}^{\tilde{s}} \tilde{V}^{\tilde{t}} \Big(\Gamma_{\otimes} \times \mathcal{M}_{\vec{p}}(s, \tilde{s}, \ldots) \Big), \tag{11.3.11}$$

$$\Gamma_{\otimes} = \frac{\Gamma[-s]\Gamma[-s+k_s]}{\Gamma[1+\tilde{s}]\Gamma[1+\tilde{s}+k_s]} \frac{\Gamma[-t]\Gamma[-t+k_t]}{\Gamma[1+\tilde{t}]\Gamma[1+\tilde{t}+k_t]} \frac{\Gamma[-u]\Gamma[-u+k_u]}{\Gamma[1+\tilde{u}]\Gamma[1+\tilde{u}+k_u]}. \tag{11.3.12}$$

Again, the sum (11.3.11) is restricted to the triangle $\tilde{s} \geq -min(0, k_s)$, $\tilde{t} \geq -min(0, k_t)$, $\tilde{u} \geq -min(0, k_u)$ due to the Γ function in the denominator of $\Gamma_{\otimes}$.

In this formalism, the correlator takes the following very compact form

$$\mathcal{M} = \frac{\delta^{12}\delta^{34}}{\mathbf{s}+1} + \frac{\delta^{14}\delta^{23}}{\mathbf{t}+1} + \frac{\delta^{13}\delta^{24}}{\mathbf{u}+1} \tag{11.3.13}$$

where the $\delta^{ij} \equiv \delta^{I_i I_j}$ are the n dimensional Kronecker deltas referred to the flavour indices I. It is easy to see that $\mathcal{M}$ manifestly respects the large p limit. In fact, the four-point scattering of tensor multiplets of $6d$ $(2,0)$ supergravity in flat space reads [141]:

$$\mathcal{A}_{\text{flat}} = \frac{\delta^{12}\delta^{34}}{s} + \frac{\delta^{14}\delta^{23}}{t} + \frac{\delta^{13}\delta^{24}}{u} \tag{11.3.14}$$

where s, t, u are the Mandelstam variables. At large p, $\mathcal{M}$ approaches $\mathcal{A}$ with the Mandelstam replaced by the bold-face variables.

What about $\mathcal{A}_{\vec{p}}^{\mathbb{A}}$? Note that the correlator $\mathcal{A}_{1111}$ only contains[3] $\mathcal{A}_{1111}^{\mathbb{S}}$. Now, as we saw already in $AdS_5 \times S^5$ and $AdS_5 \times S^3$, thanks to the hidden symmetry, we can promote the correlator with lowest charges to a generating function. This means that if we assume the existence of the hidden symmetry, then $\mathcal{A}_{\vec{p}}^{\mathbb{A}}$ is necessarily zero because it is generated

[3] In fact, $\mathcal{A}_{1111}$ is a degree zero polynomial in $y, \bar{y}$ and therefore there is no room for $\mathcal{A}_{1111}^{\mathbb{A}}$ since by construction is always accompanied by the prefactor $(x - \bar{x})(y - \bar{y})$.

from $\mathcal{A}_{1111}$ which has $\mathcal{A}^{\mathbb{A}}_{1111} = 0$. In this case, the differential operator that generates all Kaluza-Klein amplitudes is given by

$$\mathcal{A}_{\vec{p}}^{I_1 I_2 I_3 I_4} = \widehat{\mathcal{D}}_{\vec{p}}\Big[U^2 \mathcal{A}_{2222}^{I_1 I_2 I_3 I_4}\Big] \tag{11.3.15}$$

where

$$\widehat{\mathcal{D}}_{\vec{p}} = U^{-2} \sum_{\tilde{s},\tilde{t}} \left(\frac{\tilde{U}}{U}\right)^{\tilde{s}} \left(\frac{\tilde{V}}{V}\right)^{\tilde{t}} \widehat{\mathcal{D}}^{(0,0,0)}_{\vec{p},\tilde{s},\tilde{t}} \widehat{\mathcal{D}}^{(k_s,k_t,k_u)}_{\vec{p},\tilde{s},\tilde{t}} \tag{11.3.16}$$

and

$$\widehat{\mathcal{D}}^{(a,b,c)}_{\vec{p},\tilde{s},\tilde{t}} = \frac{(U\partial_U - 1 - \tilde{s} - a)_{\tilde{s}+a}}{(-)^a(\tilde{s}+a)!} \frac{(V\partial_V + 1 - \tilde{t} - b)_{\tilde{t}+b}}{(-)^b(\tilde{t}+b)!} \frac{(U\partial_U + V\partial_V)_{\tilde{u}+c}}{(\tilde{u}+c)!} \tag{11.3.17}$$

The proof that this operator generates the correct tree-amplitude for all KK modes is analogous to the one we showed for the $\mathcal{N} = 4$ case and we will skip it.

Chapter 12

The double-trace spectrum in $AdS_3 \times S^3$

With the correlator in our hand, we can now attack the mixing problem in $AdS_3 \times S^3$ in the supergravity limit. Before doing that, we need the long superblocks for this theory, which can be straightforwardly computed using the method of [107]. We recall them in section 12.1; for the explicit construction we refer to the appendix D and to [4]. Then, in section 12.2, after warming up with some explicit examples, we provide the general unmixing formulas for $AdS_3 \times S^3$.

12.1 $\mathcal{N} = (4, 4)$ superconformal symmetry and long superblocks

Our goal is to study the double-trace spectrum at large N. As we saw, a necessary ingredient are the long superblocks, which capture the contribution from all unprotected operators.

The dual conformal field theory of this system has $\mathcal{N} = (4, 4)$ superconformal symmetry in $2d$, and the relevant superconformal blocks belong to the product $(1, 1)\times\overline{(1, 1)}$, where the notation $(1, 1)$ refers to superconformal blocks of $SU(1, 1|2)$, which we can borrow from [107]. These superconformal blocks are labelled by a Young diagram $\underline{\lambda} = [\lambda, 1^{\mu}]$ with at most one row and one column, of length λ and $\mu + 1$ respectively, as shown in figure 12.1. Then, the blocks of the $(4, 4)$ theory are obtained by taking the product of two such diagrams. We refer to the appendix for the explicit construction of these blocks. Here we just give the result for the long superblocks needed in this chapter. The long blocks are labelled by four quantum numbers, two for the conformal representations and two for the internal. As for the correlator (11.2.10), they can be decomposed into a

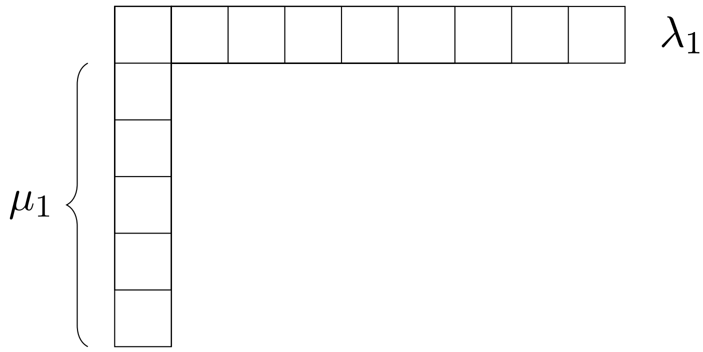


Figure 12.1: An allowed Young diagram for $SU(1,1|1)$ long reps.

symmetric and an antisymmetric part:

$$\mathbb{L}_{\vec{\tau}} = \mathbb{L}^{\mathbb{S}}_{\vec{\tau}} + (x-\bar{x})(y-\bar{y})\mathbb{L}^{\mathbb{A}}_{\vec{\tau}}. \tag{12.1.1}$$

We just need $\mathbb{L}^{\mathbb{S}}_{\vec{\tau}}$, since, as we recalled, $\mathcal{A}^{\mathbb{A}}_{\vec{p}}$ happens to be zero for these correlators. In our notation, $\mathbb{L}^{\mathbb{S}}_{\vec{\tau}}$ reads

$$\mathbb{L}^{\mathbb{S}}_{\vec{\tau}} = \mathcal{P}(x-y)(\bar{x}-y)\left(\frac{\tilde{U}}{U}\right)^{p_3} \mathcal{G}_{\tau,l}(x,\bar{x})\mathcal{H}_{b,a}(y,\bar{y})\,, \tag{12.1.2}$$

with

$$\begin{aligned}
\mathcal{G}_{\tau,l}(x,\bar{x}) &= \frac{(-1)^l}{2(1+\delta_{l,0})U^{\frac{1+p_{43}}{2}}}\Big(\mathcal{F}^{+}_{\frac{\tau}{2}+1+l}(x)\mathcal{F}^{+}_{\frac{\tau}{2}+1}(\bar{x}) + \mathcal{F}^{+}_{\frac{\tau}{2}+1}(x)\mathcal{F}^{+}_{\frac{\tau}{2}+1+l}(\bar{x})\Big), \\
\mathcal{H}_{b,a}(y,\bar{y}) &= \frac{1}{2(1+\delta_{a,0})\tilde{U}^{1-\frac{p_{43}}{2}}}\left(\mathcal{F}^{-}_{-\frac{b}{2}-a}(y)\mathcal{F}^{-}_{-\frac{b}{2}}(\bar{y}) + \mathcal{F}^{-}_{-\frac{b}{2}}(y)\mathcal{F}^{-}_{-\frac{b}{2}-a}(\bar{y})\right),
\end{aligned} \tag{12.1.3}$$

where we remind that

$$\mathcal{F}^{\pm}_{h}(x) = x^h {}_2F_1\left[h \mp \tfrac{p_{12}}{2}, h \mp \tfrac{p_{43}}{2}, 2h\right](x). \tag{12.1.4}$$

In common with the other theories, let us also introduce the $\vec{h}$ labels via

$$h = \frac{\tau}{2}+1+l, \quad \bar{h} = \frac{\tau}{2}+1, \quad j = -\frac{b}{2}-a, \quad \bar{j} = -\frac{b}{2}. \tag{12.1.5}$$

Note that $j, \bar{j}$ are nothing but (minus) the spins of the $SU(2)_L \times SU(2)_R$ R-symmetry representations.

A nice surprise, perhaps expect from the fact that $SU(1,1|2)\times SU(1,1|2)$ might contain an $SU(2,2|4)$ factor, comes from $\mathbb{L}^{\mathbb{A}}_{\vec{\tau}}$, which coincides with the long superconformal blocks of $\mathcal{N}=4$ SYM, equation (2.3.21). In appendix D we show this explicitly.

12.2 Unmixing the double-trace spectrum in $AdS_3 \times S^3$

Simiarly to $AdS_5 \times S^3$, we need to split the correlator in irreducible representation of the flavour group, as explained for example in [142, 143]. The unmixing will then be carried for each structure separately. In this case we have three channels, the singlet, $\mathbb{I}$, the symmetric, $+$, and the antisymmetric channel, $-$. Let us then rewrite the correlator according to the irreducible representation of the flavour group:

$$\mathcal{M}_{\vec{p}} = \mathcal{M}^{\text{sing}}_{\vec{p}}\delta^{12}\delta^{34} + \mathcal{M}^{+}_{\vec{p}}\left(\delta^{13}\delta^{24} + \delta^{14}\delta^{23} - \frac{2}{n}\delta^{12}\delta^{34}\right) + \mathcal{M}^{-}_{\vec{p}}\left(\delta^{14}\delta^{23} + \delta^{13}\delta^{24}\right) \tag{12.2.6}$$

where

$$\begin{aligned}
\mathcal{M}^{\text{sing}}_{\vec{p}} &= \frac{1}{n}\left[\frac{1}{\mathbf{t}+1} + \frac{1}{\mathbf{u}+1}\right] + \frac{1}{\mathbf{s}+1}, \\
\mathcal{M}^{+}_{\vec{p}} &= \frac{1}{2}\left[\frac{1}{\mathbf{t}+1} + \frac{1}{\mathbf{u}+1}\right], \\
\mathcal{M}^{-}_{\vec{p}} &= \frac{1}{2}\left[\frac{1}{\mathbf{t}+1} - \frac{1}{\mathbf{u}+1}\right],
\end{aligned} \tag{12.2.7}$$

and analogously for the disconnected correlator.

We will focus on $\mathcal{M}^{\pm}_{\vec{p}}$, since these are closed sectors. In fact, the unmixing in the singlet channel would require other correlators in the $6d$ $(2,0)$ supergravity, namely the ones involving the graviton, which has the same quantum numbers as those of the tensor multiplet and can in principle mix with it[1].

The two-particle operators we want to study are long operators exchanged in $\mathcal{M}^{\pm}_{\vec{p}}$.

$$\mathcal{O}^{\pm}_{pq} = \mathbb{P}^{\pm}_{I_1 I_2}\mathcal{O}^{I_1}_{p}\partial^{l}\square^{\frac{1}{2}(\tau-p-q)}\mathcal{O}^{I_2}_{q} \tag{12.2.8}$$

where $\mathbb{P}^{\pm}_{I_1 I_2}$ projects onto a given flavour representation. For given $SO(2,2)$ quantum numbers τ, l, and $SO(4)$ representation $\mathfrak{R} = [ab]$, the number of degenerate states is organised into the usual rectangle,

$$R_{\vec{\tau}} := \left\{ (p,q) : \begin{array}{l} p = i + a + r \\ q = i + a + b - r \end{array} , \begin{array}{l} i = 1, \ldots, (t-1) \\ r = 0, \ldots, (\mu - 1) \end{array} \right\} \tag{12.2.9}$$

that consists of $d = \mu(t-1)$ allowed lattice points where now

$$t \equiv \frac{(\tau - b)}{2} - a, \quad \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a + l \text{ even}, \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a + l \text{ odd}. \end{cases}$$

We will skip the details because the machinery is completely analogous to the other

[1]These correlators have been recently studied in [66].

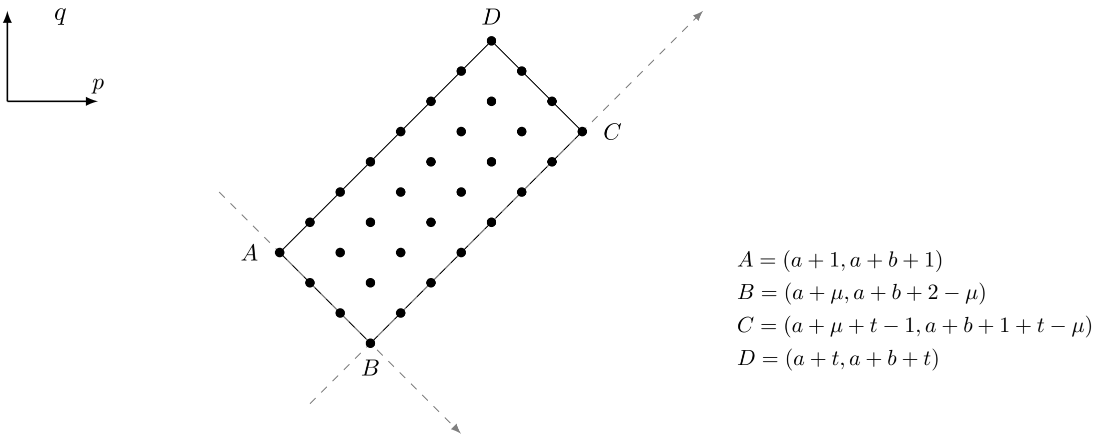


Figure 12.2: A typical rectangle in $AdS_3 \times S^3$.

cases we analysed. In short, per each channel, we need the two sets of matrix equations:

- the order $\sim N^0$ equation
$$\mathbf{C}^{\pm}_{\vec{\tau}} \mathbf{C}^{\pm\,T}_{\vec{\tau}} = \mathbf{L}^{\pm}_{\vec{\tau}}\,, \tag{12.2.10}$$
where $\mathbf{L}^{\pm}_{\vec{\tau}}$ is the block decomposition of long disconnected free theory and $\mathbf{C}^{\pm}_{\vec{\tau}}$ is the matrix of three-point couplings. $\mathbf{L}^{\pm}_{\vec{\tau}}$ is obviously diagonal since only correlators with pairwise equal charges have a non-zero contribution;

- the order $\sim N^{-1}$ equation
$$\mathbf{C}^{\pm}_{\vec{\tau}} \boldsymbol{\eta}^{\pm}_{\vec{\tau}} \mathbf{C}^{\pm\,T}_{\vec{\tau}} = \mathbf{M}^{\pm}_{\vec{\tau}} \tag{12.2.11}$$
where $\boldsymbol{\eta}^{\pm}_{\vec{\tau}}$ is diagonal and $\mathbf{M}^{\pm}_{\vec{\tau}}$ comes from the decomposition of the $\log U$ part of $\mathcal{M}^{\pm}_{\vec{p}}$.

At this point, normalising $(\mathbf{M}^{\pm}_{\vec{\tau}})^{-1}$ with $(\mathbf{L}^{\pm}_{\vec{\tau}})^{-1}$ from the right yields the *unmixing matrix* whose eigenvalues are the anomalous dimensions.

12.2.1 Unmixing examples

Before presenting the general formulae, let us give some explicit examples. We will discuss the symmetric flavor channel $+$, the antisymmetric $-$ being completely analogous. The simplest representation we can study is $\mathfrak{R} = [00]$. The first case we can look at is the unique two-particle operator[2] at $\tau = 2$ and even spin $l = 0, 2, \ldots 2\mathbb{N}$. This case has no mixing
$$\mathbf{L}^{+}_{\tau=2,l,[00]} = \tfrac{(l+1)!^2}{(2l+2)!}\,, \qquad \mathbf{M}^{+}_{\tau=2,l,[00]} = -2\tfrac{(l+1)!^2}{(2l+2)!}\,.$$

[2]Note that the lowest available twist in this theory is $\tau = 2$ and not $\tau = 4$ as it was the case for $AdS_5 \times S^5$ and $AdS_5 \times S^3$. This is because here the lowest half-BPS operator has $\tau = 1$.

The anomalous dimension are therefore

$$\eta^+_{11}\big|_{\tau=2,l,[00]} = \texttt{Eigenvalues}[\mathbf{M}^+_{\vec{\tau}}(\mathbf{L}^+_{\vec{\tau}})^{-1}] = -2, \tag{12.2.12}$$

Note that, unlike the other theories we discussed, at the lowest available twist, the anomalous dimensions are constant and do not depend on the spin. This feature has also been observed recently in [144].

The first mixing problem is at $\tau = 4$, where we find two even spin operators. The corresponding data is

$$\begin{aligned} \mathbf{L}^+_{\tau=4,l,[00]} &= \tfrac{1}{3}\tfrac{(l+2)!^2}{(2l+4)!} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4}(l+1)(l+4) \end{bmatrix} \\ \mathbf{M}^+_{\tau=4,l,[00]} &= -2\tfrac{(l+2)!^2}{(2l+4)!} \begin{bmatrix} +1 & -1 \\ -1 & \frac{1}{4}(10+5l+l^2) \end{bmatrix} \end{aligned} \tag{12.2.13}$$

which gives

$$(\eta^+_{11}, \eta^+_{22})\big|_{\tau=4,l,[00]} = \texttt{Eigenvalues}[\mathbf{M}^+_{\vec{\tau}}(\mathbf{L}^+_{\vec{\tau}})^{-1}] = \left(-6\frac{l+2}{l+4}, -6\frac{l+3}{l+1}\right). \tag{12.2.14}$$

We remind that the rep [00] is special in the sense that the rectangle $R_{\tau,l,[00]}$ is simply a line whose length is controlled by τ, and all anomalous dimensions are labelled uniquely.

Next, let us consider the rep $\mathfrak{R} = [10]$. This is analogous to [00], but for the fact that only odd spins contribute $l = 1, 3, \ldots, 2\mathbb{N}+1$. The first case is at $\tau = 4$ with one operator,

$$\begin{aligned} \mathbf{L}^+_{\tau=4,l,[10]} &= \tfrac{(l+2)!^2}{(2l+4)!} \times \tfrac{1}{12}(24+25l+5l^2) \\ \mathbf{M}^+_{\tau=4,l,[10]} &= -2\tfrac{(l+2)!^2}{(2l+4)!} \times (l+1)(l+4) \end{aligned} , \tag{12.2.15}$$

and we get

$$\eta^+_{22}\big|_{\tau=4,l,[10]} = \texttt{Eigenvalues}[\mathbf{M}^+_{\vec{\tau}}(\mathbf{L}^+_{\vec{\tau}})^{-1}] = -24\frac{(l+1)(l+4)}{24+25l+5l^2}. \tag{12.2.16}$$

Analogously, at $\tau = 6$ we have two operators, and therefore the unmixing matrix is 2×2.

We close the list of examples by illustrating a mixing problem with partial degeneracy. The simplest case of partial degeneracy appears in $\mathfrak{R} = [02]$, even spins $l = 0, 2, \ldots 2\mathbb{N}$ and $\tau = 6$, which is the second available twist in the [02] rep[3]; the rectangle $R_{\tau=6,l,[02]}$ in this case consists of four points. The CFT data we are interested in to see the partial degeneracy is

$$\mathbf{L}^+_{\tau=6,l,[02]} = \tfrac{(l+3)!(l+4)!}{2(2l+6)!}\texttt{Diag}\left(\tfrac{2(l+4)}{15}, \tfrac{3(l+1)(l+4)(l+6)}{160}, \tfrac{3(l+3)}{5}, \tfrac{(l+1)(l+3)(l+6)}{10}\right), \tag{12.2.17}$$

[3] This rep also allows for odd spins, but there is no degeneracy for odd spins, so it is not conceptually different from the cases we showed already.

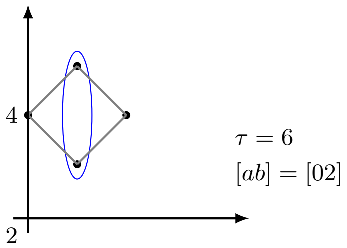
and

$$\mathbf{M}^{+}_{\tau=6,l,[02]} = \frac{(l+3)!(l+4)!}{(2l+6)!} \times \begin{pmatrix} \frac{2(15+4L^2)}{3(-1+2L)} & -\frac{23+4L^2}{2(-1+2L)} & 8 & -8 \\ -\frac{23+4L^2}{2(-1+2L)} & \frac{1715+40L^2+48L^4}{128(-1+2L)} & -8 & \frac{55+4L^2}{8} \\ 8 & -8 & \frac{5+12L^2}{1+2L} & -\frac{2(7+4L^2)}{1+2L} \\ -8 & \frac{55+4L^2}{8} & -\frac{2(7+4L^2)}{1+2L} & \frac{265-40L^2+16L^4}{8(1+2L)} \end{pmatrix}, \tag{12.2.18}$$

that gives

$$\eta^{+}_{\tau=6,l,[02]} = \texttt{Eigenvalues}[\mathbf{M}^{+}_{\vec{\tau}}(\mathbf{L}^{+}_{\vec{\tau}})^{-1}] = -10\left(\frac{(l+5)}{l+1}, \frac{(l+2)(l+5)}{(l+3)(l+4)}, \frac{(l+2)(l+5)}{(l+3)(l+4)}, \frac{(l+2)}{(l+6)}\right), \tag{12.2.19}$$

where we have defined $L = l + \frac{7}{2}$ for convenience. The figure below shows the rectangle for this case.



(12.2.20)

The leftmost root is indexed, in $R_{\tau=6,l,[02]}$, by the leftmost corner at $(pq) = (13)$, then the two (degenerate) middle ones are indexed by $(pq) = (24), (22)$, and the rightmost by the rightmost corner at $(pq) = (33)$.

Finally, when we consider the antisymmetric sector, what happens is that for given $\mathfrak{R}$ even and odd spin sectors are exchanged, but otherwise the mixing problem is the same. For example, the singlet in this case only exists for odd spins, and so on.

We are now ready to present the general formulae for the mixing problem in this theory.

12.2.2 General formulae for $AdS_3 \times S^3$

With the same procedure outlined in the previous chapters for $AdS_5 \times S^5$ and $AdS_5 \times S^3$, we can easily spot a formula for the coefficients of the long part of disconnected free theory. In this case, it takes the following form

$$L^{\pm}_{\vec{\tau}} = -\frac{1 \pm (-1)^{a+l}\delta_{pq}}{pq} A_h A_{\bar{h}} B_j B_{\bar{j}} \boldsymbol{\delta}\,, \tag{12.2.21}$$

with

$$
\begin{aligned}
A_h =& \frac{\Gamma(h+\frac{p-q}{2})\Gamma(h-\frac{p-q}{2})\Gamma(h+\frac{p+q}{2})}{\Gamma(2h-1)\Gamma(h-\frac{p+q}{2})}, \\
B_j =& \frac{\Gamma(2-2j)}{\Gamma(1-j+\frac{p-q}{2})\Gamma(1-j-\frac{p-q}{2})} \frac{1}{\Gamma(\frac{p+q}{2}+j-1)\Gamma(\frac{p+q}{2}-j)},
\end{aligned} \tag{12.2.22}
$$

and $\boldsymbol{\delta}$ this time is given by

$$
\boldsymbol{\delta} = \frac{\delta^{(4)}_{h,\bar{h},j,\bar{j}} + \delta^{(4)}_{h,\bar{h},\bar{j},j}}{\delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j}}, \qquad \delta^{(4)}_{h,\bar{h},j,\bar{j}} \equiv \delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},\bar{j}}, \qquad \delta^{(2)}_{h,j} = (h-j)(h+j-1), \tag{12.2.23}
$$

with the $\vec{h}$ labels related to $\vec{\tau}$ labels via (12.1.5).

Next, we move to the anomalous dimensions. By performing the computation for many cases, we find that they follow the pattern

$$
\eta^{\pm}_{\vec{\tau}} = -\frac{2}{N} \frac{\delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j}}{\delta^{(4)}_{h,\bar{h},j,\bar{j}} + \delta^{(4)}_{h,\bar{h},\bar{j},j}} \frac{1}{(l^{\pm}_{6d}+1)_2} \tag{12.2.24}
$$

where l_{6d} is

$$
l^{\pm}_{6d} = l + 2(p-1) + \frac{1 \mp (-1)^{a+l}}{2} - a\,. \tag{12.2.25}
$$

We end the chapter with a few comments.

- Firstly, as it was the case for the other theories, the anomalous dimensions only depend on one label. This results in a residual degeneracy;
- secondly, following [31], the partial wave decomposition of the flat $6d$ $(2,0)$ amplitude suggests that l_{6d} should be interpreted as a $6d$ spin;
- comparing the anomalous dimensions of the three theories, we see that $AdS_5 \times S^5$ and $AdS_5 \times S^3$ amplitudes are special in the sense that the numerator of $\boldsymbol{\delta}$, which shows up in disconnected free theory, simplifies in the computation and it does not appear in the anomalous dimensions. This cancellation is non-trivial, and, in fact, in $AdS_3 \times S^3$ this does not happen and the anomalous dimensions come with the denominator $\delta^{(4)}_{h,\bar{h},j,\bar{j}} + \delta^{(4)}_{h,\bar{h},\bar{j},j}$. It would be interesting to understand the reason behind this cancellation mechanism.
- Finally, note that the large spin behaviour goes like $-1/l^0$. In $AdS_5 \times S^3$ is $-1/l^1$, while in $AdS_5 \times S^5$ is $-1/l^2$.

Chapter 13

Final act: hidden symmetry across dimensions

In this last chapter we would like to show that many of the results we presented in this thesis can be nicely gathered in compact formulae which go across the different backgrounds. Let us introduce two parameters, θ_1, θ_2, that parametrise the various $AdS_{\theta_1+1} \times S^{\theta_2+1}$ backgrounds.

For future convenience, it will be useful to introduce the combinations

$$\hat{\theta} = \frac{\theta_1 + \theta_2}{2}, \qquad \check{\theta} = \frac{\theta_1 - \theta_2}{2}. \tag{13.0.1}$$

Note that $\hat{\theta}$ is the free scaling dimension of a scalar field[1] in $2\hat{\theta}+2$ spacetime dimensions.

The chapter is organised as follows. In section 13.1 we present the general formulae for disconnected free theory. Then, in section 13.2, we explore the tree-level dynamics of all three theories. Finally, we end the chapter with a more speculative section on higher derivative corrections to tree-level amplitudes. These will be responsible of the breaking of the residual degeneracy in the anomalous dimensions. In particular, we are able to give general formulae for the associated characteristic polynomials which nicely interpolate between the different theories.

13.1 Disconnected free theory across dimensions

Formulae (6.2.13), (10.2.11), (12.2.21) nicely fit into the following expression

$$L^{\pm}_{\vec{\tau}} = -\frac{1 \pm (-1)^{a+l}\delta_{pq}}{(p-\check{\theta})(q-\check{\theta})} A_h A_{\bar{h}} B_j B_{\bar{j}}\, \boldsymbol{\delta}^{(\theta_1,\theta_2)}. \tag{13.1.2}$$

[1]The axi-dilaton!

with

$$A_h = \frac{\Gamma(h+\frac{p-q}{2})\Gamma(h-\frac{p-q}{2})\Gamma(h+\frac{p+q}{2}-\check{\theta})}{\Gamma(2h-1)\Gamma(h-\frac{p+q}{2}-\check{\theta})}, \qquad (13.1.3)$$
$$B_j = \frac{\Gamma(2-2j)}{\Gamma(1-j+\frac{p-q}{2})\Gamma(1-j-\frac{p-q}{2})} \frac{1}{\Gamma(\frac{p+q}{2}+j-1)\Gamma(\frac{p+q}{2}-j)},$$

and

$$\boldsymbol{\delta}^{(2,2)} = \frac{\delta^{(4)}+\bar{\delta}^{(4)}}{\delta^{(4)}\bar{\delta}^{(4)}}, \qquad AdS_3\times S^3,$$
$$\boldsymbol{\delta}^{(4,4)} = \frac{\delta^{(4)}-\bar{\delta}^{(4)}}{\delta^{(4)}\bar{\delta}^{(4)}}, \qquad AdS_5\times S^5,$$
$$\boldsymbol{\delta}^{(4,2)} = \frac{\delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},j}}{\delta^{(2)}_{h,j}-\delta^{(2)}_{\bar{h},j}}, \qquad AdS_5\times S^3. \qquad (13.1.4)$$

where $\delta^{(4)} \equiv \delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},\bar{j}}$ and $\bar{\delta}^{(4)} \equiv \delta^{(2)}_{h,\bar{j}}\delta^{(2)}_{\bar{h},j}$. Here, the general dictionary interpolating between the different theories reads

$$h = \frac{\tau+\theta_2}{2}+l, \qquad \bar{h} = \frac{\tau}{2}+1-\check{\theta},$$
$$j = -\frac{b+\theta_2}{2}-a+1, \qquad \bar{j} = -\frac{b}{2}. \qquad (13.1.5)$$

Note that in $\mathcal{N}=4$ SYM only $L^{+}_{\vec{\tau}}$ is physical, since there is no antisymmetric sector.

We can now appreciate how the formulae written in $\vec{h}$ labels manifest a sort of universality, in the sense that in this basis the θ_1, θ_2, parameters drop out in many coefficients. In fact, the existence of such formulae for disconnected graphs interpolating between different theories turns out to be a particular case of a more general formula for all free-theory diagrams in different theories which can be proved through a Cauchy identity [107, 115]. An explicit formula was however still missing in the literature and the purpose of the last part of this section is to fill the gap.

The general idea is that superconformal blocks in generalised analytic superspace with $SU(m,m|2n)$ supersymmetry manifest a universal structure. In particular, as explained in [107], we can use $SU(m,m|0) = SU(m,m)$ bosonic blocks to compute the coefficients of any free-theory diagram in any theory with $SU(m,m|2n)$ symmetry.

A typical representation for these bosonic blocks is specified by a number, γ and a Young diagram $\underline{\lambda} = [\lambda_1, \dots \lambda_m]$ with at most m rows, as in figure 13.1. Let us thus define $A^{k,m}_{\gamma,[\lambda]}$ to be the coefficients in the superblock expansion of the generic diagram contributing to $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$, as in figure 13.2. In this notation the four point correlator has charges p_1, p_2, p_3, p_4, where they are labelled starting from the lower left corner and continuing

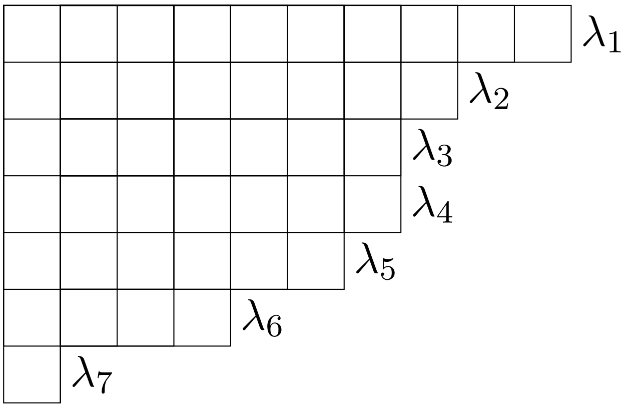
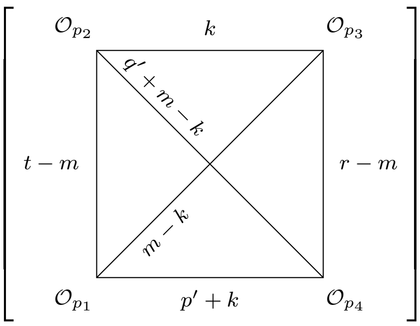


Figure 13.1: A generic $SU(m,m)$ rep.



$$= \sum_{[\underline{\lambda}]} A^{k,m}_{\gamma,[\underline{\lambda}]} \mathbb{S}_{\gamma,[\underline{\lambda}]} \,. \tag{13.1.6}$$

Figure 13.2: The superblock decomposition of a generic free-theory diagram.

clock-wise. Moreover, t is defined via $p_4 = p_1 + p_2 + p_3 - 2t$ and

$$p' = p_1 - t \,, \qquad q' = p_2 - t \,. \tag{13.1.7}$$

The labels on the lines in the diagram in eq. (13.2) denote the powers of the corresponding propagators. Having fixed the external charges (and hence t) we find that the diagram in eq. (13.2) has non-zero coefficients for Young tableaux with up to m rows.

As shown in [107], the coefficients $A^{k,m}_{\gamma,[\underline{\lambda}]}$ obey the following equation in terms of bosonic blocks,

$$\sum_{\lambda_1 \geq \ldots \geq \lambda_m} A^{k,m}_{\gamma,[\lambda_1,\lambda_2,\ldots,\lambda_m]} F^{\alpha\beta\gamma[\lambda_1,\lambda_2,\ldots,\lambda_m]}(x_1,\ldots,x_m) = \left(\frac{1}{(1-x_1)\ldots(1-x_m)} \right)^k \tag{13.1.8}$$

where m is the number of possible rows of the associated Young tableau, γ is the number of propagators going from $(\mathcal{O}_{p_1}, \mathcal{O}_{p_2})$ to $(\mathcal{O}_{p_3}, \mathcal{O}_{p_4})$, and k is the number of propagators connecting $\mathcal{O}_{p_2}$ and $\mathcal{O}_{p_3}$. The parameters α, β and γ are determined as follows,

$$\begin{aligned}\alpha &= p_2 - t + m = q' + m\\ \beta &= m\\ \gamma &= p_1 + p_2 - 2t + 2m = p' + q' + 2m.\end{aligned} \tag{13.1.9}$$

The bosonised m-row conformal blocks are given by:

$$F^{\alpha\beta\gamma[\underline{\lambda}]} = \frac{\det\left[x_i^{\lambda_j+m-j}{}_2F_1(\lambda_j+1-j+\alpha,\lambda_j+1-j+\beta,2\lambda_j+2-2j+\gamma;x_i)\right]_{1\le i,j\le m}}{\det\left(x_i^{m-j}\right)_{1\le i,j\le m}}, \tag{13.1.10}$$

and, moreover, for us $m=\beta$ which means the size of the matrix is determined by the charges in the correlation function and the number of propagators going across in the diagram under consideration. k, for a given value of $m=\beta$, stays in the range:

$$0 \le k \le m. \tag{13.1.11}$$

The coefficients A depend on q', p', β, k and the β variables $\lambda_1, \cdots, \lambda_\beta$. By performing the computation for many cases we find that the coefficients are given by the following compact formula

$$A^{p',q',k,\beta}_{[\lambda_1,\lambda_2,\ldots,\lambda_\beta]} = \left[\prod_{l=1}^{\beta-k}\frac{1}{(\beta-k-l)!(q'+l-1)!}\right]\left[\prod_{l=1}^{k}\frac{1}{(k-l)!(p'+l-1)!}\right]\left[\prod_{l=1}^{\beta}X_l\right] \tag{13.1.12}$$

$$\times \sum_{S_k\subset\{1,\ldots,\beta\}}\left[\prod_{\substack{i<j\\ i,j\in S_k}}Y_{ij}\right]\left[\prod_{\substack{i<j\\ i,j\in \overline{S}_k}}Y_{ij}\right]\left[\prod_{i\in S_k}(\lambda_i-i+\beta+p')!\right]\left[\prod_{i\in\overline{S}_k}(-1)^{\lambda_i}(\lambda_i-i+\beta+q')!\right].$$

Here we have introduced the notation

$$\begin{aligned}X_i &= \frac{(\lambda_i-i+\beta+p'+q')!}{(2\lambda_i-2i+2\beta+p'+q')!},\\ Y_{ij} &= (\lambda_i-\lambda_j-i+j)(\lambda_i+\lambda_j-i-j+2\beta+1+p'+q').\end{aligned} \tag{13.1.13}$$

The sum in (13.1.12) runs over subsets S_k of $\{1,\ldots,\beta\}$ of size k. The complement of S_k in $\{1,\ldots,\beta\}$ is denoted by $\overline{S}_k$.

We remark once again that this formula gives the block-decomposition of *all* free-theory diagrams for *all* theories with $SU(m,m|2n)$ symmetry.

For example, starting from (13.1.12), let us explicitly see how to reproduce the known formulae for disconnected free-theory in the three theories we considered in this thesis. To start with, note that, as it stands, the formula is *bosonic*, in the sense that is associated to a Young tableau with rows of length $\lambda_1, \ldots \lambda_m$ with no restriction on the length of each row[2]. However, the length of the rows depends on the theory: for example in $\mathcal{N}=4$ SYM we can have at most two rows with length greater than 2.

There are only two types of disconnected diagrams[3], and they are present only in correlators with pairwise equal charges. For concreteness we will focus on the u-channel of correlators with all equal charges, the t-channel being completely equivalent. Let us begin with $\mathcal{N}=4$ SYM.

13.1.1 Disconnected free theory in $AdS_5 \times S^5$

In $\mathcal{N}=4$ SYM we have that $\lambda_i \leq 2$ with $i \geq 3$. The u-channel diagram of $\langle \mathcal{O}_p \mathcal{O}_p \mathcal{O}_p \mathcal{O}_p \rangle$ has $t=p, p'=q'=k=0$. Moreover, we remind that

$$\lambda_1 = \frac{\tau-\gamma}{2} + l + 2, \qquad \lambda_2 = \frac{\tau-\gamma}{2} + 2. \tag{13.1.14}$$

The lowest possible charge is $p=2$ for which we have $\frac{\gamma}{2}=\beta=2$. By plugging this in (13.1.12), we precisely get the u-channel of (6.2.13). Analogously, for $p=3$ ($\frac{\gamma}{2}=\beta=3$), we have three rows, with the third row running over the range $\lambda_3 = 0,1,2$. These three values are in correspondence with the three $SU(4)$ representations exchanged in $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$. By plugging these values in (6.2.13) we again get the u-channel of (6.2.13) for the representations [000] ($\lambda_3=2$), [101] ($\lambda_3=1$), [000] ($\lambda_3=0$).

13.1.2 Disconnected free theory in $AdS_5 \times S^3$

Let us now briefly discuss $AdS_5 \times S^3$. Here we expect to reproduce only the dependence on the superconformal variables, i.e. the $h, \bar{h}, j$ dependence in (10.2.11), but not the $\bar{j}$ dependence. This is because, as we stressed in various occasions, $\bar{j}$ is a flavour variable and comes from the decomposition under $SU(2)_L$ spherical harmonics which are not part of the $4d$ $\mathcal{N}=2$ superalgebra. To check that (13.1.12) gives indeed the right coefficients, we just need to remember that in this case Young tableau can have at most one column, therefore $\lambda_i \leq 1$ with $i \geq 3$, and [107]

$$\lambda_1 = \frac{\tau-\gamma}{2} + l + 1, \qquad \lambda_2 = \frac{\tau-\gamma}{2} + 1. \tag{13.1.15}$$

Upon inspection, we see that (13.1.12) and (10.2.11) coincide up to an overall normalisation, as far as the $h, \bar{h}, j$ dependence is concerned.

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[2]There is of course the usual condition for the Young tableau, i.e. $\lambda_1 \geq \ldots \geq \lambda_m$.
[3]There are really three, the other one being the identity.

13.1.3 Disconnected free theory in $AdS_3 \times S^3$

Finally, let us see how to recover $AdS_3 \times S^3$. The blocks of this theory belong to the product $(1,1)\times\overline{(1,1)}$, where $(1,1)$ refers to superconformal blocks of $SU(1,1|2)$. We therefore need to consider the long block decomposition of $(1,1)$. This is captured by (13.1.12) upon taking $\lambda_i \leq 1$ with $i \geq 2$.[4] Re-fitting formula (13.1.12), we find, up to an overall normalisation

$$L^{\pm}_{\vec{\tau}} = A_h B_j \,\boldsymbol{\delta}^{(1,1)} \tag{13.1.16}$$

with

$$\boldsymbol{\delta}^{(1,1)} = \frac{1}{\delta^{(2)}_{h,j}}. \tag{13.1.17}$$

Analogously $\overline{(1,1)}$ gives

$$L^{\pm}_{\vec{\tau}} = A_{\bar{h}} B_{\bar{j}} \,\boldsymbol{\delta}^{\overline{(1,1)}} \tag{13.1.18}$$

with

$$\boldsymbol{\delta}^{\overline{(1,1)}} = \frac{1}{\delta^{(2)}_{\bar{h},\bar{j}}}. \tag{13.1.19}$$

Taking the product of the two we obtain the $(1,1)\times\overline{(1,1)}$ free-theory. Now, to get to the $AdS_3 \times S^3$ ones, we need to symmetrise[5] in $j \leftrightarrow \bar{j}$. Noting that

$$\frac{1}{\delta^{(2)}_{h,j}}\frac{1}{\delta^{(2)}_{\bar{h},\bar{j}}} + \frac{1}{\delta^{(2)}_{h,\bar{j}}}\frac{1}{\delta^{(2)}_{\bar{h},j}} = \frac{\delta^{(4)}_{h,\bar{h},j,\bar{j}} + \delta^{(4)}_{h,\bar{h},\bar{j},j}}{\delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j}} \tag{13.1.20}$$

we get (12.2.21).

Lastly, note that, as it was the case for the long blocks themselves, the antisymmetrisation exactly reproduces (6.2.13). In fact, from these explicit computations we can see that $\mathcal{N}=4$ and $\mathcal{N}=2$ coefficients can be written as 2×2 determinants.

13.2 Hidden symmetry at tree-level: the general formula

Let us now take one step forward in the $1/N$ expansion and look at tree-level correlators. The hidden conformal symmetry at tree level implies that all correlators are obtained by acting with certain operators on a seed function. This seed takes the same form as the correlator for minimal charges, $\mathcal{A}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}}$, which is singlet under the sphere, and the role of the aforementioned operators is to add charge in order to generate $\mathcal{A}_{\vec{p}}$. The purpose of this section is to show that all tree-level correlators are captured by a

[4] In fact, the long representations in this theory are labelled by a diagram with at most one row and one column.

[5] We remind that we are only interested in the symmetric sector of (11.2.10), because the antisymmetric is zero.

compact formula which depends on the two parameters θ_1, θ_2. The general seed is

$$\mathcal{A}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}} = \oint ds dt\, U^s V^t \tilde{U}^{\tilde{s}} \tilde{V}^{\tilde{t}} \times \Gamma[-s]^2 \Gamma[-t]^2 \Gamma[-u]^2 \times \mathcal{M}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}} \tag{13.2.21}$$

where $s+t+u = -\hat{\theta}$ and $\mathcal{M}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}}$ depends on the theory and the channel we consider. Now, note that the following Mellin amplitude

$$\mathcal{M}^{\pm}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}} = \frac{1}{(s+1)_{\hat{\theta}-2}} \left(\frac{1}{t+1} \pm \frac{1}{u+1} \right) \tag{13.2.22}$$

where $(\cdots)_i$ is the Pochhammer symbol, straightforwardly reproduces symmetric and antisymmetric sectors of $AdS_3 \times S^3$ and $AdS_5 \times S^3$ for $\hat{\theta} = 2, 3$. Moreover, plugging in $\hat{\theta} = 4$ in the symmetric amplitude, it yields

$$\frac{1}{(s+1)_2} = \left(\frac{1}{t+1} \pm \frac{1}{u+1} \right) = \frac{1}{(s+1)(t+1)(u+1)}, \qquad s+t+u = -4, \tag{13.2.23}$$

which is the $\mathcal{N} = 4$ amplitude. Note that the formula, as it stands, also predicts the existence of an odd sector for $\hat{\theta} = 4$, which is not realised in $\mathcal{N} = 4$ SYM. There might however exist another theory with $\hat{\theta} = 4$ where both sectors are physical.

Finally, the amplitude $\mathcal{A}$ for general $\vec{p}$ is generated via the differential operator,

$$\mathcal{A}_{\vec{p}} = \widehat{\mathcal{D}}_{\vec{p}} \Big[U^{\hat{\theta}} \mathcal{A}^{I_1 I_2 I_3 I_4}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}} \Big] \tag{13.2.24}$$

where

$$\widehat{\mathcal{D}}_{\vec{p}} = U^{-\hat{\theta}} \sum_{\tilde{s},\tilde{t}} \left(\frac{\tilde{U}}{U} \right)^{\tilde{s}} \left(\frac{\tilde{V}}{V} \right)^{\tilde{t}} \widehat{\mathcal{D}}^{(0,0,0)}_{\vec{p},\tilde{s},\tilde{t}} \widehat{\mathcal{D}}^{(k_s,k_t,k_u)}_{\vec{p},\tilde{s},\tilde{t}} \tag{13.2.25}$$

and

$$\widehat{\mathcal{D}}^{(a,b,c)}_{\vec{p},\tilde{s},\tilde{t}} = \frac{(U\partial_U + 1 - \hat{\theta} - \tilde{s} - a)_{\tilde{s}+a}}{(-)^a (\tilde{s}+a)!} \frac{(V\partial_V + 1 - \tilde{t} - b)_{\tilde{t}+b}}{(-)^b (\tilde{t}+b)!} \frac{(U\partial_U + V\partial_V)_{\tilde{u}+c}}{(\tilde{u}+c)!} \tag{13.2.26}$$

We recall that at the level of the Mellin amplitude, this just amounts to replacing the Mellin variables s, t, u with the bold-face variables, thus:

$$\mathcal{M}^{\pm}_{\vec{p}} = \frac{1}{(\mathbf{s}+1)_{\hat{\theta}-2}} \left(\frac{1}{\mathbf{t}+1} \pm \frac{1}{\mathbf{u}+1} \right) \tag{13.2.27}$$

with $\mathbf{s} + \mathbf{t} + \mathbf{u} = -\hat{\theta}$.

We conclude the section by mentioning one more property of these correlators. Because of the hidden symmetry, $\mathcal{A}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}}$ can be interpreted as the four-point correlator of a scalar of dimension $\hat{\theta}$ in $2\hat{\theta} + 2$ dimensions. This suggests that the correlator should then have a natural decomposition not only in long superconformal blocks for the corresponding SCFTs, but also in $SO(2\hat{\theta}+2,2)$ conformal blocks at the unitarity bound, as shown in [31] for $\mathcal{N}=4$. Inspired by that result, we find that this is indeed true for all theories with hidden symmetries with the decomposition given by

$$\mathcal{A}_{\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}\frac{\theta_1}{2}}\Bigg|_{\log U} = \sum_{\ell} \frac{\hat{\theta}\,\Gamma[\ell+\hat{\theta}]^2}{\Gamma[2\ell+2\hat{\theta}-1]} \frac{{}_2F_1[\hat{\theta}+\ell,\hat{\theta}+\ell;2\hat{\theta}+2\ell;P]}{U^{\hat{\theta}}} \tag{13.2.28}$$

where ${}_2F_1[\theta+\ell,\ldots]$ is a single normalised block in which we understand the ${}_2F_1$ as a power series with the replacement $z^n \to P_{[\theta+\ell+n,\theta]}(x,\bar{x};\theta)$, and $P(;\theta)$ being the two-variables Jack polynomial. A compact way of writing this polynomial is

$$\frac{(\theta)_k}{k!}\frac{P_{[\theta+k,\theta]}(x,\bar{x};\theta)}{U^{\theta}} = e^{-k\varphi}\sum_{j=0}^{k}\frac{(\theta)_j(\theta)_{k-j}}{j!(k-j)!}\,e^{i(k-2j)\phi} \tag{13.2.29}$$

where $x=e^{-\varphi+i\phi}$ as in [145, 146] and $k=\ell+n$.

The point here is that the $SO(2\theta+2,2)$ decomposition in (13.2.28) only runs over a single sum.

13.3 The general breaking of residual degeneracy

This last section will be more speculative. Guided by $\mathcal{N}=4$, we will seek for some tree-level higher derivative corrections to supergravity/Yang-Mills amplitudes and see whether they present some common features, such as splitting of residual degeneracy in the anomalous dimensions and so on. As the title of the section suggests, this will indeed be the case. By analogy with $\mathcal{N}=4$, we will denote the coupling parameter α', even though we do not know whether these have anything to do with string corrections.

As we reviewed in the first part, in $\mathcal{N}=4$ these corrections correspond to the α' expansion of the VS amplitude in $AdS_5\times S^5$. In particular, the structure of these amplitudes is such that it sequentially breaks the degeneracy of the supergravity anomalous dimensions and uniquely fixes the three-point functions. Moreover, despite the presence of ambiguities, the problem of computing the anomalous dimensions at the edge of the rectangle turns out to be well posed and independent of any of the ambiguities. This is due to the existence of a preferred sub-amplitude in the VS amplitude directly related to its flat space counterpart. In fact, in the language of [50], this amplitude is obtained by replacing partial derivative with suitable $AdS\times S$ ones in an effective action in $10d$, by ignoring the ordering of the derivatives.

We now want to point out that the situation in $AdS_3\times S^3$ and $AdS_5\times S^3$ is perfectly analogous, in the sense that it is possible to build polynomial amplitudes at each order in α' such that they break the residual degeneracy as in $\mathcal{N}=4$. A way to see this is to use the bootstrap approach we outlined in the first part of the thesis[6].

We will parametrise these α' deformations by

$$\mathcal{M} = \mathcal{M}_{\text{pp}} + \sum_n \mathcal{M}_n \alpha'^{n+\hat{\theta}-1}. \tag{13.3.30}$$

where $\mathcal{M}_{\text{pp}}$ is the field theory/supergravity contribution and $\mathcal{M}_n$ are polynomial Mellin amplitudes of degree n to be determined.

Note that, since some of the correlators have a flavour structure, we will have to consider two different level splitting problems, one for symmetric and another for antisymmetric amplitudes[7]. In fact, as far as the level-splitting problem is concerned, what we really need to compute is just the AdS completion of the flat space terms $t^n \pm u^n$. The reason is that, remembering the discussion in section (7.1), we expect that the anomalous dimensions that do not suffer ambiguities are those for which the higher-dimensional spin of the corresponding double-trace operators saturates the flat space bound. The latter is just n for a polynomial of degree n in t. This is best understood with an example. In $\mathcal{N} = 4$ the α'^5 amplitude is (5.4.39):

$$\tilde{\mathcal{M}}_2^{\text{ptic}} = (S^2 + \Sigma^2) + (T^2 + \Sigma^2) + (U^2 + \Sigma^2) \tag{13.3.31}$$

However, an explicit computation shows that the resulting anomalous dimensions at the edge are the same as if we performed the unmixing with the amplitude

$$(T^2 + \Sigma^2) + (U^2 + \Sigma^2) = T^2 + U^2 + 2\Sigma^2. \tag{13.3.32}$$

This is because $(S^2 + \Sigma^2)$ happens to be subleading in $10d$ spin (in this case it has $l_{10} = 0$). The idea is therefore to make an ansatz in $t-$type variables only (i.e. $\mathbf{t}$, $\tilde{t}$, c_t and Σ which is singlet under swapping $t \leftrightarrow u$), symmetrise (and antisymmetrise) in $t \leftrightarrow u$ and then solve analogous OPE equations to those in section 7.2, with the only difference that the now that there are two sets of equations, one for symmetric and another for antisymmetric amplitudes. Since the computation is the same as $\mathcal{N} = 4$ we will skipe all the details and just quote the results.

These are conveniently represented in (a suitable generalisation of) the formalism we introduced in (5.4.36), which has the role of absorbing some Pochhammers appearing in the amplitude. Let us thus define

$$\mathcal{M}_n = \frac{i}{2\pi} \int_0^\infty d\alpha \int_{\mathcal{C}} d\beta \; e^{-\alpha-\beta} \alpha^{\theta_2 - \frac{\theta_1}{2} - 1 + \Sigma} (-\beta)^{\theta_1 - \frac{\theta_2}{2} - 1 - \Sigma} \tilde{\mathcal{M}}_n(\alpha, \beta) \tag{13.3.33}$$

[6] One could equivalently use a straightforward generalisation of the effective field theory approach of [50], arriving at the same result.

[7] As before, this will implicitly define an antisymmetric sector for $\mathcal{N} = 4$ which however does not exist.

where $\mathcal{C}$ is the Hankel contour and $\tilde{\mathcal{M}}_n$ is a simplified amplitude, defined in terms of the variables

$$S = \alpha\hat{s} - \beta\check{s}, \quad \tilde{S} = \alpha\hat{s} + \beta\check{s}, \qquad \begin{cases} \hat{s} = s - \frac{1}{2}k_s + 1\,, \\ \check{s} = \tilde{s} + \frac{1}{2}k_s + 1\,, \end{cases} \tag{13.3.34}$$

and similarly for t- and u-type. We remind that, for polynomial amplitudes, such as those we will be considering here, the integrals just provide gamma functions, direct and inverse.

We can now present the results at the first few orders. The $\mathcal{N} = 4$ results can be found in chapter 5, we will therefore here just show the amplitudes for the other two theories.

Since we do not know what the UV completion of these theories is, we cannot fix the overall normalisation of the amplitudes, which is usually obtained by taking the flat space limit and matching with the corresponding flat space amplitude. We will then make an arbitrary choice such that it matches with the $\mathcal{N} = 4$ normalisation when $\theta_1 = \theta_2 = 4$, for which the overall coefficients are in fact fixed by the flat space VS amplitude[8].

13.3.1 $AdS^3 \times S^3$ higher derivative corrections

The degree zero amplitude is just

$$\mathcal{M}_0 = 2\Sigma, \tag{13.3.35}$$

therefore

$$\tilde{\mathcal{M}}_0 = 2. \tag{13.3.36}$$

Note that here obviously there is only the symmetric amplitude. The degree-1 amplitudes are

$$2\tilde{\mathcal{M}}_1^+ = T + U - 2\Sigma, \tag{13.3.37}$$

$$2\tilde{\mathcal{M}}_1^- = T - U. \tag{13.3.38}$$

Let us we stress again that these results are ambiguous in the sense that there is an infinite family of amplitudes satisfying the rank constraints. For example, in this specific case we could add a constant term to the symmetric amplitude without changing the anomalous dimensions on the edge. The antisymmetric amplitude at this order is instead unambiguously fixed.

[8] We will however drop the ζ_{n+3} coefficient present in $\mathcal{N} = 4$.

The degree two amplitude reads

$$\tilde{\mathcal{M}}_2^+ = (T-\Sigma)^2 + (U-\Sigma)^2 - 2\left(\tilde{T}+\tilde{U}\right) - \left(\frac{c_t^2}{2}+\frac{c_u^2}{2}\right), \tag{13.3.39}$$

$$\tilde{\mathcal{M}}_2^- = (T-\Sigma)^2 - (U-\Sigma)^2 - 2\left(\tilde{T}-\tilde{U}\right) - \left(\frac{c_t^2}{2}-\frac{c_u^2}{2}\right). \tag{13.3.40}$$

13.3.2 $AdS^5 \times S^3$ higher derivative corrections

Next, let us move to $AdS^5 \times S^3$. In this case the degree zero amplitude is

$$\mathcal{M}_0 = 2(\Sigma-2)_2, \tag{13.3.41}$$

and, once again,

$$\tilde{\mathcal{M}}_0 = 2. \tag{13.3.42}$$

Obviously, also here the antisymmetric amplitude is zero.

The symmetric and antisymmetric amplitudes of degree 1 are

$$2\tilde{\mathcal{M}}_1^+ = T + U + 3\Sigma, \tag{13.3.43}$$

$$2\tilde{\mathcal{M}}_1^- = T - U. \tag{13.3.44}$$

To conclude, the degree 2 amplitudes in this case read

$$2\tilde{\mathcal{M}}_2^+ = T^2 + U^2 - \Sigma(T+U) + \frac{\Sigma^2}{2} + 3(T+U) - \frac{3}{4}(c_t^2+c_u^2) - 2(\tilde{T}+\tilde{U}) - 3\Sigma, \tag{13.3.45}$$

$$2\tilde{\mathcal{M}}_2^- = T^2 - U^2 - \Sigma(T-U) + 3(T-U) - \frac{3}{4}(c_t^2-c_u^2) - 2(\tilde{T}-\tilde{U}) \tag{13.3.46}$$

We are now going to see that these amplitudes generate very similar characteristic polynomials to those we found in $\mathcal{N}=4$.

13.3.3 Anomalous dimensions on the edge for all theories

Before presenting the results, we need to refine and generalise the conjecture on the bound for the effective $(2\hat{\theta}+2)$ spin. Thus, let us first define the effective $(2\hat{\theta}+2)d$ spin

$$l^{\pm}_{(2\hat{\theta}+2)d} = l + |a| + 2m - \frac{1\pm(-1)^{a+l}}{2} - 1 \tag{13.3.47}$$

where, as usual, m measures the distance on the p axis:

$$m = p - |a| + 1 - \frac{\theta_2}{2} \tag{13.3.48}$$

Note that in $AdS_3 \times S^3$ and $AdS_5 \times S^5$ only $a > 0$ representations exist, therefore $|a| = a$. Moreover, the dependence on θ_2 in m can be somewhat justified by remembering that the lowest half-BPS in $AdS_3 \times S^3$ has $\tau = 1$ while in the other two cases has $\tau = 2$.

The amplitudes we mentioned before are such that the operators exchanged satisfy the following inequality

$$l^{\pm}_{(2\hat{\theta}+2)d} \leq n^{\pm} \tag{13.3.49}$$

and $n^{\pm}$ is the exponent of the highest power in $\mathbf{t}$ in the amplitudes at the order $\alpha'^{n+\hat{\theta}-1}$ in $\tilde{\mathcal{M}}^{\pm}_{n}$. It is easy to see that this is equal to

$$n^{\pm} = n - \frac{1 \mp (-1)^n}{2}. \tag{13.3.50}$$

As done in $\mathcal{N} = 4$, it is useful to turn (13.3.49) into an inequality for m

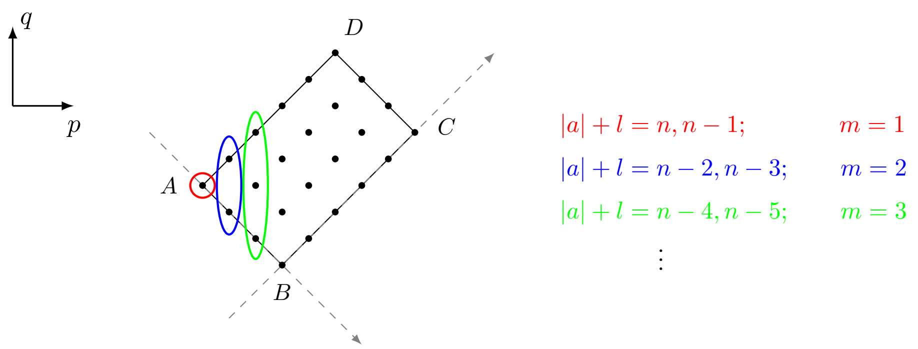
$$m \leq \frac{n^{\pm} - |a| - l - \frac{1 \mp (-1)^{a+l}}{2}}{2} + 1. \tag{13.3.51}$$

We are now ready to present the general formulas for the anomalous dimensions on the edge. Generalising the discussion of section (7.1), we can say that the anomalous dimensions which do not suffer ambiguities are those for which m satisfies

$$m = m^* = \frac{n - |a| - l - \frac{1 \mp (-1)^{a+l}}{2}}{2} + 1, \tag{13.3.52}$$

where it is understood that in (13.3.50) we need to take n even (odd) for the symmetric (antisymmetric) amplitude.

Let us start from the operators exchanged in the symmetric amplitude. As usual, we will classify the anomalous dimensions according to their value of $|a| + l$, which tells us which edge-operators acquire anomalous dimension, as shown in the picture below.



For future convenience, let us define the rescaled anomalous dimension

$$\eta_{\vec{\tau},m} = \mathcal{F}_{\vec{\tau},n}\, \tilde{\eta}_{\vec{\tau},m} \tag{13.3.53}$$

where

$$\mathcal{F}_{\tau,l,[aba],n} =$$
$$2n!\frac{(n+\hat{\theta})!}{(2(n+\hat{\theta}))!}\delta^{\theta_1,\theta_2}\left(\frac{\tau-b-a-|a|-\theta_1+2}{2}\right)_{|a|+l+\hat{\theta}-1}\left(\frac{\tau+b+a-|a|+2-2\check{\theta}}{2}\right)_{|a|+l+\hat{\theta}-1} \tag{13.3.54}$$

where $\delta^{(\theta_1,\theta_2)}$ is the rational function present in the $\alpha'=0$ anomalous dimensions i.e.

$$\begin{aligned} \delta^{(2,2)} &= \frac{\delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j}}{\delta^{(4)}_{h,\bar{h},j,\bar{j}}+\delta^{(4)}_{h,\bar{h},\bar{j},j}}, && AdS_3\times S^3,\\ \delta^{(4,4)} &= \delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j}, && AdS_5\times S^5,\\ \delta^{(4,2)} &= \delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},j}, && AdS_5\times S^3. \end{aligned} \tag{13.3.55}$$

The simplest case is when $|a|+l=n$, for which only one anomalous dimensions is turned on and we find

$$\tilde{\eta}^{(n)}_{\vec{\tau}} = -1 \tag{13.3.56}$$

Analogously, for $a+l=n-1$ (even n, therefore $a+l$ is odd) we have

$$\tilde{\eta}^{(n-1)}_{pq} = B-T \tag{13.3.57}$$

where

$$\begin{aligned} T &= \frac{1}{4}(\tau-2\check{\theta})(\tau+2l+\theta_2),\\ B &= \frac{1}{4}(b+a-|a|)(b+a+|a|+2\theta_2). \end{aligned} \tag{13.3.58}$$

The splitting starts at $|a|+l=n-2$. We find convenient to define the following quantities

$$l_{\theta_1} = l+\frac{\theta_1}{2}, \tag{13.3.59}$$

$$a_{\theta_2} = |a|+\frac{\theta_2}{2}. \tag{13.3.60}$$

The characteristic polynomial for $|a|+l=n-2$ the admits the following representation

$$\mathcal{P}^*_{\vec{\tau},2} = (\tilde{\eta}+r)^2+(\tilde{\eta}+r)\gamma_{2,1}+\gamma_{2,0} \tag{13.3.61}$$

where

$$\gamma_{2,1} = -\frac{(n+\hat{\theta}-2)(n+\hat{\theta}-1)}{2(n+\hat{\theta})-3}\Big((2l_{\theta_1}+1)B + (2a_{\theta_2}+1)T - l_{\theta_1}a_{\theta_2}\Big), \tag{13.3.62}$$

$$\gamma_{2,0} = +\frac{(n+\hat{\theta}-2)^2(n+\hat{\theta}+1)^2}{2(n+\hat{\theta})-3}BT \tag{13.3.63}$$

and the shift is

$$r = (T-B)^2 + Bl_{\theta_1} + Ta_{\theta_2}. \tag{13.3.64}$$

Finally, the characteristic polynomial for $a+l=n-3$ reads

$$\mathcal{P}^*_{\vec{\tau},2} = (\tilde{\eta}+r)^2 + (\tilde{\eta}+r)\gamma_{2,1} + \gamma_{2,0}, \tag{13.3.65}$$

where this time

$$\gamma_{2,1} = \frac{(n+\hat{\theta}-2)(n+\hat{\theta}-1)}{2(n+\hat{\theta})-3}\,(B-T) \tag{13.3.66}$$
$$\Big((2a_{\theta_2}+1)T + (2l_{\theta_1}+1)B - 3(a_{\theta_2}+1)(l_{\theta_1}+1) + 2(n+\hat{\theta})-3\Big)$$

$$\gamma_{2,0} = -\frac{(n+\hat{\theta}-2)^2(n+\hat{\theta}+1)^2}{2(n+\hat{\theta})-3}\,(B-T)^2\,(Ta_{\theta_2} + Bl_{\theta_1} - 3BT) \tag{13.3.67}$$

and the shift is

$$r = (T-B)^3 + (T-B)\left((3a_{\theta_2}+1)T + (3l_{\theta_1}+1)B\right) + (a_{\theta_2}-l_{\theta_1})\big(Bl_{\theta_1} + Ta_{\theta_2}\big). \tag{13.3.68}$$

For what concerns the antisymmetric amplitude, we have checked that all formulae are the same with the only difference that now the operators will be turned on for first time only for odd n. For example, for $n=1$ the only anomalous dimensions turned on are the ones in left-most corner. The relevant formula in this case is therefore (13.3.56) and includes the two cases $|a|=1, l=0$ and $a=0, l=1$. Then, for $n=3$, the new anomalous dimensions turned on will be the ones with labels $|a|=3, l=0$, $a=0, l=3$, $|a|=2, l=1$, $|a|=1, l=2$ which are captured by (13.3.56), $|a|=2, l=0$, $a=0, l=2$, $|a|=1, l=1$, captured by (13.3.57), $|a|=1, l=0$, $a=0, l=1$ captured by (13.3.61), (13.3.62), (13.3.64) and finally $a=0, l=0$ captured by (13.3.65), (13.3.66), (13.3.3).

Lastly, let us also point that the features we observed in $\mathcal{N}=4$ nicely generalise for all values of θ_1, θ_2. In particular, suitable generalisation of rank-reduction and low b factorisation hold. Finally, note that the Z_2 symmetry present in the $\mathcal{N}=4$ anomalous dimensions that exchanges $(B,a) \leftrightarrow (T,l)$ upgrades to a $(B, a_{\theta_2}) \leftrightarrow (T, l_{\theta_1})$ symmetry.

Epilogue II

In this second part of the thesis we have discussed several aspects of holographic correlators in $AdS_5 \times S^3$ and $AdS_3 \times S^3$ backgrounds.

In particular, in the first three chapters, we focused on the four-point function of supergluons in $AdS_5 \times S^3$, which in [67] was shown to enjoy an accidental $8d$ conformal symmetry. This suggested the existence of a generalised Mellin amplitude, along the line of that introduced by Vieira and Aprile for $\mathcal{N} = 4$ [32], which should manifestly respect the large p limit. In this formalism the correlator takes a very simple form; for example, the colour-ordered amplitude $\mathcal{M}_{\vec{p}}(1,2,3,4)$ is given by

$$\mathcal{M}_{\vec{p}}(1,2,3,4) = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)}, \qquad \mathbf{s}+\mathbf{t}+\mathbf{u} = -3. \tag{13.3.69}$$

Interestingly, we found that this generalised Mellin amplitudes satisfies a number of other properties, such as $U(1)$ decoupling identity, BCJ and double-copy relations analogous to flat space. As an example, we have

$$(\mathbf{t}+1)\mathcal{M}_{\vec{p}}(1,2,3,4) = (\mathbf{u}+1)\mathcal{M}_{\vec{p}}(1,3,4,2). \tag{13.3.70}$$

The important point is that, as a consequence of the hidden symmetry, these relations turn out to be the same for *all* Kaluza-Klein modes. Differential representations of BCJ relations have been observed recently in AdS boundary correlators in [147, 148]. In the special case of the $AdS_5 \times S^3$ background we can see that, quite nicely, they also admit a generalised Mellin space version.

The knowledge of the correlator allowed us to compute all leading order anomalous dimensions, which turn out to have the same structure as the $\mathcal{N} = 4$ ones:

$$\eta^{\pm}_{\vec{\tau}} = -\frac{2}{N}\frac{\delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},j}}{(l^{\pm}_{8d}+1)_4}, \qquad l^{\pm}_{8d} = l + 2(p-2) + \frac{1 \mp (-1)^{a+l}}{2} - |a|\,, \tag{13.3.71}$$

where the superscript $\pm$ refers to operators exchanged in symmetric and antisymmetric amplitudes. Anomalous dimensions and three-point functions are an important part of the bootstrap program, and can be used to compute higher-loop correlators for arbitrary KK modes, beyond the lowest charge correlator [69], as done in [33, 35, 36, 38] for the

$\mathcal{N}=4$ case. Because of the many similarities that this theory shares with $\mathcal{N}=4$, we expect loop corrections to follow similar patterns. This would ultimately allow to further investigate the structure of gluon amplitudes in AdS. In particular, we know that, in flat space, BCJ and double-copy relations at loop level work at the level of certain *integrands*; it would be interesting to see if and how this modifies when we consider (super)gluons in AdS.

The other theory we discussed is the D1-D5 CFT. This particular $2d$ CFT has various tractable corners (see for example [71, 149–152]), and most notably, the weak coupling regime has a (worldsheet) WZW description. However, the $2d$ theory at the boundary of $AdS_3\times S^3$ with pure RR flux, whose four-point correlators we discussed in this thesis, is strongly coupled. The bootstrap approach is therefore quite natural in this case, since it does not rely on having a weakly coupled Lagrangian description. The many clues of simplicity that we have encountered encourage the idea that the bootstrap program can tackle quantitatively this strongly coupled regime, offering new dynamical insights. The simplicity - ultimately due to an hidden $6d$ conformal symmetry [64, 66] - is captured on the one side by the amplitude which can be nicely written as

$$\mathcal{M}=\frac{\delta^{12}\delta^{34}}{\mathbf{s}+1}+\frac{\delta^{14}\delta^{23}}{\mathbf{t}+1}+\frac{\delta^{13}\delta^{24}}{\mathbf{u}+1},\qquad \mathbf{s}+\mathbf{t}+\mathbf{u}=-2, \tag{13.3.72}$$

and on the other side by the anomalous dimensions, given by

$$\eta^{\pm}_{\vec{\tau}}=-\frac{2}{N}\frac{\delta^{(4)}_{h,\bar{h},j,\bar{j}}\delta^{(4)}_{h,\bar{h},\bar{j},j}}{\delta^{(4)}_{h,\bar{h},j,\bar{j}}+\delta^{(4)}_{h,\bar{h},\bar{j},j}}\frac{1}{(l^{\pm}_{6d}+1)_2},\qquad l^{\pm}_{6d}=l+2(p-1)+\frac{1\mp(-1)^{a+l}}{2}-a\,. \tag{13.3.73}$$

In the last chapter we have shown that many of the formulae we found can be regrouped into compact expressions that cross through the different theories. We saw that the coefficients of the block decomposition of long disconnected free theory of the three theories can all be written in a similar fashion. In fact, they turn out to be particular cases of the general formula (13.1.12) that captures *all* free theory-diagrams in *all* theories with $SU(m,m|2n)$ symmetry. We find very non-trivial the existence of such a formula. We then considered the tree-level scattering and noticed that all tree-level $AdS_{\theta_1+1}\times S_2^{\theta}+1$ correlators can be gathered in the single formula

$$\mathcal{M}^{\pm}_{\vec{p}}=\frac{1}{(\mathbf{s}+1)_{\hat{\theta}-2}}\left(\frac{1}{\mathbf{t}+1}\pm\frac{1}{\mathbf{u}+1}\right),\qquad \mathbf{s}+\mathbf{t}+\mathbf{u}=-\hat{\theta}, \tag{13.3.74}$$

with $\hat{\theta}=\frac{\theta_1+\theta_2}{2}$. We find very remarkable the existence of such a formula. It would be interesting to see whether there are other values of the parameters θ_1,θ_2 for which (13.2.27) acquires a physical meaning.

Finally, inspired by $\mathcal{N}=4$ SYM, we showed that one can compute higher derivative corrections for both $AdS_5\times S^3$ and $AdS_3\times S^3$ theories. These higher derivative corrections are such that they break the residual degeneracy in the anomalous dimensions. The breaking is controlled by a characteristic polynomial and takes a similar form for all three theories and we found evidence that it is possible to write down *simple* formulae for these polynomials that interpolate between the three different cases. As an example, the general rank-1 problem for all theories is simply given by (13.3.56), (13.3.57).

We should however warn that, despite the existence of this formulae, it is unclear whether these higher derivative corrections do represent some physical UV completion (string?) of the theory, like in $\mathcal{N} = 4$ SYM. It is tempting to say, for example, that higher derivative corrections in $AdS_5 \times S^3$ correspond to the low energy expansion of a certain AdS completion of the flat space Veneziano amplitude, i.e. the scattering of four open strings. This would help to understand how the known KLT and world-sheet monodromy[9] relations, obeyed by the analogous flat space amplitudes, generalise to AdS. We hope to report on this in the future.

[9]These are the stringy versions of double-copy and BCJ relations, respectively.

Appendix A

Ansatz for the VS amplitude: iterative scheme

As stated in chapter 5, we expect to stratify the VS amplitude and accommodate each stratum in the ansatz

$$\mathcal{S}_{n,l} = \sum_{0\le d_1+d_2\le \ell} \underbrace{K^{(n)}_{\ell;\, d_1 d_2}(\tilde{s},\tilde{t},p_1p_2p_3p_4)}_{K^{(n)}_{\ell;\, d_1 d_2} = \sum_{0\le \delta_1+\delta_2\le (n-\ell)} k^{(n)}_{\ell;\, d_1 d_2,\delta_1\delta_2}(p_1p_2p_3p_4)\, \tilde{s}^{\delta_1}\tilde{t}^{\delta_2}} \mathbf{s}^{d_1}\mathbf{t}^{d_2} \tag{A.0.1}$$

at given order n in the $(\alpha')^{n+3}$ expansion.

There are nonetheless two issues. Firstly, as we mentioned already, (A.0.1) will contain the new stratum $\mathcal{M}_{n,\ell}$ we were looking for, but also pieces of the amplitude at previous orders $< n$, which we have to discard by hand. This is inevitable because of the inequalities in the sums, which allow to take into account powers of $\mathbf{u}$ and $\tilde{u}$ correctly, but introduce much more freedom than the one really contained in a stratum. Secondly, the crossing symmetric version of $\mathcal{S}_{n,\ell}$ is still written in the variables $\mathbf{s},\mathbf{t},\tilde{s},\tilde{t},p_{i=1,2,3,4}$ and we want to make crossing symmetry manifest. Therefore, we rewrite it in terms of crossing invariant combinations built out of $\mathbf{s},\mathbf{t},\mathbf{u}$, $\tilde{\mathbf{s}},\tilde{\mathbf{t}},\tilde{\mathbf{u}}$ and c_s,c_t,c_u,Σ, in practise by making a second ansatz and checking that we can map free parameters with an invertible matrix.

To fix ideas consider the case $n=3$ and $\ell=2$. After imposing crossing and rewriting the solution in terms of crossing invariant combinations, we find

$$\begin{aligned}
\mathcal{S}_{n=3,2} &= \overbrace{\mathcal{H}_{3,(2,1)} + \mathcal{M}_{n=2,\ell=2}}^{\mathcal{M}_{n=3,2}} + \mathcal{S}_{n=2,\ell=1} \\
\mathcal{H}_{n=3,(2,1)} &= a^{(1)}\left(\mathbf{s}^2\tilde{\mathbf{s}} + \mathbf{t}^2\tilde{\mathbf{t}} + \mathbf{u}^2\tilde{\mathbf{u}}\right) + a^{(2)}(\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2)\Sigma \\
\mathcal{M}_{n=2,2} &= a^{(3)}(\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2)
\end{aligned} \tag{A.0.2}$$

In (A.0.2), $\mathcal{M}_{n=3,2}$ is the stratum we are looking for, and it comes with two contributions, $\mathcal{M}_{n=2,2}$ and the polynomial $\mathcal{H}_{k,(\ell,k-\ell)}$, where

> $\mathcal{H}_{n,(\ell,n-\ell)}$ is defined to be a crossing symmetric polynomial in all its variables, of degree n, such that only monomials of degree ℓ in $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{u}$ appear.

Notice that $\mathcal{H}_{n,(\ell,n-\ell)}$ is *homogeneous* and that its contribution is genuinely the new contribution in $\mathcal{M}_{n,\ell}$. In (A.0.2) in fact, $\mathcal{M}_{2,2}$ is known from $(\alpha')^5$.

Summarising

$\mathcal{S}_{n,\ell}$	polynomial of *max* degree ℓ in $\mathbf{s}$ and *max* degree n in the large p limit
$\mathcal{M}_{n,\ell}$	polynomial of *fixed* degree ℓ in $\mathbf{s}$ and *max* degree n in the large p limit
$\mathcal{H}_{n,(\ell,n-\ell)}$	polynomial of *fixed* degree ℓ in $\mathbf{s}$ and *fixed* degree n in the large p limit

The idea is the following: assume $\mathcal{S}_{n-1,\ell}$ (the crossing symmetric version of it) is known for $\ell = 0, \ldots n-1$, then

$$\mathcal{S}_{n,\ell} = \overbrace{\mathcal{H}_{n,(\ell,n-\ell)} + \mathcal{M}_{n-1,\ell}}^{\mathcal{M}_{n,\ell}} + \mathcal{S}_{n-1,\ell-1} \tag{A.0.3}$$

This is because $\mathcal{S}_{n,\ell}$ by definition has maximum degree ℓ in $\mathbf{s}, \mathbf{t}, \mathbf{u}$ and maximum degree $n-\ell$ in $\tilde{s}, \tilde{t}, \tilde{u}$. Therefore, once we extract off $\mathcal{M}_{n,\ell}$ the remaining polynomial must have maximum degree $\ell - 1$ in $\mathbf{s}, \mathbf{t}, \mathbf{u}$ and maximum degree $n - \ell = (n-1) - (\ell - 1)$ in $\tilde{s}, \tilde{t}, \tilde{u}$, which is by definition $\mathcal{S}_{n-1,\ell-1}$. Note that the only new contribution is $\mathcal{H}_{n,(\ell,n-\ell)}$. Notice also that $\mathcal{S}_{n,0}$ is not contaminated by previous orders, and always returns the corresponding stratum.

The amplitudes $\mathcal{M}_{n,n}$ are known from covariantising the flat VS amplitude, thus we do not need to construct them. The beginning of the recursion is peculiar due to the Mandelstam-type constraints on the Mellin variables, which give $\mathcal{H}_{n=1,(1,0)} = 0$ and $\mathcal{M}_{1,1} = 0$. Then, at $(\alpha')^5$ we find

$$\mathcal{S}_{2,1} = \mathcal{H}_{2,(1,1)} + (\mathcal{M}_{1,1} = 0) + \mathcal{S}_{1,0} \qquad \rightarrow \qquad \mathcal{M}_{2,1} = \mathcal{H}_{2,(1,1)} = a_{4,1}\left(\mathbf{s}\tilde{\mathbf{s}} + \mathbf{t}\tilde{\mathbf{t}} + \mathbf{u}\tilde{\mathbf{u}}\right)$$

and $\mathcal{S}_{n=2,0} = \mathcal{M}_{2,0} = \mathcal{H}_{2,(0,2)} + \mathcal{M}_{1,0}$. For the case of $(\alpha')^6$ all terms contribute in the recursion,

$$\mathcal{S}_{3,2} = \overbrace{\mathcal{H}_{3,(2,1)} + \mathcal{M}_{2,2}}^{\mathcal{M}_{3,2}} + \mathcal{S}_{2,1}, \qquad \mathcal{S}_{3,1} = \overbrace{\mathcal{H}_{3,(1,2)} + \mathcal{M}_{2,1}}^{\mathcal{M}_{3,1}} + \mathcal{S}_{2,0} \tag{A.0.4}$$

and finally $\mathcal{S}_{3,0} = \mathcal{M}_{3,0} = \mathcal{H}_{3,(0,3)} + \mathcal{M}_{2,0}$.

Let us highlight some patterns which we tested up to $(\alpha')^9$. When we construct $\mathcal{H}_{n,(\ell,n-\ell)}$ we begin with $\mathbf{s}^\ell \times \mathcal{P}(\tilde{s}, \tilde{t}, \tilde{u}, c_s, c_t, c_u, \Sigma)$ crossing symmetrised. The overall homogeneous scaling has to be n, therefore the polynomial $\mathcal{P}$ can have the structure

- monomials of the form $(\tilde{\mathbf{s}}^{d_1} c_s^{d_2} + \texttt{crossing})$ with $d_1 + d_2 = n - \ell$,
- monomials of the form $(\tilde{\mathbf{s}}^{d_1} c_s^{d_2} + \texttt{crossing}) \times \mathcal{I}_{n-d_1-d_2}(\tilde{\mathbf{s}}, \tilde{\mathbf{t}}, \tilde{\mathbf{u}}, c_s, c_t, c_u, \Sigma)$

with $\mathcal{I}_{n-d_1-d_2}$, invariant under crossing. Then, we can also have a structure like

- products of invariants under crossing of the form $\mathfrak{I}_d(\mathbf{s}, \mathbf{t}, \mathbf{u}, \tilde{\mathbf{s}}, \tilde{\mathbf{t}}, \tilde{\mathbf{u}}, c_s, c_t, c_u)$.

Typically these invariants are found from the amplitudes at previous orders.

For $\mathcal{M}_{n,1}$ we cannot have products of invariants, because $\mathfrak{I}_{\ell=1}(\mathbf{s}, \mathbf{t}, \mathbf{u}) = \mathbf{s} + \mathbf{t} + \mathbf{u} = -4$. This feature of $\mathcal{M}_{n,1}$ offers a starting point for the analysis of the various strata. For example, at $n = 6$ the crossing invariant ansatz for $\mathcal{H}_{6,(1,5)}$ is the symmetrisation of

$$\begin{array}{llcl}
\mathbf{s}^1\otimes & \{\tilde{\mathbf{s}}^5, \ldots, \tilde{\mathbf{s}} c_s^4\} & \otimes & \{1\} \\
\mathbf{s}^1\otimes & \{\tilde{\mathbf{s}}^4, \ldots, c_s^4\} & \otimes & \{\Sigma\} \\
\mathbf{s}^1\otimes & \{\tilde{\mathbf{s}}^3, \ldots, \tilde{\mathbf{s}} c_s^2\} & \otimes & \{\Sigma^2, (c_s^2 + c_t^2 + c_u^2), (\tilde{\mathbf{s}}^2 + \tilde{\mathbf{t}}^2 + \tilde{\mathbf{u}}^2)\} \\
\mathbf{s}^1\otimes & \{\tilde{\mathbf{s}}^2, c_s^2\} & \otimes & \big\{\Sigma^3, \Sigma(c_s^2 + c_t^2 + c_u^2), c_s c_t c_u,\, (\tilde{\mathbf{s}}^3 + \tilde{\mathbf{t}}^3 + \tilde{\mathbf{u}}^3), (\tilde{\mathbf{s}} c_s^2 + \tilde{\mathbf{t}} c_t^2 + \tilde{\mathbf{u}} c_u^2)\big\} \\
\mathbf{s}^1\otimes & \{\tilde{s}, c_s\} & \otimes & \{\Sigma^4, \Sigma^2(c_s^2 + c_t^2 + c_u^2), (c_s^2 + c_t^2 + c_u^2)^2, (c_s^4 + c_t^4 + c_u^4), \Sigma\, c_s c_t c_u\}
\end{array}$$

The case $\mathcal{M}_{n,2}$ is the first case in which we can have an invariant in the boldfont variables, i.e. $\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2$. For $\mathcal{M}_{n,\ell\geq 3}$ there is a similar story. Novelties in general come from the possibility of adding products of invariants. The basis up to $(\alpha')^9$ is given in the ancillary file `calHbasis`.

Appendix B

OPE equations

Let us recall that the OPE at genus zero gives the following α'-dependent constraints

$$\mathbf{C}_{\vec{\tau}}(\alpha')\mathbf{C}^T_{\vec{\tau}}(\alpha') = \mathbf{L}_{\vec{\tau}}, \qquad \mathbf{C}_{\vec{\tau}}(\alpha')\boldsymbol{\eta}_{\vec{\tau}}(\alpha')\mathbf{C}^T_{\vec{\tau}}(\alpha') = \mathbf{M}_{\vec{\tau}}(\alpha')\,, \tag{B.0.1}$$

where $\mathbf{M}_{\vec{\tau}}(\alpha')$ is the CPW of the $\log u$ discontinuity of the VS amplitude, while $\mathbf{L}_{\vec{\tau}}$ is the CPW from disconnected free theory, in the long sector. The α' expansion reads

$$\begin{aligned} \boldsymbol{\eta} &= \boldsymbol{\eta}^{(0)} + \alpha'^3\boldsymbol{\eta}^{(3)} + \alpha'^5\boldsymbol{\eta}^{(5)} + \dots\,, \\ \mathbf{C} &= \mathbf{C}^{(0)} + \alpha'^3\mathbf{C}^{(3)} + \alpha'^5\mathbf{C}^{(5)} + \dots\,. \end{aligned} \tag{B.0.2}$$

Inserting this in the OPE we will find a tower of relations, of which the first one obviously coincides with the supergravity eigenvalue problem. At order $(\alpha')^{n+3}$ we find

$$\left(\mathbf{C}^{(n+3)}\mathbf{C}^{(0)T} + \mathbf{C}^{(0)}\mathbf{C}^{(n+3)T}\right) + \sum_{\substack{k_1+k_2=n+3\\ k_1\neq n+3}} \mathbf{C}^{(k_1)}\mathbf{C}^{(k_2)T} = 0 \tag{B.0.3}$$

$$\left(\mathbf{C}^{(0)}\boldsymbol{\eta}^{(n+3)}\mathbf{C}^{(0)T}+\mathbf{C}^{(n+3)}\boldsymbol{\eta}^{(0)}\mathbf{C}^{(0)T} + \mathbf{C}^{(0)}\boldsymbol{\eta}^{(0)}\mathbf{C}^{(n+3)T}\right) + \sum_{\substack{k_1+k_2+k_3=n+3\\ k_2\neq n+3}} \mathbf{C}^{(k_1)}\boldsymbol{\eta}^{(k_2)}\mathbf{C}^{(k_3)T} = \mathbf{M}^{(n+3)} \tag{B.0.4}$$

where we isolated the first term to emphasize that $\mathbf{C}^{(n+3)}$ is new at this order, while the other matrices in the sum already featured at previous orders (when existing). Actually the sum is over distinct permutations.

We will now rewrite the two equations in (B.0.3)-(B.0.4) by going to the eigenvector basis $\mathbf{c}^{(0)}_{\vec{\tau}} = \mathbf{L}^{-\frac{1}{2}}_{\vec{\tau}}\mathbf{C}^{(0)}_{\vec{\tau}}$, and using the resolution of the identity

$$\mathbf{C}^{(0)T}\,\mathbf{L}^{-1}\,\mathbf{C}^{(0)} = \mathbf{1} \tag{B.0.5}$$

to split matrix products of three point functions and anomalous dimensions corresponding to different orders. To do so it is convenient to introduce the matrix

$$\mathbf{D}^{(k)} = \mathbf{L}^{-\frac{1}{2}} \left(\mathbf{C}^{(k)}\mathbf{C}^{(0)T}\right) \mathbf{L}^{-\frac{1}{2}} = \mathbf{L}^{-\frac{1}{2}}\, \mathbf{C}^{(k)}\, \mathbf{c}^{(0)T} \tag{B.0.6}$$

and rewrite both as

$$\left(\mathbf{D}^{(n+3)} + \mathbf{D}^{(n+3)T}\right) + \sum_{\substack{k_1+k_2=n+3\\ k_1\neq n+3}} \mathbf{D}^{(k_1)}\mathbf{D}^{(k_2)T} = 0 \tag{B.0.7}$$

$$\left(\mathbf{c}^{(0)}\boldsymbol{\eta}^{(n+3)}\mathbf{c}^{(0)T} + \mathbf{D}^{(n+3)}\mathbf{N}^{(0)} + \mathbf{N}^{(0)}\mathbf{D}^{(n+3)T}\right) + \sum_{\substack{k_1+k_2+k_3=n+3\\ k_1,k_2,k_3\neq n+3}} \mathbf{D}^{(k_1)}\left[\mathbf{c}^{(0)}\boldsymbol{\eta}^{(k_2)}\mathbf{c}^{(0)T}\right]\mathbf{D}^{(k_3)T} = \mathbf{N}^{(n+3)} \tag{B.0.8}$$

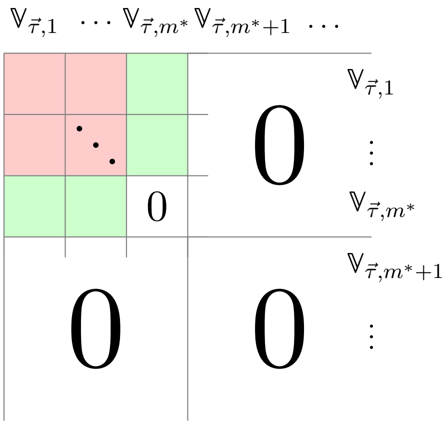
where

$$\mathbf{N}_{\vec{\tau}}^{(n+3)} = \mathbf{L}_{\vec{\tau}}^{-\frac{1}{2}}\mathbf{M}_{\vec{\tau}}^{(n+3)}\mathbf{L}_{\vec{\tau}}^{-\frac{1}{2}} \tag{B.0.9}$$

and $\mathbf{N}^{(0)}, \mathbf{N}^{(n+3)}$ are by construction symmetric.

The matrix $\mathbf{D}^{(n+3)}$ has a block structure depicted below,

$$\left(\mathbf{D}_{\vec{\tau}}^{(n+3)}\right)_{mm'} =$$



(B.0.10)

The symmetric part of $\mathbf{D}$, contained in the red block, is fully determined by previous orders,

$$\left(\mathbf{D}^{(n+3)} + \mathbf{D}^{(n+3)T}\right) = -\sum_{\substack{k_1+k_2=n+3\\ k_1\neq n+3}} \mathbf{D}^{(k_1)}\mathbf{D}^{(k_2)T}. \tag{B.0.11}$$

The anomalous dimensions $\boldsymbol{\eta}^{(n+3)}$ and the antisymmetric part of $\mathbf{D}^{(n+3)}$ are determined by the other equation, therefore by $\mathbf{N}^{(n+3)}$ on the r.h.s. and $\sum \mathbf{D}^{(k_1)}\left[\mathbf{c}^{(0)}\boldsymbol{\eta}^{(k_2)}\mathbf{c}^{(0)T}\right]\mathbf{D}^{(k_3)T}$. Note that, when $m = m^*$ (or $m' = m^*$), the second term in (B.0.7) vanishes[1] and therefore $\mathbf{D}^{(n+3)}$ is antisymmetric. Moreover, it is not difficult to see that when we sandwich (B.0.8) between two vectors belonging to $\mathbb{V}_{\vec{\tau},m^*}$, the only term that survives in the l.h.s. is $\mathbf{c}^{(0)}\boldsymbol{\eta}^{(n+3)}\mathbf{c}^{(0)T}$. As a consequence, the eigenvalues of the level-splitting matrix (7.2.14)

[1] This is because, at $m = m^*$ (remember that, at fixed a, l, m^* is fixed once n is fixed), $\mathbf{D}^{(k)}_{mm'}$ is zero if $k < n+3$.

give the anomalous dimensions η_{m^*}.

Appendix C

Spin structures in the VS amplitude

In this appendix we would like to give a formula for the $10d$ spin of all structures appearing in the VS amplitude. The value of the 10d spin of a monomial in s, t, u contributing to the α' expansion of the VS amplitude is counted by its power in t, with the constraint on u implemented. From its exponential form,, we can immediately see that the projection of the VS amplitude onto the term $\zeta_{n_1} \cdots \zeta_{n_r}$ can be parametrised in the following way

$$\mathcal{V}_{\text{flat}}\Big|_{\zeta_{n_1}\cdots\zeta_{n_r}} \propto \frac{\sigma_{n_1} \cdots \sigma_{n_r}}{stu} (\alpha')^{\sum_{i=1}^{r} n_i}, \qquad \sigma_n \equiv s^n + t^n + u^n. \tag{C.0.1}$$

Moreover, σ_n decomposes as [153]

$$\sigma_n \propto \sum_{2p+3q=n} \frac{(p+q-1)!}{p!q!} \left(\frac{\sigma_2}{2}\right)^p \left(\frac{\sigma_3}{3}\right)^q, \tag{C.0.2}$$

with

$$\sigma_2 = s^2 + t^2 + u^2, \qquad \sigma_3 = s^3 + t^3 + u^3. \tag{C.0.3}$$

Note that both σ_2 and σ_3 have spin 2 therefore σ_n has always even spin by default.

We now want to find a formula for the $10d$ spin of (C.0.1). A term $(\sigma_2)^p(\sigma_3)^q$ in (C.0.2) counts $2p + 2q = n - q$. We have a sum in (C.0.2) and therefore various possible $(\sigma_2)^p(\sigma_3)^q$, i.e. monomials in s, t, u. Consider first the the contribution of maximum $10d$ spin, which is obtained for the minimal q in the sum. This is given by terms containing single ζs The equation $2p+3q = n$ is solved by $q = 1$, $p = (n-3)/2$. Taking into account the denominator stu, we find that the spin is $n - 3$ The result can be easily generalised to products of single ζs. We have

$$l_{10} \texttt{ of the } \zeta_{n_1}\cdots\zeta_{n_r} \texttt{ contribution} \leq -2+\sum_{i=1}^{r}(n_i-1) \tag{C.0.4}$$

where the -2 comes from the denominator $stu = \frac{1}{3}\sigma_3$.

Lastly, let us consider the more general case in which σ_n decompose in various $(\sigma_2)^p(\sigma_3)^q$ terms. The spin of each term is still counted by $2p+2q = n-q$, thus the value of q parametrises the various terms. To fix ideas consider

$$\zeta_9 \times \frac{s^9+t^9+u^9}{stu} = \zeta_9 \times \left(s^6+t^6+u^6-\#(stu)^2\right) \tag{C.0.5}$$

In this case we find two spin structure, $s^6+t^6+u^6$ with $10d$ spin 6 and $(stu)^2$ with $10d$ spin 4. The latter is the same contribution as ζ_3^3. In order to generalise our previous formula we need, together with the information about the ζ_{n_i}, also the values of q_i we are looking at. Thus,

$$l_{10} \texttt{ of the } \zeta_{n_1}\cdots\zeta_{n_r}\Big|_{\{q_1,\ldots q_r\}} \texttt{ contribution} = -2+\sum_{i=1}^{r}n_i-\sum_{i=1}^{r}q_i \tag{C.0.6}$$

where $\sum_{i=1}^{r} q_i$ moves in steps of 2 in the range

$$r \leq \sum_{i=1}^{r} q_i \leq r+\frac{1}{3}\left\lfloor \sum_{i=1}^{r} n_i \right\rfloor . \tag{C.0.7}$$

In this way we span over all the different spins present in the VS amplitude at given order in α'.

Appendix D

Construction of long blocks for $AdS_3 \times S^3$

In this appendix we review the construction of the long superconformal blocks for $AdS_3 \times S^3$.

Let us first introduce the $(1,1)$ superconformal blocks. An exchanged $(1,1)$ representation is specified by a Young diagram $\underline{\lambda} = (\lambda, 1^\mu)$ with one row and one column as in figure 12.1, together with a parameter γ. The latter plays an important role for short representations. However, since we will be mainly interested in long representation, it will not be essential in our discussion. The $(1,1)$ superconformal blocks are

$$B^{(\alpha,\beta)}_{\gamma,\underline{\lambda}} = \underbrace{g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left[\frac{g_{14}}{g_{24}}\right]^{\frac{p_1-p_2}{2}} \left[\frac{g_{14}}{g_{13}}\right]^{\frac{p_4-p_3}{2}} \left(\frac{x}{y}\right)^{\frac{\gamma}{2}}}_{\texttt{prefactor}_\gamma} F^{(\alpha,\beta)}_{\gamma,\underline{\lambda}} \tag{D.0.1}$$

where $\alpha = \max(\frac{\gamma-p_{12}}{2}, \frac{\gamma-p_{43}}{2})$ and $\beta = \min(\frac{\gamma-p_{12}}{2}, \frac{\gamma-p_{43}}{2})$

$$F^{(\alpha,\beta)}_{\gamma,\underline{\kappa}} = \delta_{\underline{\lambda},\underline{0}} \left(\frac{y}{x}\right)^{\beta} + (x-y) H_{\underline{\lambda}}(x,y) \tag{D.0.2}$$

and the dependence on $\underline{\lambda}$ enter through

$$H_{\underline{\lambda}} = \begin{cases} \sum_{\lambda=0}^{\beta-1} h^{(\alpha,\beta,\gamma)}_{-\lambda}(x) h^{(-\alpha,-\beta,-\gamma)}_{\lambda+1}(y) & \underline{\lambda} = \underline{0} \\ (-)^{\mu} h^{(\alpha,\beta,\gamma)}_{\lambda}(x) h^{(-\alpha,-\beta,-\gamma)}_{\mu+1}(y) & \text{otherwise} \end{cases} \tag{D.0.3}$$

with $h^{(a,b,c)}_{\lambda}(z) = z^{\lambda-1}\ {}_2F_1(\lambda+a, \lambda+b; 2\lambda+c; z)$.

A basis for the $\mathcal{N} = (4,4)$ superconformal blocks is obtained by taking products of such

F. On the real slice, we will distinguish among,

$B_{\varnothing}(x,y)B_{\varnothing}(\bar{x},\bar{y})$	half-BPS	
$B_{\underline{\lambda}}(x,y)B_{\varnothing}(\bar{x},\bar{y})+c.c.$	short	(D.0.4)
$B_{\underline{\lambda_1}}(x,y)B_{\underline{\lambda_2}}(\bar{x},\bar{y})+c.c.$	long	

In each of these cases the result always fits into the form

$$\mathcal{B}=\mathcal{C}+\Big[(x-y)\mathcal{S}(x,y)+c.c.\Big]+(x-y)(\bar{x}-\bar{y})\mathcal{L}(x,\bar{x},y,\bar{y}) \tag{D.0.5}$$

where $\mathcal{C}$ is a constant, while $\mathcal{S}$ and $\mathcal{L}$ are the *single*- and *two*- variables contributions, respectively. Note that (D.0.5) automatically satisfies the $\mathcal{N}=(4,4)$ Ward Identity,

$$\Big[(\partial_x+\partial_y)\mathcal{B}\Big]_{x=y}=0 \quad ; \quad \Big[(\partial_{\bar{x}}+\partial_{\bar{y}})\mathcal{B}\Big]_{\bar{x}=\bar{y}}=0 \tag{D.0.6}$$

for any $\mathcal{C},\mathcal{S}$ and $\mathcal{L}$.

Long superconformal blocks factorise into their bosonic components, i.e. conformal and internal. To see this, take (D.0.4) and change basis by considering linear combinations of the form

$$\tfrac{1}{2}\left(B_{[\lambda_1,1^{\mu_1}]}\overline{B}_{[\lambda_2,1^{\mu_2}]}\pm B_{[\lambda_1,1^{\mu_2}]}\overline{B}_{[\lambda_2,1^{\mu_1}]}\right)+c.c. \tag{D.0.7}$$

This change of basis leads to the general decomposition

$$\mathcal{L}(x,\bar{x},y,\bar{y})=\mathcal{L}^{\mathbb{S}}(U,V,\tilde{U},\tilde{V})+(x-\bar{x})(y-\bar{y})\mathcal{L}^{\mathbb{A}}(U,V,\tilde{U},\tilde{V}) \tag{D.0.8}$$

where $\mathcal{L}^{\mathbb{S},\mathbb{A}}$ will now have a clear relation with bosonic blocks, since they are symmetric in $x,\bar{x}$ and $y,\bar{y}$, and therefore writable as function of U,V and $\tilde{U},\tilde{V}$. Note that the most general form of a $\mathcal{N}=(4,4)$ correlator, for four half-BPS external particles, is necessarily given by $\mathcal{G}$ in (D.0.5), with the splitting of $\mathcal{H}$ as in (D.0.8). The dynamical correlator in (11.1.2) thus admits two types of kinematics,

$$\begin{aligned}\texttt{kinematics}^{+}&=\texttt{prefactor}_{p_3+p_4}\times(x-y)(\bar{x}-\bar{y})\\ \texttt{kinematics}^{-}&=(x-\bar{x})(y-\bar{y})\times\texttt{kinematics}^{+}\end{aligned} \tag{D.0.9}$$

Thus, the structure of the long blocks is consistent with that of the correlator (11.2.10). In order to perform the block expansion, we actually only need $\mathcal{L}^{\mathbb{S}}$ because the interacting antisymmmetric part of these correlators seems to be absent [63–66].

At this point, from (D.0.1)-(D.0.3) we find

$$\mathcal{L}^{\mathrm{long},\mathbb{S}}=\left(\frac{y\bar{y}}{x\bar{x}}\right)^{\frac{\gamma}{2}}\frac{(-)^{\mu_1-\mu_2}\,\mathtt{B}^{(+p_{12},+p_{43})}_{\mu_1+1-\frac{\gamma}{2},\,\mu_2+1-\frac{\gamma}{2},}(y,\bar{y})}{\tilde{U}}\;\frac{\mathtt{B}^{(-p_{12},-p_{43})}_{\lambda_1+\frac{\gamma}{2},\,\lambda_2+\frac{\gamma}{2}}(x,\bar{x})}{U} \tag{D.0.10}$$

with the bosonic (and normalised) block [112]

$$\mathtt{B}^{(a,b)}_{\mu_1\mu_2}(z,\bar{z}) = \frac{z h^{(\frac{a}{2},\frac{b}{2},0)}_{\mu_1}(z)\,\bar{z} h^{(\frac{a}{2},\frac{b}{2},0)}_{\mu_2}(\bar{z}) + c.c.}{2(1+\delta_{\mu_1\mu_2})} \tag{D.0.11}$$

These are nothing but the blocks[1] (12.1.2), upon replacing the labels of the Young diagrams with the $SO(2,2)\times SO(4)$ quantum numbers via

$$\begin{aligned} 1+\tfrac{\tau}{2} = \tfrac{\gamma}{2}+\lambda_2 \quad &, \quad l = \lambda_1 - \lambda_2 \geq 0 \\ \tfrac{b}{2}+1 = \tfrac{\gamma}{2}-\mu_1 \quad &, \quad a = \mu_1 - \mu_2 \geq 0. \end{aligned} \tag{D.0.12}$$

Note that, as we anticipated $\mathcal{L}^{\text{long},\mathbb{S}}$ does not depend on γ anymore.

Lastly, let us point out that the combination of hypergeometrics $\mathcal{L}^{\text{long},\mathbb{A}}$ has bosonic quantum numbers identified as

$$\begin{aligned} 1+\tfrac{\tau}{2} = \tfrac{\gamma}{2}+\lambda_2 \quad &, \quad l+1 = \lambda_1 - \lambda_2 \geq 0 \\ \tfrac{b}{2}+1 = \tfrac{\gamma}{2}-\mu_1 \quad &, \quad a+1 = \mu_1 - \mu_2 \geq 0 \end{aligned} \tag{D.0.13}$$

where this time $\mu_1 - \mu_2 \geq 1$, by antisymmetry. This is precisely the same combination of hypergeometrics showing up in the long sector of $\mathcal{N}=4$ SYM [107], cf. formula (2.3.21). In the latter, Young diagrams for long representations have two rows and two columns and the translation between these and the quantum label $\vec{\tau}$ is given by table 2.2. However, it is simple to see that the arguments of the ${}_2F_1$ coincide. Thus the set of $\mathcal{L}^{\text{long},\mathbb{A}}$ is spanned by the same bosonic blocks that appear in $\mathcal{N}=4$ SYM in $4d$.

[1] To be precise we have $(x-y)(\bar{x}-\bar{y})\mathcal{L}^{\text{long},\mathbb{S}} = \mathbb{L}^{\mathbb{S}}_{\vec{\tau}}$.

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