On the virtual and residual properties of a generalization of Bestvina-Brady groups

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Abstract

Previously one of us introduced a family of groups $G_L^M(S)$, parametrized by a finite flag complex L, a regular covering M of L, and a set S of integers. We give conjectural descriptions of when $G_L^M(S)$ is either residually finite or virtually torsion-free. In the case that M is a finite cover and S is periodic, there is an extension with kernel $G_L^M(S)$ and infinite cyclic quotient that is a CAT(0) cubical group. We conjecture that this group is virtually special. We relate these three conjectures to each other and prove many cases of them.

1 Introduction

Bestvina-Brady groups are a family of infinite discrete groups that were constructed in the 1990s to answer a long standing open question in homological group theory [2]; the existence of non-finitely presented groups of type FP. In [16], one of us generalized the Bestvina-Brady construction, producing an uncountable family of groups of type FP. Further results concerning these groups can be found in [14, 5]. Our aim is to study some of the other, non-homological, properties of these groups.

It is well-known that Bestvina-Brady groups are torsion-free, residually finite and linear over $\mathbb{Z}$. We address the question of when the groups introduced in [16] have these and other related properties. In order to state our results, we first need to say a little about the construction of the groups.

Bestvina-Brady groups are parametrized by a finite flag simplicial complex, and we denote by BB_L the group corresponding to the complex L. The map $L \mapsto BB_L$ can be viewed as a functor from the category of non-empty flag complexes and simplicial maps to the category of groups.

The groups $G_L^M(S)$ introduced in [16] are parametrized by a finite connected flag simplicial complex L, together with a connected regular (possibly infinite) covering $M \to L$ of L and a set $S \subseteq \mathbb{Z}$. For the applications to homological group theory the main case of interest is when M is the universal covering, so [16] focussed mainly on that case, but see the discussion in [16, section 21] for the general case. The group of deck transformations of the regular covering $M \to L$ plays a major role in describing $G_L^M(S)$, so we introduce the notation $\pi(M, L)$ for this group. Of course, a choice of basepoints identifies $\pi(M, L)$ with the factor group $\pi_1(L)/\pi_1(M)$ of the two fundamental groups.

For fixed L and M, the groups $G_L^M(S)$ interpolate between two groups that are easily described in terms of Bestvina-Brady groups: $G_L^M(\mathbb{Z})$ is BB_L and $G_L^M(\emptyset)$ is the semidirect product $BB_M \rtimes \pi(M,L)$, where the action of $\pi(M,L)$ on M is used to define its conjugation action on BB_M. (Usually Bestvina-Brady groups are defined only for finite complexes, because this is the case in which the homological finiteness properties of BB_L are controlled by the homology of L, but the definition makes sense for arbitrary flag complexes such as M.)

The case $S = \mathbb{Z}$ is in some ways an exception, as will become apparent in the statement of some of our results. This is because for every other S, $G_L^M(S)$ contains subgroups isomorphic to $\pi(M,L)$. Another exceptional case is when M is the trivial covering of L, or equivalently $\pi(M,L) = \{1\}$; in this case $G_L^L(S) = BB_L$ is independent of S.

It is not hard to decide which of the groups $G_L^M(S)$ are torsion-free.

Proposition 1.1. *The group $G_L^M(S)$ is torsion-free if and only if either $S = \mathbb{Z}$ or $\pi(M,L)$ is torsion-free.*

We give necessary conditions for $G_L^M(S)$ to be virtually torsion-free and to be residually finite. In the statement, a subset S of $\mathbb{Z}$ is said to be periodic if there exists $n > 0$ so that $S + n = S$, and the least such n is called the period of the set S.

Theorem 1.2. *If $G_L^M(S)$ is virtually torsion-free then at least one of the following holds:*

- *$S = \mathbb{Z}$;*
- *$\pi(M,L)$ is torsion-free;*
- *$\pi(M,L)$ is virtually torsion-free and S is periodic.*

If $G_L^M(S)$ is residually finite then at least one of the following holds:

- *$S = \mathbb{Z}$;*
- *$\pi(M,L) = \{1\}$;*
- *$\pi(M,L)$ is residually finite and S is closed in the profinite topology on $\mathbb{Z}$.*

It seems plausible that these necessary conditions may also be sufficient, and so we make the following conjectures.

Conjecture 1.3. *If S is periodic and $\pi(M,L)$ is virtually torsion-free then $G_L^M(S)$ is virtually torsion-free.*

Conjecture 1.4. *If S is closed in the profinite topology on $\mathbb{Z}$ and $\pi(M,L)$ is residually finite then $G_L^M(S)$ is residually finite.*

Some cases of the first part of Theorem 1.2 appeared as [23, thm. 3.1], and conjecture 1.3 of [15] discusses another context in which groups that are parametrized by subsets of $\mathbb{Z}$ are expected to be virtually torsion-free if and only if the subset is periodic; interestingly the opposite implication is the one that remains open for those groups. In the 1970's Dyson defined a family of groups $L(S)$ for $S \subseteq \mathbb{Z}$ as amalgamations of two copies of the lamplighter group, and she showed that $L(S)$ is residually finite if and only if

S is closed in $\mathbb{Z}$ [11]. The connection between residual finiteness of a group parametrized by $S \subseteq \mathbb{Z}$ and the set S being closed arises in [11] for much the same reason as in our work.

We offer some evidence for Conjectures 1.3 and 1.4. Firstly, we offer a reduction to a smaller family of cases.

Theorem 1.5. *If Conjecture 1.3 or Conjecture 1.4 holds whenever $\pi(M, L)$ is finite and S is periodic, then it holds in all cases.*

Secondly, we establish all the conjectures under some hypotheses on the covering.

Theorem 1.6. *Let Γ be a simplicial graph, obtained by subdividing each edge of another graph into at least r pieces, and let $\Delta \to \Gamma$ be any finite regular covering of Γ. Suppose that M is a connected component of M_0, defined as the pullback*

$$\begin{array}{ccc} M_0 & \longrightarrow & \Delta \\ \downarrow & & \downarrow \\ L & \longrightarrow & \Gamma \end{array}$$

for some simplicial map $L \to \Gamma$. For $r \geq 4$ Conjecture 1.3 holds for (M, L) and for $r \geq 12$ Conjecture 1.4 holds for (M, L).

The hypotheses on the covering $M \to L$ split naturally into two parts, one topological and one combinatorial. The topological hypothesis is that there is a graph $\overline{\Gamma}$ and a finite regular covering $p : \overline{\Delta} \to \overline{\Gamma}$, together with a map $f : |L| \to |\overline{\Gamma}|$ of topological realizations, such that $|M|$ is a connected component of the pullback covering. We recall that the pullback is the regular covering of $|L|$ defined as $\{(x, y) \in |L| \times |\overline{\Delta}| \;:\; f(x) = p(y)\}$, with the covering map $(x, y) \mapsto x$. The combinatorial hypothesis is that the triangulation of L (and hence also of M) is sufficiently fine that f is homotopic to a simplicial map from L to a suitable subdivision Γ of $\overline{\Gamma}$.

In the following corollary, we replace the topological hypothesis by a hypothesis that involves only the fundamental groups $\pi_1(L)$ and $\pi_1(M)$, at the expense of making the combinatorial hypothesis far less explicit.

Corollary 1.7. *Suppose that there is a homomorphism $f : \pi_1(L) \to F$, for F a free group and a finite-index normal subgroup $N \triangleleft F$ so that $\pi_1(M) = f^{-1}(N)$. Then both conjectures 1.3 and1.4 hold for sufficiently fine subdivisions (M', L') of the pair (M, L).*

The distinction between the topological and combinatorial hypotheses is a useful one. If (M', L') is a subdivision of (M, L), the homological finiteness properties of $G_{L'}^{M'}(S)$ are similar to those of $G_L^M(S)$. So from the point of view of constructing examples, the combinatorial hypotheses that we make can be ignored. However, we warn the reader that there may be a topological obstruction to each of our conjectures, although we have been unable to construct any counterexamples. In particular, in the case when L is a flag triangulation of the projective plane $\mathbb{R}P^2$ and M its universal cover, we have been able to establish Conjecture 1.3 only for a small number of choices of S, including $S = 2\mathbb{Z}$ which is a special case of Proposition 8.1. We see $L = \mathbb{R}P^2$ as an important test case for Conjecture 1.3.

The proofs of many of our results use an action of the group $G_L^M(S)$ on a CAT(0) cubical complex $X_L^M(S)$, which generalizes the action of BB_L on the universal cover of the Salvetti complex for the right-angled Artin group A_L. The group $G_L^M(S)$ acts freely except that some vertex stabilizers are isomorphic to $\pi(M, L)$. In particular, the action is proper if and only if $\pi(M, L)$ is finite. The action of $G_L^M(S)$ on $X_L^M(S)$ has infinitely many orbits of vertices and so is never cocompact. However, in the case when S is periodic of period n there is a larger cocompact group of cubical automorphisms of $X_L^M(S)$ which we denote by $G_L^M(S) \rtimes n\mathbb{Z}$. The action of $G_L^M(S) \rtimes n\mathbb{Z}$ on $X_L^M(S)$ has n orbits of vertices and the group contains $G_L^M(S)$ as a normal subgroup with infinite cyclic quotient.

In the case when both $\pi(M, L)$ is finite and S is periodic of period $n > 0$, the group $G_L^M(S) \rtimes n\mathbb{Z}$ is CAT(0) cubical in the sense that it acts properly and cocompactly on the CAT(0) cube complex $X_L^M(S)$. We make a third conjecture concerning this case.

Conjecture 1.8. *If $\pi(M, L)$ is finite and $S + n = S$ for some $n > 0$ then the CAT(0) cubical group $G_L^M(S) \rtimes n\mathbb{Z}$ is virtually special in the sense of [12].*

Haglund and Wise showed that virtually special groups are virtually torsion-free and residually finite [12], and hence Conjecture 1.8 implies our other conjectures.

Proposition 1.9. *Conjecture 1.8 for the complex L implies Conjectures 1.3 and 1.4 for the complex L.*

Rather more surprisingly, we show that, at least from the topological viewpoint, there is no obstruction to Conjecture 1.8 provided that Conjecture 1.3 holds.

Theorem 1.10. *Suppose that $\pi(M, L)$ is finite and that $S + n = S$ for some $n > 0$. If $G_L^M(S)$ is virtually torsion-free, then for any sufficiently fine subdivision (M', L') of the pair (M, L), the group $G_{L'}^{M'}(S) \rtimes n\mathbb{Z}$ is virtually special. In particular, Conjecture 1.3 for L implies Conjectures 1.8 and 1.4 for L'. The second barycentric subdivision is sufficiently fine.*

To show that $G_L^M(S)$ is virtually torsion-free, we rely on group presentations and carefully chosen maps to finite groups. However, most of our other results rely heavily on studying the cube complex $X_L^M(S)$. For example, to prove that a non-identity element $g \in G_L^M(S)$ has non-identity image in $G_L^N(T)$ for some finite cover N and periodic $T \supseteq S$ we show that we can choose N and T so that the geodesic in $X_L^M(S)$ from a base vertex x_0 to gx_0 projects to a geodesic in $X_L^N(T)$. All our results concerning residual finiteness rely on proving cases of Conjecture 1.8. Like the action of BB_L on the universal covering of the Salvetti complex, the action of $G_L^M(S)$ on the CAT(0) cube complex $X_L^M(S)$ is never cocompact, but it does have only finitely many orbits of hyperplanes. To show that $X_L^M(S)/H$ is non-cocompact special for some torsion-free finite-index normal subgroup $H \leq G_L^M(S)$ we use the action of $Q = G_L^M(S)/H$ on the complex. Edges of $X_L^M(S)/H$ are in free Q-orbits, but some of the vertices are in non-free orbits. As an example, to show that a hyperplane in $X_L^M(S)/H$ cannot directly self-osculate we consider the stabilizer in Q of the hyperplane. Provided that this stabilizer has trivial intersection with each vertex stabilizer no direct inter-osculation can occur.

After reviewing some background material, we prove Theorem 1.2 in Section 3 and Theorem 1.5 in Section 4. Sections 5 and 6 complete the proof of Theorem 1.10 and

Section 7 completes the proof of Theorem 1.6. Sections 8 and 9 describe further examples, and in Section 10 we use our results to construct groups with surprising combinations of properties.

Much of this work was done while the second named author was working on his PhD under the supervision of the first named author; further related work appears in the second named author's PhD thesis [24] and in [23]. The authors thank the referee for their comments on an earlier version of this article.

2 Background

A flag complex is a simplicial complex L with the property that every finite clique within its edge graph spans a simplex. The barycentric subdivision of any simplicial complex is flag. The right-angled Artin group associated to a flag complex is the group with generators the vertices of L, subject only to the relations that the ends of each edge commute:

$$A_L = \langle v \in L^0 \ : \ [v,w] = 1 \ (v,w) \in L^1 \rangle.$$

This construction is functorial in L, in the sense that a simplicial map $f : M \to L$ induces a group homomorphism $f_* : A_M \to A_L$ defined on generating sets by $f_*(v) = f(v)$.

There is a good model of the Eilenberg-Mac Lane space $K(A_L, 1)$, the Salvetti complex, which we shall denote by $\mathbb{T}_L$. For $v \in L^0$, let $\mathbb{T}_v$ be a copy of the unit circle, viewed as a CW-complex with one 0-cell and one 1-cell. For each simplex σ of L, define $\mathbb{T}_\sigma$ to be the product $\prod_{v\in\sigma} \mathbb{T}_v$. This gives a functor from the simplices of L (including the empty set, viewed as the unique -1-simplex) to CW-complexes and cellular maps, and the complex $\mathbb{T}_L$ is the colimit of this functor, which is a subcomplex of the product $\prod_{v\in L^0} \mathbb{T}_v$. Equivalently, $\mathbb{T}_L$ is the polyhedral product of the pair $(\mathbb{T}, *)^L$.

The map $L \to \mathbb{T}_L$ can also be made strictly functorial, using the group structure on the circle, where we insist that the 0-cell in the CW-complex $\mathbb{T}$ is the identity element of the group. If $f : M \to L$ is a simplicial map of flag complexes, the group structure on the torus is used to define a based map $\mathbb{T}(f) : \mathbb{T}_M \to \mathbb{T}_L$ that induces $f_* : A_M \to A_L$ on fundamental groups. To describe $\mathbb{T}(f)$, view $\mathbb{T}_M$ as a subcomplex of the torus $\mathbb{T}^{M^0}$ and similarly, view $\mathbb{T}_L$ as a subcomplex of the torus $\mathbb{T}^{L^0}$. With this notation, the map $\mathbb{T}(f)$ can be defined coordinatewise. For $v \in L^0$, let $U = f^{-1}(v) \subseteq M^0$. Now the v-coordinate of $\mathbb{T}(f)$ takes $(t_1, \ldots, t_k) \in \mathbb{T}^U = \prod_{u\in U} \mathbb{T}_u$ to the product $t_1 \cdots t_k \in \mathbb{T}_v$.

The Bestvina-Brady group BB_L associated to a non-empty flag complex is the kernel of the homomorphism $A_L \to \mathbb{Z}$ that sends each vertex to $1 \in (\mathbb{Z}, +)$. Like A_L, this is functorial in the non-empty flag complex L; in particular, if $*$ denotes a 1-vertex complex then BB_L may be viewed as the kernel of the map $A_L \to A_*$ induced by the unique (simplicial) map $L \to *$. There is a good model for $K(BB_L, 1)$, defined as the infinite cyclic covering $\widetilde{\mathbb{T}}_L$ of $\mathbb{T}_L$, which comes equipped with a $\mathbb{Z}$-equivariant map to the universal cover of $\mathbb{T}_*$, which is a copy of $\mathbb{R}$.

The complex $\mathbb{T}_L$ has a single vertex. The link of this vertex is the sphericalization or octahedralization $\mathbb{S}(L)$ of L. It has two vertices v^+, v^- for each vertex $v \in L^0$, where for any choices of signs ϵ_i, the vertices $v_0^{\epsilon_0}, \ldots, v_n^{\epsilon_n}$ span an n-simplex of $\mathbb{S}(L)$ if and only if the vertices $v_0, \ldots, v_n$ span an n-simplex of L. In the case when L is itself an n-simplex, $\mathbb{S}(L)$ is an n-sphere, triangulated as the boundary of the $(n + 1)$-dimensional analogue of the octahedron. The universal covering X_L of $\mathbb{T}_L$ is a CAT(0) cubical complex, on which A_L acts freely cellularly, wtih one orbit of vertices. There is an A_L-equivariant map $X_L \to X_* \cong \mathbb{R}$ which we view as a height function on X_L; the subgroup BB_L is the subgroup of elements that act trivially on X_*, while each of the standard generators for A_L acts on $X_* \cong \mathbb{R}$ as translation by 1.

Provided that L is connected, the group BB_L is generated by elements indexed by the directed edges of L, where the directed edge a from x to y corresponds to the element $x^{-1}y$ of $BB_L \leq A_L$. The two directions of a directed edge correspond to mutually inverse elements, and for each directed cycle $(a_1, \ldots, a_l)$ and each integer n, the product $a_1^n a_2^n \cdots a_l^n$ is the identity. It can be shown that these relators, for all cycles and all non-zero integers n, suffice to present BB_L [10]. For directed cycles (a, b, c) of length 3, the relators $abc = 1 = a^{-1}b^{-1}c^{-1}$ imply that a, b and c commute and generate a group isomorphic to $\mathbb{Z}^2$, so for cycles of length 3, only the relators for $n = \pm 1$ are needed. See [10] for more details.

Now suppose that M is a connected regular covering of L, with $\pi = \pi(M, L)$ as its group of deck transformations. In this case π acts on the given presentations for A_M and BB_M by permuting the generators and the relations. Thus we can form the semi-direct products $A_M \rtimes \pi$ and $BB_M \rtimes \pi$. The action of π on M also induces an action of π on $\mathbb{T}_M$, which permutes the cells freely except that the vertex is fixed. In this way $A_M \rtimes \pi$ is realized geometrically as the group of all self-isomorphisms of X_M that lift the action of some element of π on $\mathbb{T}_M$. If H is a subgroup of $A_M \rtimes \pi$ that maps isomorphically to π under the map $A_M \rtimes \pi \to \pi$, then H fixes a vertex of X_M, and no other point of X_M. This follows from the facts that the action is by isometries, so the geodesic between two fixed points would also be fixed, and that the group H acts freely on the link of the vertex that it fixes. Thus in $A_M \rtimes \pi$, there is a bijective correspondence between the vertices of X_M and the subgroups H that map isomorphically to π. Furthermore, each such H is its own normalizer, because the normalizer of H must act on the fixed point set X_M^H but H is the entire stabilizer of this set. Since there is just one A_M-orbit of vertices in X_M, all of these subgroups are conjugate in $A_M \rtimes \pi$.

Now consider the group $BB_M \rtimes \pi$. Under the action of $BB_M \rtimes \pi$, vertices of different heights lie in different orbits, while vertices of the same height lie in the same orbit. Hence one sees that the conjugacy classes in $BB_M \rtimes \pi$ of vertex stabilizers are permuted freely transitively by $A_M/BB_M \cong \mathbb{Z}$. Choosing for once and for all an equivariant bijection between the set of conjugacy classes of vertex stabilizers in $BB_M \rtimes \pi$ and the group $A_M/BB_M \cong \mathbb{Z}$, we can index the conjugacy classes of vertex stabilizers by $\mathbb{Z}$. (Equivalently, this amounts to fixing a choice of splitting map $\pi \to BB_M \rtimes \pi$.) For $S \subseteq \mathbb{Z}$, let $N(S)$ denote the normal subgroup of $BB_M \rtimes \pi$ generated by the stabilizers of the vertices whose height lies in S. The group $G_L^M(S)$ can be defined as the factor group $BB_M \rtimes \pi / N(S)$. A geometric argument (essentially [16, lemmas 14.3,14.4]) shows that if H is the stabilizer of a vertex of height not in S, then $H \cap N(S)$ is trivial. The quotient complex $X_M/N(S)$ has vertex links $\mathbb{S}(L)$ for the vertices of height in S and $\mathbb{S}(M)$ for the vertices of height not in S. Since $N(S)$ is generated by elements that fix a vertex of X_M, the quotient complex $X_M/N(S)$ is simply connected. Hence by Gromov's criterion it is CAT(0), and we define it to be $X_L^M(S)$. The group $G_L^M(S)$ acts on it by isometries, freely except that vertices whose height is not in S have stabilizer isomorphic to π.

Since the conjugate $vN(S)v^{-1}$ of $N(S)$ by any $v \in M^0$ is equal to $N(S+1)$, for general

S the subgroup BB_M is the entire normalizer of $N(S)$ inside A_M. The only exceptions to this are subsets S that are periodic: if $S+n=S$ for some $n>0$, then the normalizer of $N(S)$ contains the index n subgroup of A_M which is the inverse image in A_M of the unique index n subgroup of $A_M/BB_M \cong \mathbb{Z}$. If we write H for this subgroup, then the quotient $H/N(S)$ is a (necessarily split) extension with kernel $G_L^M(S)$ and infinite cyclic quotient, which we will denote by $G_L^M(S)\rtimes n\mathbb{Z}$. Since H acts cocompactly on X_M (with n orbits of vertices), it follows that $G_L^M(S)\rtimes n\mathbb{Z}$ acts cocompactly on $X_L^M(S) = X_M/N(S)$. This is the group and action that feature in the statement of Conjecture 1.8.

The presentation that we gave for BB_M gives rise to a presentation for each group $G_L^M(S)$. For simplicity, we will focus mainly on the case when $0 \in S$. Since for $v \in M^0$, we have that $vN(S)v^{-1} = N(S+1)$, the only isomorphism type that is not covered by this assumption is $G_L^M(\emptyset) = BB_M \rtimes \pi$. First consider the case $S = \{0\}$. Killing the standard copy of π inside $BB_M \rtimes \pi$ has the effect of identifying directed edges of M that lie in the same π-orbit. Hence the group $G_L^M(\{0\})$ has the directed edges of L as its generators. The relators have a similar form to the relators in the presentation we described above for BB_L, except that we now have relators of the form $a_1^n a_2^n \cdots a_l^n = 1$ for all $n \in \mathbb{Z}$, but only for directed edge loops in L that lift to loops in M under the covering map. With respect to this presentation, one may readily describe a representative of each conjugacy class of vertex stabilizers. Fix a vertex $v \in L$ and a lift $v_0 \in M$ of v. For each vertex v_1 in the orbit $\pi.v_0$, pick a directed edge loop $(a_1, \ldots, a_l)$ in L that can be lifted to a path in M from v_0 to v_1. Since the composite of one such loop with the reverse of another is a directed loop in L that lifts to a loop in M, the group element defined by $a_1^n a_2^n \cdots a_l^n$ does not depend on the choice of the loop, only on v_0 and v_1. For each fixed $n \neq 0$, these elements form a subgroup of $G_L^M(S)$ that is isomorphic to π. Different values of n correspond to different conjugacy classes of subgroup.

In this way, we obtain a presentation for $G_L^M(S)$ whenever $0 \in S$. The generators are the directed edges of L. For each $n \in S$, we take the relators $a_1^n a_2^n \cdots a_l^n$ for all directed edge loops $(a_1, \ldots, a_l)$. For each $n \notin S$, we also take the relators $a_1^n a_2^n \cdots a_l^n$, but only for those directed edge loops in L that lift to loops in M.

This presentation makes clear the functoriality of the group $G_L^M(S)$. Given any commutative square of simplicial maps in which the vertical maps are coverings

$$\begin{array}{ccc} M' & \to & M \\ \downarrow & & \downarrow \\ L' & \to & L \end{array}$$

and any inclusion $S' \subseteq S \subseteq \mathbb{Z}$, there is an induced homomorphism $G_{L'}^{M'}(S') \to G_L^M(S)$. In particular this applies when $L' \to L$ is a simplicial map and M' is obtained as a connected component of the pullback of a covering M of L, and when $L' = L$ and M' is a covering of L that factors through M.

For the sake of completeness, we now give some more details about the presentation for $G_L^M(\emptyset)$ and in particular, we describe representatives of the conjugacy classes of subgroups isomorphic to $\pi = \pi(M, L)$. The generators for $G_L^M(\emptyset)$ will be the directed edges $a, b, c, \ldots$ of M together with the elements $g, h, j, \ldots$ of π. We view π as acting on the *left* of M via deck transformations, and for $g \in \pi$ and a a directed edge of M, let $g \cdot a$ be the directed edge obtained by acting on a by g. Thus the relations for $G_L^M(\emptyset) = BB_M \rtimes \pi$ are the relations previously described in the presentation for BB_M, the relations that hold between the elements of π, and the conjugation relations which have the form $gag^{-1} = g \cdot a$ for each $g \in \pi$ and each directed edge a. If $\gamma = (a_1, \ldots, a_l)$ is a directed edge path in M, it will be convenient to introduce the notation $\gamma[n]$ for the group element $a_1^n a_2^n \cdots a_l^n$.

To construct representatives of the different conjugacy classes of subgroups isomorphic to π, we first fix a vertex $v \in M$. Next, for each $g \in \pi$, choose a path γ_g from v to $g \cdot v$. If γ_g' is another such choice, then the concatenation $\gamma_g.\overline{\gamma_g'}$ of γ_g with the reverse of γ_g' is a closed loop in M. From this it follows that for each $n \in \mathbb{Z}$, $\gamma_g[n] = \gamma_g'[n]$, so the group element $\gamma_g[n]$ does not depend on the choice of path γ. For each $n \in \mathbb{Z}$, define a subset $\pi(n)$ of $G_L^M(\emptyset)$ as

$$\pi(n) := \{\gamma_g[n]g \quad g \in \pi\}.$$

We claim that $\pi(n)$ is a subgroup of $G_L^M(\emptyset)$ that is isomorphic to π. Since $\pi(n)$ maps bijectively to π under the map $G_L^M(\emptyset) \to \pi$, this claim will follow provided that $\pi(n)$ is closed under multiplication.

For any $g, h \in \pi$, the directed path $g \cdot \gamma_h$ obtained by applying g to the path γ_h is a path from $g \cdot v$ to $gh \cdot v$. Hence the concatenation $\gamma_g.(g \cdot \gamma_h)$ is a directed path from v to $gh \cdot v$. From this it follows that in $G_L^M(\emptyset)$

$$\gamma_g[n]g\gamma_h[n]h = \gamma_g[n](g\gamma_h[n]g^{-1})gh = (\gamma_g.g \cdot \gamma_h)[n]gh = \gamma_{gh}[n]gh.$$

Hence $\pi(n)$ is closed under multiplication and is a subgroup isomorphic to π as claimed. Replacing the vertex v by another vertex $w \in M$ gives rise to a group that is conjugate to the group $\pi(n)$; just conjugate by the element $\gamma[n]$, where γ is a path between v and w. To show that the groups $\pi(n)$ represent all of the different conjugacy classes one can use a geometric argument. Alternatively, the description of the conjugation action of A_M on BB_M in [10] can be used to show directly that the conjugate of $\pi(n)$ by the vertex v, viewed as one of the generators for A_M, is equal to $\pi(n+1)$.

The above description of the subgroups $\pi(n)$ of $G_L^M(\emptyset)$ gives a way to present each $G_L^M(S)$ as a quotient of $G_L^M(\emptyset)$. To the relations for $G_L^M(\emptyset)$ one adds the relation $\gamma_g[n]g = 1$ for each $n \in S$ and each $g \in \pi$.

Next we briefly recall some material concerning special cube complexes from [12]. A *hyperplane* in a non-positively curved cube complex is an equivalence class of directed edges under the relation generated by 'form the opposite directed edges of a square'. A hyperplane is *2-sided* if it does not contain any pair of directed edges associated to a single directed edge. Hyperplanes *intersect* if they contain directed edges that are adjacent sides of a square. Two directed edges *directly osculate* if they are not contained in a square and share the same terminal vertex. A hyperplane *self-intersects* if there is a square containing two directed edges of the hyperplane as adjacent sides. A hyperplane *directly self-osculates* if it contains two directed edges that directly osculate. Two hyperplanes *inter-osculate* if they intersect and also contain a pair of directed edges that directly osculate.

A locally CAT(0) cube complex is *A-special* if its hyperplanes are 2-sided and do not self-intersect, directly self-osculate or inter-osculate. The fundamental group of a finite A-special cube complex embeds in a right-angled Artin group and hence is torsion-free and linear over $\mathbb{Z}$, which implies that it is residually finite [12, thm. 1.1]. A *special* locally CAT(0) cube complex is similar to an A-special complex except that hyperplanes are not required to be 2-sided. For each of our complexes $X_L^M(S)$ the height function

$X_L^M(S) \to \mathbb{R}$ allows one to distinguish upward and downward pointing directed edges. For this reason every hyperplane in $X_L^M(S)/H$ is 2-sided for any $H \le G_L^M(S)$, and so $X_L^M(S)/H$ is special if and only if it is A-special.

We close this section with some remarks concerning the profinite topology on $\mathbb{Z}$. This is the topology in which the basic open sets are the periodic subsets. Each periodic subset is also closed, and it follows that every open set is a union of periodic sets and that every closed set is an intersection of periodic sets. We make a more precise version of this second statement below.

Lemma 2.1. *Any $S \subseteq \mathbb{Z}$ that is closed in the profinite topology is the intersection of a nested sequence $T_1 \supseteq T_2 \supseteq T_3 \cdots$ of periodic sets. Furthermore, we may suppose that $S \cap [-n, n] = T_n \cap [-n, n]$.*

Proof. Let $n_1, n_2, n_3, \ldots$ be the elements of $\mathbb{Z}-S$, enumerated so that $|n_i| \le |n_j|$ whenever $i < j$. Since S is closed, for each i we can find an open set O_i with $n_i \in O_i$ and $S \cap O_i = \emptyset$. Since the periodic sets form a basis for the topology, we may suppose in addition that O_i is periodic. If we let $F_i = \mathbb{Z} - O_i$ then F_i is periodic, $S \subseteq F_i$, and $n_i \notin F_i$. From this it follows that $\bigcap_i F_i = S$. If we define $T_n = \bigcap_{i=1}^{2n+1} F_i$ then T_n has the properties claimed in the statement. □

The natural numbers $\mathbb{N} \subset \mathbb{Z}$ is an easy example of a subset that is far from being either open nor closed: the only closed set that contains $\mathbb{N}$ is $\mathbb{Z}$ and the only open set contained in $\mathbb{N}$ is the empty set. We give a construction of a large collection of closed subsets, starting with a Gödel numbering ϕ of the finite subsets of $\mathbb{N}$. For F a finite subset of $\mathbb{N}$, define

$$\phi(F) = \sum_{n \in F} 10^n,$$

with the usual convention that the empty sum is 0. The image of ϕ is the subset $T(\mathbb{N})$ consisting of all positive integers all of whose decimal digits are equal to 0 or 1. Now for any $S \subseteq \mathbb{N}$, define $T(S) \subseteq \mathbb{Z}$ to be the image under ϕ of the finite subsets of S; equivalently $T(S)$ is the positive integers with digits $\{0, 1\}$, where the nth digit is 0 when $n \notin S$.

Proposition 2.2. *For each $S \subseteq \mathbb{N}$, the set $T(S)$ is closed. The set $T(S)$ (resp. $\mathbb{N}-T(S)$) is recursively enumerable if and only if S (resp. $\mathbb{N} - S$) is.*

Proof. For $n \ge 0$, let $F_n = (T(S) \cap [0, 2.10^n]) + 10^{n+1}\mathbb{Z}$. Each F_n is periodic and $T(S) = \bigcap_n F_n$, which implies that $T(S)$ is closed. The definition of $T(S)$ gives a recursive procedure for computing $T(S)$ from S, showing that $T(S)$ is recursively enumerable when S is. Computing $\mathbb{N}-T(S)$ from $\mathbb{N}-S$ is slightly more complicated. To compute $\mathbb{N}-T(S)$, fix an integer N and run for N steps an algorithm to generate elements of $C := \mathbb{N} - S$, keeping a list $C(N)$ of the elements of C so obtained. Then output every integer in the range $[0, N]$ that either has a decimal digit not equal to 0 or 1, or has its nth decimal digit equal to 1 for some $n \in C(N)$. Now repeat this procedure for increasing values of N. For the converse statements, note that $10^n \in T(S)$ if and only if $n \in S$. Thus a recursive enumeration of $T(S)$ (resp. $\mathbb{N} - T(S)$) gives rise to a recursive enumeration of S (resp. $\mathbb{N} - S$). □

3 A set-valued invariant

In this section we prove Theorem 1.2 using a set-valued invariant of a group and a sequence of elements, $\mathcal{R}(G, \mathbf{g})$ that was introduced in [16]. But first we prove Proposition 1.1.

Proof. (Proposition 1.1) $G_L^M(S)$ acts on the CAT(0) cubical complex $X_L^M(S)$, freely except that vertices whose height is not in S have stabilizer isomorphic to $\pi(M, L)$. Any action of an element of finite order on any CAT(0) space must fix a point, and the claim follows. $\square$

For a group G and a finite sequence $\mathbf{g} = (g_1, \ldots, g_l)$ of elements of G, the invariant $\mathcal{R}(G, \mathbf{g})$ is the subset of $\mathbb{Z}$ defined by

$$\mathcal{R}(G, \mathbf{g}) = \{n \in \mathbb{Z} \;:\; g_1^n g_2^n \cdots g_l^n = 1\}.$$

In [16, lemma 15.3], it was shown that in the case when $G = G_L^M(S)$ and $\mathbf{g}$ is the sequence of generators spelling out an edge loop in L that does not lift to a loop in M, then $\mathcal{R}(G, \mathbf{g}) = S$. (This was stated only in the case when M is the universal cover, but the same proof holds in general.) We require another property:

Proposition 3.1. *If G is finite, $\mathcal{R}(G, \mathbf{g})$ is periodic. If G is residually finite, $\mathcal{R}(G, \mathbf{g})$ is closed in the profinite topology on $\mathbb{Z}$.*

Proof. For the first statement, it suffices to make the weaker assumption that G has finite exponent, m say. In this case for any n, $g_1^{n+m} g_2^{n+m} \cdots g_l^{n+m} = g_1^n g_2^n \cdots g_l^n$, from which it follows that $\mathcal{R}(G, \mathbf{g}) = \mathcal{R}(G, \mathbf{g}) + m$.

For the second statement, whenever $f : G \to Q$ is a homomorphism from G to a finite group, the first statement implies that $\mathcal{R}(Q, f(\mathbf{g}))$ is periodic. Since G is assumed to be residually finite, $g_1^n \cdots g_l^n$ is equal to the identity if and only if its image under each such $f : G \to Q$ is. Hence $\mathcal{R}(G, \mathbf{g})$ is the intersection of a family of periodic sets of the form $\mathcal{R}(Q, f(\mathbf{g}))$. This proves the claim. $\square$

Proof. (Theorem 1.2) Suppose firstly that $G_L^M(S)$ is virtually torsion-free. If $S = \mathbb{Z}$ or $\pi(M, L)$ is torsion-free, then we have already seen that $G_L^M(S)$ is torsion-free. Thus we may suppose that $S \neq \mathbb{Z}$ and that $\pi(M, L)$ contains some torsion. Since $S \neq \mathbb{Z}$, $G_L^M(S)$ contains subgroups isomorphic to $\pi(M, L)$, and so $\pi(M, L)$ must be virtually torsion-free as claimed. Now let $\gamma = (a_1, \ldots, a_l)$ be a directed edge loop in L that represents a non-trivial torsion element in $\pi(M, L)$. Thus γ does not lift to a closed loop in M but there is some $m > 1$ the iterated loop γ^m does lift to a closed loop in M. Now suppose that $f : G_L^M(S) \to Q$ is a homomorphism to a finite group with torsion-free kernel. For each n, $a_1^n \cdots a_l^n$ is a torsion element of $G_L^M(S)$, and this element is equal to the identity if and only if $n \in S$. Since the kernel of f is torsion-free, it follows that $\mathcal{R}(Q, (f(a_1), \ldots, f(a_l)) = S$, and so by Proposition 3.1, S must be periodic.

Next suppose that $G_L^M(S)$ is residually finite. If either $S = \mathbb{Z}$ or $\pi(M, L)$ is trivial, then $G_L^M(S)$ is the Bestvina-Brady group BB_L, which is residually finite. Thus we may assume that $\pi(M, L) \neq \{1\}$ and that $S \neq \mathbb{Z}$. Since $S \neq \mathbb{Z}$, $G_L^M(S)$ contains subgroups isomorphic to $\pi(M, L)$ and so $\pi(M, L)$ must be residually finite as claimed. Now let $(a_1, \ldots, a_l)$ be a directed edge loop in L that does not lift to a loop in M. The element $a_1^n \cdots a_l^n \in G_L^M(S)$

is equal to the identity if and only if $n \in S$. Hence $S = \mathcal{R}(G_L^M(S), (a_1, \dots, a_l))$ must be closed in the profinite topology by Proposition 3.1.

$\square$

4 Reduction to finite covers and periodic sets

The two theorems in this section, Theorems 4.1 and 4.2, together imply Theorem 1.5.

Theorem 4.1. *Suppose that S is closed in the profinite topology. For any non-identity element $g \in G_L^M(S)$ there is periodic $T \supseteq S$ so that the image of g in $G_L^M(T)$ is not the identity.*

Proof. This argument is based on one in [14]. If S is periodic there is nothing to prove. If not, then $S \neq \emptyset$ and we may assume without loss of generality that $0 \in S$, and we may take as generators for $G_L^M(S)$ the directed edges of L. By Lemma 2.1 there is a nested sequence $T_1 \supseteq T_2 \supseteq \cdots$ of periodic sets with intersection S and such that $T_n \cap [-n, n] = S \cap [-n, n]$. It will suffice to show that the word length of any non-identity element of $G_L^M(S)$ that maps to the identity in $G_L^M(T_n)$ tends to infinity with n.

Since $0 \in S$, the group $G_L^M(S)$ acts freely on the vertices of height 0 in $X_L^M(S)$, and the Cayley graph for $G_L^M(S)$ embeds in the height 0 subset of $X_L^M(S)$, with each generator mapping to the diagonal of a square. Fix v_o a vertex of height 0 in $X_L^M(S)$ and let γ be the geodesic arc from v_0 to gv_0, and let f_n denote the map $f_n : X_L^M(S) \to X_L^M(T_n)$. if $f_n \circ \gamma$ is a geodesic arc in $X_L^M(T_n)$, then $f_n(gv_0) \neq f(v_0)$, which implies that $f_n(g) \neq 1$. This will happen unless γ passes through a vertex of $X_L^M(S)$ of height in $T_n - S$. By the argument used in [14, lemma 3.2], this implies that the word length of g is strictly greater than $n\sqrt{2/(d+1)}$, where d is the dimension of L. $\square$

Theorem 4.2. *If $\pi(M, L)$ is residually finite, then for any non-identity element $g \in G_L^M(S)$, there is a finite regular cover $N \to L$ lying between M and L so that the image of g in $G_L^N(S)$ is not the identity.*

Proof. Let x be a point of $X_L^M(S)$ such that $d(x, gx)$ is minimized. If $d(x, gx) = 0$, then x must be a vertex, and g is contained in a conjugate of $\pi(M, L) \leq G_L^M(S)$. By hypothesis there is a finite quotient Q of $\pi(M, L)$ in which the image of g is not the identity, and we may choose N so that $\pi(N, L) = Q$.

Otherwise, let γ be the unique geodesic arc from x to gx in $X_L^M(S)$. If $N \to L$ is any finite regular covering so that M also covers N, let $f : X_L^M(S) \to X_L^N(S)$ be the induced map of CAT(0) cubical complexes. If $f \circ \gamma$ is a geodesic arc in $X_L^N(S)$, then $f(x) \neq f(gx)$, indicating that the image of g in $G_L^N(S)$ is not the identity. Thus it suffices to show that we can choose N so that $f \circ \gamma$ is a geodesic arc. For any N, the map f will be a local isometry except at the vertices with height in S, so if the interior of γ contains no such vertices, we may take $N = L$. In any case, there are only finitely many such vertices.

For each vertex v that is contained in the interior of γ, the inward and outward pointing parts of γ define a pair of points $\gamma^-, \gamma^+ \in \mathrm{Lk}_X(v) \cong \mathbb{S}(M)$ necessarily separated by at least π in $\mathbb{S}(M)$. Now $f \circ \gamma$ is locally geodesic at $f(v)$ provided that the distance in $\mathrm{Lk}_{f(X)}(f(v)) \cong \mathbb{S}(N)$ between $f(\gamma^-)$ and $f(\gamma^+)$ is also at least π. The open ball of radius π in $\mathbb{S}(M)$ centred at γ^- contains only finitely many points $h\gamma^+$ of the orbit $\pi(M, L)\gamma^+$, and none of these h is the identity. Since there are only finitely many vertices on γ, we obtain a finite set $\{h_1, \ldots, h_m\}$ of non-identity elements of $\pi(M, L)$ with the property that $f \circ \gamma$ is a geodesic arc provided that $f(h_i) \neq 1 \in \pi(N, L)$ for each h_i. Since $\pi(M, L)$ is residually finite, we can find such an N. $\square$

5 Simplicial approximations

If L' is a subdivision of a flag complex L, a simplicial approximation to the identity is a simplicial map $f : L' \to L$ such that the induced map of topological spaces $f_* : |L'| \to |L|$ is homotopic to the identity map. If $M \to L$ is a covering, and $M' \to L'$ is the induced covering of L', then any simplicial approximation to the identity for L will lift to a $\pi(M, L)$-equivariant simplicial approximation to the identity for M.

Definition 5.1. *A subdivision L' of L is* suitable *if there is a simplicial approximation f to the identity such that for any pair u, v of adjacent vertices of L', the image $f(\mathrm{St}(u) \cup \mathrm{St}(v))$ is contained in a single simplex of L.*

Proposition 5.2. *The second barycentric subdivision of any simplicial complex is suitable.*

Proof. The vertices of the barycentric subdivision of L are indexed by the simplices of L, with an edge joining the vertices τ, σ if and only if one of τ and σ is a face of the other. There is a well-known description of a simplicial approximation in this case: fix a partial order on the vertex set of L that is total when restricted to each simplex, and send the vertex σ of the barycentric subdivision to the least vertex in L of the simplex σ.

The vertices of the second barycentric subdivision of L are indexed by chains $\underline{\sigma} = \sigma_0 < \sigma_1 < \cdots < \sigma_n$ of simplices of L, where $\underline{\sigma}$ and $\underline{\tau}$ are joined by an edge if and only if one is a subchain of the other. Write $\underline{\tau} \subseteq \underline{\sigma}$ to indicate that $\underline{\tau}$ is a subchain of $\underline{\sigma}$. The natural choice of a partial order on the vertices of the barycentric subdivision is to order by dimension of the corresponding simplex of L. With these choices, the composite simplicial approximation from the second barycentric subdivision to L sends the vertex $\underline{\sigma}$ to the least vertex of the simplex σ_0, i.e., the least vertex of the minimal simplex in the chain.

A vertex in the star $\mathrm{St}(\underline{\sigma})$ is either a subchain or a superchain of $\underline{\sigma}$. If $\underline{\tau} \subseteq \underline{\sigma}$, then the minimal simplex τ_0 of $\underline{\tau}$ is contained in σ_n, the maximal simplex of $\underline{\sigma}$. If instead $\underline{\tau} \supseteq \underline{\sigma}$, then τ_0 is contained in σ_0. In either case, the minimal vertex of the minimal simplex of $\underline{\tau}$ is a vertex of σ_n, and so $f(\mathrm{St}(\underline{\sigma}))$ is contained in the simplex σ_n.

If there is an edge in the second barycentric subdivision between $\underline{\sigma}$ and $\underline{\tau}$, then one of the two chains is a subchain of the other, so we may suppose $\underline{\tau} \subseteq \underline{\sigma}$. In this case the maximal simplex τ_p is contained in the maximal simplex σ_n, and so $f(\mathrm{St}(\underline{\tau}) \cup \mathrm{St}(\underline{\sigma}))$ is contained in the single simplex σ_n of L. $\square$

6 Special cube complexes

There is a natural identification of $X_L^M(S)/G_L^M(S)$ with X_L/BB_L, so we start by considering the cube complex X_L/BB_L, and its quotient $\mathbb{T}_L = X_L/A_L$. The Salvetti complex

$\mathbb{T}_L$ has one vertex, edges in bijective correspondence with the vertex set L^0 and squares in bijective correspondence with L^1. The map $\mathbb{T}_L \to \mathbb{T}$ described earlier lifts to a map $X_L/BB_L \to \mathbb{R}$ such that the image of each vertex is an integer, and furthermore the image of each n-cube of X_L is an interval of length n.

We may view edges of $X_L^M(S)$ as being labelled by elements of L^0 via the identification of $X_L^M(S)/G_L^M(S) = X_L/BB_L$, and the squares of $X_L^M(S)$ as being labelled by elements of L^1. The function $X_L/BB_L \to \mathbb{R}$ induces a height function on $X_L^M(S)$. This height function and the labellings discussed above are preserved by the action of $G_L^M(S)$. In X_L/BB_L there is one vertex of each height $n \in \mathbb{Z}$, and for each $x \in L^0$ there is one edge labelled x whose vertices are of heights n and $n+1$. Directed edges either point upwards or downwards, and the opposite sides of a square of $X_L^M(S)$ point the same way.

If H is a finite index normal subgroup of $G_L^M(S)$, it follows that $X_L^M(S)/H$ has finitely many vertices of each height, and finitely many edges of each height. Moreover, the group $G_L^M(S)/H$ acts freely and transitively on the edges with label $x \in L^0$ of each fixed height. This group acts transitively on the vertices of each fixed height too, but for vertices whose height is not in S, the stabilizer of a vertex is the group $\pi(M,L)/(\pi(M,L) \cap H)$.

Since the adjacent sides of each square have distinct labels in L^0, no hyperplane of $X_L^M(S)/H$ can self-intersect. Since the opposite sides of each square point either upwards or downwards, no hyperplane in $X_L^M(S)/H$ can fail to be 2-sided. Thus to establish that $X_L^M(S)/H$ is special, we only need to check that there are no direct self-osculations and no inter-osculations.

Each simplex σ of L corresponds to a coordinate subtorus $\mathbb{T}_\sigma$ of $\mathbb{T}_L$, and this lifts to a single infinite cylinder inside X_L/BB_L. If we identify the torus $\mathbb{T}_\sigma$ with the quotient $\mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$, where σ is an n-simplex, then its preimage inside $X_L/BB_L = X_L^M(S)/G_L^M(S)$ is the quotient $\mathbb{R}^{n+1}/K$, where $K = \{(m_0, \ldots, m_n) \in \mathbb{Z}^{n+1} : m_0 + \cdots m_n = 0\}$, a subgroup of $\mathbb{Z}^{n+1}$ of rank n.

The link of a vertex of $X_L^M(S)$ is either $\mathbb{S}(L)$ for a vertex of height in S or $\mathbb{S}(M)$ for a vertex of height not in S. If σ is any simplex of L, then the inverse image of σ in M is a disjoint union of finitely many simplices $\sigma_1, \ldots, \sigma_k$ of M, where k is the index of the cover. The inverse image of $\mathbb{S}(\sigma)$ in $\mathbb{S}(M)$ is thus a disjoint union of k copies of the n-sphere $\mathbb{S}(\sigma_1) \sqcup \ldots, \sqcup \mathbb{S}(\sigma_k)$.

To simplify the discussion, suppose from now on that $\pi(M,L)$ is finite and that H is torsion-free. In this case, the stabilizer in $G_L^M(S)/H$ of each vertex of $X_L^M(S)/H$ of height not in S is isomorphic to $\pi(M,L)$. By the observations above, the inverse image in $X_L^M(S)/H$ of the cylinder of X_L/BB_L labelled by σ is a disjoint union of finite covers of the cylinder, except that the vertices of the inverse image of height not in S are identified in orbits of size $\pi(M,L)$. The cylinders labelled by a given simplex are permuted by $G_L^M(S)/H$, and the stabilizer of each cylinder in this action is abelian and generated by at most n elements: if the cylinder is $\mathbb{R}^{n+1}/K$, and we view K as a subgroup of $G_L^M(S)$, then the stabilizer is the group $K/(K \cap H) \cong KH/H \leq G_L^M(S)/H$.

Lemma 6.1. *If the intersection of two cylinders contains an edge e with a given label in L^0, then it contains an edge of each height with that same label.*

Proof. If v is one of the vertices of the edge e, e defines a vertex of the link $\mathrm{Lk}_X(v)$ which is either $\mathbb{S}(L)$ or $\mathbb{S}(M)$. By [16, prop. 7.3] the antipode of this point of $\mathrm{Lk}_X(v)$ is uniquely determined, and corresponds to an edge e' of $X_L^M(S)$ with the same label as e but with height differing by one from that of e. If e is contained in a cylinder C, then so is e'. $\square$

Definition 6.2. *Say that edges e, e' of $X_L^M(S)/H$ are* cylinder equivalent *if they have the same label in L^0 and there are $r \geq 0$, edges $e_0, \dots, e_r$ and cylinders $C_1, \dots, C_r$ so that each e_i has the same label as e, and for $1 \leq i \leq r$ both e_{i-1} and e_i are contained in the cylinder C_i.*

Let $Q = G_L^M(S)/H$ be the group of deck transformations of the branched cover $X_L^M(S)/H \to X_L/BB_L$, so that Q acts freely on the edges of $X_L^M(S)/H$ and permutes the cylinders.

Proposition 6.3. *Suppose that e' is an edge of the same height as e, and let $P \leq Q$ be the subgroup generated by the stabilizers of all of the cylinders that contain e. Then e' is cylinder equivalent to e if and only if e' lies in the orbit Pe.*

Proof. By induction on the length r of the chain of cylinders used to establish the cylinder equivalence. By Lemma 6.1, if e' is cylinder equivalent to e, we may choose $e_0, \dots, e_r$ and $C_1, \dots, C_r$ as in the definition with each e_i having the same height as e. Let P_1 be the stabilizer in Q of C_1. By transitivity, there exists $g \in P_1$ so that $ge = ge_0 = e_1$. Since there is a shorter cylinder equivalence from e_1 to e_r, there exists $g' \in P$ so that $g'e_1 = e_r = e'$, and since $P_1 \leq P$, we see that $g'g \in P$ and that $(g'g)e = e'$. This argument can be reversed, giving the converse. $\square$

Proposition 6.4. *If e and e' are in the same hyperplane, then e and e' are cylinder equivalent.*

Proof. Each square of $X_L^M(S)/H$ is contained in a cylinder. $\square$

Proposition 6.5. *Suppose that $S + n = S$ and that $M \to L$ is a finite regular cover. If $G_L^M(S)$ is virtually (non-cocompact) special then $G_L^M(S) \rtimes n\mathbb{Z}$ is virtually special.*

Proof. Suppose that $H \leq G_L^M(S)$ is a finite-index subgroup such that $X_L^M(S)/H$ is a special cube complex. Since $G_L^M(S)$ contains finitely many subgroups of a given index, by passing to a subgroup if necessary we may assume that H is characteristic in $G_L^M(S)$, so that $n\mathbb{Z}$ normalizes H. The action of $n\mathbb{Z}$ on $X_L^M(S)/H$ identifies vertices of different heights, so it does not create any new pairs of edges where an osculation takes place. However, it may be that the compact cube complex $X_L^M(S)/(H \rtimes n\mathbb{Z})$ has fewer hyperplanes than $X_L^M(S)/H$, which may cause extra interosculations or self-osculations. To avoid this, note that $n\mathbb{Z}$ acts as permutations of the finitely many hyperplanes in $X_L^M(S)/H$, and so for some $m > 0$ the subgroup $mn\mathbb{Z} \leq n\mathbb{Z}$ preserves each hyperplane. For this m, $X_L^M(S)/(H \rtimes mn\mathbb{Z})$ will be special because $X_L^M(S)/H$ is by hypothesis. $\square$

Theorem 6.6. *Suppose that $M \to L$ is a finite cover, and that $\theta : G_L^M(S) \to Q$ is a homomorphism to a finite group such that the kernel of θ is torsion-free and such that for any two adjacent vertices $u, v \in L$, the image under θ of the subgroup of $G_L^M(S)$ generated by the edges of $\mathrm{St}(u) \cup \mathrm{St}(v)$ is abelian. Then $G_L^M(S)$ is (non-cocompact) virtually special.*

Proof. We construct a homomorphism to a larger finite group whose kernel will be shown to be special. The abelianization $H_1(A_L;\mathbb{Z})$ of the right-angled Artin group A_L is free, with basis the set L^0, and the abelianization $H_1(BB_L;\mathbb{Z})$ of BB_L is the codimension one summand consisting of all elements $\sum_v n_v v$ with $\sum_v n_v = 0$. Now let m be the exponent of the finite group Q, and let $H = H_1(BB_L;\mathbb{Z}/m\mathbb{Z}) = H_1(BB_L;\mathbb{Z}) \otimes \mathbb{Z}/m\mathbb{Z}$. The quotient map $\phi : G_L^M(S) \to G_L^M(\mathbb{Z}) = BB_L \to H$ and θ together give a homomorphism $(\theta, \phi) : G_L^M(S) \to Q \times H$, and it is this map whose kernel K will be shown to be special.

Note that every vertex stabilizer $\pi(M, L)$ is in the kernel of the map from $G_L^M(S)$ to BB_L, and so every copy of $\pi(M, L)$ is mapped by (θ, ϕ) to a subgroup of $Q \times \{0\}$.

As remarked earlier, hyperplanes in $X_L^M(S)/K$ are always 2-sided and never self-intersect, so we only need to rule out direct self-osculations and inter-osculations. Suppose that e and e' are adjacent edges of $X_L^M(S)/K$ of the same height that share a square, so that the hyperplanes they belong to intersect, and let x be the vertex that is incident on both e and e'. We claim that no other edge of the same height that is cylinder equivalent to e or to e' can be incident on x. Since hyperplanes are contained in cylinder equivalence classes, this will imply that there are no direct self-osculations or inter-osculations.

To establish this claim, note that the labels in L^0 attached to e and e' are adjacent vertices u, v.

The stabilizer of an n-cylinder of $X_L^M(S)$ in $G_L^M(S)/K$ is an abelian group. If the cylinder is labelled by an n-simplex σ, then its stabilizer is the image in $G_L^M(S)/K$ of the free abelian subgroup $BB_\sigma \leq G_L^M(S)$ which is of rank n. A cylinder of $X_L^M(S)/K$ that contains the edge e corresponds to a simplex τ of L that contains u and similarly, a cylinder that contains e' corresponds to a simplex τ' of L that contains v. Any such τ, τ' are contained in the subcomplex $J = \mathrm{St}(u) \cup \mathrm{St}(v)$ of L. Since J is simply connected, the subgroup of $G_L^M(S)$ generated by the edges of J is isomorphic to BB_J, and hence the inclusion $J \to L$ induces a monomorphism $H_1(J;\mathbb{Z}/m\mathbb{Z}) \to H = H_1(BB_L;\mathbb{Z}/m\mathbb{Z})$. By hypothesis $\theta(BB_J)$ is an abelian subgroup of Q, necessarily of exponent dividing m. But $\phi(BB_J)$ is the largest possible abelian quotient of BB_J of exponent m, and so it follows that the image of BB_J under (θ, ϕ) has trivial intersection with $Q \times \{0\}$.

The claim now follows, since any element of $(\theta, \phi)(G_L^M(S)) \leq Q \times H$ that fixes the vertex x must lie in $Q \times \{0\}$, whereas any element that sends either e or e' to a cylinder equivalent edge must lie in $(\theta, \phi)(BB_J)$. $\square$

Proof. (Theorem 1.10.) The kernel of the map $G_{L'}^{M'}(S) \to G_L^M(S)$ is torsion-free, and so if $G_L^M(S) \to Q$ is any homomorphism with torsion-free kernel, then the composite $G_{L'}^{M'}(S) \to Q$ also has torsion-free kernel. Since for any two adjacent vertices u, v of L', the image of $J = \mathrm{St}(u) \cup \mathrm{St}(v)$ is contained in a single simplex of L, the image in $G_L^M(S)$ of the subgroup BB_J is abelian and hence so is its image in Q. Since $G_{L'}^{M'}(S)$ is virtually special and $S + n = S$, Proposition 6.5 implies that $G_{L'}^{M'}(S) \rtimes n\mathbb{Z}$ is virtually special. $\square$

Corollary 6.7. *Suppose that M is a finite cover of L and that there is a homomorphism $G_L^M(S) \to Q$ with torsion-free kernel, with Q is a finite abelian group. Then $G_L^M(S)$ is virtually (non-cocompact) special, and if $S + n = S$ then $G_L^M(S) \rtimes n\mathbb{Z}$ is virtually special.*

Proof. Follows from Theorem 6.6 and Proposition 6.5. $\square$

7 Covers pulled back from graphs

In this section we prove Theorem 1.6. We start with a Proposition in finite group theory. Let A_N and S_N denote the alternating and symmetric groups on a finite set of size N, let C_n denote a finite cyclic group of order n, The wreath product $S_N \wr C_n$ is a finite group containing a normal subgroup isomorphic to the direct product $(S_N)^n$ of n copies of S_N, with quotient the cyclic group C_n. If ρ is a generator for the cyclic group C_n, conjugation by ρ permutes the n copies of S_N freely. If α denotes a permutation in S_N, we write α_i with $1 \leq i \leq n$ for the element of the product $(S_N)^n$ that has its ith component equal to α and the other components equal to the identity. In the usual notation for elements of a direct product, α_i would be written as $(1, \ldots, 1, \alpha, 1, \ldots, 1)$. For permutations α, β, the elements α_i and β_j commute if $i \neq j \in \mathbb{Z}/n\mathbb{Z}$, while $\alpha_i\beta_i = (\alpha\beta)_i$, and the conjugation action of ρ is given by

$$\rho\alpha_i\rho^{-1} = \begin{cases} \alpha_{i+1} & i < n, \\ \alpha_1 & i = n. \end{cases}$$

Proposition 7.1. *Let α, β be elements of S_N and let σ be the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$. For $1 \leq k < n$ define elements $a, b, c, d \in S_N \wr C_n$, depending on k as well as N and n, by the formulae*

$$a = \rho, \quad b = \alpha_n\rho^{-1}\alpha_n^{-1}, \quad c = \alpha_n\beta_k\rho\beta_k^{-1}\alpha_n^{-1}, \quad d = \beta_k\rho^{-1}\beta_k^{-1}.$$

For each k, the elements a, b, c, d all have order n, and for any integer j,

$$a^jb^jc^jd^j = \begin{cases} \sigma_k & j \equiv k \text{ modulo } n, \\ 1 & j \not\equiv k \text{ modulo } n. \end{cases}$$

Proof. Each of a, b, c, d is a conjugate of either ρ or ρ^{-1}, so each has order n as claimed. From this is follows that it suffices to check the claim concerning the order of $a^jb^jc^jd^j$ for $1 \leq j < n$. For j in this range, since α_n commutes with β_k we see that

$$\begin{aligned} a^jb^jc^jd^j &= \rho^j(\alpha_n\rho^{-j}\alpha_n^{-1})(\alpha_n\beta_k\rho^j\beta_k^{-1}\alpha_n^{-1})(\beta_k\rho^{-j}\beta_k^{-1}) \\ &= \rho^j\alpha_n\rho^{-j}(\alpha_n^{-1}\alpha_n\beta_k)\rho^j(\beta_k^{-1}\alpha_n\beta_k)\rho^{-j}\beta_k^{-1} \\ &= (\rho^j\alpha_n\rho^{-j})\beta_k(\rho^j\alpha_n^{-1}\rho^{-j})\beta_k^{-1} \\ &= \alpha_j\beta_k\alpha_j^{-1}\beta_k^{-1} = \begin{cases} \sigma_k & j = k \\ 1 & j \neq k. \end{cases} \end{aligned}$$

□

Theorem 7.2. *Let $\overline{\Delta} \to \overline{\Gamma}$ be a finite regular covering of graphs, and let Γ be a simplicial graph obtained by subdividing each edge of $\overline{\Gamma}$ into at least r parts, with Δ the corresponding covering of Γ. For any periodic set $S \subseteq \mathbb{Z}$ and any $r \geq 4$, the group $G_\Gamma^\Delta(S)$ is virtually torsion-free.*

Proof. If $S = \emptyset$ then $G_\Gamma^\Delta(S)$ is the semidirect product $BB_\Delta \rtimes \pi(\Delta, \Gamma)$ which is clearly virtually torsion-free. In all other cases, we may assume up to isomorphism that $0 \in S$, and we do so for the rest of this proof.

Embed the group $\pi(\overline{\Delta}, \overline{\Gamma}) = \pi(\Delta, \Gamma)$ into a finite alternating group A_N. Choose a maximal tree $T \subseteq \overline{\Gamma}$, and fix an orientation on the edges of $\overline{\Gamma} - T$, so that the fundamental group of $\overline{\Gamma}$ is naturally isomorphic to the free group on the set of edges of $\overline{\Gamma} - T$. The covering thus gives rise to a labelling σ of the directed edges of $\overline{\Gamma}$ by elements of A_N, with the properties that every edge of T is labelled by the identity element and that the product of the labels on the two different orientations of the same edge is the identity. The labelling σ associates an element of A_N to each directed edge path in $\overline{\Gamma}$: if the directed edge path is $e_1, e_2, \ldots, e_l$, the associated element is $\sigma(e_1)\sigma(e_2)\cdots\sigma(e_l)$. By definition, the element of A_N associated to a closed directed edge path will be the identity if and only if this path lifts to a closed path in $\overline{\Delta}$.

The relators in the presentation for $G_\Gamma^\Delta(S)$ given in Section 2 are of the form $e_1^j e_2^j \cdots e_l^j$, where $e_1, \ldots, e_j$ is a closed directed edge path in Γ and either $j \in S$ or the path lifts to a closed path in Δ. Moreover, every non-identiy element of finite order in $G_\Gamma^\Delta(S)$ is conjugate to an element of the form $e_1^j e_2^j \cdots e_l^j$, where $j \notin S$ and $e_1, \ldots, e_l$ is a closed directed edge path in Γ whose lift to Δ is not a closed path. It follows that to construct a homomorphism with torsion-free kernel from $G_\Gamma^\Delta(S)$ to a finite group, it suffices to construct a labelling μ of the directed edges of Γ by the elements of a finite group so that for $j \in S$ and for every closed directed edge path $e_1, \ldots, e_l$ in Γ, $\mu(e_1)^j \mu(e_2)^j \cdots \mu(e_l)^j = 1$, while for $j \notin S$ we have that $\mu(e_1)^j \mu(e_2)^j \cdots \mu(e_l)^j = 1$ if and only if $e_1, \ldots, e_l$ lifts to a closed path in Δ.

Fix some $n > 0$ with $S + n = S$. First we consider the case when $S = \mathbb{Z} - (k + n\mathbb{Z})$ for some k with $1 \leq k < n$. In this case, the finite group that will be the target of our labelling μ is the wreath product $S_N \wr C_n$. To ease the notation, we fix an orientation on each of the edges of $\overline{\Gamma}$ and define the labelling μ on those directed edges of Γ that are oriented in the same direction as our chosen orientation on the edge of $\overline{\Gamma}$ that they are contained in. If e is a directed edge of Γ with our chosen orientation, $\sigma(e)$ is an element of the alternating group A_N. It is known that every element of A_N is equal to the commutator of a pair of elements of S_N [21]. Hence we may choose $\alpha(e), \beta(e) \in S_N$ with $\sigma(e) = \alpha(e)\beta(e)\alpha(e)^{-1}\beta(e)^{-1}$. Define elements $a(e), b(e), c(e), d(e)$ of $S_N \wr C_n$ as in the statement of Proposition 7.1. If the directed path in Γ that maps homeomorphically to e with its given orientation is $e_1, \ldots, e_r$ where $r \geq 4$, define the labelling μ on these edges by

$$\mu(e_i) = \begin{cases} a(e) & i = 1 \\ b(e) & i = 2 \\ c(e) & i = 3 \\ d(e) & i = 4 \\ 1 & i > 4. \end{cases}$$

By Proposition 7.1, for this labelling we have that

$$\mu(e_1)^j \mu(e_2)^j \mu(e_3)^j \cdots \mu(e_r)^j = \begin{cases} \sigma(e)_k & j \equiv k \text{ modulo } n \\ 1 & j \not\equiv k \text{ modulo } n. \end{cases}$$

Here $\sigma(e)_k$ denotes the copy of $\sigma(e)$ inside the kth direct factor in $(S_N)^n < S_N \wr C_n$. This completes the proof in the case when $S = \mathbb{Z} - (k + n\mathbb{Z})$.

For the general case, rename the labelling μ used above as $\mu^{(k)}$, to emphasize the dependence on k. If S is any set with $0 \in S$ and $S + n = S$, define a finite set $\{k_1, \ldots, k_l\}$

as $\{1,\ldots,n-1\}-S$. For this S, define a new labelling $\underline{\mu}$ of the edges of Γ by elements of $(S_N \wr C_n)^l$, where the label attached to the edge e of Γ is

$$\underline{\mu}(e) = (\mu^{(k_1)}(e), \mu^{(k_2)}(e), \ldots, \mu^{(k_l)}(e)).$$

The labelling $\underline{\mu}$ has the property that the product of the jth powers of the labels around any closed path in Γ is the identity if $j \in S$, whereas for $j \in S$ the product of the jth powers of the labels around a closed path is equal to the identity if and only if the path lifts to a closed path in Δ. Hence the kernel of the corresponding homomorphism is torsion-free as required. □

Proof. (Theorem 1.6) By Theorem 7.2 we see that the group $G_\Gamma^\Delta(S)$ is virtually torsion-free in the case when S is periodic and Γ is a graph obtained from another graph $\overline{\Gamma}$ by subdividing each edge into at least four pieces. If $M \to L$ is a covering obtained by pulling back the regular covering $\Delta \to \Gamma$ along a simplicial map $f : L \to \Gamma$, then $\pi(M, L)$ is a subgroup of $\pi(\Delta, \Gamma)$, and every finite subgroup of $G_L^M(S)$ maps isomorpically to a subgroup of $G_\Gamma^\Delta(S)$ under the map induced by f. Hence in this case $G_L^M(S)$ is also virtually torsion-free.

Now suppose that $\widehat{\Gamma}$ is obtained from $\overline{\Gamma}$ by subdividing each edge into exactly 4 pieces, and that Γ is obtained from $\overline{\Gamma}$ by subdividing each edge into at least 12 pieces. In this case, there is a map from Γ to $\widehat{\Gamma}$ with the property that the image of any three consecutive edges of Γ is either a vertex or a single edge of $\widehat{\Gamma}$. In more detail, suppose that $\overline{e}$ is a directed edge of $\overline{\Gamma}$ that is subdivided into $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4 \in \widehat{\Gamma}$ and into $e_1, \ldots, e_r \in \Gamma$ with $r \geq 12$. In this case, such a map is given explicitly by mapping $e_2, e_5, e_{r-4}, e_{r-1}$ homeomorphically to the edges $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4$ respectively and collapsing each other e_j to a point. Thus if $M \to L$ is obtained by pulling back the covering $\Delta \to \Gamma$, then both the hypotheses of Theorem 6.6 and Proposition 6.5 are satisfied and $G_L^M(S) \rtimes n\mathbb{Z}$ is virtually special, which implies that Conjectures 1.3 and 1.4 hold in this case. □

Remark 7.3. *In Theorem 1.6 and Theorem 7.2, the hypothesis that each edge of $\overline{\Gamma}$ be subdivided into at least r pieces can be replaced by a slightly weaker hypothesis: it is sufficient for each edge of $\overline{\Gamma} - T$ to be subdivided into at least r pieces.*

Proof. (Corollary 1.7) Let $\overline{\Gamma}$ be a rose (i.e., a 1-dimensional CW-complex with one vertex) whose fundamental group is isomorphic to the free group F, and fix such an isomorphism. A standard argument of obstruction theory shows that the homomorphism $f : \pi_1(L) \to F$ is induced by some continuous map $\phi : |L| \to \overline{\Gamma}$ [13, Ch. 4.3]. To see this, view $|L|$ as a CW-complex, with cells the topological realizations of the simplices of L. Pick a maximal tree T in L, and send every 0-cell and every 1-cell in $|T|$ to the 0-cell of the rose. Every other 1-cell $|\sigma|$ of $|L|$ represents a unique word in the free generators of F, and this word can be used to define $\phi|_{|\sigma|}$. Assume by induction that the map ϕ has been defined on the $n-1$-skeleton of L for some $n \geq 2$. Since the universal cover of the rose $\overline{\Gamma}$ is a tree, the higher homotopy groups of $\overline{\Gamma}$ are all trivial. Thus for each n-simplex σ, the map from the $(n-1)$-sphere to $|\overline{\Gamma}|$ defined as the restriction of ϕ to the boundary of σ can be extended to a map from the n-disc to $|\overline{\Gamma}|$. By doing this for each n-simplex, one extends σ to the n-skeleton of $|L|$.

Let Γ be obtained from the rose $\overline{\Gamma}$ by subdividing each edge into 12 pieces. By the simplicial approximation theorem [13, Ch. 2.C], there is an iterated barycentric subdivision L' of L with respect to which the map $\phi : |L'| = |L| \to |\Gamma| = |\overline{\Gamma}|$ is homotopic to a simplicial map $\psi : L' \to \Gamma$. If Δ is the regular covering of Γ corresponding to the finite-index normal subgroup $N \lhd F$, the induced cover of L' is a (possibly not connected) regular covering of L' with F/N as its group of deck transformations. The fundamental group of each component of this covering is $f^{-1}(N)$, and we may take M' to be one of these components. □

8 Some torsion-free-by-cyclic examples

Proposition 8.1. *Let p be a prime and suppose that $S = p\mathbb{Z}$ and that $M \to L$ is a connected p-fold regular covering. Then $G_L^M(S) \rtimes p\mathbb{Z}$ is virtually special and so all of our conjectures hold in this case.*

Proof. The p-fold covering M is classified by an element of $H^1(L; \mathbb{Z}/p\mathbb{Z})$, so let $f : L^1 \to \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$ be a cocycle representing this cohomology class. The cocycle f extends to a group homomorphism $G_L^M(S) \to \mathbb{F}_p$ and this homomorphism is easily seen to have torsion-free kernel. The claim now follows from Corollary 6.7. □

9 An example in detail

We consider now the simplest case of Proposition 8.1 in detail; the case when $p = 2$, $S = 2\mathbb{Z}$ and L is the boundary of a square. Let the edges of L be labelled a, b, c, d so that a group presentation for $G_L^M(S)$ is

$$\langle a, b, c, d \ : \ a^{2n}b^{2n}c^{2n}d^{2n} = 1 = (a^n b^n c^n d^n)^2, \ n \in \mathbb{Z}\rangle.$$

There are fifteen index two subgroups of $G_L^M(S)$, indexed by the subset of the generators $\{a, b, c, d\}$ consisting of elements not in the subgroup. The torsion-free subgroups are those in which $abcd$ is not contained in the subgroup, or equivalently the set of generators not in the subgroup has odd cardinality. The order four rotation of L induces a group of four automorphisms of L, so up to isomorphism there are only two cases to consider: the subgroup containing b, c, d but not a and the subgroup containing d but not containing a, b, c.

The space X_L/BB_L, which is a classifying space for $BB_L = G_L^M(\mathbb{Z})$, consists of a union of four 2-dimensional cylinders. Label the four vertices of L by w, x, y, z, so that the directed edges are $a = (w, x)$, $b = (x, y)$, $c = (y, z)$ and $d = (z, w)$. Let P denote a copy of the plane $\mathbb{R}^2$, tesselated by squares with vertex set $\mathbb{Z}^2$ and 1-skeleton $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$. Each of the four cylinders making up X_L/BB_L is isomorphic to the quotient of P by the subgroup generated by $(-1, 1)$, with the height function on P and on $P/\langle(-1, 1)\rangle$ given by $(s, t) \mapsto s + t$. In $P/\langle(-1, 1)\rangle$, the images of the horizontal edges all belong to one hyperplane and the images of the vertical edges all belong to a second hyperplane.

If H is any of the eight torsion-free index two subgroups of $G_L^M(2\mathbb{Z})$, then $X_L^M(2\mathbb{Z})/H$ is a 2-fold branched covering of X_L/BB_L, with branching only at the vertices of even height. To better understand $X_L^M(2\mathbb{Z})/H$, we first describe the subcomplexes X_a, X_b, X_c and X_d consisting of the inverse images of the four cylinders of X_L/BB_L. The isomorphism type of such a subcomplex depends only on whether the letter that labels it is contained in the subgroup H or not, so consider X_h for some $h \in \{a, b, c, d\}$. The link of an unbranched vertex of $X_L^M(2\mathbb{Z})/H$ is a copy of the octahedralization of L and the link at a branched vertex is a copy of the octahedralization of M. Since the inverse image in M of each edge of L is a disjoint union of two edges, the link of a branch vertex inside X_h is a disjoint union of two squares (i.e., the octahedralization of a pair of disjoint edges). If $h \in H$, then X_h consists of two copies of $P/\langle(-1, 1)\rangle$, with each vertex of odd height in one copy identified with the vertex in the other copy of the same height. If $h \notin H$, then instead X_h consists of one copy of a larger cylinder $P/\langle(-2, 2)\rangle$, in which the two vertices of each odd height are identified with each other.

To study the hyperplanes in the whole complex, we first consider the hyperplanes in X_h. Suppose that $u, v \in \{w, x, y, z\}$ are the vertices of the edge h. If $h \in H$, then X_h contains two hyperplanes labelled u and two hyperplanes labelled v, one of each type in each of the two copies of $P/\langle(-1, 1)\rangle$. Each u-hyperplane intersects exactly one of the two v-hyperplanes. In this case, viewing it as a complex in its own right, X_h is special. If on the other hand $h \notin H$, then as before there are two u-hyperplanes and two v-hyperplanes, but this time each u-hyperplane intersects each v-hyperplane. Furthermore, at every branch vertex two u-edges and two v-edges of each height all meet. The two edges with the same label at the same height belong to different hyperplanes. Hence in the case when $h \notin H$, each u-hyperplane interosculates with each v-hyperplane, and X_h itself is not special. Note also that if we define a line to be the image in X_h of either $\mathbb{R} \times \{n\}$ or $\{n\} \times \mathbb{R}$ for some integer n, then the edges in a single line alternate between the two hyperplanes of X_h labelled by the relevant letter.

It follows from the above considerations that the complex $X_L^M(2\mathbb{Z})$ is never special. In the case when $a \notin H$ and $b, c, d \in H$, the two w-hyperplanes in X_a become identified in X_d, because the intersection of X_a and either of the cylinders of X_d consists of a line that contains edges from both w-hyperplanes of X_a. Similarly, the two x-hyperplanes in X_a become identified in X_b. Hence the whole complex contains one w-hyperplane and one x-hyperplane, each of which self-osculates. The w-hyperplane and the x-hyperplane also interosculate. There are two y-hyperplanes and two z-hyperplanes which are not involved in any self-osculation or inter-osculation.

In the case when $a, b, c \notin H$ and $d \in H$, the two z-hyperplanes in X_c become identified in X_d and the two w-hyperplanes in Z_a become identified in X_d. Thus there is just one z-hyperplane and one w-hyperplane, each of which self-osculates. There are two x-hyperplanes and two y-hyperplanes, each of which does not self-osculate. However, any pair of hyperplanes labelled by the distinct ends of an edge interosculate with each other.

Thus we see that for H any of the torsion-free index two subgroups of $G_L^M(2\mathbb{Z})$, the complex $X_L^M(2\mathbb{Z})/H$ fails to be special. The proof of Corollary 6.7 tells us that there is an index 16 normal subgroup $H \leq G_L^M(2\mathbb{Z})$ such that $X_L^M(2\mathbb{Z})/H$ is special and since the quotient group has exponent 2, it follows that this H is the kernel of the map to $H_1(G_L^M(2\mathbb{Z}); \mathbb{F}_2)$.

It can also be seen directly that this covering is special. In $X_L^M(2\mathbb{Z})$, ignoring for now the identification of vertices that is responsible for the branching, the inverse image of each of the four cylinders of X_L/BB_L consists of 8 copies of the cylinder $P/\langle(-2, 2)\rangle$. The edges of a given height labelled by each fixed letter form a single free orbit for the action of $Q = H_1(G_L^M(2\mathbb{Z}); \mathbb{F}_2) \cong (C_2)^4$. It can be shown that these edges all lie in distinct hyperplanes, so that there are 16 distinct hyperplanes labelled with each letter. The vertices of odd height form a single Q-orbit of type $Q/\langle abcd\rangle$, where we have identified the element $abcd$ of $G_L^M(2\mathbb{Z})$ and its image in Q. This already implies that no self-osculation or interosculation can occur, without considering cylinder equivalence. However, to illustrate the special case of our general argument, we discuss cylinder equivalence. The cylinder-equivalence classes of edges labelled x correspond to the cosets $Q/\langle a,b\rangle$ and the cylinder-equivalence classes of edges labelled y correspond to the cosets $Q/\langle b,c\rangle$. Since $abcd \notin \langle a,b,c\rangle$, one sees that if e, e' are incident edges labelled x and y, then no edge cylinder equivalent to e can be incident on any edge cylinder equivalent to either e or e', except for e, e' themselves. The cylinder equivalence classes for other edges are similar.

10 Two applications

In this section we use the cases of our conjectures that we have established to construct some groups with surprising combinations of properties.

Theorem 10.1. *For each $m \geq 6$ there is a finitely generated group G_m with an infinite presentation satisfying the $C'(1/m)$ small cancellation condition with the properties that G_m is residually finite, torsion-free and embeds in a finitely presented group, but the word problem for G_m is insoluble.*

Proof. Fix some integer $l \geq 2m+1$, let L be a circle triangulated as the boundary of a l-gon, and let M be the universal cover of L. The group G_m will be the group $G_L^M(T)$ for a suitable set $T \subseteq \mathbb{Z}$. Any such group is torsion-free by Proposition 1.1. As discussed above, this group has the presentation

$$G_m = \langle\, a_1, \ldots, a_l \;:\; a_1^n a_2^n \cdots a_l^n \;\, n \in T \,\rangle.$$

The choice of $l \geq 2m+1$ implies that for each T this presentation satisfies the $C'(1/m)$ condition. The boundary of the l-gon may be viewed as a subdivision of the 1-edge CW-structure on the circle, and so since $l \geq 12$ the hypotheses of Theorem 1.6 are satisfied. The subset T that we will choose will be of the form $T = T(S)$ as in the statement of Proposition 2.2, for some $S \subseteq \mathbb{N}$. Each such set is closed in the profinite topology on $\mathbb{Z}$ so by Theorem 1.6 G_m is residually finite. By [16, lemma 15.3] and the related discussion in Section 3, the element $a_1^n \cdots a_l^n$ is equal to the identity in G_m if and only if $n \in T = T(S)$. By Proposition 2.2, if S is recursively enumerable but $\mathbb{N} - S$ is not (so that S is not recursive), there can be no algorithm to decide membership of $T(S)$ and so the word problem for G_m is insoluble. Since $T(S)$ is recursively enumerable the Higman embedding theorem [19, ch. IV.7] tells us that G_m can be embedded in a finitely presented group. $\square$

Examples of residually finite groups with insoluble word problem that can be embedded in finitely presented groups were constructed in the 1970's by Dyson and by Meskin [11, 20], but their examples contain torsion.

For any L and for $M = \widetilde{L}$, it has been shown that $G_L^M(S)$, $S \neq \mathbb{Z}$, has soluble word problem if and only if $\pi_1(L)$ has soluble word problem and S is recursive [5, thm. 6.4]. A direct proof of this can be given in the case when L is the boundary of an l-gon for $l \geq 13$ as in the theorem above. Since $a_1^n \cdots a_l^n = 1$ if and only if $n \in S$, a solution to the word problem implies that S is recursive. Conversely, given a word of length N in the a_i, if S is recursive we may list the elements of $S \cap [-N/l, N/l]$ and thus list the relators in the given presentation of length at most N. Since this presentation satisfies the $C'(1/6)$ condition, any word of length N that is equal to the identity will contain more than half of a dihedral permutation of a one of these relators as a subword.

Proposition 10.2. *Let $l \geq 12$ and let G be given by the presentation*

$$G = \langle a_1, a_2, \ldots, a_l \ : \ (a_1^n a_2^n \cdots a_l^n)^2 = 1, \ n \in \mathbb{Z} \rangle.$$

Then G is residually finite, but G is not virtually torsion-free and not linear in characteristic zero. Every finite subgroup of G has order at most 2.

Proof. Let $M \to L$ be the 2-fold cover of the l-gon. The group G given above is isomorphic to $G_L^M(\{0\})$. Any finite subset of $\mathbb{Z}$ is closed in the profinite topology, and any non-empty finite subset is not periodic. Hence this group is residually finite by Theorem 1.6, and is not virtually torsion-free by Theorem 1.2. Every non-trivial finite subgroup of G is conjugate to the group generated by $a_1^n \cdots a_l^n$ for some $n \neq 0$ and has order two. Any finitely generated linear group in characteristic zero is virtually torsion-free [1], and so G cannot be linear. □

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