

Effect of the twist operator in the D1D5 CFT

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Abstract

The D1D5 CFT has been very useful in the study of black holes. The interaction in this theory involves a twist operator, which links together different copies of a free CFT. For the bosonic fields, we examine the action of this twist when it links together CFT copies with winding numbers M and N to produce a copy with winding $M + N$. Starting with the vacuum state generates a squeezed state, which we compute. Starting with an initial excitation on one of the copies gives a linear combination of excitations on the final state, which we also compute. These results generalize earlier computations where these quantities were computed for the special case $M = N = 1$. Our results should help in understanding the thermalization process in the D1D5 CFT, which gives the dual of black hole formation in the bulk.

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1 Introduction

String theory has had remarkable success in explaining the quantum physics of black holes. A very useful example has been the D1D5 system - a bound state of N_1 D1 branes and N_5 D5 branes. This bound state and its excitations give a dual description of black holes in 4+1 noncompact dimensions [1, 2].

The dynamics of the D1D5 bound state is given by a 1+1 dimensional CFT. The moduli space of couplings is believed to have an ‘orbifold point’, where the theory is essentially free [3]. At this point the CFT is described by a symmetric product of $N_1 N_5$ copies of a free CFT, where each copy contains 4 bosons and 4 fermions. Since the different copies are symmetrized, the operator content includes twist operators. A twist operator σ_n takes n different copies of the CFT and links them into a single copy living on a circle that is n times longer. We call any such linked set of copies a ‘component string’.

The ‘free’ theory at the orbifold point has been surprisingly successful in reproducing many aspects of black hole physics. The free CFT yields exact agreement with the properties of near-extremal black holes; for example the entropy and greybody factors are reproduced exactly [1, 4].

However, the full dynamics of black holes is not given by the CFT at the orbifold point; we have to deform away from this ‘free’ theory by an operator which corresponds to turning on the coupling constant of the orbifold theory. In particular the process of black hole formation is described in the CFT by the thermalization of an initially non-thermal state; such a process requires nontrivial interactions in the CFT.

The deformation operator describing these interactions has the form of a twist operator σ_2 , dressed with a supercharge: $\hat{O} \sim G_{-\frac{1}{2}} \sigma_2$. The effect of the twist is depicted in Fig. 1. Before the interaction, we have component strings, with windings M, N . The interaction links these component strings together, generating a component string with winding $M + N$.¹

In this paper we study the effects of this interaction. We carry out the following computations:

(a) Firstly, we start with the vacuum state $|0^{(1)}\rangle|0^{(2)}\rangle$ on the initial component strings in Fig. 1. We apply the twist operator σ_2 at a point w_0 . This generates a component string with winding $M + N$, but this component string will not be in its vacuum state $|0\rangle$. As argued in [5], the starting vacuum state gets converted into a state on the final component string with the schematic form

$$|0^{(1)}\rangle|0^{(2)}\rangle \rightarrow |\chi\rangle \sim e^{\sum_{k,l} \gamma_{kl}^B \alpha_{-k} \alpha_{-l}} |0\rangle \quad (1.1)$$

¹In the present paper we focus on the process of two component strings joining together. If the twist operator acts on two strands of the same component string, then it will decompose the component string into two parts. The computations for this case can be done in a similar manner to the computations presented here.

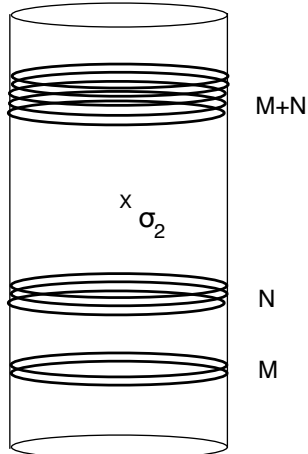


Figure 1: The cylinder with coordinate w . The state before the twist has component strings with windings M, N . The twist operator σ_2 links these into a single component string of winding $M + N$.

where α_k are the oscillator modes of a free scalar field. In [5] the coefficients γ_{kl}^B were computed, for both bosonic and fermionic excitations, for the case where the initial component strings had windings $M = N = 1$. In the present paper we will restrict attention to bosons, but find the γ_{kl}^B for arbitrary M, N .

(b) Secondly, we start with an initial excitation $\alpha_{-m}^{(i)}$ on one of the component strings before the twist; here $i = 1, 2$ labels which component string we excite. After the twist, this excitation gets converted to a linear combination of excitations above the state $|\chi\rangle$,

$$\alpha_{-m}^{(i)} |0^{(1)}\rangle |0^{(2)}\rangle \rightarrow \sum_k f_{mk}^{B(i)} \alpha_{-k} |\chi\rangle, \quad i = 1, 2 \quad (1.2)$$

on the final component string of length $M + N$. The coefficients $f_{mk}^{B(i)}$ were found for $M = N = 1$ in [6]. In the present paper, again restricting to bosonic fields, we find these coefficients for general M, N .

(c) The physics of the black hole is captured by taking $N_1 N_5 \gg 1$. In this limit, component strings will typically have windings $M \gg 1$, and the excitations α_{-m} on the component strings will typically have a wavelength that is much shorter than the length of the component string. Thus we will be interested in a ‘continuum limit’ where the excitations have $m \gg 1$; such excitations are not sensitive to the infra-red cutoff scale set by the length of the component string. We find the continuum limit approximations to the expressions for $\gamma_{kl}^B, f_{mk}^{B(i)}$.

The computation of $\gamma_{kl}^B, f_{mk}^{B(i)}$ for general M, N is somewhat more involved than the

computation for $M = N = 1$, for the following reason. For $M = N = 1$ there are two twist operators: the interaction σ_2 and a twist σ_2 required to generate the component string of length 2 in the final state. For general M, N , there are four twists: twists σ_M, σ_N for the initial component strings, the interaction σ_2 , and a twist σ_{M+N} at infinity for the final component string.

The expressions for $\gamma_{kl}^B, f_{mk}^{B(i)}$ turn out to have a nice structure given in terms of gamma functions. The continuum limit expressions are simpler, and may be of more use in understanding the thermalization problem.

Several other directions have been studied with the twist operator. In [7] intertwining relations were derived for operators before and after the twist. The effect of the twist on entanglement entropy was studied in [8]. Twist-nontwist correlators were computed in [9], and operator mixing was studied in [10]. For other related work, see [11]. Our line of enquiry complements the fuzzball program; for early work, see [12], for reviews see [13] and for recent work, see e.g. [14, 15].

This paper is organized as follows. In Section 2 we introduce the D1D5 CFT and discuss the nature of the twist interaction. In Section 3 we compute the effect of the twist on the vacuum state and in Section 4 we compute the effect of the twist on a state with an initial excitation. In Section 5 we obtain the continuum limit approximations to these quantities. In Section 6 we discuss our results.

2 The D1D5 CFT at the orbifold point

In this section we summarize some properties of the D1D5 CFT at the orbifold point and the deformation operator that we will use to perturb away from the orbifold point. For more details, see [5].

2.1 The D1D5 CFT

Consider type IIB string theory, compactified as

$$M_{9,1} \rightarrow M_{4,1} \times S^1 \times T^4. \quad (2.1)$$

Wrap N_1 D1 branes on S^1 , and N_5 D5 branes on $S^1 \times T^4$. The bound state of these branes is described by a field theory. We think of the S^1 as being large compared to the T^4 , so that at low energies we look for excitations only in the direction S^1 . This low energy limit gives a conformal field theory (CFT) on the circle S^1 .

We can vary the moduli of string theory (the string coupling g , the shape and size of the torus, the values of flat connections for gauge fields etc.). These changes move us to different points in the moduli space of the CFT. It has been conjectured that we can move

to a point called the ‘orbifold point’ where the CFT is particularly simple [3]. At this orbifold point the CFT is a 1+1 dimensional sigma model. We will work in the Euclidean theory, where the base space is a cylinder spanned by the coordinates

$$\tau, \sigma : \quad 0 \leq \sigma < 2\pi, \quad -\infty < \tau < \infty \quad (2.2)$$

The target space of the sigma model is the symmetrized product of $N_1 N_5$ copies of T^4 ,

$$(T^4)^{N_1 N_5} / S_{N_1 N_5}, \quad (2.3)$$

with each copy of T^4 giving 4 bosonic excitations X^1, X^2, X^3, X^4 . It also gives 4 fermionic excitations, which we call $\psi^1, \psi^2, \psi^3, \psi^4$ for the left movers, and $\bar{\psi}^1, \bar{\psi}^2, \bar{\psi}^3, \bar{\psi}^4$ for the right movers. The central charge of the theory with fields $X^i, \psi^i, i = 1 \dots 4$ is $c = 6$. The total central charge of the entire system is thus $6N_1 N_5$.

We will not consider the fermions in this paper; we hope to present their dynamics elsewhere. The bosons can be grouped into a matrix

$$X_{A\dot{A}} = \frac{1}{\sqrt{2}} X_i \sigma_i = \frac{1}{\sqrt{2}} \begin{pmatrix} X_3 + iX_4 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 + iX_4 \end{pmatrix} \quad (2.4)$$

where $\sigma_i = \sigma_1, \sigma_2, \sigma_3, iI$. The 2-point functions are

$$\langle \partial X_{A\dot{A}}(z) \partial X_{B\dot{B}}(w) \rangle = \frac{1}{(z-w)^2} \epsilon_{AB} \epsilon_{\dot{A}\dot{B}}. \quad (2.5)$$

In [5] it was noted that we can write the deformation operator as

$$\hat{O}_{\dot{A}\dot{B}}(w_0) = \left[\frac{1}{2\pi i} \int_{w_0} dw G_{\dot{A}}^-(w) \right] \left[\frac{1}{2\pi i} \int_{\bar{w}_0} d\bar{w} \bar{G}_{\dot{B}}^-(\bar{w}) \right] \sigma_2^{++}(w_0) \quad (2.6)$$

Since we do not consider fermions in the present paper, we ignore the spins on the twist operator and also the action of the supercharge G . That is, for the purposes of our computations, the deformation operator will be simply σ_2 . Throughout the paper we shall write expressions for left-movers only; analogous expressions hold for right-movers.

2.2 Nature of the twist interaction

To understand the effect of such a twist σ_2 , consider a discretization of a 1+1 dimensional bosonic free field X . We can model this field by a collection of point masses joined by springs. This gives a set of coupled harmonic oscillators, and the oscillation amplitude of the masses then gives the field $X(\tau, \sigma)$. Consider such a collection of point masses on two different circles, and let the state in each case be the ground state of the coupled oscillators (Fig. 2(a)). At time τ_0 and position σ_0 , we insert a twist σ_2 . The effect of this twist is to

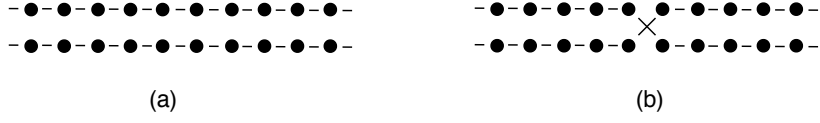


Figure 2: (a) The scalar field on the component strings modeled by point masses joined by springs. (b) The twist operator σ_2 changes the springs so that the masses are linked in a different way.

connect the masses with a different set of springs, so that the masses make a single chain of longer length (Fig. 2(b)).

To see what we might expect of such an interaction, consider a single harmonic oscillator, which starts in its ground state. The Hamiltonian can be expressed in terms of annihilation and creation operators \hat{a} , \hat{a}^\dagger , and the ground state $|0\rangle_a$ is given by $\hat{a}|0\rangle_a = 0$. At time $\tau = \tau_0$, imagine changing the spring constant to a different value. The wavefunction does not change at this instant, and the Lagrangian remains quadratic. But the ground state with this new spring constant is a different state $|0\rangle_b$, and the operator \hat{a} can be expressed as a linear combination of the new annihilation and creation operators \hat{b} , \hat{b}^\dagger :

$$\hat{a} = \alpha\hat{b} + \beta\hat{b}^\dagger \quad (2.7)$$

The wavefunction after this change of coupling can be reexpressed as

$$|0\rangle_a = C e^{-\frac{1}{2}\gamma\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_b \quad (2.8)$$

where $\gamma = \alpha^{-1}\beta$ and C is a constant. If we had a single initial excitation before the twist, it will give a single excitation after the twist, but with a nontrivial coefficient² f

$$\hat{a}^\dagger|0\rangle_a = f\hat{b}^\dagger C e^{-\frac{1}{2}\gamma\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_b. \quad (2.9)$$

Now let us return to the CFT. Regarding the scalar field on the component strings as a set of coupled harmonic oscillators, we note that the twist interaction changes the coupling matrix between the oscillators but not the wavefunction itself. Thus the effect of the twist is captured by reexpressing the state before the twist in terms of the natural oscillators after the twist,

$$\alpha_{AA,k} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi(M+N)} \partial_w X_{AA}(w) e^{\frac{k}{M+N}w} dw. \quad (2.10)$$

In analogy to (2.8) the state after the twist will then have the form

$$\begin{aligned} |\chi\rangle &\equiv \sigma_2(w_0)|0^{(1)}\rangle \otimes |0^{(2)}\rangle \\ &= C(w_0) e^{\sum_{k \geq 1, l \geq 1} \gamma_{kl}^B (-\alpha_{++}, -k\alpha_{--}, -l\alpha_{+-}, -k\alpha_{+-}, -l)} |0\rangle. \end{aligned} \quad (2.11)$$

²We have $f = \alpha^* - \beta^*\gamma$.

The index structure on the α oscillators is arranged to obtain a singlet under the group of rotations in the T^4 . This is explained in detail in [5], where we also fix the normalization $C(w_0)$ to unity after we include the fermions [5]. Since we do not consider fermions in this paper, we do not seek to determine $C(w_0)$; our goal is to find the $\gamma_{kl}^B(w_0)$.

An initial excitation on one of the component strings will transform to a linear combination of excitations on the final component string above the state $|\chi\rangle$. Analogous to (2.9), we will have

$$\sigma_2(w_0)\alpha_{AA,-m}^{(i)}|0\rangle^{(1)} \otimes |0\rangle^{(2)} = \sum_{k \geq 1} f_{mk}^{B(i)} \alpha_{AA,-k} e^{\sum_{k' \geq 1, l' \geq 1} \gamma_{k'l'}^B(-\alpha_{+, -k'} \alpha_{-, -l'} + \alpha_{-, -k'} \alpha_{+, -l'})} |0\rangle. \quad (2.12)$$

where $i = 1, 2$ for the initial component strings with windings M, N respectively. We will find the $f_{mk}^{B(i)}$.

3 Effect of the twist operator on the vacuum state

Consider the process depicted in Fig. 1. We insert the twist operator σ_2 at a location

$$w_0 = \tau_0 + i\sigma_0 \quad (3.1)$$

The twist operator changes the Lagrangian of the theory from one free Lagrangian (that describes free CFTs on circles of length $2\pi M$ and $2\pi N$) to another free Lagrangian (that for a free CFT on a circle of length $2\pi(M + N)$). As explained in [5], in this situation the vacuum state of the initial theory does not go over to the vacuum of the new theory. But the excitations must take the special form of a Gaussian (2.11), so our goal is to find the coefficients γ_{kl}^B .

The steps we follow are analogous to those in [5]. We first map the cylinder w to the complex plane through $z = e^w$. The CFT field X will be multivalued in the z plane, due to the presence of twist operators. The initial component strings with windings M, N are created by twist operators σ_M, σ_N . The interaction is another twist operator σ_2 . The point at infinity has a twist of order $M + N$ corresponding to the component string in the final state.

To handle the twist operators, we go to a covering space t where X is single valued. The twist operators become simple punctures in the t plane, with no insertions at these punctures. We can therefore trivially close these punctures. The nontrivial physics is now encoded in the definition of oscillator modes on the t plane – the creation operators on the cylinder are linear combinations of creation and annihilation operators on the t plane. Performing appropriate Wick contractions, we obtain the γ_{kl}^B . Finally, we change notation to a form that will be more useful in the situation where $k \gg 1, l \gg 1$.

3.1 Modes on the cylinder w

Let us begin by defining operator modes on the cylinder. Below the twist insertion ($\tau < \tau_0$) we have modes $\alpha_{A\dot{A},m}^{(1)}$ on the component string of winding M and modes $\alpha_{A\dot{A},n}^{(2)}$ on the component string of winding N :

$$\alpha_{A\dot{A},m}^{(1)} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi M} \partial_w X_{A\dot{A}}^{(1)}(w) e^{\frac{m}{M}w} dw \quad (3.2)$$

$$\alpha_{A\dot{A},n}^{(2)} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi N} \partial_w X_{A\dot{A}}^{(2)}(w) e^{\frac{n}{N}w} dw \quad (3.3)$$

From (2.5), we find that the commutation relations are

$$[\alpha_{A\dot{A},m}^{(i)}, \alpha_{B\dot{B},n}^{(j)}] = -\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\delta^{ij}m\delta_{m+n,0} \quad (3.4)$$

Above this twist insertion ($\tau > \tau_0$) we have a single component string of winding $M + N$. The modes are

$$\alpha_{A\dot{A},k} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi(M+N)} \partial_w X_{A\dot{A}}(w) e^{\frac{k}{M+N}w} dw \quad (3.5)$$

The commutation relations are

$$[\alpha_{A\dot{A},k}, \alpha_{B\dot{B},l}] = -\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}k\delta_{k+l,0} \quad (3.6)$$

3.2 Modes on the z plane

We wish to go to a covering space where the field $X_{A\dot{A}}$ will be single valued. As a preparatory step, it is convenient to map the cylinder with coordinate w to the plane with coordinate z ,

$$z = e^w \quad (3.7)$$

Under this map the operator modes change as follows. Before the insertion of the twist ($|z| < e^{\tau_0}$) we have, using a contour circling $z = 0$

$$\alpha_{A\dot{A},m}^{(1)} \rightarrow \frac{1}{2\pi} \int_{\arg(z)=0}^{2\pi M} \partial_z X_{A\dot{A}}^{(1)}(z) z^{\frac{m}{M}} dz \quad (3.8)$$

$$\alpha_{AA,n}^{(2)} \rightarrow \frac{1}{2\pi} \int_{\arg(z)=0}^{2\pi N} \partial_z X_{AA}^{(2)}(z) z^{\frac{n}{N}} dz \quad (3.9)$$

After the twist ($|z| > e^{\tau_0}$) we have, using a contour circling $z = \infty$

$$\alpha_{AA,k} \rightarrow \frac{1}{2\pi} \int_{\arg(z)=0}^{2\pi(M+N)} \partial_z X_{AA}(z) z^{\frac{k}{M+N}} dz. \quad (3.10)$$

3.3 Modes on the covering space t

We now proceed to the covering space t where X_{AA} will be single-valued. Consider the map

$$z = t^M (t - a)^N \quad (3.11)$$

The various operator insertions map as follows:

(i) The initial component strings were at $w \rightarrow -\infty$ on the cylinder, which corresponds to $z = 0$ on the z plane. In the t plane, the component string of winding M maps to $t = 0$, while the component string with winding N maps to $t = a$.

(ii) The final component string state is at $w \rightarrow \infty$ on the cylinder, which corresponds to $z = \infty$. This maps to $t = \infty$.

(iii) The twist operator σ_2 is at w_0 on the cylinder, which corresponds to e^{w_0} on the z plane. To find its location on the covering space t , we note that $\frac{dz}{dt}$ should vanish at the location of every twist, since these are ramification points of map (3.11) to the covering space. We find that apart from the ramification points at $t = 0, a, \infty$, the function $\frac{dz}{dt}$ also vanishes at

$$t = \frac{aM}{M+N} \quad (3.12)$$

which corresponds to the following value of z :

$$z_0 = a^{M+N} \frac{M^M N^N}{(M+N)^{M+N}} (-1)^N. \quad (3.13)$$

To solve for a , we must specify how to deal with the fractional exponent. We do this by choosing

$$z_0 = a^{M+N} \frac{M^M N^N}{(M+N)^{M+N}} e^{i\pi N} \quad (3.14)$$

which determines the quantity a in terms of $z_0 = e^{w_0}$, as

$$a = e^{-i\pi \frac{N}{M+N}} \left(\frac{z_0}{M^M N^N} \right)^{\frac{1}{M+N}} (M+N). \quad (3.15)$$

Now let us consider the modes in the t plane. Before the twist we have

$$\alpha_{AA,m}^{(1)} \rightarrow \frac{1}{2\pi} \oint_{t=0} dt \partial_t X_{A\dot{A}}(t) (t^M(t-a)^N)^{\frac{m}{M}} \quad (3.16)$$

$$\alpha_{AA,n}^{(2)} \rightarrow \frac{1}{2\pi} \oint_{t=a} dt \partial_t X_{A\dot{A}}(t) (t^M(t-a)^N)^{\frac{n}{N}} \quad (3.17)$$

After the twist we have

$$\alpha_{AA,k} \rightarrow \frac{1}{2\pi} \oint_{t=\infty} dt \partial_t X_{A\dot{A}}(t) (t^M(t-a)^N)^{\frac{k}{M+N}} \quad (3.18)$$

We also define mode operators that are natural to the t plane, as follows:

$$\tilde{\alpha}_{AA,m} \equiv \frac{1}{2\pi} \oint_{t=0} dt \partial_t X_{A\dot{A}}(t) t^m \quad (3.19)$$

The commutation relations are

$$[\tilde{\alpha}_{AA,k}, \tilde{\alpha}_{B\dot{B},l}] = -\epsilon_{AB} \epsilon_{\dot{A}\dot{B}} k \delta_{k+l,0} \quad (3.20)$$

3.4 Method for finding the γ_{kl}^B

Let us consider the amplitude

$$\begin{aligned} \mathcal{A}_1 &= \langle 0 | \sigma_2(w_0) | 0 \rangle^{(1)} \otimes | 0 \rangle^{(2)} \\ &= C(w_0) \langle 0 | e^{\sum_{k \geq 1, l \geq 1} \gamma_{kl}^B [-\alpha_{++}, -k\alpha_{--}, -l\alpha_{-+}, -k\alpha_{+-}, -l]} | 0 \rangle \\ &= C(w_0) \end{aligned} \quad (3.21)$$

where we assume that the vacuum is normalized as $\langle 0 | 0 \rangle = 1$.

We compare this to the amplitude

$$\begin{aligned} \mathcal{A}_2 &= \langle 0 | \left(\alpha_{++l} \alpha_{--k} \right) \sigma_2(w_0) | 0 \rangle^{(1)} \otimes | 0 \rangle^{(2)} \\ &= C(w_0) \langle 0 | \left(\alpha_{++l} \alpha_{--k} \right) e^{\sum_{k' \geq 1, l' \geq 1} \gamma_{k'l'}^B [-\alpha_{++}, -k'\alpha_{--}, -l'+\alpha_{-+}, -k'\alpha_{+-}, -l']} | 0 \rangle \\ &= -C(w_0) kl \gamma_{kl}^B \langle 0 | 0 \rangle = -C(w_0) kl \gamma_{kl}^B \end{aligned} \quad (3.22)$$

Thus we see that

$$\gamma_{kl}^B = -\frac{1}{kl} \frac{\mathcal{A}_2}{\mathcal{A}_1}. \quad (3.23)$$

To compute \mathcal{A}_1 we map the cylinder w to the plane z and then to the cover t . In this cover the locations of the twist operators are just punctures with no insertions. Thus these

punctures can be closed, making the t space just a sphere. Closing the punctures involves normalization factors, so we write

$$\mathcal{A}_1 = \langle 0 | \sigma_2(w_0) | 0 \rangle^{(1)} \otimes | 0 \rangle^{(2)} = D(z_0) {}_t \langle 0 | 0 \rangle_t \quad (3.24)$$

Factors like $D(z_0)$ were computed in [16], but here we do not need to compute $D(z_0)$ since it will cancel in the ratio $\mathcal{A}_2/\mathcal{A}_1$. We have

$$\mathcal{A}_2 = \langle 0 | \left(\alpha_{++ , l} \alpha_{-- , k} \right) \sigma_2(w_0) | 0 \rangle^{(1)} \otimes | 0 \rangle^{(2)} = D(z_0) {}_t \langle 0 | \left(\alpha'_{++ , l} \alpha'_{-- , k} \right) | 0 \rangle_t \quad (3.25)$$

where the primes on the operators on the RHS signify the fact that these operators arise from the unprimed operators by the various maps leading to the t plane description. Thus we have

$$\gamma_{kl}^B = -\frac{1}{kl} \frac{\mathcal{A}_2}{\mathcal{A}_1} = -\frac{1}{kl} \frac{{}_t \langle 0 | \left(\alpha'_{++ , l} \alpha'_{-- , k} \right) | 0 \rangle_t}{{}_t \langle 0 | 0 \rangle_t}. \quad (3.26)$$

3.5 Computing the γ_{kl}^B

The operators α' are given by contour integrals at large t :

$$\begin{aligned} & {}_t \langle 0 | \left(\alpha'_{++ , l} \alpha'_{-- , k} \right) | 0 \rangle_t = \\ & {}_t \langle 0 | \left(\frac{1}{2\pi} \int dt_1 \partial_t X_{++}(t_1) (t_1^M (t_1 - a)^N)^{\frac{l}{M+N}} \right) \left(\frac{1}{2\pi} \int dt_2 \partial_t X_{--}(t_2) (t_2^M (t_2 - a)^N)^{\frac{k}{M+N}} \right) | 0 \rangle_t \end{aligned} \quad (3.27)$$

with $|t_1| > |t_2|$. We have³

$$(t_1^M (t_1 - a)^N)^{\frac{l}{M+N}} = t_1^l (1 - at_1^{-1})^{\frac{lN}{M+N}} = t_1^l \sum_{p \geq 0} \frac{lN}{M+N} C_p(-a)^p t_1^{-p} = \sum_{p \geq 0} \frac{lN}{M+N} C_p(-a)^p t_1^{l-p} \quad (3.28)$$

$$(t_2^M (t_2 - a)^N)^{\frac{k}{M+N}} = t_2^k (1 - at_2^{-1})^{\frac{kN}{M+N}} = t_2^k \sum_{q \geq 0} \frac{kN}{M+N} C_q(-a)^q t_2^{-q} = \sum_{q \geq 0} \frac{kN}{M+N} C_q(-a)^q t_2^{k-q} \quad (3.29)$$

Thus

$$\begin{aligned} \frac{1}{2\pi} \int dt_1 \partial_t X_{++}(t_1) (t_1^M (t_1 - a)^N)^{\frac{l}{M+N}} &= \sum_{p \geq 0} \frac{lN}{M+N} C_p(-a)^p \tilde{\alpha}_{++ , l-p} \\ \frac{1}{2\pi} \int dt_2 \partial_t X_{--}(t_2) (t_2^M (t_2 - a)^N)^{\frac{k}{M+N}} &= \sum_{q \geq 0} \frac{kN}{M+N} C_q(-a)^q \tilde{\alpha}_{-- , k-q} \end{aligned} \quad (3.30)$$

³The symbol ${}^n C_m$ is the binomial coefficient, also written $\binom{n}{m}$.

We then find

$$\gamma_{kl}^B = -\frac{1}{kl} \sum_{p \geq 0} \sum_{q \geq 0} \frac{t^{lN}}{M+N} C_p \frac{t^{kN}}{M+N} C_q (-a)^{p+q} \frac{{}_t \langle 0 | \tilde{\alpha}_{++ , l-p} \tilde{\alpha}_{--, k-q} | 0 \rangle_t}{{}_t \langle 0 | 0 \rangle_t} \quad (3.31)$$

Using the commutation relations (3.20) we get

$$k - q = -(l - p) \Rightarrow q = (k + l) - p \quad (3.32)$$

This gives

$$\gamma_{kl}^B = \frac{(-a)^{k+l}}{kl} \sum_{p \geq 0} \frac{t^{lN}}{M+N} C_p \frac{t^{kN}}{M+N} C_{(k+l)-p} (l - p) \quad (3.33)$$

Note that in order to give a non-zero contribution, $\tilde{\alpha}_{++ , l-p}$ needs to be an annihilation operator, so we require $p \leq l$. Thus we have

$$\gamma_{kl}^B = \frac{(-a)^{k+l}}{kl} \sum_{p=0}^l \frac{t^{lN}}{M+N} C_p \frac{t^{kN}}{M+N} C_{(k+l)-p} (l - p) \quad (3.34)$$

Evaluating this sum gives

$$\gamma_{kl}^B = -\frac{(-a)^{k+l} \sin\left[\frac{\pi M k}{M+N}\right] \sin\left[\frac{\pi M l}{M+N}\right]}{\pi^2} \frac{MN}{(M+N)^2 (k+l)} \frac{1}{\Gamma[k]} \frac{\Gamma\left[\frac{Mk}{M+N}\right] \Gamma\left[\frac{Nk}{M+N}\right] \Gamma\left[\frac{Ml}{M+N}\right] \Gamma\left[\frac{Nl}{M+N}\right]}{\Gamma[l]}. \quad (3.35)$$

3.6 Expressing γ_{kl}^B in final form

It will be convenient to write the expression for γ_{kl}^B in a slightly different notation. We make the following changes:

(i) We can replace the parameter a by the variable $z_0 = e^{w_0}$ through (3.15).

(ii) In addressing the continuum limit it is useful to use fractional mode numbers defined as

$$s = \frac{k}{M+N}, \quad s' = \frac{l}{M+N} \quad (3.36)$$

The parameters s, s' directly give the physical wavenumbers of the modes on the cylinder with coordinate w . When using s, s' in place of k, l , we will write

$$\gamma_{kl}^B \rightarrow \tilde{\gamma}_{ss'}^B. \quad (3.37)$$

(iii) We define the useful shorthand notation

$$1 - e^{2\pi i M s} = \mu_s. \quad (3.38)$$

Note that

$$(-1)^k \sin\left[\frac{\pi Mk}{M+N}\right] = (-1)^{(M+N)s} \sin(\pi Ms) = \frac{i}{2}(-1)^{Ns} (1 - e^{2\pi i Ms}) = \frac{i}{2}(-1)^{Ns} \mu_s. \quad (3.39)$$

With these changes of notation we find

$$\tilde{\gamma}_{ss'}^B = \frac{1}{4\pi^2} z_0^{s+s'} \frac{\mu_s \mu_{s'}}{s+s'} \frac{MN}{(M+N)^3} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{s+s'} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} \frac{\Gamma[Ms']\Gamma[Ns']}{\Gamma[(M+N)s']}. \quad (3.40)$$

3.7 The case $M = N = 1$

In [5] the $\gamma_{ss'}^B$ were computed for the case $M = N = 1$. Let us check that our general result reduces to the result in [5] for these parameters.

Our general expression for $\gamma_{ss'}^B$ is given in (3.40). For $M = N = 1$ we have

$$\Gamma[(M+N)s] = \Gamma[2s] = \frac{2^{2s-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \Gamma[s] \Gamma\left[s + \frac{1}{2}\right] \quad (3.41)$$

Substituting in (3.40) we get

$$\tilde{\gamma}_{ss'}^B = \frac{z_0^{s+s'}}{2\pi(s+s')} \frac{\Gamma[s]}{\Gamma\left[s + \frac{1}{2}\right]} \frac{\Gamma[s']}{\Gamma\left[s' + \frac{1}{2}\right]} \quad (3.42)$$

which agrees with the result in [5].

4 Effect of the twist operator on an initial excitation

We now turn to the case where one of the initial component strings has an oscillator excitation. We denote this excitation by $\alpha_{AA,-m}^{(1)}$ for the component string with winding M and by $\alpha_{AA,-m}^{(2)}$ for the component string with winding N . We write

$$\sigma_2(w_0) \alpha_{AA,-m}^{(1)} |0\rangle^{(1)} \otimes |0\rangle^{(2)} = \sum_{k \geq 1} f_{mk}^{B(1)} \alpha_{AA,-k} e^{\sum_{k' \geq 1, l' \geq 1} \gamma_{k'l'}^B (-\alpha_{++,-k'} \alpha_{--,-l'} + \alpha_{+,-k'} \alpha_{+,-l'})} |0\rangle \quad (4.1)$$

$$\sigma_2(w_0) \alpha_{AA,-m}^{(2)} |0\rangle^{(1)} \otimes |0\rangle^{(2)} = \sum_{k \geq 1} f_{mk}^{B(2)} \alpha_{AA,-k} e^{\sum_{k' \geq 1, l' \geq 1} \gamma_{k'l'}^B (-\alpha_{++,-k'} \alpha_{--,-l'} + \alpha_{+,-k'} \alpha_{+,-l'})} |0\rangle \quad (4.2)$$

In this section we find the functions $f_{mk}^{B(i)}$.

4.1 Method for finding the $f_{mk}^{B(i)}$

Analogously to the computation of γ_{kl}^B , let us consider the amplitude

$$\begin{aligned}
\mathcal{A}_3 &= \langle 0 | \alpha_{--,k} \sigma_2(w_0) \alpha_{++,-m}^{(1)} | 0 \rangle^{(1)} \otimes | 0 \rangle^{(2)} \\
&= C(w_0) \sum_{l \geq 1} f_{ml}^{B(1)} \langle 0 | \alpha_{--,k} \alpha_{++,-l} e^{\sum_{k' \geq 1, l' \geq 1} \gamma_{k'l'}^B (-\alpha_{++,-k'} \alpha_{--,-l'} + \alpha_{-+,-k'} \alpha_{+-,-l'})} | 0 \rangle \\
&= C(w_0) \sum_{l \geq 1} f_{ml}^{B(1)} (-k) \delta_{kl} \\
&= -C(w_0) k f_{mk}^{B(1)}. \tag{4.3}
\end{aligned}$$

In the second step above, we note that there is also a contribution when $\alpha_{--,k}$ contracts with the terms in the exponential, but this contribution consists of $\alpha_{++,-k'}$ with $k' > 0$. Such oscillators annihilate the vacuum $\langle 0 |$, and so this contribution in fact vanishes.

Thus we see that

$$f_{mk}^{B(1)} = -\frac{1}{k} \frac{\mathcal{A}_3}{\mathcal{A}_1}. \tag{4.4}$$

We have

$$\mathcal{A}_3 = \langle 0 | \alpha_{--,k} \sigma_2(w_0) \alpha_{++,-m}^{(1)} | 0 \rangle^{(1)} \otimes | 0 \rangle^{(2)} = D(z_0) {}_t \langle 0 | \alpha'_{--,k} \alpha'_{++,-m}^{(1)} | 0 \rangle_t \tag{4.5}$$

Thus we obtain

$$f_{mk}^{B(1)} = -\frac{1}{k} \frac{\mathcal{A}_3}{\mathcal{A}_1} = -\frac{1}{k} \frac{{}_t \langle 0 | \alpha'_{--,k} \alpha'_{++,-m}^{(1)} | 0 \rangle_t}{{}_t \langle 0 | 0 \rangle_t}. \tag{4.6}$$

4.2 Computing $f_{mk}^{B(1)}$

Let us now carry out the details of the computation we outlined above.

The operator $\alpha'_{--,k}$ is applied at $w = \infty$, and is thus given in the t plane by a contour at large t . The operator $\alpha'_{++,-m}$ on the other hand is applied at $w = -\infty$ to the component string with winding M , and is thus given in the t plane by a contour around $t = 0$. Thus we get

$$\begin{aligned}
{}_t \langle 0 | \alpha'_{--,k} \alpha'_{++,-m}^{(1)} | 0 \rangle_t &= {}_t \langle 0 | \left(\frac{1}{2\pi} \int_{t_1=\infty} dt_1 \partial_t X_{--}(t_1) (t_1^M (t_1 - a)^N)^{\frac{k}{M+N}} \right) \\
&\quad \times \left(\frac{1}{2\pi} \int_{t_2=0} dt_2 \partial_t X_{++}(t_2) (t_2^M (t_2 - a)^N)^{-\frac{m}{M}} \right) | 0 \rangle_t. \tag{4.7}
\end{aligned}$$

Since t_1 is large, we expand as

$$(t_1^M (t_1 - a)^N)^{\frac{k}{M+N}} = \sum_{p' \geq 0} \frac{Nk}{M+N} C_{p'} (-a)^{p'} t_1^{k-p'} \tag{4.8}$$

Thus we find

$$\alpha'_{A\dot{A},k} = \sum_{p' \geq 0} \frac{Nk}{M+N} C_{p'} (-a)^{p'} \tilde{\alpha}_{A\dot{A},k-p'} . \quad (4.9)$$

Next, since $t_2 \approx 0$ we expand as

$$\begin{aligned} (t_2^M (t_2 - a)^N)^{-\frac{m}{M}} &= t_2^{-m} (t_2 - a)^{-\frac{mN}{M}} = t_2^{-m} (-a)^{-\frac{mN}{M}} \left(1 - \frac{t_2}{a}\right)^{-\frac{mN}{M}} \\ &= t_2^{-m} (-a)^{-\frac{mN}{M}} \sum_{p \geq 0} \frac{-\frac{mN}{M}}{p} C_p (-a)^{-p} t_2^p = \sum_{p \geq 0} \frac{-\frac{mN}{M}}{p} C_p (-a)^{-\frac{mN}{M}-p} t_2^{p-m} \end{aligned} \quad (4.10)$$

Thus we find

$$\alpha'^{(1)}_{A\dot{A},-m} = \sum_{p \geq 0} \frac{-\frac{mN}{M}}{p} C_p (-a)^{-\frac{mN}{M}-p} \tilde{\alpha}_{A\dot{A},p-m} \quad (4.11)$$

Since $\tilde{\alpha}_{A\dot{A},p-m}|0\rangle_t = 0$ for $p - m \geq 0$, we get a contribution only from $p < m$ in the above sum. Thus we have

$$\alpha'^{(1)}_{A\dot{A},-m} = \sum_{p=0}^{m-1} \frac{-\frac{mN}{M}}{p} C_p (-a)^{-\frac{mN}{M}-p} \tilde{\alpha}_{A\dot{A},p-m} \quad (4.12)$$

Using these expansions, we obtain

$$\begin{aligned} & {}_t\langle 0 | \alpha'_{--,k} \alpha'^{(1)}_{++, -m} | 0 \rangle_t \\ &= \sum_{p=0}^{m-1} \sum_{p' \geq 0} \frac{-\frac{mN}{M}}{p} C_p \frac{Nk}{M+N} C_{p'} (-a)^{-\frac{mN}{M}-p+p'} {}_t\langle 0 | \tilde{\alpha}_{--,k-p'} \tilde{\alpha}_{++,p-m} | 0 \rangle_t \\ &= \sum_{p=0}^{m-1} \sum_{p' \geq 0} \frac{-\frac{mN}{M}}{p} C_p \frac{Nk}{M+N} C_{p'} (-a)^{-\frac{mN}{M}-p+p'} (p' - k) \delta_{k-p'+p-m,0} {}_t\langle 0 | 0 \rangle_t \\ &= (-a)^{k - \frac{(M+N)m}{M}} \sum_{p=\max(m-k,0)}^{m-1} \frac{-\frac{mN}{M}}{p} C_p \frac{Nk}{M+N} C_{k+p-m} (p - m) {}_t\langle 0 | 0 \rangle_t \\ &= -\frac{(-1)^m k \sin(\pi \frac{Mk}{M+N})}{\pi(M+N)} \frac{(-a)^{k - \frac{(M+N)m}{M}}}{\frac{k}{M+N} - \frac{m}{M}} \frac{\Gamma[\frac{(M+N)m}{M}]}{\Gamma[m] \Gamma[\frac{Nm}{M}]} \frac{\Gamma[\frac{Nk}{M+N}] \Gamma[\frac{-Mk}{M+N}]}{\Gamma[k]} {}_t\langle 0 | 0 \rangle_t \end{aligned} \quad (4.13)$$

and using (4.6), we obtain

$$f_{mk}^{B(1)} = \frac{(-1)^m \sin(\pi \frac{Mk}{M+N})}{\pi(M+N)} \frac{(-a)^{k - \frac{(M+N)m}{M}}}{\frac{k}{M+N} - \frac{m}{M}} \frac{\Gamma[\frac{(M+N)m}{M}]}{\Gamma[m] \Gamma[\frac{Nm}{M}]} \frac{\Gamma[\frac{Nk}{M+N}] \Gamma[\frac{Mk}{M+N}]}{\Gamma[k]} . \quad (4.14)$$

Note that in the above expression, when we have

$$\frac{k}{M+N} = \frac{m}{M}, \quad (4.15)$$

both numerator and denominator vanish, since

$$\sin\left(\pi \frac{Mk}{M+N}\right) = \sin(\pi m) = 0. \quad (4.16)$$

Since the above expression for $f_{mk}^{B(1)}$ is indeterminate in this situation, we return to the sum in the third line of (4.13), and take parameter values

$$m = Mc, \quad k = (M+N)c. \quad (4.17)$$

Here $c = \frac{j}{Y}$, where j is a positive integer and $Y = \gcd(M, N)$. Then we have

$$\begin{aligned} {}_t\langle 0 | \alpha'_{--,k} \alpha'^{(1)}_{++,-m} | 0 \rangle_t &= (-a)^{-\frac{(M+N)m}{M} + k} \sum_{p=\max(m-k,0)}^{m-1} {}^{-\frac{mN}{M}} C_p {}^{\frac{Nk}{M+N}} C_{k+p-m} (p-m) \\ &= \sum_{p=0}^{m-1} {}^{-Nc} C_p {}^{Nc} C_{Nc+p} (p-Mc). \end{aligned} \quad (4.18)$$

We note that since Nc is a positive integer,

$${}^{Nc} C_{Nc+p} = 0, \quad p > 0 \quad (4.19)$$

Thus only the $p = 0$ term survives in the above sum, and we get

$${}_t\langle 0 | \alpha'_{--,k} \alpha'^{(1)}_{++,-m} | 0 \rangle_t = -Mc \quad (4.20)$$

Thus

$$f_{mk}^{B(1)} = \frac{Mc}{k} = \frac{Mc}{(M+N)c} = \frac{M}{M+N}. \quad (4.21)$$

So for this special case, we write

$$f_{mk}^{B(1)} \Big|_{\frac{k}{M+N} = \frac{m}{M}} = \frac{M}{M+N}. \quad (4.22)$$

Using the gamma function identity (3.41), one can check that the results (4.14), (4.22) agree with the corresponding expression obtained in [6] for the case $M = N = 1$.

4.3 Expressing $f_{mk}^{B(1)}$ in final form

We make the same changes of notation that we did for the γ_{lk}^B :

- (i) We replace a by z_0 .
- (ii) We use fractional modes

$$q = \frac{m}{M}, \quad s = \frac{k}{M+N}. \quad (4.23)$$

When expressing our result using q and s in place of m and k , we will write

$$f_{mj}^{B(1)} \rightarrow \tilde{f}_{qs}^{B(1)} \quad (4.24)$$

- (iii) We again use the shorthand $\mu_s = (1 - e^{2\pi i Ms})$. In doing this we note that

$$(-1)^m \sin\left(\pi \frac{Mk}{M+N}\right) = (-1)^{Mq} \sin(\pi Ms) = \frac{i}{2} e^{-i\pi M(s-q)} (1 - e^{2\pi i Ms}). \quad (4.25)$$

With these changes of notation we get, for $s \neq q$,

$$\tilde{f}_{qs}^{B(1)} = \frac{i}{2\pi} z_0^{s-q} \frac{\mu_s}{s-q} \frac{1}{(M+N)} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{(s-q)} \frac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} \quad (4.26)$$

and for $s = q$ we have

$$\tilde{f}_{qs}^{B(1)}|_{q=s} = \frac{M}{M+N}. \quad (4.27)$$

4.4 Computing $f_{mk}^{B(2)}$

The expression for $f_{mk}^{B(2)}$ can be obtained by a similar computation. There are only a few changes, which we mention here. In place of (4.7) we have

$$\begin{aligned} {}_t\langle 0 | \alpha'_{--,k} \alpha'^{(2)}_{++,-m} | 0 \rangle_t &= {}_t\langle 0 | \left(\frac{1}{2\pi} \int_{t_1=\infty} dt_1 \partial_t X_{--}(t_1) (t_1^M (t_1 - a)^N)^{\frac{k}{M+N}} \right) \\ &\quad \times \left(\frac{1}{2\pi} \int_{t_2=a} dt_2 \partial_t X_{++}(t_2) (t_2^M (t_2 - a)^N)^{-\frac{m}{N}} \right) | 0 \rangle_t \end{aligned} \quad (4.28)$$

The power $\frac{m}{M}$ has been replaced by $\frac{m}{N}$, and the t_2 contour is now around $t_2 = a$ instead of around $t_2 = 0$. We define a shifted coordinate in the t plane

$$t' = t - a \quad (4.29)$$

which gives

$$\begin{aligned}
{}_t\langle 0|\alpha'_{--,k}\alpha'_{++, -m}|0\rangle_t &= {}_t\langle 0|\left(\frac{1}{2\pi}\int_{t'_1=\infty} dt'_1\partial_t X_{--}(t'_1)\left(t'_1{}^N(t'_1+a)^M\right)^{\frac{k}{M+N}}\right) \\
&\quad \times \left(\frac{1}{2\pi}\int_{t'_2=0} dt'_2\partial_t X_{++}(t'_2)\left(t'_2{}^N(t'_2+a)^M\right)^{-\frac{m}{N}}\right)|0\rangle_t. \quad (4.30)
\end{aligned}$$

This expression is the same as (4.7) with the replacements

$$M \rightarrow N \quad N \rightarrow M \quad a \rightarrow -a. \quad (4.31)$$

Thus $f_{mk}^{B(2)}$ can be obtained from (4.26) with the above replacements. For $\frac{k}{M+N} \neq \frac{m}{N}$ this gives:

$$f_{mk}^{B(2)} = \frac{(-1)^m \sin\left(\pi\frac{Nk}{M+N}\right)}{\pi(M+N)} \frac{a^{k-\frac{(M+N)m}{N}}}{\frac{k}{M+N} - \frac{m}{N}} \frac{\Gamma\left[\frac{(M+N)m}{N}\right]}{\Gamma[m]\Gamma\left[\frac{Mm}{N}\right]} \frac{\Gamma\left[\frac{Nk}{M+N}\right]\Gamma\left[\frac{Mk}{M+N}\right]}{\Gamma[k]} \quad (4.32)$$

and we have the special case

$$f_{mk}^{B(2)}\Big|_{\frac{k}{M+N}=\frac{m}{N}} = \frac{N}{M+N}. \quad (4.33)$$

We can again use fractional modes

$$r = \frac{m}{N}, \quad s = \frac{k}{M+N} \quad (4.34)$$

When expressing our result using r and s in place of m and k , we will write

$$f_{mk}^{B(2)} \rightarrow \tilde{f}_{rs}^{B(2)}. \quad (4.35)$$

We then have for $r \neq s$:

$$\tilde{f}_{rs}^{B(2)} = -\frac{i}{2\pi} z_0^{s-r} \frac{\mu_s}{s-r} \frac{1}{(M+N)} \left(\frac{(M+N)^{M+N}}{M^M N^N}\right)^{(s-r)} \frac{\Gamma[(M+N)r]}{\Gamma[Mr]\Gamma[Nr]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} \quad (4.36)$$

while for $r = s$:

$$\tilde{f}_{rs}^{B(2)}\Big|_{r=s} = \frac{N}{M+N}. \quad (4.37)$$

4.5 Summarizing the result for $\tilde{f}_{qs}^{B(1)}$, $\tilde{f}_{rs}^{B(2)}$

For convenient reference, we record the result for $\tilde{f}_{qs}^{B(1)}$, $\tilde{f}_{rs}^{B(2)}$ for general M, N :

$$\tilde{f}_{qs}^{B(1)} = \begin{cases} \frac{M}{M+N} & q = s \\ -\frac{i}{2\pi} z_0^{s-q} \frac{\mu_s}{s-q} \frac{1}{(M+N)} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{(s-q)} \frac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} & q \neq s \end{cases} \quad (4.38)$$

$$\tilde{f}_{rs}^{B(2)} = \begin{cases} \frac{N}{M+N} & r = s \\ -\frac{i}{2\pi} z_0^{s-r} \frac{\mu_s}{s-r} \frac{1}{(M+N)} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{(s-r)} \frac{\Gamma[(M+N)r]}{\Gamma[Mr]\Gamma[Nr]} \frac{\Gamma[M_s]\Gamma[N_s]}{\Gamma[(M+N)s]} & r \neq s \end{cases} \quad (4.39)$$

5 The continuum limit

While we have obtained the exact expressions for γ_{kl}^B , $f_{qs}^{B(1)}$, $f_{rs}^{B(2)}$, the nature of the physics implied by these expressions may not be immediately clear because the expressions look somewhat involved. We will comment on the structure of these expressions in the discussion section below. But first we note that these expressions simplify considerably in the limit where the arguments q, r, s are much larger than unity. We call the resulting approximation the ‘continuum limit’, since large mode numbers correspond to short wavelengths, and at short wavelength the physics is not sensitive to the finite length of the component string. Thus the expressions in this continuum limit describe the results obtained in the limit of *infinite* component strings. Such expressions are useful for the following reason. When the D1D5 system is used to describe a black hole, then the total winding $N_1 N_5$ of the component strings is very large, and so one expects the individual component strings to have large winding as well. Having long component strings (M, N much larger than unity) is approximately equivalent to holding M, N fixed and taking q, r, s large.

5.1 Continuum limit for the γ_{kl}^B

Let us start by looking at γ_{kl}^B . We have the exact expression

$$\tilde{\gamma}_{ss'}^B = \frac{1}{4\pi^2} z_0^{s+s'} \frac{\mu_s \mu_{s'}}{s+s'} \frac{MN}{(M+N)^3} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{s+s'} \frac{\Gamma[M_s]\Gamma[N_s]}{\Gamma[(M+N)s]} \frac{\Gamma[M_{s'}]\Gamma[N_{s'}]}{\Gamma[(M+N)s']}. \quad (5.1)$$

We wish to find an approximation for this expression when

$$s \gg 1, \quad s' \gg 1 \quad (5.2)$$

We have the basic identity, for positive integer K :

$$\Gamma[x]\Gamma\left[x + \frac{1}{K}\right]\Gamma\left[x + \frac{2}{K}\right] \dots \Gamma\left[x + \frac{K-1}{K}\right] = (2\pi)^{\frac{K-1}{2}} K^{\frac{1}{2}-Kx} \Gamma[Kx] \quad (5.3)$$

Using this identity, we get

$$\Gamma[M s] = \frac{1}{(2\pi)^{\frac{M-1}{2}} M^{\frac{1}{2}-Ms}} \Gamma[s]\Gamma\left[s + \frac{1}{M}\right] \dots \Gamma\left[s + \frac{M-1}{M}\right] \quad (5.4)$$

$$\Gamma[N s] = \frac{1}{(2\pi)^{\frac{N-1}{2}} N^{\frac{1}{2}-Ns}} \Gamma[s]\Gamma\left[s + \frac{1}{N}\right] \dots \Gamma\left[s + \frac{N-1}{N}\right] \quad (5.5)$$

$$\Gamma[(M+N)s] = \frac{1}{(2\pi)^{\frac{M+N-1}{2}} (M+N)^{\frac{1}{2}-(M+N)s}} \Gamma[s]\Gamma\left[s + \frac{1}{M+N}\right] \dots \Gamma\left[s + \frac{M+N-1}{M+N}\right] \quad (5.6)$$

We have, for $s \gg 1$, $x \ll s$

$$\frac{\Gamma[s+x]}{\Gamma[s]} \approx s^x \quad (5.7)$$

which gives

$$\Gamma[s+x] \approx \Gamma[s] s^x \quad (5.8)$$

We use the above approximations in the expression

$$\frac{\Gamma[M s]\Gamma[N s]}{\Gamma[(M+N)s]} \quad (5.9)$$

There are an equal number of factors $\Gamma[s]$ at the numerator and denominator, so they cancel out. We can now collect the powers of s . In the numerator we have the power

$$\left(\frac{1}{M} + \frac{2}{M} + \dots + \frac{M-1}{M}\right) + \left(\frac{1}{N} + \frac{2}{N} + \dots + \frac{N-1}{N}\right) = \frac{M+N-2}{2} \quad (5.10)$$

In the denominator we have the power

$$\frac{1}{N+M} + \frac{2}{N+M} + \dots + \frac{N+M-1}{N+M} = \frac{(M+N-1)}{2} \quad (5.11)$$

Thus overall we get

$$\frac{\Gamma[M s]\Gamma[N s]}{\Gamma[(M+N)s]} \approx (2\pi)^{\frac{1}{2}} \left(\frac{M^M N^N}{(M+N)^{M+N}}\right)^s \sqrt{\frac{M+N}{MN}} s^{-\frac{1}{2}} \quad (5.12)$$

Similarly, we get

$$\frac{\Gamma[M s']\Gamma[N s']}{\Gamma[(M+N)s']} \approx (2\pi)^{\frac{1}{2}} \left(\frac{M^M N^N}{(M+N)^{M+N}}\right)^{s'} \sqrt{\frac{M+N}{MN}} s'^{-\frac{1}{2}} \quad (5.13)$$

Using these approximations in (5.1), we find

$$\tilde{\gamma}_{ss'}^B \approx \frac{1}{2\pi} \frac{1}{(M+N)^2} z_0^{s+s'} \frac{\mu_s \mu_{s'}}{\sqrt{ss'}} \frac{1}{s+s'}. \quad (5.14)$$

5.2 Continuum limit for the $f_{qs}^{B(i)}$

Let us also obtain the continuum limit for the $f_{qs}^{B(i)}$. We have the exact expression

$$\tilde{f}_{qs}^{B(1)} = \frac{i}{2\pi} z_0^{s-q} \frac{\mu_s}{s-q} \frac{1}{(M+N)} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{(s-q)} \frac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]}. \quad (5.15)$$

We have, from (5.12)

$$\frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} \approx (2\pi)^{\frac{1}{2}} \left(\frac{M^M N^N}{(M+N)^{M+N}} \right)^s \sqrt{\frac{M+N}{MN}} s^{-\frac{1}{2}} \quad (5.16)$$

Similarly,

$$\frac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]} \approx (2\pi)^{-\frac{1}{2}} \left(\frac{M^M N^N}{(M+N)^{M+N}} \right)^{-q} \sqrt{\frac{MN}{M+N}} q^{\frac{1}{2}} \quad (5.17)$$

Thus we obtain

$$\tilde{f}_{qs}^{B(1)} \approx \frac{i}{2\pi} \frac{1}{(M+N)} z_0^{s-q} \mu_s \sqrt{\frac{q}{s}} \frac{1}{s-q}. \quad (5.18)$$

Similarly, for $\tilde{f}_{rs}^{B(2)}$ we obtain

$$\tilde{f}_{rs}^{B(2)} \approx -\frac{i}{2\pi} \frac{1}{(M+N)} z_0^{s-r} \mu_s \sqrt{\frac{r}{s}} \frac{1}{s-r}. \quad (5.19)$$

6 Discussion

We have considered the effect of the twist operator σ_2 when it links together component strings of windings M, N into a single component string of length $M+N$. We have found the final state in the case where the initial state on the component strings was the vacuum $|0\rangle^{(1)} \otimes |0\rangle^{(2)}$, and also in the case where one of the strings had an initial excitation $\alpha_{-m}^{(i)}$.

While we have discussed this problem in the context of the D1D5 CFT, we note that this is a problem that could arise in other areas of physics. Each component string describes a free field theory on a circle, and the twist interaction joins these circles. In this process the vacuum state of the initial field theory goes over to a ‘squeezed state’ of the final theory; the coefficients γ_{ss}^B describe this squeezed state.

It is interesting to analyze the structure of the results that we have found. Consider the expression for γ_{kl}^B given in (3.35). This contains a factor

$$\frac{1}{\Gamma[k]} \frac{1}{\Gamma[l]} \quad (6.1)$$

which vanishes when either of the integers k, l is zero or negative. From (2.11) we see that this implies that the γ_{kl}^B will multiply only creation operators. Similarly, in the expression (4.14) for $f_{mk}^{B(1)}$ we have the factor

$$\frac{1}{\Gamma[m]} \frac{1}{\Gamma[k]} \quad (6.2)$$

which vanishes when either of the integers m or k is zero or negative. Thus the function $f_{mk}^{B(i)}$ vanishes unless we have creation operators in the initial state, and it yields only creation operators in the final state.

A second observation is that the expression for $\gamma_{ss'}^B$ almost separates into a product of terms corresponding to s and s' . Only the term $\frac{1}{s+s'}$ fails to separate in this manner. The part corresponding to s contains the beta function $\frac{\Gamma[M s] \Gamma[N s]}{\Gamma[(M+N)s]}$ and the part for s' contains the beta function $\frac{\Gamma[M s'] \Gamma[N s']}{\Gamma[(M+N)s']}$.

Similarly, $f_{qs}^{B(i)}$ almost separates into a product of terms corresponding to q and s . Only the term $\frac{1}{s-q}$ fails to separate. The part for s has the beta function $\frac{\Gamma[M s] \Gamma[N s]}{\Gamma[(M+N)s]}$, and the part for q has the reciprocal of the beta function $\frac{\Gamma[(M+N)q]}{\Gamma[M q] \Gamma[N q]}$. The appearance of the reciprocal here may be attributed to the fact that the index q corresponds to an operator in the initial state while the index s appears in the final state. (In the case of γ_{kl}^B , both operators appear in the final state.) In particular, the appearance of the reciprocal of the beta function ensures that we get the Γ functions in the form (6.2) required to ensure that only creation operators are involved in the effect of $f_{mk}^{B(i)}$.

We have considered only bosonic excitations in this paper. A similar analysis can be done for the fermionic excitations, and the supercharge G^- in (2.6) should then be applied to the overall state for bosons and fermions. These steps were carried out for the case $M = N = 1$ in [5, 6]; for the case of general M, N we hope to present the results elsewhere.

The expressions for $\gamma_{ss'}^B, f_{qs}^{B(i)}$ simplify considerably in the continuum limit. This limit may be more useful for obtaining the qualitative dynamics of thermalization, which is expected to be dual to the process of black hole formation. It would therefore be helpful to have a way of obtaining the continuum limit expressions directly, without having to obtain the exact expressions first. We hope to return to this issue elsewhere.

In general, it is hoped that by putting together knowledge of the fuzzball construction (which gives the gravity description of individual black hole microstates) and dynamical processes in the interacting CFT (which include black hole formation), we will arrive at a deeper understanding of black hole dynamics.

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