# Free particles from Brauer algebras in complex matrix models 

Yusuke Kimura ${ }^{1}$, Sanjaye Ramgoolam ${ }^{2}$ and David Turton ${ }^{3}$<br>Queen Mary University of London<br>Centre for Research in String Theory<br>Department of Physics<br>Mile End Road<br>London E1 4NS UK


#### Abstract

The gauge invariant degrees of freedom of matrix models based on an $N \times N$ complex matrix, with $U(N)$ gauge symmetry, contain hidden free particle structures. These are exhibited using triangular matrix variables via the Schur decomposition. The Brauer algebra basis for complex matrix models developed earlier is useful in projecting to a sector which matches the state counting of $N$ free fermions on a circle. The Brauer algebra projection is characterized by the vanishing of a scale invariant laplacian constructed from the complex matrix. The special case of $N=2$ is studied in detail: the ring of gauge invariant functions as well as a ring of scale and gauge invariant differential operators are characterized completely. The orthonormal basis of wavefunctions in this special case is completely characterized by a set of five commuting Hamiltonians, which display free particle structures. Applications to the reduced matrix quantum mechanics coming from radial quantization in $\mathcal{N}=4 \mathrm{SYM}$ are described. We propose that the string dual of the complex matrix harmonic oscillator quantum mechanics has an interpretation in terms of strings and branes in $2+1$ dimensions.


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## 1 Introduction

### 1.1 Background and motivations

There is a class of Gaussian matrix models in $D$ spacetime dimensions ${ }^{1} x^{\mu}$ where the two-point function of the matrix $Z\left(x^{\mu}\right)$, up to a trivial spacetime dependence, is

$$
\begin{equation*}
\left\langle Z_{j}^{i} Z^{\dagger k}{ }_{l}\right\rangle=\delta_{l}^{i} \delta^{k}{ }_{j} \tag{1.1}
\end{equation*}
$$

and where there is an adjoint $U(N)$ action

$$
\begin{equation*}
Z \rightarrow g Z g^{\dagger}, \quad g \in U(N) \tag{1.2}
\end{equation*}
$$

Our main interest is in $D=4, \mathcal{N}=4$ superconformal Yang-Mills (SYM) with gauge group $U(N)$ at zero coupling and its dimensional reduction on $S^{3}$ to a $D=1$ matrix harmonic oscillator quantum mechanics. Some of our results apply more generally and in particular we make connections to the $D=0$ Gaussian complex matrix model considered by Ginibre [1].
$\mathcal{N}=4$ super Yang-Mills contains three complex scalar fields in the adjoint of the gauge group. The states built from holomorphic functions in one of these fields, say $Z$, comprise the half-BPS sector in which the two-point function is diagonalised by the Schur polynomials $\chi_{R}(Z)$ [2]:

$$
\begin{equation*}
\left\langle\chi_{S}\left(Z^{\dagger}\right) \chi_{R}(Z)\right\rangle \propto \delta_{R S} \tag{1.3}
\end{equation*}
$$

The Schur polynomials were identified as gauge theory duals of giant gravitons [3], generalizing the proposal of [4] which associated determinants and sub-determinants to giant gravitons expanding in the $S^{5}$ of the $A d S_{5} \times S^{5}$ dual of $\mathcal{N}=4 \mathrm{SYM}$. This is manifestation of the 'stringy exclusion principle' $[5,6]$.

[^1]The Schur polynomial basis allowed the identification of gauge theory duals for multiple giants as well as giants expanding in the $A d S_{5}$ directions. Extensive evidence for the proposed map has been found, see for example the review [7]. It was also observed that Schur Polynomials were related to wavefunctions of free fermions which are the eigenvalues of the matrix $Z[2,8]$. The free fermions were subsequently identified as arising from supergravity solutions [9]. Non-renormalization theorems [10, 11] allow a comparison of zero coupling to strong coupling; the diagonalisation of the two-point function was a crucial step enabling this comparison.

Motivated by the goal of understanding non-supersymmetric sectors of the $A d S_{5} / C F T_{4}$ gauge-string duality, [12] undertook the study of gauge invariant non-holomorphic functions of $Z$. Two-point functions of operators which are polynomials of degree $m$ in $Z$ and degree $n$ in $Z^{\dagger}$ in the $U(N)$ theory were diagonalised in terms of a Brauer basis which was constructed systematically using the representation theory of the Brauer algebra $B_{N}(m, n)$, previously studied in $[13,14,15,16]$. This non-holomorphic sector is arguably the simplest non-supersymmetric sector, yet contains many of the subtleties of multi-matrix combinatorics, since in general $Z$ does not commute with its conjugate transpose $Z^{\dagger}$.

The interest in a detailed study at zero Yang-Mills coupling is two-fold. Firstly, the strong form of the Maldacena conjecture [17] implies that there is a string theory dual to zero coupling Yang-Mills theory. At zero coupling, the sector containing only $Z, Z^{\dagger}$ forms a consistent truncation and we may ask whether there is a string dual of the quantum mechanics of the resulting complex matrix quantum mechanics.

Secondly, while there is undoubtedly less control in comparing zero coupling with strong coupling and thus to semiclassical brane solutions and supergravity solutions it is possible that some qualitative features uncovered might, in appropriate large charge regimes, survive in the strong coupling limit. This line of reasoning is used, for example, in [18] where black hole entropy is investigated from the counting of gauge invariant operators at zero coupling. For some earlier works on complex matrix models, see for example [19, 20, 21, 22].

In this paper we always work at finite $N$. There is an important distinction between

$$
\begin{equation*}
N \geq m+n \quad \text { and } \quad N<m+n . \tag{1.4}
\end{equation*}
$$

The condition $N \geq m+n$ may be read as a condition that $N$ be larger than the lengths of operators one wishes to discuss; for example taking the planar limit $N \rightarrow \infty$ achieves this trivially. For fixed finite $N$, this is the regime in which lengths of operators are less than or equal to $N$. The opposite regime $N<m+n$ is relevant to studies of heavy operators in $\mathcal{N}=4$ super Yang-Mills such as conjectured duals of multi-branes and black holes in $A d S_{5} \times S^{5}$.

The representation theory of Brauer algebras, and thus the construction of the Brauer basis, is well understood for $N \geq m+n$ however there are interesting subtleties for $N<m+n$ (see for example [23]).

In single hermitian or unitary (or more generally, normal) matrix models, the unitary group action (1.2) is sufficient to diagonalise the matrix, leading to a free particle description in terms of the $N$ eigenvalues as we shall review. In an unrestricted complex matrix model, this is not sufficient to diagonalise $Z$; a generic complex matrix may at best be put into triangular form.

More precisely, using the Schur decomposition $Z=U T U^{\dagger}$ where $T$ is upper triangular, the space $g l(N ; \mathbb{C})$ may be decomposed into a parameter space of inequivalent orbits $\mathcal{M}_{N}$ and the orbits of the $U(N)$ action. $\mathcal{M}_{N}$ has real dimension $N^{2}+1$ and is a fibration over the symmetric product $\operatorname{Sym}^{N}(\mathbb{C})$ :

$$
\begin{align*}
& \mathcal{M}_{N}  \tag{1.5}\\
& \quad \downarrow \\
& \operatorname{Sym}^{N}(\mathbb{C})=\mathbb{C}^{N} / S_{N}
\end{align*}
$$

The eigenvalues are however coupled to the off-diagonal triangular entries and so cannot represent positions of free particles.

We shall show that free particles arise in a non-trivial way by exploiting a map identified in [12] between the $k=0$ sector (to be defined later) of the Brauer basis and a unitary matrix model, providing in turn a map to $N$ free fermions on a circle. In this paper we will give evidence for the following conjecture: that these $N$ free fermions of the $k=0$ sector can be constructed from degrees of freedom which are composed of eigenvalues as well as off-diagonal elements of the matrix $Z$. We also observe a different emergence of free particles in the $m=n=k$ sector $^{2}$. While we start with the gauge invariant sector of a Gaussian complex matrix model, which is a system of $N^{2}$ particles constrained by the gauge invariance condition, the emergent particles are $N$ free fermions without constraints.

Much of our work has been carried out $N=2$ as this allows explicit calculations. Many of our $N=2$ results extend to general $N$ as discussed in Section 5; in particular the key point of free particles emerging from a Matrix model from degrees of freedom beyond eigenvalues is valid for any $N$.

### 1.2 Outline of paper

The structure of the paper is as follows. Section 2 reviews the emergence of free particles from the eigenvalues of hermitian and unitary matrix models and explains the new features

[^2]arising for a complex matrix model. Section 3 reviews the Schur decomposition, gauged matrix quantum mechanics and describes the orbits over $\mathcal{M}_{N}$ in general and over $\mathcal{M}_{2}$ in detail. In Section 3.5 we describe the ring of functions on $\mathcal{M}_{2}$, which come from gauge invariant polynomials on $g l(2, \mathbb{C})$.

In Section 4 we describe the ring of Casimir operators studied in [25], we present computational results on the Brauer basis counting at $N=2$ and state a conjecture for the complete solution. The conjectured counting can be elegantly described in terms of five integer labels and is the first main result of this paper.

We derive explicit expressions for the Casimir operators as differential operators on $\mathcal{M}_{2}$ and express the integer labels as functions of the Casimirs. We define free particle momentum operators and express these operators as functions of differential operators on $\mathcal{M}_{2}$. Amongst these operators are the conjectured $k=0$ sector free fermion momenta on a circle. This is the second main result of this paper.

Section 5 presents a conjecture that the $k=0$ sector is the kernel of a scale-invariant laplacian on $\mathcal{M}_{N}$. We give expressions for a class of three-point functions of operators in the $k=0$ sector in terms of unitary matrix integrals, which provides further evidence for the conjectured equivalence to $N$ free fermions on a circle. We extend some remarks on the counting of states at $N=2$ to higher $N$.

We also find free particle structures in the $m=n=k$ sector, by observing that this sector consists of multi-traces of the combination $Z^{\dagger} Z$. We show that this sector may be identified with the kernel of a differential operator on $\mathcal{M}_{N}$.

Most of the material in Sections 3-5 refers to the properties related to invariants under the $U(N)$ action (1.2) and as such is relevant to general complex matrix models with $U(N)$ symmetry. Section 6 deals specifically with the matrix quantum mechanics obtained by dimensional reduction of SYM and draws a connection with Ginibre's $D=0$ matrix model. Using the higher conserved charges to define new Hamiltonians, and the expressions from Section 4, we find non-holomorphic generalizations of the Calogero-Sutherland models at special couplings. Technical details are presented in the Appendices.

## 2 Review of free particles in Matrix models

We briefly review examples of hermitian and unitary matrix models which arise in the context of string theory, in particular string theory in two dimensions. This review is not intended to be complete in any sense but rather to provide the reader with context for the current work.

### 2.1 Hermitian matrix quantum mechanics

Let us consider the Gaussian hermitian matrix model defined by the Lagrangian

$$
\begin{equation*}
\mathcal{L}=\operatorname{tr}\left(\frac{1}{2} \dot{\Phi}^{2}-\frac{1}{2} \Phi^{2}\right) \tag{2.1}
\end{equation*}
$$

which is invariant under the $U(N)$ action

$$
\begin{equation*}
\Phi \rightarrow g \Phi g^{\dagger}, \quad g \in U(N) . \tag{2.2}
\end{equation*}
$$

We follow the treatment in $[26,27,28]$ restricting attention to the theory with quadratic potential. The Hamiltonian of this model is

$$
\begin{equation*}
H=\operatorname{tr}\left(-\frac{1}{2} \frac{\partial^{2}}{\partial \Phi \partial \Phi}+\frac{1}{2} \Phi^{2}\right) \tag{2.3}
\end{equation*}
$$

Introducing the annihilation and creation operators

$$
\begin{equation*}
A=\frac{1}{\sqrt{2}}\left(\Phi+\frac{\partial}{\partial \Phi}\right) \quad A^{\dagger}=\frac{1}{\sqrt{2}}\left(\Phi-\frac{\partial}{\partial \Phi}\right) \tag{2.4}
\end{equation*}
$$

and using the usual convention for matrix indices

$$
\begin{equation*}
\left(\frac{\partial}{\partial \Phi}\right)_{j}^{i}=\frac{\partial}{\partial \Phi_{i}^{j}} \tag{2.5}
\end{equation*}
$$

we have $\left[A_{j}^{i}, A^{\dagger k}\right]=\delta^{k}{ }_{j} \delta_{l}^{i}$ and the Hamiltonian can be rewritten as

$$
\begin{equation*}
H=\operatorname{tr}\left(A^{\dagger} A\right)+\frac{N^{2}}{2} \tag{2.6}
\end{equation*}
$$

The ground state has energy $\frac{N^{2}}{2}$ and its wavefunction is

$$
\begin{equation*}
\Phi_{0}=\langle\Phi \mid 0\rangle=e^{-\frac{1}{2} \operatorname{tr} \Phi^{2}} . \tag{2.7}
\end{equation*}
$$

$U(N)$ singlet excited states are obtained by acting on $\Phi_{0}$ with $U(N)$ invariant functions of $A^{\dagger}$, or by absorbing factors of $\sqrt{2}$, multiplying by $U(N)$ invariant functions of $\Phi$. A basis for such functions is given by the Schur polynomials, which are polynomials of degree $n$ labelled by a representation $R$ of $S_{n}$,

$$
\begin{equation*}
\chi_{R}(\Phi)=\sum_{\sigma \in S_{n}} \chi_{R}(\sigma) \Phi_{i_{\sigma_{1}}}^{i_{1}} \cdots \Phi_{i_{\sigma_{n}}}^{i_{n}}, \tag{2.8}
\end{equation*}
$$

where $\chi_{R}(\sigma)$ is the character of $\sigma$ in the representation $R$. The associated wavefunction

$$
\begin{equation*}
\Psi_{R}=\chi_{R}(\Phi) e^{-\frac{1}{2} \operatorname{tr} \Phi^{2}} \tag{2.9}
\end{equation*}
$$

has energy $\frac{N^{2}}{2}+n$.
A hermitian matrix $\Phi$ may be decomposed as

$$
\begin{equation*}
\Phi=U \Lambda U^{\dagger}, \quad \Lambda=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{N}\right), \quad U \in U(N) \tag{2.10}
\end{equation*}
$$

under which we get

$$
\begin{equation*}
\operatorname{tr}\left(\dot{\Phi}^{2}\right)=\operatorname{tr}\left(\dot{\Lambda}^{2}\right)+\operatorname{tr}\left[\Lambda, U^{\dagger} \dot{U}\right]^{2} \tag{2.11}
\end{equation*}
$$

The anti-hermitian matrix $U^{\dagger} \dot{U}$ may be expanded in generators of $U(N)$ as

$$
U^{\dagger} \dot{U}=\sum_{i} \alpha_{i} H_{i}+\frac{i}{\sqrt{2}} \sum_{j<k}\left(\dot{\alpha}_{j k} T_{j k}+\dot{\beta}_{j k} \widetilde{T}_{j k}\right)
$$

where $H_{i}$ are the diagonal generators of the Cartan subalgebra, $T_{j k}$ is the matrix $M$ such that $M_{j k}=M_{k j}=1$ and all other entries are 0 , and $\widetilde{T}_{i j}$ is the matrix $M$ such that $M_{i j}=-M_{j i}=-i$ and all other entries are 0 . This gives

$$
\operatorname{tr}\left[\Lambda, U^{\dagger} \dot{U}\right]^{2}=\sum_{i<j}\left(\lambda_{i}-\lambda_{j}\right)^{2}\left(\dot{\alpha}_{i j}^{2}+\dot{\beta}_{i j}^{2}\right)
$$

and so the Lagrangian becomes

$$
\begin{equation*}
L=\sum_{i}\left(\frac{1}{2} \dot{\lambda}_{i}^{2}+\frac{1}{2} \lambda_{i}^{2}\right)+\frac{1}{2} \sum_{i<j}\left(\lambda_{i}-\lambda_{j}\right)^{2}\left(\dot{\alpha}_{i j}^{2}+\dot{\beta}_{i j}^{2}\right) . \tag{2.12}
\end{equation*}
$$

Under the transformation (2.2) the measure becomes

$$
\begin{equation*}
\mathcal{D} \Phi=\mathcal{D} \Omega \prod_{i} d \lambda_{i} \Delta^{2}(\lambda) \tag{2.13}
\end{equation*}
$$

where $\Delta(\lambda)$ is the Vandermonde determinant $\prod_{i<j}\left(\lambda_{i}-\lambda_{j}\right)$, the kinetic term for the eigenvalues becomes

$$
\begin{equation*}
-\frac{1}{2} \sum_{i} \frac{1}{\Delta^{2}(\lambda)} \frac{d}{d \lambda_{i}} \Delta^{2}(\lambda) \frac{d}{d \lambda_{i}}=-\frac{1}{2 \Delta(\lambda)} \sum_{i} \frac{d^{2}}{d \lambda_{i}^{2}} \Delta(\lambda) \tag{2.14}
\end{equation*}
$$

and so the Hamiltonian is

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i}\left(-\frac{1}{\Delta(\lambda)} \frac{\partial^{2}}{\partial \lambda_{i}^{2}} \Delta(\lambda)+\lambda_{i}^{2}\right)-\frac{1}{2} \sum_{i<j} \frac{1}{\left(\lambda_{i}-\lambda_{j}\right)^{2}}\left(\frac{\partial^{2}}{\partial \alpha_{i j}^{2}}+\frac{\partial^{2}}{\partial \beta_{i j}^{2}}\right) \tag{2.15}
\end{equation*}
$$

Wavefunctions which are singlet under (2.2) are symmetric functions of the eigenvalues, $\chi_{\text {sym }}(\lambda)$. On these wavefunctions the Hamiltonian simplifies to

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i}\left(-\frac{1}{\Delta(\lambda)} \frac{\partial^{2}}{\partial \lambda_{i}^{2}} \Delta(\lambda)+\lambda_{i}^{2}\right) \tag{2.16}
\end{equation*}
$$

One may simplify further the analysis by defining the antisymmetric wavefunction

$$
\begin{equation*}
\Psi^{f}(\lambda)=\Delta(\lambda) \chi_{s y m}(\lambda) \tag{2.17}
\end{equation*}
$$

and the modified Hamiltonian

$$
\begin{equation*}
H^{f}=\Delta(\lambda) H \frac{1}{\Delta(\lambda)}=\frac{1}{2} \sum_{i}\left(-\frac{d^{2}}{d \lambda_{i}^{2}}+\lambda_{i}^{2}\right) \tag{2.18}
\end{equation*}
$$

which is a sum of one particle harmonic oscillator Hamiltonians. Then $H^{f}$ has eigenstates $\Psi^{f}(\lambda)$ with the same eigenvalues as $H$ :

$$
\begin{align*}
H \Psi(\lambda) & =E \Psi(\lambda)  \tag{2.19}\\
\Rightarrow H^{f} \Psi^{f}(\lambda) & =E \Psi^{f}(\lambda) \tag{2.20}
\end{align*}
$$

The ground state wavefunction of $H^{f}$ is

$$
\begin{equation*}
\Psi_{0}^{f}=\Delta e^{-\frac{1}{2} \operatorname{tr} \Phi^{2}} \tag{2.21}
\end{equation*}
$$

excited states are given by Slater determinants

$$
\begin{equation*}
\Psi_{\overrightarrow{\mathcal{E}}}^{f}=\operatorname{det}_{i, j} \lambda_{i}^{\mathcal{E}_{j}} e^{-\frac{1}{2} \operatorname{tr} \Phi^{2}}=\Delta(\lambda) \Psi_{R}(U) \tag{2.22}
\end{equation*}
$$

and so the $U(N)$ singlet sector is equivalent to $N$ non-interacting fermions in a harmonic oscillator potential, where the fermion energies $\mathcal{E}_{i}$ are related to the integer row lengths $r_{i}$ of $R$ by

$$
\begin{equation*}
\mathcal{E}_{i}=r_{i}+(N-i) \tag{2.23}
\end{equation*}
$$

### 2.2 Unitary matrix quantum mechanics

We next review the unitary matrix quantum mechanics which arises in the study of twodimensional Yang-Mills, which is given by the Hamiltonian [29, 30]:

$$
\begin{equation*}
H=\operatorname{tr}\left(U \frac{\partial}{\partial U}\right)^{2}=\sum_{a} E^{a} E^{a} \tag{2.24}
\end{equation*}
$$

where $E^{a}$ generates left rotations of $U$ :

$$
\begin{equation*}
E^{a}=\operatorname{tr} t^{a} U \frac{\partial}{\partial U} \tag{2.25}
\end{equation*}
$$

The form of $H$ means that acting on a wavefunction which is a matrix element of an irreducible representation $R$,

$$
\begin{equation*}
\left(\psi_{R}\right)_{i j}(U)=D_{i j}^{R}(U) \tag{2.26}
\end{equation*}
$$

it measures the quadratic Casimir of the representation $R$,

$$
\begin{equation*}
H \psi_{R}(U)=C_{2}(R) D_{i j}^{R}(U) \tag{2.27}
\end{equation*}
$$

Representations are classified by their characters, the Schur polynomials

$$
\begin{equation*}
\chi_{R}(U)=\operatorname{tr} D^{R}(U) \tag{2.28}
\end{equation*}
$$

which form an orthonormal basis for wavefunctions invariant under the $U(N)$ action

$$
\begin{equation*}
U \rightarrow g U g^{\dagger}, \quad g \in U(N) \tag{2.29}
\end{equation*}
$$

This may be used to express any unitary matrix $U$ as

$$
\begin{equation*}
U=g D g^{\dagger}, \quad D=\operatorname{diag}\left(e^{i \theta_{1}}, \ldots, e^{i \theta_{N}}\right), \quad g \in U(N) \tag{2.30}
\end{equation*}
$$

On functions invariant under (2.29), performing the change of variables (2.30) the Hamiltonian becomes [29]:

$$
\begin{equation*}
H=-\sum_{i}\left[\frac{1}{\tilde{\Delta}} \frac{d^{2}}{d \theta_{i}^{2}} \tilde{\Delta}\right]-\frac{1}{12} N\left(N^{2}-1\right) \tag{2.31}
\end{equation*}
$$

where denoting the eigenvalues by $u_{i}=e^{i \theta_{i}}$,

$$
\begin{equation*}
\tilde{\Delta}=\prod_{i<j} \sin \frac{\theta_{i}-\theta_{j}}{2}=\frac{\Delta(u)}{\prod_{i} u_{i}^{\frac{N-1}{2}}}=\frac{\Delta(u)}{(\operatorname{det} U)^{\frac{N-1}{2}}} \tag{2.32}
\end{equation*}
$$

and where

$$
\begin{equation*}
\Delta(u)=\prod_{i<j}\left(u_{i}-u_{j}\right) \tag{2.33}
\end{equation*}
$$

Absorbing $\tilde{\Delta}$ into the wavefunctions and the Hamiltonian,

$$
\begin{equation*}
\psi_{f}=\tilde{\Delta} \psi, \quad H_{f}=\tilde{\Delta} H \frac{1}{\tilde{\Delta}}=\sum_{i} \frac{\partial}{\partial \theta_{i}^{2}}-\frac{1}{12} N\left(N^{2}-1\right) \tag{2.34}
\end{equation*}
$$

the wavefunctions become antisymmetric under exchange of any pair $\theta_{i} \leftrightarrow \theta_{j}$. The oneparticle wavefunctions with quantized momentum $p$ are $\psi_{p}=e^{i p \theta}$ and the Slater determinants

$$
\begin{equation*}
\psi_{\vec{p}}=\operatorname{det}_{i, j} u_{i}^{p_{j}} \tag{2.35}
\end{equation*}
$$

are eigenfunctions of $H_{f}$ with energy $E=\sum_{i} p_{i}^{2}-N\left(N^{2}-1\right) / 12$, so the sector of this theory invariant under (2.29) is equivalent to a theory of $N$ free fermions on a circle. The ground state has fermion with momenta distributed symmetrically about $n=0$, and energy zero, so the Fermi energy is $n_{F}=\frac{N-1}{2}$ and there are Fermi surfaces at $\pm n_{F}$. The Slater determinants are related to the Schur polynomials via

$$
\begin{equation*}
\psi_{\vec{p}}=\Delta(u) \chi_{R}(U) \tag{2.36}
\end{equation*}
$$

where the momenta $p_{i}$ are related to the integer row lengths $r_{i}$ of $R$ by

$$
\begin{equation*}
p_{i}=r_{i}+\left(n_{F}+1-i\right) \tag{2.37}
\end{equation*}
$$

### 2.3 Complex matrix models

Previous studies of complex matrix models have centred on models in which there is enough symmetry to diagonalise the matrix. This can be achieved by studying a normal matrix $\left(\left[Z, Z^{\dagger}\right]=0\right)$ with $U(N)$ symmetry (see e.g. [31, 32])

$$
\begin{equation*}
Z \rightarrow g Z g^{\dagger}, \quad g \in U(N) \tag{2.38}
\end{equation*}
$$

or by studying an unrestricted complex $Z$ with $U(N) \times U(N)$ symmetry (see e.g. [33])

$$
\begin{equation*}
Z \rightarrow g Z h^{\dagger}, \quad g, h \in U(N) \tag{2.39}
\end{equation*}
$$

In this paper, motivated by gauge-gravity duality we study an unrestricted complex matrix $Z$ with a single $U(N)$ symmetry (2.38). This requires us to go beyond an eigenvlue description and take into account off-diagonal degrees of freedom.

Due to the off-diagonal degrees of freedom we do not expect a straightforward transformation to a description in terms of free particles for complex matrix models with unitary symmetry. Nevertheless, our investigations of the $k=0$ sector of the Brauer basis indicate that the free particles on a circle of the Unitary matrix model can be constructed from the degrees of freedom of the complex matrix model. These free particles on a circle are emergent degrees of freedom arising from eigenalues and off-diagonal elements constrained by equations which define the $k=0$ sector.

In passing we note that the Gaussian complex matrix model with $U(N)$ symmetry (2.38) may be written as a two-Hermitian matrix model [34] using

$$
\begin{equation*}
X=\frac{1}{2}(Z+\bar{Z}), \quad Y=-\frac{i}{2}(Z-\bar{Z}) \tag{2.40}
\end{equation*}
$$

where $\bar{Z}$ denotes complex conjugate of $Z$. Studies of the same model in terms of two hermitian matrices are done in $[35,36]$.

## 3 Orbits and parameter spaces

The relation between $g l(N, \mathbb{C})$, the space of complex matrices $Z$ and the space $\mathcal{M}_{N}$, of orbits under the adjoint action (1.2), is given by the Schur decomposition.

### 3.1 Orbits and the structure of $\mathcal{M}_{N}$

Schur's decomposition (see e.g. [37]) allows one to write any complex matrix $Z$ as

$$
\begin{equation*}
Z=U T U^{\dagger} \tag{3.1}
\end{equation*}
$$

where $U \in U(N)$ and $T$ is upper triangular. It has been used previously in the context of the complex matrix model in $[38,39]$. The eigenvalues $z_{i}$ of $Z$ become the diagonal entries (and hence the eigenvalues) of $T$. There are also off-diagonal elements $t_{i j}$ for $i<j$. The equation (3.1) can be viewed as describing a map from the pair $(U, T)$ to complex matrices. The map is onto, but not one-to-one. Pairs $(U, T)$ and $\left(e^{i \theta} U, T\right)$ describe the same $Z$. There is a $U(1)^{N}$ action

$$
\begin{align*}
& U \rightarrow U^{\prime}=U H, \quad H=\operatorname{diag}\left(e^{i \theta_{1}}, \ldots, e^{i \theta_{N}}\right) \\
& T \rightarrow T^{\prime}=H^{\dagger} T H \tag{3.2}
\end{align*}
$$

which leaves $Z$ unchanged. The diagonal $e^{i \theta}$ acts trivially on $T$ but the $U(1)^{N-1}$ part defined by $\sum \theta_{j}=0$ mixes non-trivially with the angles in $T$.

We can parameterize the coset $U(N) / U(1)^{N}$ using the variable $L$ and decomposing $U=L H$ (as for example in [40]) leading to

$$
\begin{equation*}
Z=L\left(H T H^{\dagger}\right) L^{\dagger}=L \tilde{T} L^{\dagger} \tag{3.3}
\end{equation*}
$$

where $\tilde{T} \equiv H T H^{\dagger}$. It is also convenient to use the $U(1)^{N-1}$ part of (3.2) to set the $N-1$ entries on the superdiagonal of $T$ (namely $t_{j, j+1}$ ) to be real, and to use $(U, T)$.

There is also the freedom, for fixed $Z$, to rearrange the eigenvalues in any order on the diagonal of $T$ by altering $U$. This freedom exists because there is a Schur decomposition for each possible ordering of eigenvalues on the diagonal of $T$. Given

$$
\begin{equation*}
Z=U_{1} T_{1} U_{1}^{\dagger}=U_{2} T_{2} U_{2}^{\dagger} \tag{3.4}
\end{equation*}
$$

where $T_{1}$ and $T_{2}$ have different orderings of diagonal entries, we have

$$
\begin{equation*}
T_{2}=\left(U_{2}^{\dagger} U_{1}\right) T_{1}\left(U_{2}^{\dagger} U_{1}\right)^{\dagger}=U_{12} T_{1} U_{12}^{\dagger} \tag{3.5}
\end{equation*}
$$

where $U_{12} \equiv U_{2}^{\dagger} U_{1}$.
We have thus derived the construction mentioned in the introduction of $\mathcal{M}_{N}$ as a fibration over the symmetric product $S y m^{N}(\mathbb{C})$ :

$$
\begin{align*}
& \mathcal{M}_{N}  \tag{3.6}\\
& \stackrel{\downarrow}{\operatorname{Sym}^{N}}(\mathbb{C})=\mathbb{C}^{N} / S_{N} .
\end{align*}
$$

The set of eigenvalues $z_{1}, z_{2}, \ldots, z_{N}$ of $Z$ modulo permutations in $S_{N}$ forms the space $\operatorname{Sym}^{N}(\mathbb{C})$. Local coordinates on the fibre of $\mathcal{M}_{N}$ over $\operatorname{Sym}^{N}(\mathbb{C})$ are obtained from the upper triangular elements $t_{i j}$, with $i<j$, appearing in $T$.

Functions of degree $n$ on $\mathbb{R}^{N} / S_{N}$ and natural inner products on the space of functions, which are expressible in terms of integrals, are organised by the symmetric group
$S_{n}$. Since $n$ can be arbitrarily large, we may say that $S_{\infty}$, defined as an inductive limit from finite symmetric groups (see e.g. [41]), is the symmetry organising the space of functions on $\mathbb{R}^{N} / S_{N}$. In the case of $\mathcal{M}_{N}$ there is an infinite-dimensional underlying Brauer algebra constructed as a limit of finite algebras $B_{N}(m, n)$.

The space of $N \times N$ complex matrices $g l(N, \mathbb{C})$ consists of orbits generated by the $U(N)$ action $Z \rightarrow U Z U^{\dagger}$. Due to the trivial $U(1)$ action the real dimension of the parameter space of orbits $\mathcal{M}_{N}$ is $N^{2}+1=2 N^{2}-\left(N^{2}-1\right)$.

This suggests that the number of generators of ring of functions on $\mathcal{M}_{N}$ should be $N^{2}+1$. This works in a straightforward way at $N=2$, but in a nontrivial way at $N=3$. We will come back to this in Section 5 .

Local coordinates on $\mathcal{M}_{N}$ are given by $z_{i}$ and variables $t_{i j}$. At generic $z_{i}, t_{i j}$ the orbits are topologically $U(N) / U(1)=S U(N) / Z_{N}$. At $t_{i j}=0$, the parameter space $\mathcal{M}_{N}$ becomes $S^{\prime} m^{N}(\mathbb{C})$. The orbit is then generically $S U(N) / U(1)^{N-1}$. Note that, when $U(N)$ acts on its Lie algebra, the adjoint orbits are always Kähler (and hence even dimensional) [42]. This is no longer the case for orbits in the complexified Lie algebra $g l(N, \mathbb{C})$.

### 3.2 Differential Gauss's law

Dimensional reduction of $\mathcal{N}=4$ SYM onto $\mathbb{R}_{t} \times S^{3}$ yields a $U(N)$ gauged matrix quantum mechanics involving a complex matrix $Z(t)$ in the adjoint coupled to a gauge field $A_{0}(t)$. The action takes the form

$$
\begin{equation*}
\mathcal{S}=\int d t \operatorname{tr}\left(D_{0} Z\left(D_{0} Z\right)^{\dagger}-Z Z^{\dagger}\right) \tag{3.7}
\end{equation*}
$$

where $D_{0} Z=\partial_{0} Z+i\left[A_{0}, Z\right]$.
Using the above change of variables (3.1), (3.3) we may derive an expression for the 1-form on $g l(N ; \mathbb{C})$

$$
\begin{equation*}
d Z=U(d T+[\omega, T]) U^{\dagger}=L(d \tilde{T}+[V, \tilde{T}]) L^{\dagger} \tag{3.8}
\end{equation*}
$$

where $\omega=U^{\dagger} d U$ and $V=L^{\dagger} d L$. This allows us to write the line element $\operatorname{tr}\left(d Z d Z^{\dagger}\right)$ in terms of the structure constants of the Lie algebra, with a choice of decomposition into coset and sub-algebra.

As an aside, it is interesting to note that, as a consequence of (3.8) we can write a quantum mechanics theory with $U(N)$ global symmetry

$$
\begin{equation*}
\mathcal{S}=\int d t \operatorname{tr}\left(\partial_{0} Z \partial_{0} Z^{\dagger}\right)=\int d t \operatorname{tr}\left(\partial_{0} \tilde{T}+[V, \tilde{T}]\right)\left(\partial_{0} \tilde{T}^{\dagger}+\left[V, \tilde{T}^{\dagger}\right]\right) \tag{3.9}
\end{equation*}
$$

as a quantum mechanics with gauged $U(1)^{N}$ symmetry and charged matter fields $\tilde{T}$ where the one form $V$ on the coset couples as a gauge field. The gauge symmetry is $\tilde{T} \rightarrow h \tilde{T} h^{-1}$
and $V(y) \rightarrow h V h^{-1}+h \partial_{0} h^{-1}$, under which $\left(\partial_{0} \tilde{T}+[V, \tilde{T}]\right)$ transforms covariantly and the action is invariant.

We next review remarks contained in [25] and introduce notation we shall use later. A convenient gauge fixing choice is to set $A_{0}=0$. The equation of motion for $A_{0}$ must still be imposed, leading to Gauss's Law:

$$
\begin{equation*}
Z^{\dagger} \dot{Z}+Z \dot{Z}^{\dagger}-\dot{Z} Z^{\dagger}-\dot{Z}^{\dagger} Z=0 \tag{3.10}
\end{equation*}
$$

Upon canonical quantization this leads to the differential form of Gauss's Law, which can be written as

$$
\begin{equation*}
G=G_{1}+G_{2}+G_{3}+G_{4}=0 \tag{3.11}
\end{equation*}
$$

where $G_{i}$ are defined as:

$$
\begin{align*}
\left(G_{1}\right)_{j}^{i} & =Z_{k}^{\dagger i}\left(\frac{\partial}{\partial Z^{\dagger}}\right)_{j}^{k} & \left(G_{2}\right)_{j}^{i} & =Z_{k}^{i}\left(\frac{\partial}{\partial Z}\right)_{j}^{k} \\
\left(G_{3}\right)_{j}^{i} & =-Z_{j}^{\dagger k}\left(\frac{\partial}{\partial Z^{\dagger}}\right)_{k}^{i} & \left(G_{4}\right)_{j}^{i} & =-Z_{j}^{k}\left(\frac{\partial}{\partial Z}\right)_{k}^{i} \tag{3.12}
\end{align*}
$$

and we use the usual convention for matrix indices given in (2.5). Note that in $G_{1}$ and $G_{2}$ the ordering of indices is that of usual matrix multiplication, while for $G_{3}$ and $G_{4}$ the opposite is the case. The $G_{i}$ correspond respectively to each of the terms in (3.10). The operator $G$ is the infinitesimal generator of the adjoint action

$$
\begin{equation*}
Z \rightarrow U Z U^{\dagger}, \quad Z^{\dagger} \rightarrow U Z^{\dagger} U^{\dagger} \tag{3.13}
\end{equation*}
$$

and invariance under this action restricts gauge invariant operators to be products of traces of the matrices $Z$ and $Z^{\dagger}$.

### 3.3 Geometry of $\mathcal{M}_{2}$ : coordinates

In this section and in Section 4 we perform explicit calculations at $N=2$. The motivation for considering small values of $N$ is to perform explicit calculations which shed light on the harder (and more interesting) task of obtaining results at arbitrary finite $N$, a task we return to in Section 5.

We start from the Schur decomposition as discussed in Section 3.1,

$$
\begin{equation*}
Z=U T U^{\dagger}=L \tilde{T} L^{\dagger} \tag{3.14}
\end{equation*}
$$

In the $N=2$ case $U(2) / U(1) \cong S U(2) / \mathbb{Z}_{2} \cong S O(3)$. We can specify explicit coordinates

$$
\begin{align*}
U & =\left(\begin{array}{cc}
\cos \frac{\theta}{2} e^{\frac{i}{2}(\phi+\psi)} & \sin \frac{\theta}{2} e^{\frac{i}{2}(\phi-\psi)} \\
-\sin \frac{\theta}{2} e^{-\frac{i}{2}(\phi-\psi)} & \cos \frac{\theta}{2} e^{-\frac{i}{2}(\phi+\psi)}
\end{array}\right)  \tag{3.15}\\
T & =\left(\begin{array}{cc}
z_{1} & t_{0} \\
0 & z_{2}
\end{array}\right) . \tag{3.16}
\end{align*}
$$

The angles $\theta, \phi, \psi$ are the Euler angles of $S U(2) / \mathbb{Z}_{2} \cong S O(3)$. With these coordinates $L$ and $\tilde{T}$ take the form

$$
\begin{align*}
L & =\left(\begin{array}{cc}
\cos \frac{\theta}{2} e^{\frac{i}{2} \phi} & \sin \frac{\theta}{2} e^{\frac{i}{2} \phi} \\
-\sin \frac{\theta}{2} e^{-\frac{i}{2} \phi} & \cos \frac{\theta}{2} e^{-\frac{i}{2} \phi}
\end{array}\right)  \tag{3.17}\\
\tilde{T} & =\left(\begin{array}{cc}
z_{1} & t_{0} e^{i \psi} \\
0 & z_{2}
\end{array}\right) . \tag{3.18}
\end{align*}
$$

The ranges of the coordinates are

$$
\begin{array}{ll}
z_{1}, z_{2} \in \mathbb{C}, & 0 \leq t_{0}<\infty \\
0 \leq \theta \leq \pi, & 0 \leq \phi<2 \pi, \tag{3.20}
\end{array} \quad 0 \leq \psi<2 \pi
$$

The Jacobian for the change of variables from $Z_{i j}$ to those above is

$$
\begin{equation*}
J=\left|z_{1}-z_{2}\right|^{2} t_{0} \sin \theta \tag{3.21}
\end{equation*}
$$

and so we have

$$
\begin{equation*}
\int \prod_{i, j} d Z_{i j} d \bar{Z}_{i j}=\int d z_{1} d z_{2} d t_{0} t_{0} d U\left|z_{1}-z_{2}\right|^{2} \tag{3.22}
\end{equation*}
$$

Note the factor of $t_{0}$ here which is analogous to the $\int r d r$ one gets when using plane polar coordinates. Here $d U$ is the Haar measure on $S U(2)$ which we integrate out and normalise to 1 in the definition of the measure. The implication of the measure is that the region $t_{0}=0$, where the orbit structure changes compared to that at $t_{0} \neq 0$, has measure zero. Likewise the collision of points $z_{1}=z_{2}$ in $\operatorname{Sym}^{N}(\mathbb{C})$ has measure zero.

The invariant line element on $g l(2, \mathbb{C})$ is given by

$$
\begin{equation*}
d s^{2}=\operatorname{tr} d Z d Z^{\dagger} \tag{3.23}
\end{equation*}
$$

We introduce the notation

$$
\omega=U^{-1} d U=\left(\begin{array}{cc}
\omega_{11} & \omega_{12}  \tag{3.24}\\
-\bar{\omega}_{12} & -\omega_{11}
\end{array}\right)
$$

and using $\omega^{\dagger}=-\omega$ we expand $d Z=U(d T+[\omega, T]) U^{\dagger}$.
The line element is then expressible as

$$
\begin{align*}
d s^{2}= & \operatorname{tr}(d T+[\omega, T])\left(d T^{\dagger}+\left[\omega, T^{\dagger}\right]\right)  \tag{3.25}\\
= & \left|d z_{1}+t_{0} \bar{\omega}_{12}\right|^{2}+\left|d z_{2}-t_{0} \bar{\omega}_{12}\right|^{2} \\
& +\left|d t_{0}+2 t_{0} \omega_{11}-\left(z_{1}-z_{2}\right) \omega_{12}\right|^{2}+\left|\left(z_{1}-z_{2}\right) \omega_{12}\right|^{2} \tag{3.26}
\end{align*}
$$

Using the Cartan one-forms $\omega_{i}$ on $S U(2)$ (see e.g. [43]),

$$
\begin{equation*}
\omega=U^{-1} d U=-\omega_{i} T_{i}, \quad T_{j}=\frac{i}{2} \sigma_{j} \tag{3.27}
\end{equation*}
$$

one may read off the metric on the orbit; we shall do this in the next section.
As an aside, we note that $U_{12}$ defined below 3.5 is not a standard permutation matrix in $U(N)$ (the reader may check that the standard permutation matrices in $U(N)$ do not preserve the triangular form). For concreteness we now exhibit this at $N=2$. Consider the two matrices

$$
\begin{align*}
& T_{1}=\left(\begin{array}{cc}
z_{1} & t_{0} \\
0 & z_{2}
\end{array}\right)  \tag{3.28}\\
& T_{2}=\left(\begin{array}{cc}
z_{2} & t_{0} \\
0 & z_{1}
\end{array}\right) \tag{3.29}
\end{align*}
$$

where we have chosen $t_{0} \in \mathbb{R}$.
Defining $D=\sqrt{t_{0}^{2}+\left|z_{1}-z_{2}\right|^{2}}$, we then have $T_{2}=U_{12} T_{1} U_{12}^{\dagger}$ with

$$
U_{12}=\frac{1}{D}\left(\begin{array}{cc}
t_{0} & -\left(\bar{z}_{1}-\bar{z}_{2}\right)  \tag{3.30}\\
z_{1}-z_{2} & t_{0}
\end{array}\right)
$$

Clearly this is not the standard permutation matrix $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$, but it performs the permutation transformation $z_{1} \leftrightarrow z_{2}$ while preserving the triangular structure. For $N>2$ the analogous transformation does not just permute the $z_{i}$ entries but transforms the $t_{i j}$ nontrivially.

### 3.4 Differential Gauss's law and orbits at $N=2$

Using a change of variables, one may express the Gauss Law operator $G$ (3.11-3.12) in the coordinates defined in (3.15-3.16). This results in the following form of the Gauss's Law operator:

$$
G=\left(\begin{array}{cc}
-i \frac{\partial}{\partial \phi} & i e^{i \psi}\left(-\frac{\partial}{\partial \theta}-i \cot \theta \frac{\partial}{\partial \phi}+i \csc \theta \frac{\partial}{\partial \psi}\right)  \tag{3.31}\\
i e^{-i \psi}\left(\frac{\partial}{\partial \theta}-i \cot \theta \frac{\partial}{\partial \phi}+i \csc \theta \frac{\partial}{\partial \psi}\right) & i \frac{\partial}{\partial \phi}
\end{array}\right)
$$

This must vanish on gauge invariant wavefunctions, which must therefore be functions only of $z_{i}, t_{0}$. We will show in Section 3.5 that the ring of gauge invariant polynomials has five generators.

The Gauss's Law reduces the 8 D space $g l(2, \mathbb{C})$ to the 5 D space parametrized by $\left(z_{1}, z_{2}, t_{0}\right)$. We shall find it convenient to define

$$
\begin{equation*}
z_{c}=z_{1}+z_{2}, \quad z=z_{1}-z_{2} \tag{3.32}
\end{equation*}
$$

As we have seen, we can exchange $z_{1}, z_{2}$ while leaving $t_{0}$ invariant; this means mapping $z \rightarrow-z$, and so the space of inequivalent orbits is

$$
\begin{equation*}
\mathcal{M}_{2}=\mathbb{C} \times\left(\mathbb{C} / \mathbb{Z}_{2}\right) \times \mathbb{R}^{+} \tag{3.33}
\end{equation*}
$$

From the metric (3.26) expressed in terms of $z_{i}, t_{0}$ we see that the nature of the orbits changes as we move in the space $\left(\mathbb{C} / \mathbb{Z}_{2}\right) \times \mathbb{R}^{+}$. The centre of mass coordinate $z_{c}$ does not affect the nature of the orbits and so we restrict our attention to a $\mathbb{Z}_{2}$ quotient of the $z, t_{0}$ space. Let us define

$$
\begin{align*}
X & =\left(\mathbb{C} / \mathbb{Z}_{2}\right) \times \mathbb{R}^{+} \\
& =X_{0} \cup X_{1} \cup X_{2} \cup X_{3} \tag{3.34}
\end{align*}
$$

where $X$ is the region in $\left(z, t_{0}\right)$ space where $t_{0} \geq 0, \operatorname{Re}(z) \geq 0$, and the subregions $X_{i}$ are defined as follows:

- $X_{0}$ is the subregion $t_{0}>0, z \neq 0$
- $X_{1}$ is the subregion $t_{0}>0, z=0$
- $X_{2}$ is the subregion $t_{0}=0, z \neq 0$
- $X_{3}$ is the point $t_{0}=0, z=0$.

The metric on the gauge orbit is determined by fixing $z_{i}, t_{0}$ in (3.26). On $X_{0}$ and $X_{1}$ the orbit is topologically $S O(3)$; the metric is complicated in general but on $X_{1}$ it qualitatively resembles the round three-sphere metric. On $X_{2}$ the orbit is a round $S^{2}$, while on $X_{3}$ the orbit is a point. This completes the global description of the parameter space and the orbits. Note that on $X_{0}$ the metric is regular but on $X_{1}, X_{2}$ and $X_{3}$, the determinant of the metric is zero.

### 3.5 The algebra of functions on $\mathcal{M}_{2}$

The algebra of functions on $\mathcal{M}_{N}$ is generated by single trace polynomials in $Z, Z^{\dagger}$. In the $N \rightarrow \infty$ limit any word in the two letters $Z, Z^{\dagger}$, up to cyclic permutations, corresponds to a single-trace gauge-invariant function and hence to a function on $\mathcal{M}_{\infty}$. At finite $N$, traces of long words can be expressed in terms of products of traces of shorter words and so the ring of gauge invariant functions has a finite set of generators.

In [12] this truncation of the generators was discussed in terms of degenerations of Brauer algebra projectors. Here we investigate these finite $N$ truncations in detail at $N=2$ and find that it suffices to apply the Cayley-Hamilton theorem to obtain the necessary relations.

The Cayley-Hamilton theorem states that a matrix satisfies its own characteristic polynomial. At $N=2$ this means that

$$
\begin{equation*}
Z^{2}-(\operatorname{tr} Z) Z+(\operatorname{det} Z) 1_{2}=0 \tag{3.35}
\end{equation*}
$$

Taking the trace of this equation gives a relation between $\operatorname{tr} Z^{2}, \operatorname{tr} Z$ and $\operatorname{det} Z$, only two of which are thus algebraically independent as polynomials in the matrix entries. We
choose $\operatorname{tr} Z^{2}$ and $\operatorname{tr} Z$ to be independent, and write

$$
\begin{equation*}
\operatorname{det} Z=\frac{1}{2}\left[\operatorname{tr} Z \operatorname{tr} Z-\operatorname{tr} Z^{2}\right] \tag{3.36}
\end{equation*}
$$

We also have the corresponding equation for $Z^{\dagger}$.
We claim that the ring of multi-trace GIOs in $Z, Z^{\dagger}$ at $N=2$ is the polynomial ring generated by the set

$$
\begin{equation*}
\mathcal{B}=\left\{\operatorname{tr} Z, \quad \operatorname{tr} Z^{2}, \quad \operatorname{tr} Z^{\dagger}, \quad \operatorname{tr} Z^{\dagger 2}, \quad \operatorname{tr} Z Z^{\dagger}\right\} \tag{3.37}
\end{equation*}
$$

In order to prove this, it is enough to show that all other single trace operators are algebraically dependent on the operators above.

We prove this in an inductive fashion. Let $W$ to denote any matrix word made from $Z$ and $Z^{\dagger}$, e.g. $W=Z Z Z^{\dagger} Z$. Multiply (3.35) by $W$ and take the trace. This yields the relation

$$
\begin{equation*}
\operatorname{tr}\left(Z^{2} W\right)-(\operatorname{tr} Z) \operatorname{tr}(Z W)+\frac{1}{2}\left[\operatorname{tr} Z \operatorname{tr} Z-\operatorname{tr} Z^{2}\right] \operatorname{tr} W=0 \tag{3.38}
\end{equation*}
$$

This shows that $\operatorname{tr}\left(Z^{2} W\right)$ is algebraically dependent on $\operatorname{tr}(Z W), \operatorname{tr} W$ and the operators in $\mathcal{B}$, and similarly, $\operatorname{tr}\left(Z^{\dagger 2} W\right)$ is algebraically dependent on $\operatorname{tr}\left(Z^{\dagger} W\right), \operatorname{tr} W$ and the operators in $\mathcal{B}$.

Replacing $Z$ by $Z Z^{\dagger}$ in (3.35) and using $\operatorname{det} Z Z^{\dagger}=\operatorname{det} Z \operatorname{det} Z^{\dagger}$ gives

$$
\begin{equation*}
\operatorname{tr}\left(Z Z^{\dagger}\right)^{2}=\left(\operatorname{tr} Z Z^{\dagger}\right)^{2}-\frac{1}{2}\left[\operatorname{tr} Z \operatorname{tr} Z-\operatorname{tr} Z^{2}\right]\left[\operatorname{tr} Z^{\dagger} \operatorname{tr} Z^{\dagger}-\operatorname{tr} Z^{\dagger 2}\right] \tag{3.39}
\end{equation*}
$$

This shows us that $\operatorname{tr}\left(Z Z^{\dagger}\right)^{2}$ is algebraically dependent on the operators in the set $\mathcal{B}$. Similarly, for any word $W_{2}$ of length at least two, $\operatorname{tr} W_{2}^{2}$ is algebraically dependent on $\operatorname{tr} W_{2}$ and the operators in the set $\mathcal{B}$.

We conclude that a single trace operator consisting of the trace of a word made from $Z$ and $Z^{\dagger}$ is algebraically dependent on single trace operators of shorter length iff it contains one of the following combinations as part of the word:

$$
\begin{equation*}
Z^{2} W, \quad Z^{\dagger 2} W, \quad \text { or } \quad W_{2}^{2} \tag{3.40}
\end{equation*}
$$

where as above $W$ stands for any (non-zero length) word in $Z$ and $Z^{\dagger}$, and $W_{2}$ stands for such a word of length at least two.

Iterating the above results, a single trace operator containing one of the combinations in (3.40) can be expressed as sums of products of shorter and shorter single trace operators until it is expressed as a sum of products of single trace operators containing none of the combinations in (3.40). A maximal set of algebraically independent operators is therefore
given by those single trace operators which do not contain any of the expressions in (3.40). As claimed this is the set $\mathcal{B}$.

It is worth remarking that we start with a description of the space $g l(2, \mathbb{C})$ in terms of polynomials in $z_{1}, z_{2}, t_{0}, \theta, \phi, \psi$. The differential Gauss Law (3.31) removes the angular variables leaving the ring of polynomials in the remaining variables, which we denote

$$
\begin{equation*}
\left\langle z_{1}, z_{2}, \bar{z}_{1}, \bar{z}_{2}, t_{0}\right\rangle . \tag{3.41}
\end{equation*}
$$

Invariance under large gauge transformations reduces the ring of gauge invariant polynomials to the polynomial ring generated by $\mathcal{B}$. Recalling the definitions $z_{c}=z_{1}+z_{2}, z=$ $z_{1}-z_{2}$ and defining

$$
\begin{equation*}
\mathcal{Z}=z^{2}, \quad \overline{\mathcal{Z}}=\bar{z}^{2}, \quad T_{0}=t_{0}^{2}+\frac{z \bar{z}}{2} \tag{3.42}
\end{equation*}
$$

the ring of gauge invariant polynomials is equivalently the polynomial ring

$$
\begin{equation*}
\left\langle z_{c}, \bar{z}_{c}, \mathcal{Z}, \overline{\mathcal{Z}}, T_{0}\right\rangle \tag{3.43}
\end{equation*}
$$

This is analogous to $U(N)$ gauged Hermitian matrix quantum mechanics where the differential Gauss Law reduces to polynomials in the eigenvalues

$$
\begin{equation*}
\left\langle x_{1}, x_{2}, \ldots, x_{N}\right\rangle \tag{3.44}
\end{equation*}
$$

and invariance under the $S_{N}$ residual Weyl transformations reduces the gauge invariant polynomials to symmetric polynomials in $x_{1}, x_{2}, \cdots, x_{N}$, equivalently polynomials in the variables

$$
\begin{equation*}
\left\langle\left(x_{1}+x_{2}+\cdots+x_{N}\right),\left(x_{1}^{2}+x_{2}^{2}+\cdots+x_{N}^{2}\right), \ldots,\left(x_{1}^{N}+x_{2}^{N}+\cdots+x_{N}^{N}\right)\right\rangle . \tag{3.45}
\end{equation*}
$$

In the hermitian case, we are going from a ring to a sub-ring, which corresponds to going from the space $\mathbb{R}^{N}$ to its quotient space $\mathbb{R}^{N} / S_{N}$. In our model, we are going from the ring (3.41) to the sub-ring (3.43), and correspondingly from the $\mathbb{R}^{4} \times \mathbb{R}^{+}=\mathbb{C}^{2} \times \mathbb{R}^{+}$ parametrized by the five coordinates $z_{i}, t_{0}$ to $\mathcal{M}_{2}$. Because of the off-diagonal degrees of freedom, $\mathcal{M}_{2}$ is not a straightforward quotient of $\mathbb{R}^{4} \times \mathbb{R}^{+}$.

A full investigation of finite $N$ relations for $N>2$ is left for the future. We expect it will be useful to combine the Cayley-Hamilton approach with the the vanishing of the Brauer projectors, such as in equation (8.16) of [12].

## 4 Free particle structures and counting on $\mathcal{M}_{2}$

The remainder of the paper involves the Brauer basis for complex matrix models constructed in [12]. A Brauer basis operator is a linear combination of multi-trace operators; it is a polynomial of degree $m$ in $Z$ and degree $n$ in $Z^{\dagger}$. A brief review of the essential properties of the basis and some simple examples are given in Appendix A, where references to existing literature are also given.

A Brauer basis operator is written as

$$
\begin{equation*}
\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(Z, Z^{\dagger}\right) \tag{4.1}
\end{equation*}
$$

where $\alpha$ and $\beta$ are Young diagrams with $m$ and $n$ boxes and $\gamma=\left(k, \gamma_{+}, \gamma_{-}\right)$where $k$ is an integer in the range $0 \leq k \leq \min (m, n)$ and $\gamma_{+}, \gamma_{-}$are Young diagrams with $m-k$ and $n-k$ boxes respectively. For a more complete explanation of the labels please see Appendix A.

As an example for the reader to bear in mind, when $(m, n)=(1,1)$, suppressing non-essential labels the Brauer basis is

$$
\begin{align*}
\mathcal{O}_{[11][1]}^{k=0}\left(Z, Z^{\dagger}\right) & =\operatorname{tr} Z \operatorname{tr} Z^{\dagger}-\frac{1}{N} \operatorname{tr} Z Z^{\dagger}  \tag{4.2}\\
\mathcal{O}_{[1],[1]}^{k=1}\left(Z, Z^{\dagger}\right) & =\frac{1}{N} \operatorname{tr} Z Z^{\dagger} \tag{4.3}
\end{align*}
$$

Since we discuss in particular the label $k$ throughout the rest of this paper, the following comment will be useful to the uninitiated reader. In the construction of the Brauer basis, a term with a single ' $Z Z^{\dagger}$ ' inside the same trace, such as $\operatorname{tr} Z Z Z^{\dagger}$, involves a single 'Brauer contraction'. Terms such as $\operatorname{tr} Z Z^{\dagger} \operatorname{tr} Z Z^{\dagger} Z^{\dagger}$ or $\operatorname{tr} Z Z^{\dagger} Z Z Z^{\dagger}$ involve two such Brauer contractions, etc.

The label $k$ is related to the number of contractions as follows. If one writes a Brauer basis operator as a sum of terms in order of increasing contractions, as the two operators above are written, an operator with label $k$ begins with a term involving $k$ Brauer contractions. We have not proved this, but we believe it to be true from all the examples we know. Thus the leading term in a $k=0$ operator is the product of a purely holomorphic operator and a purely anti-holomorphic operator, while all terms in a $m=n=k$ operator involve $k$ contractions.

The first result of this section is a conjecture for the solution to the $N=2$ counting of the operators of the Brauer basis, for which which we provide numerical evidence.

The second result will be to find evidence of a 'free fermions on a circle' structure in the $k=0$ sector. This generalizes to any $N$, as discussed in Section 5.1. We show that the Brauer basis at $N=2$ can be neatly expressed in terms of five integers and observe the correspondence between states in the $k=0$ sector and two free fermions on a circle. In these developments a crucial role is played by the structure of the ring of Casimirs.

The third result in this section is to show that the momenta of the free fermions of the $k=0$ sector can be constructed from differential operators in variables which include both eigenvalues and off-diagonal elements of $Z$. This leads us to observe that the complex matrix model contains free fermions arising in a novel way, different from the way they arise in hermitian or unitary models.

### 4.1 Casimir operators and a ring of degree-preserving differential operators

The differential operators introduced in equation (3.12) were studied in [25] as generalized Casimirs commuting with the scaling operator for $Z, Z^{\dagger}$, which is the Hamiltonian for zero coupling SYM. This ring is analogous to the ring generated by $\mathcal{B}$ in Section 3.5; at $N=2$ the generating set is

$$
\begin{equation*}
\mathcal{D}=\left\{\operatorname{tr} G_{2}, \quad \operatorname{tr} G_{2}^{2}, \quad \operatorname{tr} G_{3}, \quad \operatorname{tr} G_{3}^{2}, \quad \operatorname{tr} G_{2} G_{3}\right\} \tag{4.4}
\end{equation*}
$$

where $G_{2}, G_{3}$ were defined in (3.12)

$$
\begin{equation*}
\left(G_{2}\right)_{j}^{i}=Z_{k}^{i}\left(\frac{\partial}{\partial Z}\right)_{j}^{k} \quad\left(G_{3}\right)_{j}^{i}=-Z_{j}^{\dagger k}\left(\frac{\partial}{\partial Z^{\dagger}}\right)_{k}^{i} \tag{4.5}
\end{equation*}
$$

Defining

$$
\begin{equation*}
G_{L}=G_{2}+G_{3}, \tag{4.6}
\end{equation*}
$$

we introduce the Hamiltonians

$$
\begin{array}{ll}
H_{1}=\operatorname{tr} G_{2} & H_{2}=\operatorname{tr} G_{2}^{2} \\
\bar{H}_{1}=\operatorname{tr} G_{3} & \bar{H}_{2}=\operatorname{tr} G_{3}^{2} \tag{4.7}
\end{array} \quad H_{L}=\operatorname{tr} G_{L}^{2} .
$$

Each of these operators commutes with the scaling operator for $Z$ and $Z^{\dagger}$, which is $H=H_{1}+\bar{H}_{1}$. The operators in $\mathcal{D}$ generate a ring of commuting Hamiltonians related to the integrability of the system. We have defined $H_{L}$ for later convenience; its name derives from the fact that the operator $G_{2}+G_{3}$ is the infinitesimal generator of the left action of $U(N)$ [25]:

$$
\begin{equation*}
Z \rightarrow U Z, \quad Z^{\dagger} \rightarrow Z^{\dagger} U^{\dagger} \tag{4.8}
\end{equation*}
$$

It was shown in [25] that the five operators defined in (4.7),

$$
\mathcal{H}_{A}=\left\{\begin{array}{lllll}
H_{1}, & \bar{H}_{1}, & H_{2}, & \bar{H}_{2}, & H_{L} \tag{4.9}
\end{array}\right\}
$$

measure respectively the Casimirs

$$
\mathcal{C}_{A}=\left\{\begin{array}{lllll}
C_{1}(\alpha), & C_{1}(\beta), & C_{2}(\alpha), & C_{2}(\beta), & C_{2}(\gamma) \tag{4.10}
\end{array}\right.
$$

Generalized Casimir operators such as $\operatorname{tr}\left(G_{2}^{2} G_{3}\right)$ were investigated in [25] and were shown to be sensitive to the labels $i, j$ in (4.1). Since the matrix elements of $G_{2}$ and $G_{3}$ commute, we may regard $G_{2}$ and $G_{3}$ as matrices of c-numbers and apply the Cayley-Hamilton theorem as in Section 3.5 to show that the set $\mathcal{D}$ is a maximal algebraically independent set of degree-preserving gauge invariant differential operators.

This observation implies that the generalized Casimir operators such as $\operatorname{tr} G_{2}^{2} G_{3}$ do not yield independent information about the wavefunctions at $N=2$, i.e. that all the information in the labels $\{\alpha, \beta, \gamma, i, j\}$ is in fact contained only in $\{\alpha, \beta, \gamma\}$. We can interpret this fact in terms of Brauer algebra representation theory as follows.

In the restriction of an irreducible representation $\gamma$ of the Brauer algebra to the representation $A=(\alpha, \beta)$ of $\mathbb{C}\left[S_{m} \times S_{n}\right]$, there enters an integer multiplicity $M_{A}^{\gamma ; N}$ defined by

$$
\begin{equation*}
V_{\gamma}^{B_{N}(m, n)}=\bigoplus_{A} M_{A}^{\gamma: N} V_{A}^{\mathbb{C}\left(S_{m} \times S_{n}\right)} \tag{4.11}
\end{equation*}
$$

For large $N$, i.e. $m+n<N$, we denote this multiplicity by $M_{A}^{\gamma}$ or $M_{\alpha, \beta}^{\gamma}$ and using $\delta \vdash k$ to denote that $\delta$ is a partition of $k$ we have the formula [14]

$$
\begin{equation*}
M_{A}^{\gamma}=M_{\alpha, \beta}^{\gamma}=\sum_{\delta \vdash k} \sum_{\delta} g\left(\gamma_{+}, \delta ; \alpha\right) g\left(\gamma_{-}, \delta, \beta\right) \tag{4.12}
\end{equation*}
$$

where $g\left(\gamma_{+}, \delta ; \alpha\right)$ is a Littlewood-Richardson coefficient.
As reviewed in Appendix A the indices $i, j$ on a Brauer operator range over the values $\left\{1, \ldots, M_{A}^{\gamma ; N}\right\}$, and so the redundancy of the $i, j$ labels at $N=2$ means that $M_{A}^{\gamma ; N=2}$ is either 0 or 1 for all $\gamma, A$. A direct proof of this by using the finite $N$ constraints on the states of the Brauer representation in [15] would be interesting to obtain. At this point we will take a more pragmatic perspective, assume it is true, and will find that it leads to a consistent counting of states of the complex matrix model at $N=2$.

### 4.2 Counting of states at $N=2$ and Brauer basis labels

The ring of gauge invariant operators at $N=2$ is generated by five single trace operators (3.37). Hence the number of linearly independent multi-trace operators $Q_{m t}^{N=2}(m, n)$ for fixed $(m, n)$ is counted by the generating function

$$
\begin{equation*}
\frac{1}{(1-x)(1-y)\left(1-x^{2}\right)\left(1-y^{2}\right)(1-x y)}=\sum_{m, n} Q_{m t}^{N=2}(m, n) x^{m} y^{n} \tag{4.13}
\end{equation*}
$$

This is the Plethystic Exponential [44, 45] of the single trace generating function

$$
\begin{equation*}
\sum_{m, n} Q_{s t}^{N=2}(m, n) x^{m} y^{n}=1+x+y+x^{2}+y^{2}+x y \tag{4.14}
\end{equation*}
$$

derived from the independent single traces in the basis $\mathcal{B}$ (3.37).
Having found the $N=2$ counting of multi-traces, we can express it in terms of constraints on the large $N$ Brauer counting. The obvious constraint $c_{1}\left(\gamma_{+}\right)+c_{1}\left(\gamma_{-}\right) \leq 2$ is not sufficient. We have argued above that the multiplicities $M_{\alpha, \beta}^{\gamma ; N=2}$ are either 0 or 1 . We first set

$$
M_{\alpha, \beta}^{\gamma ; N=2}= \begin{cases}1 & \text { if } M_{\alpha, \beta}^{\gamma}>0  \tag{4.15}\\ 0 & \text { otherwise }\end{cases}
$$

where $M_{\alpha, \beta}^{\gamma}$ is given by (4.12). Having done this we also find it necessary to impose extra constraints on the labels $\alpha, \beta$ for agreement with (4.13).

The constraints on $\alpha, \beta$ are as follows. Denoting the length of the $p^{\text {th }}$ column of a Young diagram $R$ by $c_{p}(R)$, we constrain:

1. $c_{1}(\alpha)+c_{1}(\beta) \leq N+k$
2. $\left[c_{1}(\alpha)+c_{1}(\beta)\right]+\left[c_{2}(\alpha)+c_{2}(\beta)\right] \leq 2 N+k$
and in general for each $p=1,2, \ldots, \min (m, n)$, constrain

$$
\begin{equation*}
\sum_{r=1}^{p}\left(c_{r}(\alpha)+c_{r}(\beta)\right) \leq p N+k \tag{4.16}
\end{equation*}
$$

We have used SAGE and Mathematica to enumerate all possible Brauer basis operators subject to the constraint (4.16) and to compare with the Trace basis generating function. The two agree up to $(m, n)=(15,15)$ which is the practical limit for a desktop computer. This conjecture generalizes the 'Non-chiral Stringy Exclusion Principle' introduced in [12]. This counting of operators at $N=2$ implies a result for the reduction multiplicities $M_{A}^{\gamma, N=2}$, namely that

$$
M_{\alpha, \beta}^{\gamma ; N=2}= \begin{cases}1 & \text { if } M_{\alpha, \beta}^{\gamma}>0 \text { and (4.16) holds }  \tag{4.17}\\ 0 & \text { otherwise }\end{cases}
$$

We will re-state this result after simplifying the condition (4.16).
We can use the fact that the Brauer basis diagonalises the five Casimirs $\mathcal{H}_{A}$ to explicitly enumerate the independent Brauer operators at $N=2$. Fixing ( $m, n$ ) we pick a basis of multi-traces. Acting with the explicit form of the Casimirs (4.41), we can construct linear combinations which are eigenstates of the Casimirs. Because the eigenvalues of the five Casimirs determine the labels $\alpha, \beta, \gamma$ uniquely, we can also read off the labels of the allowed operators. We have carried out this procedure for selected values of $(m, n)$ up to $(m, n)=(4,3)$.

### 4.3 The Brauer basis labels at $N=2$ in terms of five integers

In Section 3.5 we described the states of the $N=2$ theory as generated by a finite set of traces. In this section we will obtain the description in terms of the Brauer basis for multi-traces. For general $N$, we give a review of the Brauer basis states in Section A. For ease of notation we denote $r_{i}=r_{i}(\alpha)$ and $\bar{r}_{i}=r_{i}(\beta)$.

We can choose different sets of five integers to parameterise the states, such as

$$
\begin{align*}
& r_{1}, r_{2}, \bar{r}_{1}, \bar{r}_{2}, r_{1}^{\gamma}  \tag{4.18}\\
& r_{1}^{\gamma}, r_{2}^{\gamma}, k, r_{1}, \bar{r}_{1}  \tag{4.19}\\
& r_{1}^{\gamma}, r_{2}^{\gamma}, k, r_{1}, \bar{r}_{2} . \tag{4.20}
\end{align*}
$$

We will show that each of the above sets of five integers determines a state uniquely, and we will give the constraints on the integers.

A state is determined uniquely at $N=2$ by $\alpha, \beta, \gamma$, containing the set of integers

$$
\begin{equation*}
\left\{r_{1}, r_{2} ; \bar{r}_{1}, \bar{r}_{2} ; k, r_{1}^{\gamma}, r_{2}^{\gamma}\right\} \tag{4.21}
\end{equation*}
$$

From the Brauer algebra representation theory briefly reviewed in Appendix A, we have the following relations :

$$
\begin{array}{ll}
\sum_{i} r_{i}=m, & \sum_{i} \bar{r}_{i}=n, \\
\sum_{i} r_{i}\left(\gamma_{+}\right)=m-k, & \sum_{i} r_{i}\left(\gamma_{-}\right)=n-k .
\end{array}
$$

Using the relationship between $r_{i}(\gamma), r_{i}\left(\gamma_{+}\right)$and $r_{i}\left(\gamma_{-}\right)$we have

$$
\begin{equation*}
\sum_{i} r_{i}(\gamma)=\sum_{i} r_{i}\left(\gamma_{+}\right)-\sum_{i} r_{i}\left(\gamma_{-}\right)=m-n \tag{4.24}
\end{equation*}
$$

which at $N=2$ reads

$$
\begin{equation*}
r_{1}^{\gamma}+r_{2}^{\gamma}=m-n \tag{4.25}
\end{equation*}
$$

Adding the two expressions in (4.23) we find that

$$
\begin{equation*}
\sum_{i}\left|r_{i}(\gamma)\right|=\sum_{i} r_{i}\left(\gamma_{+}\right)+\sum_{i} r_{i}\left(\gamma_{-}\right)=m+n-2 k \tag{4.26}
\end{equation*}
$$

which at $N=2$ gives

$$
\begin{equation*}
k=\frac{1}{2}\left(m+n-\left|r_{1}^{\gamma}\right|-\left|r_{2}^{\gamma}\right|\right) . \tag{4.27}
\end{equation*}
$$

We now show that each of (4.18)-(4.20) are enough to determine the state via (4.21):

1. Starting from the five integers in (4.18), we deduce $m, n$ from (4.22), $r_{2}^{\gamma}$ from (4.25) and $k$ from (4.27).
2. Starting from (4.19) we read off $r_{i}\left(\gamma_{+}\right)$and $r_{i}\left(\gamma_{-}\right)$by inspecting whether $r_{1}^{\gamma}$ and $r_{1}^{\gamma}$ are positive or negative. We then deduce $m$ and $n$ from (4.23) and $r_{2}$ and $\bar{r}_{2}$ from (4.22).
3. Starting from (4.20) we proceed as in point 2 above.

This shows that each of the three sets of five integers identified are sufficient to identify any state.

### 4.3.1 $N=2$ constraints in terms of five integers

Let us consider the case where $k$ is one of our five integers. We rewrite the $N=2$ constraint (4.16) as a lower bound on $k$ :

$$
\begin{equation*}
k \geq \sum_{r=1}^{p}\left(c_{r}(\alpha)+c_{r}(\beta)\right)-2 p \quad \text { for each } p=1, \ldots \min (m, n) \tag{4.28}
\end{equation*}
$$

Note that as $p$ increases the lower bound on $k$ gets stronger only when

$$
\begin{equation*}
c_{p}(\alpha)+c_{p}(\beta)>2 . \tag{4.29}
\end{equation*}
$$

Before presenting a general expression for the lower bound on $k$ we examine in detail the case

$$
\begin{equation*}
0<r_{2}<\bar{r}_{2}<r_{1}<\bar{r}_{1} \tag{4.30}
\end{equation*}
$$

We observe that

- For $1 \leq p \leq r_{2}$ we have $c_{p}(\alpha)+c_{p}(\beta)=4$
- For $r_{2}<p \leq \bar{r}_{2}$ we have $c_{p}(\alpha)+c_{p}(\beta)=3$
- For $p>\bar{r}_{2}$ we have $c_{p}(\alpha)+c_{p}(\beta) \leq 2$

The strongest lower bound on $k$ is therefore at $p=\bar{r}_{2}$ where we have

$$
\begin{align*}
k & \geq 4 r_{2}+3\left(\bar{r}_{2}-r_{2}\right)-2 \bar{r}_{2} \\
\Rightarrow k & \geq r_{2}+\bar{r}_{2} . \tag{4.31}
\end{align*}
$$

Proceeding similarly we find a general expression for the lower bound on $k$. For simplicity, wlog suppose $r_{2} \leq \bar{r}_{2}$. There are three cases to consider:

1. $r_{2} \leq r_{1} \leq \bar{r}_{2} \leq \bar{r}_{1} \quad \Rightarrow \quad k \geq r_{1}+r_{2}$
2. $r_{2} \leq \bar{r}_{2} \leq r_{1} \leq \bar{r}_{1} \quad \Rightarrow \quad k \geq r_{2}+\bar{r}_{2}$
3. $r_{2} \leq \bar{r}_{2} \leq \bar{r}_{1} \leq r_{1} \quad \Rightarrow \quad k \geq r_{2}+\bar{r}_{2}$.

Combining these we obtain the lower bound

$$
\begin{equation*}
k \geq \min \left(r_{2}, \bar{r}_{2}\right)+\min \left(\min \left(r_{1}, \bar{r}_{1}\right), \max \left(r_{2}, \bar{r}_{2}\right)\right) \tag{4.32}
\end{equation*}
$$

which is equivalent to (4.16). We can also express the constraint (4.32) in terms of the five integers in (4.18) by substituting for $k$ from (4.27) to find

$$
\begin{equation*}
\frac{1}{2}\left(m+n-\left|r_{1}^{\gamma}\right|-\left|m-n-r_{1}^{\gamma}\right|\right) \geq \min \left(r_{2}, \bar{r}_{2}\right)+\min \left(\min \left(r_{1}, \bar{r}_{1}\right), \max \left(r_{2}, \bar{r}_{2}\right)\right) \tag{4.33}
\end{equation*}
$$

We can now re-state the result (4.17) for the $N=2$ reduction multiplicities:

$$
M_{\alpha, \beta}^{\gamma ; N=2}= \begin{cases}1 & \text { if } M_{\alpha, \beta}^{\gamma}>0 \text { and (4.32) holds }  \tag{4.34}\\ 0 & \text { otherwise }\end{cases}
$$

### 4.4 The Casimirs as differential operators in $z_{i}, t_{0}$

In this section we express the Casimir operators / Hamiltonians from Section 4.1 as differential operators on $\mathcal{M}_{2}$.

Below are calculated expressions in the coordinates $z_{i}, t_{0}$ for the Hamiltonians defined in (4.7). For convenience define

$$
\begin{array}{lll}
L_{1}=z_{1} \frac{\partial}{\partial z_{1}} & \bar{L}_{1}=\bar{z}_{1} \frac{\partial}{\partial \bar{z}_{1}} & \\
L_{2}=z_{2} \frac{\partial}{\partial z_{2}} & \bar{L}_{2}=\bar{z}_{2} \frac{\partial}{\partial \bar{z}_{2}} & L_{t}=\frac{t_{0}}{2} \frac{\partial}{\partial t_{0}} \tag{4.36}
\end{array}
$$

and recall the notation $z_{c}=z_{1}+z_{2}, z=z_{1}-z_{2}$.
Recalling the definition $G_{L}=G_{2}+G_{3}$ from above (4.7), we find the following expressions:

$$
\begin{gather*}
H_{1}=\operatorname{tr} G_{2}=L_{1}+L_{2}+L_{t}  \tag{4.37}\\
\bar{H}_{1}=-\operatorname{tr} G_{3}=\bar{L}_{1}+\bar{L}_{2}+\bar{L}_{t}  \tag{4.38}\\
H_{2}=\operatorname{tr} G_{2}^{2}= \\
+\frac{2}{z}\left(z_{1}^{2} L_{1}-L_{2}^{2}+\left(1-\frac{2 z_{1} z_{2} \bar{z}}{z t_{0}^{2}}\right) L_{t}+\frac{z_{c}}{z}\left(L_{1}-L_{2}\right)+L_{t}\right.  \tag{4.39}\\
H_{3}=\operatorname{tr} G_{3}^{2}=  \tag{4.40}\\
\begin{aligned}
& \operatorname{tr}\left(G_{2}^{2}\right) \\
& H_{L}=\operatorname{tr} G_{L}^{2}=\left(L_{1}-\bar{L}_{1}\right)^{2}+\left(L_{2}-\bar{L}_{2}\right)^{2}+\frac{z_{c}}{z}\left(L_{1}-L_{2}\right)+\frac{\bar{z}_{c}}{\bar{z}}\left(\bar{L}_{1}-\bar{L}_{2}\right) \\
&- \frac{2}{|z|^{2}}\left\{t_{0}^{2}\left(L_{1}-L_{2}\right)\left(\bar{L}_{1}-\bar{L}_{2}\right)+\frac{1}{t_{0}^{2}}\left(z_{1} \bar{z}_{1}-z_{2} \bar{z}_{2}\right)^{2} L_{t}^{2}\right. \\
&\left.-\left(z_{1} \bar{z}_{1}-z_{2} \bar{z}_{2}\right)\left[\left(L_{1}-L_{2}\right)+\left(\bar{L}_{1}-\bar{L}_{2}\right)\right] L_{t}-\left(z_{1} \bar{z}_{1}+z_{2} \bar{z}_{2}\right) L_{t}\right\}
\end{aligned}
\end{gather*}
$$

Some useful formulae in doing these calculations are now given. Recall from (3.8) the definition $V=L^{\dagger} d L$ and the expression

$$
\begin{equation*}
d Z=L(d \tilde{T}+[V, \tilde{T}]) L^{\dagger} \tag{4.42}
\end{equation*}
$$

Defining

$$
\begin{equation*}
d \tilde{X}=d \tilde{T}+[V, \tilde{T}] \quad \text { and } \quad\left(\tilde{G}_{2}\right)_{j}^{i}=\tilde{T}_{p}^{i}\left(\frac{\partial}{\partial \tilde{X}}\right)_{j}^{p} \tag{4.43}
\end{equation*}
$$

one may derive

$$
\begin{align*}
d Z_{j}^{i} & =L_{p}^{i} d \tilde{X}_{q}^{p} L_{j}^{\dagger q} \\
\left(\frac{\partial}{\partial Z}\right)_{j}^{i} & =L_{p}^{i} L_{j}^{\dagger q}\left(\frac{\partial}{\partial \tilde{X}}\right)_{q}^{p} \\
\left(G_{2}\right)_{j}^{i} & =L_{p}^{i} L_{j}^{q}\left(\tilde{G}_{2}\right)^{p}{ }_{q} . \tag{4.44}
\end{align*}
$$

The computation of $\left(\tilde{G}_{2}\right)^{p}$ shows that it contains angular derivatives. When we calculate

$$
\begin{equation*}
\operatorname{tr} G_{2}^{2}=L_{p}^{i} L_{j}^{q}\left(\tilde{G}_{2}\right)^{p}{ }_{q} L_{r}^{i} L_{j}^{s}\left(\tilde{G}_{2}\right)^{r}{ }_{s} \tag{4.45}
\end{equation*}
$$

it is important not to neglect the terms obtained from the action of these angular derivatives from $\left(\tilde{G}_{2}\right)^{p}{ }_{q}$ on $L_{r}^{i} L^{s}{ }_{j}$.

### 4.4.1 The Casimirs as operators on polynomial rings

We observed in equation (3.43) that the multi-trace operators built from $Z, Z^{\dagger}$ form a polynomial ring whose generators we may take to be

$$
\begin{equation*}
z_{c}, \quad \bar{z}_{c}, \quad \mathcal{Z}=z^{2}, \quad \overline{\mathcal{Z}}=\bar{z}^{2}, \quad T_{0}=t_{0}^{2}+\frac{z \bar{z}}{2} \tag{4.46}
\end{equation*}
$$

The above differential operators $H_{2}, H_{3}, H_{L}$ map polynomials in these variables to polynomials. Changing variables to these generators makes this manifest:

$$
\begin{align*}
H_{2}= & 2 L\left(L+\frac{1}{2}\right)+\frac{1}{2} L_{c}\left(L_{c}+3\right)+L_{0}\left(L_{0}+1\right)+\frac{2 z_{c}^{2}}{\mathcal{Z}} L+\frac{z_{c}^{2}}{\mathcal{Z}}(2 L-1) L \\
& +\frac{\mathcal{Z}}{2 z_{c}^{2}} L_{c}\left(L_{c}-1\right)+\frac{\overline{\mathcal{Z}}}{8 T_{0}^{2}}\left(z_{c}^{2}-\mathcal{Z}\right) L_{0}\left(L_{0}-1\right)+2\left(1+\frac{z_{c}^{2}}{\mathcal{Z}}\right) L L_{0}+2 L_{0} L_{c}+4 L L_{c} \tag{4.47}
\end{align*}
$$

where $L=\mathcal{Z} \frac{\partial}{\partial \mathcal{Z}}, L_{0}=T_{0} \frac{\partial}{\partial T_{0}}$ and $L_{c}=z_{c} \frac{\partial}{\partial z_{c}} . H_{3}$ is obtained by complex conjugation and the same exercise can also be done for $H_{L}$ to illustrate that they are operators that map polynomials to polynomials.

### 4.5 Eigenvalues of the Casimir operators

As reviewed in Section 2, a Young diagram $R$ with non-negative row lengths $r_{i}$ labels energies $\mathcal{E}_{i}$ of $N$ fermions in a one-dimensional harmonic oscillator potential, given by

$$
\begin{equation*}
\mathcal{E}_{i}=r_{i}+(N-i) \tag{4.48}
\end{equation*}
$$

and a Young diagram ( $N$-staircase) $R$ with arbitrary integer $r_{i}$ labels momenta $p_{i}$ of $N$ free fermions on a circle given in terms of the Fermi energy $n_{F}=\frac{N-1}{2}$ by

$$
\begin{equation*}
p_{i}=r_{i}+\left(n_{F}+1-i\right) . \tag{4.49}
\end{equation*}
$$

In this section we review the fact that the values of

- the $N$ independent $U(N)$ Casimirs $C_{i}(R)$ of the representation $R$
- the $N$ row lengths $r_{i}$, and
- the $N$ corresponding fermion momenta $p_{i}$ are equivalent data.

The same remark holds for non-negative $r_{i}$ with $p_{i}$ replaced by $\mathcal{E}_{i}$.
In Section 4.1 we introduced differential operators studied in [25] which when acting on a Brauer basis function $\mathcal{O}_{\alpha \beta}^{\gamma}\left(Z, Z^{\dagger}\right)$ measure the quadratic Casimir of the Young diagrams $\alpha, \beta, \gamma$. Given a $U(N)$ Young diagram $R$, its linear and quadratic Casimirs are

$$
\begin{align*}
C_{1}(R) & =\sum_{i} r_{i}=n  \tag{4.50}\\
C_{2}(R) & =n N+\sum_{i} r_{i}\left(r_{i}-2 i+1\right) \tag{4.51}
\end{align*}
$$

Using the definition of $p_{i}$ (4.49) we can write $C_{2}$ as

$$
\begin{equation*}
C_{2}(R)=\sum_{i=1}^{N} p_{i}^{2}-\frac{N}{12}\left(N^{2}-1\right) \tag{4.52}
\end{equation*}
$$

which agrees with (2.34). Using the definition of $\mathcal{E}_{i}$ (4.48) we can also write $C_{2}$ as

$$
\begin{equation*}
C_{2}(R)=\sum_{i=1}^{N} \mathcal{E}_{i}^{2}-(N-1) n-\frac{N}{6}(N-1)(2 N-1) . \tag{4.53}
\end{equation*}
$$

For general $N$, knowledge of the values of the $N$ independent Casimir invariants $C_{i}$ determine the values of the power sum symmetric polynomials

$$
\begin{equation*}
\mathcal{P}_{a}=p_{1}^{a}+p_{2}^{a}+\ldots . .+p_{N}^{a} \tag{4.54}
\end{equation*}
$$

which in turn for $a=1, \ldots, N$ enables us to solve for $p_{i}$ or respectively $\mathcal{E}_{i}$ (see e.g. [46]).
We now demonstrate this in the $N=2$ theory. The free fermions on a circle have ground state with energy $p_{1}=\frac{1}{2}, p_{2}=-\frac{1}{2}$ and in general we have

$$
\begin{equation*}
p_{1}=r_{1}+\frac{1}{2}, \quad p_{2}=r_{2}-\frac{1}{2} . \tag{4.55}
\end{equation*}
$$

Setting $N=2$ in (4.51) gives

$$
\begin{equation*}
C_{2}=r_{1}\left(r_{1}+1\right)+r_{2}\left(r_{2}-1\right) \tag{4.56}
\end{equation*}
$$

and so we may express $C_{1}$ and $C_{2}$ in terms of $p_{i}$ as

$$
\begin{align*}
C_{1} & =p_{1}+p_{2} \\
C_{2} & =p_{1}^{2}+p_{2}^{2}-\frac{1}{2} \tag{4.57}
\end{align*}
$$

The resulting quadratic equations for $p_{i}$ in terms of $C_{1}$ and $C_{2}$ have solution

$$
\begin{align*}
& p_{1}=\frac{C_{1}}{2}+\sqrt{\frac{C_{2}}{2}-\frac{C_{1}^{2}}{4}+\frac{1}{4}} \\
& p_{2}=C_{1}-p_{1} \tag{4.58}
\end{align*}
$$

### 4.6 The $k=0$ sector

In the $k=0$ sector $\gamma=(0, \alpha, \beta)$ so operators are labelled simply by $\alpha$ and $\beta$ which are representations of $S_{m}$ and $S_{n}$ respectively. To connect with the notation of the unitary matrix model, we write $\alpha=R$ and $\beta=S$. If $S=\emptyset$, then the $k=0$ operator is the holomorphic Schur polynomial corresponding to the representation $R$ :

$$
\begin{equation*}
\mathcal{O}_{R, \emptyset}^{k=0}\left(Z, Z^{\dagger}\right)=\chi_{R}(Z) \tag{4.59}
\end{equation*}
$$

If $R=\emptyset$, then the $k=0$ operator is the anti-holomorphic Schur polynomial corresponding to the representation $\bar{S}$ :

$$
\begin{equation*}
\mathcal{O}_{\emptyset, \bar{S}}^{k=0}\left(Z, Z^{\dagger}\right)=\chi_{S}\left(Z^{\dagger}\right) \tag{4.60}
\end{equation*}
$$

and if both $\alpha$ and $\beta$ are nontrivial, the leading order term in the expansion of $\mathcal{O}^{k=0}$ begins with the product of the holomorphic and antiholomorphic Schur polynomials:

$$
\begin{equation*}
\mathcal{O}_{R, \bar{S}}^{k=0}\left(Z, Z^{\dagger}\right)=\chi_{R}(Z) \chi_{S}\left(Z^{\dagger}\right)+\cdots, \tag{4.61}
\end{equation*}
$$

where the dots denote terms with at least one $Z Z^{\dagger}$ inside a trace as discussed at the start of Section 4.

There is an isomorphism between the $k=0$ sector and the states of the Unitary matrix model [12]:

$$
\begin{equation*}
\mathcal{O}_{R \bar{S}}^{k=0}\left(Z, Z^{\dagger}\right) \longleftrightarrow \chi_{R \bar{S}}(U) \tag{4.62}
\end{equation*}
$$

which is obtained by replacing $Z$ with a unitary matrix:

$$
\begin{equation*}
\mathcal{O}_{R \bar{S}}^{k=0}\left(U, U^{\dagger}\right)=d_{R} d_{S} \chi_{R \bar{S}}(U) \tag{4.63}
\end{equation*}
$$

The two point functions of both sets of operators are diagonal; up to a choice of normalisation,

$$
\begin{equation*}
\left\langle\mathcal{O}_{R \bar{S}}^{\dagger k=0}\left(Z, Z^{\dagger}\right) \mid \mathcal{O}_{R^{\prime} \bar{S}^{\prime}}^{k=0}\left(Z, Z^{\dagger}\right)\right\rangle=\left\langle\chi_{R \bar{S}}^{\dagger}(U) \mid \chi_{R^{\prime} \bar{S}^{\prime}}(U)\right\rangle=\delta_{R R^{\prime}} \delta_{\bar{S} \bar{S}^{\prime}} \tag{4.64}
\end{equation*}
$$

and the reader familiar with the 'coupled characters' studied in two-dimensional YangMills will notice that the structure of (4.61) is of the same form as the coupled character $\chi_{R \bar{S}}$. The $k=0$ states are thus isomorphic to the states of $N$ free fermions on a circle via the map given in the same section.

At $N=2$, the label $\gamma_{c}$ as defined in (A.4) may have at most two rows, $r_{1}^{\gamma}, r_{2}^{\gamma}$ and so the integers $\left(k=0, r_{1}^{\gamma}, r_{2}^{\gamma}\right)$ are enough to specify an operator. The list of all $N=2$ operators for given $(m, n)$ in Appendix B shows that:

- If $r_{1}^{\gamma}>0, r_{2}^{\gamma} \geq 0$, then $\beta=\emptyset$ and we have a holomorphic Schur polynomial.
- If $r_{1}^{\gamma} \leq 0, r_{2}^{\gamma}<0$ then $\alpha=\emptyset$ and we have an antiholomorphic Schur polynomial.
- If $r_{1}^{\gamma}>0, r_{2}^{\gamma}<0$ then the operator is of the form (4.61). At $N=2$ there is a unique such operator.

Since row lengths and fermion momenta are equivalent data in specifying a state, the above constraints may be rewritten in terms of fermion momenta $p_{i}^{\gamma}$. In the next section, we will see how the momenta of these fermions can be expressed in terms of differential operators in $z_{i}, t_{0}$.

### 4.7 Free particle momenta as functions of differential operators

As noted in (4.9), when applied to an $N=2$ Brauer basis operator $\mathcal{O}_{\alpha, \beta}^{\gamma}$, the differential operators

$$
\mathcal{H}_{A}=\left\{\begin{array}{lllll}
H_{1}, & \bar{H}_{1}, & H_{2}, & \bar{H}_{2}, & H_{L} \tag{4.65}
\end{array}\right\}
$$

measure the values of the Casimirs

$$
\begin{equation*}
\mathcal{C}_{A}=\left\{C_{1}(\alpha), \quad C_{1}(\beta), \quad C_{2}(\alpha), \quad C_{2}(\beta), \quad C_{2}(\gamma)\right\} \tag{4.66}
\end{equation*}
$$

respectively. We also have the fact that $C_{1}(\gamma)$ is measured by $H_{1}-\bar{H}_{1}$. We define fermion momentum operators

$$
\hat{p}_{A}=\left\{\begin{array}{llllll}
\hat{p}_{1}, & \hat{p}_{2}, & \hat{\bar{p}}_{1}, & \hat{\bar{p}}_{2}, & \hat{p}_{1}^{\gamma}, & \hat{p}_{2}^{\gamma} \tag{4.67}
\end{array}\right\}
$$

whose eigenvalues are $p_{1}, p_{2}, \bar{p}_{1}, \bar{p}_{2}, p_{1}^{\gamma}, p_{2}^{\gamma}$ respectively. We now repeatedly apply (4.58) to each of $\alpha, \beta, \gamma$ in turn which enables us to derive expressions for these operators in terms of the basic gauge invariant operators $\mathcal{H}_{A}$.

Applying (4.58) to the label $\alpha$ and promoting to an operator equation we obtain

$$
\begin{align*}
& \hat{p}_{1}=\frac{H_{1}}{2}+\sqrt{\frac{H_{2}}{2}-\frac{H_{1}^{2}}{4}+\frac{1}{4}} \\
& \hat{p}_{2}=H_{1}-\hat{p}_{1} \tag{4.68}
\end{align*}
$$

Applying (4.58) to the label $\beta$ we obtain analogous expressions for $\hat{\bar{p}}_{1}, \hat{\bar{p}}_{2}$ in terms of $\bar{H}_{1}, \bar{H}_{2}$.

Applying (4.58) to the label $\gamma$, promoting to an operator equation and defining $\hat{d}=$ $H_{1}-\bar{H}_{1}$ we obtain

$$
\begin{align*}
& \hat{p}_{1}^{\gamma}=\frac{\hat{d}}{2}+\sqrt{\frac{H_{L}}{2}-\frac{\hat{d}^{2}}{4}+\frac{1}{4}} \\
& \hat{p}_{2}^{\gamma}=\hat{d}-\hat{p}_{1}^{\gamma} \tag{4.69}
\end{align*}
$$

As noted in Section 4.6, in the $k=0$ sector a state is specified simply by the values of the row lengths $r_{1}^{\gamma}, r_{2}^{\gamma}$, or equivalently by the values of the fermion momenta $p_{1}^{\gamma}, p_{2}^{\gamma}$ and so we now identify $\hat{p}_{1}^{\gamma}, \hat{p}_{2}^{\gamma}$ as formal expressions for the momenta of the $k=0$ fermions on a circle. We shall extend this result to arbitrary $N$ in the next section.

Comparing to the explicit expressions for $\mathcal{H}_{A}$ obtained in Section 4.4, we see that these fermion momenta are functions of differential operators in both the eigenvalues $z_{i}$ and the off-diagonal element $t_{0}$. In hermitian matrix models and unitary matrix models, the emergent fermions are the eigenvalues of the relevant matrix. Here, however, the $k=0$ emergent fermions have no such direct connection to eigenvalues of $Z$.

## 5 Free particle structures and counting on $\mathcal{M}_{N}$

In this section we extend aspects of our $N=2$ discussion of the algebra of gauge invariant functions and the rings of scale invariant and gauge invariant differential operators to the case of general $N$.

Following our considerations for the $k=0$ sector from Section 4, we show that the momenta of the free fermions are determined in terms of differential operators on $\mathcal{M}_{N}$. We show that the correspondence between states of the $k=0$ sector and the unitary matrix model (and hence free fermion wavefunctions) extends beyond two-point functions to a class of three-point functions.

At $N=2$ we have shown that the number of generators of the ring of gauge invariant functions on $\mathcal{M}_{N}$ will be $N^{2}+1$. At $N=3$, we will show that there is an interesting twist but that the above statement remains true in a refined form.

Finally we study the $m=n=k$ sector. This is the maximum possible value of $k$, in contrast to our studies of $k=0$ which is the minimum possible value. This sector consists of traces and multi-traces of $Z^{\dagger} Z$ and we show that it may be mapped to $N$ free fermions in a one-dimensional harmonic oscillator potential. This is a second, distinct appearance of free particles in complex matrix models.

### 5.1 The $k=0$ sector revisited

We first observe that our construction of free fermion momenta as functions of differential operators in $z_{i}, t_{i j}$ may be extended to general $N$ in a slightly weaker form as follows.

The construction in the previous section may be carried out for general $N$ by identifying differential operators which measure higher order Casimirs. These will be traces of higher powers of the $G_{i}$. We have not found closed form expressions analogous to (4.57) for higher $N$ since this would require us to solve arbitrary order polynomials. However, since the $p_{i}$ are integer or half-integer, they may always be determined in terms of the eigenvalues of the Hamiltonians [46], and hence implicitly in terms of differential operators in $z_{i}, t_{i j}$. We have thus identified an implicit map from $k=0$ operators to fermions on a circle for all finite $N$.

We next conjecture that the $k=0$ sector may be descibed as the kernel of the differential operator $\operatorname{tr} G_{2} G_{3}$. Let us recall from Section 4.1 that the differential operator

$$
\begin{equation*}
\operatorname{tr}\left(G_{2}+G_{3}\right)^{2}=\operatorname{tr}\left(G_{2}^{2}+2 G_{2} G_{3}+G_{3}^{2}\right) \tag{5.1}
\end{equation*}
$$

measures $C_{2}(\gamma)$, and so $\operatorname{tr} G_{2} G_{3}$ measures

$$
\begin{equation*}
\frac{1}{2}\left(C_{2}(\gamma)-C_{2}(\alpha)-C_{2}(\beta)\right) \tag{5.2}
\end{equation*}
$$

Since for a $k=0$ operator $\gamma=(0, \alpha, \beta)$, we have that

$$
\begin{equation*}
C_{2}(\gamma)=C_{2}(\alpha)+C_{2}(\beta) \tag{5.3}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left(\operatorname{tr} G_{2} G_{3}\right) O^{k=0}\left(Z, Z^{\dagger}\right)=0 \tag{5.4}
\end{equation*}
$$

As a brief aside, note that the action of the Brauer contraction element $C_{1 \overline{1}}$ on $Z_{j}^{i} Z^{\dagger k}$ is as follows [12]:

$$
\begin{equation*}
C_{1 \overline{1}}\left(Z_{j}^{i} Z_{l}^{k}\right)=\delta_{l}^{i}\left(Z^{\dagger} Z\right)^{k}{ }_{j} . \tag{5.5}
\end{equation*}
$$

Since

$$
\begin{equation*}
\left(G_{2}\right)^{p}{ }_{q} Z_{j}^{i}=\delta_{q}^{i} Z_{j}^{p} \quad \text { and } \quad-\left(G_{3}\right)_{p}^{q} Z_{l}^{\dagger k}=\delta_{l}^{q} Z^{\dagger k} \tag{5.6}
\end{equation*}
$$

we have

$$
\begin{equation*}
-\operatorname{tr} G_{2} G_{3}\left(Z_{j}^{i} Z_{l}^{\dagger k}\right)=\delta_{l}^{i}\left(Z^{\dagger} Z\right)_{j}^{k} \tag{5.7}
\end{equation*}
$$

and since $\operatorname{tr} G_{2} G_{3}$ acts via the Leibniz rule, the action of $-\operatorname{tr} G_{2} G_{3}$ on

$$
\begin{equation*}
\mathcal{O}=Z_{j_{1}}^{i_{1}} Z_{j_{2}}^{i_{2}} \cdots Z_{j_{m}}^{i_{m}} Z_{q_{1}}^{\dagger p_{1}} Z_{q_{2}}^{\dagger p_{2}} \cdots Z_{q_{n}}^{\dagger p_{n}} \tag{5.8}
\end{equation*}
$$

is that of the sum over all individual contractions

$$
\begin{equation*}
C=\sum_{r=1}^{m} \sum_{s=1}^{n} C_{r \bar{s}} . \tag{5.9}
\end{equation*}
$$

Similarly the action of the laplacian

$$
\begin{equation*}
\square=\operatorname{tr}\left(\frac{\partial}{\partial Z} \frac{\partial}{\partial Z^{\dagger}}\right) \tag{5.10}
\end{equation*}
$$

on $Z_{j}^{i} Z^{\dagger}{ }_{l}$ is given by

$$
\begin{equation*}
\square\left(Z_{j}^{i} Z^{\dagger k}{ }_{l}\right)=\delta_{l}^{i} \delta^{k}{ }_{j} . \tag{5.11}
\end{equation*}
$$

which is a Wick contraction using the two point function (1.1), and as before extends via the Leibniz rule. It was noted in [12] that the $k=0$ operators have no self Wick contractions and so we have

$$
\begin{equation*}
\square O^{k=0}\left(Z, Z^{\dagger}\right)=0 \tag{5.12}
\end{equation*}
$$

a result we shall use later in Section 6. For now we simply note that it is possible to construct simple examples which show that the $k=0$ operators do not comprise the full kernel of $\square$.

We expect however that the converse of (5.4) is true for any $N$

$$
\begin{equation*}
\operatorname{tr}\left(G_{2} G_{3}\right) \mathcal{O}=0 \quad \Rightarrow \quad \mathcal{O}=\mathcal{O}^{k=0} \tag{5.13}
\end{equation*}
$$

meaning that the kernel of $\operatorname{tr}\left(G_{2} G_{3}\right)$ is exactly the $k=0$ sector. As a differential operator, $\operatorname{tr}\left(G_{2} G_{3}\right)$ can be viewed as a modification of the laplacian which is invariant under scalings of $Z$ and $Z^{\dagger}$.

It is instructive to try and construct a counterexample to (5.13). From (5.2) we know that $\operatorname{tr}\left(G_{2} G_{3}\right) \mathcal{O}=0$ is equivalent to

$$
\begin{equation*}
C_{2}(\gamma)=C_{2}(\alpha)+C_{2}(\beta) \tag{5.14}
\end{equation*}
$$

The operator with labels

$$
\begin{equation*}
\alpha=[1,1], \quad \beta=[1,1], \quad \gamma=\left(k=1, \gamma_{+}=[1], \gamma_{-}=[1]\right) \tag{5.15}
\end{equation*}
$$

has Casimirs

$$
\begin{equation*}
C_{2}(\alpha)=2, \quad C_{2}(\beta)=2, \quad C_{2}(\gamma)=4 \tag{5.16}
\end{equation*}
$$

however this operator in fact does not exist since it fails our $N=2$ constraint (4.16) in the form:

$$
\begin{equation*}
c_{1}(\alpha)+c_{1}(\beta) \leq N+k . \tag{5.17}
\end{equation*}
$$

This example supports (5.13) and shows that it is sensitive to finite $N$ constraints of the Brauer basis.

### 5.2 Three-point functions of $k=0$ operators

In Section 4.6 we reviewed the map between $k=0$ operators and Unitary matrix model operators, and the result that the two point functions on both sides of the correspondence agree. Here we show that the same is true for a class of three-point functions. The correlators we consider are a subclass of the correlators

$$
\begin{equation*}
\left\langle\mathcal{O}_{A_{1}}\left(Z, Z^{\dagger}\right) \mathcal{O}_{A_{2}}\left(Z, Z^{\dagger}\right) \mathcal{O}_{A_{3}}^{\dagger}\left(Z, Z^{\dagger}\right)\right\rangle \tag{5.18}
\end{equation*}
$$

where $A_{1}=R_{1} \bar{S}_{1}$ is a short notation for the labels of the operators in the $k=0$ sector, and similarly for $A_{2}$ and $A_{3}$. In terms of Brauer projectors the operators are defined by

$$
\begin{equation*}
\mathcal{O}_{R \bar{S}}\left(Z, Z^{\dagger}\right)=\operatorname{tr}_{m, n}\left(P_{R \bar{S}} Z \otimes Z^{\dagger}\right) \tag{5.19}
\end{equation*}
$$

where $P_{R \bar{S}}$ is defined in Appendix A. The subclass we study is that in which $m_{1}+m_{2}=m_{3}$ and $n_{1}+n_{2}=n_{3}$. Performing the Wick contractions, we get

$$
\begin{equation*}
\left\langle\mathcal{O}_{A_{1}}\left(Z, Z^{\dagger}\right) \mathcal{O}_{A_{2}}\left(Z, Z^{\dagger}\right) \mathcal{O}_{A_{3}}^{\dagger}\left(Z, Z^{\dagger}\right)\right\rangle=m_{3}!n_{3}!\operatorname{tr}_{m_{3}, n_{3}}\left(\left(P_{A_{1}} \circ P_{A_{2}}\right) P_{A}\right) \tag{5.20}
\end{equation*}
$$

The calculation is very similar to those in [2, 47]. It is convenient to express projectors as an integral over the $U(N)$ group as

$$
\begin{equation*}
P_{\gamma}=\operatorname{Dim} \gamma \int d U \chi_{\gamma}\left(U^{\dagger}\right) U \tag{5.21}
\end{equation*}
$$

where $\operatorname{Dim} \gamma$ is the dimension of the $U(N)$ representation $\gamma$; this follows from Schur-Weyl duality. We can therefore calculate (5.20) via

$$
\begin{align*}
\operatorname{tr}_{m, n}\left(\left(P_{A_{1}} \circ P_{A_{2}}\right) P_{A_{3}}\right) & =\operatorname{DimA}_{3} \int d U_{3} \chi_{A_{3}}\left(U_{3}^{\dagger}\right) \operatorname{tr}_{m_{3}, n_{3}}\left(\left(P_{A_{1}} \circ P_{A_{2}}\right) U_{3}\right) \\
& =\operatorname{DimA}_{3} d_{A_{1}} d_{A_{2}} \int d U_{3} \chi_{A_{1}}\left(U_{3}\right) \chi_{A_{2}}\left(U_{3}\right) \chi_{A_{3}}\left(U_{3}^{\dagger}\right) \\
& =\operatorname{DimA}_{3} d_{A_{1}} d_{A_{2}} g\left(A_{1}, A_{2} ; A_{3}\right) \\
& =\operatorname{DimA}_{3} d_{R_{1}} d_{R_{2}} d_{S_{1}} d_{S_{2}} g\left(R_{1}, R_{2} ; R_{3}\right) g\left(S_{1}, S_{2} ; S_{3}\right) \tag{5.22}
\end{align*}
$$

where $d_{A} \equiv d_{R \bar{S}}=d_{R} d_{S}$ and the following has been used to get the second equality:

$$
\begin{equation*}
\operatorname{tr}_{m, n}\left(\left(P_{A_{1}} \circ P_{A_{2}}\right) U_{3}\right)=d_{A_{1}} d_{A_{2}} \chi_{A_{1}}\left(U_{3}\right) \chi_{A_{2}}\left(U_{3}\right) \tag{5.23}
\end{equation*}
$$

and

$$
\begin{equation*}
g\left(A_{1}, A_{2} ; A_{3}\right)=\int d U_{3} \chi_{A_{1}}\left(U_{3}\right) \chi_{A_{2}}\left(U_{3}\right) \chi_{A_{3}}\left(U_{3}^{\dagger}\right) \tag{5.24}
\end{equation*}
$$

is the Littlewood-Richardson coefficient which counts the number of $A_{3}$ in the tensor product $A_{1} \otimes A_{2}$. This derivation shows that integration over $\mathcal{M}_{N}$ can be done using Brauer algebras.

### 5.3 Finite $N$ counting of single traces and multi-traces

Since $Z$ and $Z^{\dagger}$ do not in general commute, enumerating multi-trace operators in one complex matrix is an equivalent problem to enumerating multi-trace operators in two hermitian matrix models.

Therefore an important check on the Brauer basis is that the counting of Brauer basis operators agrees with that of the counting of operators in two-matrix models [48, 49, 50, 51, 52].

At finite $N$, we denote the counting of multi-trace operators built from two matrices with $m$ of one type and $n$ of the other as $Q_{m t}^{N}(m, n)$. The following two expressions determine $Q_{m t}^{N}(m, n)$ in terms of group theoretical quantities [53, 48, 49]:

$$
\begin{equation*}
Q_{m t}^{N}(m, n)=\sum_{\substack{R \vdash m+n, \Lambda \vdash m+n \\ c_{1}(R) \leq N, c_{1}(\Lambda) \leq 2}} C(R, R, \Lambda) g([m],[n] ; \Lambda) \tag{5.25}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{m t}^{N}(m, n)=\sum_{\substack{R \vdash m+n \\ c_{1}(R) \leq N}} \sum_{\substack{R_{1} \vdash m \\ c_{1}\left(R_{1}\right) \leq N}} \sum_{\substack{R_{2} \vdash n \\ c_{1}\left(R_{2}\right) \leq N}} g\left(R_{1}, R_{2} ; R\right)^{2} . \tag{5.26}
\end{equation*}
$$

Here $C(R, R ; \Lambda)$ is the multiplicity of the irreducible representation $\Lambda$ of $S_{m+n}$ appearing in the tensor product of irreducible representations $R \otimes R$ of $S_{m+n}$, and $g(\cdot, \cdot ; \cdot)$ is a Littlewood-Richardson coefficient.

Defining the finite $N$ multi-trace generating function

$$
\begin{equation*}
Z_{m t}^{N}(x, y)=\sum_{m, n} Q_{m t}^{N}(m, n) x^{m} y^{n} \tag{5.27}
\end{equation*}
$$

the relation between the counting of single-traces and multi-traces is given by the Plethystic Logarithm [44, 45]:

$$
\begin{equation*}
Z_{s t}^{N}(x, y)=\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \left(Z_{m t}^{N}\left(x^{k}, y^{k}\right)\right) \tag{5.28}
\end{equation*}
$$

At $N=2$ this leads to the five single traces identified in equation (3.37).
At $N=3$ the Plethystic Logarithm gives the single-trace generating function

$$
\begin{align*}
\sum_{m, n} Q_{s t}^{N=3}(m, n) x^{m} y^{n}= & 1+x+y+x^{2}+y^{2}+x y+x^{3}+y^{3} \\
& +x^{2} y+x y^{2}+x^{2} y^{2}+x^{3} y^{3}-x^{6} y^{6} \tag{5.29}
\end{align*}
$$

The interpretation of this generating function is that there are 11 independent single trace operators along with one syzygy (algebraic relation) which occurs at order $(m, n)=(6,6)$. It would be interesting to find this relation explicitly.

We expect that the reduction multiplicities for $B_{N}(m, n)$ to $S_{m} \times S_{n}$, which we have denoted by $M_{\alpha, \beta}^{\gamma ; N}$, satisfy the counting

$$
\begin{equation*}
\sum_{\gamma, \alpha, \beta}\left(M_{\alpha, \beta}^{\gamma ; N}\right)^{2}=Q_{m t}^{N}(m, n) \tag{5.30}
\end{equation*}
$$

We show in the next section that this is true for $N>m+n$ and it is also consistent with our calculations at $N=2$ in Section 4.2. It would be interesting to prove it for general $N$, with or without using the connections to matrix models.

### 5.4 Large $N$ Brauer basis counting

For $N$ sufficiently large, i.e $N>m+n$, we denote the counting of multi-trace operators by $Q_{m t}(m, n)$. The formulae in (5.25), (5.26) give rise to this quantity $Q_{m t}(m, n)$ when $N>m+n$. By Pólya counting $Q_{m t}(m, n)$ is also given by (see [48, 49] \& refs within)

$$
\begin{equation*}
\prod_{r=1}^{\infty} \frac{1}{1-\left(x^{r}+y^{r}\right)}=\sum_{m, n=0}^{\infty} Q_{m t}(m, n) x^{m} y^{n} \tag{5.31}
\end{equation*}
$$

In equation (177) of [49] the following expression was derived:

$$
\begin{equation*}
Q_{m t}(m, n)=\sum_{c_{l}(1): \sum_{l} c_{l}(1) l=m} \sum_{c_{l}(2): \sum_{l} c_{l}(2) l=n} \prod_{l} \frac{\left(c_{l}(1)+c_{l}(2)\right)!}{c_{l}(1)!c_{l}(2)!} \tag{5.32}
\end{equation*}
$$

The counting of Brauer basis operators is denoted $N_{s b}(m, n)$ and was shown in [12] to be given by

$$
\begin{equation*}
N_{s b}(m, n)=\sum_{\gamma, A}\left(M_{A}^{\gamma}\right)^{2} \tag{5.33}
\end{equation*}
$$

In [12] it was argued that this formula correctly counts multi-traces at large $N$, namely that

$$
\begin{equation*}
N_{s b}(m, n)=Q_{m t}(m, n) \tag{5.34}
\end{equation*}
$$

In Appendices C and D we give two proofs of this fact, firstly by direct comparison to (5.32) and secondly by enumerating invariants in the reduction $G L(N) \times G L(N) \rightarrow G L(N)$.

### 5.5 The $m=n=k$ sector: Operators and free fermions

We recall from the discussion at the start of Section 4 that the integer $k$ is directly related to the minimum number of Brauer contractions involved in the terms which are summed to make up an operator in the Brauer basis.

For $m=n=k$, all terms in an operator involve the maximum number of contractions, which translates into the fact that these operators are multi-traces of the matrix $Y=Z^{\dagger} Z$. Since $Y$ is hermitian we find the $N$ fermions of the hermitian matrix model emerging in this sector, as follows.

In this sector we have $\gamma=\left(k=m, \gamma_{+}=\emptyset, \gamma_{-}=\emptyset\right)$ and $\alpha=\beta$, so the projectors $Q_{\alpha, \beta}^{\gamma}$ (defined in Appendix A) are in this sector labelled by $\alpha$ alone. We write

$$
\begin{equation*}
P_{\alpha}^{k=m}=Q_{\alpha, \alpha}^{\gamma} \quad \text { with } \gamma \text { as above. } \tag{5.35}
\end{equation*}
$$

The projector is written in terms of the $k$-contraction operator $C_{(k)}$ defined by

$$
\begin{equation*}
C_{(k)}=\sum_{\sigma \in S_{k}} C_{\sigma(1) \overline{1}} \cdots C_{\sigma(k) \bar{k}} \tag{5.36}
\end{equation*}
$$

and the projector $p_{\alpha}$ which projects the holomorphic half of $V^{\otimes k} \otimes \bar{V}^{\otimes k}$ to the representation $\alpha$. It is proved in Appendix E that the projector takes the form

$$
\begin{equation*}
P_{\alpha}^{k=m}=\frac{d_{\alpha}}{k!\operatorname{Dim\alpha }} C_{(k)} p_{\alpha} \tag{5.37}
\end{equation*}
$$

and that the operator satisfies the following required properties:

$$
\begin{equation*}
\left(P_{\alpha}^{k=m}\right)^{2}=P_{\alpha}^{k=m} \quad \text { and } \quad \operatorname{tr}_{k, k}\left(P_{\alpha}^{k=m}\right)=\left(d_{\alpha}\right)^{2} \tag{5.38}
\end{equation*}
$$

where $d_{\alpha}$ is the dimension of the $S_{k}$ representation $\alpha$. The operators in the $m=n=k$ sector therefore take the explicit form:

$$
\begin{align*}
& \operatorname{tr}_{k, k}\left(P_{\alpha}^{k=m} Z^{\otimes k} \otimes Z^{* \otimes k}\right) \\
= & \frac{d_{\alpha}}{k!D i m \alpha} \operatorname{tr}_{k, k}\left(C_{(k)} p_{\alpha} Z^{\otimes k} \otimes Z^{* \otimes k}\right) \\
= & \frac{d_{\alpha}}{k!D i m \alpha} \sum_{\sigma \in S_{k}} \operatorname{tr}_{k, k}\left(\sigma C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma^{-1} p_{\alpha} Z^{\otimes k} \otimes Z^{* \otimes k}\right) \\
= & \frac{d_{\alpha}}{\operatorname{Dim\alpha }} \operatorname{tr}_{k, k}\left(C_{1 \overline{1}} \cdots C_{k \bar{k}} p_{\alpha} Z^{\otimes k} \otimes Z^{* \otimes k}\right) \\
= & \frac{d_{\alpha}}{\operatorname{Dim\alpha }} \operatorname{tr}_{k}\left(p_{\alpha} Y^{\otimes k}\right) \tag{5.39}
\end{align*}
$$

where $Y=Z^{\dagger} Z$. So operators in the $m=n=k$ sector are Schur polymonials constructed from $Y$.

We may understand these results in the following way. First observe that $H_{L}$ annihilates $\left(Z^{\dagger} Z\right)^{i}$, since $H_{L}=G_{2}+G_{3}$ generates the $U(N)$ action on the lower index of $Z^{\dagger}$ and the upper index of $Z$,

$$
\begin{equation*}
Z \rightarrow U Z, \quad Z^{\dagger} \rightarrow Z^{\dagger} U^{\dagger} \tag{5.40}
\end{equation*}
$$

and that the product $\left(Z^{\dagger} Z\right)^{i}{ }_{j}$ is invariant under this action. Traces of powers of $Y$ are thus also invariant under (5.40).
$H_{L}$ measures $C_{2}(\gamma)$ which implies that $C_{2}(\gamma)=0$ for all operators built from $Y_{j}^{i}$. This is consistent with the fact that in the $m=n=k$ sector $\gamma=\left(k=m, \gamma_{+}=\emptyset, \gamma_{-}=\emptyset\right)$ and so $C_{2}(\gamma)=0$. We can consider a Casimir of the form $\operatorname{tr}\left(Y \frac{\partial}{\partial Y}\right)^{2}$ which measures the labels of the Young diagram.

By the map discussed in Section 2.1, Schur polynomials in a hermitian matrix correspond to the states of $N$ free fermions in a harmonic oscillator potential. The harmonic oscillator fermions observed here are a second emergence of free particles, distinct from those of the $k=0$ sector.

## 6 Applications to integrable quantum mechanics

### 6.1 Review of matrix harmonic oscillator quantum mechanics

We now return to the dimensional reduction of $\mathcal{N}=4$ Super Yang-Mills on $R \times S^{3}$ in the zero coupling limit, truncated to the sector of one complex matrix. As noted in Section 3.2 one may choose the $A_{0}=0$ gauge while imposing Gauss's Law, yielding the quantum mechanics for the matrix $Z(t)$ defined by the following action [54]:

$$
\begin{equation*}
\mathcal{S}=\int d t \operatorname{tr}\left(\dot{Z} \dot{Z}^{\dagger}-Z Z^{\dagger}\right) \tag{6.1}
\end{equation*}
$$

It is well known that the holomorphic sector of the theory is equivalent to a system of non-interacting fermions in a one-dimensional harmonic oscillator potential [2, 8, 39]. As a subsector of $\mathcal{N}=4$ Super Yang-Mills extremal correlators in this sector are protected by supersymmetry $[10,11]$ and the states of this sector are dual to the LLM supergravity geometries [9].

Going beyond the holomorphic sector, we no longer have non-renomalization theorems so the connection to supergravity is not straightforward. Based on the following investigations, we will infer properties of any candidate string dual of the complex matrix model sector at zero coupling in Section 7.

We first review the previous analysis of the above theory [2]. The momenta conjugate to $Z^{i}{ }_{j}$ and $Z^{\dagger i}{ }_{j}$ are

$$
\begin{equation*}
\Pi_{i}^{j} \equiv \Pi_{Z_{j}^{i}}=\frac{\partial L}{\partial \dot{Z}_{j}^{i}}=\dot{Z}_{i}^{\dagger j}, \quad \quad \Pi_{i}^{\dagger j} \equiv \Pi_{Z^{\dagger i}}=\frac{\partial L}{\partial \dot{Z}_{j}^{\dagger i}}=\dot{Z}_{i}^{j} \tag{6.2}
\end{equation*}
$$

The equal time canonical commutation relations are

$$
\begin{equation*}
\left[Z^{p}, \Pi_{i}^{j}\right]=i \delta_{q}^{j} \delta_{i}^{p} \quad\left[Z^{\dagger}{ }_{q}, \Pi^{\dagger j}{ }_{i}\right]=i \delta^{j}{ }_{q} \delta_{i}^{p} \tag{6.3}
\end{equation*}
$$

so we can identify the conjugate momenta with matrix derivatives in the usual way using (2.5). We define the creation and annihilation operators:

$$
\begin{array}{ll}
A^{\dagger}=\frac{1}{\sqrt{2}}\left(Z-i \Pi^{\dagger}\right)=\frac{1}{\sqrt{2}}\left(Z-\frac{\partial}{\partial Z^{\dagger}}\right) & A=\frac{1}{\sqrt{2}}\left(Z^{\dagger}+i \Pi\right)=\frac{1}{\sqrt{2}}\left(Z^{\dagger}+\frac{\partial}{\partial Z}\right) \\
B^{\dagger}=\frac{1}{\sqrt{2}}\left(Z^{\dagger}-i \Pi\right)=\frac{1}{\sqrt{2}}\left(Z^{\dagger}-\frac{\partial}{\partial Z}\right) & B=\frac{1}{\sqrt{2}}\left(Z+i \Pi^{\dagger}\right)=\frac{1}{\sqrt{2}}\left(Z+\frac{\partial}{\partial Z^{\dagger}}\right) \tag{6.4}
\end{array}
$$

Importantly, the dagger on $A^{\dagger}$ does not signify hermitian conjugate of $A$. It signifies purely that this is a creation operator. The hermitian conjugate of $A^{\dagger i}{ }_{j}$ is $A_{i}^{j}$. The canonical commutation relations become

$$
\begin{equation*}
\left[A_{j}^{i}, A^{\dagger k}\right]=\delta_{l}^{i} \delta^{k}{ }_{j} \quad\left[B_{j}^{i}, B_{l}^{\dagger k}\right]=\delta_{l}^{i} \delta^{k}{ }_{j} \tag{6.5}
\end{equation*}
$$

The Hamiltonian and $U(1)$ current take the form

$$
\begin{align*}
\hat{H} & =\operatorname{tr}\left(-\frac{\partial^{2}}{\partial Z \partial Z^{\dagger}}+Z Z^{\dagger}\right)=\operatorname{tr}\left(A^{\dagger} A+B^{\dagger} B\right)+N^{2} \\
\hat{J} & =\operatorname{tr}\left(Z \frac{\partial}{\partial Z}-Z^{\dagger} \frac{\partial}{\partial Z^{\dagger}}\right)=\operatorname{tr}\left(A^{\dagger} A-B^{\dagger} B\right) \tag{6.6}
\end{align*}
$$

where $N^{2}$ is the zero point energy for $N^{2}$ harmonic oscillators in two dimensions.
The ground state of this system satisfies $A|0\rangle=B|0\rangle=0$. The corresponding (nonnormalised) wavefunction $\Psi_{0}=\langle Z, \bar{Z} \mid 0\rangle$ is

$$
\begin{equation*}
\Psi_{0}\left(Z, Z^{\dagger}\right)=e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.7}
\end{equation*}
$$

Holomorphic gauge invariant excitations of this system are defined by the constraint $B|\mathcal{O}\rangle=0$ and consist of operators built from $A^{\dagger}$ acting on the ground state. These may be written as

$$
\begin{equation*}
\operatorname{tr}_{n}\left(\sigma\left(A^{\dagger}\right)^{\otimes m}\right)|0\rangle \tag{6.8}
\end{equation*}
$$

where $\sigma$ is an element of $S_{n}$, and controls how the indices are contracted to form either a single or multi-trace operator. A more convenient basis for operators of the form (6.8) is the Schur polynomial basis (for details see [2]):

$$
\begin{equation*}
\left|\Psi_{R}\right\rangle=\chi_{R}\left(A^{\dagger}\right)|0\rangle \tag{6.9}
\end{equation*}
$$

where $\chi_{R}$ is the character of the $U(N)$ representation $R$. Since

$$
\begin{equation*}
A^{\dagger} e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)}=\sqrt{2} Z e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.10}
\end{equation*}
$$

we may write

$$
\begin{equation*}
\Psi_{R}\left(Z, Z^{\dagger}\right)=\chi_{R}(\sqrt{2} Z) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.11}
\end{equation*}
$$

This state has $E=m+N^{2}$ and $J=m$ and is holomorphic in $Z$ up to the exponential factor. If we triangularize $Z$ and redefine the wavefunction by absorbing the Jacobian of the transformation into the definition of the wavefunction, it becomes a wavefuction for $N$ fermions in the Lowest Landau Level of the Quantum Hall system [2, 8, 55].

### 6.2 Non-holomorphic sector

The most general eigenstate can be constructed by acting with both $A^{\dagger}$ and $B^{\dagger}$ on the ground state,

$$
\begin{equation*}
\left|\Psi_{\mathcal{O}}\right\rangle=\mathcal{O}\left(A^{\dagger}, B^{\dagger}\right)|0\rangle \tag{6.12}
\end{equation*}
$$

where $\mathcal{O}\left(A^{\dagger}, B^{\dagger}\right)$ is a gauge invariant polynomial constructed from $m A^{\dagger}$ 's and $n B^{\dagger}$ 's.
The wavefunction of such a state may be written as

$$
\begin{equation*}
\Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)=\left\langle Z, Z^{\dagger} \mid \Psi_{\mathcal{O}}\right\rangle=\mathcal{O}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.13}
\end{equation*}
$$

The Brauer Algebra may be used to organise the states above. Such states are analogous to those used in Section 4 and take the form

$$
\begin{equation*}
\left|\Psi_{\alpha, \beta ; i, j}^{\gamma}\right\rangle=\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(A^{\dagger}, B^{\dagger}\right)|0\rangle \tag{6.14}
\end{equation*}
$$

where the labels are explained in Appendix A. This state has $E=m+n+N^{2}$ and $J=m-n$.

Unlike for the holomorphic sector wavefunctions, we have

$$
\begin{equation*}
\mathcal{O}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \neq \mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.15}
\end{equation*}
$$

because the derivative of $Z$ inside $A^{\dagger}$ acts on $Z$ which comes from the action of $B^{\dagger}$ on the exponential factor. For example we have

$$
\begin{equation*}
\operatorname{tr}\left(A^{\dagger} B^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)}=\left(2 \operatorname{tr} Z Z^{\dagger}-N^{2}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.16}
\end{equation*}
$$

and in general the correct relation is

$$
\begin{equation*}
\Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)=\mathcal{O}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)}=\left[e^{-\frac{\square}{2}} \mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right] e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.17}
\end{equation*}
$$

where $\square$ is the laplacian $\operatorname{tr} \frac{\partial}{\partial Z} \frac{\partial}{\partial Z^{\dagger}}$ and the brackets indicate that the derivatives in $\square$ act only on $\mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)$ and not on the exponential. $e^{-\frac{\square}{2}}$ is defined by its series expansion; it was observed in (5.11) that the laplacian generates Wick contractions and so here $e^{-\frac{0}{2}}$ performs a normal ordering, subtracting terms in which pairs of $\sqrt{2} Z$ and $\sqrt{2} Z^{\dagger}$ have been contracted (c.f. [56]).

Note however that in a $k=0$ operator we have from (5.12) that

$$
\begin{equation*}
\square \mathcal{O}^{k=0}=0 \tag{6.18}
\end{equation*}
$$

and so we can replace $A^{\dagger}$ and $B^{\dagger}$ with $\sqrt{2} Z$ and $\sqrt{2} Z^{\dagger}$ respectively without worrying about the above subtlety.

We can define operators corresponding to the $G_{i}$ in (3.12) as follows.

$$
\begin{array}{ll}
\left(\hat{G}_{1}\right)_{j}^{i}=\left(B^{\dagger} B\right)_{j}^{i} & \left(\hat{G}_{2}\right)_{j}^{i}=\left(A^{\dagger} A\right)_{j}^{i} \\
\left(\hat{G}_{3}\right)_{j}^{i}=-B^{\dagger k}{ }_{j} B_{k}^{i} & \left(\hat{G}_{4}\right)_{j}^{i}=-A^{\dagger t}{ }_{j} A_{k}^{i} \tag{6.19}
\end{array}
$$

Defining $\left|A_{j}^{i}\right\rangle=A^{i}{ }_{j}|0\rangle$ and so on, using the commutation relations we find

$$
\begin{align*}
\left(\hat{G}_{1}\right)_{j}^{i}\left|B^{\dagger p}\right\rangle & =\delta_{j}^{p}\left|B^{\dagger i}\right\rangle & \left(\hat{G}_{2}\right)_{j}^{i}\left|A^{\dagger p}\right\rangle & =\delta^{p}{ }_{j}\left|A^{\dagger i}{ }_{q}\right\rangle \\
\left(\hat{G}_{3}\right)_{j}^{i}\left|B^{\dagger p}\right\rangle & =-\delta_{q}^{i}\left|B^{\dagger p}\right\rangle & \left(\hat{G}_{4}\right)^{i}{ }_{j}^{i}\left|A^{\dagger p}\right\rangle & =-\delta_{q}^{i}\left|A^{\dagger p}\right\rangle \tag{6.20}
\end{align*}
$$

which is the same as the adjoint action of the operators $G_{i}$ defined in (3.12) on the matrices $Z, Z^{\dagger}$ (see equation (11) of [25]).

The result is that we can define harmonic oscillator Casimir operators

$$
\hat{\mathcal{H}}_{A}=\left\{\begin{array}{lllll}
\hat{H}_{1}, & \hat{H}_{2}, & \hat{\bar{H}}_{1}, & \hat{\bar{H}}_{2}, & \hat{H}_{L} \tag{6.21}
\end{array}\right\}
$$

by replacing $G_{i}$ in (4.7) with $\hat{G}_{i}$. The eigenvalues of hatted Casimirs acting on $\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(A^{\dagger}, B^{\dagger}\right)|0\rangle$ are the same as those of the corresponding unhatted Casimirs acting on $\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(Z, Z^{\dagger}\right)$. This is because the same commutator manipulations can be done to evaluate both, and the arguments which prove that $\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(Z, Z^{\dagger}\right)$ are eigenstates of the Casimirs in (4.7) also prove that $\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(A^{\dagger}, B^{\dagger}\right)|0\rangle$ are eigenstates of the hatted versions.

We can take this one step further. Noting that

$$
\begin{align*}
{\left[Z_{j}^{i},-\frac{\square}{2}\right] } & =\frac{1}{2}\left(\frac{\partial}{\partial Z^{\dagger}}\right)_{j}^{i}  \tag{6.22}\\
\Rightarrow\left[Z_{j}^{i}, e^{-\frac{\square}{2}}\right] & =\frac{1}{2}\left(\frac{\partial}{\partial Z^{\dagger}}\right)_{j}^{i} e^{-\frac{\square}{2}} \tag{6.23}
\end{align*}
$$

and similarly

$$
\begin{equation*}
\left[Z^{\dagger i}, e^{-\frac{\square}{2}}\right]=\frac{1}{2}\left(\frac{\partial}{\partial Z}\right)_{j}^{i} e^{-\frac{\square}{2}} \tag{6.24}
\end{equation*}
$$

then using (6.17) we derive

$$
A^{\dagger i}{ }_{j} \Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)=A^{\dagger i}{ }_{j} \mathcal{O}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)}
$$

$$
\begin{align*}
& =A^{\dagger i}\left[e^{-\frac{\square}{2}} \mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right] e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \\
& =\left[e^{-\frac{\square}{2}}\left(\sqrt{2} Z_{j}^{i}\right) \mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right] e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.25}
\end{align*}
$$

where again the brackets indicate that the derivatives act only on $\mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)$ and not on the exponential. Similarly

$$
\begin{equation*}
A_{j}^{i} \Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)=\left[e^{-\frac{\square}{2}}\left(\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial Z}\right)_{j}^{i}\right) \mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right] e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.26}
\end{equation*}
$$

implying the following relation between $\hat{G}_{2}$ and $G_{2}$ :

$$
\begin{equation*}
\left(\hat{G}_{2}\right)_{j}^{i} \Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)=\left[e^{-\frac{\square}{2}}\left(G_{2}\right)_{j}^{i} \mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right] e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.27}
\end{equation*}
$$

Similar results apply to the remaining $\hat{G}_{i}$, the Hamiltonians $\hat{H}_{i}$ as well as the canonical Hamiltonian

$$
\begin{equation*}
\hat{H}=\hat{H}_{1}+\hat{\bar{H}}_{1}+N^{2}=\operatorname{tr}\left(A^{\dagger} A+B^{\dagger} B\right)+N^{2} \tag{6.28}
\end{equation*}
$$

whose action on wavefunctions $\Psi\left(Z, Z^{\dagger}\right)$ can be written in terms of the (first-order) scaling operator $H$ :

$$
\begin{equation*}
H=H_{1}+\bar{H}_{1}+N^{2}=\operatorname{tr}\left(Z \frac{\partial}{\partial Z}+Z^{\dagger} \frac{\partial}{\partial Z^{\dagger}}\right)+N^{2} \tag{6.29}
\end{equation*}
$$

Applying (6.27) and the corresponding relation for $\hat{G}_{3}$ we find that

$$
\begin{equation*}
\hat{H} \Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)=\left[e^{-\frac{\square}{2}} H \mathcal{O}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right] e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} . \tag{6.30}
\end{equation*}
$$

A similar manipulation in the holomorphic sector was performed in Appendix A of [57]. Note that for a $k=0$ operator we have $\square \mathcal{O}^{k=0}=0$ and so the above analysis gives

$$
\begin{equation*}
\hat{H}\left[\mathcal{O}^{k=0}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)}\right]=\left[H \mathcal{O}^{k=0}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right] e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.31}
\end{equation*}
$$

The inner product on wavefunctions may be derived using

$$
\begin{equation*}
\int\left[d Z d Z^{\dagger}\right]\left|Z, Z^{\dagger}\right\rangle\left\langle Z, Z^{\dagger}\right|=1 \tag{6.32}
\end{equation*}
$$

where $\left[d Z d Z^{\dagger}\right]=\prod_{i, j} d Z_{i j} d Z_{i j}^{\dagger}$, as follows:

$$
\begin{align*}
\left\langle\Psi_{\mathcal{O}_{1}} \mid \Psi_{\mathcal{O}_{2}}\right\rangle & =\frac{1}{\pi^{N^{2}}} \int\left[d Z d Z^{\dagger}\right]\left\langle\mathcal{O}_{1}\left(A^{\dagger}, B^{\dagger}\right) \mid Z, Z^{\dagger}\right\rangle\left\langle Z, Z^{\dagger} \mid \mathcal{O}_{2}\left(A^{\dagger}, B^{\dagger}\right)\right\rangle \\
& =\frac{1}{\pi^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\Psi_{\mathcal{O}_{1}}\left(Z, Z^{\dagger}\right)} \Psi_{\mathcal{O}_{2}}\left(Z, Z^{\dagger}\right) \tag{6.33}
\end{align*}
$$

where $\pi^{N^{2}}$ compensates for using non-normalised wavefunctions, and is found by imposing

$$
\begin{equation*}
\left\langle\Psi_{0} \mid \Psi_{0}\right\rangle=1 \tag{6.34}
\end{equation*}
$$

Using (6.17), the above expression (6.33) becomes

$$
\begin{align*}
\left\langle\Psi_{\mathcal{O}_{1}} \mid \Psi_{\mathcal{O}_{2}}\right\rangle & =\frac{1}{\pi^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\mathcal{O}_{1}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr} Z Z^{\dagger}} \mathcal{O}_{2}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr} Z Z^{\dagger}}} \\
& =\frac{1}{\pi^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\left(e^{-\frac{\square}{2}} \mathcal{O}_{1}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right)}\left(e^{-\frac{\square}{2}} \mathcal{O}_{2}\left(\sqrt{2} Z, \sqrt{2} Z^{\dagger}\right)\right) e^{-2 \operatorname{tr} Z Z^{\dagger}} \tag{6.35}
\end{align*}
$$

and rescaling factors of two we have the result

$$
\begin{equation*}
\left\langle\Psi_{\mathcal{O}_{1}} \mid \Psi_{\mathcal{O}_{2}}\right\rangle=\frac{1}{(2 \pi)^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\left(e^{-\square} \mathcal{O}_{1}\left(Z, Z^{\dagger}\right)\right)}\left(e^{-\square} \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right) e^{-\operatorname{tr} Z Z^{\dagger}} \tag{6.36}
\end{equation*}
$$

which is the non-holomorphic generalisation of (A.12) of [57].
In the next section we use the right hand side of the above equation to define an inner product on gauge invariant polynomials $\mathcal{O}\left(Z, Z^{\dagger}\right)$ rather than the harmonic oscillator wavefunctions $\Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)$ which contain exponentials.

### 6.3 Related integrable quantum mechanics models

In the discussion above we related the action of the Hamiltonian $\hat{H}$ in terms of the (firstorder) scaling operator $H$ : We thus have an explicit map (6.30) relating the the action of $\hat{H}$ on its eigenstates

$$
\begin{equation*}
\Psi_{\mathcal{O}}\left(Z, Z^{\dagger}\right)=\mathcal{O}\left(A^{\dagger}, B^{\dagger}\right) e^{-\operatorname{tr}\left(Z Z^{\dagger}\right)} \tag{6.37}
\end{equation*}
$$

to the action of $H$ on its eigenstates

$$
\begin{equation*}
\mathcal{O}\left(Z, Z^{\dagger}\right) \tag{6.38}
\end{equation*}
$$

The right hand side of (6.36) can be used to define an inner product on polynomial functions of $Z, Z^{\dagger}$. Explicitly this inner product is

$$
\begin{equation*}
\left(\mathcal{O}_{1}\left(Z, Z^{\dagger}\right), \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right)=\frac{1}{(2 \pi)^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\left(e^{-\square \mathcal{O}_{1}\left(Z, Z^{\dagger}\right)}\right)}\left(e^{-\square} \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right) e^{-\operatorname{tr} Z Z^{\dagger}} \tag{6.39}
\end{equation*}
$$

Note that the two inner products $(\cdot, \cdot)$ and $\langle\cdot \mid \cdot\rangle$ are defined on two different Hilbert spaces:

$$
(\cdot, \cdot):\left\{\text { Polynomials in } Z, Z^{\dagger}\right\} \rightarrow \mathbb{R}
$$

$$
\begin{equation*}
\langle\cdot \mid \cdot\rangle:\{\text { Harmonic oscillator states }\} \rightarrow \mathbb{R} \tag{6.40}
\end{equation*}
$$

By construction the inner products satisfy

$$
\begin{equation*}
\left(\mathcal{O}_{1}\left(Z, Z^{\dagger}\right), \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right)=\left\langle\Psi_{\mathcal{O}_{1}} \mid \Psi_{\mathcal{O}_{2}}\right\rangle \tag{6.41}
\end{equation*}
$$

and each inner product is diagonalised by the corresponding Brauer basis

$$
\begin{equation*}
\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(Z, Z^{\dagger}\right) \quad \text { and } \quad\left|\Psi_{\alpha, \beta ; ;, j}^{\gamma}\right\rangle=\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(A^{\dagger}, B^{\dagger}\right)|0\rangle . \tag{6.42}
\end{equation*}
$$

The relevant Hamiltonians have the properties that

- $\hat{\mathcal{H}}_{A}$ are hermitian with the inner product $\langle\cdot \mid \cdot\rangle$
- $\mathcal{H}_{A}$ are hermitian with the inner product $(\cdot, \cdot)$

The inner product $(\cdot, \cdot)$ can in fact be constructed by starting from the inner product arising from the zero-dimensional complex matrix model and requiring $H_{i}$ to be hermitian. This involves subtracting $Z, Z^{\dagger}$ contractions and is discussed in Appendix F.

In addition to the original Hamiltonian $\hat{H}$, it is natural to consider the conserved charges e.g $\operatorname{tr} \hat{G}_{2}^{2}$ as Hamiltonians. Since these higher Hamiltonians were constructed to be simultaneously diagonalised with $\hat{H}$ in the Brauer algebra basis, we know these are solvable Hamiltonians related to the Brauer algebra. Related to these higher Hamiltonians are simpler ones which are obtained by replacing that hatted $G$ 's with unhatted ones. For example $H_{2}=\operatorname{tr} G_{2}^{2}$ or $H_{L}=\operatorname{tr}\left(G_{2}+G_{3}\right)^{2}$ define solvable quantum mechanics models. They are second order in derivatives as opposed to fourth order like $\hat{H}_{2}, \hat{H}_{L}$, and hermitian in the inner product (6.39).

Considering the expressions in (4.41) and comparing with equation (2.4) of [58] we see that these integrable quantum mechanics models are non-holomorphic generalizations of the Calogero-Sutherland model at a specific coupling (see also [59, 60] for related literature). A natural question is whether the Calogero-Sutherland model at generic coupling has such integrable non-holomorphic generalizations. Although we have not written out the Hamiltonians for general $N$ explicitly as differential operators in terms of coordinates on $\mathcal{M}_{N}$ it is clear that this can be done by changing variables from $g l(N, \mathbb{C})$ to $z_{i}, t_{i j}$.

## 7 Summary and outlook

We described free particle structures hidden in matrix models of an $N \times N$ complex matrix $Z$. We related these structures to the geometry of the configuration space $\mathcal{M}_{N}$ of gauge-inequivalent configurations, a space of dimension $N^{2}+1$. We showed that $\mathcal{M}_{N}$ supports an interesting class of functions, obtained from the gauge invariant functions of $Z$. The Schur decomposition gives coordinates $z_{i}, t_{i j}$ useful for describing $\mathcal{M}_{N}$. Integrals
over complex matrices give a measure of integration on $\mathcal{M}_{N}$ which can be used to define an inner product on gauge invariant functions of $z_{i}, t_{i j}$. Following [12], Brauer algebras $B_{N}(m, n)$ give orthogonal bases which diagonalise the inner products which arise. Higher Casimirs constructed from the Brauer algebra, which resolve these orthogonal bases [25], give rise to a complete set of scale and gauge invariant differential operators on $\mathcal{M}_{N}$.

Among the labels of the Brauer basis is a non-negative integer $k$. For any $N$ the $k=0$ sector has states in one-to-one correspondence with those of $N$ free fermions on a circle. These states correspond to the composite representations $R \bar{S}$ which play a role in two dimensional Yang Mills. The differential operators which measure Casimirs are polynomials in the free fermion momenta; for $N=2$ we inverted these relations to write the momenta as algebraic functions of the differential operators. We conjectured that the $k=0$ sector is the kernel of a scale invariant version of the laplacian, the operator $\operatorname{tr}\left(G_{2} G_{3}\right)$. We also gave an equality of correlators between the unitary matrix model and the complex matrix model for a class of three-point functions of $k=0$ operators. It is important to note that while the usual emergence of free fermions in matrix models can be seen from a change of variables to eigenvalues [38], here the $k=0$ sector has no direct relation to the eigenvalues of $Z$. Indeed the operator $\operatorname{tr}\left(G_{2} G_{3}\right)$ characterizing it involves derivatives with respect to $z_{i}$ as well as the off-diagonal $t_{i j}$.

Another interesting sector where states are counted by Young diagrams is the sector $m=n=k$. This is a sector of gauge invariant functions of $\left(Z^{\dagger} Z\right)^{i}{ }_{j}$ which is the kernel of another second order operator on $\mathcal{M}_{N}$, namely $\operatorname{tr} G_{L}^{2}$ as defined above (4.7). We observe that $k$ appears to interpolate between radial and angular free particle systems on a plane. It would be interesting to further elucidate this in a stringy context.

A precise understanding of the commutative ring of scale and gauge invariant differential operators led us to computational results on the reduction multiplicities of representations of $B_{N}(m, n)$ to $S_{m} \times S_{n}$ for $N=2$.

The connection between $\mathcal{M}_{N}$ and the space of gauge-invariant poynomial functions of $Z, Z^{\dagger}$ is a more intricate version of the connection between $\mathbb{R}^{N} / S_{N}$ and symmetric polynomials. Likewise the connection between Brauer algebras and gauge-invariant differential operators on $\mathcal{M}_{N}$ is a generalization of the connection between symmetric groups and differential operators on $\mathbb{R}^{N} / S_{N}$.

We present some avenues for future research :

1. We wonder if some of these free particle structures can be obtained from the dual supergravity side of AdS/CFT in the sector which is $S O(4) \times S O(4)$ invariant. This would be a non-supersymmetric generalization of the LLM [9] discovery of supergravity geometries corresponding to the free fermions of the holomorphic sector of the complex matrix model $[2,8]$.
2. We have given explicit expressions for the free fermion momenta for the $k=0$ sector of the $N=2$ matrix theories in terms of the original matrix variables. It is an open
problem to find explicit expressions for the coordinates of the fermions, and the wavefunctions as Slater determinants. It is also interesting to explore whether this would be useful for the computation of correlators.
3. We have found a non-holomorphic generalization of the Calogero-Sutherland Hamiltonian at a fixed coupling. What is the physics of these non-holomorphic models? Can we observe Brauer Algebra wavefunctions in the laboratory?
4. We have presented results on finite $N$ counting of complex matrix model states in terms of Brauer algebras at $N=2$. These are related to reduction multiplicities for $B_{N=2}(m, n)$ irreps into $S_{m} \times S_{n}$ irreps. What are these finite $N$ reduction multiplicities for general $m, n, N$, in particular for $N<m+n$ ?
5. Our analysis has developed integrable quantum mechanics models for the space $\mathcal{M}_{N}$ and exploited (Brauer) algebras to identify and organise interesting spaces of functions and differential operators on these spaces. $\mathcal{M}_{N}$ is a fibration over $\mathbb{R}^{N} / S_{N}$ which arises in hermitian matrix models and, like $\mathbb{R}^{N} / S_{N}$, has different strata where the orbits qualitatively change their structure. While symmetric groups $S_{n}$ or their inductive limit $S_{\infty}$ organise functions and differential operators on the symmetric product, the Brauer algebras $B_{N}(m, n)$ or similarly their inductive limit $B_{N}(\infty, \infty)$ organise $\mathcal{M}_{N}$. Results in matrix models, especially multimatrix models [61, 12, 49, 50, 51, 25, 62, 63] can give analogous results for other stratified spaces which arise as the space of inequivalent configurations. Is it possible to understand the role of algebras, integrable structures and hidden free particle systems intrinsically from the stratified geometries? What is the intrinsic characterization of stratified geometries which allow such structures? Studies of Hilbert space structures which mirror the strata in certain stratified spaces have been done [64]. Studying $\mathcal{M}_{N}$ from a similar point of view and finding its relations to the Brauer algebra description of functions and differential operators would be very interesting.
6. There is a substantial literature discussing consistent truncations of the Maldacena duality. For example, it is known that the $S U(2)$ sector defines a consistent truncation to all orders in perturbation theory [65]. Sectors such as the $Z, Z^{\dagger}$ sector are well-defined truncations at zero coupling. Assuming the strong finite $N$ form of the Maladacena conjecture, and making the plausible assumption that consistent quantum truncations of a quantum field theory with a string dual have a string dual, we are led to ask: What is the gauge-string theory dual of one free complex matrix in four dimensions? Similarly what is the dual of the quantum mechanics from reduction of the $Z, Z^{\dagger}$ sector on $\mathbb{R} \times S^{3}$ ? For the large $N$ Gaussian Hermitian matrix model (without the quadratic potential) there is the non-critical string considered in the old matrix model literature [27]. For double scaling limits
of the complex matrix models, we have the Type-0 string backgrounds [33]. For the large $N$ hermitian matrix oscillator quantum mechanics, which is also a consistent truncation of the s-wave sector of $\mathcal{N}=4 \mathrm{SYM}$ in radial quantization, there is the proposal [66]. A well-known example of a duality between matrix quantum mechanics and M-theory is given by [67].

We do not have a clear answer to the last question, but the following remarks are suggested by the investigations in this paper. We conjecture that there exists a string dual of the matrix harmonic oscillator quantum mechanics discussed in Section 6 which has a $2+1$ dimensional space-time and whose physics involves interacting strings and branes. The $z_{i}$ coordinates are positions of $N$ branes in 2 space dimensions. By analogy to the treatment in [68] we expect the variables $t_{i j}$ of the Schur decomposition to describe strings connecting brane $i$ to $j$; here the triangular constraint ( $t_{i j}=0$ for $i>j$ ) will make the dual qualitatively different from the standard system of strings and branes at weak coupling. This ought to be explained by an explicit construction of the string theory. The Hamiltonian $H$ contains terms $t \frac{\partial}{\partial t}$ along with $z_{i} \frac{\partial}{\partial z_{i}}$. Excitations involving polynomials in $z_{i}$ have energies comparable to excitations involving $t$. This means that strings and branes have comparable masses. Usually string states have masses of order 1 (with $l_{s}=1$ ) whereas branes have masses of order $1 / g_{s}$. In this sense, the model at hand appears to have $g_{s} \sim 1$. An interesting problem is to construct this strings and branes model in detail and to provide a physical interpretation for the labels of the Brauer algebra basis, in particular $k$, and their constraints at finite $N$.

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## A The Brauer algebra basis

In this appendix we briefly introduce the Brauer algebra basis for gauge invariant polynomials in $Z, Z^{\dagger}$. The Brauer algebra $B_{N}(m, n)$ is used to construct a basis of these polynomials of degree $m$ in $Z$ and degree $n$ in $Z^{\dagger}$. We will not need a precise definition of these algebras here, rather we will recall below how their representation theoretic data are used to label a useful basis. The definition of these Brauer algebras and their use in constructing a basis of gauge invariant operators is found in the original paper [12]. A more detailed review of the of the construction may be found in Section 2 of [63].

A Brauer basis operator is a linear combination of multi-trace operators built from $m$ $Z$ 's and $n Z^{\dagger}$ 's and is written as

$$
\begin{equation*}
\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(Z, Z^{\dagger}\right) . \tag{A.1}
\end{equation*}
$$

These operators are constructed by viewing $Z^{\otimes m} \otimes\left(Z^{*}\right)^{\otimes n}$ as operators on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$, composing them with elements in the Brauer algebra $Q_{\alpha, \beta ; i, j}^{\gamma}$ and taking a trace [12]:

$$
\begin{equation*}
\mathcal{O}_{\alpha, \beta ; i, j}^{\gamma}\left(Z, Z^{\dagger}\right)=\operatorname{tr}_{m, n}\left(Q_{\alpha, \beta ; i, j}^{\gamma}\left(\mathbf{Z} \otimes \mathbf{Z}^{*}\right)\right) \tag{A.2}
\end{equation*}
$$

The same construction can be done with the creation operators of the matrix quantum mechanics by replacing $Z$ with $A^{\dagger}$ and $Z^{*}$ with $\left(B^{\dagger}\right)^{T}$ where $T$ denotes matrix transpose. These operators diagonalize the two-point function for $Z, Z^{\dagger}$ at zero Yang-Mills coupling or the Fock space inner product for the states created by the $A^{\dagger}, B^{\dagger}$ of Section 6 .

The labels on the operator are as follows:

1. $\alpha, \beta$ are Young diagrams with $m$ and $n$ boxes respectively, with $c_{1}(\alpha) \leq N$ and $c_{1}(\beta) \leq N$. They label representations of $U(N)$ as well as $S_{m}$ and $S_{n}$ respectively.
2. $\gamma=\left(k, \gamma_{+}, \gamma_{-}\right)$where
(a) $k$ is an integer in the range $0 \leq k \leq \min (m, n)$
(b) $\gamma_{+}, \gamma_{-}$are Young diagrams with $m-k$ and $n-k$ boxes respectively, with $c_{1}\left(\gamma_{+}\right)+c_{1}\left(\gamma_{-}\right) \leq N$.
$\gamma$ labels a representation of the (walled) Brauer algebra $B_{N}(m, n)$.
3. $i, j$ are indices which run from 1 to the multiplicity $M_{\alpha \beta}^{\gamma}$ of the representation $(\alpha, \beta)$ of $\mathbb{C}\left[S_{m} \times S_{n}\right]$ in the representation $\gamma$ of the Brauer algebra.

The Brauer representation labelled by $\gamma=\left(k, \gamma_{+}, \gamma_{-}\right)$has an associated $U(N)$ composite representation labelled by $\gamma_{c}$ which is defined as follows. Using the usual notation in which a Young diagram with row lengths $r_{i}$ is written $\left[r_{1}, r_{2}, \ldots, r_{N}\right]$, let

$$
\begin{equation*}
\gamma_{+}=\left[r_{1}, r_{2}, \ldots, r_{p}\right], \quad \gamma_{-}=\left[s_{1}, s_{2}, \ldots, s_{q}\right] \tag{A.3}
\end{equation*}
$$

then providing $p+q \leq N, \gamma_{c}$ is given by

$$
\begin{equation*}
\gamma_{c}=\left[r_{1}, r_{2}, \ldots, r_{p}, 0,0, \ldots, 0,-s_{q},-s_{q-1}, \ldots,-s_{1}\right] \tag{A.4}
\end{equation*}
$$

where there are $N-(p+q)$ zeroes inserted. In the language of the mathematics literature $\gamma_{c}$ is an $N$-staircase with positive part $\gamma_{+}$and negative part $\gamma_{-}[13,15]$.

When we discuss Casimir operators we use the shorthand $C_{2}(\gamma)$ for the $U(N)$ quadratic Casimir of the representation $\gamma_{c}$, and similarly $r_{p}(\gamma)$ or $r_{p}^{\gamma}$ for the $p$-th row of $\gamma_{c}$.

When $k=0$ the $i, j$ labels are trivial and we have $\alpha=\gamma_{+}, \beta=\gamma_{-}$. Thus $\gamma$ is given by ( $k=0, \alpha, \beta$ ) and so the $k=0$ operators are thus determined by two Young diagrams $\alpha$ and $\beta$. To connect with the notation of 'coupled representations' in the two-dimensional Yang-Mills literature (see e.g. [69]), we rename $\gamma^{+}=R, \gamma_{-}=S$. If we substitute a unitary matrix in place of $Z$, the $k=0$ polynomials coincide with the 'coupled characters' $\chi_{R \bar{S}}(U)$ studied in the context of the string theory two-dimensional Yang-Mills [69].

At $(m, n)=(1,1)$, suppressing non-essential labels, the Brauer basis is

$$
\begin{align*}
\mathcal{O}_{[1],[1]}^{k=0}\left(Z, Z^{\dagger}\right) & =\operatorname{tr} Z \operatorname{tr} Z^{\dagger}-\frac{1}{N} \operatorname{tr} Z Z^{\dagger}  \tag{A.5}\\
\mathcal{O}_{[1],[1]}^{k=1}\left(Z, Z^{\dagger}\right) & =\frac{1}{N} \operatorname{tr} Z Z^{\dagger} \tag{A.6}
\end{align*}
$$

Here we have suppressed $\gamma_{+}$and $\gamma_{-}$since for a $k=0$ operator it is always the case that $\alpha=\gamma_{+}$and $\beta=\gamma_{-}$, and since for the above $k=1$ operator, $\gamma_{+}$and $\gamma_{-}$are both the empty diagram. The multiplicity indices $i, j$ are not relevant for this example. For further examples of Brauer basis operators see Appendix A. 4 of [12].

## B List of $\gamma_{+}$and $\gamma_{-}$at $N=2$ for given $(m, n)$

Given $(m, n)$ the possible $\gamma_{+}$and $\gamma_{-}$are listed below, along with $\gamma_{c}$ as defined in equation (A.4). Note that $r_{1}^{\gamma}$ (and hence $p_{1}^{\gamma}$ ) distinguishes operators, as does $r_{2}^{\gamma}$. We use the shorthand $C_{2}(\gamma)$ for the $U(N)$ quadratic Casimir of the representation labelled by $\gamma_{c}$.

List of $\gamma_{+}$and $\gamma_{-}$when $m \geq n$ using $d=m-n$

| $\gamma_{+}$ | $\gamma_{-}$ | $\gamma_{c}$ | $k$ | $C_{2}(\gamma)$ |
| :---: | :---: | :---: | :---: | :---: |
| $[m]$ | $[n]$ | $[m,-n]$ | 0 | $m(m+1)+n(n+1)$ |
| $[m-1]$ | $[n-1]$ | $[m-1,-(n-1)]$ | 1 | $(m-1)(m)+(n-1)(n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $[d+1]$ | $[1]$ | $[d+1,-1]$ | $n-1$ | $(d+1)(d+2)+2$ |
| $[d]$ | $\emptyset$ | $[d, 0]$ | $n$ | $d(d+1)$ |
| $[d-1,1]$ | $\emptyset$ | $[d-1,1]$ | $n$ | $(d-1)(d)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\left[\left\lceil\frac{d}{2}\right\rceil,\left\lfloor\frac{d}{2}\right\rfloor\right]$ | $\emptyset$ | $\left[\left\lceil\frac{d}{2}\right\rceil,\left\lfloor\frac{d}{2}\right\rfloor\right]$ | $n$ | $\left\lceil\frac{d}{2}\right\rceil\left(\left\lceil\frac{d}{2}\right\rceil+1\right)+\left\lfloor\frac{d}{2}\right\rfloor\left(\left\lfloor\frac{d}{2}\right\rfloor-1\right)$ |

List of $\gamma_{+}$and $\gamma_{-}$when $m<n$ using $\tilde{d}=n-m$

| $\gamma_{+}$ | $\gamma_{-}$ | $\gamma_{c}$ | $k$ | $C_{2}(\gamma)$ |
| :---: | :---: | :---: | :---: | :---: |
| $[m]$ | $[n]$ | $[m,-n]$ | 0 | $m(m+1)+n(n+1)$ |
| $[m-1]$ | $[n-1]$ | $[m-1,-(n-1)]$ | 1 | $(m-1)(m)+(n-1)(n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $[1]$ | $[\tilde{d}+1]$ | $[1,-(\tilde{d}+1)]$ | $m-1$ | $(\tilde{d}+1)(\tilde{d}+2)+2$ |
| $\emptyset$ | $[\tilde{d}]$ | $[0,-\tilde{d}]$ | $m$ | $\tilde{d}(\tilde{d}+1)$ |
| $\emptyset$ | $[\tilde{d}-1,1]$ | $[-1,-(\tilde{d}-1)]$ | $m$ | $(\tilde{d}-1)(\tilde{d})$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\emptyset$ | $\left[\left\lceil\frac{\tilde{d}}{2}\right\rceil,\left\lfloor\frac{\tilde{d}}{2}\right\rfloor\right]$ | $\left[-\left\lfloor\frac{\tilde{d}}{2}\right\rfloor,-\left\lceil\frac{\tilde{d}}{2}\right\rceil\right]$ | $m$ | $\left\lceil\frac{\tilde{d}}{2}\right\rceil\left(\left\lceil\frac{\tilde{d}}{2}\right\rceil+1\right)+\left\lfloor\frac{\tilde{d}}{2}\right\rfloor\left(\left\lfloor\frac{\tilde{d}}{2}\right\rfloor-1\right)$ |

## C Brauer counting at large $N$ from Clebsch counting

In this section we show that $N_{s b}(m, n)$ as defined in (5.33),

$$
\begin{equation*}
N_{s b}(m, n)=\sum_{\gamma, A}\left(M_{A}^{\gamma}\right)^{2} \tag{C.1}
\end{equation*}
$$

agrees with equation (5.32),

$$
\begin{equation*}
Q_{m t}(m, n)=\sum_{c_{l}(1): \sum_{l} c_{l}(1) l=m} \sum_{c_{l}(2): \sum_{l} c_{l}(2) l=n} \prod_{l} \frac{\left(c_{l}(1)+c_{l}(2)\right)!}{c_{l}(1)!c_{l}(2)!} \tag{C.2}
\end{equation*}
$$

We first expand

$$
\begin{align*}
& \sum_{\substack{\gamma \\
\min (m, n)}} \sum_{A}\left(M_{A}^{\gamma}\right)^{2} \\
= & \sum_{k=0} \sum_{\gamma_{+} \vdash(m-k)} \sum_{\gamma-\vdash(n-k)} \sum_{\alpha \vdash m} \sum_{\beta \vdash n}\left(\sum_{\delta \vdash k} g\left(\delta, \gamma_{+} ; \alpha\right) g\left(\delta, \gamma_{-} ; \beta\right)\right)^{2} \\
= & \sum_{k} \sum_{\gamma_{+}, \gamma_{-}} \sum_{\alpha, \beta}\left(\sum_{\delta \vdash k} g\left(\delta, \gamma_{+} ; \alpha\right) g\left(\delta, \gamma_{-} ; \beta\right)\right)\left(\sum_{\delta^{\prime} \vdash k} g\left(\delta^{\prime}, \gamma_{+} ; \alpha\right) g\left(\delta^{\prime}, \gamma_{-} ; \beta\right)\right) . \tag{C.3}
\end{align*}
$$

Here $g$ is the Littlewood-Richardson coefficient which is defined by

$$
\begin{equation*}
g\left(\delta, \gamma_{+} ; \alpha\right)=\frac{1}{k!} \frac{1}{(m-k)!} \sum_{\sigma_{1} \in S_{k}} \sum_{\sigma_{2} \in S_{m-k}} \chi_{\delta}\left(\sigma_{1}\right) \chi_{\gamma_{+}}\left(\sigma_{2}\right) \chi_{\alpha}\left(\sigma_{1} \circ \sigma_{2}\right) \tag{C.4}
\end{equation*}
$$

To simplify the expresssion (C.3), we use the orthogonality of characters

$$
\begin{equation*}
\sum_{R} \chi_{R}(\sigma) \chi_{R}(\tau)=\delta_{T_{\sigma}, T_{\tau}} \frac{n!}{\left|T_{\sigma}\right|}=\delta_{T_{\sigma}, T_{\tau}} \operatorname{Sym}\left(T_{\sigma}\right) \tag{C.5}
\end{equation*}
$$

where $T_{\sigma}$ is the size of the conjugacy class which contains $\sigma$, and $\operatorname{Sym}\left(T_{\sigma}\right)$ represents the number of elements which commute with $\sigma$ :

$$
\begin{align*}
\operatorname{Sym}\left(T_{\sigma}\right) & =c_{1}(\sigma)!1^{c_{1}(\sigma)} c_{2}(\sigma)!2^{c_{2}(\sigma)} \cdots c_{n}(\sigma)!n^{c_{n}(\sigma)} \\
& =\prod_{i} c_{i}(\sigma)!i^{c_{i}(\sigma)} \tag{C.6}
\end{align*}
$$

where $c_{i}(\sigma)$ represents the number of an $i$-cycle in $\sigma$. Since $\sigma$ is an element of $S_{n}$, we have $\sum_{i=1}^{n} i c_{i}(\sigma)=n$.

Using (C.5), some factors in (C.3) can be rearranged as follows:

$$
\sum_{\gamma_{+}} \sum_{\alpha} g\left(\delta, \gamma_{+} ; \alpha\right) g\left(\delta^{\prime}, \gamma_{+} ; \alpha\right)
$$

$$
\begin{align*}
& =\sum_{\gamma_{+}} \sum_{\alpha}\left(\frac{1}{k!(m-k)!}\right)^{2} \sum_{\sigma_{1}, \sigma_{2}, \tau_{1}, \tau_{2}} \chi_{\delta}\left(\sigma_{1}\right) \chi_{\gamma_{+}}\left(\sigma_{2}\right) \chi_{\alpha}\left(\sigma_{1} \circ \sigma_{2}\right) \chi_{\delta^{\prime}}\left(\tau_{1}\right) \chi_{\gamma_{+}}\left(\tau_{2}\right) \chi_{\alpha}\left(\tau_{1} \circ \tau_{2}\right) \\
& =\left(\frac{1}{k!(m-k)!}\right)^{2} \sum_{\sigma_{1}, \sigma_{2}, \tau_{1}, \tau_{2}} \chi_{\delta}\left(\sigma_{1}\right) \chi_{\delta^{\prime}}\left(\tau_{1}\right) \delta_{T_{\sigma_{2},}, T_{\tau_{2}}} \frac{(m-k)!}{\left|T_{\sigma_{2}}\right|} \delta_{T_{\sigma_{1} \circ \sigma_{2}, T T_{\tau_{1} \circ \tau_{2}}} \frac{m!}{\left|T_{\sigma_{1} \circ \sigma_{2}}\right|}}^{=\frac{m!}{(k!)^{2}(m-k)!} \sum_{\tau_{1}, \tau_{2}} \chi_{\delta}\left(\tau_{1}\right) \chi_{\delta^{\prime}}\left(\tau_{1}\right) \frac{1}{\left|T_{\tau_{2}}\right|} \frac{1}{\left|T_{\tau_{1} \circ \tau_{2}}\right|}\left|T_{\tau_{1}}\right|\left|T_{\tau_{2}}\right|} \\
& =\frac{m!}{(k!)^{2}(m-k)!} \sum_{\tau_{1}, \tau_{2}} \chi_{\delta}\left(\tau_{1}\right) \chi_{\delta^{\prime}}\left(\tau_{1}\right) \frac{1}{\left|T_{\tau_{1} \circ \tau_{2}}\right|}\left|T_{\tau_{1}}\right|
\end{align*}
$$

Then we get the following expression for (C.3)

$$
\begin{aligned}
& N_{s b}(m, n)=\sum_{k=0}^{\min (m, n)} \frac{m!n!}{(k!)^{4}(m-k)!(n-k)!} \\
& \sum_{\delta, \delta^{\prime}} \sum_{\tau_{1}, \tau_{2}} \chi_{\delta}\left(\tau_{1}\right) \chi_{\delta^{\prime}}\left(\tau_{1}\right) \frac{\left|T_{\tau_{1}}\right|}{\left|T_{\tau_{1} \circ \tau_{2}}\right|} \sum_{\rho_{1}, \rho_{2}} \chi_{\delta}\left(\rho_{1}\right) \chi_{\delta^{\prime}}\left(\rho_{1}\right) \frac{\left|T_{\rho_{1}}\right|}{\left|T_{\rho_{1} \circ \rho_{2}}\right|} \\
& =\sum_{k}\binom{m}{k}\binom{n}{k} \sum_{\tau_{1}, \tau_{2}} \sum_{\rho_{1}, \rho_{2}} \delta_{T_{\tau_{1}}, T_{\rho_{1}}} \delta_{T_{\tau_{1}}, T_{\rho_{1}}} \frac{1}{\left|T_{\tau_{1}}\right|} \frac{\left|T_{\tau_{1}}\right|}{\left|T_{\tau_{1} \circ \tau_{2}}\right|} \frac{\left|T_{\rho_{1}}\right|}{\left|T_{\rho_{1} \circ \rho_{2}}\right|} \\
& =\sum_{k}\binom{m}{k}\binom{n}{k} \sum_{\tau_{1} \in S_{k}} \sum_{\tau_{2} \in S_{m-k}} \sum_{\rho_{2} \in S_{n-k}} \frac{1}{\left|T_{\tau_{1} \circ \tau_{2}}\right|} \frac{\left|T_{\tau_{1}}\right|}{\left|T_{\tau_{1} \circ \rho_{2}}\right|} \\
& =\sum_{k}\binom{m}{k}\binom{n}{k} \sum_{T_{\tau_{1}} \in S_{k}} \sum_{T_{\tau_{2}} \in S_{m-k}} \sum_{T_{\rho_{2}} \in S_{n-k}}\left|T_{\tau_{1}}\right|^{2}\left|T_{\tau_{2}}\right|\left|T_{\rho_{2}}\right| \frac{1}{\left|T_{\tau_{1} \circ \tau_{2}}\right|} \frac{1}{\left|T_{\tau_{1} \circ \rho_{2}}\right|} \\
& =\sum_{k} \sum_{T_{\tau_{1}} \in S_{k}} \sum_{T_{\tau_{2}} \in S_{m-k}} \sum_{T_{\rho_{2}} \in S_{n-k}} \frac{\operatorname{Sym}\left(T_{\tau_{1} \circ \tau_{2}}\right)}{\operatorname{Sym}\left(T_{\tau_{1}}\right) \operatorname{Sym}\left(T_{\tau_{2}}\right)} \frac{\operatorname{Sym}\left(T_{\tau_{1} \circ \rho_{2}}\right)}{\operatorname{Sym}\left(T_{\tau_{1}}\right) \operatorname{Sym}\left(T_{\rho_{2}}\right)} \\
& =\sum_{k} \sum_{T_{\tau_{1}} \in S_{k}} \sum_{T_{\tau_{2}} \in S_{m-k}} \sum_{T_{\rho_{2}} \in S_{n-k}} \prod_{l}\binom{c_{l}\left(\tau_{1}\right)+c_{l}\left(\tau_{2}\right)}{c_{l}\left(\tau_{1}\right)}\binom{c_{l}\left(\tau_{1}\right)+c_{l}\left(\rho_{2}\right)}{c_{l}\left(\tau_{1}\right)} .
\end{aligned}
$$

The above can be rewritten as

$$
\begin{equation*}
\sum_{c_{l}(1): \sum l c_{l}(1)=m} \sum_{c_{l}(2): \sum l c_{l}(2)=n} \prod_{l} \sum_{k} \sum_{c_{l}(3): \sum l c_{l}(3)=k}^{\min \left(c_{l}(1) c_{l}(2)\right)} \frac{c_{l}(1)!}{\left(c_{l}(1)-c_{l}(3)\right)!c_{l}(3)!} \frac{c_{l}(2)!}{\left(c_{l}(2)-c_{l}(3)\right)!c_{l}(3)!} . \tag{C.8}
\end{equation*}
$$

We now compare to (5.32), which is the expression

$$
\begin{equation*}
Q_{m t}(m, n)=\sum_{c_{l}(1): \sum_{l} c_{l}(1) l=m} \sum_{c_{l}(2): \sum_{l} c_{l}(2) l=n} \prod_{l} \frac{\left(c_{l}(1)+c_{l}(2)\right)!}{c_{l}(1)!c_{l}(2)!} \tag{C.9}
\end{equation*}
$$

For any fixed cycle length in $S_{m} \times S_{n}$ consider the conjugacy class with $c_{l}(1)$ cycles in $S_{m}$ and $c_{l}(2)$ cycles in $S_{n}$. The factor $\frac{\left(c_{1}(1)+c_{l}(2)\right)!}{c_{l}(1) \cdot c_{l}(2)!}$ is the number of ways of arrangements of $\left(c_{1}(1)+c_{l}(2)\right)$ objects with $c_{l}(1)$ of one kind (say red) and $c_{l}(2)$ of another kind (say blue). Suppose we lay our the objects in a line. We can take the first arrangemnet to be the one with $c_{l}(1)$ reds on the left and $c_{l}(2)$ blues on the right. Then we permute to generate the rest. A general arrangement will have $c_{l}(3)$ blues on the left among $c_{l}(1)-c_{l}(3)$ red objects and $c_{l}(3)$ reds among $c_{l}(2)-c_{l}(3)$ blues on the right. Of these we have $\frac{c_{l}(1)!}{k!\left(c_{l}(1)-c_{l}(3)\right)!} \times \frac{c_{l}(2)!}{k!\left(c_{l}(2)-c_{l}(3)\right)!}$ arrangements. Hence we get

$$
\begin{equation*}
\frac{\left(c_{1}(1)+c_{l}(2)\right)!}{c_{l}(1)!c_{l}(2)!}=\sum_{c_{l}(3)=0}^{\min \left(c_{l}(1), c_{l}(2)\right)} \frac{c_{l}(1)!}{c_{l}(3)!\left(c_{l}(1)-c_{l}(3)\right)!} \times \frac{c_{l}(2)!}{c_{l}(3)!\left(c_{l}(2)-c_{l}(3)\right)!} . \tag{C.10}
\end{equation*}
$$

This proves the desired equality between (C.8) and (C.9), and so we conclude that the two countings (5.32) and (5.33) agree:

$$
\begin{equation*}
N_{s b}(m, n)=Q_{m t}(m, n) \tag{C.11}
\end{equation*}
$$

## D Brauer counting from $G L(N) \times G L(N) \rightarrow G L(N)$ reduction

We now show that the Brauer basis correctly counts invariants under the adjoint $U(N)$ action

$$
\begin{equation*}
Z \rightarrow U Z U^{\dagger} \tag{D.1}
\end{equation*}
$$

We consider invariants under (D.1) constructed from objects of the form:

$$
\begin{equation*}
Z_{j_{1}}^{i_{1}} \cdots Z_{j_{m}}^{i_{m}} Z_{l_{1}}^{\dagger k_{1}} \cdots Z_{l_{n}}^{\dagger k_{n}} . \tag{D.2}
\end{equation*}
$$

As far as counting invariants under $U(N)$ action is concerned, the problem is equivalent to counting invariants under $G L(N)$.

The Lie algebra of $G L(N)$ is just the full Matrix algebra $M(N, \mathbb{C})$ and the symmetric algebra over $S(M(N, \mathbb{C}))$ is decomposed into the direct sums [14]

$$
\begin{equation*}
S(M(N, \mathbb{C}))=\sum_{\lambda} V_{\lambda, N} \otimes\left(V_{\lambda, N}\right)^{*} \tag{D.3}
\end{equation*}
$$

as $G L(N) \otimes G L(N)$ modules. The sum is over partitions with length at most $N$, i.e. Young diagrams with first column no longer than $N$. Restricting to $S_{m}(M(N, \mathbb{C}))$ leads to the restriction $|\lambda|=m$, i.e we are looking at the case of Young diagrams with $m$ boxes.

Decomposing the $G L(N) \times G L(N)$ into $G L(N)$ we have

$$
\begin{equation*}
S_{m}(M(N, \mathbb{C}))=\sum_{\tau, \eta, \nu, \lambda} g(\tau, \eta ; \lambda) g(\tau, \nu ; \lambda) V_{\eta, \nu} \tag{D.4}
\end{equation*}
$$

Here $V_{\eta, \mu}$ is a composite representation of $G L(N)$. Therefore the set of invariants in $S_{m}(M(N, \mathbb{C})) \otimes S_{n}(M(N, \mathbb{C}))$, is

$$
\begin{equation*}
\operatorname{Inv}\left\{\sum_{\tau, \eta, \nu, \lambda} \sum_{\lambda^{\prime}, \tau^{\prime}, \eta^{\prime}, \nu^{\prime}} g(\tau, \eta ; \lambda) g(\tau, \nu ; \lambda) g\left(\tau^{\prime}, \eta^{\prime} ; \lambda^{\prime}\right) g\left(\tau^{\prime}, \nu^{\prime} ; \lambda^{\prime}\right) V_{\eta, \nu} \otimes V_{\eta^{\prime}, \nu^{\prime}}\right\} \tag{D.5}
\end{equation*}
$$

which is nonempty only if $\eta=\nu^{\prime}, \nu=\eta^{\prime}$. Hence the number of invariants is

$$
\begin{equation*}
\sum_{\tau, \eta, \nu, \lambda, \tau^{\prime}, \lambda^{\prime}} g(\tau, \eta, \lambda) g(\tau, \nu, \lambda) g\left(\tau^{\prime}, \nu, \lambda^{\prime}\right) g\left(\tau^{\prime}, \eta, \lambda^{\prime}\right) \tag{D.6}
\end{equation*}
$$

We relabel

$$
\begin{array}{lll}
\lambda \rightarrow \alpha & \tau \rightarrow \gamma_{+} & \eta \rightarrow \delta \\
\lambda^{\prime} \rightarrow \beta & \tau^{\prime} \rightarrow \gamma_{-} & \nu \rightarrow \delta^{\prime} \tag{D.7}
\end{array}
$$

to get

$$
\begin{align*}
& \sum_{\alpha, \beta, \gamma_{+}, \gamma_{-}, \delta, \delta^{\prime}} g\left(\gamma_{+}, \delta ; \alpha\right) g\left(\gamma_{+}, \delta^{\prime}, \alpha\right) g\left(\gamma_{-}, \delta^{\prime}, \beta\right) g\left(\gamma_{-}, \delta, \beta\right) \\
= & \sum_{\alpha, \beta, \gamma_{+}, \gamma_{-}}\left(\sum_{\delta} g\left(\gamma_{+}, \delta ; \alpha\right) g\left(\gamma_{-}, \delta, \beta\right)\right)\left(\sum_{\delta^{\prime}} g\left(\gamma_{+}, \delta^{\prime}, \alpha\right) g\left(\gamma_{-}, \delta^{\prime}, \beta\right)\right) \tag{D.8}
\end{align*}
$$

and using the definition of $M_{\alpha \beta}^{\gamma}$ (4.12) this becomes

$$
\begin{equation*}
\sum_{\gamma, \alpha, \beta}\left(M_{\alpha \beta}^{\gamma}\right)^{2} \tag{D.9}
\end{equation*}
$$

which is $N_{s b}(m, n)$ from (5.33).

## E Proofs for $m=n=k$ projectors

In this appendix, we shall show the operator (5.37) satisfies the following properties:

$$
\begin{equation*}
\left(P_{\alpha}^{k=m}\right)^{2}=P_{\alpha}^{k=m} \tag{E.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{tr}_{k, k}\left(P_{\alpha}^{k=m}\right)=\left(d_{\alpha}\right)^{2} . \tag{E.2}
\end{equation*}
$$

The second equation follows from the Schur-Weyl duality;

$$
\begin{align*}
V^{\otimes k} \otimes \bar{V}^{\otimes k} & =\bigoplus_{\gamma} V_{\gamma}^{U(N)} \otimes V_{\gamma}^{B_{N}(k, k)} \\
& =\bigoplus_{\gamma, A} V_{\gamma}^{U(N)} \otimes V_{A}^{\mathbb{C}\left(S_{k} \times S_{k}\right)} \otimes V_{\gamma \rightarrow A}^{B_{N}(k, k) \rightarrow \mathbb{C}\left(S_{k} \times S_{k}\right)} \tag{E.3}
\end{align*}
$$

In the second line, we have decomposed each irreducible representation $\gamma$ of the Brauer algebra into irreducible representations $A$ of the group algebra of $S_{m} \times S_{n}$. Acting with the projector $P_{\alpha}^{k=m}$ on this equation and taking a trace in $V^{\otimes k} \otimes \bar{V}^{\otimes k}$, we get

$$
\begin{equation*}
\operatorname{tr}_{k, k}\left(P_{\alpha}^{k=m}\right)=d_{(\alpha, \alpha)}=\left(d_{\alpha}\right)^{2} \tag{E.4}
\end{equation*}
$$

where we have used $\operatorname{Dim} \gamma=1$ and $M_{A}^{\gamma}=1$ for $\gamma=(\emptyset, \emptyset, k=m)$.
The $k$-contraction operator $C_{(k)}$ can be written in many ways, for example

$$
\begin{align*}
C_{(k)} & =\sum_{\sigma \in S_{k}} C_{\sigma(1) \overline{1}} \cdots C_{\sigma(k) \bar{k}} \\
& =\sum_{\sigma \in S_{k}} \sigma C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma^{-1} \\
& =\sum_{\bar{\sigma} \in \bar{S}_{k}} \bar{\sigma} C_{1 \overline{1}} \cdots C_{k \bar{k}} \bar{\sigma}^{-1} \tag{E.5}
\end{align*}
$$

The second equality follows from

$$
\begin{equation*}
\sigma C_{i \bar{j}}=C_{\sigma(i) \bar{j}} \sigma \tag{E.6}
\end{equation*}
$$

In order to show (E.1), we first calculate $\left(C_{(k)}\right)^{2}$ :

$$
\begin{align*}
\left(C_{(k)}\right)^{2} & =\sum_{\rho, \sigma \in S_{k}} \rho C_{1 \overline{1}} \cdots C_{k \bar{k}} \rho^{-1} \sigma C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma^{-1} \\
& =\sum_{\rho, \sigma \in S_{k}} \operatorname{tr}_{k}\left(\rho^{-1} \sigma\right) \rho C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma^{-1} \\
& =\sum_{\rho, \sigma \in S_{k}} N^{C_{\rho^{-1}} \sigma} \rho C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma^{-1} \\
& =\sum_{\tau, \sigma \in S_{k}}^{N_{\tau} \tau \sigma C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma^{-1}} \\
& =N^{k} \Omega_{k} C_{(k)} \tag{E.7}
\end{align*}
$$

where $\Omega_{k}$ is the Omega factor defined by

$$
\begin{equation*}
\Omega_{k}=\sum_{\sigma \in S_{k}} N^{C_{\sigma}-k} \sigma \tag{E.8}
\end{equation*}
$$

where $C_{\sigma}$ is the number of cycles in $\sigma$. Using the equation (E.7), we can easily show that the projector (5.37) satisfies (E.1).

We also have another interesting equation for $C_{(k)}$ :

$$
\begin{equation*}
C_{(k)} p_{\alpha}=C_{(k)} \bar{p}_{\alpha} \tag{E.9}
\end{equation*}
$$

which is a consequence of

$$
\begin{equation*}
C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma=C_{1 \overline{1}} \cdots C_{k \bar{k}} \bar{\sigma}^{-1} \tag{E.10}
\end{equation*}
$$

We finally prove (E.2):

$$
\begin{align*}
\operatorname{tr}_{k, k}\left(P_{\alpha}^{k=m}\right) & =\frac{d_{\alpha}}{k!D_{i m \alpha}} \operatorname{tr}_{k, k}\left(C_{(k)} p_{\alpha}\right) \\
& =\frac{d_{\alpha}}{k!\operatorname{Dim\alpha } \alpha} \sum_{\sigma \in S_{k}} \operatorname{tr}_{k, k}\left(\sigma C_{1 \overline{1}} \cdots C_{k \bar{k}} \sigma^{-1} p_{\alpha}\right) \\
& =\frac{d_{\alpha}}{\operatorname{Dim\alpha }} \operatorname{tr}_{k, k}\left(C_{1 \overline{1}} \cdots C_{k \bar{k}} p_{\alpha}\right) \\
& =\frac{d_{\alpha}}{\operatorname{Dim\alpha }} \operatorname{tr}_{k}\left(p_{\alpha}\right) \\
& =\frac{d_{\alpha}}{D_{\alpha} \alpha} d_{\alpha} \operatorname{Dim\alpha } \\
& =\left(d_{\alpha}\right)^{2} . \tag{E.11}
\end{align*}
$$

## F Constructing an inner product on polynomials

The inner product on gauge invariant polynomials $\mathcal{O}\left(Z, Z^{\dagger}\right)$ given in (6.39),

$$
\begin{equation*}
\left(\mathcal{O}_{1}\left(Z, Z^{\dagger}\right), \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right)=\frac{1}{(2 \pi)^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\left(e^{\left.-\square \mathcal{O}_{1}\left(Z, Z^{\dagger}\right)\right)}\left(e^{-\square} \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right) e^{-\operatorname{tr} Z Z^{\dagger}}, ~\right.} \tag{F.1}
\end{equation*}
$$

was introduced by identifying it with the integral representation of the inner product on matrix harmonic oscillator states $|\Psi\rangle$.

In this appendix we show that this inner product can be derived by

- Starting from the inner product arising from the two-point function of the zerodimensional complex matrix model of Ginibre [1],

$$
\begin{equation*}
\left(\mathcal{O}_{1}\left(Z, Z^{\dagger}\right), \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right)_{G}=\frac{1}{(2 \pi)^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\mathcal{O}_{1}\left(Z, Z^{\dagger}\right)} \mathcal{O}_{2}\left(Z, Z^{\dagger}\right) e^{-\operatorname{tr} Z Z^{\dagger}} \tag{F.2}
\end{equation*}
$$

where the normalisation factor is the value of the integral with no insertions;

- Requiring $\mathcal{H}_{A}$ to be hermitian.

We will see that this leads us to the inner product (F.1).
The construction proceeds as follows. We know that $H_{1}, \bar{H}_{1}, H_{2}, \bar{H}_{2}, H_{L}$ have eigenstates given by the Brauer basis polynomials $\mathcal{O}_{\alpha \beta}^{\gamma}(Z, \bar{Z})$ with real eigenvalues. These eigenstates are a complete set of gauge invariant polynomials. So in fact any inner product diagonal in these labels $\gamma, \alpha, \beta$

$$
\begin{equation*}
\left(\mathcal{O}_{\alpha_{1} \beta_{1}}^{\gamma_{1}}, \mathcal{O}_{\alpha_{2} \beta_{2}}^{\gamma_{2}}\right)=f_{\alpha_{1} \beta_{1}}^{\gamma_{1}} \delta^{\gamma_{1} \gamma_{2}} \delta_{\alpha_{1} \alpha_{2}} \delta_{\beta_{1} \beta_{2}} \tag{F.3}
\end{equation*}
$$

for some real $f_{\alpha_{1} \beta_{1}}^{\gamma_{1}}$ will guarantee that $H_{1}, \bar{H}_{1}, H_{2}, \bar{H}_{2}, H_{L}$ are hermitian.
We now give an explicit construction of such an inner product, which is also welldefined for general polynomials in $Z, Z^{\dagger}$, not just gauge-invariant ones.

The basic idea is to define our inner product on degree 1 monomials in $Z, Z^{\dagger}$ and then extend to arbitrary monomials by Wick's theorem.

$$
\begin{align*}
\left(Z^{i}{ }_{j}, Z_{l}^{k}\right) & =\delta^{i k} \delta^{j l} \\
\left(Z^{\dagger i}, Z_{l}^{\dagger}{ }_{l}\right) & =\delta^{i k} \delta_{j l} \\
\left(Z_{j}^{i}, Z^{\dagger k}{ }_{l}\right) & =0 \tag{F.4}
\end{align*}
$$

We generalize to higher degree monomials

$$
\begin{equation*}
\left(Z_{j_{1}}^{i_{1}} \cdots Z_{j_{m}}^{i_{m}} Z_{q_{1}}^{\dagger p_{1}} \cdots Z_{q_{n}}^{\dagger p_{n}}, Z_{l_{1}}^{k_{1}} \cdots Z_{l_{m}}^{k_{m}} Z_{s_{1}}^{\dagger r_{1}} \cdots Z_{s_{n}}^{\dagger r_{n}}\right) \tag{F.5}
\end{equation*}
$$

by summing over different possible pairings of the $Z$ on the left with the $Z$ on the right, and the $Z^{\dagger}$ on the left with the $Z^{\dagger}$ on the right, with each individual pairing being given by (F.4). Very importantly we do not include contractions between pairs $Z$ and $\bar{Z}$ both on the left or both on the right. In the Ginibre inner product (F.2), we have $(1, Z \bar{Z}) \neq 0$ which shows the inner product under construction is different to (F.2).

To prove $G_{2}$ is hermitian with this inner product we first work with the basic pairing.

$$
\begin{align*}
& \left(Z_{j}^{i},\left(G_{2}\right)^{p}{ }_{q} Z_{l}^{k}\right)=\delta^{k}{ }_{q}\left(Z_{j}^{i}, Z_{l}^{p}\right)=\delta^{k}{ }_{q} \delta^{i p} \delta_{j l} \\
& \left(\left(G_{2}\right)^{p}{ }_{q} Z_{j}^{i}, Z_{l}^{k}\right)=\delta_{q}^{i}\left(Z^{p}{ }_{j}, Z_{l}^{k}\right)=\delta_{q}^{i} \delta^{p k} \delta_{j l} \tag{F.6}
\end{align*}
$$

We thus find, on these degree 1 monomials

$$
\begin{equation*}
\left(\left(G_{2}\right)_{q}^{p}\right)^{h}=\left(G_{2}\right)_{p}^{q} \tag{F.7}
\end{equation*}
$$

where $h$ denotes hermitian conjugate. When we consider the action of $\left(G_{2}\right)_{q}^{p}$ on a general pairing

$$
\begin{equation*}
\left(Z_{j_{1}}^{i_{1}} \cdots Z_{j_{m}}^{i_{m}} Z_{q_{1}}^{\dagger p_{1}} \cdots Z_{q_{n}}^{\dagger p_{n}},\left(G_{2}\right)_{q}^{p} Z_{l_{1}}^{k_{1}} \cdots Z_{l_{m}}^{k_{m}} Z_{s_{1}}^{\dagger r_{1}} \cdots Z_{s_{n}}^{\dagger r_{n}}\right) \tag{F.8}
\end{equation*}
$$

we can use the fact that $\left(G_{2}\right)^{p}{ }_{q}$ acts as a derivation, so that the right factor becomes a sum of terms with the $\left(G_{2}\right)^{p}{ }_{q}$ acting on each successive $Z$ or $Z^{\dagger}$. The action on $Z^{\dagger}$ gives zero. For each term in this sum, the inner product is a sum over Wick contractions. For each Wick contraction of the form

$$
\begin{equation*}
(Z, G Z)(Z, Z) \cdots\left(Z^{\dagger}, Z^{\dagger}\right) \cdots \tag{F.9}
\end{equation*}
$$

we can move the $\left(G_{2}\right)^{p}{ }_{q}$ over to the left to give $\left(G_{2}\right)^{q}{ }_{p}$ using (F.6). We can recollect the sum over Wick contractions to get

$$
\begin{equation*}
\left(\left(G_{2}\right)_{p}^{q} Z_{j_{1}}^{i_{1}} \cdots Z_{j_{m}}^{i_{m}} Z_{q_{1}}^{\dagger p_{1}} \cdots Z_{q_{n}}^{\dagger p_{n}}, Z_{l_{1}}^{k_{1}} \cdots Z_{l_{m}}^{k_{m}} Z_{s_{1}}^{\dagger r_{1}} \cdots Z_{s_{n}}^{\dagger r_{n}}\right) . \tag{F.10}
\end{equation*}
$$

This establishes for any monomial in $Z, Z^{\dagger}$ that $\left(\left(G_{2}\right)_{q}^{p}\right)^{h}=\left(G_{2}\right)_{p}^{q}$, and by linearity this extends to any polynomial. Having established

$$
\begin{equation*}
\left(\left(G_{2}\right)_{q}^{p}\right)^{h}=\left(G_{2}\right)_{p}^{q} \tag{F.11}
\end{equation*}
$$

it easily follows that

$$
\begin{equation*}
\left(\operatorname{tr} G_{2}\right)^{h}=\operatorname{tr} G_{2} \quad \text { and } \quad\left(\operatorname{tr} G_{2}^{2}\right)^{h}=\operatorname{tr} G_{2}^{2} \tag{F.12}
\end{equation*}
$$

and similarly we find

$$
\begin{align*}
\left(\left(G_{3}\right)_{j}^{i}\right)^{h} & =\left(G_{3}\right)_{i}^{j} \\
\left(\operatorname{tr} G_{3}\right)^{h} & =\operatorname{tr} G_{3} \\
\left(\operatorname{tr} G_{3}^{2}\right)^{h} & =\operatorname{tr} G_{3}^{2} \\
\left(\operatorname{tr}\left(G_{2} G_{3}\right)\right)^{h} & =\operatorname{tr}\left(G_{3} G_{2}\right)=\operatorname{tr}\left(G_{2} G_{3}\right) \tag{F.13}
\end{align*}
$$

where the last equality follows since the entries of $G_{2}$ and $G_{3}$ commute.
We can also derive the above relations by noting that

$$
\begin{align*}
\left(Z_{j}^{i}\right)^{h} & =\left(\frac{\partial}{\partial Z}\right)_{i}^{j} \\
\left(Z_{j}^{\dagger i}\right)^{h} & =\left(\frac{\partial}{\partial Z^{\dagger}}\right)_{i}^{j} \\
\Rightarrow\left(\left(G_{2}\right)_{q}^{p}\right)^{h} & =\left(Z_{i}^{p}\left(\frac{\partial}{\partial Z}\right)_{q}^{i}\right)^{h}=Z_{i}^{q}\left(\frac{\partial}{\partial Z}\right)_{p}^{i}=\left(G_{2}\right)_{p}^{q} \tag{F.14}
\end{align*}
$$

and similarly for $G_{3}$ etc.
The above proofs work by construction since we have defined our inner product to have the same properties as the oscillator inner product for $A^{\dagger} B^{\dagger}$ and exploited the similarities

$$
\begin{equation*}
\left(G_{2}\right)_{j}^{i} \simeq\left(A^{\dagger} A\right)_{j}^{i}, \quad\left(G_{3}\right)_{j}^{i} \simeq\left(B^{\dagger} B\right)_{j}^{i} . \tag{F.15}
\end{equation*}
$$

We now derive an integral form of this inner product. We begin with (F.2) and normal order by removing all contributions to the inner product from self-contractions in the wavefunctions. It was observed in (5.11) that the laplacian generates Wick contractions so we define (c.f. [56])

$$
\begin{equation*}
: \mathcal{O}\left(Z, Z^{\dagger}\right):=\left(1-\square+\frac{\square^{2}}{2}+\cdots\right) \mathcal{O}\left(Z, Z^{\dagger}\right)=e^{-\square} \mathcal{O}\left(Z, Z^{\dagger}\right) \tag{F.16}
\end{equation*}
$$

and our inner product becomes the following modification of (F.2):

$$
\begin{align*}
\left(\mathcal{O}_{1}\left(Z, Z^{\dagger}\right), \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right) & =\left(: \mathcal{O}_{1}\left(Z, Z^{\dagger}\right):,: \mathcal{O}_{2}\left(Z, Z^{\dagger}\right):\right)_{G} \\
& =\frac{1}{(2 \pi)^{N^{2}}} \int\left[d Z d Z^{\dagger}\right] \overline{\left(e^{-\square} \mathcal{O}_{1}\left(Z, Z^{\dagger}\right)\right)}\left(e^{-\square} \mathcal{O}_{2}\left(Z, Z^{\dagger}\right)\right) e^{-\operatorname{tr} Z Z^{\dagger}} \tag{F.17}
\end{align*}
$$

which is (F.1) as we set out to show.

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[^0]:    ${ }^{1}$ y.kimura@qmul.ac.uk
    ${ }^{2}$ s.ramgoolam@qmul.ac.uk
    ${ }^{3}$ d.j.turton@qmul.ac.uk

[^1]:    ${ }^{1}$ By 'matrix models in $D$ spacetime dimensions' we include random matrix models ( $D=0$ ), matrix quantum mechanics $(D=1)$ or field theories. By 'Gaussian' we mean a quadratic Lagrangian.

[^2]:    ${ }^{2}$ As this paper was being written up, we became aware of [24] which studies this sector and the associated free fermions using a matrix polar decomposition.

