Highlights

Aerodynamic Sound Generation in Thermoviscous Fluids: A Canonical Problem Revisited

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- Aerodynamic sound production near solid boundaries in thermoviscous fluids
- Extension of aeroacoustic theory to nonlinear viscous flows with heat conduction
- Unsteady viscous dissipation as a sound generation mechanism in thermoviscous flows
- Use of Neumann Green's functions for aeroacoustic problems in thermoviscous fluids.

Aerodynamic Sound Generation in Thermoviscous Fluids: A Canonical Problem Revisited

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Abstract

Although the Lighthill–Curle acoustic analogy theory is formally exact, the presence of linear source terms related to viscous stresses and non-isentropic density changes makes it unsuitable for studying aerodynamic sound generation in low Reynolds number thermoviscous flows. Here we use an extension of the Ffowcs Williams and Hawkings formulation, with thermoviscous effects explicitly included, to find an analytical solution to the canonical problem of sound radiation from a circular cylinder immersed in a viscous heat-conducting fluid and rotating sinusoidally about its axis. Existing published solutions are compared and an earlier null result is explained. The new analysis reveals the dominant source of sound at low Mach numbers to be unsteady viscous dissipation rather than Reynolds-stress quadrupoles, unless the fluid parameter $B = \alpha c^2/c_p$ is zero.

Keywords: Aeroacoustics theory, low Reynolds number flows, Green's functions, unsteady dissipation

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Notation

a	Thermal expansion parameter $a = \alpha/\rho c_p$
A	Dimensionless thermodynamic property $A = \alpha T$
B	Dimensionless thermodynamic property $B = \alpha c^2/c_p$
c	Sound speed $c = (\partial p/\partial \rho)_s^{1/2}$
c_p	Constant-pressure specific heat
$\dot{D}$	instantaneous rate of energy dissipation per unit volume due to viscous stresses
$\dot{D}_T$	instantaneous rate of energy dissipation per unit volume due to heat conduction $\dot{D}_T = \kappa\lvert\nabla T\rvert^2/T$
$f(\mathbf{x}, t)$	indicator function, positive in $\mathcal{V}$ and negative in $\overline{\mathcal{V}}$, with $\lvert\nabla f\rvert = 1$ on $\overline{\mathcal{S}}$
$g(r \mid \xi)$	Green's functions for the Helmholtz operator in cylindrical coordinates (1D)
$\hat{h}$	Complex amplitude of outward heat flux from the cylinder
$H(f)$	Heaviside (unit step) function acting as a spatial window
$H_n^{(1)}(\xi)$	Hankel function of the first kind $H_n^{(1)}(\xi) = J_n(\xi) + \mathrm{i}Y_n(\xi)$
i	Imaginary unit
$J_n(\xi)$	Bessel function of the first kind of order n
k	Radial wavenumber (Sec. 3); lossless acoustic wavenumber $2\Omega/c_0$ (Sec. 5)
k_a	Complex wavenumber for the acoustic mode (Sec. 5, $k_a = k\left[1 + \frac{1}{2}\mathrm{i}\varepsilon_L + \frac{1}{2}\mathrm{i}(\gamma - 1)\varepsilon_\kappa + O(\varepsilon^2)\right]$)

k_w	Characteristic wavenumber for the vorticity mode $k_w = (\mathrm{i}\Omega/\nu_0)^{1/2}$
K	Helmholtz number $K = \Omega r_0/c_0$
M	Mach number $M = U/c_0$
n	Azimuthal order in cylindrical coordinates
$\hat{\mathbf{n}}$	Unit vector normal to surface $\overline{\mathcal{S}}$, pointing outward from excluded region $\overline{\mathcal{V}}$
p	Thermodynamic pressure of fluid
p''	Scaled pressure perturbation $p'' = p/p_0 - 1$ (Sec. 3)
p'_{mod}	Acoustic mode pressure variable, $p'_{\text{mod}} = c_0^2(\rho^\star - \rho_0) - \overline{\mu}\Theta_s$
$\hat{p}_1, \hat{p}_2, \hat{p}_3$	Complex amplitudes of radiated pressure contributions from q, $q_{\phi\phi}$, $\mathbf{q}\cdot\hat{\mathbf{n}}$ respectively
P	Dimensionless quantity $P = Pr\,S$
Pr	Prandtl number of fluid $Pr = c_p\mu/\kappa$
$q, q_{\phi\phi}$	Monopole, quadrupole source density for the acoustic mode variable p'_{mod}
$\mathbf{q}$	Heat flux vector
r	Radial coordinate in cylindrical coordinates
r_0	Radius of rotationally oscillating cylinder
r''	Scaled radial coordinate Ur/ν_0 (Sec. 3)
r''_0	Scaled cylinder radius $r''_0 = Ur_0/\nu_0$
s	Specific entropy
S	Stokes number $S = \Omega r_0^2/\nu_0$
$\overline{\mathcal{S}}$	Surface separating excluded region $\overline{\mathcal{V}}$ from region $\mathcal{V}$, defined by $f(\mathbf{x}) = 0$

t	Time
t''	Scaled time $t'' = U^2 t/\nu_0$
T	Thermodynamic temperature
$\mathbf{u}$	Fluid velocity vector
u_r, u_ϕ	Fluid velocity components
v_r, v_ϕ	Scaled fluid velocity components $v_r = u_r/U$, $v_\phi = u_\phi/U$ (Sec. 3)
$\mathcal{V}$	Fluid region
$\overline{\mathcal{V}}$	Excluded region, adjacent to $\mathcal{V}$
$\mathbf{w}$	Vector potential in the representation $\mathbf{u} = \nabla\varphi + \operatorname{curl}\mathbf{w}$
w_x	Component of $\mathbf{w}$ in x direction
$W_{\rm rad}$, $W_{\rm diss}$	Power radiated, power dissipated (per unit length of cylinder)
x	Axial coordinate in cylindrical coordinates
$\mathbf{x}$, x_i	Position vector, cartesian components ($i = 1, 2, 3$)
α	Volume thermal expansivity of fluid
β	Thermal admittance of cylinder boundary $\beta = -\hat{h}/\hat{T}(r_0)$
γ	Ratio of specific heats $\gamma = c_p/c_v$, equal to the isentropic index in the case of an ideal gas
$\Gamma_{\rm vol}$, $\Gamma_{\rm surf}$	Volume and surface sources in an acoustic analogy
$\delta(f)$	Dirac delta function
δ	Viscous length scale $(\nu_0/\Omega)^{1/2}$
Δ	Relative error in asymptotic approximations
ε	$\max(\varepsilon_\kappa, \varepsilon_L)$
ε_κ	Thermal-diffusion parameter $\varepsilon_\kappa = 2\Omega\chi_0/c_0^2$
ε_L	Longitudinal-viscosity parameter $\varepsilon_L = 2\Omega\mu_L/\rho_0 c_0^2$
ε_μ	Viscosity parameter $\varepsilon_\mu = K^2/S = \Omega\nu/c_0^2$

$\eta_{\rm ac}$	Acoustic conversion efficiency $W_{\rm rad}/W_{\rm diss}$
Θ	Fluid dilatation rate $\Theta = \operatorname{div} \mathbf{u}$
κ	Thermal conductivity
λ	Acoustic wavelength
μ	Shear viscosity of fluid
μ_B	Bulk viscosity of fluid
μ_L	Longitudinal viscosity of fluid $\mu_L = \mu_B + \frac{4}{3}\mu$
$\overline{\mu}$	Thermoviscous coefficient $\overline{\mu} = \mu_B + \frac{4}{3}\mu - \kappa/c_{p0}$
ν	Kinematic viscosity of fluid $\nu = \mu/\rho$
ξ	Radial source position in cylindrical coordinates
$\xi_{\rm max}$	Displacement amplitude of cylinder surface
ρ	Fluid density
$\rho^\star$	Isentropic density $\rho^\star = \rho(p, s_0)$
ρ''	Scaled density perturbation $\rho'' = \rho/\rho_0 - 1$ (Sec. 3)
$\hat{\tau}_{r\phi}$	Complex amplitude of viscous shear stress at cylinder surface
φ	Scalar potential of velocity in the representation $\mathbf{u} = \nabla\varphi + \operatorname{curl} \mathbf{w}$
ϕ	Azimuthal coordinate in (x, r, ϕ) system
χ	Thermal diffusivity $\chi = \kappa/\rho c_p$
ω_0	Scaled angular frequency $\omega_0 = \Omega\nu_0/U^2$ (Sec. 3)
ω_x	Axial component of vorticity
$\boldsymbol{\omega}$	Fluid vorticity vector $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$
$\sim$	Varies asymptotically as
$\simeq$	Asymptotically equals
$(\cdot)_0$	Uniform unperturbed value of any local property of the fluid (ρ, p, s, T, etc.)

$(\cdot)'$	Departure of local property from its unperturbed value
$(\hat{\cdot})$	Complex amplitude of a sinusoidally varying quantity (phasor)
$(\cdot)^*$	Complex conjugate

1. Introduction

In the early days of theoretical aeroacoustics, following the pioneering work of Lighthill [1] and Curle [2], the debate over the role of solid boundaries in aerodynamic sound generation led to a demand for analytically-solvable model problems able to provide insight into this issue. The presence of an infinite plane boundary, either rigid or pressure-release, was shown by Powell [3] to act as a simple reflector for the quadrupole sources in Lighthill's acoustic analogy [1]. This idea was later extended by Ffowcs Williams [4] to a plane boundary with a uniform locally-reacting impedance. Lauvstad and Meecham [5], however, recognized that the Lighthill–Curle theory could be applied to any localized unsteady flow with solid boundaries for which an analytical solution existed in the incompressible limit, and that this would enable prediction of the surface and volume radiated sound at low Mach numbers. They proposed to use this approach to examine the role of the surface dipoles in Curle's formulation [2].

To this end (and following an earlier attempt by Meecham [6]), Lauvstad and Meecham [5] applied the Lighthill–Curle theory to the idealized problem of sound generation by a long circular cylinder, rotating sinusoidally about its axis and surrounded by an infinite uniform viscous fluid initially at rest. In the limit $M \to 0$, where $M = U/c_0$ is the ratio of the surface velocity amplitude to the unperturbed sound speed, the unsteady flow field can be described analytically provided the flow remains laminar and stable. The incompressible velocity field in the viscous boundary layer, and the incompressible pressure on the cylinder boundary, then provide the necessary source terms for the sound to be predicted using Ref. [2]. A distinctive feature of the solution is that because of the rotational symmetry, no sound is radiated in the linear approximation: to obtain the sound field one has to proceed to second order. In this respect the rotating-cylinder problem deviates from the general rule that tangential oscillations of a solid boundary next to a viscous fluid generate a linear acoustic response [7, 8].

Surprisingly, the calculation in [5] yielded a null result, with the normal-dipole surface term in Curle's expression for the far-field density exactly cancelling the volume quadrupole term. This outcome contradicted a previous estimate by Lauvstad [9], based on the entirely different approach of matched asymptotic expansions. It also left unanswered the question of which approach (if either) is valid for problems in viscous-fluid aeroacoustics.

The aim of the present paper is to resolve the issue and to show that the problem chosen is more complex than was allowed for in Refs. [5] and [9]. Our approach is a nonlinear extension of earlier work by Pierce [10, 11], who used an asymptotic approximation to describe the excitation of linear modes in thermoviscous fluids by external sources. It can be viewed as a generalization to arbitrary fluids of the bilateral mode-interaction analysis in Chu and Kovásznay [12], with the addition of boundary source terms. The restriction in [12] to an ideal gas with constant specific heats and Prandtl number $\frac{3}{4}$ is removed, and we focus on the sound produced by vorticity–vorticity interaction. The results provide new insights into aerodynamic sound generation in such fluids, along with a benchmark analytical solution that may have value in validating numerical codes for nonlinear thermoviscous acoustics.

Our objectives are (a) to adapt the Lighthill–Curle acoustic analogy formulation in its generalized form due to Ffowcs Williams and Hawkings [13], to better handle flows that involve viscosity and heat conduction; (b) to use the modified analogy to predict the sound field generated by the oscillating cylinder described above; (c) to account for the apparent zero radiation found in Lauvstad and Meecham [5]; (d) to rework the matched-expansion calculation of Lauvstad [9] and thereby clarify the assumptions involved; and (e) to show how the sound output from the oscillating cylinder is affected by the thermal boundary condition at the cylinder wall.

The outline of the article is as follows. Section 2 provides a statement of the problem. In Section 3 we re-examine Lauvstad's matched asymptotic solution, under his isothermal condition. In Section 4 we introduce a thermoviscous acoustic analogy that can be used to solve the problem for a real fluid (i.e. without assuming an ideal gas with $\gamma = 1$, so that heat conduction is important as well as viscosity); results of this procedure are given in Section 5. Finally, the effects of non-adiabatic boundary conditions at the cylinder wall are discussed in Section 6.

2. Statement of problem

The problem to be solved is that defined in Lauvstad [9], to find the sound radiated by a long circular cylinder rotating sinusoidally around its axis in a viscous compressible fluid. Thus Figure 1 shows a cross-section through an infinitely long rigid cylinder of radius r_0, whose axis lies along the x-axis of a cylindrical coordinate system (x, r, ϕ). The cylinder is surrounded by an unbounded fluid with pressure p, density ρ and temperature T. Fluid properties are represented as $(\cdot) = (\cdot)_0 + (\cdot)'$, where $(\cdot)_0$ is the uniform unperturbed value and $(\cdot)'$ is the perturbation. The fluid has constant shear viscosity μ.

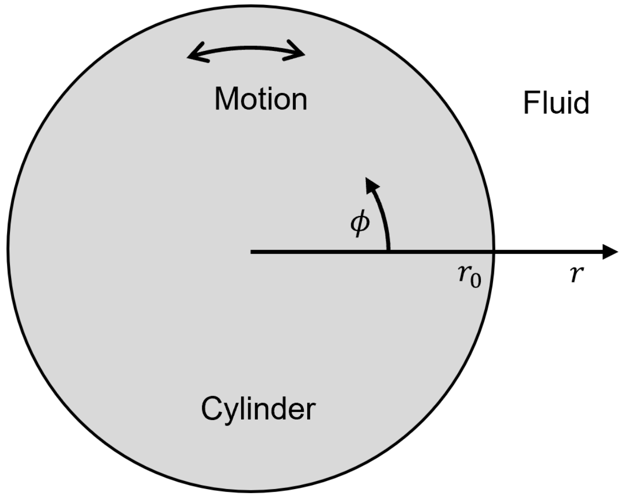

Figure 1: Diagram of Lauvstad's rotating-cylinder problem.

The cylinder rotates sinusoidally about its axis at angular frequency Ω with alternating clockwise and anticlockwise motion, U being the amplitude of the velocity at the cylinder's surface where $r = r_0$. We assume, with Lauvstad [9], that $U \ll c_0$, the speed of sound in the fluid, and is also low enough that transition to turbulent flow does not occur; conditions for this are discussed in Appendix A. In the limit $U/c_0 \to 0$ the density perturbations will be proportional to the unsteady component of the pressure whose gradient balances the acceleration field induced by the cylinder's motion. When heat conduction and thermal expansivity are present, however, the fluid will be heated by the work done in shearing it, causing it to dilate, and this dissipative heating will be a significant sound-generation mechanism, at least in comparison with the isothermal case assumed in [9].

The fluid velocity field $\mathbf{u} = \left[0, u_r(r,t), u_\phi(r,t)\right]$ is subject to the following boundary conditions:

- The no-slip condition at the cylinder's surface gives $u_\phi(r_0, t) = \mathrm{Re}\,[U \exp(-\mathrm{i}\Omega t)]$ (we follow Lauvstad [9] in using complex exponentials despite the nonlinearity of the problem).

- The rigidity and axisymmetry of the cylinder give $u_r(r_0, t) = 0$.

- The decay of disturbances with distance from the cylinder gives $u_r(\infty, t) = u_\phi(\infty, t) = 0$.

Note that we omit dependence on x and ϕ when writing the components of the velocity field.

In [9] Lauvstad implicitly assumes that the fluid is an ideal gas with adiabatic index $\gamma = 1$, meaning that the speed of sound can be taken as its isothermal value $c_0 = \sqrt{p_0/\rho_0}$, the temperature T never departs from T_0 and no heat conduction occurs; in addition the bulk viscosity is assumed to be zero. We retain these assumptions in Sec. 3 where we reproduce Lauvstad's matched-asymptotic analysis. Thereafter, however, we relax them and allow the fluid to have thermal expansivity α, and thermal conductivity κ and bulk viscosity μ_B, both constant.

The governing equations conserving mass and momentum can be written [14]:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) = 0, \tag{1a}$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} - \frac{u_\phi^2}{r}\right) = -\frac{\partial p}{\partial r} + \frac{4}{3}\mu\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(ru_r), \tag{1b}$$

$$\rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_r u_\phi}{r}\right) = \mu\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi). \tag{1c}$$

Under the isothermal assumption of Lauvstad [9] the energy equation is not required. The problem then is to find the far-field pressure fluctuations $p'(r,t)$ for large r.

3. Lauvstad's matched asymptotic solution revisited

In this section we reproduce Lauvstad's matched asymptotic analysis of the problem described above. Lauvstad used dashed variables for physical quantities so that the scaled variables in which he analysed the problem were plain. Although we follow his scaling for ease of comparison, we prefer to use plain physical variables, and where scaled variables using the same symbols appear in Sec. 3 they will be given double dashes. In what follows, Lauvstad's dimensionless radial coordinate is introduced as $r'' = Ur/\nu_0$ and $t'' = U^2 t/\nu_0$ denotes the dimensionless time.

The dimensional radius of the cylinder r_0 is converted to non-dimensional form as $r_0'' = Ur_0/\nu_0$. The instantaneous density is scaled as $\rho/\rho_0 = 1 + \rho''$, and the velocity field is scaled as $v_r = u_r/U$, $v_\phi = u_\phi/U$. The fluid is modelled as an ideal gas with the pressure scaled as $p/p_0 = 1+p''$, and the scaled pressure perturbation is related to the density perturbation by $p'' = \rho''$, equivalent to assuming the flow is isothermal. Finally the Mach number is introduced based on the isothermal sound speed as $M = U/(p_0/\rho_0)^{1/2} = U/c_0$. Then the system of governing equations (1) becomes, in terms of scaled perturbation variables,

$$\frac{\partial \rho''}{\partial t''} + \frac{1}{r''}\frac{\partial}{\partial r''}(r''\rho'' v_r) = 0, \tag{2a}$$

$$\rho''\left(\frac{\partial v_r}{\partial t''} + v_r\frac{\partial v_r}{\partial r''} - \frac{v_\phi^2}{r''}\right) = -M^{-2}\frac{\partial \rho''}{\partial r''} + \frac{4}{3}\frac{\partial}{\partial r''}\frac{1}{r''}\frac{\partial}{\partial r''}(r'' v_r), \tag{2b}$$

$$\rho''\left(\frac{\partial v_\phi}{\partial t''} + v_r\frac{\partial v_\phi}{\partial r''} - \frac{v_r v_\phi}{r''}\right) = \frac{\partial}{\partial r''}\frac{1}{r''}\frac{\partial}{\partial r''}(r'' v_\phi). \tag{2c}$$

To facilitate comparison of these equations with Eq. (7) in [9], we note that Lauvstad used primes to designate dimensional variables and no primes for dimensionless scaled variables. For consistency, we also note that the factor $\frac{4}{3}$ is missing from the final term in his Eq. (7b), which is the counterpart of our Eq. (2b). The same applies to the final term in Lauvstad's Eq. (13). However, these errors do not affect the end result Eq. (43) in [9] because the solution for the outer field does not use these equations.

To solve the scaled equations, Lauvstad used the method of matched asymptotic expansions. We follow him in solving the inner problem. The inner expansion is expanded in ascending powers of M^2 as

$$\rho''(r'', t''; M) \simeq M^2 \rho_1''(r'', t'') + M^4 \rho_2''(r'', t'') + \cdots, \tag{3a}$$

$$v_r(r'', t''; M) \simeq v_{r0} + M^2 v_{r1}(r'', t'') + \cdots, \tag{3b}$$

$$v_\phi(r'', t''; M) \simeq v_{\phi 0} + M^2 v_{\phi 1}(r'', t'') + \cdots. \tag{3c}$$

This solution ansatz is substituted into Eqs. (2) and the chain rule is used to solve problems of ascending order in the parameter M. This will lead to a first-order inner solution for the radial velocity, which is then matched to an outer acoustic field.

The zeroth-order equation for the radial velocity is

$$\frac{1}{r''} \frac{\partial}{\partial r''}(r'' v_{r0}) = 0. \tag{4}$$

Since at the surface of the cylinder ($r'' = r_0''$) this velocity component equals zero, i.e. $v_{r0}(r_0'', t'') = 0$ it also vanishes in the whole fluid volume, giving $v_{r0}(r'', t'') = 0$.

The first-order equations for the density disturbances are

$$\frac{\partial \rho_1''}{\partial t''} + \frac{1}{r''} \frac{\partial}{\partial r''}(r'' v_{r1}) = 0 \quad \text{and} \quad \frac{\partial \rho_1''}{\partial r''} = \frac{v_{\phi 0}^2}{r''}. \tag{5}$$

Density fluctuations are readily eliminated in this system to yield the equation for the first non-vanishing component of the radial velocity:

$$\frac{\partial}{\partial r''}\left[\frac{1}{r''}\frac{\partial}{\partial r''}(r'' v_{r1})\right] = -\frac{1}{r''}\frac{\partial}{\partial t''}(v_{\phi 0}^2). \tag{6}$$

To find the right-hand side of Eq. (6) we solve the zeroth-order problem for the circumferential velocity component:

$$\frac{\partial v_{\phi 0}}{\partial t''} = \frac{\partial}{\partial r''}\frac{1}{r''}\frac{\partial}{\partial r''}(r'' v_{\phi 0}). \tag{7}$$

Hereafter the rotating motion of the cylinder is assumed time-harmonic so that $v_{\phi 0}(r'', t'') = \operatorname{Re}\left[\hat{v}_{\phi 0} \mathrm{e}^{-\mathrm{i}\omega_0 t''}\right]$. We note that the second equation from the set Eq. (1) in [9], which is meant to be written in dimensional form, uses ω_0 for the dimensional angular frequency; we use Ω instead and reserve ω_0 for the scaled frequency, as later adopted by Lauvstad in his Eq. (16). Thus, the frequency scaling is $\omega_0 = \Omega \nu_0 / U^2$, and Eq. (7) becomes

$$-\mathrm{i}\omega_0 \hat{v}_{\phi 0} = \frac{\mathrm{d}}{\mathrm{d}r''}\frac{1}{r''}\frac{\mathrm{d}}{\mathrm{d}r''}(r'' \hat{v}_{\phi 0}), \tag{8}$$

with boundary condition $\hat{v}_{\phi 0}(r''_0) = 1$. The solution is

$$\hat{v}_{\phi 0} = \frac{H_1^{(1)}(r'' \sqrt{\mathrm{i}\omega_0})}{H_1^{(1)}(r''_0 \sqrt{\mathrm{i}\omega_0})}. \tag{9}$$

In this equation, and throughout the article, the principal value of the complex square root is taken. The nonlinear term $v_{\phi 0}^2(r'', t'')$ from (6) is represented in terms of $\hat{v}_{\phi 0}$ by writing

$$v_{\phi 0}^2(r'', t'') = \frac{1}{4}\left[\hat{v}_{\phi 0}(r'')\mathrm{e}^{-\mathrm{i}\omega_0 t''} + \hat{v}_{\phi 0}^*(r'')\mathrm{e}^{\mathrm{i}\omega_0 t''}\right]^2 = \tfrac{1}{2}\operatorname{Re}[\hat{v}_{\phi 0}(r'')\mathrm{e}^{-\mathrm{i}\omega_0 t''}]^2 + \tfrac{1}{2}|\hat{v}_{\phi 0}(r'')|^2. \tag{10}$$

This shows that the quadratic quantity $v_{\phi0}^2(r'', t'')$ should be represented by the phasor $\frac{1}{2}[\hat{v}_{\phi0}(r'')\mathrm{e}^{-\mathrm{i}\omega_0 t''}]^2$, rather than $[\hat{v}_{\phi0}(r'')\mathrm{e}^{-\mathrm{i}\omega_0 t''}]^2$ as in [9].

Combining Eq. (6) with (9) and (10) gives a frequency-domain equation for the first-order radial component of velocity,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r''^2} + \frac{1}{r''}\frac{\mathrm{d}}{\mathrm{d}r''} - \frac{1}{r''^2}\right)\hat{v}_{r1}(r'') = \frac{\mathrm{i}\omega_0}{r''}\hat{v}_{\phi0}^2(r'') = \frac{\mathrm{i}\omega_0}{r''}\left[\frac{H_1^{(1)}(r''\sqrt{\mathrm{i}\omega_0})}{H_1^{(1)}(r_0''\sqrt{\mathrm{i}\omega_0})}\right]^2, \quad (11)$$

with Dirichlet boundary condition $\hat{v}_{r1}(r_0'') = 0$. From the discussion above it follows that when returning to the time domain the radial velocity is $\mathrm{Re}\left[\hat{v}_{r1}(r'')\mathrm{e}^{-2\mathrm{i}\omega_0 t''}\right]$.

The operator in (11) is a limiting case of the Helmholtz radial operator in cylindrical coordinates, with azimuthal order $n = 1$ and eigenvalue $k^2 \to 0$. In the general case, the one-dimensional Green's function for this operator can be defined as the solution of

$$\left(\frac{\partial^2}{\partial r''^2} + \frac{1}{r''}\frac{\partial}{\partial r''} - \frac{n}{r''^2} + k^2\right)g(r''\,|\,\xi) = -\frac{1}{r''}\delta(r'' - \xi), \quad (12)$$

that satisfies the Dirichlet boundary condition $g^- = 0$ at $r'' = r_0''$ and outgoing-wave condition $(\partial/\partial r'')g^+ = \mathrm{i}kg^+$ as $r'' \to \infty$. This solution is given by

$$g^+(r''\,|\,\xi) = \frac{\pi}{2}\left[J_n(kr_0'')Y_n(k\xi) - Y_n(kr_0'')J_n(k\xi)\right]\frac{H_n^{(1)}(kr'')}{H_n^{(1)}(kr_0'')} \quad (r'' > \xi \geq r_0''), \quad (13a)$$

$$g^-(r''\,|\,\xi) = g^+(\xi\,|\,r'') \quad (\xi > r'' \geq r_0''). \quad (13b)$$

It is specialized for the case $n = 1$, $k = 0$ as:

$$g_1^+(r''\,|\,\xi) = \frac{\xi}{2r''} - \frac{r_0''^2}{2r''\xi} \quad (r'' > \xi \geq r_0''), \quad (14a)$$

$$g_1^-(r''\,|\,\xi) = \frac{r''}{2\xi} - \frac{r_0''^2}{2r''\xi} \quad (\xi > r'' \geq r_0''). \quad (14b)$$

Then

$$\hat{v}_{r1}(r'') = \int_0^\infty F(z) g_1(r'' \,|\, z)\,\mathrm{d}z, \quad F(z) = -\mathrm{i}\omega_0 \left[\frac{H_1^{(1)}(z\sqrt{\mathrm{i}\omega_0})}{H_1^{(1)}((r_0''\sqrt{\mathrm{i}\omega_0})} \right]^2, \tag{15}$$

which gives

$$\begin{aligned} \hat{v}_{r1}(r'') = &-\frac{\mathrm{i}\omega_0}{2r''\left[H_1^{(1)}\left(r_0''\sqrt{\mathrm{i}\omega_0}\right)\right]^2} \left\{ \int_{r_0''}^{r''} z \left[H_1^{(1)}\left(z\sqrt{\mathrm{i}\omega_0}\right)\right]^2 \mathrm{d}z \right. \\ &\left. + r''^2 \int_{r''}^{\infty} \frac{1}{z}\left[H_1^{(1)}\left(z\sqrt{\mathrm{i}\omega_0}\right)\right]^2 \mathrm{d}z - r_0''^2 \int_{r_0''}^{\infty} \frac{1}{z}\left[H_1^{(1)}\left(z\sqrt{\mathrm{i}\omega_0}\right)\right]^2 \mathrm{d}z \right\}. \end{aligned} \tag{16}$$

Next we use the large-argument asymptotic form of the Hankel function to write

$$\left[H_1^{(1)}\left(y\sqrt{\mathrm{i}\omega_0}\right)\right]^2 \simeq \frac{2}{\pi y\sqrt{\mathrm{i}\omega_0}} \exp\left(2\mathrm{i}y\sqrt{\mathrm{i}\omega_0}\right), \quad (y = r_0'', z). \tag{17}$$

This substitution allows the integrals in (16) to be evaluated analytically for $S^{1/2} \gg 1$, and the product of radial velocity and radial distance can be written in terms of re-scaled radial distance $r''/r_0'' = r/r_0$ as follows:

$$\begin{aligned} r''\hat{v}_{r1}(r'') = &\frac{1}{4} - \frac{3(1+\mathrm{i})\sqrt{2}}{16\sqrt{S}} \\ &+ \frac{(1+\mathrm{i})\left[3\sqrt{2} - 2(1-\mathrm{i})(r''/r_0'')\sqrt{S}\right]\exp\left[\sqrt{2S}(1 - r''/r_0'')(1-\mathrm{i})\right]}{16\sqrt{S}(r''/r_0'')^2}, \end{aligned} \tag{18}$$

where $S = \Omega r_0^2/\nu_0 = \omega_0 r_0''^2$ is the conventional Stokes number. This function vanishes at $r'' = r_0''$ by virtue of the Green's function (14b). For large Stokes numbers and outside the viscous boundary layer, the first term dominates, and this formula becomes:

$$r''\hat{v}_{r1}(r'') \simeq \frac{1}{4}. \tag{19}$$

Examples of inner-solution velocity profiles are presented in Fig. 2.

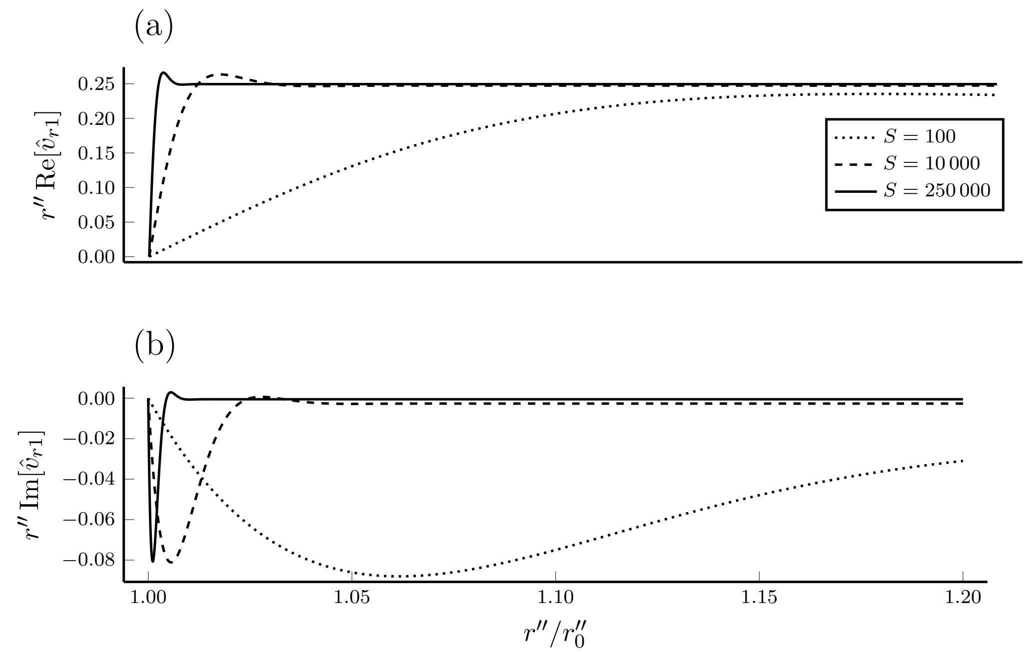

Figure 2: (a) Real and (b) imaginary parts of inner-solution radial velocity multiplied by radial distance r'' and plotted against scaled radial distance r''/a, for three values of Stokes number S.

Naturally, this solution is for the inner zone, and the thickness of the boundary layer collapses as the Stokes number increases. Ep. (19) gives the purely real velocity at the outer edge of the boundary layer, which is the asymptote for velocity profiles given by (18) at all Stokes numbers. Imaginary parts of velocities from (18) attain the same maximum at different positions for different Stokes numbers and vanish at different rates.

Substituting (19) into Eq. (3b) and converting to dimensional form gives the inner solution for the radial velocity outside the viscous boundary layer to order M^2,

$$u_r(r,t) = \frac{1}{4}\frac{\nu_0}{r}\left(\frac{U}{c_0}\right)^2 \mathrm{e}^{-2\mathrm{i}\Omega t} = \frac{1}{4}c_0\frac{r_0}{r}\frac{KM^2}{S}\mathrm{e}^{-2\mathrm{i}\Omega t}, \tag{20}$$

where $K = \Omega r_0/c_0$ is the Helmholtz number and the real part of phasor quantities is henceforth implied.

The radial velocity extrapolated back to the surface of the cylinder is predicted by (20) as

$$u_r(r_0, t) \simeq \frac{1}{4} c_0 \frac{KM^2}{S} \mathrm{e}^{-2\mathrm{i}\Omega t}, \tag{21}$$

Obviously, it does not satisfy the Dirichlet boundary condition, and Lauvstad [9] followed the rigorous procedure of matching the inner (nonlinear viscous) and outer (linear acoustic) solutions inside and outside the boundary layer.

However, inspection of the velocity profiles in Fig. 2 provides an incentive to use a much simpler method of patching. We consider (21) as a source of the acoustic far field pressure, and assume that the viscous boundary layer thickness is so small that the formula for $u_r(r_0, t)$ may be used just as a boundary condition for the conventional linear acoustics problem. We also follow Lauvstad in assuming $K \ll 1$; this assumption is introduced at Eq. (34) in [9]. Then the dimensional time-harmonic velocity potential is:

$$\varphi(r, t) = -\frac{\pi \mathrm{i}}{8} \frac{K^2 M^2}{S} \frac{c_0^2}{\Omega} H_0^{(1)} \left(\frac{2\Omega r}{c_0} \right) \mathrm{e}^{-2\mathrm{i}\Omega t}. \tag{22}$$

The acoustic pressure becomes

$$p'(r, t) = \frac{\pi}{4} \rho_0 c_0^2 \frac{K^2 M^2}{S} H_0^{(1)} \left(\frac{2\Omega r}{c_0} \right) \mathrm{e}^{-2\mathrm{i}\Omega t}. \tag{23}$$

In the far field we have

$$p'(r, t) = \frac{\sqrt{\pi}}{4} \rho_0 c_0^2 \left(\frac{\Omega r_0}{c_0} \right)^{3/2} \frac{\nu_0}{\Omega r_0^2} \left(\frac{U}{c_0} \right)^2 \sqrt{\frac{r_0}{r}} \exp\left[-\mathrm{i} \left(2\Omega(t - r/c_0) + \pi/4 \right) \right]. \tag{24}$$

Then

$$\frac{|p'(r, t)|}{\rho_0 U^2} = \frac{\sqrt{\pi}}{4} \left(\frac{\Omega r_0}{c_0} \right)^{3/2} \frac{\nu_0}{\Omega r_0^2} \sqrt{\frac{r_0}{r}} = \frac{1}{4} \sqrt{\frac{\pi c_0}{\Omega r}} \frac{\Omega \nu_0}{c_0^2}. \tag{25}$$

This result perfectly agrees with Eq. (59) in Sec. 5.6.

In Lauvstad [9], the classical van Dyke method is used to match the solution of the inner problem to the conventional solution of the outer linear acoustic problem. The possible problem with this method, pointed out by Lesser and Crighton [15], of logarithmic gauge functions and 'switchback' does not arise in this problem. The end result is given in dimensional form as Eq. (43) by Lauvstad [9]. When corrected by a factor of $\frac{1}{2}$, as has been explained, this becomes:

$$\rho(r,t) - \rho_0 = \frac{\sqrt{\pi}}{4}\rho_0 M^2 \varepsilon_\mu \left(\frac{1}{\sqrt{k_0 r}}\right) \exp\left[2\mathrm{i}k_0(r - c_0 t) + \frac{3\mathrm{i}\pi}{4}\right], \tag{26}$$

where $\varepsilon_\mu = \nu_0\Omega/c_0^2$ corresponds to symbol S in [9], where it is referred to as the modified Stokes number. To avoid confusion with our Stokes number defined following Eq. (18), we have replaced Lauvstad's S with the symbol ε_μ in the present main text.

The pressure fluctuations are related to the density fluctuations in the conventional way,

$$p'(r,t) = c_0^2[\rho(r,t) - \rho_0]. \tag{27}$$

The acoustic far-field pressure corresponding to (26) is then

$$p'(r,t) = \rho_0 c_0^2 \frac{1}{4}\sqrt{\frac{\pi c_0}{\Omega r}} \left(\frac{U}{c_0}\right)^2 \frac{\Omega\nu_0}{c_0^2} \exp\left[2\mathrm{i}\frac{\Omega}{c_0}(r - c_0 t) + \frac{3\mathrm{i}\pi}{4}\right], \tag{28}$$

giving the non-dimensional pressure amplitude for $K \ll 1$ as

$$\frac{|p'(r,t)|}{\rho_0 U^2} = \frac{1}{4}\sqrt{\frac{\pi c_0}{\Omega r}} \frac{\Omega\nu_0}{c_0^2}. \tag{29}$$

Lauvstad's corrected pressure amplitude in (29) agrees with the result in (25) and with Eq. (59) in Sec. (5.6). However there is a sign difference between Eq. (28) and the corrsponding result from Sec. 5 which remains unexplained.

4. Outline of the weakly thermoviscous acoustic analogy equation

An acoustic analogy consists of an exact rearrangement of the governing equations with a propagation operator applied to the variable of interest on the left-hand side, and with the remaining terms interpreted as equivalent sources for that variable. In Lighthill's original acoustic analogy [1] the propagation operator was the lossless wave operator and the variable of interest was the density fluctuation, and this was also the case for its extensions to bounded domains by Curle [2] and by Ffowcs Williams and Hawkings [13].

As remarked in the introduction, the application by Lauvstad and Meecham [5] of Curle's form of Lighthill's acoustic analogy failed to replicate the matched asymptotic result of Lauvstad [9]. This failure follows from their use of the large Stokes number approximation $S \gg 1$ which, when combined with Curle's formulation [2], leads to volume and surface terms that cancel to lowest order. We show below that the problem disappears if one uses the more general Ffowcs Williams and Hawkings formulation [13] together with a Neumann Green's function, rather than the free-field Green's function implicit in [2]; there is then no surface term, and the lowest-order large-S approximation gives a finite result.

If, however, we wish to generalize the problem to allow for the fluid not being an ideal gas, or even for realistic gases with $\gamma > 1$, the Lighthill analogy runs into difficuties because the flow is no longer isothermal. The resulting temperature perturbations lead to linear heat conduction terms in the apparent source distribution, violating the basic premise of the acoustic analogy approach that no linear volume terms should appear. In this section we describe a thermoviscous-fluid acoustic analogy that leads to purely nonlinear volume source terms. Applying the modified analogy allows us in the remaining sections to solve the generalized version of Lauvstad's rotating-cylinder problem.

Infinitesimal disturbances to a uniform thermoviscous fluid at rest have three modes of propagation: acoustic, entropy and vorticity [10, 11, 12, 16]. Our analogy uses the acoustic-mode wave operator, together with a version of the acoustic-mode variable that corresponds to the weakly-thermoviscous approximation as developed in Pierce [11], Sec. 10-3. Our acoustic wave variable will be defined by

$$p'_{\text{mod}} = c_0^2(\rho^\star - \rho_0) - \overline{\mu}\Theta_s, \tag{30}$$

where $\overline{\mu} = (\mu_L - \kappa/c_{p0})$ and μ_L is the longitudinal viscosity $\frac{4}{3}\mu + \mu_B$; κ is the thermal conductivity, and c_p is the constant-pressure specific heat. Symbol $\rho^\star$ is the isentropic density $\rho^\star = \rho(p, s_0)$, with s denoting the specific entropy, and the quantity $\Theta_s = (\alpha T/c_p)\mathrm{D}s'/\mathrm{D}t$ is the dilatation rate associated with entropy changes following a fluid particle.

Note that for a fluid with Prandtl number $\frac{3}{4}$ and $\mu_B = 0$, as assumed in [12], $\overline{\mu}$ vanishes. The subtracted term in Eq. (30) represents the entropy-mode component of the pressure perturbation. Outside any thermoviscous boundary layers, the acoustic-mode pressure variable $p'_{\text{mod}} \to p'$ in the limit of small pressure perturbations $p' \ll \rho_0 c_0^2$.

The equations of motion for a bounded compressible fluid, with μ, μ_B and κ all assumed constant [14], can be rearranged without approximation to give a forced acoustic wave equation,

$$\left\{\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - (1 + \mathcal{D})\nabla^2\right\}(p'_{\text{mod}} H) = \Gamma_{\text{vol}} + \Gamma_{\text{surf}}, \tag{31}$$

where $\mathcal{D}$ is a damping operator defined below. The left-hand side of Eq. (31) corresponds to the linearized acoustic mode equation in a uniform thermoviscous fluid at rest with sound speed c_0. The wave operator in Eq. (31) is equivalent to that used in Eq. (10-3.13) of Pierce [11], which can be recovered by dividing Eq. (31) by $(1 + \mathcal{D})$ and dropping $O(\varepsilon^2)$ terms.

The spatial window function H is the unit step function $H(f)$, whose derivative is the delta function $\delta(f)$. Their argument $f(\mathbf{x}, t)$ is an indicator function, positive in the fluid region $\mathcal{V}$ and negative in the complementary region $\overline{\mathcal{V}}$, with $f = 0$ on the common interface $\overline{\mathcal{S}}$; thus $H = 1$ in $\mathcal{V}$ and $H = 0$ in $\overline{\mathcal{V}}$. Ep. (31) differs in two main respects from the windowed-variable acoustic analogy introduced by Ffowcs Williams and Hawkings [13]. First, p'_{mod} contains an additional term [10, 11] that cancels the entropy-mode component of p' in the weakly thermoviscous limit, thus ensuring that Γ_{vol} contains no linear terms. Secondly, the acoustic wave operator on the left allows for long-range attenuation of sound, via the thermoviscous damping operator

$$\mathcal{D} = \left[\mu_L + (\gamma_0 - 1)\kappa/c_{p0}\right] \frac{1}{\rho_0 c_0^2} \frac{\partial}{\partial t}. \tag{32}$$

In order to apply Eq. (31) to the oscillating-cylinder problem where the sound field has angular frequency 2Ω, the thermoviscous frequency parameters $\varepsilon_L = 2\Omega\mu_L/\rho_0 c_0^2$ and $\varepsilon_\kappa = 2\Omega\chi_0/c_0^2$ need to be much less than 1, where $\chi = \kappa/\rho c_p$ denotes the thermal diffusivity of the fluid. The relative error in Eq. (31) is then $\Delta = O(\varepsilon)$, where $\varepsilon = \max(\varepsilon_L, \varepsilon_\kappa)$.

To within relative error $\Delta = O(\varepsilon)$, the volume source distribution in Eq. (31) is free of terms linear in perturbation variables, in contrast to earlier acoustic analogy formulations. Thus the Γ_{vol} source terms in Eq. (31) are of second or higher order in the perturbation quantities $\mathbf{u}$, p', s', T' etc. If one limits attention to second-order terms, then of the six possible bilateral combinations among the acoustic, entropy and vorticity modes only one is relevant to the present problem, since the linear field is confined to the vorticity mode. A detailed analysis in the time domain yields the dominant vorticity–vorticity interaction terms responsible for acoustic mode generation as follows:

$$\Gamma_{\mathrm{vol}} \simeq \rho_0 \frac{\partial}{\partial t}(a\dot{D}H) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho_0 u_i u_j H). \tag{33}$$

Here $\dot{D}$ is the viscous dissipation rate per unit volume, and a is the fluid thermal-expansion parameter $\alpha/\rho c_p$, where α is the fluid's thermal expansivity. In order to arrive at Eq. (33), each term in the exact Γ_{vol} expression has been scaled using a quasilinear assumption; the far-field pressures contributed by each term can then be ordered with respect to the small parameters ε_L and ε_κ defined earlier. The Γ_{vol} expression above is an asymptotic approximation, based on the vorticity–vorticity terms whose contributions to p' in the far field are of lowest order in $(\varepsilon_L, \varepsilon_\kappa)$.

The surface source distribution Γ_{surf}, on the other hand, does contain linear components. We choose $|\nabla f| = 1$ on $\overline{\mathcal{S}}$, so that

$$\frac{\partial H}{\partial x_i} = \hat{n}_i \delta(f) \tag{34}$$

where $\hat{\mathbf{n}}$ is the unit normal to $\overline{\mathcal{S}}$ pointing into $\mathcal{V}$. For the rotating-cylinder application, $\overline{\mathcal{S}}$ is chosen to coincide with the cylinder surface, so may be taken as impermeable. Then Γ_{surf} is given by

$$\Gamma_{\mathrm{surf}} \simeq \rho_0 \frac{\partial}{\partial t}\left[(\mathbf{u}\cdot\hat{\mathbf{n}} + a\mathbf{q}\cdot\hat{\mathbf{n}})\delta(f)\right] + \operatorname{div}\left[\mu(\boldsymbol{\omega}\times\hat{\mathbf{n}})\delta(f)\right] - \operatorname{div}\left[p'_{\mathrm{mod}}\hat{\mathbf{n}}\delta(f)\right], \tag{35}$$

where $\mathbf{q}$ is the heat flux and $a = \alpha/\rho c_p$. The $\mu(\boldsymbol{\omega}\times\hat{\mathbf{n}})$ term, where $\boldsymbol{\omega}$ is the vorticity, corresponds to Section 10-6 of Pierce's textbook [11], where the Kirchhoff–Helmholtz representation is modified to include viscosity effects. A linearized version of (31) and (35), with $\Gamma_{\mathrm{vol}} = 0$, was developed by Morfey et al.

[8] with no restriction placed on ε_L. However, their result was limited to fluids with zero thermal expansivity α; the heat flux term in (35) was therefore absent.

The $\mathbf{u}\cdot\hat{\mathbf{n}}$ term in Eq. (35) vanishes for the rotating-cylinder problem, since the normal velocity on the boundary is zero. Likewise the tangential-dipole term $\operatorname{div}\left[\mu(\boldsymbol{\omega}\times\hat{\mathbf{n}})\delta(f)\right]$ is zero by symmetry, while the normal-dipole term in p'_{mod} can be eliminated by using the Neumann Green's function to solve for the pressure field. This leaves the $\mathbf{q}\cdot\hat{\mathbf{n}}$ term, which we can remove only by choosing an adiabatic boundary condition at the cylinder surface. Any alternative requires formulating an extra equation for the entropy perturbation s', which is possible but more complicated and is discussed in Sec. 6.

5. Acoustic analogy results for the oscillating cylinder

5.1. Vorticity equation for axisymmetric two-dimensional flow

Provided rotational symmetry about the cylinder's axis is maintained, with velocity $\mathbf{u} = \left[0, u_r(r,t), u_\phi(r,t)\right]$, vorticity $\boldsymbol{\omega} = [\omega_x(r,t), 0, 0]$ and $\rho = \rho(r,t)$ in cylindrical (x, r, ϕ) coordinates, the nonlinear vorticity equation for axisymmetric flow of a compressible fluid with constant shear and bulk viscosity is given by

$$\frac{\partial \omega_x}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r \omega_x) = \mu\left[\frac{\partial \rho^{-1}}{\partial r}\frac{\partial \omega_x}{\partial r} + \rho^{-1}\left(\frac{\partial^2 \omega_x}{\partial r^2} + \frac{1}{r}\frac{\partial \omega_x}{\partial r}\right)\right]. \qquad (36)$$

This equation follows from Eqs. (1b, c) on dividing by ρ and taking the curl.

In the limiting case of small perturbations to a uniform fluid at rest, with initial density $\rho = \rho_0$, Eq. (36) reduces to a linear equation describing the diffusion of vorticity,

$$\frac{\partial \omega_x}{\partial t} - \nu_0\left(\frac{\partial^2 \omega_x}{\partial r^2} + \frac{1}{r}\frac{\partial \omega_x}{\partial r}\right) = 0, \quad (\nu_0 = \mu/\rho_0). \qquad (37)$$

5.2. *Linear solution for the velocity field*

The velocity field corresponding to Eq. (37) is solenoidal, with the velocity expressible in terms of a vector potential $\mathbf{w}$:

$$\mathbf{u} = \operatorname{curl}\mathbf{w}, \quad \operatorname{div}\mathbf{u} = 0. \tag{38}$$

The vorticity follows as $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u} = -\nabla^2\mathbf{w}$. In (x, r, ϕ) cylindrical coordinates, the symmetry of the problem gives the flow field as

$$\mathbf{w} = \mathbf{w}(r,t) = (w_x, 0, 0); \quad \mathbf{u} = \mathbf{u}(r,t) = (0, 0, u_\phi). \tag{39}$$

For harmonic oscillations with time factor $\mathrm{e}^{-\mathrm{i}\Omega t}$, the vector potential $\mathbf{w}$ is the real part of the phasor $\hat{\mathbf{w}}\mathrm{e}^{-\mathrm{i}\Omega t}$. The complex amplitude $\hat{\mathbf{w}}(r > r_0)$, where r_0 is the cylinder radius, satisfies a Helmholtz equation that follows from Eq. (37):

$$\left(\nabla^2 + k_w^2\right)\hat{\mathbf{w}} = 0, \quad k_w^2 = \frac{\mathrm{i}\Omega}{\nu_0}. \tag{40}$$

The component of (40) in the x direction is the homogeneous Bessel equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\hat{w}_x}{\mathrm{d}r}\right) + k_w^2\hat{w}_x = 0, \quad r > r_0, \tag{41}$$

with outgoing-wave solution $\hat{w}_x \propto H_0^{(1)}(k_w r)$. The fluid circumferential velocity $u_\phi(r,t) = \operatorname{Re}\left(\hat{u}_\phi \mathrm{e}^{-\mathrm{i}\Omega t}\right)$ then follows as

$$\hat{u}_\phi(r) = -\frac{\mathrm{d}\hat{w}_x}{\mathrm{d}r} = U\frac{H_1^{(1)}(k_w r)}{H_1^{(1)}(k_w r_0)}, \quad r > r_0, \tag{42}$$

where U (taken as real) is the amplitude of the tangential velocity at the cylinder surface.

5.3. *Acoustic analogy formulation*

The weakly thermoviscous acoustic analogy equation can be used to relate the sound field of the oscillating cylinder to the velocity field in the viscous boundary layer. The final result depends on the thermal boundary condition applied at the cylinder surface. We assume initially for simplicity that the boundary is adiabatic, since this eliminates the one significant boundary source term (for details, see Sec. 4). The dominant volume sources of sound then follow from Eq. (33). They comprise a quadrupole density $q_{\phi\phi}$ and a monopole density q, given to leading order by

$$q_{\phi\phi}(r,t) \simeq \rho_0 u_\phi^2; \quad q(r,t) \simeq \rho_0 a_0 \frac{\partial \dot{D}}{\partial t}. \tag{43}$$

The role of unsteady dissipation in sound generation, represented by the monopole term in Eq. (43), was discussed theoretically and demonstrated experimentally by Kambe and Minota [17] and Minota and Kambe [18]. Viscous energy dissipation in the flow around the cylinder is confined to the viscous boundary layer, and is related to the velocity field by

$$\dot{D} = \mu \left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right)^2. \tag{44}$$

In (43) u_ϕ^2 and $\dot{D}$ both contain a mean component which does not radiate, plus a component at angular frequency 2Ω. It follows from Eqs. (42–44) that the relevant source terms are $q_{\phi\phi}(r,t) = \mathrm{Re}\left(\hat{q}_{\phi\phi} \mathrm{e}^{-2\mathrm{i}\Omega t}\right)$ and $q(r,t) = \mathrm{Re}\left[\hat{q} \mathrm{e}^{-2\mathrm{i}\Omega t}\right]$, where

$$\hat{q}_{\phi\phi}(r) = \tfrac{1}{2}\rho_0 U^2 \left\{ \frac{H_1^{(1)}\left[(\mathrm{i}S)^{1/2}\, r/r_0\right]}{H_1^{(1)}\left[(\mathrm{i}S)^{1/2}\right]} \right\}^2, \quad \hat{q}(r) = \rho_0^2 a_0 \Omega^2 U^2 \left\{ \frac{H_2^{(1)}\left[(\mathrm{i}S)^{1/2}\, r/r_0\right]}{H_1^{(1)}\left[(\mathrm{i}S)^{1/2}\right]} \right\}^2. \tag{45}$$

The relation $(k_w r_0)^2 = \mathrm{i}S$ has been used in (45), where S is the Stokes number $\Omega r_0^2/\nu_0$.

5.4. Green's function solutions

The quadrupole component of the radiated pressure outside the boundary layer is represented in what follows by $p'_1 = \mathrm{Re}\left[\hat{p}_1(r)\mathrm{e}^{-2\mathrm{i}\Omega t}\right]$, and the monopole component by $p'_2 = \mathrm{Re}\left[\hat{p}_2(r)\mathrm{e}^{-2\mathrm{i}\Omega t}\right]$. To relate these to $\hat{q}_{\phi\phi}$ and $\hat{q}$ respectively, we use the Neumann acoustic Green's function $g(r\,|\,\xi; n = 0)$ whose value g^+ in $r > \xi$ represents axisymmetric outgoing waves and has gradient $\partial g^+/\partial\xi = 0$ in the limit $\xi \to r_0$. The governing equation is

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + k_a^2\right\} g(r\,|\,\xi) = -\frac{1}{r}\delta(r-\xi), \quad (r,\xi) \geq r_0. \tag{46}$$

The acoustic wavenumber k_a in (46) can be substituted by$2\Omega/c_0 = k$ in the present weakly thermoviscous approximation, provided r is not so large that attenuation due to viscosity and heat conduction becomes significant. The resulting relative error in the Green's function is of order $\varepsilon = \max(\varepsilon_L, \varepsilon_\kappa)$ where $\varepsilon_L = 2\Omega\mu_L/\rho_0 c_0^2$ and $\varepsilon_\kappa = 2\Omega\chi_0/c_0^2$ are the dimensionless frequency parameters defined in Sec. 4. The solution of (46) for $r > \xi$ is then

$$g^+(r\,|\,\xi) = \frac{\pi}{2}\left[J_1(kr_0)Y_0(k\xi) - Y_1(kr_0)J_0(k\xi)\right]\frac{H_0^{(1)}(kr)}{H_1^{(1)}(kr_0)}. \tag{47}$$

With $\xi = (1+y)r_0$, $g^+(r\,|\,\xi)$ may be expanded in powers of $k(\xi - r_0) = 2Ky$ to give

$$g^+(r\,|\,\xi) \simeq \frac{H_0^{(1)}(kr)}{2KH_1^{(1)}(2K)}\left[1 - 2K^2y^2 + \tfrac{2}{3}K^2y^3 - \tfrac{1}{2}K^2y^4 + \cdots\right], \quad K = \Omega r_0/c_0; \tag{48}$$

$$\frac{\partial}{\partial\xi}g^+(r\,|\,\xi) \simeq \frac{2\Omega}{c_0}\frac{H_0^{(1)}(kr)}{H_1^{(1)}(2K)}\left[-y + \tfrac{1}{2}y^2 - \tfrac{1}{2}y^3 + \cdots\right]. \tag{49}$$

For points ξ within the boundary layer, the parameter Ky is of order $\frac{1}{2}k\delta = \left(\Omega\nu_0/c_0^2\right)^{1/2}$, where $\delta = (\nu_0/\Omega)^{1/2}$ is a measure of the viscous boundary-layer thickness. Since $\Omega\nu_0/c_0^2 = K^2/S$ has to be small for the weakly thermoviscous approximation to be valid, convergence of the series in (48) and (49) is assured for all values of ξ within the boundary layer. However, several terms may be required if δ is comparable with r_0 , which would lead to y being of order 1.

For purposes of this discussion we follow Lauvstad [9] and restrict attention from here on to large Stokes numbers ($S \gg 1$), so the boundary layer thickness is much less than the cylinder radius. The pressure field outside the boundary layer due to each source component is given by

$$\hat{p}_1(r) \simeq \int_{r_0}^{\infty} \hat{q}_{\phi\phi}(\xi)\frac{\partial}{\partial\xi}g^+(r\,|\,\xi)\,\mathrm{d}\xi, \quad \hat{p}_2(r) \simeq \int_{r_0}^{\infty} \xi\hat{q}(\xi)g^+(r\,|\,\xi)\,\mathrm{d}\xi, \tag{50}$$

with relative error $O(\varepsilon)$. Substituting $\hat{q}_{\phi\phi}$ and $\hat{q}$ from (45), with the expansions of $g^+(r\,|\,\xi)$ and $\partial g^+/\partial\xi$ truncated after the first term, gives

$$\frac{\hat{p}_1(r)}{\rho_0 U^2} \simeq -J(S)\frac{K}{H_1^{(1)}(2K)}H_0^{(1)}\left(\frac{2\Omega r}{c_0}\right), \tag{51}$$

$$\frac{\hat{p}_2(r)}{\rho_0 U^2} \simeq \tfrac{1}{2}B_0 I(S)\frac{K}{H_1^{(1)}(2K)}H_0^{(1)}\left(\frac{2\Omega r}{c_0}\right). \tag{52}$$

Here B is the non-dimensional fluid property $\alpha c^2/c_p$, equal to $(\gamma-1)$ for an ideal gas. The coefficients $J(S)$, $I(S)$ are the non-dimensional integrals defined below:

$$J(S) = \int_0^{\infty}\left\{\frac{H_1^{(1)}\left[(1+y)(\mathrm{i}S)^{1/2)}\right]}{H_1^{(1)}\left[(\mathrm{i}S)^{1/2)}\right]}\right\}^2 y\,\mathrm{d}y, \quad I(S) = \int_0^{\infty}\left\{\frac{H_2^{(1)}\left[(1+y)(\mathrm{i}S)^{1/2)}\right]}{H_1^{(1)}\left[(\mathrm{i}S)^{1/2)}\right]}\right\}^2 \mathrm{d}y. \tag{53}$$

5.5. Asymptotic approximations for large S

For $S \gg 1$, replacement of the Hankel functions in (53) by their large-argument asymptotic forms gives

$$J(S) \simeq \int_0^{\infty}\left(\frac{y}{1+y}\right)\mathrm{e}^{2\mathrm{i}(\mathrm{i}S)^{1/2}y}\mathrm{d}y, \quad I(S) \simeq \int_0^{\infty}\left(\frac{-1}{1+y}\right)\mathrm{e}^{2\mathrm{i}(\mathrm{i}S)^{1/2}y}\mathrm{d}y. \tag{54}$$

The integrals in (54) may be expanded in inverse powers of $S^{1/2}$ by using the binomial theorem to expand $(1+y)^{-1}$, and integrating by parts to obtain a recurrence relation between successive terms. Thus for integrals of this form with exponent by, provided $\mathrm{Re}(b) < 0$, we have

$$\int_0^\infty \left(\frac{y}{1+y}\right) \mathrm{e}^{by} \mathrm{d}y = \frac{1}{b^2} + \frac{2}{b^3} + \frac{3}{b^4} + \cdots, \quad \int_0^\infty \left(\frac{-1}{1+y}\right) \mathrm{e}^{by} \mathrm{d}y = \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots, \tag{55}$$

which with $b = 2\mathrm{i}(\mathrm{i}S)^{1/2}$ yields

$$J(S) \simeq \frac{1}{4}\left(\frac{\mathrm{i}}{S}\right) - \frac{1}{4}\left(\frac{\mathrm{i}}{S}\right)^{3/2} + \frac{3}{16}\left(\frac{\mathrm{i}}{S}\right)^2 - \cdots, \quad I(S) \simeq -\frac{1}{2}\left(\frac{\mathrm{i}}{S}\right)^{1/2} + \frac{1}{4}\left(\frac{\mathrm{i}}{S}\right) - \frac{1}{8}\left(\frac{\mathrm{i}}{S}\right)^{3/2} + \cdots. \tag{56}$$

5.6. Far-field radiated pressure

In the acoustic far field ($2\Omega r/c_0 \gg 1$), Eqs. (51), (52) and (56) give the quadrupole and monopole complex pressure amplitudes as follows, in the limit $S \gg 1$.

$$\frac{\hat{p}_1(r)}{\rho_0 U^2} \simeq \frac{-1}{4\pi^{1/2}} \left(\frac{r_0}{r}\right)^{1/2} \frac{K^{1/2}/S}{H_1^{(1)}(2K)} \mathrm{e}^{\mathrm{i}(kr+\pi/4)}, \tag{57}$$

$$\frac{\hat{p}_2(r)}{\rho_0 U^2} \simeq \frac{-1}{4\pi^{1/2}} B_0 \left(\frac{r_0}{r}\right)^{1/2} \frac{(K/S)^{1/2}}{H_1^{(1)}(2K)} \mathrm{e}^{\mathrm{i}kr}. \tag{58}$$

Comparison of (57) and (58) shows that $|\hat{p}_2(r)|$ is larger than $|\hat{p}_1(r)|$ by a factor $B_0 S^{1/2}$. The reason for the difference is that radiation from the $\phi\phi$ volume quadrupole distribution is partially suppressed by the Neumann boundary condition at $r = r_0$, as follows from Eqs. (49) and (50) above. Had we used a free-field rather than a Neumann Green's function to calculate the quadrupole sound, the same end result would have been obtained, but the source terms would have included a normal-dipole term related to the pressure on the boundary at $r = r_0$ (see Sec. 5.8 and Sec. 4). However the normal-dipole term and the volume quadrupole term cancel to lowest order in $S^{-1/2}$ (the ratio of viscous penetration depth to cylinder radius). Thus the free-field Green's function can be misleading when approximations are used, as was found earlier by Lauvstad and Meecham [5].

For the acoustically compact special case $K \ll 1$, where the cylinder radius is much smaller than the sound wavelength λ, Eqs. (57) and (58) reduce to

$$\left\{\frac{\hat{p}_1(r)}{\rho_0 U^2}, \frac{\hat{p}_2(r)}{\rho_0 U^2}\right\} \simeq \frac{1}{4}\pi^{1/2}\left(\frac{r_0}{r}\right)^{1/2} \mathrm{e}^{\mathrm{i}kr}\left\{(K^{3/2}/S)\mathrm{e}^{-\mathrm{i}\pi/4}, B_0 K^{1/2}\varepsilon_\mu^{1/2}\mathrm{e}^{-\mathrm{i}\pi/2}\right\}, \quad (59)$$

where the parameter $\varepsilon_\mu = \Omega\nu_0/c_0^2 = (\pi\delta/\lambda)^2 = K^2/S$ has to be small for the present acoustic analogy to be valid.

5.7. *Radiated sound power and acoustic conversion efficiency*

A useful parameter for characterizing the far-field radiation from aeroacoustic sources is the ratio of the mean sound power output to the mean power dissipated by viscous stresses. For oscillatory rotation of an infinite cylinder, the power dissipated per unit surface area can be obtained from Eq. (42) by defining the tangential impedance z_t as

$$z_t = (-\hat{\tau}_{r\phi}/\hat{u}_\phi)_{r=r_0}, \quad \hat{\tau}_{r\phi} = \mu\left(\frac{\partial \hat{u}_\phi}{\partial r} - \frac{\hat{u}_\phi}{r}\right). \quad (60)$$

The mean dissipated power per unit cylinder surface area is $\frac{1}{2}U^2\,\mathrm{Re}(z_t)$. Expressed per unit length, the power dissipated follows from (60) and (42) as

$$W_{\mathrm{diss}} = \pi r_0 U^2\,\mathrm{Re}(z_t) = \pi\mu U^2\,\mathrm{Re}\left[k_w r_0\, H_2^{(1)}(k_w r_0)/H_1^{(1)}(k_w r_0)\right], \quad (61)$$

where $k_w = (\mathrm{i}\Omega/\nu_0)^{1/2}$. In the limit $S^{1/2} \gg 1$, (61) becomes

$$W_{\mathrm{diss}} \simeq \frac{\pi}{2}\mu U^2 (2S)^{1/2}. \quad (62)$$

The radiated sound power per unit length of cylinder is

$$W_{\text{rad}} \simeq \pi r \left|\hat{p}(r)\right|^2 / \rho_0 c_0 \tag{63}$$

where $\hat{p}(r)$ is the far-field pressure, given when $S^{1/2} \gg 1$ by Eq. (57) for the quadrupole mechanism and by Eq. (58) for the monopole mechanism. Acoustic attenuation due to viscosity and heat conduction is here neglected for purposes of estimating the sound power output, as was done for estimating the Green's function.

The acoustic conversion efficiencies associated with the two mechanisms follow from (62) and (63). Assuming $K^2 \ll 1$ as well as $S^{1/2} \gg 1$ gives

$$\left(\frac{W_{\text{rad}}}{W_{\text{diss}}}\right)_1 = \eta_{\text{ac},1} \simeq \frac{\pi\sqrt{2}}{8} M^2 K^2 S^{-3/2}, \quad \left(\frac{W_{\text{rad}}}{W_{\text{diss}}}\right)_2 = \eta_{\text{ac},2} \simeq \frac{\pi\sqrt{2}}{8} B_0^2 M^2 K^2 S^{-1/2}. \tag{64}$$

In the opposite case ($K^2 = \varepsilon_\mu S \gg 1$) where the cylinder radius r_0 is large compared with the acoustic wavelength, Eqs. (57) and (58 shows that $\eta_{\text{ac},2}$ becomes independent of r_0, while $\eta_{\text{ac},1}$ tends to zero in the flat-plate limit:

$$\eta_{\text{ac},1} \simeq \frac{\sqrt{2}}{16} M^2 \varepsilon_\mu^{1/2} S^{-1}, \quad \eta_{\text{ac},2} \simeq \frac{\sqrt{2}}{16} B_0^2 M^2 \varepsilon_\mu^{1/2}. \tag{65}$$

It is interesting to rewrite the conversion efficiencies in Eq. (64) in terms of the nonlinearity parameter $N = \xi_{\text{max}}/\delta = M/\varepsilon_\mu^{1/2}$ where ξ_{max} is the maximum displacement U/Ω of the cylinder surface. The N value determines the stability of the viscous boundary layer in the flat-plate limit $S^{1/2} \gg 1$ (see Appendix A). Ep. (64) becomes

$$\eta_{\text{ac},1} \simeq \frac{\sqrt{2}}{8} \pi N^2 \varepsilon_\mu^2 S^{-1/2}, \quad \eta_{\text{ac},2} \simeq \frac{\sqrt{2}}{8} \pi B_0^2 N^2 \varepsilon_\mu^{3/2} K \quad (K^2 = \varepsilon_\mu S \ll 1, S^{1/2} \gg 1). \tag{66}$$

Once N reaches around 350, the viscous boundary layer becomes turbulent and the present description of the sound radiation breaks down.

5.8. *Comparison with previous results*

In Lauvstad's matched asymptotic expansion calculation [9], reviewed in Sec. 3, it is assumed that $K^2 \ll 1$ as well as ($S^{1/2} \gg 1$, $M^2 \ll 1$). With the correction factor $\frac{1}{2}$ noted in Sec. 3, Lauvstad's far-field pressure agrees with our quadrupole result in Eq. (59) apart from the sign.

In a later paper, Meecham [19] adopted the simple-source approach of Ribner [20, 21, 22] and treated the cylinder as a line monopole source in a general fluid, again assuming $K^2 \ll 1$ as well as ($S^{1/2} \gg 1$, $M^2 \ll 1$). By combining this source model with a free-field Green's function, and doubling the source strength to allow for reflection by the cylinder, a far-field pressure result was found that agrees with the quadrupole component in Eq. (59). However in the limit $K^2 \ll 1$, the Neumann and free-field Green's functions for radiation from sources on the boundary are the same. The source strength should therefore not have been doubled, and the agreement in [19] appears to result from two errors cancelling.

As mentioned in Sec. 5.6, Lauvstad and Meecham [5] failed to replicate the result of Ref. [9] by using Curle's acoustic analogy formulation [2]. If we had followed Ref. [2] and used a free-field Green's function when applying our thermoviscous acoustic analogy to the rotating cylinder problem, the $\hat{p}_1$ expression in Eq. (50) above would have been replaced by

$$\hat{p}_1(r) \simeq \int_{r_0}^{\infty} \hat{q}_{\phi\phi}(\xi)\frac{\partial}{\partial\xi}g_\infty^+(r\,|\,\xi)\,\mathrm{d}\xi + r_0\hat{p}_{\mathrm{mod}}(r_0)\left.\frac{\partial g_\infty^+}{\partial\xi}\right|_{\xi=r_0} = \hat{p}_{1,V} + \hat{p}_{1,S}. \quad (67)$$

Here the free-field acoustic Green's function $g_\infty^+(r\,|\,\xi)$ replaces the Neumann Green's function of Eq. (50), and a surface term $\hat{p}_{1,S}$ appears due to the $p'_{\mathrm{mod}}\hat{\mathbf{n}}$ normal-dipole distribution on the cylinder boundary, as shown in Eq. (35).

Evaluating the surface term $\hat{p}_{1,S}$ can be done in two ways: either one uses an incompressible estimate of the surface pressure to write $\hat{p}_{\mathrm{mod}}(r_0) \simeq \hat{p}_{\mathrm{inc}}(r_0)$ as in [5], or one solves directly for $\hat{p}_{\rm mod}(r_0)$ by means of a Neumann Green's function. In the limit $K \ll 1$ these give the same result, because the surface pressure in this case is dominated by the quadrupole component $\hat{p}_1$. Assuming the boundary is adiabatic,

$$\frac{\hat{p}_{\rm mod}(r_0)}{\rho_0 U^2} \simeq -\frac{1}{4}\left(\frac{\rm i}{S}\right)^{1/2} \qquad (K \ll 1, S \gg 1). \tag{68}$$

If K is not small Eq. (68) is not valid and the dissipation-generated pressure has to be taken into account, as Eq. (82) below indicates. For $K \ll 1$, however, it is interesting to examine the sound field prediction obtained from (67) and (68) since it casts light on the failure of Ref. [5] to obtain a result from Curle's formulation [2].

The Green's function derivative $(\partial/\partial\xi)g_\infty^+(r\,|\,\xi)$ for $r > \xi$ appearing in the integral of Eq. (67) is

$$\frac{\partial g_\infty^+}{\partial \xi} = -{\rm i}\frac{\pi}{2}kJ_1(k\xi)\,H_0^{(1)}(kr). \tag{69}$$

Comparison of $\partial g_\infty^+/\partial\xi$ with $\partial g^+/\partial\xi$ from Eq. (49) gives their ratio as $1/(2y)$ where $y = (\xi/r_0 - 1) \simeq S^{-1/2}$, to lowest order in y. Thus provided $S^{1/2} \gg 1$, the integral in (67) resembles the $\hat{p}_1(r)$ integral in (50), but with $J(S)$ replaced by $-\frac{1}{2}I(S)$. It follows from Eqs. (54) and (56) that $\hat{p}_{1,V}(r) \simeq ({\rm i}/S)^{-1/2}\hat{p}_1(r)$ to lowest order in $S^{-1/2}$, so from Eq. (59)

$$\frac{\hat{p}_{1,V}(r)}{\rho_0 U^2} \simeq \frac{1}{4}\pi^{1/2}\left(\frac{r_0}{r}\right)^{1/2}{\rm e}^{{\rm i}(kr-\pi/2)}K^{3/2}S^{-1/2}. \tag{70}$$

The surface term $\hat{p}_{1,S}$ in (67) is given by putting $\xi = r_0$ in Eq. (67) and using (68):

$$\frac{\hat{p}_{1,S}(r)}{\rho_0 U^2} \simeq -\frac{1}{4}\pi^{1/2}\left(\frac{r_0}{r}\right)^{1/2}{\rm e}^{{\rm i}(kr-\pi/2)}K^{3/2}S^{-1/2}. \tag{71}$$

Equations (70) and (71) show that in the present approximation, the volume and surface sources $\hat{p}_{1,V}$ and $\hat{p}_{1,S}$ do indeed cancel.

6. Effect of a non-adiabatic boundary condition at the cylinder wall

On removal of the restriction to an ideal gas with $\gamma = 1$ implied in Lauvstad's analysis [9], not only does dissipative heating take over as the dominant mechanism of sound generation, but the far-field pressure amplitude now depends on the thermal boundary condition imposed at the cylinder wall $r = r_0$. To demonstrate the influence of a non-adiabatic boundary, we recalculate the sound field using the isothermal boundary condition $T' = 0$ in place of the adiabatic condition $\mathbf{q} \cdot \hat{\mathbf{n}} = 0$ assumed in Sec. 5. This leads to an additional boundary source term involving the normal heat flux $\mathbf{q} \cdot \hat{\mathbf{n}}$, as indicated in Eq. (35).

6.1. Nonlinear equation for the entropy mode

In order to apply a boundary condition on T' at the cylinder wall, we require a solution for $\hat{s}(r_0)$ as well as $\hat{p}_{\text{mod}}(r_0)$. The linearized equation that describes the entropy mode to lowest order in ε in a general thermoviscous fluid is [11, 10]

$$\frac{\partial s'}{\partial t} - \chi_0 \nabla^2 s' = 0, \tag{72}$$

but it ceases to be accurate when perturbations from the uniform reference state are no longer small. In this section we outline the development of a nonlinear version of (72) that describes the generation of the entropy mode in bounded regions with relative error $\Delta = O(\varepsilon)$, analogous to the weakly thermoviscous acoustic mode equation in Sec. 4. As in that case, the resulting equation contains volume source terms arising from second-order interactions between first-order perturbations.

The exact entropy equation for a compressible thermoviscous fluid is given in [14], Eq. (49.4) as

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \operatorname{div} \mathbf{q}, \tag{73}$$

where τ_{ij} are the components of the viscous stress tensor $\boldsymbol{\tau}$, and $\mathbf{q}$ is the heat flux vector. The first term on the right is the rate of viscous energy dissipation, denoted by $\dot{D}$. We also define the instantaneous rate of energy dissipation per unit volume associated with heat flux down a temperature gradient,

$$\dot{D}_T = -(\mathbf{q} \cdot \nabla T)/T = \kappa |\nabla T|^2 / T, \tag{74}$$

where the second expression applies to a fluid with constant thermal conductivity κ. If we also assume that the shear viscosity μ and the bulk viscosity μ_B are constant,

$$\dot{D} = \dot{D}_{\text{shear}} + \dot{D}_{\text{vol}}, \quad \text{with} \quad \dot{D}_{\text{shear}} = \frac{1}{2}\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\Theta \delta_{ij} \right)^2 \quad \text{and} \quad \dot{D}_{\text{vol}} = \mu_B \Theta^2. \tag{75}$$

Ep. (73) can then be written

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \kappa T \frac{\partial}{\partial x_i} \left(\frac{1}{T} \frac{\partial T}{\partial x_i} \right) + \dot{D}_T + \dot{D}. \tag{76}$$

Note that both $\dot{D}_T$ and $\dot{D}$ are positive definite, as shown by (74) and (75).

Ep. (76) can be rearranged without approximation to give a forced entropy mode equation for the windowed variable $s' H(f)$, with the same operator on the left as Eq. (72) above:

$$\left\{ \frac{\partial}{\partial t} - \chi_0 \nabla^2 \right\} (s' H) = \Gamma_{\text{vol}} + \Gamma_{\text{surf}}. \tag{77}$$

To solve for the entropy field in the boundary layer, we note that in the linear approximation only the vorticity mode is present, and the dominant vorticity-vorticity volume source term is due to dissipative heating. Thus in Eq. (77) we have

$$\Gamma_{\text{vol}} \simeq \frac{1}{\rho_0 T_0} \dot{D} H, \tag{78}$$

provided the convective term in $\mathrm{D}s/\mathrm{D}t$ (namely $u_i\partial s/\partial x_i = u_r\partial s'/\partial r$) can be shown to contribute a vanishingly small fraction of the total material derivative[1]. The weakly thermoviscous asymptotic description on which Eq. (78) is based requires $\varepsilon = \max(\varepsilon_L, \varepsilon_\kappa) \ll 1$, as in Eq. (31).

As in Sec. 4 the boundary $\overline{\mathcal{S}}$ is chosen to coincide with the cylinder surface, so may be taken as impermeable. Then Γ_{surf} is given by

$$\Gamma_{\text{surf}} \simeq \frac{1}{\rho T}\mathbf{q}\cdot\hat{\mathbf{n}}\delta(f) - \left(\frac{\alpha\chi}{\rho}\right)_0 \left[\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\cdot\hat{\mathbf{n}}\delta(f) + \operatorname{div}\left[\mu(\boldsymbol{\omega}\times\hat{\mathbf{n}})\delta(f)\right]\right] - \chi_0 \operatorname{div}\left[s'\hat{\mathbf{n}}\delta(f)\right]. \tag{79}$$

The factor $(\mathrm{D}\mathbf{u}/\mathrm{D}t)\cdot\hat{\mathbf{n}}$ in Eq. (79) may be replaced by $-u_\phi^2/r$, since the normal velocity on the boundary is zero. The tangential-dipole term $\operatorname{div}\left[\mu(\boldsymbol{\omega}\times\hat{\mathbf{n}})\delta(f)\right]$, where $\boldsymbol{\omega}$ is the vorticity, is zero by symmetry; and the normal-dipole term in s' can be eliminated by using the Neumann Green's function to solve for the entropy field. This leaves the $\mathbf{q}\cdot\hat{\mathbf{n}}$ term as an unknown, to be determined by setting a thermal boundary condition at $r = r_0$.

The solution procedure for Eq. (77) is similar to that used for the acoustic mode in Sec. 5, with the main difference being that we wish to solve for the fluctuating entropy at the cylinder boundary, $s'(r_0, t)$. We shall also need the acoustic-mode solution for $p'_{\text{mod}}(r_0, t)$. A thermal boundary condition can then be imposed and the unknown heat flux determined.

6.2. *Results for an isothermal cylinder boundary*

To describe the boundary heat flux and associated second-order temperature fluctuations associated with a non-adiabatic cylinder boundary, we write

$$T'(r,t) = \operatorname{Re}\left[\hat{T}(r)\mathrm{e}^{-2\mathrm{i}\Omega t}\right], \quad (\mathbf{q}\cdot\hat{\mathbf{n}})(r_0,t) = \operatorname{Re}\left[\hat{h}\mathrm{e}^{-2\mathrm{i}\Omega t}\right]. \tag{80}$$

[1] This is shown to be the case for the generalized oscillating-cylinder problem in Sec. 6.4 below.

The complex amplitude $\hat{T}(r)$ is a combination of acoustic-mode and entropy-mode components, given with relative error $\Delta = O(\varepsilon)$ by

$$\frac{\hat{T}(r)}{T_0} \simeq \frac{B_0}{\rho_0 c_0^2}\hat{p}_{\text{mod}}(r) + \frac{1}{c_{p0}}\hat{s}(r). \tag{81}$$

Solving the acoustic-mode equation for $\hat{p}_{\text{mod}}(r_0)$ and the entropy-mode equation for $\hat{s}(r_0)$ gives, assuming $S^{1/2} \gg 1$ and $P^{1/2} \gg 1$,

$$\frac{\hat{p}_{\text{mod}}(r_0)}{\rho_0 U^2} \simeq -\frac{1}{4}\left(\frac{\mathrm{i}}{S}\right)^{1/2} - B_0\left[\frac{1}{4}\left(\frac{\mathrm{i}}{S}\right)^{1/2} K + \frac{\mathrm{i}\hat{h}}{\rho_0 c_0 U^2}\right]\frac{H_0^{(1)}(2K)}{H_1^{(1)}(2K)}, \tag{82}$$

and

$$\frac{\hat{s}(r_0)}{c_{p0}} \simeq \frac{1}{\kappa T_0}\left[\frac{1}{4}\frac{X}{S}\Omega\rho_0 U^2 r_0^2 + \left(\frac{\mathrm{i}}{2P}\right)^{1/2} r_0\hat{h}\right]. \tag{83}$$

Here $P = (Pr)S$ and $X = \left[Pr + (2\,Pr)^{1/2}\right]^{-1}$, where $Pr = c_{p0}\,\mu/\kappa$ is the fluid Prandtl number.

Imposing an isothermal boundary condition $\hat{T}(r_0) = 0$ allows the boundary heat flux $\hat{h}$ to be determined from Eqs. (81)–(83). It turns out that regardless of K, the $\hat{p}_{\text{mod}}(r_0)$ contribution to the boundary temperature may be neglected for purposes of estimating $\hat{h}$, provided $(P^{1/2}, S^{1/2}) \gg 1$. Specifically, if we define $A = \alpha T = (\gamma - 1)/B$ the relative error in $\hat{h}$ is of order $A_0(P^{-1/2} + S^{-1/2})$ for $K \ll 1$ and $(\gamma_0 - 1)(\varepsilon_\mu^{1/2} + \varepsilon_\kappa^{1/2})$ for $K \gg 1$. Then

$$\frac{\hat{h}}{\rho_0 c_0 U^2} \simeq -\tfrac{1}{4}\mathrm{e}^{-\mathrm{i}\pi/4}\overline{X}\varepsilon_\mu^{1/2} \quad \text{(isothermal boundary)} \tag{84}$$

where $\overline{X} = (2\,Pr)^{1/2}X$.

The acoustic consequence of (84) is that the sound pressure radiated outside the boundary layer acquires an extra term

$$\hat{p}_3(r) = -2\mathrm{i}\Omega\rho_0 a_0 r_0 \hat{h} g^+(r\,|\,r_0), \tag{85}$$

where g^+ is the outgoing-wave acoustic Green's function introduced in Eq. (47). Thus for $K \ll 1$, the adiabatic-boundary results in Eq. (59) for the far-field pressure are supplemented by a heat-flux term

$$\frac{\hat{p}_3(r)}{\rho_0 U^2} \simeq \frac{1}{4}\pi^{1/2}\left(\frac{r_0}{r}\right)^{1/2} \mathrm{e}^{\mathrm{i}(kr+\pi/2)} B_0 K^{1/2} \overline{X} \varepsilon_\mu^{1/2}. \tag{86}$$

Comparison of $\hat{p}_3(r)$ from (86) with $\hat{p}_2(r)$ from Eq. (59) shows the two pressure components to be of opposite sign, with $|\hat{p}_3|/|\hat{p}_2| = \overline{X} = \left(1 + \sqrt{Pr/2}\right)^{-1}$. As $Pr \to 0$, $\overline{X} \to 1$ and these two contributions cancel, leaving the much smaller quadrupole component $\hat{p}_1(r)$.

6.3. Generalized thermal boundary condition

Our two assumptions of an adiabatic or isothermal rigid boundary at $r = r_0$ may be viewed as limiting cases of a solid cylinder that is heat-conducting, but does not expand on heating (to ensure $u_r = 0$ at the boundary). If the thermal boundary condition for this generalized case is written as $\hat{h} = -\beta \hat{T}(r_0)$, and we assume the thermal penetration depth in the solid to be small compared with the cylinder radius, it follows that

$$\beta \simeq \kappa_s \left(\frac{2\Omega}{\chi_s}\right) \mathrm{e}^{-\mathrm{i}\pi/4}, \tag{87}$$

where κ_s and χ_s are respectively the thermal conductivity and thermal diffusivity of the solid material. Solving for $\hat{h}$ then gives

$$\hat{h} = Z\hat{h}_{\text{isothermal}}, \quad \text{with} \quad Z = \left[1 + \frac{\kappa}{\kappa_s}\left(\frac{\chi_s}{\chi_0}\right)^{1/2}\right]^{-1}. \tag{88}$$

The sum $\hat{p}_2 + \hat{p}_3$ is now $(1 - Z\overline{X})$ times $\hat{p}_2$. The isothermal result $Z = 1$ is recovered in the limit $(\kappa\rho c_p)_s/(\kappa\rho c_p)_0 \to \infty$.

6.4. *Source terms omitted from the nonlinear entropy-mode equation*

Results have been derived in this section based on a restricted set of entropy-mode source terms, namely

$$\Gamma_{\text{vol}} \simeq \frac{1}{\rho_0 T_0}\dot{D}\,H \quad \text{and} \quad \Gamma_{\text{surf}} \simeq \frac{1}{\rho T}\mathbf{q}\cdot\hat{\mathbf{n}}\delta(f). \tag{89}$$

Although other terms are present in general, in the present context these can mostly be seen to either vanish or be small. The two terms whose omission is not obviously justified are the surface term in $(\mathrm{D}\mathbf{u}/\mathrm{D}t)\cdot\hat{\mathbf{n}}\delta(f)$ that appears in Eq. (79), and the volume term in $u_r\partial s'/\partial r$ that arises from the difference between $\mathrm{D}s'/\mathrm{D}t$ and $\partial s'/\partial t$.

The first of these can be assessed by noting that the radial component of $\mathrm{D}\mathbf{u}/\mathrm{D}t$ at the cylinder boundary is $-u_\phi^2/r_0$, since $u_r(r_0,t) = 0$. The ratio of the $\mathrm{D}\mathbf{u}/\mathrm{D}t$ source term to the $\mathbf{q}\cdot\hat{\mathbf{n}}$ source term in (89) then follows from (84) as

$$|\text{ratio of terms}| \sim A_0\frac{\overline{X}}{Pr}S^{-1/2}. \tag{90}$$

Since we are assuming $S^{1/2} \gg 1$, dropping the $\mathrm{D}\mathbf{u}/\mathrm{D}t$ surface term is justified.

The discarded volume source term $u_r\partial s'/\partial r$ is potentially significant on account of the large factor $\varepsilon_\kappa^{-1/2}$ introduced by taking the gradient of the fluctuating entropy. To assess this term, one should in principle calculate the amplitude of $u_r\partial s'/\partial r$ within the boundary layer. This involves solving the nonlinear entropy mode equation to find $\hat{s}(r)$, and likewise the acoustic mode equation to find $\hat{p}(r) \simeq \hat{p}_{\text{mod}}(r)$. Using the notation $u_r(r,t) = \mathrm{Re}\left[\hat{u}_r(r)\mathrm{e}^{-2\mathrm{i}\Omega t}\right]$, the continuity equation then gives

$$\hat{u}_r(r) = \frac{1}{r}\int_{r_0}^{r}\xi\hat{\Theta}(\xi)\,\mathrm{d}\xi, \quad \text{where} \quad \hat{\Theta} = 2\mathrm{i}\Omega\left(\frac{1}{\rho_0 c_0^2}\hat{p} - \frac{A_0}{c_{p0}}\hat{s}\right). \tag{91}$$

The amplitude of $u_r \partial s'/\partial r$ follows as $\frac{1}{2}\,|\hat{u}_r \partial \hat{s}/\partial r|$. To justify neglecting this term, its maximum value through the boundary layer, $|u_r \partial s'/\partial r|_{\max}$, needs to be small compared with the maximum of the dissipation source term $\dot{D}/\rho_0 T_0$.

Alternatively one can avoid the detailed calculation above by using scaling arguments, based on the solutions already presented for $\hat{s}(r)$ and $\hat{p}(r)$. These give the amplitude ratio of the two entropy-mode source terms as

$$|u_r \partial s'/\partial r|_{\max} / \frac{1}{\rho_0 T_0} |\dot{D}|_{\max} \sim B_0 F(Pr) M^2, \tag{92}$$

where $F(Pr)$ is a function of the Prandtl number $Pr = c_{p0}\mu/\kappa$. Provided $F(Pr)$ is of order 1, dropping the $u_r \partial s'/\partial r$ term is justified as long as $M^2 \ll 1$.

7. Conclusions

The following conclusions can be drawn regarding the aeroacoustic sound output of a rotationally oscillating infinite circular cylinder in the stable oscillatory flow regime with $S^{1/2} \gg 1$, for a cylinder whose boundary is adiabatic:

- Sound is generated in the viscous boundary layer by two different mechanisms. In the present acoustic analogy formulation these appear as a ρu_ϕ^2 quadrupole distribution, and a monopole distribution due to viscous dissipation. The latter mechanism depends on the thermal expansivity of the fluid.

- The quadrupole term $q_{\phi\phi}$ is equivalent, in the present approximation with $M^2 \ll 1$ and $\varepsilon \ll 1$, to a centrifugal body force field $g_r = u_\phi^2/r$; it is a nonlinear (second order) source, quadratic in the vorticity-mode velocity field. Likewise the monopole dissipation term is also a second-order source, equivalent to a heat input distribution $\dot{D}$ per unit volume.

- For the special case of a fluid with $B_0 = 0$, where B is the dimensionless quantity $\alpha c^2/c_p$ based on the thermal expansivity α, sound speed c, and constant-pressure specific heat c_p, only the first mechanism operates. This was effectively the case considered by Lauvstad [9], who modelled the fluid as an ideal gas with density changes occurring isothermally, corresponding to a specific heat ratio $\gamma = 1$.

- For a general fluid, the sound power output due to the viscous-dissipation mechanism is greater by a factor $B_0^2 S$ than that due to the quadrupole mechanism.

- The acoustic analogy calculations of Lauvstad and Meecham [5] and Meecham [19] were not restricted to fluids with $B_0 = 0$. However only the quadrupole mechanism was considered.

- The attempt by Lauvstad and Meecham [5] to calculate the radiated quadrupole field at large Stokes numbers S using surface and volume terms based on Curle's free-field formulation [2] was unsuccessful, because to lowest order in $S^{-1/2}$ the two contributions cancel as demonstrated in Sec. 5.8. It appears that for the present problem the formulation in [5] is ill-conditioned, whereas the Neumann formulation used in Sec. 5 is much less sensitive to large-S approximations.

- It is interesting to note that Doak [23], writing at the same time as Powell [3], was already making the point that one can choose any of a wide range of Green's functions to represent sound radiation in the presence of boundaries, and not just the free-field Green's function implicit in Curle's formulation of Lighthill's theory [2]. Thus the split between surface and volume contributions to the radiated sound is to a large extent arbitrary.

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Appendix A. Stability of the oscillatory boundary layer

The transition to turbulence in an oscillatory boundary layer has been studied experimentally by Akhavan et al. [24] for the special case of incompressible rectilinear flow, corresponding to $S \to \infty$, $M \to 0$ in the present problem where S is the Stokes number $\Omega r_0^2/\nu_0$ and ν_0 is the kinematic viscosity. They showed that in pulsatile fully-developed pipe flow, with a sinusoidally-oscillating volume flowrate proportional to $\sin\Omega t$ and zero mean flow, bursts of turbulence appear in the boundary layer when the amplitude U of the cross-sectional mean velocity reaches $U \approx 350(\Omega\nu_0)^{1/2}$. Their experiments used two pipe diameters, 16 and 30 times the viscous length scale $(\nu_0/\Omega)^{1/2} = \delta$, with no significant difference in the onset of turbulent flow. The criterion $U_{\text{crit}} \approx 350(\Omega\nu_0)^{1/2}$ for laminar flow breakdown corresponds to a value of 350 for the parameter $N = \xi_{\max}/\delta$, where $\xi_{\max}$ is the relative displacement amplitude between the wall and the fluid outside the boundary layer.

In a second paper, Akhavan et al. [25] carried out numerical simulations of oscillatory two-dimensional flow in a plane channel of width 28δ. The results were consistent with the findings from the pipe-flow experiments, and showed that the onset of turbulence was due to the nonlinear growth of three-dimensional disturbances, rather than to linear instability. We conclude that for the present cylinder problem with $S \gg 1$, the oscillatory boundary layer remains stable as long as N is less than around 300.

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