A persistency of excitation condition for continuous-time systems

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***Abstract*—We study identifiability for linear, continuous time-invariant systems. We state sufficient conditions on an input trajectory $\widehat{\mathrm{u}}$ and a finite number of its derivatives, in order to be able to deduce all differential equations describing the data-generating system from any corresponding input-output sequence $(\widehat{\mathrm{u}}, \widehat{\mathrm{y}})$ on a finite interval.**

***Index Terms*—Persistency of excitation, identifiability, continuous-time linear time-invariant systems**

I. Introduction

We consider solutions of systems of constant-coefficient linear differential equations, called *linear differential systems* in the following. We illustrate some results related to persistency of excitation, parallel to those established for discrete-time systems in [15] and [14]. We answer the following question:

> *Let $(\widehat{u}, \widehat{y}) : \mathbb{R} \to \mathbb{R}^{m+p}$ be an inputs-output (i-o) trajectory generated by a linear differential system.*
> *Under which conditions on $(\widehat{u}, \widehat{y})$ is it possible to recover from it all differential equations that describe the system?*

Our solution to this problem is based on state-space representations, and we closely follow the technique used in [14] for the discrete-time case. In this adaptation we exploit two analogies: that between differentiation in the continuous-time domain, and shift in the discrete-time one; and that between vector of functions and their annihilators, and Hankel matrices and their left annihilators. In [15] the same problem was solved in discrete-time using only higher-order polynomial difference operators (i.e. adopting the tenets of the "behavioral approach").

Notwithstanding the technical analogies, a fundamental conceptual difference exists between the results presented here and those of [14]. The authors of [14] aimed at stating sufficient conditions for the parametrization of the restrictions of all *trajectories* generated by a linear, time-invariant discrete-time system. They used the fact that linear combinations of the time-shifts of restrictions of a given system trajectory provide a convenient representation to generate (restrictions of) other system trajectories. This has important consequences in simulation and data-driven control (see e.g. [1], [7], [9] and the more recent literature on the subject).

A continuous-time analogue to the results of [14], [15] appears in [5], where a parametrization of restrictions of admissible continuous-time trajectories is obtained solving a system of linear, time-varying differential equations determined from the data $(\widehat{u}, \widehat{y})$ (see Theorem 2 therein).

The problem we solve in this paper is *dual* to that analysed in [5], [14]: we aim at finding sufficient conditions for the parametrization of all *dynamical laws* (differential equations) that describe the system, rather than at generating all system trajectories. Instead of a *descriptive* point of view (parametrizing finite-length system trajectories in terms of a given one), we adopt a *prescriptive* one (parametrizing all annihilators of the system trajectories). We solve an *identifiability* problem: how "rich" must a given system trajectory be in order for a unique model of the system to be computed from the data. Our point of view is analogous to that adopted in [6], and to some of the results in [15]. We discuss these relations more extensively in Section IV.

The paper is organized as follows. In Section II we define the property of persistency of excitation, and we state some preliminary results. In Section III we state a sufficient condition for identifiability, based on the concept of persistency of excitation and controllability. In Section IV we discuss the relation of our conditions with those for discrete-time systems. In Section V we illustrate an application of our results to continuous-time system identification. Section VI contains some final remarks.

II. Preliminary results

In the rest of this paper, given a j-times continuously differentiable function $f : \mathbb{R} \to \mathbb{R}^q$ and $j \in \mathbb{N}$, the notation $f^{(j)}$ denotes the j-th derivative of f.

The following is a fundamental definition.

Definition 1 (Persistency of excitation): Let $\mathbb{I} = (t_0, t_1) \subseteq \mathbb{R}$. $f : \mathbb{I} \to \mathbb{R}^m$ is *persistently exciting of order* k if

a) f is $(k-1)$-times continuously differentiable in $\mathbb{I}$;
b) For every $v := \begin{bmatrix} v_0 & \dots & v_{k-1} \end{bmatrix} \in \mathbb{R}^{1 \times km}$ it holds that

$$v \begin{bmatrix} f(t) \\ f^{(1)}(t) \\ \vdots \\ f^{(k-1)}(t) \end{bmatrix} = 0 \ \forall\ t \in \mathbb{I} \implies v_i = 0_{1 \times m} \ , \ i = 0, \dots, k-1 \ . \quad (1)$$

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It is straightforward to verify that if f is persistently exciting of order k, then it does not satisfy any constant-coefficient differential equation of order less than or equal to $k-1$. For example, if $m=1$ a linear combination of k linearly independent exponential functions is persistently exciting of order k on every interval $\mathbb{I} \subseteq \mathbb{R}$.

We establish a consequence of this property.

Proposition 1: Let (x, u) be a state-input trajectory generated by a controllable continuous-time input-state system

$$\frac{d}{dt} x = Ax + Bu \ , \tag{2}$$

where u is m-dimensional and x is n-dimensional.

Let $\mathbb{I} = (t_0, t_1) \subseteq \mathbb{R}$; assume that u is persistently exciting of order at least $n+L$ in $\mathbb{I}$. Then for every $\zeta \in \mathbb{R}^{1\times n}$, $\eta := \begin{bmatrix} \eta_0 & \dots & \eta_{L-1} \end{bmatrix} \in \mathbb{R}^{1\times mL}$ the following implication holds:

$$\begin{bmatrix} \zeta & \eta \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \\ u^{(1)}(t) \\ \vdots \\ u^{(L-1)}(t) \end{bmatrix} = 0 \ \text{ for all } t \in \mathbb{I} \tag{3}$$
$$\Longrightarrow \zeta = 0 \ , \ \eta_i = 0 \ , \ i = 0, \dots, L-1 \ .$$

Proof: Let $\zeta \in \mathbb{R}^{1\times n}$, $\eta \in \mathbb{R}^{1\times (n+L)m}$, $t \in \mathbb{I}$ be such that (3) holds. Differentiating this expression i times, $i = 0, \dots, n$ one obtains

$$\begin{bmatrix} \zeta & \eta \end{bmatrix} \begin{bmatrix} x^{(i)}(t) \\ u^{(i)}(t) \ u^{(i+1)}(t) \\ \vdots \\ u^{(L-1+i)}(t) \end{bmatrix} = 0 \ , \ i = 0, \dots, n \ .$$

Differentiating i times the state equation $\frac{d}{dt} x = Ax + Bu$, one obtains the expression $x^{(i)} = A^i x + \sum_{j=0}^{i-1} A^{i-1-j} B u^{(j)}$ for $i = 1, \dots$. It follows that for every $t \in \mathbb{I}$ the following equation holds:

$$\begin{bmatrix} \zeta & \eta \end{bmatrix} \begin{bmatrix} x^{(i)}(t) \\ u^{(i)}(t) \ u^{(i+1)}(t) \\ \vdots \\ u^{(L-1+i)} \end{bmatrix}$$
$$= \begin{bmatrix} \zeta & \eta \end{bmatrix} \begin{bmatrix} A^i x(t) + \sum_{j=0}^{i-1} A^{i-1-j} B u^{(j)}(t) \\ u^{(i)}(t) \\ u^{(i+1)}(t) \\ \vdots \\ u^{(L-1+i)}(t) \end{bmatrix} = 0 \ .$$

Consequently for $i = 1, \dots, n$ it holds that

$$0 = \begin{bmatrix} \zeta A^i & \dots & \zeta B & \eta_0 & \dots & \eta_{L-1} \end{bmatrix} \begin{bmatrix} x(t) \ u(t) \\ \vdots \\ u^{(i)}(t) \\ \vdots \\ u^{(L-1+i)}(t) \end{bmatrix} \ .$$

Define

$$\begin{aligned} w_0 &:= \begin{bmatrix} \zeta & \eta_0 & \dots & \eta_{L-1} & 0_{nm} \end{bmatrix} \\ w_1 &:= \begin{bmatrix} \zeta A & \zeta B & \eta_0 & \dots & \eta_{L-1} & 0_{(n-1)m} \end{bmatrix} \\ &\ \vdots \\ w_n &:= \begin{bmatrix} \zeta A^n \dots & \zeta B & \eta_0 & \dots & \eta_{L-1} \end{bmatrix} \ . \end{aligned}$$

From the equations (4) conclude that the w_i's are left-annihilators on $\mathbb{I}$ of the $(n+(n+L)m)$-dimensional vector of functions

$$\begin{bmatrix} x^\top & u\top & u^{(1)\top} & \dots & u^{(n+L-1)\top} \end{bmatrix}^\top \ . \tag{4}$$

Because of property b) in Def. 1, this vector-valued function has at most n annihilators on $\mathbb{I}$, and consequently the $n+1$ vectors w_i are linearly dependent. Given the structure of the w_i's, $i = 0, \dots, n$, i.e. the presence of zeros in the last components, we conclude that $\eta_{L-1} = 0_{1\times m}$, then $\eta_{L-2} = 0$, and so on until $\eta_0 = 0$. This proves $\eta = 0$.

It follows that

$$\begin{aligned} w_0 &= \begin{bmatrix} \zeta & 0_{1\times (n+L)m} \end{bmatrix} \\ w_1 &= \begin{bmatrix} \zeta A & \zeta B & 0_{1\times (n+L-1)m} \end{bmatrix} \\ &\ \vdots \\ w_n &= \begin{bmatrix} \zeta A^n & \zeta A^{n-1} B & \dots & \zeta B & 0_{1\times Lm} \end{bmatrix} \ , \end{aligned}$$

are annihilators of (4). Denote by α_i, $i = 0 \dots, n$ the coefficients of the characteristic polynomial of A, and observe that $\sum_{i=0}^n w_i \alpha_i$ equals

$$\begin{aligned} &\zeta \begin{bmatrix} \sum_{i=0}^n A^i \alpha_i & \sum_{i=1}^n \alpha_i A^{i-1} B & \dots & B & 0_{1\times Lm} \end{bmatrix} \\ &= \zeta \begin{bmatrix} 0_{1\times n} & \sum_{i=1}^n \alpha_i A^{i-1} B & \dots & B & 0_{1\times Lm} \end{bmatrix} \ ; \end{aligned}$$

note that the first n entries of $\sum_{i=0}^n w_i \alpha_i$ are zero since $\sum_{i=0}^n A^i \alpha_i = 0$. By construction, $\forall\ t \in \mathbb{I}$ it holds that

$$\zeta \begin{bmatrix} \sum_{i=1}^n \alpha_i A^{i-1} B & \dots & \alpha_n B \end{bmatrix} \begin{bmatrix} u(t) \\ u^{(1)}(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix} = 0 \ ;$$

given the assumption (1), we conclude that

$$\zeta \begin{bmatrix} \sum_{i=1}^n \alpha_i A^{i-1} B & \sum_{i=2}^n \alpha_i A^{i-2} B & \dots & \alpha_n B \end{bmatrix} = 0 \ .$$

It follows from the last m equations that $\alpha_n \zeta B = 0$; since the highest coefficient α_n of the characteristic polynomial of A equals 1, we conclude that $\zeta B = 0$. The previous m-dimensional block-entry of the vector is $\alpha_{n-1}\zeta B + \alpha_n \zeta AB = 0 + \alpha_n \zeta AB = 0$. We conclude that $\zeta AB = 0$. The same argument can be used to prove $\zeta A^i B = 0$, $i = 0, \dots, n-1$. Given the controllability of the pair (A, B) we conclude that $\zeta = 0$. ■

Remark 1: As in the case of discrete-time systems, under the controllability assumption persistency of excitation is not necessary for (3) to hold: condition (3) also depends on the initial conditions of the system, which may excite all system dynamics even in the case of an insufficiently rich input signal. Consider for example the controllable system with $n = 1$ described by $\dot{x} = \alpha x + u$, with $\alpha \in \mathbb{R}$. Let $u(t) = e^{\lambda t}$

for $t \in [0,+\infty)$, $\lambda \neq \alpha$ and $x(0) = c$; straightforward computations show that $x(t) = \left(c + \frac{1}{\alpha-\lambda}\right) e^{\alpha t} - \frac{1}{\alpha-\lambda} e^{\lambda t}$. Let $L = 1$. Then for every $c \neq -\frac{1}{\alpha-\lambda}$ statement (3) holds; since $\lambda \neq \alpha$ it holds that if $\zeta \left(\left(c + \frac{1}{\alpha-\lambda}\right) e^{\alpha t} - \frac{1}{\alpha-\lambda} e^{\lambda t}\right) + \eta e^{\lambda t} = 0$, then $\zeta = 0$, $\eta = 0$. However, u satisfies the equality $\begin{bmatrix} -\lambda & 1 \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \end{bmatrix} = 0$, and u is not persistently exciting of order 2.

We state a straightforward consequence of Proposition 1.

Corollary 1: Let (x, u) be a state-input trajectory generated by a controllable continuous-time system (2). If u is persistently exciting of order at least n, then

$$\zeta \in \mathbb{R}^{1\times n} \text{ and } \zeta x(t) = 0 \text{ for all } t \in \mathbb{I} \Longrightarrow \zeta = 0 \,.$$

III. Main results

We illustrate a consequence of condition (1) of Proposition 1 on vectors

$$\begin{bmatrix} u^\top & \dots & u^{(L-1)\top} & y^\top & \dots & y^{(L-1)\top} \end{bmatrix}^\top , \tag{5}$$

computed from i-o trajectories (u, y) of a linear differential system.

Theorem 1: Let

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \,, \end{aligned} \tag{6}$$

be an i-s-o representation with m inputs, p outputs, and n state variables. Let $(\widehat{u}, \widehat{x}, \widehat{y})$ satisfy (6). Assume that (A, B) is controllable, and that $\widehat{u}$ is persistently exciting of order at least $n + L$ on $\mathbb{I}$. Let $\eta \in \mathbb{R}^{1\times m}[s]$, $\xi \in \mathbb{R}^{1\times p}[s]$, deg η, deg $\xi \leq L - 1$. The following statements are equivalent:

a) $\begin{bmatrix} \eta\left(\frac{d}{dt}\right) & \xi\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{y} \end{bmatrix} = 0$ on $\mathbb{I}$;

b) The differential operator $\begin{bmatrix} \eta\left(\frac{d}{dt}\right) & \xi\left(\frac{d}{dt}\right) \end{bmatrix}$ annihilates every continuously differentiable i-o trajectory of (6) on $\mathbb{I}$.

Proof: Evidently statement b) implies statement a).

To show that a) $\Longrightarrow$ b), we introduce some notation. Define the *k-th observability matrix*, the *k-th controllability matrix*, and the *Markov matrix*, respectively, of (6) by

$$\mathcal{O}_k := \begin{cases} C & \text{if } k = 0 \\ \begin{bmatrix} \mathcal{O}_{k-1} \\ CA^k \end{bmatrix} & \text{if } k \geq 1 \end{cases} ,$$

$$\mathcal{C}_k := \begin{cases} B & \text{if } k = 0 \\ \begin{bmatrix} A^k B & \mathcal{C}_{k-1} \end{bmatrix} & \text{if } k \geq 1 \end{cases} ,$$

$$\mathcal{T}_k := \begin{cases} D & \text{if } k = 0 \\ \begin{bmatrix} \mathcal{T}_{k-1} & 0 \\ C\mathcal{C}_{k-1} & D \end{bmatrix} & \text{if } k \geq 1 \end{cases} . \tag{7}$$

Let (u, x, y) be an arbitrary trajectory of (6), with u continuously differentiable at least $(L - 1)$ times. Use successive differentiations of $\dot{x} = Ax+Bu$ and $y = Cx+Du$ to conclude that

$$\begin{bmatrix} u \\ \vdots \\ u^{(L-1)} \\ y \\ \vdots \\ y^{(L-1)} \end{bmatrix} = \begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_{L-1} & \mathcal{T}_{L-1} \end{bmatrix} \begin{bmatrix} x \\ u \\ \vdots \\ u^{(L-1)} \end{bmatrix} . \tag{8}$$

Denote the coefficients of $\eta(s)$ and $\xi(s)$ by η_i and ξ_i, i.e. $\eta(s) = \sum_{i=0}^{L-1} \eta_i s^i$ and $\xi(s) = \sum_{i=0}^{L-1} \xi_i s^i$, and define q by

$$q := \begin{bmatrix} \eta_0 & \dots & \eta_{L-1} & \xi_0 & \dots & \xi_{L-1} \end{bmatrix} \in \mathbb{R}^{L(m+p)} \,.$$

From statement a), using (8) on $(\widehat{u}, \widehat{x}, \widehat{y})$, conclude that

$$q \begin{bmatrix} \widehat{u} \\ \vdots \\ \widehat{u}^{(L-1)} \\ \widehat{y} \\ \vdots \\ \widehat{y}^{(L-1)} \end{bmatrix} = q \begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_{L-1} & \mathcal{T}_{L-1} \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{u} \\ \vdots \\ \widehat{u}^{(L-1)} \end{bmatrix} = 0 \,. \tag{9}$$

Under the assumptions (1) on $\widehat{u}$ and its derivatives, the result of Proposition 1 holds and zero is the only left annihilator of $\begin{bmatrix} \widehat{x}^\top & \widehat{u}^\top & \dots & \widehat{u}^{(L-1)\top} \end{bmatrix}^\top$. Consequently, (9) holds if and only if

$$q \begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_{L-1} & \mathcal{T}_{L-1} \end{bmatrix} = 0 \,. \tag{10}$$

Now let (u', x', y') be an arbitrary input-state-output trajectory of (6), with u' at least $(L-1)$-continuously differentiable; then

$$q \begin{bmatrix} u' \\ \vdots \\ u'^{(L-1)} \\ y' \\ \vdots \\ y'^{(L-1)} \end{bmatrix} = q \begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_{L-1} & \mathcal{T}_{L-1} \end{bmatrix} \begin{bmatrix} x' \\ u' \\ \vdots \\ u'^{(L-1)} \end{bmatrix} = 0 \,.$$

Rewrite this equation as

$$\begin{bmatrix} \sum_{i=0}^{L-1} \eta_i \frac{d^i}{dt^i} & \sum_{i=0}^{L-1} \xi_i \frac{d^i}{dt^i} \end{bmatrix} \begin{bmatrix} u' \\ y' \end{bmatrix} = 0 \,,$$

to conclude that statement b) holds. ■

In the following corollary we formalize a partial answer to the question posed in Section I.

Corollary 2: Let $(\widehat{u}, \widehat{x}, \widehat{y})$ be a trajectory satisfying (6). Assume that (A, B) is controllable and that $\widehat{u}$ is persistently exciting of order at least $L + n$ on $\mathbb{I}$. Let $\eta(s) \in \mathbb{R}^{1\times m}[s]$, $\xi(s) \in \mathbb{R}^{1\times p}[s]$, with $\deg(\eta), \deg(\xi) \leq L - 1$.

The following two statements are equivalent:

1) (u, x, y) satisfy (6) $\Longrightarrow \begin{bmatrix} \eta\left(\frac{d}{dt}\right) & \xi\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = 0$ on $\mathbb{I}$;

2) $\begin{bmatrix} \eta\left(\frac{d}{dt}\right) & \xi\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{y} \end{bmatrix} = 0$ on $\mathbb{I}$.

Corollary 2 states that under controllability and sufficiently high persistent excitation of the input trajectory $\widehat{u}$, the set of *all* differential equations *of order up to* L satisfied by *all* i-o system trajectories coincides with the set of all differential equations of order up to L satisfied by the *particular* trajectory $(\widehat{u}, \widehat{y})$. This result provides only a *partial* answer to the question in Section I, since it does not state how much persistently exciting an input trajectory needs to be to determine *all* differential equations satisfied by the system trajectories. To fully answer this question we need to introduce some additional concepts and notation.

Definition 2 (System lag): Define Ω_k by (7). The *system lag*, denoted by $\ell(C, A)$, is defined by

$$\ell(C, A) := \min\{k \in \mathbb{N} \mid \text{rank } \Omega_k = \text{rank } \Omega_{k-1}\} . \tag{11}$$

It follows from Definition 2 that $\ell(C, A) \le n$, and that if (C, A) is observable, then $\ell(C, A)$ is the observability index of the pair. We denote by $\mathfrak{B}$ the space consisting of all infinitely-differentiable i-o trajectories (u, y) for which there exists a state-trajectory x such that (u, x, y) satisfies (6):

$$\begin{aligned} \mathfrak{B} := \quad & \big\{(u, y) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{m+p}) \mid \exists\, x : \mathbb{R} \to \mathbb{R}^n \\ & \text{such that } \ (u, x, y) \text{ satisfies (6)}\big\} . \end{aligned} \tag{12}$$

The *set of annihilators* of $\mathfrak{B}$, denoted by $\mathcal{N}(\mathfrak{B})$ is defined by

$$\begin{aligned} \mathcal{N}(\mathfrak{B}) := \quad & \Big\{\begin{bmatrix}\eta(s) & \xi(s)\end{bmatrix} \in \mathbb{R}^{1\times(m+p)}[s] \mid \\ & \eta\left(\frac{d}{dt}\right) u + \xi\left(\frac{d}{dt}\right) y = 0 \ \forall\ (u, y) \in \mathfrak{B}\Big\} . \end{aligned} \tag{13}$$

$\mathcal{N}(\mathfrak{B})$ is a *module* of $\mathbb{R}^{1\times(m+p)}[s]$: if $\begin{bmatrix}\eta_i(s) & \xi_i(s)\end{bmatrix} \in \mathcal{N}(\mathfrak{B})$, $i = 1, 2$, then for every $p_1, p_2 \in \mathbb{R}[s]$ also

$$p_1(s)\begin{bmatrix}\eta_1(s) & \xi_1(s)\end{bmatrix} + p_2(s)\begin{bmatrix}\eta_2(s) & \xi_2(s)\end{bmatrix} \in \mathcal{N}(\mathfrak{B}) .$$

We denote by $\mathcal{N}(\mathfrak{B})^L$ the set of annihilators of $\mathfrak{B}$ with degree less than or equal to L:

$$\begin{aligned} \mathcal{N}(\mathfrak{B})^L := \quad & \big\{\begin{bmatrix}\eta(s) & \xi(s)\end{bmatrix} \in \mathcal{N}(\mathfrak{B}) \mid \\ & \deg\ \begin{bmatrix}\eta(s) & \xi(s)\end{bmatrix} \le L\big\} , \end{aligned}$$

and by $\langle\mathcal{N}(\mathfrak{B})^L\rangle$ the module generated by its elements:

$$\begin{aligned} \langle\mathcal{N}(\mathfrak{B})^L\rangle := \big\{ & p_1(s)\begin{bmatrix}\eta_1(s) & \xi_1(s)\end{bmatrix} + p_2(s)\begin{bmatrix}\eta_2(s) & \xi_2(s)\end{bmatrix} \mid \\ & \begin{bmatrix}\eta_i(s)\xi_i(s)\end{bmatrix} \in \mathcal{N}(\mathfrak{B})^L \ \text{ and } p_i \in \mathbb{R}[s]\ , i = 1, 2\big\} . \end{aligned}$$

Note that since there is no restriction on the degree of p_i, $i = 1, 2$, $\langle\mathcal{N}(\mathfrak{B})^L\rangle$ contains elements of degree larger than L.

We state the following straightforward consequence of Theorem 1, fully answering the question posed in Section I.

Corollary 3: Let $(\widehat{u}, \widehat{x}, \widehat{y}) : \mathbb{R} \to \mathbb{R}^{m+n+p}$ be a trajectory of (6). Assume that (A, B) is controllable and that $\widehat{u}$ is persistently exciting of order at least $\ell(C, A) + n$ on $\mathbb{I}$. Define

$$\begin{aligned} \mathcal{N}(\widehat{u}, \widehat{y})^{\ell(C,A)} \quad & := \Big\{\begin{bmatrix}\eta(s) & \xi(s)\end{bmatrix} \in \mathbb{R}^{1+(m+p)}[s] \mid \\ & \eta\left(\frac{d}{dt}\right)\widehat{u} + \xi\left(\frac{d}{dt}\right)\widehat{y} = 0 \text{ on } \mathbb{I} \\ & \text{and } \ \deg\begin{bmatrix}\eta(s) & \xi(s)\end{bmatrix} \le \ell(C, A)\Big\} . \end{aligned}$$

Then the module $\langle\mathcal{N}(\widehat{u}, \widehat{y})^{\ell(C,A)}\rangle$ of $\mathbb{R}^{1\times(m+p)}[s]$ satisfies the equality $\langle\mathcal{N}(\widehat{u}, \widehat{y})^{\ell(C,A)}\rangle = \mathcal{N}(\mathfrak{B})$.

Proof: The proof makes use of the following fact (see statement $viii.$ of Theorem 6 p. 570 of [16]): the system lag $\ell(C, A)$ is such that

$a)$ $\mathcal{N}(\mathfrak{B}) = \langle\mathcal{N}(\mathfrak{B})^{\ell(C,A)}\rangle$; and

$b)$ $\ell(C, A) = \min\big\{L \in \mathbb{N} \mid \mathcal{N}(\mathfrak{B}) = \langle\mathcal{N}(\mathfrak{B})^L\rangle\big\}$.

It follows that $\mathcal{N}(\mathfrak{B})$ is finitely generated: there exist $g \in \mathbb{N}$ and $r_i(s) \in \mathbb{R}^{1\times(m+p)}[s]$, $i = 1, \ldots, g$ such that $\deg\ r_i(s) \le \ell(C, A)$ and the module $\langle r_1, \ldots, r_g\rangle$ generated by the $r_i(s)$ equals $\mathcal{N}(\mathfrak{B})$. Now apply Corollary 2 to conclude that $\begin{bmatrix}\eta(s) & \xi(s)\end{bmatrix} \in \mathcal{N}(\widehat{u}, \widehat{y})^{\ell(C,A)}$ if and only if it has degree $\le \ell(C, A)$ and it annihilates all trajectories in $\mathfrak{B}$:

$$\mathcal{N}(\widehat{u}, \widehat{y})^{\ell(C,A)} = \mathcal{N}(\mathfrak{B})^{\ell(C,A)} .$$

This equality implies that $\langle\mathcal{N}(\widehat{u}, \widehat{y})^{\ell(C,A)}\rangle = \langle\mathcal{N}(\mathfrak{B})^{\ell(C,A)}\rangle$. Consider a set of generators $\{r_i(s)\}_{i=1,\ldots,g}$ of $\mathcal{N}(\mathfrak{B})$ such that $\deg\ r_i(s) \le \ell(C, A)$; note that $r_i \in \mathcal{N}(\mathfrak{B})^{\ell(C,A)}$, $i = 1, \ldots, g$. Conclude that $\langle\mathcal{N}(\mathfrak{B})^{\ell(C,A)}\rangle = \mathcal{N}(\mathfrak{B})$, from which the claim follows. ■

Corollary 3 gives a full answer to the question posed in Section I: under the controllability assumption, if $\widehat{u}$ is persistently exciting of order at least $n + \ell(C, A)$, then any i-o trajectory $(\widehat{u}, \widehat{y})$ corresponding to it is sufficiently informative about the system dynamics.

IV. Relation with discrete-time approaches

We discuss how our approach relates to the discrete-time ones of [6], [14], [15]. Two conceptual analogies between discrete- and continuous-time underlie our results.

The first one is the correspondence between the data Hankel matrix in discrete-time, and the vector of continuous time functions (5). Dynamical laws satisfied by the data correspond to left annihilators of the Hankel matrix in the discrete-time case, and to annihilators of the vector of functions (5) in the continuous-time case.

The second crucial analogy between the discrete- and the continuous-time case is that between time-shift and differentiation. This correspondence is ultimately based on the “injective cogenerator property” (see [8]) that allows to translate statements and properties established on suitable solution spaces of linear differential or difference equations, into properties expressed in terms of modules of differential or difference operators. For discrete-time systems, the i-th shift of a state trajectory can be written as a linear combination of the k-th shift of the input trajectory, $k = 0, \ldots, i-1$, and the state trajectory itself. In continuous time, the i-th derivative of the state can be written as a linear combination of the k-th derivative of the input, $k = 0, \ldots, i-1$, and the state.

With the first conceptual analogy in mind, and considering that if $L > \ell(A, C)$ then $\langle\mathcal{N}(\widehat{u}, \widehat{y})^{\ell(C,A)}\rangle = \mathcal{N}(\mathfrak{B})$ (see Corollary 3), it is straightforward to verify the following analogies between our results and those of [15]:

- Corollary 1 is equivalent to statement (i) in Corollary 2 p. 328 in [15];
- Corollary 2 is the continuous-time equivalent to equation (K) in Theorem 1 p. 327 in [15].

Regarding the relations between our results and those in [14], we observe the following:

- The argument used in proving Proposition 1 exploits the analogy between left annihilators of the data Hankel matrix and of (5); and the consequences of the correspondence between time-shift and differentiation;
- The result of Proposition 1 is dual (in the sense illustrated in Section I) to that of Theorem 1 p. 603 in [14];
- The argument used in proving Theorem 1 exploits the standard relation (8) between Markov parameters and input-state-output trajectories to relate left annihilators of (5) with annihilators of (10).

Finally, we discuss connections with relevant results in [6]. The setting therein is more general: neither knowledge of an i-o partition of the measured variables, nor of the number of input or output variables, are postulated. Moreover, controllability is not assumed a priori, and the data may consist of multiple finite time-series. Some of the conditions provided in [6] (see Theorem 15 p. 8) are also *necessary*, not only sufficient for identifiability.

Algorithm 1 p. 7 of [6] iteratively constructs a basis for the module of annihilators of the *most unfalsified model* (MPUM) (see Definition 11 p. 6 therein) for the data, in the process computing also an i-o partition of the variables. Theorem 14 p. 7 of [6] states two sufficient conditions for identifiability, based on the outcome of such algorithm.

The first condition (see the first formula in $(ID1)$) is that the system lag is smaller than the depth of the Hankel matrix. In our notation, this is equivalent to the requirement that the order of differentiation L in (5) satisfies $L > \ell(A, C)$. This requirement is implied by Definition 1 and the condition in Corollary 3 that $\widehat{u}$ is persistently exciting of order at least $\ell(C, A) + n$.

The second condition is that the number of outputs of the MPUM computed by the algorithm equals the number of outputs of the data-generating system (see the second formula in $(ID1)$). In our setting, this corresponds to the condition that the set of left-annihilators of (5) is equal to the set of annihilators of the data-generating system. Under the assumptions of controllability and persistent excitation of order at least $n + \ell(A, C)$, this is precisely the statement of Corollary 3.

V. Application to identification

The design of identification algorithms that, under the assumption of persistent excitation, start from an i-o trajectory and identify the continuous-time system that generated it, is beyond the scope of the present article. In this section we briefly sketch one of the possible approaches to continuous-time system identification opened up by the results presented in this paper, we put it in the context of continuous-time identification, and we discuss some of its evident limitations.

The following is an instrumental result.

Proposition 2: Let $(\widehat{u}, \widehat{x}, \widehat{y})$ satisfy (6). Assume that (A, B) is controllable, that (A, C) is observable, and that $\widehat{u}$ is persistently exciting of order $n+L$ on $\mathbb{I}$. Assume that $L \geq \ell(C, A)$; then there exist $s \in \mathbb{N}$ and $t_i \in \mathbb{I}$, $i = 1, \dots, s$, such that

$$\operatorname{rank} \begin{bmatrix} \widehat{u}(t_1) & \widehat{u}(t_2) & \dots & \widehat{u}(t_s) \\ \widehat{u}^{(1)}(t_1) & \widehat{u}^{(1)}(t_2) & \dots & \widehat{u}^{(1)}(t_s) \\ \vdots & \vdots & \dots & \vdots \\ \widehat{u}^{(L-1)}(t_1) & \widehat{u}^{(L-1)}(t_2) & \dots & \widehat{u}^{(L-1)}(t_s) \\ \widehat{y}(t_1) & \widehat{y}(t_2) & \dots & \widehat{y}(t_s) \\ \widehat{y}^{(1)}(t_1) & \widehat{y}^{(1)}(t_2) & \dots & \widehat{y}^{(1)}(t_s) \\ \vdots & \vdots & \dots & \vdots \\ \widehat{y}^{(L-1)}(t_1) & \widehat{y}^{(L-1)}(t_2) & \dots & \widehat{y}^{(L-1)}(t_s) \end{bmatrix} = mL+n \, . \tag{14}$$

Proof: Denote the matrix on the left-hand side of (14) by $H_{\{t_i\}_{i=1,\dots,s}}(\widehat{u}, \widehat{y})$. Denote by $H_{\{t_i\}_{i=1,\dots,s}}(\widehat{x}, \widehat{u})$ the matrix

$$\begin{bmatrix} \widehat{x}(t_1) & \widehat{x}(t_2) & \dots & \widehat{x}(t_s) \\ \widehat{u}(t_1) & \widehat{u}(t_2) & \dots & \widehat{u}(t_s) \\ \widehat{u}^{(1)}(t_1) & \widehat{u}^{(1)}(t_2) & \dots & \widehat{u}^{(1)}(t_s) \\ \vdots & \vdots & \dots & \vdots \\ \widehat{u}^{(L-1)}(t_1) & \widehat{u}^{(L-1)}(t_2) & \dots & \widehat{u}^{(L-1)}(t_s) \end{bmatrix} ,$$

and use (8) to conclude that

$$H_{\{t_i\}_{i=1,\dots,s}}(\widehat{u}, \widehat{y}) = \begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_L & \mathcal{T}_L \end{bmatrix} H_{\{t_i\}_{i=1,\dots,s}}(\widehat{x}, \widehat{u}) \, .$$

Since $L \geq \ell(C, A)$, it follows that

$$\operatorname{rank} \begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_L & \mathcal{T}_L \end{bmatrix} = mL + \operatorname{rank} \mathcal{O}_L = mL + n \, ,$$

and $\operatorname{rank} H_{\{t_i\}_{i=1,\dots,s}}(\widehat{u}, \widehat{y}) = \operatorname{rank} H_{\{t_i\}_{i=1,\dots,s}}(\widehat{x}, \widehat{u})$. Now consider the one-parameter family of $(Lm+n)$-dimensional vectors defined by

$$v_t := \begin{bmatrix} \widehat{x}(t)^\top & \widehat{u}(t)^\top & \dots & \widehat{u}^{(L-1)}(t)^\top \end{bmatrix} , \; t \in \mathbb{I} \, .$$

Since $\widehat{u}$ is persistently exciting of order $Lm+n$, it follows from Proposition 1 that the family $\{v_t \mid t \in \mathbb{I}\}$ is not contained in any fixed, non-trivial hyperplane of $\mathbb{R}^{Lm+n}$. Consequently, s and $\{t_i\}_{i=1,\dots,s} \subset \mathbb{I}$ exist, such that $\operatorname{rank} H_{\{t_i\}_{i=1,\dots,s}}(\widehat{x}, \widehat{u}) = n + mL$; note that necessarily $s \geq Lm + n$.

The same integer $s \in \mathbb{N}$ and choice of sampling instants $t_i \in \mathbb{I}$, $i = 1, \dots, s$ is such that $\operatorname{rank} H_{\{t_i\}_{i=1,\dots,s}}(\widehat{u}, \widehat{y}) = mL + n$, as was to be proved. ∎

To state our next result we need to introduce the concept of (algebraic) *genericity*. Let $\mathcal{L}$ be a d-dimensional linear space over $\mathbb{R}$, with a basis $\{\ell_i\}_{i=1,\dots,d}$. If $\ell \in \mathcal{L}$, then there exist $x_i \in \mathbb{R}$, $i = 1, \dots, d$, such that $\ell = \sum_{i=1}^{d} x_i \ell_i$. A map $p : \mathcal{L} \to \mathbb{R}$ is a *polynomial* if $p(\ell)$ is a polynomial in the coefficients x_i's. An *algebraic variety* is a subset $\mathcal{V}$ of $\mathcal{L}$ consisting of all solutions to a system of algebraic equations defined by polynomials p_i, $i = 1, \dots, g$. A subset $\mathcal{S} \subset \mathcal{L}$ is called *generic* if there is a proper algebraic variety $\mathcal{V} \subsetneq \mathcal{L}$ such that $\mathcal{S} \supset (\mathcal{L} \setminus \mathcal{V})$. If a generic set $\mathcal{S}$ consists of all point in $\mathcal{L}$ that have a certain property, then the property is called generic.

Proposition 3: Let $(\widehat{u}, \widehat{x}, \widehat{y})$ satisfy (6). Assume that (A, B) is controllable, that (A, C) is observable, and that $\widehat{u}$ is persistently exciting of order $n + L$ on $\mathbb{I}$, that $L \geq \ell(C, A)$ and that $s \geq Lm + n$. Then generically for every choice of $\{t_i\}_{i=1,\dots,s} \subset \mathbb{I}$ rank $H_{\{t_i\}_{i=1,\dots,s}}(\widehat{u}, \widehat{y}) = mL + n$.

Proof: Use the argument in Proposition 2 to conclude that for every choice of the t_i, $i = 1, \ldots, s$, the equality rank $H_{\{t_i\}_{i=1,\ldots,s}}(\widehat{u}, \widehat{y}) =$ rank $H_{\{t_i\}_{i=1,\ldots,s}}(\widehat{x}, \widehat{u})$ holds.

Consider the one-parameter family of $(Lm+n)$-dimensional vectors $\{v_t\}$ introduced in the proof of Proposition 2. Since $\widehat{u}$ is persistently exciting of order $Lm+n$, $\{v_t \mid t \in \mathbb{I}\}$ is not contained in any fixed, non-trivial hyperplane of $\mathbb{R}^{Lm+n}$. Consequently, generically any choice of $s \geq Lm+n$ values t_i of the parameter t yields s vectors that do not all lie on any hyperplane. Conclude that rank $H_{\{t_i\}_{i=1,\ldots,s}}(\widehat{x}, \widehat{u}) = Lm+n$, from which the claim follows. ∎

Based on the result of Proposition 3 we sketch the following procedure to identify a set of i-o differential equations describing the generating system.

Assume that upper bounds on the system lag $\ell(C, A)$ and the state dimension n_{min} of any observable and controllable i-s-o representation of the generating system are known. Choose arbitrarily $s \geq \ell(C, A)m + n_{\text{min}}$ time instants $t_i \in \mathbb{I}$, $i = 1, \ldots, s$; then generically rank $H_{\{t_i\}_{i=1,\ldots,s}}(\widehat{u}, \widehat{y}) = \ell(C, A)m + n_{\text{min}}$ (see Proposition 3). Assume that such rank condition holds for the particular choice of $\{t_i\}_{i=1,\ldots,s}$, and compute a basis for the subspace of $\mathbb{R}^{1\times L(m+p)}$ consisting of all left-annihilators of $H_{\{t_i\}_{i=1,\ldots,s}}(\widehat{u}, \widehat{y})$. Each element of this basis corresponds to a $(m+p)$-dimensional polynomial row-vector associated with a differential operator annihilating all system trajectories (see Corollary 3). A basis for the module $\mathcal{N}(\mathfrak{B})$ of annihilators can be computed using standard polynomial matrix computations (e.g. see Theorem 6.3.2 p. 375 of [4]). From such module basis, an i-s-o representation of the system can be straightforwardly computed; see [11] for details.

We conclude setting our conceptual approach in the context of continuous-time system identification (see [2], [3], [10], [13]), and discussing its limitations and some of the challenges to make it viable. Our abstract procedure is based on the unrealistic assumption that higher derivatives of inputs and outputs are measured. Consequently, a solution must be found to the "time-derivative measurement problem" (see section 2 of [2]), i.e. the computation or approximation of such derivatives. To be theoretically consistent with our approach, the solution must also be compatible with the algebraic theory establishing the conceptual equivalence between modules of differential operators and spaces of signals. We discuss some issues and opportunities that arise in connection with standard approaches.

Approximate differentiation based on difference schemes, and low-pass filtering are among the solutions adopted to solve the time-derivative measurement problem (see [2], [13]). In both cases the end result is that one associates with samples of the original trajectories, samples of other ones unrelated to them by differentiation. Consequently, in a behavioral framework such trajectories may be considered to be *auxiliary variables* ones. To adopt such solutions in our setting, the relation of the original behavior with such augmented set of trajectories should be put on a sound theoretical footing, to fit into the exact modelling framework of [17].

Some recent work by the authors (see [12]) exploited orthogonal basis function (OBF) representations for data-driven continuous-time control in a state-space setting (OBFs are also used in continuous-time system identification, see [13]). Differentiation and integration of signals are formalized as the product of an (infinite) vector of coefficients times an (infinite) differentiation or integration matrix. Depending on the OBFs, the derivative is exactly computable even from a finite number of coefficients; in this case, the original system parameters can be associated to a "transformed" system operating on OBF representations of signals. It is a matter of current investigation whether the use of OBFs, successful in data-driven control applications, can be extended to our conceptual procedure to provide a more realistic one, consistent with the basic tenets of behavioral identification.

VI. Conclusions

In Corollary 3 of Section III we established a sufficient condition on an input trajectory $\widehat{u}$ and a finite number of its derivatives, to generate "sufficiently informative" i-o trajectories $(\widehat{u}, \widehat{y})$, from which all differential equations relating inputs and outputs can be deduced. In Section IV we connected our results and techniques to the literature on identifiability for discrete-time systems. In Section V we used our results to devise a conceptual procedure that generically computes a set of i-o differential equations for the generating system.

References

[1] Berberich, J. and Allgöwer, F., "A trajectory-based framework for data-driven system analysis and control", in *Proc. 2020 European Control Conference (ECC)*, pp. 1365–1370, 2020.

[2] Garnier, H., Mensler, M. and Richard, A., "Continuous-time model identification from sampled data: Implementation issues and performance evaluation", *Int. J. Contr.*, vol. 76, no. 13, pp. 1337–1357, 2003.

[3] Garnier, H., "Direct continuous-time approaches to system identification. Overview and benefits for practical applications", *European Journal of Control*, vol. 24, pp. 50–62, 2015.

[4] Kailath, T., *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.

[5] Lopez, V.G. and Müller, M.A., "On a Continuous-Time Version of Willems' Lemma", , arXiv, `https://doi.org/10.48550/arXiv.2203.03702`.

[6] Markovsky, I. and Dörfler, F. "Identifiability in the Behavioral Setting", Tech. Rep., 2020.

[7] Markovsky, I. and Rapisarda, P., "Data-driven simulation and control", *Int. J. Contr.*, vol. 81, no. 12, pp. 1946–1959, 2008.

[8] Oberst, U., "Multidimensional constant linear systems", *Acta Applicandae Mathematicae*, vol. 20, pp. 1–175, 1990.

[9] De Persis, C. and Tesi, P., "Formulas for data-driven control: Stabilization, optimality, and robustness", *IEEE Trans. Aut. Contr.*, vol. 65, no. 3, pp. 909–924, 2020.

[10] Rao, G.P., and Unbehauen, H., "Identification of continuous-time systems". *IEE Proceedings - Control Theory and Applications*, vol. 153, no. 2, pp. 185–220, 2006.

[11] Rapisarda, P. and Willems, J.C., "State maps for linear systems", *SIAM J. Contr. Opt.*, vol. 35, pp. 1053-1091, 1997.

[12] Rapisarda, P., Çamlibel, M.K. and van Waarde, H.J., "Orthogonal bases for data-driven analysis and control of continuous-time systems", manuscript, 2022.

[13] Unbehauen, H., and Rao, G.P., "A review of identification in continuous-time systems", *Annual Reviews in Control*, vol. 22, pp. 145-171, 1998.

[14] van Waarde, H.J., De Persis, C., Çamlibel, M.K. and Tesi, P., "Willems' Fundamental Lemma for State-Space Systems and Its Extension to Multiple Datasets", *IEEE Control Systems Letters*, vol. 4, no. 3, pp. 602-607, 2020.

[15] Willems, J.C., Rapisarda, P., Markovsky, I. and De Moor, B.L., "A note on persistency of excitation", *Syst. Control Lett.*, vol. 54, no. 4, pp. 325-329, 2005.

[16] Willems, J.C., "From time series to linear system, Part I. Finite dimensional linear time-invariant systems", *Automatica*, vol. 22, no. 5, pp. 561–580, 1986.

[17] Willems, J.C., "From time series to linear system, Part II. Exact modelling", *Automatica*, vol. 22, no. 6, pp. 675–694, 1986.