Effect of the deformation operator in the D1D5 CFT

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Abstract

The D1D5 CFT gives a holographic dual description of a near-extremal black hole in string theory. The interaction in this theory is given by a marginal deformation operator, which is composed of supercharges acting on a twist operator. The twist operator links together different copies of a free CFT. We study the effect of this deformation operator when it links together CFT copies with winding numbers M and N to produce a copy with winding $M + N$, populated with excitations of a particular form. We compute the effect of the deformation operator in the full supersymmetric theory, firstly on a Ramond-Ramond ground state and secondly on states with an initial bosonic or fermionic excitation. Our results generalize recent work which studied only the bosonic sector of the CFT. Our findings are a step towards understanding thermalization in the D1D5 CFT, which is related to black hole formation and evaporation in the bulk.

1 Introduction

String theory has successfully described many aspects of the quantum physics of black holes. In string theory we have an explicit microscopic description of a near-extremal black hole, given by the D1D5 system – a bound state of N_1 D1 branes and N_5 D5 branes. This bound state and its excitations reproduce the entropy of black holes in 4+1 noncompact dimensions [1], and provides one of the original examples of AdS/CFT duality [2].

The gauge theory on the D1D5 bound state flows in the infrared to a (4,4) SCFT. It is conjectured that there is a point in the moduli space where this SCFT is a symmetric product orbifold theory, consisting of $N_1 N_5$ symmetrized copies of a free $c = 6$ SCFT [3].

At this 'orbifold point' in moduli space, the gravity description is strongly coupled. Despite this fact, much has been achieved by studying the orbifold CFT. There is a well-studied AdS/CFT dictionary for two-charge BPS states [4, 5, 6], as well as three-charge BPS states arising from spectral flow [7]. More recently, gravity solutions corresponding to the symmetry algebra generators of the CFT have been constructed [8, 9], and CFT duals of the most general two-center D1D5P gravity solutions [10, 11] have been identified [12]. The orbifold CFT has even been shown to describe aspects of non-BPS states, including reproducing the radiation rate from non-BPS gravity solutions [10, 13, 14].

There are of course physical processes which are not well described by the orbifold CFT. In particular, it is interesting to consider the process of black hole formation in the bulk, which is dual to a process of thermalization in the CFT[1]. To study thermalization, we must use the deformed theory. The operator content of the undeformed theory includes twist operators; a twist operator σ_M takes M copies of the $c = 6$ CFT and joins them into a single copy living on a circle which is M times longer. We denote any such linked set of copies a 'component string'. The marginal deformation is composed of a twist operator σ_2^{++} dressed with left- and right-moving supercharges: $\hat{O} \sim G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} \sigma_2^{++}$ (see e.g. [17]).

We are interested in the process in which the deformation operator joins component strings of length M and N to make a component string of length $M + N$[2]. This process is depicted in Fig. 1. For $M = N = 1$, the effect of the deformation operator has has been computed, firstly in the case where the operator acts on the vacuum state of the two initial component strings [18], and also in the case where the operator acts on excited states [19]. Recently, focusing on bosonic fields only and using the bosonic part of the twist operator σ_2, this process was studied for general M, N, both using exact CFT techniques [20] and also using an alternative method involving Bogoliubov coefficients, which obtained the effect of the twist in a 'continuum limit' approximation [21]. We shall discuss this approximation

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[1]Here we are using the term 'thermalization' in a somewhat broad sense; it has been argued [15] that in global AdS, below the Hawking-Page transition, black hole formation should correspond to prethermalization [16], and the subsequent evaporation should correspond to actual thermalization.

[2]The twist operator σ_2^{++} may also act on two strands of the same component string, whereupon it will divide the component string into two parts. The computations for this case can be carried out in a similar manner to the calculations in the present paper.
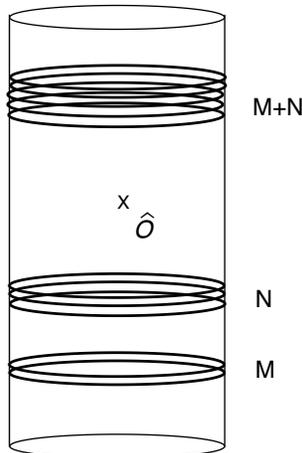
Figure 1: The process of interest, sketched on the cylinder. The state before the insertion of the deformation operator has component strings with windings M, N. The deformation operator $\hat{O}$ links these into a single component string of winding $M + N$.

in more detail later.

In this paper we compute the effect of the deformation operator in the above process, in the full supersymmetric theory. As well as analyzing the effect of the deformation operator on the fermionic fields, we compute the overall prefactor of the final state, which was not computed in the bosonic case of [20].

In more detail, we carry out the following calculations:

(a) Firstly, we begin with a given Ramond-Ramond (R-R) ground state $|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}$ on the initial component strings depicted in Fig. 1 (we will define all notation in due course). We apply the deformation operator at the point w_0. The deformation operator consists of supercharges acting on a twist operator σ_2^{++}; we first apply the twist operator. This generates a component string with winding $M + N$. Following the arguments in [18], we make the ansatz that the state produced by the twist σ_2^{++} on the final component string has the schematic form of an exponential acting on the R-R ground state $|0_R^{--}\rangle$:

$$
\begin{aligned}
|\chi\rangle &\equiv \sigma_2^{++}(w_0)|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)} \\
&= C_{MN}\left[\exp\left(\sum_{k,l}\gamma_{kl}^B\alpha_{-k}\alpha_{-l} + \sum_{k,l}\gamma_{kl}^F d_{-k}d_{-l}\right) \times [\text{antihol.}]\right]|0_R^{--}\rangle \quad (1.1)
\end{aligned}
$$

where α_k and d_k are the oscillator modes of a free boson and fermion respectively. For the case where the initial component strings had windings $M = N = 1$, the coefficients γ_{kl}^B, γ_{kl}^F and $C_{11} = 1$ were computed in [18], and the exponential ansatz was verified up to fourth order in the bosonic oscillators. For general M, N, the bosonic coefficients γ_{kl}^B were calculated in [20]. In the present paper we will compute the fermionic coefficients γ_{kl}^F and the overall prefactor C_{MN} for arbitrary M, N. The calculation of C_{MN} is quite nontrivial

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and involves the methods of [22, 23]. We then apply the supercharge to obtain the full effect of the deformation operator on the R-R ground state $|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}$.

(b) Secondly, we begin with an initial excitation $\alpha_{-m}^{(i)}$ or $d_{-m}^{(i)}$ on one of the component strings before the twist; here the index $i = 1, 2$ labels the component string which we excite. After the action of the twist, each such excitation is converted into a linear combination of excitations above the state $|\chi\rangle$,

$$
\begin{aligned}
\sigma_2^{++}(w_0)\alpha_{-m}^{(i)}|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)} &= \sum_k f_{mk}^{B(i)}\, \alpha_{-k}|\chi\rangle, & i = 1, 2 \\
\sigma_2^{++}(w_0)d_{-m}^{(i)}|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)} &= \sum_k f_{mk}^{F(i)}\, d_{-k}^{(i)}|\chi\rangle, & i = 1, 2
\end{aligned}
\tag{1.2}
$$

on the final component string of length $M + N$. (Due to the SU(2) indices on the fermions, which are suppressed in the above schematic expressions, there are two fermionic quantities $f^{F\pm(i)}$ which we will define in due course). For $M = N = 1$, the coefficients $f_{mk}^{B(i)}$ and $f_{mk}^{F(i)}$ were found in [19]. For general M, N, the coefficients $f_{mk}^{B(i)}$ were calculated in [20]. In the present paper, for general M, N, we compute the coefficients $f_{mk}^{F(i)}$ and apply the supercharge in both bosonic and fermionic cases, thus obtaining the full effect of the deformation operator on an initial bosonic or fermionic excitation.

(c) The regime of parameters of interest for black hole physics is $N_1 N_5 \gg 1$. In this regime, component strings typically have windings $M \gg 1$, which is the main physical reason for studying the present general M, N problem. In this limit, the excitations on such component strings typically have wavelengths much shorter than the length of the component string. Thus we are interested in taking a 'continuum limit' in which the excitations $\alpha_{-m}^{(i)}$, $d_{-m}^{(i)}$ have $m \gg 1$; such excitations are not sensitive to the infra-red cutoff scale set by the length of the component string. We find the continuum limit approximations to the expressions for $\gamma^F, f^{F(i)}$.

The deformation operator has recently been studied in various other works. Intertwining relations for operators before and after a twist insertion were derived in [24]. The effect of the twist on entanglement entropy was studied in [25]. Twist-nontwist correlators were studied in [26], and operator mixing was computed in [27]. For other related work, see [28]. This line of enquiry complements the fuzzball program [29, 30]; for recent work in this area, see e.g. [31, 32, 33].

This paper is organized as follows. In Section 2 we introduce the D1D5 CFT. In Section 3 we introduce the computation. Section 4 describes the method we use. In Section 5 we compute the effect of the deformation operator on the R-R ground state $|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}$. In Section 6 we compute the effect of the deformation operator on states with initial excitations. In Section 7 we obtain the continuum limit approximations to the quantities $\gamma^F, f^{F(i)}$. In Section 8 we discuss our results.

2 The D1D5 CFT

Here we review some of the properties of the D1D5 CFT. First, consider type IIB string theory compactified as:

$$M_{9,1} \to M_{4,1} \times S^1 \times T^4. \tag{2.1}$$

We take the S^1 to be large compared to the T^4, which we consider to be string-scale. We wrap N_1 D1 branes on S^1, and N_5 D5 branes on $S^1 \times T^4$.

At low energies, the gauge theory on the bound state flows to a $(4, 4)$ SCFT. It is conjectured that there is a point in moduli space where this SCFT is a symmetric product orbifold theory, consisting of $N_1 N_5$ symmetrized copies of a free $c = 6$ SCFT [3].

We work in a Euclideanized theory where the base space is a cylinder, which we parameterize via:

$$w = \tau + i\sigma, \quad 0 \le \sigma < 2\pi, \quad -\infty < \tau < \infty. \tag{2.2}$$

The target space is the symmetrized product of $N_1 N_5$ copies of T^4:

$$(T_4)^{N_1 N_5}/S_{N_1 N_5}. \tag{2.3}$$

Each copy of T^4 gives 4 bosonic fields X^1, X^2, X^3, X^4, along with 4 left-moving fermionic excitations $\psi^1, \psi^2, \psi^3, \psi^4$ and the corresponding right-moving excitations, which we denote with a bar ($\bar{\psi}^1$, etc.). The central charge of the theory on each copy of T^4 is $c = 6$, which yields a total central charge of $6N_1 N_5$.

2.1 Symmetries of the CFT

The D1D5 CFT has a (small) $\mathcal{N} = 4$ superconformal symmetry in both the left and right-moving sectors. The generators and their OPEs are given in Appendix A.

Each superconformal algebra contains an R symmetry $SU(2)$. Thus we have the global symmetry $SU(2)_L \times SU(2)_R$, with quantum numbers:

$$SU(2)_L : \ (j, m); \qquad SU(2)_R : \ (\bar{j}, \bar{m}). \tag{2.4}$$

In addition there is an $SO(4) \simeq SU(2) \times SU(2)$ symmetry, coming from rotations in the four directions of the T^4, which is broken by the fact that we have compactified these directions into a torus. However, the symmetry still provides a useful organizing tool for states. We label this symmetry by

$$SU(2)_1 \times SU(2)_2. \tag{2.5}$$

We use indices $\alpha, \dot{\alpha}$ for $SU(2)_L$ and $SU(2)_R$ respectively, and indices $A, \dot{A}$ for $SU(2)_1$ and $SU(2)_2$ respectively. The 4 real fermion fields of the left sector are grouped into complex fermions $\psi^{\alpha A}$. The right fermions are grouped into fermions $\bar{\psi}^{\dot{\alpha} A}$. The boson

fields X^i are a vector in T^4 and have no charge under $SU(2)_L$ or $SU(2)_R$, so are grouped as $X_{A\dot{A}}$. Different copies of the $c = 6$ CFT are denoted with a copy label in brackets, e.g.

$$X^{(1)}, \ X^{(2)}, \ \cdots, \ X^{(N_1 N_5)}. \tag{2.6}$$

Further details are given in Appendix A.

2.2   NS and R vacua

Consider a single copy of the $c = 6$ CFT. The lowest energy state of the left-moving sector is the NS vacuum,

$$|0_{NS}\rangle, \qquad h = 0, \quad m = 0 \tag{2.7}$$

where h denotes the L_0 eigenvalue. However, we study the CFT in the R sector. In particular we are interested in the R vacua denoted by[3]

$$|0_R^\pm\rangle, \qquad h = \frac{1}{4}, \quad m = \pm\frac{1}{2}. \tag{2.8}$$

We can relate the NS and R sectors using spectral flow. In particular, spectral flow by $\alpha = \pm 1$ in the left-moving sector produces the transformations

$$
\begin{aligned}
\alpha = 1: \quad & |0_R^-\rangle \to |0_{NS}\rangle, \quad |0_{NS}\rangle \to |0_R^+\rangle \\
\alpha = -1: \quad & |0_R^+\rangle \to |0_{NS}\rangle, \quad |0_{NS}\rangle \to |0_R^-\rangle
\end{aligned}
\tag{2.9}
$$

Similar relations hold for the right-moving sector, which we denote with bars. In many places we will write expressions only for the left-moving sector, as the two sectors are on an equal footing in this paper. However, in certain places we will write expressions involving the full vacuum of both sectors. We therefore introduce the notation

$$|0_R^{--}\rangle \ \equiv \ |0_R^-\rangle|\bar{0}_R^-\rangle. \tag{2.10}$$

Following the notation of [18], we denote the hermitian conjugate of $|0_R^{--}\rangle$ as

$$\langle 0_{R,--}| . \tag{2.11}$$

2.3   Twist operators and twisted R vacua

The symmetric orbifold theory contains twist operators, as well as more general spin-twist operators in the twisted sector. Let us recall their definition and, in doing so, we will define the initial and final states of the amplitude we compute. We will mostly follow the content of [22, 23], but with a different presentation which is more appropriate for our purposes[4].

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[3]There are two other left-moving R ground states, which may be obtained by acting with fermion zero modes. Including right-movers then gives a total of 16 R-R ground states; see e.g. [34].

[4]See also the discussion in [35].

The bare twist operator σ_M is defined as follows. Let us work on the cylinder, and insert σ_M at a point w_*. As the fields circle this point, the fields transform as

$$X^{(1)} \to X^{(2)} \to \cdots \to X^{(M)} \to X^{(1)} \tag{2.12}$$

$$\psi^{(1)} \to \psi^{(2)} \to \cdots \to \psi^{(M)} \to -\psi^{(1)}. \tag{2.13}$$

Note the minus sign in the fermionic expression. These boundary conditions are achieved as follows. First, we map the problem to the plane with coordinate z via

$$z = e^w \tag{2.14}$$

and then to a local covering plane with coordinate t via a map which has the local form

$$z - z_* \approx b_* (t - t_*)^M \tag{2.15}$$

where z_* and t_* are the respective images of the insertion point w_* in the z plane and the t plane. In the t plane, we insert the identity operator at the point t_*. The M bosonic fields in (2.12) map to one single-valued bosonic field $X(t)$ in the t plane, and similarly for the fermions.

In the absence of other insertions, we end up with an empty t plane, i.e. in the NS vacuum. Equivalently, one can think in terms of the 'covering cylinder' with coordinate u, defined by

$$t = e^u. \tag{2.16}$$

In the NS vacuum on the u cylinder, the fermions are antiperiodic, which agrees with (2.13).

In the present paper we are interested in an initial state which involves component strings of length M, N in R vacuum states. Thus, we next define the (spin-)twist operator $\sigma_M^\pm$ as follows. Follow the procedure used to define the bare twist σ_M, but in the covering t plane, insert a spin field $S^\pm$ at t_* (if $b_* \neq 1$ in (2.15), we also need to include a normalization factor, which we will ignore here, but which will be important in Section 5.2). If we take $t_* = 0$, we obtain the R vacuum $|0_R^\pm\rangle_t$ of the t plane. We write

$$\sigma_M^\alpha = S_M^\alpha \sigma_M, \qquad \alpha = +, -. \tag{2.17}$$

Back on the original cylinder, with coordinate w, as the fields circle the operator $\sigma_M^\pm$, they transform as

$$X^{(1)} \to X^{(2)} \to \cdots \to X^{(M)} \to X^{(1)} \tag{2.18}$$

$$\psi^{(1)} \to \psi^{(2)} \to \cdots \to \psi^{(M)} \to +\psi^{(1)}. \tag{2.19}$$

Note the plus sign in the fermionic expression. On the covering cylinder with coordinate u, the fermion field is periodic, and if the operator is inserted at past infinity, we obtain the R vacuum $|0_R^\pm\rangle_u$ of the covering cylinder. We write the corresponding state on the original cylinder (with coordinate w) as $|0_R^\pm\rangle_M$.

Adding in the right-moving sector, we obtain the full spin-twist field

$$\sigma_M^{\alpha\dot\alpha} = S_M^\alpha \bar{S}_M^{\dot\alpha} \sigma_M \tag{2.20}$$

and the corresponding state $|0_R^{\alpha\dot\alpha}\rangle_M$.

We can now write our full initial state as $|0_R^{--}\rangle_M^{(1)}|0_R^{--}\rangle_N^{(2)}$. For ease of notation however, we shall immediately suppress the subscripts M, N on the states, thus writing our full initial state as

$$|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}. \tag{2.21}$$

We shall denote states on the final component string of length $M+N$ without a copy label, and we will express the states produced by the action of the deformation operator in terms of excitations above the R-R ground state

$$|0_R^{--}\rangle. \tag{2.22}$$

For the remainder of the paper, $|0_R^{--}\rangle$ and its hermitian conjugate $\langle 0_{R,--}|$ will denote states on the final component string of length $M + N$.

2.4 The deformation operator

In the previous subsection, we defined the operators $\sigma_M^{\alpha\dot\alpha}$. When $M = 2$, the operator σ_2^{++} is a chiral primary in both left- and right-moving sectors, with quantum numbers

$$h = j = m = \frac{1}{2}, \qquad \bar{h} = \bar{j} = \bar{m} = \frac{1}{2}. \tag{2.23}$$

The twist operator $\sigma_2^{\alpha\dot\alpha}$ is normalized to have a unit OPE with its conjugate $\sigma_{2,\alpha\dot\alpha}$,

$$\sigma_{2,\alpha\dot\alpha}(w,\bar{w})\sigma_2^{\alpha\dot\alpha}(w',\bar{w}') \sim \frac{1}{|w - w'|^2} \tag{2.24}$$

where the above indices α, $\dot\alpha$ are *not* summed over.

The deformation operator is given by (see e.g. the discussion in [18])

$$\hat{O}_{\dot{A}\dot{B}}(w_0,\bar{w}_0) = \left[\frac{1}{2\pi i}\int_{w_0} dw\, G_{\dot{A}}^-(w)\right]\left[\frac{1}{2\pi i}\int_{\bar{w}_0} d\bar{w}\, \bar{G}_{\dot{B}}^-(\bar{w})\right]\sigma_2^{++}(w_0,\bar{w}_0) \tag{2.25}$$

The $\dot{A}, \dot{B}$ indices can be contracted to rewrite the above four operators as a singlet and a triplet of $SU(2)_2$.

Let us now switch to writing only the holomorphic parts of states and operators, for ease of notation. The deformation operator involves the application of a supercharge G^-, which is integrated around the twist operator σ_2^+. This can be written as a combination
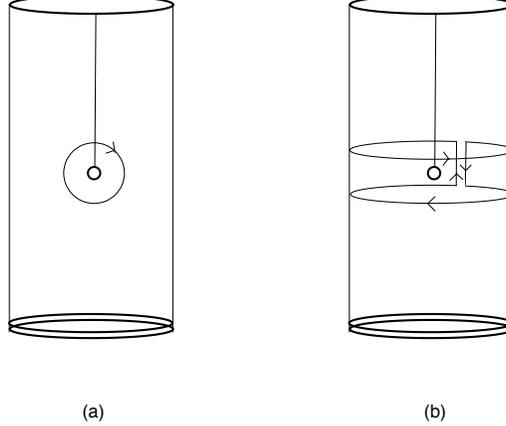
Figure 2: The deformation operator involves the application of a supercharge, integrated around the twist operator. The contour may be stretched as shown, so that the deformation operator may be written in terms of a combination of applying the supercharge before and after the twist.

of applying the supercharge before and after the twist (see Fig. 2); the upper part of the contour gives

$$
\frac{1}{2\pi i} \int\limits_{w=\tau_0+\epsilon}^{\tau_0+\epsilon+2\pi i(M+N)} G_{\dot{A}}^-(w)\, dw \;=\; G_{\dot{A},0}^- \tag{2.26}
$$

and the lower part of the contour gives

$$
-\frac{1}{2\pi i}\left[\int\limits_{w=\tau_0-\epsilon}^{\tau_0-\epsilon+2\pi iM} G_{\dot{A}}^{(1)-}(w)\, dw + \int\limits_{w=\tau_0-\epsilon}^{\tau_0-\epsilon+2\pi iN} G_{\dot{A}}^{(2)-}(w)\, dw \right] \;=\; -\left(G_{\dot{A},0}^{(1)-} + G_{\dot{A},0}^{(2)-} \right). \tag{2.27}
$$

Thus we obtain

$$
\hat{O}_{\dot{A}} = G_{\dot{A},0}^- \sigma_2^+ - \sigma_2^+ \left(G_{\dot{A},0}^{(1)-} + G_{\dot{A},0}^{(2)-} \right) \tag{2.28}
$$

and similarly for the right sector. When acting upon a state, we can use the fact that the supercharge annihilates the vacuum to simplify our expressions. For the vacuum, we have

$$
\hat{O}_{\dot{A}}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \;=\; G_{\dot{A},0}^- \sigma_2^+ |0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \tag{2.29}
$$

while for some operator $\hat{Q}$, we have

$$
\begin{aligned}
\hat{O}_{\dot{A}}\hat{Q}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \;&=\; G_{\dot{A},0}^- \sigma_2^+ \hat{Q}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} - \sigma_2^+ \left[\left(G_{\dot{A},0}^{(1)-} + G_{\dot{A},0}^{(2)-} \right), \hat{Q} \right] |0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \\
&=\; G_{\dot{A},0}^- \sigma_2^+ \hat{Q}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} - \sigma_2^+ \left\{ \left(G_{\dot{A},0}^{(1)-} + G_{\dot{A},0}^{(2)-} \right), \hat{Q} \right\} |0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)}
\end{aligned}
\tag{2.30}
$$

The commutation/anticommutation relations of the supercharge with basic operators such as the creation and annihilation operators for the bosonic and fermionic fields are known. As such, the bulk of our computations focus on the effects of the twist.

9

3 Introducing the computation

3.1 Mode Expansions on the cylinder

In order to set up the computation and present the explicit form of the exponential ansatz, we now give the mode expansions of the fundamental fields on the cylinder. There are two component strings before the twist; string (1) is wound M times, while string (2) is wound N times.

Before the twist, the mode expansions are

$$\alpha^{(1)}_{A\dot{A},m} = \frac{1}{2\pi i} \int\limits_{\sigma=0}^{2\pi M} \partial X^{(1)}_{A\dot{A}}(w) e^{\frac{m}{M}w} dw \tag{3.1}$$

$$\alpha^{(2)}_{A\dot{A},m} = \frac{1}{2\pi i} \int\limits_{\sigma=0}^{2\pi N} \partial X^{(2)}_{A\dot{A}}(w) e^{\frac{m}{N}w} dw \tag{3.2}$$

$$d^{(1)\alpha A}_{m} = \frac{1}{2\pi i\sqrt{M}} \int\limits_{\sigma=0}^{2\pi M} \psi^{(1)\alpha A}(w) e^{\frac{m}{M}w} dw \tag{3.3}$$

$$d^{(2)\alpha A}_{m} = \frac{1}{2\pi i\sqrt{N}} \int\limits_{\sigma=0}^{2\pi N} \psi^{(2)\alpha A}(w) e^{\frac{m}{N}w} dw \,. \tag{3.4}$$

From the two-point functions, the commutation/anticommutation relations are

$$\begin{aligned}
\left[\alpha^{(i)}_{A\dot{A},m}, \alpha^{(j)}_{B\dot{B},n}\right] &= -m\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\delta^{ij}\delta_{m+n,0} \\
\left\{d^{(i)\alpha A}_{m}, d^{(j)\beta B}_{n}\right\} &= -\varepsilon^{\alpha\beta}\varepsilon^{AB}\delta^{ij}\delta_{m+n,0} \,.
\end{aligned} \tag{3.5}$$

After the twist, there is a single component string of length $M + N$. We thus have modes

$$\alpha_{A\dot{A},m} = \frac{1}{2\pi i} \int\limits_{\sigma=0}^{2\pi(M+N)} \partial X_{A\dot{A}}(w) e^{\frac{m}{M+N}w} dw \tag{3.6}$$

$$d^{\alpha A}_{k} = \frac{1}{2\pi i\sqrt{M+N}} \int\limits_{\sigma=0}^{2\pi(M+N)} \psi^{\alpha A}(w) e^{\frac{k}{M+N}w} dw \tag{3.7}$$

with commutation relations

$$\left[\alpha_{A\dot{A},m}, \alpha_{B\dot{B},n}\right] = -m\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\delta_{m+n,0} \tag{3.8}$$

$$\left\{d^{\alpha A}_{k}, d^{\beta B}_{l}\right\} = -\varepsilon^{\alpha\beta}\varepsilon^{AB}\delta_{k+l,0} \,. \tag{3.9}$$

3.2 The exponential ansatz

We begin our computations by working firstly with the twist operator σ_2^{++}. After determining the effects of the twist operator, we use the known commutation relations of the supercharge to determine the effects of the full deformation operator. We consider separately the effects of this operator on the vacuum, and on a state with a single bosonic or fermionic excitation.

Let us now record the exponential ansatz with all notation explicit:

$$
\begin{aligned}
|\chi\rangle &\equiv \sigma_2^{++}(w_0)|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)} \\
&= C_{MN}\left[\exp\left(\sum_{m\geq 1, n\geq 1}\gamma_{mn}^{B}\left[-\alpha_{++,-m}\alpha_{--,-n}+\alpha_{+-,-m}\alpha_{-+,-n}\right]\right)\right. \\
&\qquad\left.\times\exp\left(\sum_{m\geq 0, n\geq 1}\gamma_{mn}^{F}\left[d_{-m}^{++}d_{-n}^{--}-d_{-m}^{+-}d_{-n}^{-+}\right]\right)\times[\text{antihol.}]\right]|0_R^{--}\rangle. \quad (3.10)
\end{aligned}
$$

This ansatz is a generalization of that made in [18] in the case of $M = N = 1$, where the exponential ansatz was verified up to fourth order in the bosonic oscillators.

The main novel feature of the above ansatz relative to the $M = N = 1$ case is the non-trivial overall factor C_{MN}. A priori, this could be a function of w_0, but will turn out to be a pure c-number independent of w_0. We observe that

$$
C_{MN} = \langle 0_{R,--}|\sigma_2^{++}(w_0)|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}. \quad (3.11)
$$

In the case of $M = N = 1$, the normalization of the twist σ_2^{++} implies that $C_{11} = 1$ [18]. We compute the general factor C_{MN} in Section 5.2.

In the case of an initial bosonic or fermionic excitation, the above exponential ansatz implies [19, 21] that the twist operator converts an initial excitation into a linear combination of excitations above the state $|\chi\rangle$, which we write as

$$
\sigma_2^{++}(w_0)\alpha_{A\dot{A},-m}^{(i)}|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)} = \sum_k f_{mk}^{B(i)}\,\alpha_{A\dot{A},-k}|\chi\rangle, \qquad i = 1,2 \quad (3.12)
$$

$$
\sigma_2^{++}(w_0)d_{-m}^{(i)\pm A}|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)} = \sum_k f_{mk}^{F(i)\pm}\,d_{-k}^{(i)\pm A}|\chi\rangle, \qquad i = 1,2. \quad (3.13)
$$

4 The method of computation

4.1 Mapping to a covering plane

In order to relate states before the twist to states after the twist, we first map the cylinder to the complex plane via:

$$
z = e^w = e^{\tau+i\sigma}. \quad (4.1)
$$

The point w_0 maps to a point z_0. We next map the problem to a covering plane, with coordinate t, via the map

$$z = t^M (t-a)^N .\tag{4.2}$$

This map contains an M fold bifurcation at $t = 0$, an N fold bifurcation at $t = a$, and an $M + N$ fold bifurcation at $t = \infty$ (see Fig. 3). The bosonic fields $X^{(1)}$, $X^{(2)}$ map to one single-valued bosonic field $X(t)$ in the t plane, and similarly for the fermions. The pre-twist component string (1) with winding M maps to contours around the origin in the t-plane, while the pre-twist component string (2) with winding N maps to contours around the point $t = a$. The post-twist component string of winding $M + N$ maps to contours around $t = \infty$.

The map also contains a two-fold bifurcation point at the other solution of $\frac{dz}{dt} = 0$, which is the location of the twist σ_2^{++}. It is convenient to write the derivative $\frac{dz}{dt}$ as

$$\frac{dz}{dt} = (M+N)\frac{z}{t(t-a)}\left(t - \frac{M}{M+N}a\right) .\tag{4.3}$$

We see that

$$t_0 = \frac{M}{M+N}a\tag{4.4}$$

which corresponds via (4.2) to the following value of z,

$$z_0 = a^{M+N}\frac{M^M N^N}{(M+N)^{M+N}}(-1)^N .\tag{4.5}$$

To invert this for a, we must specify how to deal with the fractional exponent. We choose

$$z_0 = a^{M+N}\frac{M^M N^N}{(M+N)^{M+N}}e^{i\pi N}\tag{4.6}$$

which determines a in terms of $z_0 = e^{w_0}$ to be

$$a = e^{-i\pi\frac{N}{M+N}}\left(\frac{z_0}{M^M N^N}\right)^{\frac{1}{M+N}}(M+N) .\tag{4.7}$$

We will perform our calculations using the parameter a, and then use this relation to re-express our results in terms of the insertion point z_0.

4.2 Spectral flowing to an empty covering plane

In the t plane, there are no twist insertions. However, we have spin field insertions at each of the bifurcation points:

$$S^-(t=0) , \quad S^+(t = \tfrac{M}{M+N}a) , \quad S^-(t=a) , \quad S^+(t=\infty)\tag{4.8}$$

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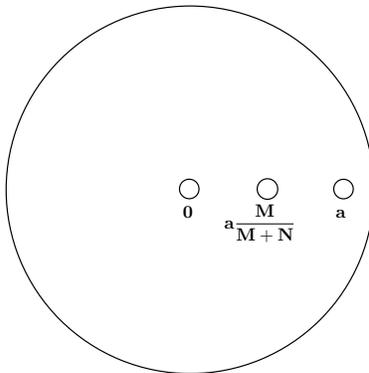
Figure 3: The covering plane with coordinate t. The bifurcation points of the map are shown.

along with appropriate normalization factors. To compute γ^F and $f^{F(i)\pm}$, we shall take a ratio of amplitudes, and so it it will not be necessary to keep account of the normalization factors of the spin fields at this point; such factors will cancel out. However, these normalization factors will later be important when we come to calculate the overall coefficient C_{MN}.

Our method of computing γ^F and $f^{F(i)\pm}$ is to perform a sequence of spectral flows and coordinate transformations to remove the spin fields, and thus to map the problem to an empty t plane.

The sequence is as follows (we write only the left-moving part, the right-moving part is identical):

i) Spectral flow by $\alpha = 1$ in the t plane.

ii) Change coordinate to $t' = t - \frac{M}{M+N}a$.

iii) Spectral flow by $\alpha = -1$ in the t' plane.

iv) Change coordinate to $\hat{t} = t' - \frac{N}{M+N}a = t - a$.

v) Spectral flow by $\alpha = 1$ in the $\hat{t}$ plane.

vi) Change coordinate back to $t = \hat{t} + a$ (when necessary).

From the action of spectral flow on the R ground states in (2.9) one sees that this sequence indeed maps the problem to one with no spin field insertions.

The action of spectral flow on operators is straightforward for those operators where the fermion content may be expressed as a simple exponential in the language in which the fermions are bosonized. For such operators with charge j, spectral flow with parameter α gives rise to the transformation

$$\hat{O}_j(t) \to t^{-\alpha j} \hat{O}_j(t).$$

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The fermion fields and spin fields are of this form.

Since we take a ratio of amplitudes to compute γ^F and $f^{F(i)\pm}$, we will not need to take account of the transformation of the spin fields (again, these will however be important for the computation of C_{MN}).

By contrast, an essential part of the computation of γ^F and $f^{F(i)\pm}$ is to follow the transformation of the modes of the fermion fields through the above sequence of spectral flows and coordinate transformations. Here we present only the final expressions for these modes; details of the derivation are given in Appendix B.

In terms of the $\hat{t}$-plane, the cylinder modes (3.3), (3.4) and (3.7) map to the following expressions. For the modes before the twist, we have

$$
d_m^{(1)+A} \;\;\to\;\; \hat{d}_m'^{(1)+A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{\hat{t}=-a} d\hat{t}\, \psi^{(1)+A}\left(\hat{t}\right)\left[\left(\hat{t}+a\right)^{m-1}\left(\hat{t}+\tfrac{Na}{M+N}\right)\hat{t}^{\frac{Nm}{M}-1}\right]
$$

$$
d_m^{(2)+A} \;\;\to\;\; \hat{d}_m'^{(2)+A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{\hat{t}=0} d\hat{t}\, \psi^{(2)+A}\left(\hat{t}\right)\left[\left(\hat{t}+a\right)^{\frac{Mm}{N}-1}\left(\hat{t}+\tfrac{Na}{M+N}\right)\hat{t}^{m-1}\right]
$$

$$
d_m^{(1)-A} \;\;\to\;\; \hat{d}_m'^{(1)-A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{\hat{t}=-a} d\hat{t}\, \psi^{(1)-A}\left(\hat{t}\right)\left[\left(\hat{t}+a\right)^{m}\hat{t}^{\frac{Nm}{M}}\right]
$$

$$
d_m^{(2)-A} \;\;\to\;\; \hat{d}_m'^{(2)-A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{\hat{t}=0} d\hat{t}\, \psi^{(2)-A}\left(\hat{t}\right)\left[\left(\hat{t}+a\right)^{\frac{Mm}{N}}\hat{t}^{m}\right] \tag{4.9}
$$

After the twist, we obtain

$$
d_k^{+A} \;\;\to\;\; \hat{d}_k'^{+A} \;=\; \frac{1}{2\pi i} \oint_{\hat{t}=\infty} d\hat{t}\, \psi^{+A}\left(\hat{t}\right)\left[\left(\hat{t}+a\right)^{\frac{Mk}{M+N}-1}\left(\hat{t}+\tfrac{N}{M+N}a\right)\hat{t}^{\frac{Nk}{M+N}-1}\right]
$$

$$
d_k^{-A} \;\;\to\;\; \hat{d}_k'^{-A} \;=\; \frac{1}{2\pi i} \oint_{\hat{t}=\infty} d\hat{t}\, \psi^{-A}\left(\hat{t}\right)\left[\left(\hat{t}+a\right)^{\frac{Mk}{M+N}}\hat{t}^{\frac{Nk}{M+N}}\right] \tag{4.10}
$$

For later use, let us also rewrite these modes in the t plane. The modes before the twist then become

$$
d_m^{(1)+A} \;\;\to\;\; d_m'^{(1)+A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{t=0} dt\, \psi^{(1)+A}\left(t\right)\left[t^{m-1}\left(t-\tfrac{M}{M+N}a\right)\left(t-a\right)^{\frac{Nm}{M}-1}\right]
$$

$$
d_m^{(2)+A} \;\;\to\;\; d_m'^{(2)+A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{t=a} dt\, \psi^{(2)+A}\left(t\right)\left[t^{\frac{Mm}{N}-1}\left(t-\tfrac{M}{M+N}a\right)\left(t-a\right)^{m-1}\right]
$$

$$
d_m^{(1)-A} \;\;\to\;\; d_m'^{(1)-A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{t=0} dt\, \psi^{(1)-A}(t)\left[t^{m}\left(t-a\right)^{\frac{Nm}{M}}\right]
$$

$$
d_m^{(2)-A} \;\;\to\;\; d_m'^{(2)-A} \;=\; \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{t=a} dt\, \psi^{(2)-A}(t)\left[t^{\frac{Mm}{N}}\left(t-a\right)^{m}\right] \tag{4.11}
$$

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and after the twist we obtain

$$d_k^{+A} \rightarrow d_k'^{+A} = \frac{1}{2\pi i} \oint_{t=\infty} dt\, \psi^{+A}(t) \left[t^{\frac{Mk}{M+N}-1} \left(t - \frac{M}{M+N}a \right)(t-a)^{\frac{Nk}{M+N}-1} \right]$$

$$d_k^{-A} \rightarrow d_k'^{-A} = \frac{1}{2\pi i} \oint_{t=\infty} dt\, \psi^{-A}(t) \left[t^{\frac{Mk}{M+N}} (t-a)^{\frac{Nk}{M+N}} \right]. \tag{4.12}$$

For later use, we also define modes natural to the t and $\hat{t}$ planes,

$$\tilde{d}_r^{\alpha A} = \frac{1}{2\pi i} \oint_{t=0} \psi^{\alpha A}(t) t^{r-\frac{1}{2}} \tag{4.13}$$

$$\hat{d}_r^{\alpha A} = \frac{1}{2\pi i} \oint_{\hat{t}=0} \psi^{\alpha A}\left(\hat{t}\right) \hat{t}^{r-\frac{1}{2}}. \tag{4.14}$$

The anticommutation relations for these modes are

$$\{\tilde{d}_r^{\alpha A}, \tilde{d}_s^{\beta B}\} = \{\hat{d}_r^{\alpha A}, \hat{d}_s^{\beta B}\} = -\epsilon^{\alpha\beta}\epsilon^{AB}\delta_{r+s,0}. \tag{4.15}$$

5 Effect of the deformation operator on the vacuum

We now find the effect of the deformation operator on the R-R ground state $|0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)}$. We start by computing the effect of the twist operator σ_2^{++}, and we then apply the super-charge.

The effect of the twist operator σ_2^{++} on the state $|0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)}$ is described in terms of the quantities γ^B, γ^F and C_{MN}. The quantity γ^B was computed in [20] (and we will recall it later); we now proceed to calculating γ^F and C_{MN}.

5.1 Computing the Bogoliubov coefficients γ^F

We now compute γ_{kl}^F. Let us consider the ratio of amplitudes

$$\frac{\mathcal{A}_2}{\mathcal{A}_1} = \frac{\langle 0_{R,--}| \left(d_l^{++}d_k^{--}\right) \sigma_2^+(w_0) |0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)}}{\langle 0_{R,--}| \sigma_2^+(w_0) |0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)}}. \tag{5.1}$$

From the exponential ansatz (3.10) we observe that

$$\mathcal{A}_2 = \langle 0_{R,--}| d_l^{++}d_k^{--} |\chi\rangle$$

$$= C_{MN}\langle 0_{R,--}| d_l^{++}d_k^{--} \exp\left(\sum_{m\geq 0, n\geq 1} \gamma_{mn}^F \left[d_{-m}^{++}d_{-n}^{--} - d_{-m}^{+-}d_{-n}^{-+}\right] \right) |0_R^{--}\rangle$$

$$= C_{MN}\, \gamma_{kl}^F \tag{5.2}$$

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and so using (3.11) we see that

$$\gamma_{kl}^F = \frac{\mathcal{A}_2}{\mathcal{A}_1}. \tag{5.3}$$

We now map this ratio of amplitudes to the empty $\hat{t}$ plane using the sequence of spectral flows and coordinate transformations discussed in Section 4.2, obtaining

$$\gamma_{kl}^F = \frac{\langle 0_{R,--}|\left(d_l^{++}d_k^{--}\right)\sigma_2^+\left(w_0\right)|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}}{\langle 0_{R,--}|\sigma_2^+\left(w_0\right)|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}} = \frac{{}_{\hat{t}}\langle 0_{NS}|\hat{d}_l'^{++}\hat{d}_k'^{--}|0_{NS}\rangle_{\hat{t}}}{{}_{\hat{t}}\langle 0_{NS}|0_{NS}\rangle_{\hat{t}}} \tag{5.4}$$

where $\hat{d}_l'^{++}$ and $\hat{d}_k'^{--}$ are the modes after all spectral flows and coordinate transformations given in (4.10).

Let us now evaluate the $\hat{t}$ plane amplitude in the numerator of (5.4). To do so, we expand the transformed modes $\hat{d}'$ in terms of the natural modes $\hat{d}$ on the $\hat{t}$ plane (4.14), whose commutation relations are known. From (4.10) we have

$$\hat{d}_l'^{++} = \frac{1}{2\pi i}\oint_{\hat{t}=\infty}d\hat{t}\,\psi^{++}\left(\hat{t}\right)\left[\left(\hat{t}+a\right)^{\frac{Ml}{M+N}-1}\left(\hat{t}+\frac{N}{M+N}a\right)\hat{t}^{\frac{Nl}{M+N}-1}\right] \tag{5.5}$$

Since we are at large $\hat{t}$ we expand

$$\left(\hat{t}+a\right)^{\frac{Ml}{M+N}-1}\hat{t}^{\frac{Nl}{M+N}-1} = \hat{t}^{l-2}\left(1+a\hat{t}^{-1}\right)^{\frac{Ml}{M+N}-1} = \sum_{q\geq 0}{}^{\frac{Ml}{M+N}-1}C_q\,a^q\,\hat{t}^{l-q-2} \tag{5.6}$$

where ${}^xC_y$ is the binomial coefficient. This gives

$$\hat{d}_l'^{++} = \sum_{q\geq 0}{}^{\frac{Ml}{M+N}-1}C_q\left(a^q\,\hat{d}_{l-q-\frac{1}{2}}^{++} + \frac{N}{M+N}\,a^{q+1}\,\hat{d}_{l-q-\frac{3}{2}}^{++}\right). \tag{5.7}$$

Similarly, for $\hat{d}_k'^{--}$ we obtain the relation

$$\hat{d}_k'^{--} = \sum_{p\geq 0}{}^{\frac{Mk}{M+N}}C_p\,a^p\,\hat{d}_{k-p+\frac{1}{2}}^{--}. \tag{5.8}$$

We then obtain

$$
\begin{aligned}
{}_{\hat{t}}\langle 0_{NS}|&\,d_l'^{++}d_k'^{--}|0_{NS}\rangle_{\hat{t}} \\
&= \sum_{p\geq 0}\sum_{q\geq 0}a^{p+q}\,{}^{\frac{Mk}{M+N}}C_p\,{}^{\frac{Ml}{M+N}-1}C_q\,{}_{\hat{t}}\langle 0_{NS}|\hat{d}_{l-q-\frac{1}{2}}^{++}\hat{d}_{k-p+\frac{1}{2}}^{--}|0_{NS}\rangle_{\hat{t}} \\
&\quad + \frac{N}{M+N}\sum_{p\geq 0}\sum_{q\geq 0}a^{p+q+1}\,{}^{\frac{Mk}{M+N}}C_p\,{}^{\frac{Ml}{M+N}-1}C_q\,{}_{\hat{t}}\langle 0_{NS}|\hat{d}_{l-q-\frac{3}{2}}^{++}\hat{d}_{k-p+\frac{1}{2}}^{--}|0_{NS}\rangle_{\hat{t}} \tag{5.9}
\end{aligned}
$$

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We have two terms. For the first term, utilizing commutation relations (4.15) we obtain

$$l - q - \frac{1}{2} = -\left(k - p + \frac{1}{2}\right) \quad \Rightarrow \quad p = k + l - q \, . \tag{5.10}$$

Similarly, for the second term we obtain

$$p = k + l - q - 1 \, . \tag{5.11}$$

Note that in the first term, $\hat{d}^{++}_{l-q-\frac{1}{2}}$ must be an annihilation operator, so we have nonzero contributions only from modes with

$$q \leq l - 1 \tag{5.12}$$

and similarly for the second term we have nonzero contributions only from

$$q \leq l - 2 \, . \tag{5.13}$$

Then (5.9) becomes

$$
\begin{aligned}
{}_{\hat{t}}\langle 0_{NS}| \, d'^{++}_{l} d'^{--}_{k} \, |0_{NS}\rangle_{\hat{t}} \; = \; \\
-a^{k+l} \left[\sum_{q=0}^{l-1} \tfrac{Ml}{M+N}{-1} C_q \tfrac{Mk}{M+N} C_{l+k-q} + \frac{N}{M+N} \sum_{q=0}^{l-2} \tfrac{Mk}{M+N}{-1} C_q \tfrac{Ml}{M+N} C_{l+k-q-1} \right] {}_{\hat{t}}\langle 0_{NS}|0_{NS}\rangle_{\hat{t}}
\end{aligned}
$$

which gives

$$\gamma^F_{kl} \; = \; -a^{k+l} \left[\sum_{q=0}^{l-1} \tfrac{Ml}{M+N}{-1} C_q \tfrac{Mk}{M+N} C_{l+k-q} + \frac{N}{M+N} \sum_{q=0}^{l-2} \tfrac{Mk}{M+N}{-1} C_q \tfrac{Ml}{M+N} C_{l+k-q-1} \right] . \tag{5.14}$$

Evaluating the sums in $Mathematica$ we find

$$\gamma^F_{kl} \; = \; \frac{a^{k+l}}{\pi^2} \sin\left[\tfrac{N\pi k}{M+N}\right] \sin\left[\tfrac{N\pi l}{M+N}\right] \frac{MN}{(M+N)^2} \frac{k}{k+l} \frac{\Gamma\left[\tfrac{Mk}{M+N}\right] \Gamma\left[\tfrac{Nk}{M+N}\right]}{\Gamma[k]} \frac{\Gamma\left[\tfrac{Ml}{M+N}\right] \Gamma\left[\tfrac{Nl}{M+N}\right]}{\Gamma[l]} \, . \tag{5.15}$$

5.1.1 Writing γ^F in final form

We next write γ^F in final form by making the following changes.

(i) We replace the parameter a by $z_0 = e^{w_0}$, so that our result is expressed in terms of the insertion point of the twist, using (4.7).

(ii) We define fractional modes:

$$s = \frac{k}{M+N}, \qquad s' = \frac{l}{M+N} \, . \tag{5.16}$$

The parameters s and s' then give directly the physical wavenumbers of the modes on the cylinder with coordinate w.

(iii) We define the shorthand notation

$$\mu_s \equiv 1 - e^{2\pi iMs}.\tag{5.17}$$

Note that

$$\sin\left[\frac{\pi Nk}{M+N}\right] = \sin(\pi Ns) = -\frac{i}{2}e^{i\pi Ns}\left(1 - e^{-2\pi iNs}\right) = -\frac{i}{2}e^{i\pi Ns}\mu_s.\tag{5.18}$$

In this notation, γ^F becomes

$$\tilde{\gamma}^F_{ss'} = -\frac{1}{4\pi^2}z_0^{s+s'}\mu_s\mu_{s'}\frac{s}{s+s'}\frac{MN}{(M+N)^2}\left(\frac{(M+N)^{M+N}}{M^M N^N}\right)^{s+s'}\frac{\Gamma[Ms]\,\Gamma[Ns]}{\Gamma[(M+N)s]}\frac{\Gamma[Ms']\,\Gamma[Ns']}{\Gamma[(M+N)s']}.\tag{5.19}$$

In order to verify that in the case of $M = N = 1$ this expression agrees with that computed in [18], note that due to different choices in normalization of the modes of the fermions, the expression $\frac{1}{2}\gamma^F$ in this paper should agree with the γ^F in [18]. One can check that this is indeed the case.

5.2   Computing the overall prefactor C_{MN}

We next compute the overall prefactor C_{MN}. Recall that from the exponential ansatz (3.10) we have

$$C_{MN} = \langle 0_{R,--}|\,\sigma_2^{++}(w_0)\,|0_R^{--}\rangle^{(1)}\,|0_R^{--}\rangle^{(2)}.\tag{5.20}$$

We use the methods developed in [22, 23] to compute this correlator. The full calculation is somewhat lengthy and is presented in Appendix C; here we summarize the main steps.

(i) We first lift to the z plane. Since σ_2^{++} has weight $(1/2,1/2)$, we obtain the Jacobian factor contribution

$$\left|\frac{dz}{dw}\right|_{z=z_0} = |a|^{M+N}\frac{M^M N^N}{(M+N)^{M+N}}\tag{5.21}$$

(ii) We compute the above correlator of spin-twist fields following the method of [22, 23] as follows. We work in a path integral formalism, and we define regularized spin-twist operators by cutting circular holes in the z plane. We lift to the covering space t where the fields become single-valued. In the t plane there is a non-trivial metric; we take account of this by defining a fiducial metric and computing the Liouville action. The Liouville action terms give

$$2^{-\frac{5}{4}}|a|^{-\frac{3}{4}(M+N)+\frac{1}{2}\left(\frac{M}{N}+\frac{N}{M}+1\right)}M^{-\frac{3}{4}M-\frac{1}{4}}N^{-\frac{3}{4}N-\frac{1}{4}}(M+N)^{\frac{3}{4}(M+N)-\frac{1}{4}}.\tag{5.22}$$

(iii) In the covering space, we must insert spin fields, each within an appropriate normalization to take account of the local form of the map near each insertion, $(z - z_*) \approx b_* (t - t_*)^n$ [23]. These normalization factors give the contribution

$$
|b_\infty|^{-\frac{1}{2(M+N)}} |b_{t_0}|^{-\frac{1}{4}} |b_a|^{-\frac{1}{2N}} |b_0|^{-\frac{1}{2M}} =
$$
$$
2^{\frac{1}{4}} |a|^{-\frac{1}{2}\left(\frac{M}{N}+\frac{N}{M}+\frac{M}{2}+\frac{N}{2}-1\right)} M^{-\frac{1}{4}(M-1)} N^{-\frac{1}{4}(N-1)} (M+N)^{\frac{1}{4}(M+N-3)} . \quad (5.23)
$$

(iv) Finally, the correlator of the spin fields in the t plane gives

$$
\frac{\langle S^+(\infty) S^+(t_0) S^-(a) S^-(0) \rangle}{\langle S^+(\infty) S^-(0) \rangle} = \frac{(M+N)^2}{MN} \frac{1}{|a|} . \quad (5.24)
$$

These four results combine to give the final result

$$
C_{MN} = \frac{M+N}{2MN} . \quad (5.25)
$$

In order to write the full expression for the effect of the twist operator σ_2^{++} on the state $|0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)}$, let us recall the expression for γ^B computed in [20],

$$
\gamma_{kl}^B = -\frac{(-a)^{k+l}}{\pi^2} \sin\left[\frac{\pi Mk}{M+N}\right] \sin\left[\frac{\pi Ml}{M+N}\right] \frac{MN}{(M+N)^2} \frac{1}{k+l} \frac{\Gamma[\frac{Mk}{M+N}]\Gamma[\frac{Nk}{M+N}]}{\Gamma[k]} \frac{\Gamma[\frac{Ml}{M+N}]\Gamma[\frac{Nl}{M+N}]}{\Gamma[l]} \quad (5.26)
$$

which since k and l are integers can be rewritten using

$$
(-1)^k \sin\left[\frac{\pi Mk}{M+N}\right] = -\sin\left[\frac{\pi Nk}{M+N}\right] , \quad (5.27)
$$

which gives

$$
\gamma_{kl}^B = -\frac{a^{k+l}}{\pi^2} \sin\left[\frac{\pi Nk}{M+N}\right] \sin\left[\frac{\pi Nl}{M+N}\right] \frac{MN}{(M+N)^2} \frac{1}{k+l} \frac{\Gamma[\frac{Mk}{M+N}]\Gamma[\frac{Nk}{M+N}]}{\Gamma[k]} \frac{\Gamma[\frac{Ml}{M+N}]\Gamma[\frac{Nl}{M+N}]}{\Gamma[l]} . \quad (5.28)
$$

Therefore, the full effect of the twist operator $\sigma_2^{++}(w_0, \bar{w}_0)$ on the state $|0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)}$ is

$$
\begin{aligned}
|\chi\rangle &\equiv \sigma_2^{++}(w_0) |0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)} \\
&= C_{MN} \left[\exp\left(\sum_{m\geq 1, n\geq 1} \gamma_{mn}^B \left[-\alpha_{++,-m}\alpha_{--,-n} + \alpha_{+-,-m}\alpha_{-+,-n} \right] \right) \right. \\
&\qquad \left. \times \exp\left(\sum_{m\geq 0, n\geq 1} \gamma_{mn}^F \left[d_{-m}^{++} d_{-n}^{--} - d_{-m}^{+-} d_{-n}^{-+} \right] \right) \times \text{[antihol.]} \right] |0_R^{--}\rangle \quad (5.29)
\end{aligned}
$$

where

$$
C_{MN} = \frac{M+N}{2MN}
$$

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$$
\begin{aligned}
\gamma_{kl}^{B} &= -\frac{a^{k+l}}{\pi^2}\sin\left[\tfrac{\pi Nk}{M+N}\right]\sin\left[\tfrac{\pi Nl}{M+N}\right]\frac{MN}{(M+N)^2}\frac{1}{k+l}\frac{\Gamma[\frac{Mk}{M+N}]\Gamma[\frac{Nk}{M+N}]}{\Gamma[k]}\frac{\Gamma[\frac{Ml}{M+N}]\Gamma[\frac{Nl}{M+N}]}{\Gamma[l]} \\
\gamma_{kl}^{F} &= \frac{a^{k+l}}{\pi^2}\sin\left[\tfrac{\pi Nk}{M+N}\right]\sin\left[\tfrac{\pi Nl}{M+N}\right]\frac{MN}{(M+N)^2}\frac{k}{k+l}\frac{\Gamma[\frac{Mk}{M+N}]\Gamma[\frac{Nk}{M+N}]}{\Gamma[k]}\frac{\Gamma[\frac{Ml}{M+N}]\Gamma[\frac{Nl}{M+N}]}{\Gamma[l]}.
\end{aligned}
\tag{5.30}
$$

5.3 Applying the Supercharge

We now apply the supercharge to obtain the full effect of the deformation operator on the state $|0_R^{--}\rangle^{(1)}|0_R^{--}\rangle^{(2)}$. For ease of notation, we introduce notation for the holomorphic and antiholomorphic parts of the state $|\boldsymbol{\chi}\rangle$,

$$
|\boldsymbol{\chi}\rangle = |\chi\rangle\,|\bar{\chi}\rangle,
\tag{5.31}
$$

where we divide the prefactor C_{MN} equally between the holomorphic and antiholomorphic parts, i.e.

$$
\begin{aligned}
|\chi\rangle = \sqrt{C_{MN}}\Bigg[&\exp\left(\sum_{m\geq 1,n\geq 1}\gamma_{mn}^{B}\left[-\alpha_{++,-m}\alpha_{--,-n}+\alpha_{+-,-m}\alpha_{-+,-n}\right]\right) \\
&\times\exp\left(\sum_{m\geq 0,n\geq 1}\gamma_{mn}^{F}\left[d_{-m}^{++}d_{-n}^{--}-d_{-m}^{+-}d_{-n}^{-+}\right]\right)\Bigg]|0_R^{-}\rangle.
\end{aligned}
\tag{5.32}
$$

Similarly, we write the final state obtained by acting with holomorphic and antiholomorphic supercharges as

$$
|\Psi\rangle = |\psi\rangle\,|\bar{\psi}\rangle
\tag{5.33}
$$

where

$$
|\psi\rangle = G_{\dot{A},0}^{-}|\chi\rangle
\tag{5.34}
$$

and similarly for the antiholomorphic part. We now compute the state $|\psi\rangle$.

From (2.26) we have

$$
G_{\dot{A},0}^{-} = \frac{1}{2\pi i}\int\limits_{w=\tau_0+\epsilon}^{w=\tau_0+\epsilon+2\pi i(M+N)} G_{\dot{A}}^{-}(w)\,dw = \frac{i}{\sqrt{M+N}}\sum_{n=-\infty}^{\infty}d_n^{-A}\alpha_{A\dot{A},-n}.
\tag{5.35}
$$

We wish to write (5.34) with only negative index modes acting on $|0_R^{-}\rangle$. We thus write

$$
G_{\dot{A},0}^{-} = \frac{i}{\sqrt{M+N}}\left(\sum_{l>0}^{\infty}d_{-l}^{-A}\alpha_{A\dot{A},l}+\sum_{l>0}^{\infty}d_l^{-A}\alpha_{A\dot{A},-l}+d_0^{-A}\alpha_{A\dot{A},0}\right)
\tag{5.36}
$$

We note e.g. from (5.28) that γ^B_{kl} is symmetric, so we have

$$
\frac{i}{\sqrt{M+N}} \sum_{l \geq 1}^{\infty} d^{-A}_{-l} \alpha_{A\dot{A},l} |\chi\rangle = \frac{i}{\sqrt{M+N}} \sum_{l \geq 1} \sum_{l \geq 1} l \gamma^B_{kl} d^{-A}_{-l} \alpha_{A\dot{A},-k} |\chi\rangle
$$

$$
\frac{i}{\sqrt{M+N}} \sum_{k \geq 1}^{\infty} d^{-A}_{k} \alpha_{A\dot{A},-k} |\chi\rangle = -\frac{i}{\sqrt{M+N}} \sum_{k \geq 1} \sum_{l \geq 1} \gamma^F_{kl} d^{-A}_{-l} \alpha_{A\dot{A},-k} |\chi\rangle
$$

$$
\frac{i}{\sqrt{M+N}} d^{-A}_0 \alpha_{A\dot{A}0} |\chi\rangle = 0 \tag{5.37}
$$

where we have used the commutation relations in (3.5). Thus

$$
G^-_{\dot{A},0} |\chi\rangle = \frac{i}{\sqrt{M+N}} \sum_{k \geq 1, l \geq 1} \left(l \gamma^B_{kl} - \gamma^F_{kl} \right) d^{-A}_{-l} \alpha_{A\dot{A},-k} |\chi\rangle . \tag{5.38}
$$

We observe that the l and k sums factorize and we obtain

$$
|\psi\rangle = G^-_{\dot{A},0} |\chi\rangle = -\frac{i}{\pi^2} \frac{MN}{(M+N)^{\frac{5}{2}}} \left(\sum_{l \geq 1} a^l \sin \left[\frac{N\pi l}{M+N} \right] \frac{\Gamma \left[\frac{Ml}{M+N} \right] \Gamma \left[\frac{Nl}{M+N} \right]}{\Gamma [l]} d^{-A}_{-l} \right)
$$

$$
\times \left(\sum_{k \geq 1} a^k \sin \left[\frac{N\pi k}{M+N} \right] \frac{\Gamma \left[\frac{Mk}{M+N} \right] \Gamma \left[\frac{Nk}{M+N} \right]}{\Gamma [k]} \alpha_{A\dot{A},-k} \right) |\chi\rangle . \tag{5.39}
$$

One can check[5] that in the case of $M = N = 1$ this expression agrees with that computed in [18].

Analogous expressions hold for

$$
|\bar{\psi}\rangle = \bar{G}^-_{\dot{B},0} |\bar{\chi}\rangle , \tag{5.40}
$$

and the complete final state is given by

$$
|\Psi\rangle = |\psi\rangle |\bar{\psi}\rangle . \tag{5.41}
$$

6 Effect of the deformation operator on excited states

We now consider the case where one of the initial component strings has an initial oscillator excitation. We first consider the fermionic excitations. From (3.13) we recall the definition

$$
\sigma_2^{++}(w_0) d^{(i)\pm A}_{-m} |0^{--}_R\rangle^{(1)} |0^{--}_R\rangle^{(2)} = \sum_k f^{F(i)\pm}_{mk} d^{(i)\pm A}_{-k} |\chi\rangle , \qquad i = 1, 2 . \tag{6.1}
$$

In this section we compute $f^{F(i)\pm}_{mk}$.

---

[5]In order to verify this, note that due to different choices in normalization of the modes of the fermions, one should replace $d^{-A} \to \frac{1}{\sqrt{2}} d^{-A}$; in addition, there is an overall minus sign due to the different directionality of the contour for $G^-_{\dot{A},0}$.

6.1 Computing $f_{mk}^{F(1)+}$

Let us start with $f_{mk}^{F(1)+}$. Consider the ratio of amplitudes

$$\frac{\mathcal{A}_3}{\mathcal{A}_1} = \frac{\langle 0_{R,--}| \, d_k^{--} \sigma_2^{++}(w_0) \, d_{-m}^{(1)++} \, |0_R^{--}\rangle^{(1)} \, |0_R^{--}\rangle^{(2)}}{\langle 0_{R,--}| \, \sigma_2^{++}(w_0) \, |0_R^{--}\rangle^{(1)} \, |0_R^{--}\rangle^{(2)}} \, . \tag{6.2}$$

From (6.1) we observe that

$$\begin{aligned}
\mathcal{A}_3 &= \langle 0_{R,--}| \, d_k^{--} \sigma_2^{++}(w_0) \, d_{-m}^{(1)++} \, |0_R^{--}\rangle^{(1)} \, |0_R^{--}\rangle^{(2)} \\
&= \sum_{l \geq 1} f_{ml}^{F(1)+} \langle 0_{R,--}| \, d_k^{--} d_{-l}^{++} \, |\chi\rangle \\
&= -C_{MN} f_{mk}^{F(1)+} \tag{6.3}
\end{aligned}$$

So we have

$$f_{mk}^{F(1)+} = -\frac{\mathcal{A}_3}{\mathcal{A}_1} = -\frac{{}_t\langle 0_{NS}| \, d_k'^{--} d_{-m}'^{(1)++} \, |0_{NS}\rangle_t}{{}_t\langle 0_{NS}|0_{NS}\rangle_t} \tag{6.4}$$

where on the RHS, the t-plane amplitude is after all spectral flows and coordinate shifts have been carried out for the copy 1 quantities $f^{F(1)\pm}$. For this calculation, we use the t coordinate in which copy 1 is at the origin. When we calculate the copy 2 quantities $f^{F(2)\pm}$ we will use the $\hat{t}$ coordinate.

We now compute the empty t plane amplitude

$$-\mathcal{A}_3 = -{}_t\langle 0_{NS}| \, d_k'^{--} d_{-m}'^{(1)++} \, |0_{NS}\rangle_t \tag{6.5}$$

using the mode expansions in (4.11) and (4.12). Let us first expand $d_k'^{--}$ around $t = \infty$. From (4.12) we have

$$d_k'^{--} = \frac{1}{2\pi i} \oint_{t=\infty} dt \, \psi^{--}(t) \left[t^{\frac{Mk}{M+N}} (t-a)^{\frac{Nk}{M+N}} \right] . \tag{6.6}$$

We therefore expand

$$t^{\frac{Mk}{M+N}} (t-a)^{\frac{Nk}{M+N}} = t^k \left(1 - at^{-1}\right)^{\frac{Nk}{M+N}} = \sum_{p \geq 0} {}^{\frac{Nk}{M+N}}C_p (-a)^p t^{k-p} . \tag{6.7}$$

For our purposes, the sum is truncated by the requirement that $\tilde{d}_{k-p+\frac{1}{2}}^{--}$ be an annihilation operator. So in terms of modes natural to the t-plane we find

$$d_k'^{--} \rightarrow \sum_{p=0}^{k} {}^{\frac{Nk}{M+N}}C_p (-a)^p \tilde{d}_{k-p+\frac{1}{2}}^{--} . \tag{6.8}$$

Next, using (4.11), we expand

$$d_{-m}^{\prime(1)++} = \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{t=0} dt\, \psi^{(1)++}(t) \left[t^{-m-1} \left(t - \frac{Ma}{M+N} \right) (t-a)^{-\frac{Nm}{M}-1} \right] \tag{6.9}$$

We find

$$t^{-m-1}\left(t - \frac{M}{M+N}a\right)(t-a)^{-\frac{Nm}{M}-1}$$
$$= (-a)^{-\frac{Nm}{M}-1}\left(t^{-m} - \frac{M}{M+N}at^{-m-1}\right) \sum_{p'\geq 0} {}^{-\frac{Nm}{M}-1}C_{p'}(-a)^{-p'}t^{p'}$$
$$= (-a)^{-\frac{Nm}{M}-1}\left[\sum_{p'\geq 0} {}^{-\frac{Nm}{M}-1}C_{p'}(-a)^{-p'}t^{p'-m} + \sum_{p'\geq 0} {}^{-\frac{Nm}{M}-1}C_{p'}\frac{M}{M+N}(-a)^{1-p'}t^{p'-m-1}\right] \tag{6.10}$$

This gives

$$d_{-m}^{\prime(1)++} \rightarrow \frac{\sqrt{M+N}}{\sqrt{M}} \sum_{p'=0}^{m-1} {}^{-\frac{Nm}{M}-1}C_{p'}(-a)^{-\frac{Nm}{M}-p'-1}\tilde{d}_{p'-m+\frac{1}{2}}^{++}$$
$$+ \frac{\sqrt{M}}{\sqrt{M+N}} \sum_{p'=0}^{m} {}^{-\frac{Nm}{M}-1}C_{p'}(-a)^{-\frac{Nm}{M}-p'}\tilde{d}_{p'-m-\frac{1}{2}}^{++} \tag{6.11}$$

where again the upper limits on the sums are determined by the requirement that the operators on the RHS be creation operators. We now use the mode expansions to compute

$$-\,{}_t\langle 0_{NS}|\, d_k^{\prime++}d_{-m}^{\prime(1)--}\,|0_{NS}\rangle_t \tag{6.12}$$

Since the expansion of $d_{-m}^{\prime(1)++}$ involves two separate terms, each involving a sum, we will separately calculate the contributions from these terms and add the resulting expressions to find the above amplitude. The contribution to the amplitude from the first term in the expansion of $d_m^{\prime(1)++}$ in (6.11) is

$$-\sum_{p=0}^{k}\sum_{p'=0}^{m-1} \frac{\sqrt{M+N}}{\sqrt{M}} {}^{\frac{Nk}{M+N}}C_p\, {}^{-\frac{Nm}{M}-1}C_{p'}(-a)^{-\frac{Nm}{M}+(p-p')-1}{}_t\langle 0_{NS}|\,\tilde{d}_{k-p+\frac{1}{2}}^{--}\tilde{d}_{p'-m+\frac{1}{2}}^{++}\,|0_{NS}\rangle_t \tag{6.13}$$

where we have used the modes natural to the t plane given in (4.13). Using the anticommutation relations in (4.15) we have the following constraints on the mode numbers

$$k - p + \frac{1}{2} = -\left(p' - m + \frac{1}{2}\right) \tag{6.14}$$

This gives

$$\frac{\sqrt{M+N}}{\sqrt{M}}(-a)^{k-\frac{m(M+N)}{M}} \sum_{p'=\max(m-k-1,0)}^{m-1} {}^{\frac{Nk}{M+N}}C_{k-m+p'+1}\, {}^{-\frac{Nm}{M}-1}C_{p'}\,{}_t\langle 0_{NS}|0_{NS}\rangle_t \tag{6.15}$$

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For the second term we find

$$-\frac{\sqrt{M}}{\sqrt{M+N}}\sum_{p=0}^{k}\sum_{p'=0}^{m}{}^{\frac{Nk}{M+N}}C_p{}^{-\frac{Nm}{M}-1}C_{p'}(-a)^{-\frac{Nm}{M}+(p-p')}{}_t\langle 0_{NS}|\,\tilde{d}^{--}_{k-p+\frac{1}{2}}\tilde{d}^{++}_{p'-m-\frac{1}{2}}\,|0_{NS}\rangle_t \quad (6.16)$$

which upon using the commutation relations becomes

$$\frac{\sqrt{M}}{\sqrt{M+N}}(-a)^{k-\frac{m(M+N)}{M}}\sum_{p'=\max(m-k,0)}^{m}{}^{\frac{Nk}{M+N}}C_{k-m+p'}{}^{-\frac{Nm}{M}-1}C_{p'}\,{}_t\langle 0_{NS}|0_{NS}\rangle_t. \quad (6.17)$$

Adding together (6.15) and (6.17), and using (6.4), we find

$$\begin{aligned}
f^{F(1)+}_{mk} &= \frac{\sqrt{M+N}}{\sqrt{M}}(-a)^{k-\frac{m(M+N)}{M}}\sum_{p'=\max(m-k-1,0)}^{m-1}{}^{\frac{Nk}{M+N}}C_{k-m+p'+1}{}^{-\frac{Nm}{M}-1}C_{p'} \\
&\quad + \frac{\sqrt{M}}{\sqrt{M+N}}(-a)^{k-\frac{m(M+N)}{M}}\sum_{p'=\max(m-k,0)}^{m}{}^{\frac{Nk}{M+N}}C_{k-m+p'}{}^{-\frac{Nm}{M}-1}C_{p'} \quad (6.18)
\end{aligned}$$

Evaluating the sums, we obtain

$$f^{F(1)+}_{mk} = \frac{(-1)^m\sqrt{M}\sin\left(\pi\frac{Mk}{M+N}\right)}{\pi(M+N)^{\frac{3}{2}}}\frac{(-a)^{k-\frac{m(M+N)}{M}}}{\frac{k}{M+N}-\frac{m}{M}}\frac{k}{m}\frac{\Gamma\left[\frac{Mk}{M+N}\right]\Gamma\left[\frac{Nk}{M+N}\right]}{\Gamma[k]}\frac{\Gamma\left[\frac{(M+N)m}{M}\right]}{\Gamma[m]\Gamma\left[\frac{Nm}{M}\right]}. \quad (6.19)$$

Note that in the above expression, when we have

$$\frac{k}{M+N}=\frac{m}{M} \quad (6.20)$$

both numerator and denominator vanish, since

$$\sin\left(\pi\frac{Mk}{M+N}\right)=\sin\left(\pi m\right)=0. \quad (6.21)$$

Since the above expression for $f^{F(1)+}_{mk}$ is indeterminate in this situation, we return to the sum in (6.18), and take parameter values

$$m = Mc, \qquad k = (M+N)c, \qquad c = \frac{j}{Y} \quad (6.22)$$

where j is a positive integer and $Y=\gcd(M,N)$. Since $m < k$, we have

$$\begin{aligned}
f^{F(1)+}_{mk} &= \frac{\sqrt{M+N}}{\sqrt{M}}(-a)^{k-\frac{m(M+N)}{M}}\sum_{p'=0}^{m-1}{}^{\frac{Nk}{M+N}}C_{k-m+p'+1}{}^{-\frac{Nm}{M}-1}C_{p'} \\
&\quad + \frac{\sqrt{M}}{\sqrt{M+N}}(-a)^{k-\frac{m(M+N)}{M}}\sum_{p'=0}^{m}{}^{\frac{Nk}{M+N}}C_{k-m+p'}{}^{-\frac{Nm}{M}-1}C_{p'} \\
&= \frac{\sqrt{M+N}}{\sqrt{M}}\sum_{p'=0}^{Mc-1}{}^{Nc}C_{Nc+p'+1}{}^{-Nc-1}C_{p'} + \frac{\sqrt{M}}{\sqrt{M+N}}\sum_{p'=0}^{Mc}{}^{Nc}C_{Nc+p'}{}^{-Nc-1}C_{p'} \\
&= \frac{\sqrt{M}}{\sqrt{M+N}}. \quad (6.23)
\end{aligned}$$

6.2 Computing $f_{mk}^{F(1)-}$

Next we compute $f^{F(1)-}$. The method is entirely analogous; this time, we take the amplitude in the numerator to be

$$
\begin{aligned}
\mathcal{A}_4 &= \langle 0_{R,--} | \, d_k^{++} \sigma_2^+ \, (w_0) \, d_{-m}^{\prime(1)--} \, |0_R^{--}\rangle^{(1)} \, |0_R^{--}\rangle^{(2)} \\
&= -C_{MN} f_{mk}^{F(1)-},
\end{aligned}
\tag{6.24}
$$

so we have

$$
f_{mk}^{F(1)-} = -\frac{\mathcal{A}_4}{\mathcal{A}_1} = -\frac{{}_t\langle 0_{NS} | \, d_k^{\prime++} d_{-m}^{\prime(1)--} \, |0_{NS}\rangle_t}{{}_t\langle 0_{NS} | 0_{NS}\rangle_t}
\tag{6.25}
$$

where again on the RHS the amplitudes are in the empty t plane after all spectral flows.

Expanding the modes as before, we obtain

$$
\begin{aligned}
f_{mk}^{F(1)-} &= \frac{1}{\sqrt{M(M+N)}} (-a)^{k - \frac{(M+N)m}{M}} \\
&\quad \times \Bigg[(M+N) \sum_{p'=\max(m-k,0)}^{m-1} {}^{\frac{Nk}{M+N}-1}C_{k-m+p'} \, {}^{-\frac{Nm}{M}}C_{p'} \\
&\qquad + M \sum_{p'=\max(m-k+1,0)}^{m-1} {}^{\frac{Nk}{M+N}-1}C_{k-m+p'-1} \, {}^{-\frac{Nm}{M}}C_{p'} \Bigg].
\end{aligned}
\tag{6.26}
$$

Evaluating the sums, we find

$$
f_{mk}^{F(1)-} = \frac{(-1)^m \sin\left(\pi \frac{Mk}{M+N}\right)}{\pi \sqrt{M(M+N)}} \frac{(-a)^{k - \frac{m(M+N)}{M}}}{\frac{k}{M+N} - \frac{m}{M}} \frac{\Gamma\left[\frac{Mk}{M+N}\right] \Gamma\left[\frac{Nk}{M+N}\right]}{\Gamma[k]} \frac{\Gamma\left[\frac{(M+N)m}{M}\right]}{\Gamma[m] \Gamma\left[\frac{Nm}{M}\right]}.
\tag{6.27}
$$

As before, for the special case of

$$
\frac{k}{M+N} = \frac{m}{M}
\tag{6.28}
$$

we have an indeterminate expression for $f_{mk}^{F(1)-}$ and using (6.26) we find

$$
\begin{aligned}
f_{mk}^{F(1)-} &= \frac{1}{\sqrt{M(M+N)}} \Bigg[(M+N) \sum_{p'=0}^{Mc-1} {}^{Nc-1}C_{Nc+p'} \, {}^{-Nc}C_{p'} + M \sum_{p'=0}^{Mc-1} {}^{Nc-1}C_{Nc+p'-1} \, {}^{-Nc}C_{p'} \Bigg] \\
&= \frac{\sqrt{M}}{\sqrt{M+N}}.
\end{aligned}
\tag{6.29}
$$

6.3 Computing $f_{mk}^{F(2)+}$ and $f_{mk}^{F(2)-}$

For an initial excitation on copy 2, we calculate $f_{mk}^{F(2)+}$. Similar to the calculation of $f_{mk}^{F(1)+}$, we consider the amplitude

$$
\begin{aligned}
\mathcal{A}_5 &= \langle 0_{R,--}| \, d_k^{--} \sigma_2^{++}(w_0) \, d_{-m}^{(2)++} |0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)} \\
&= \sum_{l \geq 1} f_{ml}^{F(2)+} \langle 0_{R,--}| \, d_k^{--} d_{-l}^{++} |\chi\rangle \\
&= -C_{MN} f_{mk}^{F(2)+}
\end{aligned}
\tag{6.30}
$$

So we have

$$
f_{mk}^{F(2)+} = -\frac{\mathcal{A}_4}{\mathcal{A}_1} = -\frac{{}_t\langle 0_{NS}| \, d_k'^{--} d_{-m}'^{(2)++} |0_{NS}\rangle_t}{{}_t\langle 0_{NS}|0_{NS}\rangle_t}. \tag{6.31}
$$

Let us now compute the empty $\hat{t}$ plane amplitude

$$
\begin{aligned}
&{}_{\hat{t}}\langle 0_{NS}| \, d_k'^{--} d_{-m}'^{(2)++} |0_{NS}\rangle_{\hat{t}} \\
&= {}_{\hat{t}}\langle 0_{NS}| \frac{1}{2\pi i} \oint_{\hat{t}=\infty} d\hat{t} \psi^{--}(\hat{t}) \left[(\hat{t}+a)^{\frac{Mk}{M+N}} \hat{t}^{\frac{Nk}{M+N}} \right] \\
&\quad \times \frac{\sqrt{M+N}}{\sqrt{N}} \frac{1}{2\pi i} \oint_{\hat{t}=0} d\hat{t} \psi^{++}(\hat{t}) \left[(\hat{t}+a)^{-\frac{Mm}{N}-1} \left(\hat{t} + \frac{Na}{M+N} \right) \hat{t}^{-m-1} \right] |0_{NS}\rangle_{\hat{t}} \quad (6.32)
\end{aligned}
$$

Looking at (6.32), (6.6) and (6.9), we notice that if we make following interchanges

$$
M \leftrightarrow N, \quad a \leftrightarrow -a \tag{6.33}
$$

then

$$
f_{mk}^{F(2)+} \leftrightarrow f_{mk}^{F(1)+} \tag{6.34}
$$

Therefore we can use the results obtained for the expression $f_{mk}^{F(1)+}$ and make the interchanges given in (6.33) to obtain the expression for $f_{mk}^{F(2)+}$. The same applies for

$$
f_{mk}^{F(2)-} \leftrightarrow f_{mk}^{F(1)-}. \tag{6.35}
$$

Then from (6.19), (6.33) we find that for the case $\frac{k}{M+N} \neq \frac{m}{N}$ we have

$$
f_{mk}^{F(2)+} = \frac{(-1)^m \sqrt{N} \sin\left(\pi \frac{Nk}{M+N}\right)}{\pi (M+N)^{\frac{3}{2}}} \frac{a^{k-\frac{m(M+N)}{N}}}{\frac{k}{M+N} - \frac{m}{N}} \frac{k}{m} \frac{\Gamma\left[\frac{Mk}{M+N}\right] \Gamma\left[\frac{Nk}{M+N}\right]}{\Gamma[k]} \frac{\Gamma\left[\frac{(M+N)m}{N}\right]}{\Gamma[m] \Gamma\left[\frac{Mm}{N}\right]} \tag{6.36}
$$

$$
f_{mk}^{F(2)-} = \frac{(-1)^m \sin\left(\pi \frac{Nk}{M+N}\right)}{\pi \sqrt{N(M+N)}} \frac{a^{k-\frac{m(M+N)}{N}}}{\frac{k}{M+N} - \frac{m}{N}} \frac{\Gamma\left[\frac{Mk}{M+N}\right] \Gamma\left[\frac{Nk}{M+N}\right]}{\Gamma[k]} \frac{\Gamma\left[\frac{(M+N)m}{N}\right]}{\Gamma[m] \Gamma\left[\frac{Mm}{N}\right]} \tag{6.37}
$$

and for the case of $\frac{k}{M+N} = \frac{m}{N}$ we have

$$
f_{mk}^{F(2)+} = f_{mk}^{F(2)-} = \frac{\sqrt{N}}{\sqrt{M+N}}. \tag{6.38}
$$

6.4 Expressing $f_{mk}^{F(i)\pm}$ in final form

To express $f_{mk}^{F(i)\pm}$ in final form, we now make analogous changes of notation as done for γ_{kl}^F.

(i) Using (4.7), we replace the parameter a by $z_0 = e^{w_0}$.

(ii) For component string (1) we use fractional modes

$$q = \frac{m}{M}, \qquad s = \frac{k}{M+N} \tag{6.39}$$

and for component string (2) we use fractional modes

$$r = \frac{m}{N}, \qquad s = \frac{k}{M+N} . \tag{6.40}$$

When expressing our result for component string (1) using indices q and s, we write

$$f_{mk}^{F(1)\pm} \to \tilde{f}_{qs}^{F(1)\pm} \tag{6.41}$$

and for component string (2) using indices r and s, we write

$$f_{mk}^{F(2)\pm} \to \tilde{f}_{rs}^{F(2)\pm} . \tag{6.42}$$

(iii) We use the shorthand $\mu_s = (1 - e^{2\pi i Ms})$. In doing this we note that

$$\sin(\pi Ms) = \frac{i}{2} e^{-i\pi Ms} \mu_s . \tag{6.43}$$

With these changes in notation, for component string (1) with $s \neq q$ we have

$$
\begin{aligned}
\tilde{f}_{qs}^{F(1)+} &= \frac{i}{2\pi} z_0^{s-q} \frac{\mu_s}{\sqrt{M(M+N)}} \frac{1}{s-q} \frac{s}{q} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{s-q} \frac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} \\
\tilde{f}_{qs}^{F(1)-} &= \frac{i}{2\pi} z_0^{s-q} \frac{\mu_s}{\sqrt{M(M+N)}} \frac{1}{s-q} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{s-q} \frac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]}
\end{aligned}
\tag{6.44}
$$

For component string (2) with $s \neq r$ we obtain

$$
\begin{aligned}
\tilde{f}_{rs}^{F(2)+} &= -\frac{i}{2\pi} z_0^{s-r} \frac{\mu_s}{\sqrt{N(M+N)}} \frac{1}{s-r} \frac{s}{r} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{s-r} \frac{\Gamma[(M+N)r]}{\Gamma[Mr]\Gamma[Nr]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} \\
\tilde{f}_{rs}^{F(2)-} &= -\frac{i}{2\pi} z_0^{s-r} \frac{\mu_s}{\sqrt{N(M+N)}} \frac{1}{s-r} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{s-r} \frac{\Gamma[(M+N)r]}{\Gamma[Mr]\Gamma[Nr]} \frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]}
\end{aligned}
\tag{6.45}
$$

For $M = N = 1$, given the normalizations chosen, the quantities $\frac{1}{\sqrt{2}} f^{F(i)+}$ in this paper should agree with the $f^{F(i)+}$ computed in [19]. One can check that this is indeed the case.

6.5 Applying the supercharge for an initial fermionic excitation

In this section we include the supercharge, in order to obtain the full effect of the deformation operator on an initial fermionic excitation. We introduce the notation

$$|\Psi^{F(i)\pm}\rangle = \hat{O}_{\dot{A}}(w_0)d^{(i)\pm B}_{-m}|0^{--}_R\rangle^{(1)}|0^{--}_R\rangle^{(2)}. \tag{6.46}$$

As before we introduce separate notation for the holomorphic and antiholomorphic parts,

$$|\Psi^{F(i)\pm}\rangle = |\psi^{F(i)\pm}\rangle |\bar{\psi}^{F(i)\pm}\rangle. \tag{6.47}$$

Since we now have have an initial excitation, we must compute the contribution of the supercharge operator above and below the twist insertion. Writing only the holomorphic parts, recall that

$$\hat{O}_{\dot{A}} = G^-_{\dot{A},0}\sigma^+_2 - \sigma^+_2\left(G^{(1)-}_{\dot{A},0} + G^{(2)-}_{\dot{A},0}\right) \tag{6.48}$$

where from (2.27) we recall

$$-G^{(1)-}_{\dot{A},0} = -\frac{1}{2\pi i}\int\limits_{w=\tau_0-\epsilon}^{\tau_0-\epsilon+2\pi iM} G^{(1)-}_{\dot{A}}(w)dw = -\frac{i}{\sqrt{M}}\sum_n d^{(1)-A}_n \alpha^{(1)}_{A\dot{A},-n} \tag{6.49}$$

and similarly for component string (2). There four cases to compute: $|\psi^{F(1)+}\rangle$, $|\psi^{F(1)-}\rangle$, $|\psi^{F(2)+}\rangle$, and $|\psi^{F(2)-}\rangle$.

Before proceeding, we record the expression for $f^{B(1)}_{mk}$ calculated in [20]:

$$f^{B(1)}_{mk} = \frac{(-1)^m\sin\left(\pi\frac{Mk}{M+N}\right)}{\pi(M+N)}\frac{(-a)^{k-\frac{m(M+N)}{M}}}{\frac{k}{M+N}-\frac{m}{M}}\frac{\Gamma\left[\frac{Mk}{M+N}\right]\Gamma\left[\frac{Nk}{M+N}\right]}{\Gamma[k]}\frac{\Gamma\left[\frac{(M+N)m}{M}\right]}{\Gamma[m]\Gamma\left[\frac{Nm}{M}\right]}. \tag{6.50}$$

6.5.1 $|\psi^{F(1)+}\rangle$

Let us begin with $|\psi^{F(1)+}\rangle$. Our final state is given by (we write only the holomorphic parts)

$$|\psi^{F(1)+}\rangle = \left(G^-_{\dot{A},0}\sigma^+_2(w_0)d^{(1)+B}_{-m} - \sigma^+_2 G^{(1)-}_{\dot{A},0}d^{(1)+B}_{-m}\right)|0^-_R\rangle^{(1)}|0^-_R\rangle^{(2)} \tag{6.51}$$

Using (3.13) the first term is given by

$$G^-_{\dot{A},0}\sum_{k\geq 1}f^{F(1)+}_{mk}d^{+B}_{-k}|\chi\rangle = \frac{i\epsilon^{AB}}{\sqrt{M+N}}\sum_{k\geq 1}f^{F(1)+}_{mk}\alpha_{A\dot{A},-k}|\chi\rangle + \sum_{k\geq 1}f^{F(1)+}_{mk}d^{+B}_{-k}|\psi\rangle \tag{6.52}$$

where we recall the notation $|\psi\rangle = G^-_{\dot{A},0}|\chi\rangle$. $|\psi\rangle$ is given explicitly in (5.39).

For the second term, using $G^{(1)-}_{\dot{A},0} |0^-_R\rangle^{(1)} = 0$ we have

$$\sigma^+_2(w_0) G^{(1)-}_{\dot{A},0} d^{(1)+B}_{-m} |0^-_R\rangle^{(1)} |0^-_R\rangle^{(2)} = \sigma^+_2(w_0)\{G^{(1)-}_{\dot{A},0}, d^{(1)+B}_{-m}\} |0^-_R\rangle^{(1)} |0^-_R\rangle^{(2)} \quad (6.53)$$

$$= \frac{i\epsilon^{AB}}{\sqrt{M}} \sum_{k\geq 1} f^{B(1)}_{mk} \alpha_{A\dot{A},-k} |\chi\rangle . \quad (6.54)$$

Combining both terms we obtain

$$|\psi^{F(1)+}\rangle = i\epsilon^{AB} \sum_{k\geq 1} \left(\frac{f^{F(1)+}_{mk}}{\sqrt{M+N}} - \frac{f^{B(1)}_{mk}}{\sqrt{M}} \right) \alpha_{A\dot{A},-k} |\chi\rangle + \sum_{k\geq 1} f^{F(1)+}_{mk} d^{+B}_{-k} |\psi\rangle \quad (6.55)$$

where we note that

$$\frac{f^{F(1)+}_{mk}}{\sqrt{M+N}} - \frac{f^{B(1)}_{mk}}{\sqrt{M}}$$

$$= \frac{1}{\pi} \frac{\sqrt{M}}{M+N} \frac{(-1)^m}{m} \sin\left(\pi \frac{Mk}{M+N}\right) (-a)^{k-\frac{m(M+N)}{M}} \frac{\Gamma\left[\frac{Mk}{M+N}\right] \Gamma\left[\frac{Nk}{M+N}\right]}{\Gamma[k]} \frac{\Gamma\left[\frac{(M+N)m}{M}\right]}{\Gamma[m]\,\Gamma\left[\frac{Nm}{M}\right]} .$$
$$(6.56)$$

6.5.2 $|\psi^{F(1)-}\rangle$

Here our final state is given by

$$|\psi^{F(1)-}\rangle = \left(G^-_{\dot{A},0}\sigma^+_2(w_0) d^{(1)-B}_{-m} - \sigma^+_2 G^{(1)-}_{\dot{A},0} d^{(1)-B}_{-m} \right) |0^-_R\rangle^{(1)} |0^-_R\rangle^{(2)} \quad (6.57)$$

For the first term we find

$$G^-_{\dot{A},0}\sigma^+_2 d^{(1)-B}_{-m} |0^-_R\rangle^{(1)} |0^-_R\rangle^{(2)} = G^-_{\dot{A},0} \sum_{k\geq 1} f^{F(1)-}_{mk} d^{-B}_{-k} |\chi\rangle = \sum_{k\geq 1} f^{F(1)-}_{mk} d^{-B}_{-k} |\psi\rangle \quad (6.58)$$

and for the second term, using $G^{(1)-}_{\dot{A},0} |0^-_R\rangle^{(1)} = 0$ we have

$$\sigma^+_2 G^{(1)-}_{\dot{A},0} d^{(1)-B}_{-m} |0^-_R\rangle^{(1)} |0^-_R\rangle^{(2)} = 0 . \quad (6.59)$$

Therefore (6.57) becomes

$$|\psi^{F(1)-}\rangle = -\sum_{k\geq 1} f^{F(1)-}_{mk} d^{-B}_{-k} |\psi\rangle . \quad (6.60)$$

6.5.3 $|\psi^{F(2)+}\rangle$ and $|\psi^{F(2)-}\rangle$

The states $|\psi^{F(2)+}\rangle$ and $|\psi^{F(2)+}\rangle$ may be computed by modifying (6.55) and (6.60), respectively, by $M \leftrightarrow N$, $a \to -a$, and $(1) \to (2)$.

6.6 Applying the supercharge for an initial bosonic excitation

We now compute the state produced when the deformation operator acts on an initial bosonic excitation. In the same way done for an initial fermionic excitation, we introduce the notation

$$|\Psi^{B(i)}\rangle = \hat{O}_{\dot{A}}(w_0)\, \alpha^{(i)}_{B\dot{B},-m} |0_R^{--}\rangle^{(1)} |0_R^{--}\rangle^{(2)} = |\psi^{B(i)}\rangle\, |\bar{\psi}^{B(i)}\rangle. \tag{6.61}$$

6.6.1 $|\psi^{B(1)}\rangle$

We first compute the state $|\psi^{B(1)}\rangle$. We have

$$|\psi^{B(1)}\rangle = \left(G^-_{\dot{A},0}\sigma_2^+(w_0)\alpha^{(1)}_{B\dot{B},-m} - \sigma_2^+ G^{(1)-}_{\dot{A},0} \alpha^{(1)}_{B\dot{B},-m} \right) |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)} \tag{6.62}$$

The first term becomes

$$G^-_{\dot{A},0}\sum_{k\geq 1} f^{B(1)}_{mk}\alpha_{B\dot{B},-k} |\chi\rangle = i\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\sum_{k\geq 1}\left(\frac{k\, f^{B(1)}_{mk}}{\sqrt{M+N}}\right) d^{-A}_{-k} |\chi\rangle + \sum_{k\geq 1} f^{B(1)}_{mk}\alpha_{B\dot{B},-k} |\psi\rangle$$

For the second term, we find

$$\sigma_2^+ G^{(1)-}_{\dot{A},0} \alpha^{(1)}_{B\dot{B},-m} |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)} = i\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\frac{m}{\sqrt{M}}\sigma_2^+ d^{(1)-A}_{-m} |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)}$$

$$= i\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\sum_{k\geq 1}\left(\frac{m\, f^{F(1)-}_{mk}}{\sqrt{M}}\right) d^{-A}_{-k} |\chi\rangle \tag{6.63}$$

Combining both terms, we obtain

$$|\psi^{B(1)}\rangle = i\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\sum_{k\geq 1}\left(\frac{k\, f^{B(1)}_{mk}}{\sqrt{M+N}} - \frac{m\, f^{F(1)-}_{mk}}{\sqrt{M}}\right) d^{-A}_{-k} |\chi\rangle + \sum_{k\geq 1} f^{B(1)}_{mk}\alpha_{B\dot{B},-k} |\psi\rangle \tag{6.64}$$

where we note that[6]

$$\frac{k\, f^{B(1)}_{mk}}{\sqrt{M+N}} - \frac{m\, f^{F(1)-}_{mk}}{\sqrt{M}} = \frac{(-1)^m \sin\left(\pi\frac{Mk}{M+N}\right)}{\pi\sqrt{M+N}}(-a)^{k-\frac{m(M+N)}{M}}\frac{\Gamma\left[\frac{Mk}{M+N}\right]\Gamma\left[\frac{Nk}{M+N}\right]}{\Gamma[k]}\frac{\Gamma\left[\frac{(M+N)m}{M}\right]}{\Gamma[m]\Gamma\left[\frac{Nm}{M}\right]}. \tag{6.65}$$

6.6.2 $|\psi^{B(1)}\rangle$

To find $|\psi^{B(2)}\rangle$ one simply modifies (6.64) by $M \leftrightarrow N$, $a \to -a$, and $(1) \to (2)$.

---

[6]To compare to [19] in the limit of $M = N = 1$, one should take into account the different conventions on fermion modes. Note also that there is a typo in the last term of equation (7.5) of [19]; this term should resemble the last term in (6.64) above.

6.7 Summary of Results

For convenient reference, here we record the results for γ^F, C_{MN} and $f^{(i)\pm}$. For completeness we include also the bosonic quantities γ^B, f^B computed in [20].

$$
\begin{aligned}
\tilde{\gamma}^B_{ss'} &= \frac{z_0^{s+s'}}{4\pi^2}\,\mu_s\mu_{s'}\,\frac{1}{s+s'}\,\frac{MN}{(M+N)^3}\left(\frac{(M+N)^{M+N}}{M^M N^N}\right)^{s+s'}\frac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]}\,\frac{\Gamma[Ms']\Gamma[Ns']}{\Gamma[(M+N)s']} \\
\tilde{\gamma}^F_{ss'} &= -\frac{z_0^{s+s'}}{4\pi^2}\mu_s\mu_{s'}\,\frac{s}{s+s'}\,\frac{MN}{(M+N)^2}\left(\frac{(M+N)^{M+N}}{M^M N^N}\right)^{s+s'}\frac{\Gamma[Ms]\,\Gamma[Ns]}{\Gamma[(M+N)s]}\,\frac{\Gamma[Ms']\,\Gamma[Ns']}{\Gamma[(M+N)s']} \\
C_{MN} &= \frac{M+N}{2MN}
\end{aligned}
\tag{6.66}
$$

$$
\tilde{f}^{B(1)}_{qs} = \begin{cases} \dfrac{M}{M+N} & q=s \\[2mm] \dfrac{i}{2\pi}z_0^{s-q}\dfrac{\mu_s}{s-q}\dfrac{1}{(M+N)}\left(\dfrac{(M+N)^{M+N}}{M^M N^N}\right)^{(s-q)}\dfrac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]}\dfrac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} & q\neq s \end{cases}
$$

$$
\tilde{f}^{B(2)}_{rs} = \begin{cases} \dfrac{N}{M+N} & r=s \\[2mm] -\dfrac{i}{2\pi}z_0^{s-r}\dfrac{\mu_s}{s-r}\dfrac{1}{(M+N)}\left(\dfrac{(M+N)^{M+N}}{M^M N^N}\right)^{(s-r)}\dfrac{\Gamma[(M+N)r]}{\Gamma[Mr]\Gamma[Nr]}\dfrac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} & r\neq s \end{cases}
\tag{6.67}
$$

$$
\tilde{f}^{F(1)+}_{qs} = \begin{cases} \dfrac{\sqrt{M}}{\sqrt{M+N}} & q=s \\[2mm] \dfrac{i}{2\pi}z_0^{s-q}\dfrac{\mu_s}{\sqrt{M(M+N)}}\dfrac{1}{s-q}\dfrac{s}{q}\left(\dfrac{(M+N)^{M+N}}{M^M N^N}\right)^{s-q}\dfrac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]}\dfrac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} & q\neq s \end{cases}
$$

$$
\tilde{f}^{F(1)-}_{qs} = \begin{cases} \dfrac{\sqrt{M}}{\sqrt{M+N}} & q=s \\[2mm] \dfrac{i}{2\pi}z_0^{s-q}\dfrac{\mu_s}{\sqrt{M(M+N)}}\dfrac{1}{s-q}\left(\dfrac{(M+N)^{M+N}}{M^M N^N}\right)^{s-q}\dfrac{\Gamma[(M+N)q]}{\Gamma[Mq]\Gamma[Nq]}\dfrac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} & q\neq s \end{cases}
\tag{6.68}
$$

$$
\tilde{f}^{F(2)+}_{rs} = \begin{cases} \dfrac{\sqrt{N}}{\sqrt{M+N}} & r=s \\[2mm] -\dfrac{i}{2\pi}z_0^{s-r}\dfrac{\mu_s}{\sqrt{N(M+N)}}\dfrac{1}{s-r}\dfrac{s}{r}\left(\dfrac{(M+N)^{M+N}}{M^M N^N}\right)^{s-r}\dfrac{\Gamma[(M+N)r]}{\Gamma[Mr]\Gamma[Nr]}\dfrac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} & r\neq s \end{cases}
$$

$$
\tilde{f}^{F(2)-}_{rs} = \begin{cases} \dfrac{\sqrt{N}}{\sqrt{M+N}} & r=s \\[2mm] -\dfrac{i}{2\pi}z_0^{s-r}\dfrac{\mu_s}{\sqrt{N(M+N)}}\dfrac{1}{s-r}\left(\dfrac{(M+N)^{M+N}}{M^M N^N}\right)^{s-r}\dfrac{\Gamma[(M+N)r]}{\Gamma[Mr]\Gamma[Nr]}\dfrac{\Gamma[Ms]\Gamma[Ns]}{\Gamma[(M+N)s]} & r\neq s \end{cases}
\tag{6.69}
$$

Since the expressions for the final states $|\psi^{F(i)\pm}\rangle$ and $|\psi^{B(i)}\rangle$ are somewhat unwieldy, we will not repeat their expressions here, and we instead refer the reader to equations (6.55), (6.60), (6.64).

7 Continuum Limit

Thus far, our computations have been exact, and the results are somewhat involved. To study black hole physics however, one would like to take the limit of large $N_1 N_5$. In this limit, typical component strings have parametrically large winding numbers, and typical mode numbers q, r, s are much larger than the spacing of modes on the respective component string. We denote this limit by the 'continuum limit'. Large mode numbers correspond to short wavelengths, which are not sensitive to the finite length of the component string. In this limit, the expressions for γ^F and $f^{F(i)\pm}$ simplify considerably, as we now show.

7.1 Continuum Limit for γ^F_{kl}

We start by taking the continuum limit of γ^F. From (5.19) we have

$$\tilde{\gamma}^F_{ss'} = -\frac{1}{4\pi^2} z_0^{s+s'} \mu_s \mu_{s'} \frac{s}{s+s'} \frac{MN}{(M+N)^2} \left(\frac{(M+N)^{M+N}}{M^M N^N} \right)^{s+s'} \frac{\Gamma[Ms]\,\Gamma[Ns]}{\Gamma[(M+N)s]} \frac{\Gamma[Ms']\,\Gamma[Ns']}{\Gamma[(M+N)s']} \tag{7.1}$$

We wish to find the approximation to this expression when

$$s >> \frac{1}{M+N}, \quad s >> \frac{1}{M}, \quad s >> \frac{1}{N} \tag{7.2}$$

and likewise for s'. We use Stirling's formula,

$$\Gamma[x] \sim \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e} \right)^x \tag{7.3}$$

for $x >> 1$. Using this, we get

$$\Gamma[Ms] \sim \sqrt{\frac{2\pi}{Ms}} \left(\frac{Ms}{e} \right)^{Ms}. \tag{7.4}$$

For the Gamma function terms of (7.1) with variable s, we then have

$$\frac{\Gamma[Ms]\,\Gamma[Ns]}{\Gamma[(M+N)s]} \approx \sqrt{\frac{M+N}{MN}} \sqrt{\frac{2\pi}{s}} \left(\frac{M^M N^N}{(M+N)^{M+N}} \right)^s \tag{7.5}$$

and likewise for the terms with variable s'.

Inserting these approximations in (7.1), we find

$$\tilde{\gamma}^F_{ss'} \approx -\frac{1}{2\pi} z_0^{s+s'} \frac{\mu_s \mu_{s'}}{(M+N)} \frac{1}{s+s'} \sqrt{\frac{s}{s'}}. \tag{7.6}$$

Let us also obtain the continuum limit for $f^{F(i)\pm}$. We need only consider the cases where $s \neq q$ for $f^{F(1)\pm}_{qs}$ and $s \neq r$ for $f^{F(2)\pm}_{qs}$ because for the cases $s = q$ and $s = r$ the continuum limit expression is identical to the exact expression.

For $f^{F(1)+}$, from (6.44) we have the exact expression

$$\tilde{f}_{qs}^{F(1)+} = \frac{i}{2\pi}z_0^{s-q}\frac{\mu_s}{\sqrt{M(M+N)}}\frac{1}{s-q}\frac{s}{q}\left(\frac{(M+N)^{M+N}}{M^M N^N}\right)^{s-q}\frac{\Gamma\left[Ms\right]\Gamma\left[Ns\right]}{\Gamma\left[(M+N)s\right]}\frac{\Gamma\left[(M+N)q\right]}{\Gamma\left[Mq\right]\Gamma\left[Nq\right]}$$

(7.7)

thus we obtain

$$\tilde{f}_{qs}^{F(1)+} \approx \frac{i}{2\pi}z_0^{s-q}\frac{\mu_s}{\sqrt{M(M+N)}}\frac{1}{s-q}\sqrt{\frac{s}{q}}\,.$$

(7.8)

Similarly, we obtain

$$\tilde{f}_{qs}^{F(1)-} \approx \frac{i}{2\pi}z_0^{s-q}\frac{\mu_s}{\sqrt{M(M+N)}}\frac{1}{s-q}\sqrt{\frac{q}{s}}$$

(7.9)

and the component string (2) quantities can be found in the same way.

8   Discussion

In this paper we have studied the effect of the deformation operator when it joins together two component strings of length M, N into a single component string of winding $M+N$. We computed the final state produced from starting with a R-R vacuum state on the original component strings, and also in the case of an initial bosonic or fermionic excitation.

Our results generalize those recently reported in [20], which studied bosonic fields only, in two ways. Firstly, we have extended the analysis to the fermionic fields. Secondly, we have computed the overall prefactor on the final state. The calculation of this prefactor is quite nontrivial, but the answer is compact: $C_{MN} = (M + N)/(2MN)$. If one considers the special case $M = N$, this becomes $C_{MM} = 1/M$, which agrees with the fact that the twist operator σ_2^{++} responsible for this coefficient has weight $(1/2, 1/2)$.

It was noted in [20] that the bosonic quantities γ^B, $f^{B(i)}$ have an interesting structure involving Gamma functions, which ensures that γ^B multiplies only creation operators and that $f^{B(i)}$ vanishes unless there is a creation operator in the initial state, and a creation operator in the final state. As one would expect, we find that these characteristics are shared by the fermionic quantities γ^F, $f^{F(i)}$.

In addition, the fermionic quantities γ^F, $f^{F(i)}$ share the property of 'almost factorization' observed for the bosonic quantities γ^B, $f^{B(i)}$ in [20]. For $\gamma_{ss'}^F$, the only part which does not factorize into a product of terms corresponding to s and s' is the factor $\frac{s}{s+s'}$. For $f_{qs}^{F(1)}$, the only part which does not factorize in this way is the factor $\frac{1}{s-q}$.

For applications to black hole physics, one is interested in the limit of large $N_1 N_5$. In this limit, component strings typically have parametrically large winding; this is the main physical reason for studying the general M, N problem. Thus, as well as obtaining our exact

results, we have extracted the behaviour of our results in the continuum limit, in which the mode numbers are large compared to the spacing of modes on the component string. We found significant simplification to the various quantities, similar to that observed in the bosonic case. These simplified expressions may prove useful in extracting the qualitative dynamics of thermalization in this theory.

The results of this paper further our understanding of the effect of the deformation operator in the D1D5 CFT. It is hoped that these results will lead to a better understanding of the process of thermalization in this theory, and thereby shed further light on aspects of black hole dynamics, in particular the processes of black hole formation and evaporation.

Acknowledgements

A   Notation and conventions

We follow the conventions of [18, 19], which we record here for convenience. We have 4 real left moving fermions $\psi_1, \psi_2, \psi_3, \psi_4$ which we group into doublets $\psi^{\alpha A}$ as follows:

$$\begin{pmatrix} \psi^{++} \\ \psi^{-+} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 + i\psi_2 \\ \psi_3 + i\psi_4 \end{pmatrix} \tag{A.1}$$

$$\begin{pmatrix} \psi^{+-} \\ \psi^{--} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_3 - i\psi_4 \\ -(\psi_1 - i\psi_2) \end{pmatrix}. \tag{A.2}$$

Here $\alpha = (+,-)$ is an index of the subgroup $SU(2)_L$ of rotations on S^3 and $A = (+,-)$ is an index of the subgroup $SU(2)_1$ from rotations in T^4. The reality conditions on the individual fermions are

$$(\psi_i)^\dagger = \psi_i \qquad \Rightarrow \qquad (\psi^{\alpha A})^\dagger = -\epsilon_{\alpha\beta}\epsilon_{AB}\psi^{\beta B}. \tag{A.3}$$

One can introduce doublets $\psi^\dagger$, whose components are given by

$$(\psi^\dagger)_{\alpha A} = (\psi^{\alpha A})^\dagger \tag{A.4}$$

in terms of which the reality condition is

$$(\psi^\dagger)_{\alpha A} = -\epsilon_{\alpha\beta}\epsilon_{AB}\psi^{\beta B}. \tag{A.5}$$

The 2-point functions are

$$< \psi^{\alpha A}(z)(\psi^\dagger)_{\beta B}(w) >= \delta^\alpha_\beta \delta^A_B \frac{1}{z-w}, \quad < \psi^{\alpha A}(z)\psi^{\beta B}(w) >= -\epsilon^{\alpha\beta}\epsilon^{AB}\frac{1}{z-w} \qquad (\text{A.6})$$

where we have

$$\epsilon_{12} = 1, \quad \epsilon^{12} = -1, \quad \psi_A = \epsilon_{AB}\psi^B, \quad \psi^A = \epsilon^{AB}\psi_B. \qquad (\text{A.7})$$

There are 4 real left moving bosons X_1, X_2, X_3, X_4 which can be grouped into a matrix

$$X_{A\dot{A}} = \frac{1}{\sqrt{2}}X_i\sigma_i = \frac{1}{\sqrt{2}}\begin{pmatrix} X_3 + iX_4 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 + iX_4 \end{pmatrix} \qquad (\text{A.8})$$

where $\sigma_i = (\sigma_a, iI)$. The reality condition on the individual bosons is

$$(X_i)^\dagger = X_i \quad \Rightarrow \quad (X_{A\dot{A}})^\dagger = -\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}X_{B\dot{B}}. \qquad (\text{A.9})$$

One can introduce a matrix $X^\dagger$ with components

$$(X^\dagger)^{A\dot{A}} = (X_{A\dot{A}})^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} X_3 - iX_4 & X_1 + iX_2 \\ X_1 - iX_2 & -X_3 - iX_4 \end{pmatrix} \qquad (\text{A.10})$$

in terms of which the reality condition is

$$(X^\dagger)^{A\dot{A}} = -\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}X_{B\dot{B}}. \qquad (\text{A.11})$$

The 2-point functions are

$$< \partial X_{A\dot{A}}(z)(\partial X^\dagger)^{B\dot{B}}(w) >= -\frac{1}{(z-w)^2}\delta^B_A\delta^{\dot{B}}_{\dot{A}}, \quad < \partial X_{A\dot{A}}(z)\partial X_{B\dot{B}}(w) >= \frac{1}{(z-w)^2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}. \qquad (\text{A.12})$$

The chiral algebra is generated by the operators

$$J^a = -\frac{1}{4}(\psi^\dagger)_{\alpha A}(\sigma^{Ta})^\alpha{}_\beta \psi^{\beta A} \qquad (\text{A.13})$$

$$G^\alpha_{\dot{A}} = \psi^{\alpha A}\partial X_{A\dot{A}}, \quad (G^\dagger)^{\dot{A}}_\alpha = (\psi^\dagger)_{\alpha A}\partial(X^\dagger)^{A\dot{A}} \qquad (\text{A.14})$$

$$T = -\frac{1}{2}(\partial X^\dagger)^{A\dot{A}}\partial X_{A\dot{A}} - \frac{1}{2}(\psi^\dagger)_{\alpha A}\partial\psi^{\alpha A} \qquad (\text{A.15})$$

$$(G^\dagger)^{\dot{A}}_\alpha = -\epsilon_{\alpha\beta}\epsilon^{\dot{A}\dot{B}}G^\beta_{\dot{B}}, \quad G^\alpha_{\dot{A}} = -\epsilon^{\alpha\beta}\epsilon_{\dot{A}\dot{B}}(G^\dagger)^{\dot{B}}_\beta. \qquad (\text{A.16})$$

These operators generate the algebra

$$J^a(z)J^b(z') \sim \delta^{ab}\frac{\frac{1}{2}}{(z-z')^2} + i\epsilon^{abc}\frac{J^c}{z-z'} \qquad (\text{A.17})$$

$$J^a(z)G^\alpha_{\dot{A}}(z') \sim \frac{1}{(z-z')}\frac{1}{2}(\sigma^{aT})^\alpha{}_\beta G^\beta_{\dot{A}} \qquad (\text{A.18})$$

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$$G_{\dot{A}}^{\alpha}(z)(G^{\dagger})_{\dot{\beta}}^{\dot{B}}(z') \sim -\frac{2}{(z-z')^3}\delta_{\beta}^{\alpha}\delta_{\dot{A}}^{\dot{B}} - \delta_{\dot{A}}^{\dot{B}}(\sigma^{Ta})^{\alpha}{}_{\beta}[\frac{2J^a}{(z-z')^2} + \frac{\partial J^a}{(z-z')}] - \frac{1}{(z-z')}\delta_{\beta}^{\alpha}\delta_{\dot{A}}^{\dot{B}}T$$

(A.19)

$$T(z)T(z') \sim \frac{3}{(z-z')^4} + \frac{2T}{(z-z')^2} + \frac{\partial T}{(z-z')}$$

(A.20)

$$T(z)J^a(z') \sim \frac{J^a}{(z-z')^2} + \frac{\partial J^a}{(z-z')}$$

(A.21)

$$T(z)G_{\dot{A}}^{\alpha}(z') \sim \frac{\frac{3}{2}G_{\dot{A}}^{\alpha}}{(z-z')^2} + \frac{\partial G_{\dot{A}}^{\alpha}}{(z-z')} \, .$$

(A.22)

Note that

$$J^a(z)\psi^{\gamma C}(z') \sim \frac{1}{2}\frac{1}{z-z'}(\sigma^{aT})^{\gamma}{}_{\beta}\psi^{\beta C} \, .$$

(A.23)

The above OPE algebra gives the commutation relations

$$[J_m^a, \, J_n^b] \;\; = \;\; \frac{m}{2}\delta^{ab}\delta_{m+n,0} + i\epsilon^{ab}{}_c J_{m+n}^c$$

(A.24)

$$[J_m^a, \, G_{\dot{A},n}^{\alpha}] \;\; = \;\; \frac{1}{2}(\sigma^{aT})^{\alpha}{}_{\beta}G_{\dot{A},m+n}^{\beta}$$

(A.25)

$$\{G_{\dot{A},m}^{\alpha}, G_{\dot{B},n}^{\beta}\} \;\; = \;\; \epsilon_{\dot{A}\dot{B}}\left[(m^2 - \frac{1}{4})\epsilon^{\alpha\beta}\delta_{m+n,0} + (m-n)(\sigma^{aT})^{\alpha}{}_{\gamma}\epsilon^{\gamma\beta}J_{m+n}^a + \epsilon^{\alpha\beta}L_{m+n}\right]$$

(A.26)

$$[L_m, \, L_n] \;\; = \;\; \frac{m(m^2 - \frac{1}{4})}{2}\delta_{m+n,0} + (m-n)L_{m+n}$$

(A.27)

$$[L_m, \, J_n^a] \;\; = \;\; -nJ_{m+n}^a$$

(A.28)

$$[L_m, \, G_{\dot{A},n}^{\alpha}] \;\; = \;\; \left(\frac{m}{2} - n\right)G_{\dot{A},m+n}^{\alpha} \, .$$

(A.29)

B Mapping modes to the covering plane

In this appendix we compute the transformations of the fermion modes under the sequence of spectral flow transformations and coordinate changes described in Section 4.2.

B.1 Modes on the cylinder

We start by recalling from Section 3.1 the mode expansions on the cylinder.

Below the twist, we have:

$$d_m^{(1)\alpha A} \;\; = \;\; \frac{1}{2\pi i\sqrt{M}}\int\limits_{\sigma=0}^{2\pi M} \psi^{(1)\alpha A}(w)e^{\frac{m}{M}w}dw$$

(B.1)

$$d_m^{(2)\alpha A} \;\; = \;\; \frac{1}{2\pi i\sqrt{N}}\int\limits_{\sigma=0}^{2\pi N} \psi^{(2)\alpha A}(w)e^{\frac{m}{N}w}dw,$$

(B.2)

while above the twist, we have:

$$d_k^{\alpha A} \;\; = \;\; \frac{1}{2\pi i\sqrt{M+N}} \int\limits_{\sigma=0}^{2\pi(M+N)} \psi^{\alpha A}(w) e^{\frac{k}{M+N}w} dw, \tag{B.3}$$

which gives

$$\psi^{\alpha A}(w) = \frac{1}{\sqrt{M+N}} \sum_k d_k^{\alpha A} e^{-\frac{k}{M+N}w}. \tag{B.4}$$

The anti-commutation relations are

$$\left\{ d_k^{\alpha A}, d_l^{\beta B} \right\} \;\; = \;\; -\varepsilon^{\alpha\beta} \varepsilon^{AB} \delta_{k+l,0}. \tag{B.5}$$

B.2   Modes on the z plane

We map the cylinder to the plane with coordinate z via

$$z \;\; = \;\; e^w. \tag{B.6}$$

The operator modes transform as follows. Before the twist, i.e. $|z| < e^{\tau_0}$, using a contour circling $z = 0$, we have

$$d_m^{(1)\alpha A} \;\; \to \;\; \frac{1}{2\pi i\sqrt{M}} \int\limits_{\arg(z)=0}^{2\pi M} \psi^{(1)\alpha A}(z) z^{\frac{m}{M}-\frac{1}{2}} dz$$

$$d_m^{(2)\alpha A} \;\; \to \;\; \frac{1}{2\pi i\sqrt{N}} \int\limits_{\arg(z)=0}^{2\pi N} \psi^{(2)\alpha A}(z) z^{\frac{m}{N}-\frac{1}{2}} dz. \tag{B.7}$$

After the twist, i.e. $|z| > e^{\tau_0}$, using a contour circling $z = \infty$, we have

$$d_k^{\alpha A} \;\; \to \;\; \frac{1}{2\pi i\sqrt{M+N}} \int\limits_{\arg(z)=0}^{2\pi(M+N)} \psi^{\alpha A}(z) z^{\frac{k}{M+N}-\frac{1}{2}} dz. \tag{B.8}$$

B.3   Modes on the covering space

We now map the problem to the covering space with coordinate t, where the fields are single-valued. We use the map

$$z \;\; = \;\; t^M (t-a)^N. \tag{B.9}$$

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It is convenient to write the derivative as

$$\frac{dz}{dt} = (M+N)\frac{z}{t(t-a)}\left(t - \tfrac{M}{M+N}a\right). \tag{B.10}$$

Under this map, the modes before the twist become:

$$d_m^{(1)\alpha A} \rightarrow \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{t=0} dt\, \psi^{\alpha A}(t) \left[t^{m-\frac{1}{2}}\left(t - \tfrac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{\frac{Nm}{M}-\frac{1}{2}}\right]$$

$$d_m^{(2)\alpha A} \rightarrow \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{t=a} dt\, \psi^{\alpha A}(t) \left[t^{\frac{Mm}{N}-\frac{1}{2}}\left(t - \tfrac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{m-\frac{1}{2}}\right] \tag{B.11}$$

and after the twist, we have:

$$d_k^{\alpha A} \rightarrow \frac{1}{2\pi i} \oint_{t=\infty} dt\, \psi^{\alpha A}(t) \left[t^{\frac{Mk}{M+N}-\frac{1}{2}}\left(t - \tfrac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{\frac{Nk}{M+N}-\frac{1}{2}}\right]. \tag{B.12}$$

B.4 Modes after the first spectral flow

We now perform the sequence of spectral flow transformations and coordinate changes described in Section 4.2. As explained there, at this point we ignore normalization factors of spin fields and transformation properties of spin fields under spectral flow. We spectral flow in the same way on left- and right-moving sectors, but write only the holomorphic expressions.

We first spectral flow by $\alpha = 1$ in the t plane. Before the spectral flow, we have the amplitude

$$_t\langle 0_{R,-}|\, S^-(a)S^+\left(\tfrac{M}{M+N}a\right)|0_R^-\rangle_t. \tag{B.13}$$

The spectral flow has the following effects:

(a) The R ground states $|0_R^-\rangle_t$ and $_t\langle 0_{R,-}|$ map to the NS ground states in the t plane:

$$|0_R^-\rangle_t \rightarrow |0_{NS}\rangle_t, \qquad _t\langle 0_{R,-}| \rightarrow {}_t\langle 0_{NS}| \tag{B.14}$$

(b) The modes of the fermion fields change as follows (the bosonic modes are unaffected):

$$\psi^{\pm A}(t) \rightarrow t^{\mp\frac{1}{2}}\psi^{\pm A}(t). \tag{B.15}$$

Then the modes before the twist become:

$$d_m^{(1)+A} \rightarrow \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{t=0} dt\psi^{(1)+A}(t) \left[t^{m-1}\left(t - \tfrac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{\frac{Nm}{M}-\frac{1}{2}}\right]$$

$$d_m^{(2)+A} \;\to\; \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{t=a} dt\,\psi^{(2)+A}(t) \left[t^{\frac{Mm}{N}-1}\left(t-\frac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{m-\frac{1}{2}}\right]$$

$$d_m^{(1)-A} \;\to\; \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{t=0} dt\,\psi^{(1)-A}(t) \left[t^{m}\left(t-\frac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{\frac{Nm}{M}-\frac{1}{2}}\right]$$

$$d_m^{(2)-A} \;\to\; \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{t=a} dt\,\psi^{(2)-A}(t) \left[t^{\frac{Mm}{N}}\left(t-\frac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{m-\frac{1}{2}}\right] \tag{B.16}$$

and the modes after the twist become:

$$d_k^{+A} \;\to\; \frac{1}{2\pi i} \oint_{t=\infty} dt\,\psi^{+A}(t) \left[t^{\frac{Mk}{M+N}-1}\left(t-\frac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{\frac{Nk}{M+N}-\frac{1}{2}}\right]$$

$$d_k^{-A} \;\to\; \frac{1}{2\pi i} \oint_{t=\infty} dt\,\psi^{-A}(t) \left[t^{\frac{Mk}{M+N}}\left(t-\frac{M}{M+N}a\right)^{\frac{1}{2}}(t-a)^{\frac{Nk}{M+N}-\frac{1}{2}}\right]. \tag{B.17}$$

B.5 Modes after the second spectral flow

Next, we change coordinate to

$$t' \;=\; t-\frac{M}{M+N}a, \tag{B.18}$$

and we spectral flow by $\alpha = -1$ in the t' plane. Before this second spectral flow, we have the amplitude

$$_{t'}\langle 0_{NS}|\, S^{-}\left(\frac{N}{M+N}a\right) |0_R^+\rangle_{t'}. \tag{B.19}$$

The spectral flow has the following effects:

(a) The R ground state $|0_R^+\rangle_{t'}$ maps to the NS vacuum in the t' plane:

$$|0_R^+\rangle_{t'} \;\to\; |0_{NS}\rangle_{t'}. \tag{B.20}$$

(b) The modes of the fermions change as follows:

$$\psi^{\pm A}(t') \;\to\; (t')^{\pm\frac{1}{2}}\psi^{\pm A}(t'). \tag{B.21}$$

Then the modes before the twist become:

$$d_m^{(1)+A} \;\to\; \frac{\sqrt{M+N}}{2\pi i\sqrt{M}} \oint_{t'=-\frac{Ma}{M+N}} dt'\,\psi^{(1)+A} \left[\left(t'+\frac{M}{M+N}a\right)^{m-1} t'\left(t'-\frac{N}{M+N}a\right)^{\frac{Nm}{M}-\frac{1}{2}}\right]$$

$$d_m^{(2)+A} \;\to\; \frac{\sqrt{M+N}}{2\pi i\sqrt{N}} \oint_{t'=\frac{Na}{M+N}} dt'\,\psi^{(2)+A} \left[\left(t'+\frac{M}{M+N}a\right)^{\frac{Mm}{N}-1} t'\left(t'-\frac{N}{M+N}a\right)^{m-\frac{1}{2}}\right]$$

$$d_m^{(1)-A} \rightarrow \frac{\sqrt{M+N}}{2\pi i \sqrt{M}} \oint_{t'=-\frac{Ma}{M+N}} dt' \psi^{(1)-A} \left[\left(t' + \tfrac{M}{M+N}a \right)^m \left(t' - \tfrac{N}{M+N}a \right)^{\frac{Nm}{M}-\frac{1}{2}} \right]$$

$$d_m^{(2)-A} \rightarrow \frac{\sqrt{M+N}}{2\pi i \sqrt{N}} \oint_{t'=\frac{Na}{M+N}} dt' \psi^{(2)-A} \left[\left(t' + \tfrac{M}{M+N}a \right)^{\frac{Mm}{N}} \left(t' - \tfrac{N}{M+N}a \right)^{m-\frac{1}{2}} \right] \quad \text{(B.22)}$$

and the modes after the twist become:

$$d_k^{+A} \rightarrow \frac{1}{2\pi i} \oint_{t'=\infty} \psi^{+A}(t') \left[\left(t' + \tfrac{M}{M+N}a \right)^{\frac{Mk}{M+N}-1} t' \left(t' - \tfrac{N}{M+N}a \right)^{\frac{Nk}{M+N}-\frac{1}{2}} \right]$$

$$d_k^{-A} \rightarrow \frac{1}{2\pi i} \oint_{t'=\infty} \psi^{-A}(t') \left[\left(t' + \tfrac{M}{M+N}a \right)^{\frac{Mk}{M+N}} \left(t' - \tfrac{N}{M+N}a \right)^{\frac{Nk}{M+N}-\frac{1}{2}} \right]. \quad \text{(B.23)}$$

B.6 Modes after the third spectral flow

Next, we change the coordinate to

$$\hat{t} = t' - \tfrac{N}{M+N}a, \quad \text{(B.24)}$$

and we spectral flow by $\alpha = 1$ in the $\hat{t}$ plane. The spectral flow has the following effects

(a) The R ground state maps to the NS vacuum in the $\hat{t}$ plane:

$$|0_R^-\rangle_{\hat{t}} \rightarrow |0_{NS}\rangle_{\hat{t}}. \quad \text{(B.25)}$$

(b) The modes of the fermions change as follows:

$$\psi^{\pm A}(\hat{t}) \rightarrow (\hat{t})^{\mp \frac{1}{2}} \psi^{\pm A}(\hat{t}) \quad \text{(B.26)}$$

Then the modes before the twist become:

$$d_m^{(1)+A} \rightarrow \hat{d}_m^{\prime(1)+A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{M}} \oint_{\hat{t}=-a} d\hat{t}\, \psi^{(1)+A}(\hat{t}) \left[(\hat{t}+a)^{m-1} \left(\hat{t} + \tfrac{Na}{M+N} \right) \hat{t}^{\frac{Nm}{M}-1} \right]$$

$$d_m^{(2)+A} \rightarrow \hat{d}_m^{\prime(2)+A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{N}} \oint_{\hat{t}=0} d\hat{t}\, \psi^{(2)+A}(\hat{t}) \left[(\hat{t}+a)^{\frac{Mm}{N}-1} \left(\hat{t} + \tfrac{Na}{M+N} \right) \hat{t}^{m-1} \right]$$

$$d_m^{(1)-A} \rightarrow \hat{d}_m^{\prime(1)-A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{M}} \oint_{\hat{t}=-a} d\hat{t}\, \psi^{(1)-A}(\hat{t}) \left[(\hat{t}+a)^m \hat{t}^{\frac{Nm}{M}} \right]$$

$$d_m^{(2)-A} \rightarrow \hat{d}_m^{\prime(2)-A} = \frac{\sqrt{M+N}}{2\pi i \sqrt{N}} \oint_{\hat{t}=0} d\hat{t}\, \psi^{(2)-A}(\hat{t}) \left[(\hat{t}+a)^{\frac{Mm}{N}} \hat{t}^m \right] \quad \text{(B.27)}$$

After the twist, we obtain

$$
\begin{aligned}
d_k^{+A} \quad &\to \quad \hat{d}_k'^{+A} = \frac{1}{2\pi i} \oint_{\hat{t}=\infty} d\hat{t}\, \psi^{+A}\left(\hat{t}\right) \left[\left(\hat{t}+a\right)^{\frac{Mk}{M+N}-1} \left(\hat{t}+\tfrac{N}{M+N}a\right) \hat{t}^{\frac{Nk}{M+N}-1} \right] \\
d_k^{-A} \quad &\to \quad \hat{d}_k'^{-A} = \frac{1}{2\pi i} \oint_{\hat{t}=\infty} d\hat{t}\, \psi^{-A}\left(\hat{t}\right) \left[\left(\hat{t}+a\right)^{\frac{Mk}{M+N}} \hat{t}^{\frac{Nk}{M+N}} \right].
\end{aligned}
\tag{B.28}
$$

C   Calculation of the overall prefactor C_{MN}

In this appendix we compute the prefactor C_{MN} given by

$$
C_{MN} = \langle 0_{R,--} | \, \sigma_2^{++}\left(w_0\right) \, |0_R^{--}\rangle^{(1)} \, |0_R^{--}\rangle^{(2)}.
\tag{C.1}
$$

We will have four contributions to our factor of C_{MN}.

First we lift the correlator to the z plane. The conformal weight of the twist operator then gives a Jacobian factor, which is our first contribution.

We then have the z plane correlator

$$
\langle 0_{R,--} | \, \sigma_2^{++}\left(z_0\right) \, |0_R^{--}\rangle^{(1)} \, |0_R^{--}\rangle^{(2)} = \langle \sigma_{M+N}^{++}(\infty)\sigma_2^{++}(z_0)\sigma_N^{--}(0)\sigma_M^{--}(0)\rangle.
\tag{C.2}
$$

We employ the methods developed in [22, 23] to compute this correlator. Introducing the notation that σ_{M+N}^{++} has dimension $(\Delta_{M+N}^{++}, \Delta_{M+N}^{++})$, we have

$$
\begin{aligned}
\langle \sigma_{M+N}^{++}(\infty)\sigma_2^{++}(z_0)\sigma_N^{--}(0)\sigma_M^{--}(0)\rangle &\equiv \lim_{|z|\to\infty} |z|^{4\Delta_{M+N}^{++}} \langle \sigma_{M+N}^{++}(z)\sigma_2^{++}(z_0)\sigma_N^{--}(0)\sigma_M^{--}(0)\rangle \\
&= \lim_{|z|\to\infty} \frac{\langle \sigma_{M+N}^{++}(z)\sigma_2^{++}(z_0)\sigma_N^{--}(0)\sigma_M^{--}(0)\rangle}{\langle \sigma_{M+N}^{++}(z)\sigma_{M+N}^{--}(0)\rangle}.
\end{aligned}
\tag{C.3}
$$

The computation of this ratio of correlators factorizes [23] into a product of terms coming from the Liouville action, which depend only on the 'bare twist' part of the above spin-twist fields, and t plane correlators of appropriately normalized spin fields. We denote the image of z and z_0 in the t plane by $t(z)$ and t_0 respectively. Then we have

$$
\langle \sigma_{M+N}^{++}(\infty)\sigma_2^{++}(z_0)\sigma_N^{--}(0)\sigma_M^{--}(0)\rangle = \lim_{|z|\to\infty} \frac{\langle \sigma_{M+N}(z)\sigma_2(z_0)\sigma_N(0)\sigma_M(0)\rangle}{\langle \sigma_{M+N}(z)\sigma_{M+N}(0)\rangle} \frac{\langle \mathcal{S}_4(t(z),t_0)\rangle}{\langle \mathcal{S}_2(t(z))\rangle}
\tag{C.4}
$$

where $\langle \mathcal{S}_4(t(z),t_0)\rangle$ and $\langle \mathcal{S}_2(t(z))\rangle$ are t plane correlators (four-point and two-point respectively) of appropriately normalized spin fields.

Our second contribution will be the ratio of correlators of bare twist fields in (C.4); our third contribution will be the various normalization factors for the t plane spin field correlators, and our final contribution will be the ratio of t plane spin field correlators.

We now compute these four contributions in turn.

C.1 Jacobian factor for lifting from cylinder to plane

We first lift the correlator to the z plane. Since σ_2^{++} has weight $(1/2,1/2)$, we obtain the Jacobian factor contribution

$$\left|\frac{dz}{dw}\right|_{z=z_0} = |a|^{M+N}\frac{M^M N^N}{(M+N)^{M+N}}\,. \tag{C.5}$$

C.2 Liouville action terms

In this subsection we deal only with the contribution from the bare twist fields, i.e. we compute

$$\langle\sigma_{M+N}(\infty)\sigma_2(z_0)\sigma_N(0)\sigma_M(0)\rangle = \lim_{|z|\to\infty}\frac{\langle\sigma_{M+N}(z)\sigma_2(z_0)\sigma_N(0)\sigma_M(0)\rangle}{\langle\sigma_{M+N}(z)\sigma_{M+N}(0)\rangle} \tag{C.6}$$

where the bare twist operators are normalized as

$$\langle\sigma_M(z)\sigma_M(0)\rangle = \frac{1}{|z|^{4\Delta_M}}\,. \tag{C.7}$$

C.2.1 Defining regulated twist operators

We work in a path integral formulation, and we define regularized bare twist operators by cutting a small circular hole of radius $\epsilon \ll 1$ around each finite insertion point,

$$\sigma_2^\epsilon(z_0),\quad \sigma_M^\epsilon(0),\quad \sigma_N^\epsilon(0)\,. \tag{C.8}$$

From (C.7) we have the following relationship between the regularized twist operators (whose normalization depends on the cutoff) and the canonically normalized twist operators:

$$\sigma_M = \frac{1}{\sqrt{\langle\sigma_M^\epsilon(0)\sigma_M^\epsilon(1)\rangle}}\sigma_M^\epsilon\,. \tag{C.9}$$

We also define a regularized twist operator at infinity by cutting a large circular hole of radius $1/\tilde\delta$ with $\tilde\delta \ll 1$. This operator is denoted

$$\sigma_{M+N}^{\tilde\delta}(\infty)\,. \tag{C.10}$$

Since in (C.6) we have a ratio of amplitudes, we do not need to worry about the normalization of $\sigma_{M+N}^{\tilde\delta}(\infty)$, and we obtain

$$\lim_{|z|\to\infty}\frac{\langle\sigma_{M+N}(z)\sigma_2(z_0)\sigma_N(0)\sigma_M(0)\rangle}{\langle\sigma_{M+N}(z)\sigma_{M+N}(0)\rangle} = \frac{\langle\sigma_{M+N}^{\tilde\delta}(\infty)\sigma_2(z_0)\sigma_N(0)\sigma_M(0)\rangle}{\langle\sigma_{M+N}^{\tilde\delta}(\infty)\sigma_{M+N}(0)\rangle}\,. \tag{C.11}$$

From (C.9) we then obtain

$$
\begin{aligned}
\langle \sigma_{M+N}(\infty)\sigma_2(z_0)\sigma_N(0)\sigma_M(0)\rangle &= \\
\frac{\langle \sigma_{M+N}^{\tilde{\delta}}(\infty)\sigma_2^\epsilon(z_0)\sigma_N^\epsilon(0)\sigma_M^\epsilon(0)\rangle}{\langle \sigma_{M+N}^{\tilde{\delta}}(\infty)\sigma_{M+N}^\epsilon(0)\rangle} &\sqrt{\frac{\langle \sigma_{M+N}^\epsilon(0)\sigma_{M+N}^\epsilon(1)\rangle}{\langle \sigma_2^\epsilon(0)\sigma_2^\epsilon(1)\rangle\langle \sigma_N^\epsilon(0)\sigma_N^\epsilon(1)\rangle\langle \sigma_M^\epsilon(0)\sigma_M^\epsilon(1)\rangle}}
\end{aligned}
\quad (\text{C.12})
$$

and so our four-point function is now expressed entirely in terms of correlators of regularized twist operators, which we now compute.

C.2.2 Lifting to the t plane

We now lift the problem to the t-plane using the map (4.2),

$$
z = t^M(t-a)^N. \quad (\text{C.13})
$$

In the z plane we have cut out various circular holes, leaving a path integral over an open set. The image of this open set in the t plane will be denoted by Σ. In the t plane, we have single-valued fields, with no twist operators, but with appropriately normalized spin field insertions (which we deal with later). There is also a non-trivial metric, which we deal with in the present section.

To take account of the non-trivial metric on the t plane, we define a fiducial metric on the cover space Σ and compute the Liouville action. Replacing the z-plane by a closed surface (a sphere), we define the z-plane metric g via

$$
\begin{aligned}
ds^2 &= \begin{cases} dz d\bar{z} & |z| < \frac{1}{\delta} \\ d\tilde{z} d\bar{\tilde{z}} & |\tilde{z}| < \frac{1}{\delta} \end{cases} \\
\tilde{z} &= \frac{1}{\delta^2}\frac{1}{z}
\end{aligned}
\quad (\text{C.14})
$$

where δ is a cutoff which will not play a role in our computation.

We define the fiducial metric on the t plane $\hat{g}$, to be flat so the curvature term in the Liouville action vanishes. We define $\hat{g}$ via

$$
\begin{aligned}
d\hat{s}^2 &= \begin{cases} dt d\bar{t} & |t| < \frac{1}{\delta'} \\ d\tilde{t} d\bar{\tilde{t}} & |\tilde{t}| < \frac{1}{\delta'} \end{cases} \\
\tilde{t} &= \frac{1}{\delta'^2}\frac{1}{t}.
\end{aligned}
\quad (\text{C.15})
$$

where δ' is another cutoff which will not play a role in our computation.

In lifting to the t-plane from the z-plane our metric will change by

$$
ds^2 = e^\phi d\hat{s}^2 \quad (\text{C.16})
$$

This corresponds to a transformation of our path integral of the form

$$Z^{(g)} = e^{S_L} Z^{(\hat{g})} \tag{C.17}$$

where S_L is the Liouville action given by (see e.g. [36])

$$S_L = \frac{c}{96\pi} \int d^2 t \sqrt{-\hat{g}} \left[\partial_\mu \phi \partial_\nu \phi \, \hat{g}^{\mu\nu} + 2R^{(\hat{g})} \phi \right]. \tag{C.18}$$

Since we map from the z space to the t space we have an induced metric given by

$$ds^2 = dz d\bar{z} = \frac{dz}{dt} \frac{d\bar{z}}{d\bar{t}} dt d\bar{t} \tag{C.19}$$

Comparing this equation with (C.16) we find that the Liouville field is given by

$$\phi = \log \frac{dz}{dt} + \log \frac{d\bar{z}}{d\bar{t}} = \log \left(\left| \frac{dz}{dt} \right|^2 \right). \tag{C.20}$$

Since the fiducial metric is flat and $\partial_\mu \partial^\mu \phi = 0$, the Liouville action becomes a boundary term, which can be written as

$$S_L = \frac{c}{96\pi} \left[i \int_\partial dt \phi \partial_t \phi + c.c \right] \tag{C.21}$$

where $c.c.$ denotes the complex conjugate and where ∂ is the boundary of Σ, comprising the images in the t plane of the circular holes defined in the z plane.

The contribution of correlation function reduces to calculating the contributions to the Liouville action from the various holes comprising the boundary of Σ, which in our problem are:

- <u>Type 1</u>: The holes whose images are at finite points in the t-plane. In our problem these holes are from finite points within the z-plane.

- <u>Type 2</u>: The hole whose image is at $t = \infty$. This hole comes from a cut in the z-plane at $\frac{1}{\tilde{\delta}}$ upon taking $\tilde{\delta} \to 0$. Since all $M + N$ copies of the CFT are twisted together at $z = \infty$, this is the only hole that appears at $t = \infty$.

Then our four-point correlation function (C.12) can be written as

$$\langle \sigma_{M+N}(\infty) \sigma_2(z_0) \sigma_N(0) \sigma_M(0) \rangle =$$
$$\frac{e^{S_{L_1}[\sigma^{\tilde{\delta}}_{M+N} \sigma^\epsilon_2 \sigma^\epsilon_N \sigma^\epsilon_M] + S_{L_2}[\sigma^{\tilde{\delta}}_{M+N} \sigma^\epsilon_2 \sigma^\epsilon_N \sigma^\epsilon_M]}}{e^{S_{L_1}[\sigma^{\tilde{\delta}}_{M+N} \sigma^\epsilon_{M+N}] + S_{L_2}[\sigma^{\tilde{\delta}}_{M+N} \sigma^\epsilon_{M+N}]}} \sqrt{\frac{\langle \sigma^\epsilon_{M+N}(0) \sigma^\epsilon_{M+N}(1) \rangle}{\langle \sigma^\epsilon_2(0) \sigma^\epsilon_2(1) \rangle \langle \sigma^\epsilon_N(0) \sigma^\epsilon_N(1) \rangle \langle \sigma^\epsilon_M(0) \sigma^\epsilon_M(1) \rangle}} \tag{C.22}$$

where $S_{L_1}[\sigma^{\tilde{\delta}}_{M+N} \sigma^\epsilon_2 \sigma^\epsilon_N \sigma^\epsilon_M]$ and $S_{L_2}[\sigma^{\tilde{\delta}}_{M+N} \sigma^\epsilon_2 \sigma^\epsilon_N \sigma^\epsilon_M]$ are the Liouville contributions to the four-point function from holes of Type 1 and Type 2 respectively, and similarly for the two-point function.

C.2.3 The four-point function of regularized twist operators

We now calculate the contribution to the Liouville action from the four-point function of regularized twist operators,

$$S_{L,4} \equiv S_{L_1}[\sigma^{\tilde{\delta}}_{M+N}\sigma^{\epsilon}_2\sigma^{\epsilon}_N\sigma^{\epsilon}_M] + S_{L_2}[\sigma^{\tilde{\delta}}_{M+N}\sigma^{\epsilon}_2\sigma^{\epsilon}_N\sigma^{\epsilon}_M] . \tag{C.23}$$

Let us begin by looking at the contribution to the Liouville action from the branch point at $z = 0$, $t = 0$. Near this point we have:

$$z = t^M(t-a)^N \approx t^M(-a)^N; \quad \frac{dz}{dt} \approx Mt^{M-1}(-a)^N; \quad t \approx \left(\frac{z}{(-a)^N}\right)^{1/M} ; \tag{C.24}$$

$$\phi = \log\left[\frac{dz}{dt}\right] + c.c. \approx 2\log\left[M|t|^{M-1}|a|^N\right]; \quad \partial_t\phi \approx \frac{M-1}{t} \tag{C.25}$$

From (C.21), the contribution to the Liouville action from this point is given by (we note that the central charge is $c = 6$)

$$S_{L_1}(z = 0, t = 0) = \frac{6}{96\pi}\left[i\int dt\phi\partial_t\phi + c.c.\right] \tag{C.26}$$

We perform branch point integration introducing the following coordinates

$$z \approx \epsilon e^{i\theta}; \qquad t \approx \left(\frac{\epsilon}{(-a)^N}\right)^{1/M}e^{i\theta'}; \qquad \theta' = \frac{\theta}{M} \tag{C.27}$$

Making this change of variables for the integral, we obtain the four-point function's contribution to the Liouville action from this region:

$$\begin{aligned}
S_{L_1,4}(z = 0, t = 0) &= \frac{6}{96\pi}\left[i\int_0^{2\pi} d\theta' it \frac{M-1}{t} 2\log[M|t|^{M-1}|a|^N] + c.c.\right] \\
&= -\frac{1}{2}(M-1)\left(\log\left[M\epsilon^{\frac{M-1}{M}}|a|^{\frac{N}{M}}\right]\right)
\end{aligned} \tag{C.28}$$

The contributions from the other branch points in the four-point function are calculated in the same manner. Here we present only the results.

$$\begin{aligned}
S_{L_1,4}(z = 0, t = a) &= -\frac{1}{2}(N-1)\left(\log\left[N\epsilon^{\frac{N-1}{N}}|a|^{\frac{M}{N}}\right]\right) \\
S_{L_1,4}(z = z_0, t = t_0) &= -\frac{1}{4}\log\left[4|b_{t_0}|\epsilon\right] \\
S_{L_2,4}(z = \infty, t = \infty) &= \frac{1}{2}(M+N-1)\log\left[(M+N)\tilde{\delta}^{-\frac{M+N-1}{M+N}}\right]
\end{aligned} \tag{C.29}$$

where

$$|b_{t_0}| = \frac{|a|^{M+N-2}}{2}\frac{M^M N^N}{(M+N)^{M+N}}\left(\frac{(M+N)^3}{MN}\right) . \tag{C.30}$$

Combining the Liouville terms for the regularized four-point function we find a total contribution of

$$
\begin{aligned}
S_{L,4} &= -\frac{1}{2}(M-1)\log\left[M\epsilon^{\frac{M-1}{M}}|a|^{\frac{N}{M}}\right] - \frac{1}{2}(N-1)\log\left[N\epsilon^{\frac{N-1}{N}}|a|^{\frac{M}{N}}\right] - \frac{1}{4}\log\left[4\,|b_{t_0}|\,\epsilon\right] \\
&\quad + \frac{1}{2}(M+N-1)\log\left[(M+N)\tilde{\delta}^{-\frac{M+N-1}{M+N}}\right].
\end{aligned}
\tag{C.31}
$$

C.2.4   The two-point function of regularized twist operators

Let us also calculate the contribution to the Liouville action from the regularized two-point function,

$$
S_{L,2} = S_{L_1}[\sigma^{\tilde{\delta}}_{M+N}\sigma^{\epsilon}_{M+N}] + S_{L_2}[\sigma^{\tilde{\delta}}_{M+N}\sigma^{\epsilon}_{M+N}]
\tag{C.32}
$$

We use the map

$$
\begin{aligned}
z = t^{M+N}, \qquad t = z^{\frac{1}{M+N}}, \qquad \frac{dz}{dt} = (M+N)t^{M+N-1} \\
\phi \approx \log\left[(M+N)t^{M+N-1}\right] + c.c. \qquad \partial_t\phi = \frac{M+N-1}{t}
\end{aligned}
\tag{C.33}
$$

We have branch points at $z = 0, t = 0$ and $z = \infty, t = \infty$. The contribution to the Liouville action from each of these points is calculated in the same manner as the contribution from the four-point function. We thus find:

$$
\begin{aligned}
S_{L_1,2}(t=0, z=0) &= -\frac{1}{2}(M+N-1)\log\left[(M+N)\epsilon^{\frac{M+N-1}{M+N}}\right] \\
S_{L_2,2}(t=\infty, z=\infty) &= \frac{1}{2}(M+N-1)\log\left[(M+N)\tilde{\delta}^{\frac{1-M-N}{M+N}}\right]
\end{aligned}
\tag{C.34}
$$

Combining the Liouville terms for both branch points of the two-point function then gives

$$
S_{L,2} = \frac{M+N-1}{2}\log\left[\tilde{\delta}^{\frac{1-M-N}{M+N}}\right] - \frac{M+N-1}{2}\log\left[\epsilon^{\frac{M+N-1}{M+N}}\right].
\tag{C.35}
$$

C.2.5   Normalization of the twist operators

In [22], it was shown that the two-point function of regularized twist operators at finite separation is given by

$$
\langle\sigma^{\epsilon}_n(0)\sigma^{\epsilon}_n(a)\rangle = a^{-4\Delta_n}\left(n^2\epsilon^{A_n}Q^{B_n}\right)
\tag{C.36}
$$

where

$$
\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right), \qquad A_n = -\frac{(n-1)^2}{n}, \qquad B_n = 1 - n
\tag{C.37}
$$

and where Q is a quantity that is regularization-dependent and which cancels out.

The contribution from normalization term is then

$$\sqrt{\frac{\langle\sigma^\epsilon_{M+N}(0)\sigma^\epsilon_{M+N}(1)\rangle}{\langle\sigma^\epsilon_2(0)\sigma^\epsilon_2(1)\rangle\langle\sigma^\epsilon_N(0)\sigma^\epsilon_N(1)\rangle\langle\sigma^\epsilon_M(0)\sigma^\epsilon_M(1)\rangle}} = \sqrt{\frac{(M+N)^2\epsilon^{-\frac{(M+N-1)^2}{(M+N)}}}{(2MN)^2\epsilon^{-\frac{(N-1)^2}{N}-\frac{(M-1)^2}{M}-\frac{1}{2}}}} \qquad (C.38)$$

Combining (C.31), (C.35) and (C.38), one can check that the regularization terms cancel, and so (C.22) becomes

$$\langle\sigma_{M+N}(\infty)\sigma_2(z_0)\sigma_N(0)\sigma_M(0)\rangle$$
$$= 2^{-\frac{5}{4}}|a|^{-\frac{3}{4}(M+N)+\frac{1}{2}\left(\frac{M}{N}+\frac{N}{M}+1\right)}M^{-\frac{3}{4}M-\frac{1}{4}}N^{-\frac{3}{4}N-\frac{1}{4}}(M+N)^{\frac{3}{4}(M+N)-\frac{1}{4}}. \qquad (C.39)$$

C.3 Normalization factors for spin field insertions

We next turn to the contributions to our amplitude (C.4) coming from the spin fields,

$$\lim_{|z|\to\infty}\frac{\langle\mathcal{S}_4(t(z),t_0)\rangle}{\langle\mathcal{S}_2(t(z))\rangle} \qquad (C.40)$$

where $\langle\mathcal{S}_4(t(z),t_0)\rangle$ and $\langle\mathcal{S}_2(t(z))\rangle$ are t plane correlators (four-point and two-point respectively) of appropriately normalized spin fields, which we now define.

The normalization coefficient for a spin field $S^\pm_n(z_*)$ depends on the local form of the cover map at the insertion point,

$$(z-z_*)\approx b_*\,(t-t_*)^n\,. \qquad (C.41)$$

The corresponding spin field insertion in the t plane is given by [23]

$$b_*^{-\frac{1}{4n}}S^\pm(t_*)\,. \qquad (C.42)$$

We shall first collect all the normalization terms b_*, and we shall compute the spin field correlator in the next subsection.

For the two-point function in the denominator, we use the map

$$z = t^{M+N} \qquad (C.43)$$

and so all the normalization factors are trivial.

For the four-point function, taking the appropriate limits of our map

$$z = t^M(t-a)^N \qquad (C.44)$$

we find the normalization coefficients b_* to be

$$\begin{aligned}
z=0, t=0; &\quad z\approx(-a)^N t^M \rightarrow &\quad b_0=(-a)^N \rightarrow &\quad b_0^{-\frac{1}{4M}}=(-a)^{-\frac{N}{4M}} \\
z=0, t=a; &\quad z\approx a^M(t-a)^N \rightarrow &\quad b_a=a^M \rightarrow &\quad b_a^{-\frac{1}{4N}}=a^{-\frac{M}{4N}} \\
z=z_0, t=t_0=\frac{aM}{M+N}; &\quad z-z_0\approx b_{t_0}(t-t_0)^2 \rightarrow &\quad b_{t_0}^{-\frac{1}{8}} \\
z=\infty, t=\infty; &\quad z\approx t^{M+N} \rightarrow &\quad b_\infty=1 \rightarrow &\quad b_\infty^{-\frac{1}{4(M+N)}}=1
\end{aligned} \qquad (C.45)$$

where b_{t_0} was given in (C.30). We can therefore write the normalized spin field correlator as (writing only holomorphic parts)

$$\lim_{|z|\to\infty} \frac{\langle \mathcal{S}_4(t(z), t_0)\rangle}{\langle \mathcal{S}_2(t(z))\rangle} = \left(b_\infty^{-\frac{1}{4(M+N)}} b_{t_0}^{-\frac{1}{8}} b_a^{-\frac{1}{4N}} b_0^{-\frac{1}{4M}} \right) \left(\frac{\langle S^+(\infty)S^+(t_0)S^-(a)S^-(0)\rangle}{\langle S^+(\infty)S^-(0)\rangle} \right). \quad (C.46)$$

Combining the holomorphic and antiholomorphic parts, the product of all spin field normalization factors is

$$|b_\infty|^{-\frac{1}{2(M+N)}} |b_{t_0}|^{-\frac{1}{4}} |b_a|^{-\frac{1}{2N}} |b_0|^{-\frac{1}{2M}} =$$
$$2^{\frac{1}{4}} |a|^{-\frac{1}{2}\left(\frac{M}{N}+\frac{N}{M}+\frac{M}{2}+\frac{N}{2}-1\right)} M^{-\frac{1}{4}(M-1)} N^{-\frac{1}{4}(N-1)} (M+N)^{\frac{1}{4}(M+N-3)}. \quad (C.47)$$

C.4 Spin field correlator

Here we show the calculation of the spin field correlator term, using two different methods: firstly via bosonization and secondly via spectral flow.

C.4.1 Method 1: Bosonization

Here we shall use bosonization to calculate the spin field correlators given in (C.46). We define the bosonized fermion fields as

$$\psi_1 = e^{i\phi_5}, \qquad \psi_2 = e^{i\phi_6} \quad (C.48)$$

We can therefore write the spin fields as

$$S^\pm(z) = e^{\pm\frac{i}{2}e_a\Phi^a(z)} \quad (C.49)$$

where

$$e_a\Phi^a(z) = \phi_5(z) - \phi_6(z). \quad (C.50)$$

The OPE is

$$e^{i\alpha\phi(z)} e^{i\beta\phi(w)} \sim e^{i(\alpha\phi(z)+\beta\phi(w))} (z-w)^{\alpha\beta}. \quad (C.51)$$

Thus we have

$$\frac{\langle S^+(\infty)S^+(t_0)S^-(a)S^-(0)\rangle}{\langle S^+(\infty)S^-(0)\rangle} = \lim_{t\to\infty} \frac{\langle S^+(t)S^+(t_0)S^-(a)S^-(0)\rangle}{\langle S^+(t)S^-(0)\rangle}$$
$$= \lim_{t\to\infty} \frac{\langle :\exp\left(\frac{i}{2}e_a\Phi^a(t)\right)::\exp\left(\frac{i}{2}e_a\Phi^a(t_0)\right)::\exp\left(-\frac{i}{2}e_a\Phi^a(a)\right)::\exp\left(-\frac{i}{2}e_a\Phi^a(0)\right):\rangle}{\langle :\exp\left(\frac{i}{2}e_a\Phi^a(t)\right)::\exp\left(-\frac{i}{2}e_a\Phi^a(0)\right):\rangle}.$$
$$(C.52)$$

For the four-point correlator, we obtain

$$\langle :\exp\left(\frac{i}{2}e_a\Phi^a(t)\right)::\exp\left(\frac{i}{2}e_a\Phi^a(t_0)\right)::\exp\left(-\frac{i}{2}e_a\Phi^a(a)\right)::\exp\left(-\frac{i}{2}e_a\Phi^a(0)\right):\rangle$$

$$= (t - t_0)^{\frac{1}{2}}(t - a)^{-\frac{1}{2}}(t_0 - a)^{-\frac{1}{2}}t^{-\frac{1}{2}}t_0^{-\frac{1}{2}}a^{\frac{1}{2}}. \tag{C.53}$$

For the two-point correlator, we obtain

$$\langle :\exp\left(\frac{i}{2}e_a\Phi^a(t)\right) : :\exp\left(-\frac{i}{2}e_a\Phi^a(0)\right) :\rangle = t^{-\frac{1}{2}}. \tag{C.54}$$

Combining (C.53) and (C.54), we find that (C.52) is given by

$$
\begin{aligned}
\frac{\langle S^+(\infty)S^+(t_0)S^-(a)S^-(0)\rangle}{\langle S^+(\infty)S^-(0)\rangle} &= \lim_{t\to\infty}(t - t_0)^{\frac{1}{2}}(t - a)^{-\frac{1}{2}}(t_0 - a)^{-\frac{1}{2}}t_0^{-\frac{1}{2}}a^{\frac{1}{2}} \\
&= (t_0 - a)^{-\frac{1}{2}}t_0^{-\frac{1}{2}}a^{\frac{1}{2}} \\
&= (-a)^{-\frac{1}{2}}N^{-\frac{1}{2}}M^{-\frac{1}{2}}(M + N). \tag{C.55}
\end{aligned}
$$

Combining (C.55) with the antiholomorphic part, we obtain

$$\frac{\langle S^+(\infty)S^+(t_0)S^-(a)S^-(0)\rangle}{\langle S^+(\infty)S^-(0)\rangle} = \frac{(M + N)^2}{MN}\frac{1}{|a|}. \tag{C.56}$$

C.4.2 Method 2: Spectral Flow

We now compute the spin field correlator using the sequence of spectral flow transformations and coordinate changes described in Section 4.2. This serves as a cross-check of the above calculation via bosonization.

Having lifted to the t plane, and taken care of the normalization factors of the spin field insertions, we are left with the t plane spin field correlator (writing holomorphic fields only)

$$_t\langle 0_{R,-}|\, S^-(a)S^+\left(\tfrac{M}{M+N}a\right)\,|0_R^-\rangle_t. \tag{C.57}$$

We recall that the action of spectral flow is straightforward for operators where the fermion content may be expressed as a simple exponential in the language in which the fermions are bosonized. For such operators with charge j, spectral flow with parameter α gives rise to the transformation

$$\hat{O}_j(t) \to t^{-\alpha j}\hat{O}_j(t).$$

We first spectral flow by $\alpha = 1$ in the t plane. This gives

$$\sqrt{\frac{M + N}{M}}\, _t\langle 0_{NS}|\, S^-(a)S^+\left(\tfrac{M}{M+N}a\right)\,|0_{NS}\rangle_t. \tag{C.58}$$

Next, we change coordinate to

$$t' = t - \tfrac{M}{M+N}a. \tag{C.59}$$

This gives the t' plane correlator

$$\sqrt{\frac{M+N}{M}} \,_{t'}\langle 0_{NS}| \, S^- \left(\tfrac{N}{M+N}a\right) |0_R^+\rangle_{t'} \, . \tag{C.60}$$

We then spectral flow by $\alpha = -1$ in the t' plane, yielding

$$\sqrt{\frac{M+N}{M}} \left(\frac{N}{M+N}a\right)^{-\frac{1}{2}} \,_{t'}\langle 0_{R,-}| \, S^- \left(\tfrac{N}{M+N}a\right) |0_{NS}\rangle_{t'} \tag{C.61}$$

$$= \left(\frac{M+N}{\sqrt{MN}}a^{-\frac{1}{2}}\right) \,_{t'}\langle 0_{R,-}| \, S^- \left(\tfrac{N}{M+N}a\right) |0_{NS}\rangle_{t'}. \tag{C.62}$$

Next, we change the coordinate to

$$\hat{t} = t' - \tfrac{N}{M+N}a \, . \tag{C.63}$$

This gives

$$\left(\frac{M+N}{\sqrt{MN}}a^{-\frac{1}{2}}\right) \,_{\hat{t}}\langle 0_{R,-}|0_R^-\rangle_{\hat{t}} \;=\; \frac{M+N}{\sqrt{MN}}a^{-\frac{1}{2}} \, . \tag{C.64}$$

The final spectral flow by $\alpha = 1$ in the $\hat{t}$ plane has no effect.

Adding in the anti-holomorphic factors, we obtain

$$\frac{(M+N)^2}{MN}\frac{1}{|a|}, \tag{C.65}$$

in agreement with (C.56).

Combining the Jacobian factor (C.5), the Liouville action contribution (C.39), the spin field normalization factors (C.47), and the spin field correlator (C.65), we find

$$C_{MN} = \frac{M+N}{2MN} \, . \tag{C.66}$$

References

[1] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," *Phys. Lett.* **B379** (1996) 99–104, `hep-th/9601029`.

[2] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `hep-th/9711200`; E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `hep-th/9802150`; S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, `hep-th/9802109`.

[3] G. Arutyunov and S. Frolov, "Virasoro amplitude from the S**N R**24 orbifold sigma model," *Theor.Math.Phys.* **114** (1998) 43–66, `hep-th/9708129`; G. Arutyunov and S. Frolov, "Four graviton scattering amplitude from S**N R**8 supersymmetric orbifold sigma model," *Nucl.Phys.* **B524** (1998) 159–206, `hep-th/9712061`; J. de Boer, "Six-dimensional supergravity on S**3 x AdS(3) and 2d conformal field theory," *Nucl. Phys.* **B548** (1999) 139–166, `hep-th/9806104`; R. Dijkgraaf, "Instanton strings and hyperKahler geometry," *Nucl.Phys.* **B543** (1999) 545–571, `hep-th/9810210`; N. Seiberg and E. Witten, "The D1/D5 system and singular CFT," *JHEP* **04** (1999) 017, `hep-th/9903224`; F. Larsen and E. J. Martinec, "U(1) charges and moduli in the D1-D5 system," *JHEP* **06** (1999) 019, `hep-th/9905064`; J. R. David, G. Mandal, S. Vaidya, and S. R. Wadia, "Point mass geometries, spectral flow and AdS(3) - CFT(2) correspondence," *Nucl.Phys.* **B564** (2000) 128–141, `hep-th/9906112`; A. Jevicki, M. Mihailescu, and S. Ramgoolam, "Gravity from CFT on S**N(X): Symmetries and interactions," *Nucl.Phys.* **B577** (2000) 47–72, `hep-th/9907144`; O. Lunin and S. D. Mathur, "Rotating deformations of AdS(3) x S(3), the orbifold CFT and strings in the pp-wave limit," *Nucl. Phys.* **B642** (2002) 91–113, `hep-th/0206107`; J. Gomis, L. Motl, and A. Strominger, "PP wave / CFT(2) duality," *JHEP* **0211** (2002) 016, `hep-th/0206166`; E. Gava and K. Narain, "Proving the PP wave / CFT(2) duality," *JHEP* **0212** (2002) 023, `hep-th/0208081`.

[4] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri, and S. F. Ross, "Supersymmetric conical defects: Towards a string theoretic description of black hole formation," *Phys. Rev.* **D64** (2001) 064011, `hep-th/0011217`; J. M. Maldacena and L. Maoz, "De-singularization by rotation," *JHEP* **12** (2002) 055, `hep-th/0012025`.

[5] O. Lunin and S. D. Mathur, "AdS/CFT duality and the black hole information paradox," *Nucl. Phys.* **B623** (2002) 342–394, `hep-th/0109154`; O. Lunin, S. D. Mathur, and A. Saxena, "What is the gravity dual of a chiral primary?," *Nucl. Phys.* **B655** (2003) 185–217, `hep-th/0211292`.

[6] K. Skenderis and M. Taylor, "Fuzzball solutions and D1-D5 microstates," *Phys. Rev. Lett.* **98** (2007) 071601, `hep-th/0609154`; I. Kanitscheider, K. Skenderis, and M. Taylor, "Holographic anatomy of fuzzballs," *JHEP* **04** (2007) 023,

`hep-th/0611171`; I. Kanitscheider, K. Skenderis, and M. Taylor, "Fuzzballs with internal excitations," *JHEP* **06** (2007) 056, `0704.0690`.

[7] O. Lunin, "Adding momentum to D1-D5 system," *JHEP* **04** (2004) 054, `hep-th/0404006`; S. Giusto, S. D. Mathur, and A. Saxena, "Dual geometries for a set of 3-charge microstates," *Nucl. Phys.* **B701** (2004) 357–379, `hep-th/0405017`; S. Giusto, S. D. Mathur, and A. Saxena, "3-charge geometries and their CFT duals," *Nucl. Phys.* **B710** (2005) 425–463, `hep-th/0406103`.

[8] S. D. Mathur and D. Turton, "Microstates at the boundary of AdS," *JHEP* **05** (2012) 014, `1112.6413`; S. D. Mathur and D. Turton, "Momentum-carrying waves on D1-D5 microstate geometries," *Nucl.Phys.* **B862** (2012) 764–780, `1202.6421`; O. Lunin, S. D. Mathur, and D. Turton, "Adding momentum to supersymmetric geometries," *Nucl.Phys.* **B868** (2013) 383–415, `1208.1770`.

[9] S. Giusto and R. Russo, "Superdescendants of the D1D5 CFT and their dual 3-charge geometries," *JHEP* **1403** (2014) 007, `1311.5536`.

[10] V. Jejjala, O. Madden, S. F. Ross, and G. Titchener, "Non-supersymmetric smooth geometries and D1-D5-P bound states," *Phys. Rev.* **D71** (2005) 124030, `hep-th/0504181`.

[11] I. Bena and N. P. Warner, "Bubbling supertubes and foaming black holes," *Phys. Rev.* **D74** (2006) 066001, `hep-th/0505166`; P. Berglund, E. G. Gimon, and T. S. Levi, "Supergravity microstates for BPS black holes and black rings," *JHEP* **0606** (2006) 007, `hep-th/0505167`.

[12] S. Giusto, O. Lunin, S. D. Mathur, and D. Turton, "D1-D5-P microstates at the cap," *JHEP* **1302** (2013) 050, `1211.0306`.

[13] V. Cardoso, O. J. C. Dias, J. L. Hovdebo, and R. C. Myers, "Instability of non-supersymmetric smooth geometries," *Phys. Rev.* **D73** (2006) 064031, `hep-th/0512277`.

[14] B. D. Chowdhury and S. D. Mathur, "Radiation from the non-extremal fuzzball," *Class. Quant. Grav.* **25** (2008) 135005, `0711.4817`; B. D. Chowdhury and S. D. Mathur, "Pair creation in non-extremal fuzzball geometries," *Class. Quant. Grav.* **25** (2008) 225021, `0806.2309`; B. D. Chowdhury and S. D. Mathur, "Non-extremal fuzzballs and ergoregion emission," *Class. Quant. Grav.* **26** (2009) 035006, `0810.2951`.

[15] F. V. Dimitrakopoulos, B. Freivogel, M. Lippert, and I.-S. Yang, "Instability corners in AdS space," `1410.1880`.

[16] J. Berges, S. Borsanyi, and C. Wetterich, "Prethermalization," *Phys.Rev.Lett.* **93** (2004) 142002, `hep-ph/0403234`.

[17] J. R. David, G. Mandal, and S. R. Wadia, "Microscopic formulation of black holes in string theory," *Phys. Rept.* **369** (2002) 549–686, `hep-th/0203048`.

[18] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Deforming the D1D5 CFT away from the orbifold point," *JHEP* **06** (2010) 031, `1002.3132`.

[19] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Excitations in the deformed D1D5 CFT," *JHEP* **06** (2010) 032, 1003.2746.

[20] Z. Carson, S. Hampton, S. D. Mathur, and D. Turton, "Effect of the twist operator in the D1D5 CFT," *JHEP* **1408** (2014) 064, 1405.0259.

[21] Z. Carson, S. D. Mathur, and D. Turton, "Bogoliubov coefficients for the twist operator in the D1D5 CFT," *Nucl. Phys.* **B** *In Press* (2014) 10.1016/j.nuclphysb.2014.10.018, 1406.6977.

[22] O. Lunin and S. D. Mathur, "Correlation functions for M(N)/S(N) orbifolds," *Int. J. Mod. Phys.* **A16S1C** (2001) 967–969, hep-th/0006196.

[23] O. Lunin and S. D. Mathur, "Three-point functions for M(N)/S(N) orbifolds with N = 4 supersymmetry," *Commun. Math. Phys.* **227** (2002) 385–419, hep-th/0103169.

[24] S. G. Avery and B. D. Chowdhury, "Intertwining Relations for the Deformed D1D5 CFT," *JHEP* **1105** (2011) 025, 1007.2202.

[25] C. T. Asplund and S. G. Avery, "Evolution of Entanglement Entropy in the D1-D5 Brane System," *Phys.Rev.* **D84** (2011) 124053, 1108.2510.

[26] B. A. Burrington, A. W. Peet, and I. G. Zadeh, "Twist-nontwist correlators in M^N/S_N orbifold CFTs," *Phys.Rev.* **D87** (2013) 106008, 1211.6689.

[27] B. A. Burrington, A. W. Peet, and I. G. Zadeh, "Operator mixing for string states in the D1-D5 CFT near the orbifold point," *Phys.Rev.* **D87** (2013) 106001, 1211.6699.

[28] A. Pakman, L. Rastelli, and S. S. Razamat, "Diagrams for Symmetric Product Orbifolds," *JHEP* **0910** (2009) 034, 0905.3448; A. Pakman, L. Rastelli, and S. S. Razamat, "Extremal Correlators and Hurwitz Numbers in Symmetric Product Orbifolds," *Phys.Rev.* **D80** (2009) 086009, 0905.3451; A. Pakman, L. Rastelli, and S. S. Razamat, "A Spin Chain for the Symmetric Product CFT(2)," *JHEP* **1005** (2010) 099, 0912.0959; C. A. Keller, "Phase transitions in symmetric orbifold CFTs and universality," *JHEP* **1103** (2011) 114, 1101.4937; C. Cardona, "Pulsating strings from two dimensional CFT on $(T^4)^N/S(N)$," 1408.5035.

[29] S. D. Mathur, "The fuzzball proposal for black holes: An elementary review," *Fortsch. Phys.* **53** (2005) 793–827, hep-th/0502050; I. Bena and N. P. Warner, "Black holes, black rings and their microstates," *Lect. Notes Phys.* **755** (2008) 1–92, hep-th/0701216; K. Skenderis and M. Taylor, "The fuzzball proposal for black holes," *Phys. Rept.* **467** (2008) 117–171, 0804.0552; V. Balasubramanian, J. de Boer, S. El-Showk, and I. Messamah, "Black Holes as Effective Geometries," *Class. Quant. Grav.* **25** (2008) 214004, 0811.0263; B. D. Chowdhury and A. Virmani, "Modave Lectures on Fuzzballs and Emission from the D1-D5 System," 1001.1444.

[30] S. D. Mathur, "Black Holes and Beyond," *Annals Phys.* **327** (2012) 2760–2793, 1205.0776; I. Bena and N. P. Warner, "Resolving the Structure of Black Holes: Philosophizing with a Hammer," 1311.4538.

[31] I. Bena, J. de Boer, M. Shigemori, and N. P. Warner, "Double, Double Supertube Bubble," *JHEP* **10** (2011) 116, 1107.2650; S. Giusto, R. Russo, and D. Turton, "New D1-D5-P geometries from string amplitudes," *JHEP* **11** (2011) 062, 1108.6331; G. Gibbons and N. Warner, "Global structure of five-dimensional fuzzballs," *Class.Quant.Grav.* **31** (2014) 025016, 1305.0957.

[32] S. D. Mathur and D. Turton, "Oscillating supertubes and neutral rotating black hole microstates," *JHEP* **1404** (2014) 072, 1310.1354; I. Bena, S. F. Ross, and N. P. Warner, "On the Oscillation of Species," 1312.3635.

[33] I. Bena, M. Shigemori, and N. P. Warner, "Black-Hole Entropy from Supergravity Superstrata States," 1406.4506; F. Chen, B. Michel, J. Polchinski, and A. Puhm, "Journey to the Center of the Fuzzball," 1408.4798; E. J. Martinec, "The Cheshire Cap," 1409.6017.

[34] S. G. Avery, "Using the D1D5 CFT to Understand Black Holes," 1012.0072.

[35] S. G. Avery and B. D. Chowdhury, "Emission from the D1D5 CFT: Higher Twists," *JHEP* **1001** (2010) 087, 0907.1663.

[36] D. Friedan, "Introduction To Polyakov's String Theory", in 'Recent Advances in Field Theory and Statistical Mechanics', ed. by J. B. Zuber and R. Stora. North Holland, 1984.