A Proportional Pricing Mechanism for Ridesharing Services With Meeting Points

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Abstract. Ridesharing is a promising approach for reducing congestion and pollution, and many variants have been studied in the literature over the past decades. In this paper, we consider a novel setting where individuals walk to a common pick-up point and ride together to a single drop-off point from where they walk to their final destination. This setting requires finding the optimal composition of riders and pick-up and drop-off meeting points, as well as an equitable distribution of the costs whereby riders are incentivised to participate. Based on game-theoretic principles, we propose a methodology to determine the optimal pick-up and drop-off points, together with a cost allocation method that is equitable in the sense that it ensures proportionality for sharing the costs, i.e., those who walk more should pay less. We present a formal evaluation of our cost allocation method and empirical evaluation against the Shapley value using real-world and simulated data. Our results show that our approach is computationally more tractable than the Shapley value, as it is linear in time while guaranteeing individual rationality under certain conditions.

Keywords: Ridesharing · Cost Sharing · Smart Mobility · Multi-Agent Systems for Transportation.

1 Introduction

Ridesharing services allow individuals with similar itineraries to share a car and split the cost of the trip. As a mobility option, ridesharing presents advantages on several fronts. From a social point of view, it contributes to lower congestion, bringing a positive externality to traffic [25,21]. From the user's standpoint, ridesharing is usually more affordable than travelling alone [14]. For these reasons, ridesharing services have become a popular mobility option across big cities. In recent years, several commercial platforms have started to offer ridesharing services, such as Uber[3], Lyft[4] and DiDi[5], as

[3] https://www.uber.com
[4] https://www.lyft.com
[5] http://www.didiglobal.com

a sign of the changing landscape in mobility patterns. According to the *US Census Bureau* [27], by 2019, ridesharing represented approximately 8–11% of the transportation modality in Canada and the USA.

Our approach considers a ridesharing setting where users walk to a common pick-up point and take a car to a single drop-off point, from where they walk to their own destinations. The walking requirement is a common practice in ridesharing services, as it provides efficiency to the system by optimizing the car route and avoiding detours [25]. Example scenarios include the case of friends arranging a private trip and meeting at the car rental place; commuters using private cars (a similar case is considered in [30]); or companies providing ridesharing services to their facilities (as considered by [12]). In all these cases, there are pre-specified pick-up and drop-off points for the car and a walking effort required from passengers to walk to/from those points. Our proposed methodology starts by determining these pick-up and drop off points in a way such that the total walking time of individuals is minimized. Additionally, we consider the walking time as a non-transferable cost for individuals. As such, we propose a method where we factor the walking time into the total trip cost using the concept of value-of-time [22] to account for the monetary value of walking. In this way, the total cost of a trip is composed of the car cost and the walking-time cost. With the travel cost defined in this manner, we find the socially optimal car allocation of passengers such that the travel cost is minimized. A central aspect we consider is the distribution of the trip cost among ridesharing passengers. For this, we propose an *equitable* cost allocation method that explicitly recognizes the walking time as a cost and we propose a cost allocation method that compensates users for their walking time when splitting the car cost.

Our research problem is two-fold: given the origin and destination coordinates for each rider (from the set of riders), we first determine the pick up and drop-off points such that the walking time is minimized and we obtain the socially optimal way to allocate riders to different cars such that the total travel cost is minimized (including the walking and car cost).[6]. Secondly, we focus on the final cost-allocation method for the trip cost. In this part, the goal is to provide an equitable cost allocation mechanism for the socially optimal coalition structure while being computationally tractable.

1.1 Introductory Example

Consider a group of friends who want to share a ride after a night out, or colleagues who live close-by and want to share a ride. In both cases, riders meet at a pick-up point and ride together to a drop-off point. Given a set of riders, and assuming an infinite number of cars, there are many possible ways to allocate riders into cars. Our premise is that individuals choose with whom to share a ride based on the cost of the trip, including riding and walking costs. We illustrate the interplay between walking and riding costs below.

As an example, consider Figure 1 below. There are 3 individuals $\{A, B, C\}$ who choose to share a car. In the left panel, each individual walks from their individual origins to a common departure point **O** and then drive to a single drop-off point **D**, from where they walk to their individual destinations. This is just one example of a

[6] This problem will be formally presented in Section3, Definition 5.

possible arrangement of riders, but there can be many others. For example, on the right panel $\{A, B\}$ ride-share together while $\{C\}$ travels alone (and therefore has no need for walking). Indeed, there are five possible combinations in which three individuals can be arranged. The formation of these possible travel arrangements, their cost optimality, and the fair cost distribution is the subject of this paper.

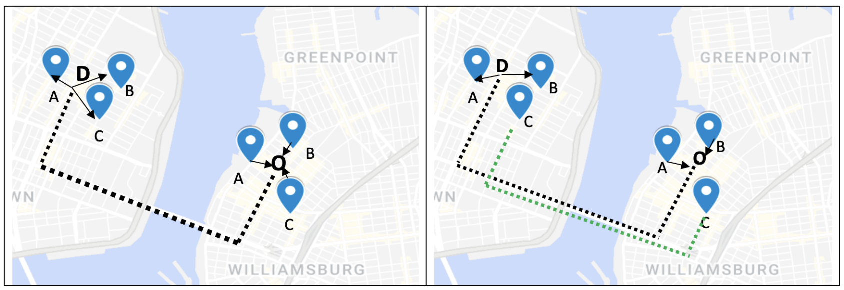

Fig. 1: Example of a ridesharing formation for three individuals. The arrows show their walking distance to the common O-D points and the dotted line the distance travelled by the car. On the left panel, the three riders share a car, while on the right panel A,B share a car.

1.2 Related Literature

Ridesharing has motivated the production of academic studies on its different aspects. Several authors have approached the rider-matching problem using game-theoretic solutions by modelling ridesharing as a cooperative game. Under this framework, a set of trip requests is partitioned as a coalition structure, where each subset is a coalition whose members will share a car based on some cost-optimization rule [15,11,10]. Another traditional area of study has been the matching problem between trip requests and available cars. For example, the work by [2] uses cooperative game theory to find the optimal combination of riders that minimises the system-wide travel cost. These approaches, like many others, assume that cars make multiple stops and riders are not required to walk, which is unrealistic in a ridesharing setting. Recently, there has been an increasing body of literature recognizing the importance of the walking requirement for the efficiency of the ridesharing system. For example, authors like [25] highlight the benefits of introducing meeting points in a ridesharing system. They show that allowing for meeting and drop-off points introduces flexibility in the system and increases the feasible matches between riders and drivers (i.e., demand and supply). More recently, the work of [9] shows that the introduction of a small walking requirement of about one minute increases the system's efficiency (as considered by the matches of trip demand and supply) by 80%. While these works highlight the importance of walking options in the efficiency of ridesharing, they focus on the matching problem between riders and drivers. Our approach deviates from these approaches, which are rooted in matching theory, and is the first one to capture walking in a game-theoretical model for ridesharing by integrating walking time as a non-transferable cost in the so-called *ridesharing games*.

Another related question is how to distribute the trip cost among riders sharing a car. It is a common approach to use game-theoretic solutions to guarantee desirable theoretical properties of the proposed cost allocation method. A traditional method is the Shapley value [18], which compensates each player by the weighted average of his/her marginal contributions. It is a widely accepted solution concept, as it is the only payment rule satisfying the four properties of Efficiency, Symmetry, Linearity and Null player [24]. The main caveat of the Shapley value is that its runtime increases exponentially with the number of players. More precisely, it can be shown that computing the Shapley value is an NP-complete problem, becoming quickly intractable for ridesharing settings [6,8]. In the context of ridesharing, there exists literature applying the Shapley value to distribute the car cost. For example, the works of [29] and [13] use the Shapley value to distribute the cost in a rider-driver allocation problem. However, none of these methods include walking as a cost to consider in the allocation mechanism, which seems an unrealistic setting, given the mentioned relevance of walking in the efficiency of ridesharing. Additionally, as it will be shown in this work, Shapley does not guarantee proportionality of the cost allocation with respect to the walking effort.

1.3 Contributions

Our contributions are as follows:

1. Existing work on ride sharing assume the meeting points for pick-up and drop-off are fixed. To address this, we are the first to propose a method to determine the pick-up and drop-off points based on the geometric median of coordinates, which minimizes the walking distance of riders [7].
2. For the first time in the ridesharing literature, our methodology models the walking cost as a non-transferable cost to determine the optimal allocation of riders.
3. We are the first to propose a cost allocation methodology that is equitable in the sense that it compensates riders for the walking effort to join the coalition.
4. The proposed cost allocation method is individually rational (under certain conditions) and has linear time complexity.

The next sections describe our proposed methodology for walking cost and the payment vector. Lastly, we evaluate the payment-allocation method theoretically and empirically and compare it with a Shapley-based cost allocation method.

2 Formal Preliminaries

This section introduces the notation for cooperative games and successive sections expand on the definitions used within the context of ridesharing. Following [19] and [16],

[7] The idea behind this approach was presented in an early pre-print [3].

in cooperative games, a set of agents $A = \{a_1, ..., a_n\}$ work together by forming coalitions and taking joint actions to maximize their utility. The goal is to find what is the best coalition structure to form. A coalitional cost game can be described in terms of its *characteristic function*, which expresses the cost of each coalition. Intuitively, this is the cost that a subset of agents face and need to pay collectively.

Definition 1. *A coalitional cost game with transferable utilities (TU) is the tuple (A, C) where A is the set of agents and C is a characteristic function $C : 2^A \to \mathbb{R}$ that returns the cost that a subset $S \subseteq A$ of agents face on their own regardless of what the remaining agents do (i.e., without any externality present).*

Given a coalitional cost game (A, C), the Coalition Structure Generation (CSG) problem focuses on generating a coalition structure (as a partition of the set of agents A) with desirable properties, e.g., those that yield a minimum cost.

Definition 2. *Given a coalitional cost game (A, C), a coalition structure $\mathbb{C} = \{S_1, \dots, S_m\}$, is a partition of A, with $m \leq n$. That is, for arbitrary distinct $1 \leq k, l \leq m$ with $k \neq l$, we have that $S_k \subseteq A$, $S_l \subseteq A$, $S_k \cap S_l = \emptyset$, and $\bigcup_{k=1}^{m} S_k = A$. The set of all possible coalition structures is denoted by $\mathscr{C}$.*

2.1 Ridesharing as a Coalitional Cost Game

A ridesharing game is a tuple (A, C, M), where $A = \{a_1, .., a_n\}$ is the set of riders, $C : 2^A \to \mathbb{R}$ is the characteristic function, and $M : A \to \mathbb{R}^2 \times \mathbb{R}^2$ is the coordinate function that yields the origin and destination coordinates for each rider, i.e., $M(a_i) = (o_i, d_i)$, where $o_i \in \mathbb{R}^2$ and $d_i \in \mathbb{R}^2$ are the origin and destination coordinates of a rider, respectively. The objective in a ridesharing game is to find the socially-optimal coalition structure with the minimum travel cost. For simplicity, we consider an unbounded supply of cars with up to four seats each. The following sections cover how to calculate the coalition cost and how the optimal coalition structure is obtained.

2.2 Cost Calculation

In contrast to the most common coalition structure generation literature, where a cost table is given, our model provides a contextual function to calculate ridesharing costs. In particular, the main contribution of our cost calculation methodology is the explicit modelling of walking costs on top of the car cost. In a sense, the walking cost can be viewed as the *cost of entry* into a coalition, as it represents the time investment required to join a ridesharing car. The walking *distance* of each individual depends on the central origin and destination points, while the walking *cost* depends on the monetary value of time, as explained below.

Determination of the Central O-D Points. The first step is to calculate the pick-up and drop-off points for all riders sharing a car. Our approach determines these points using the geometric median of the individuals' coordinates (as given by the function M).[8] We

[8] The geometric median of a set of points is defined as the point minimizing the sum of Euclidean distances to all points [5]. For its implementation we used the Weiszfeld algorithm [28].

choose this approach since it minimizes the distance walked by all individuals, while being computationally tractable.[9] Once the walking points are set, the next step is to assign a (monetary) cost to the walking time.

Cost of Walking. In order to assign a monetary value to the walking effort, we use the concept of value of time (VoT), which accounts for the opportunity cost of the walking time. Following [9], we assume that there is a walking threshold for reasonable walking distance, beyond which individuals prefer to take a car (typically, $800m$ is accepted as the walking threshold [23]). We model this trade off between walking and riding using a function similar to a Cobb-Douglas utility function [4] as in Equation 1 to express the fact that the cost of walking increases more than proportional to the distance walked. Lastly, the VoT is re-scaled by a constant representing the taxi cost per distance, to account for the fact that after a certain tolerance threshold, walking is more expensive (in terms of VoT) than taking a car. The VoT of walking from point o to d is modeled as:

$$VoT(o,d) = dist(o,d)^{\alpha}.K \tag{1}$$

where K is a constant representing the taxi fare, $dist(\cdot)$ is a distance function in the Cartesian plane, and $\alpha > 1$ is an arbitrary parameter related to the function's convexity and the walking threshold. The higher the α value, the higher the opportunity cost of walking compared to taking a car.

Cost of Riding. The cost of riding a car from coordinates o to d is simply the distance travelled times a constant K accounting for the taxi fare, i.e.:

$$P_{car}(o,d) = dist(o,d).K \tag{2}$$

Coalition Cost. The first step to obtain the travel cost for a set of riders (i.e., a coalition) $S \subseteq A$ is to calculate the central pick-up and drop-off coordinates as explained in Section 2.2. From there, one can obtain the walking and car costs using Equations 1 and 2 above. The total cost of the trip for this coalition (i.e., the characteristic function of the ridesharing game) is defined as the sum of walking and riding costs, as expressed in Equation 3. For individuals traveling alone, the cost of travel is the minimum between the taxi and walking, while for riders in a coalition, we assume that they choose to share a car instead of walking to their destination.

$$C(S) = \begin{cases} min\{VoT(o_i,d_i), P_{car}(o_i,d_i)\}, & \text{if } |S| = 1 \\ P_{car}(\bar{o},\bar{d}) + \sum_{i\in S}[VoT(o_i,\bar{o}) + VoT(d_i,\bar{d})], & \text{otherwise} \end{cases} \tag{3}$$

where $(\bar{o},\bar{d})$ are the pick-up and drop-off coordinates of the shared car, calculated as the geometric median of the riders' origin and destination coordinates respectively and (o_i,d_i) are the origin and destination coordinates of individual i. Lastly, from Definition 2, the total cost of a coalition structure $\mathbb{C}$ is simply the sum of the costs of its component coalitions:

[9] Note that for simplicity, we assume that the O-D points can lie anywhere on the Cartesian plane. In a real system, one could use the nearest feasible points on the actual road network.

$$\mathbb{C}_{cost}(\mathbb{C}) = \sum_{S \in \mathbb{C}} C(S) \tag{4}$$

3 Optimal Coalition Structure

Next, we explain how to find the optimal coalition structure that minimises the total travel cost given by Equation 4.

3.1 Dimensionality Reduction

The problem of searching for optimal structures is generally computationally intractable because the number of different possible coalitions is exponential in the number of coalition members (since there are 2^n possible coalitions for a set of n riders)[10]. Additionally, ridesharing presents the complexity of a high volume of trip requests in a short period of time, especially when considering the trips demand in densely populated areas. To reduce the dimensionality of the problem, it is common within the ridesharing literature to introduce a series of constraints and heuristics, composed of certain spatio-temporal rules that make the coalition generation problem tractable [15]. Following those approaches, we define a number of constraints that help reduce the search space. The first constraint is a spatial constraint guaranteeing that only riders within a certain radius are considered for riding together (i.e., for forming a coalition), which leads to the spatial clustering of trips.

Definition 3 (ϵ-Feasible Coalition). *In a ridesharing game (A, C, M), we say a coalition $S \subseteq A$ is ϵ-feasible if all of its members are interior points of a closed ball $N_\epsilon(p) = \{q \in \mathbb{R}^4 \mid dist(p, q) \leq \epsilon\}$, where the $dist(\cdot)$ function is the Euclidean distance between points in a four-dimensional space, $q = \{[o_i^\top, d_i^\top]^\top : o_i \in M(a_i), d_i \in M(a_i)\}$ where o_i, d_i are the individual's origin and destination vectors respectively, ϵ is an arbitrary radius in 4D, and p is an arbitrary point in this space.*

Definition 4 (ϵ-Feasible Coalition Structures). *By $\mathscr{C}^\epsilon$, we denote the set of coalition structures that only contain ϵ-feasible coalitions.*

To exemplify definitions 3 and 4, consider Figure 1 where riders $\{A, B, C\}$ have close origin and destination points, up to a walking radius of 800m. Then they will be allocated into the same ϵ-feasible coalition structure. Any other rider farther apart, will be placed on a different coalition structure.

Constraint 1 (Car capacity) *We assume the maximum capacity of a car is four passengers.*

The next definition formalizes the idea that the optimal arrangement of riders in cars is the one that minimizes the travel cost across riders.

[10] The number of coalition structures is given by the Bell number [1], while the number of possible coalitions is 2^n.

Definition 5 (Optimal Coalition Structure of a Ridesharing Game). *Within the set of ϵ-feasible coalitions structures $\mathscr{C}^{\epsilon}$, satisfying the car capacity constraint, the optimal coalition structure is* $\mathbb{C}^{*} = \arg\min_{\mathbb{C}\in\mathscr{C}^{\epsilon}}\{\mathbb{C}_{cost}(\mathbb{C})\}$.

The next section describes the steps to calculate the optimal coalition structure of a ridesharing game.

3.2 Optimal Coalition Structures in a Ridesharing Game

The objective is to find the $\mathscr{C}^{\epsilon}$ coalition structure that minimizes the total traveling cost. Below we outline the procedure to find such structure and provide a concrete example of the steps.

- The first step is to group riders by proximity on their origin and destination coordinates and obtain the ϵ-feasible coalitions as in Definition 3. This can be achieved with clustering algorithms such as DBSCAN [7].
- The next step is to calculate all the possible coalition structures among the riders of a cluster. One can use exhaustive search or other coalition formation algorithms (see for example [17]). Regardless of the partition algorithm, the idea is to input the list of individuals in a cluster and return a set of coalition structures alongside their total riding cost, as in Equation 4.
- The last step is to follow Definition 5 and select the coalition structure with the minimum travel cost per cluster. If there is more than one coalition structure with the minimum cost, we (randomly) select one with the lowest number of coalitions, as this means more members per coalition.

Example 1. Consider for example riders A, B, and C on Figure 1 and note that they are within 800m of each other's origin-destination, falling into the same cluster. The first step is to calculate the travel cost of all possible coalition structures in the cluster. Following Definition 2, there are five possible coalition structures for three riders $\{A,B,C\}$, $\{\{A,B\},\{C\}\}$, $\{\{A,C\},\{B\}\}$, $\{\{B,C\},\{A\}\}$, $\{\{A\},\{B\},\{C\}\}$. On the left-side panel of Figure 1 we present one example of such structures for the grand coalition $\{A,B,C\}$ together with its pick-up and drop-off points, calculated using the geometric median. The right-side panel of the same figure presents the coalition structure $\{\{A,B\},\{C\}\}$ with its respective pick up-and drop-off points. We then calculate the total travel cost of each coalition structure following Equations 3 and 4. For example, let the total travel cost of each of the five coalition structure be: 50, 55, 60, 65, 70, then following Definition 5, the optimal coalition structure is $\{A,B,C\}$ with a travel cost of 50.

4 Cost Allocation Problem

After obtaining the socially optimal coalition structure C^{*}, the next question is how to split the cost of the car across the coalition members. This aspect is important, as it determines whether individuals would prefer ridesharing over traveling alone and, thus, whether a coalition can be formed. The goal is to calculate a payoff vector $\vec{c}$, representing the share of the car cost allocated to riders of $S, \forall S \in C^*$. Following [16], we define a cost allocation as follows:

Definition 6. *A cost allocation is a vector* $\vec{c} = (c_1, ..., c_n)$*, such that* $c_i \geq 0$*. Moreover, if* $\sum_{i=1}^{n} c_i = P_{car}$*, then* $\vec{c}$ *is efficient.*

In the context of ridesharing, an allocation method is individually rational if the travel cost of joining a coalition, as given by Equation 3 (with its corresponding walking cost plus the share of the car fare) is less or equal than traveling alone (which does not involve walking).

Definition 7. *A cost allocation* $\vec{c}$ *is individually rational if* $c_{total_i} \leq C(\{i\})$*.*

Here, C is the coalition cost from Equation 3 and c_{total_i} is the total travel cost composed by the share of the car cost (Equation 2) plus the walking cost (Equation 1). The challenge in distributing the cost of the car in ridesharing games is that the walking part of the cost is a non-transferable utility (NTU) component and the only part of the cost that has transferable utilities is the cost of the car. Our method resolves this by linking the individual share of the car with the walking effort, as explained in the next section.

4.1 Inversely-Proportional Cost Allocation

Once the optimal coalition has been determined and the central O-D points have been established, the only difference on the cost contribution across riders is their walking effort. As such, our method seeks to compensate riders for this cost in an equitable way.

To allocate P_{car}, we start by splitting the total amount in two, as: $\gamma P_{car} + (1 - \gamma) P_{car}$, with $0 < \gamma < 1$. The first quantity, γP_{car}, can be seen as the flag-fall price and will be equally allocated across all drivers[11]. The remaining amount, $(1-\gamma)P_{car}$, will be distributed proportionally to the walking effort done by each rider, in what we call the inversely-proportional cost allocation method. The idea of the inversely-proportional cost allocation method is to translate the walking effort into a payment reduction, so those who walked relatively more in the coalition will pay relatively less of the car share. Let $w_i = \frac{VoT_i}{\sum_{i \in S} VoT_i}$ be the proportion of walking done by rider a_i after joining coalition S. Let w'_i be the proportion of the car price paid by rider a_i. The goal is to find a proportional split such that the usual condition holds:

$$\sum_{a_i \in S} w'_i = 1 \tag{5}$$

As an additional condition, we want the proportion of the car price paid by rider a_i to be inversely proportional to the walking effort, i.e.,[12]:

$$w'_i = \frac{1}{w_i} \cdot \kappa \tag{6}$$

[11] A ‘flag fall’ is a fixed initial charge incurred at the start of a taxi journey, as part of the overall fare.

[12] Although a zero walking cost is unlikely, in our implementation we have added a small constant of 0.01% in the denominator to avoid the division by zero.

where the κ term is a re-scaling term needed for Equation 5 to hold. To determine κ we plug Equation 6 into Equation 5, obtaining $\kappa = \frac{1}{\sum \frac{1}{w_i}}$. Finally, the payment allocation of the car price by each member a_i of coalition S is:

$$c_i = (1 - \gamma) \cdot P_{car}(\bar{o}, \bar{d}) \cdot w'_i \tag{7}$$

The total cost of ridesharing for an individual a_i in coalition S is the summation of the car cost and the walking cost as follows:

$$c_{total_i} = \frac{1}{|S|} \cdot \gamma \cdot P_{car}(\bar{o}, \bar{d}) + (1 - \gamma) \cdot P_{car}(\bar{o}, \bar{d}) \cdot w'_i + VoT_i(o_i, d_i) \tag{8}$$

5 Formal Evaluation and Properties

This section analyses the properties of the inversely-proportional cost allocation algorithm. Equation 5 guarantees that the inverse-walking payment allocation is efficient. We say that this cost allocation method is equitable in the sense that it compensates individuals for their walking effort, which is seen as the cost of entry into a coalition. We outline two properties of our method. (a) *Equal treatment*: if two individuals have walked in the same proportion, their cost allocation will be equal (i.e. if $w_i = w_j$ then $w'_i = w'_j$) and also (b) *Proportionality to walking effort*: if the VoT of individual i is p times more than individual j, then $w'_i = \frac{1}{p} w'_j$. The formal results for the desired properties of our approach are further described in Section 5.1.

5.1 Conditions for the Individual Rationality of the Inversely-Proportional Method

This section analyzes the conditions that have to be met for the socially-optimal coalition to be individually rational. In particular, the method presents stronger guarantees when individuals are closer in origin and destination and the distance travelled by car justifies the walking effort. Below we formalize this idea.

Theorem 1. *In a ridesharing game, the inversely-proportional cost allocation method guarantees individual rationality if* $dist(\bar{o}, \bar{d}) \geq \frac{2}{\gamma}(\epsilon + \epsilon^{\alpha})$.

Proof. We derive the conditions under which the proposed allocation method is individually rational. From Definition 7, a payment allocation c_{total_i} is individually rational if: $c_{total_i} \leq C(\{i\})$. Using Eq. 8, and assuming the car price is 1, the cost equations are:

$$\frac{1}{|S|} \cdot \gamma \cdot dist(\bar{o}, \bar{d}) + (1 - \gamma) \cdot dist(\bar{o}, \bar{d}) \cdot w'_i + \left[VoT(o_i, \bar{o}) + VoT(d_i, \bar{d})\right] \leq dist(o_i, d_i)$$

Expanding the VoT by its expression in Equations 1, we obtain:
$\frac{1}{|S|} \cdot \gamma . dist(\bar{o}, \bar{d}) + (1 - \gamma) . dist(\bar{o}, \bar{d}) \cdot w'_i + \left[dist^{\alpha}(o_i, \bar{o}) + dist^{\alpha}(d_i, \bar{d})\right] \leq dist(o_i, d_i)$.
The strategy is to write the LHS as a function of the geometric median and find an upper bound depending on the choice of ϵ (which acts a universal constant arbitrarily

chosen). By the spatial constraint 3, the *total* walking distance of the riders in S is bounded by ϵ (for example, individuals are required to walk no more than 800m in total, considering the sum of the walking at origin and destination). Then, the VoT is upper-bounded by ϵ^{α}, therefore:

$$\frac{1}{|S|}.\gamma.dist(\bar{o},\bar{d}) + (1-\gamma).dist(\bar{o},\bar{d}) \cdot w'_i + \left[dist^{\alpha}(o_i,\bar{o}) + dist^{\alpha}(d_i,\bar{d})\right] \leq \frac{1}{|S|}.\gamma.dist(\bar{o},\bar{d}) + (1-\gamma).dist(\bar{o},\bar{d}) \cdot w'_i + \epsilon^{\alpha}$$

Rearranging terms and noting that $|S| \geq 2$ and $w'_i \leq 1$, the upper bound on the LHS is: $dist(\bar{o},\bar{d}) \cdot \left(\frac{1}{|S|} \cdot \gamma \cdot + (1-\gamma) \cdot w'_i\right) + \epsilon^{\alpha} \leq dist(\bar{o},\bar{d}) \cdot \left(1 - \frac{1}{2} \cdot \gamma\right) + \epsilon^{\alpha}$.

The RHS is lower-bounded by using the triangle inequality: $dist(\bar{o},\bar{d}) \leq d(\bar{o},o_i) + d(d_i,\bar{d}) + d(o_i,d_i)$. The lower bound for the RHS is: $dist(\bar{o},\bar{d}) - \epsilon \leq dist(o_i,d_i)$.

Rearranging the LHS and RHS, and noting that $|S| \geq 2$ we obtain $dist(\bar{o},\bar{d}) \cdot \left(1 - \frac{1}{2} \cdot \gamma\right) + \epsilon^{\alpha} \leq dist(\bar{o},\bar{d}) - \epsilon$. Equivalently, $dist(\bar{o},\bar{d}) \geq \frac{2}{\gamma}\left(\epsilon + \epsilon^{\alpha}\right)$. □

The theorem shows that the inversely-proportional cost allocation scheme achieves individual rationality in settings with maximum utilisation of car capacity, relatively small epsilon and relatively small walking distance (as compared with the trip's length) and overall, the distance of the shared car compensates for the walking cost. The next section provides an evaluation analysis of the inversely-proportional allocation method.

6 Evaluation of the Cost Allocation Method

The aim of this section is to study the performance of the proposed cost allocation method under different configuration settings and compare it against established methods such as an even split and the Shapley value. Section 6.1 uses simulated trip data to ablate the performance of the proposed cost allocation method under different value of time settings. Section 6.2 studies the performance of the method under different trip lengths using real New York City taxi data, [26] as used by [9]. We compared the inversely-proportional cost allocation method to the Shapley value. The Shapley value ϕ_i of rider i in a ridehsaring game (S, C, M), which takes place among riders in S who are sharing a ride, is i's average marginal contribution to the game. Formally, it is specified as $\phi_i(C)$ which is equal to $\sum_{T \subseteq S \setminus \{i\}} \frac{t!(s-t-1)!}{s!}(C(T \cup \{i\}) - C(T))$ where t and s represent the cardinality of T and S, respectively [16]. Due to the non transferable part of ridesharing games (i.e., the walking costs), there are several alternative ways to obtain the share of the car from the Shapley value. The first variant takes the total cost $C(S)$ of the coalition as input of the ϕ function and subtracts the individual cost of walking to arrive to the car cost allocation.

$$\textbf{Shapley-total-input: } c_{car_i} = \phi_i(C) - \left[VoT(o_i,\bar{o}) + VoT(d_i,\bar{d})\right] \tag{9}$$

An alternative way of using the Shapley value to split the car cost is to use the coalition's car price as input (i.e., $C(S)$ without considering the walking cost). This variation is calculated as:

$$\textbf{Shapley-car-input: } c_{car_i} = \phi_i \left(P_{car} \right) \tag{10}$$

Lastly, following the rationale from the inversely-proportional method, the Shapley value (calculated with walking and car costs as inputs) can be used to weight the car cost as below:

$$\textbf{Shapley-weighted: } c_{car_i} = P_{car} \cdot \frac{\phi_i(C)}{C(S)} \tag{11}$$

The next sections provides further detail on the numerical experiments and their results.

6.1 Simulated Network

The first experiment analyzes the equitability of different cost allocation methods. For a given optimal coalition structure, and given car trip, we study how each method compensates riders for their walking effort. Table 1 shows the car split for the trips depicted in Figure 1, on its left panel, Figure 2 shows the optimal coalition structure where $\{A, B\}$ rideshare together and $\{C\}$ travels alone (thus, no walking). In the case of a two-person coalition, the central O-D coordinates (pick-up and drop-off points) are half-way between both members $\{A, B\}$; therefore, the walking cost is the same for both. Since each coalition member walks the same amount of time and they travel the same car distance (as there is a single stop), it is expected that both pay an equal share of the car price. We see that the inversely-proportional allocation method yields an equal payment for $\{A, B\}$ given equal travel costs (i.e., each pays 50% of the trip) which is not the case with any of the Shapley variants. The reason for this difference is that Shapley responds to different desiderata, as it determines the cost allocation based on the individual trip distance, which is more applicable in a multiple-stop taxi problem with no walking cost (see for example, the taxi problem in [20]).

On Table 3, the optimal coalition structure is for $\{A, B, C\}$ to rideshare together. Our desideratum of equitability seeks to compensate B for its walking cost, since it has to bear the most walking effort (48.47%), then C has walked the less (12.44%) so we expect this will be reflected in a higher percentage of the car share, In other words, the walking effort is ordered as: $VoT_B > VoT_A > VoT_C$ and the car cost allocation given by the inversely proportional method is ordered as: $c_C > c_A > c_B$, as expected. None of the Shapley methods have such ordering in the car cost allocation, and thus, they are not equitable.

Next we compare the performance of the different allocation methods with respect to the individual rationality property 7 (i.e., the individual cost of riding alone versus the cost of ridesharing) for a three-rider coalitions. On this particular case, we have increased the convexity of the VoT from *Weak* $\alpha = 1.008$ to *Medium* $\alpha = 1.21$, to *Strong* $\alpha = 1.45$ convexity, to increment the monetary value of the walking effort. Our results show that as the cost of walking increases, the inversely-proportional method compensates individuals for the walking cost, holding the individual rationality property for the three riders until the *Medium* VoT convexity, whereas when the cost of walking is too onerous, Shapley methods become less applicable to ridesharing games, as they

Table 1: Car cost split under different methodologies. The share of car cost is expressed as percentage of the total fare.

Table 2: A, B share a car, C travels alone

	A	B	C
Walking cost	50%	50%	0%
Inv-proportional	50%	50%	100%
Even split	50%	50%	100%
Shapley-tot-input	52.07%	47.93%	100%
Shapley-car	52.07%	47.93%	100%
Shapley-weighted	51.31%	48.69%	100%

Table 3: Riders A, B, C share a car

	A	B	C
Walking cost	39.09%	48.47%	12.44 %
Inv-proportional	20.87%	17.15%	61.98%
Even split	33%	33%	33%
Shapley-tot-input	31.63%	33.89%	34.48%
Shapley-car	31.82%	34.36%	33.82%
Shapley-weighted	32.06%	34.72%	33.23%

are unable to compensate for the NTU cost of those who walk more in the car share. For all the Shapley methods, one out of three riders had car costs that were not individually rational when the cost of walking is beyond weakly convex.

On the next experiment, we simulated $10K$ random coordinates uniformly in the square $[0, 300] \times [0, 300]$, as shown on Table 4. We run the different cost allocation methods under varied VoT settings to show the sensitivity of the individual rationality property to the convexity of the walking function. The configuration for the experiment is: car price K = 1, $\gamma = 0.05$ and the DBSCAN's $\epsilon = 25$.[13] The Euclidean distance was used across all experiments. For each run, we calculated the percentage of riders who satisfied the individual rationality condition under each method. The results of our simulations are as expected, the higher the cost of walking, the more costly it is to join a coalition, and thus, the individual rationality decreases across all methods. The said table shows that all methods perform similarly; however, the individual rationality decreases on the Shapley-car and Shapley-weighted at much faster pace than the inversely-proportional method. Notwithstanding the Shapley's onerous computational time, which will be analyzed on section 6.3.

6.2 NYC Network

The NYC dataset allows us to test the performance of the cost allocation methods under different trip lengths. The length of a trip plays a relevant role in the determination of the individual rationality of the inversely-proportional method, as shown in the proof in Section 5.1 which links the individual rationality condition to the length of the trip with the ϵ and γ parameters of the walking cost. Given a fixed ϵ of $800m$, when the trip length increases, the proportion of car cost over the walking cost increases as well, making the cost allocation individually rational. As shown on Table 5 the scenario 'NYC' includes a random selection of 15,000 taxi trips across all boroughs. This scenario contains a mix of trips leaning towards shorter trips. The 'Manhattan' scenario contains medium distance trips from lower downtown to the Central Park area, which are the common

[13] The choice of ϵ in the case of random points is proportional to the point's standard deviation.

Table 4: Percentage of individually rational riders for different walking costs. Mean over 10K random coordinates.

	Convexity of the VoT function		
Allocation method	**Weak**	**Medium**	**Strong**
Inv-proportional	$99.59 \pm 0.1\%$	$99.09 \pm 0.19\%$	$98.2 \pm 0.26\%$
Even split	$99.53 \pm 0.1\%$	$98.36 \pm 0.23\%$	$96.31 \pm 0.35\%$
Shapley-tot-input	$100 \pm 0\%$	$99.87 \pm 0.08\%$	$99.84 \pm 0.08\%$
Shapley-car-input	$99.87 \pm 0.1\%$	$98.94 \pm 0.20\%$	$97.67 \pm 0.28\%$
Shapley-weighted	$99.72 \pm 0.1\%$	$98.16 \pm 0.27\%$	$96.12 \pm 0.37\%$

trips expected during rush hours. As shown in the said table, for the given ϵ an average trip of 7 miles is enough to guarantee all methods to be individually rational. The last scenario 'Airport trips' includes long-distance trip requests from Manhattan's Midtown to Newark and LaGuardia airports. As the average trip length increases, all methods are individually rational. The configuration for the experiment for all the three scenarios is $\alpha = 1.0085$, which corresponds to peak of the VoT at the $800m$ mark. The price of car = 1 and $\gamma = 0.05$ for the portion of the price related to the flag-fall.

Table 5: Average of individual rationality per rider across different methodologies for 15K NYC trips. In parentheses is the 95% confidence interval.

	Average trip length		
Allocation method	**Entire NYC (4mi avg trip)**	**Manhattan only (7mi avg trip)**	**Airport trips (12mi avg trip)**
Inv-proportional	$99.07 \pm 0.04\%$	$100 \pm 0\%$	$100 \pm 0\%$
Even split	$99.0 \pm 0.05\%$	$100 \pm 0\%$	$100 \pm 0\%$
Shapley-tot-input	$99.89 \pm 0.02\%$	$100 \pm 0\%$	$100 \pm 0\%$
Shapley-car-input	$99.71 \pm 0.02\%$	$100 \pm 0\%$	$100 \pm 0\%$
Shapley-weighted	$99.32 \pm 0.03\%$	$100 \pm 0\%$	$100 \pm 0\%$

6.3 Time Complexity Comparison of the Cost Allocation Method

The inversely-proportional cost allocation method is linear ($O(n)$) in the number of agents, since their inputs are the individuals' value of time (which is $O(n)$ as per Equation 1), the rescaling term κ and the inversely proportional payment terms which are also linear in the number of riders as can be seen from Equations 5 - 8. This is a more tractable computational time as compared with Shapley's time complexity, which is exponential, as it requires the calculation of the characteristic function of all sub sets within a coalition.

Table 6 shows the result of an example study on the running time comparison for different number of riders between the inversely proportional cost allocation method and the Shapley-total-input. As it can be seen, for a three-person coalition our algorithm takes 20% less time to run, and as the number of riders increases, the time difference between both methods increases as well [14]. Building on these preliminary results, we aim for further investigations to establish time-savings in different settings and against other forms of Shapley-based cost allocation.

Table 6: Average *time saving* between the inversely proportional cost allocation method and Shapley *total input* method for different number of riders.

Number of riders	**n = 3**	**n=10**	**n=20**
Inv-proportional (in secs)	0.0010	0.00081	0.0014
Shapley-total-input (in secs)	0.0014	0.0028	0.026
Inv-prop vs Shapley-total-input	20%	71%	94%

7 Conclusions and Future Work

In this work we show that the socially-optimal ridesharing problem can be modelled as a coalition formation problem and solved within the framework of game theory. We propose a method to determine the optimal pick-up and drop-off points so that the distance walked by all coalition members is minimal. We describe a methodology to include the cost of walking as part of the coalition's cost together with a methodology to allocate the car cost among coalition members, called the inversely-proportional method. This method is computationally tractable, yet effective on its intended application. We provide conditions under which individual rationality is guaranteed and tested it using randomly generated data as well as the NYC taxi dataset. We show that this method is equitable, as it allocates costs proportionally to the walking effort. The equitability, individual rationality (under conditions) and computational tractability make this method a valuable alternative to the traditional Shapley value, which does not hold the said conditions. Table 7 summarizes the performance of different methods across the experiments performed. The individual rationality condition on the second column is guaranteed under certain conditions for the presented methods, for Shapley it is the non-empty core condition [16] and for the inversely-proportional method it is the condition from Section 5.1

In this work we compared the inversely-proportional cost allocation method against Shapley, as an established method. However, we have not investigated the stability of the coalitions formed, since the focus was put on the equitability of the method. For future work, we would like to compare against additional cost allocation methods, and

[14] Results calculated on a $2.3GHz$ Quad-Core Intel Core $i7$.

Table 7: Comparison of properties across different allocation methods.

Allocation Method	Efficiency	Indiv. Rationality	Equitability	Time Complexity
Inv-proportional	YES	Conditional	YES	Linear
Even allocation	YES	NO	YES	Linear
Shapley-total-input	YES	Conditional	NO	Exponential
Shapley-car-input	YES	Conditional	NO	Exponential
Shapley-weighted	YES	Conditional	NO	Exponential

test stability concepts like the core or kernel. One extension could also be to include multiple stops in the car route.

Data access statement. This study was a re-analysis of data that are publicly available from the *TLC Trip Record Data* dataset [26].

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