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Incentive ratio: A game theoretical analysis of market equilibria ☆

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A B S T R A C T

In a Fisher market, the market maker sells m products to n potential agents. The agents submit their utility functions and money endowments to the market maker, who, upon receiving submitted information, derives market equilibrium prices and allocations of the products. Agents are self-interested entities who wish to maximize their utility, and they may misreport their private information for this purpose. The *incentive ratio* characterizes the extent to which strategic plays can increase an agent's utility. While agents do benefit by misreporting their private information, we show that the ratio of improvement by a unilateral strategic play is no more than two in markets with gross substitute utilities for the agents. Moreover, it can be pinned down to $e^{1/e} \approx 1.445$ in Cobb-Douglas markets. For the Leontief markets in which products are complementary, we show that the incentive ratio is at most two as well.

1. Introduction

The Internet and world wide web have created a possibility for buyers and sellers to meet at a marketplace where pricing and allocations are determined more efficiently and effectively than ever before. Market equilibrium, a vital notion in classic economic theory, ensures optimum fairness and efficiency and has become a paradigm for practical applications. Understanding its properties and computation has been one of the central questions in computational economics and algorithmic game theory. In this paper, we consider the Fisher market model [8], in which a market maker sells m divisible items of unit supply each to n potential buyers, each with an initially endowed amount of cash e_i and with a utility function $u_i : [0, 1]^m \to \mathbb{R}$. At market equilibrium, all products are sold out, and buyers spend all their endowment; most importantly, the set of items purchased by each buyer maximizes his utility for the given equilibrium prices constrained by his initial endowment. It is well-known that a market equilibrium always exist given mild assumptions on the utility functions [4,8].

Notably, a significant issue with the application of market equilibrium is that it does not take strategic behaviors of the participants into consideration: In a Fisher market, market equilibrium prices and the associated allocations, computed in

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terms of utility functions and money endowments, may change even if only one buyer changes its utility function or endowment. Hence, one may misreport his private information if it results in a favorable solution. Adsul et al. [1] observed this phenomenon in Fisher markets when buyers are equipped with linear utility functions. The following examples show such a utility gain caused by manipulations may occur in other utilities as well, including Cobb-Douglas and Leontief functions.

- (Manipulation in a Cobb-Douglas market). In a Cobb-Douglas market, there are two items and two buyers with endowments $e_1 = \frac{1}{2}, e_2 = \frac{1}{2}$ and utility functions $u_1(x, y) = x^{\frac{1}{4}} y^{\frac{3}{4}}, u_2(x, y) = x^{\frac{3}{4}} y^{\frac{1}{4}}$, respectively. When both buyers bid their utility functions and endowments truthfully, the equilibrium price is $\mathbf{p} = (\frac{1}{2}, \frac{1}{2})$, and the equilibrium allocations are $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{3}{4}, \frac{1}{4})$; their utilities are $u_1 = u_2 = (\frac{1}{4})^{\frac{1}{4}} (\frac{3}{4})^{\frac{3}{4}} \approx 0.570$. If the first buyer strategically reports $u'_1(x, y) = x^{\frac{1}{2}} y^{\frac{1}{2}}$, then the equilibrium price is $\mathbf{p}' = (\frac{5}{8}, \frac{3}{8})$, and the equilibrium allocations are $(\frac{2}{5}, \frac{2}{3})$ and $(\frac{3}{5}, \frac{1}{3})$; their utilities are $u'_1 = (\frac{2}{5})^{\frac{1}{4}} (\frac{2}{3})^{\frac{3}{4}} \approx 0.586$ and $u'_2 = (\frac{3}{5})^{\frac{3}{4}} (\frac{1}{3})^{\frac{1}{4}}$. Hence $u'_1 > u_1$ and the first buyer gets a strictly larger utility.
- (Manipulation in a Leontief market). In a Leontief market, there are two items and two buyers with endowments $e_1 = \frac{2}{3}, e_2 = \frac{1}{3}$ and utility functions $u_1(x, y) = \min\left\{\frac{x}{2/3}, \frac{y}{1/3}\right\}, u_2(x, y) = \min\left\{\frac{x}{1/3}, \frac{y}{2/3}\right\}$, respectively. When both buyers bid their utility functions and endowments truthfully, then in an equilibrium output with price vector $\mathbf{p} = (1, 0)$ and allocations $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$, the utilities are $u_1 = 1$ and $u_2 = 1$. Now if the first buyer strategies in the market and reports $u'_1(x, y) = \min\left\{\frac{x}{5/9}, \frac{y}{4/9}\right\}$, price vector $\mathbf{p}' = (0, 1)$ and allocations $(\frac{5}{6}, \frac{2}{3})$ and $(\frac{1}{6}, \frac{1}{3})$ give an equilibrium for the new setting. Now the utilities are $u'_1 = \frac{5}{4}$ and $u'_2 = \frac{1}{2}$, where buyer 1 gets a strictly larger utility.

In a market, if buyers can increase their utility to a large extent, they would have a big incentive to do so, and the market will become volatile. For this reason, it is essential to characterize the extent to which strategic manipulations can increase utilities. The *incentive ratio* [13] was introduced to quantify agents' incentive to deviate from reporting their actual private information. Informally, it is the factor of the largest possible utility gain that an agent achieves by behaving strategically, given that all other agents have their strategies fixed.

We show the incentive ratio upper bounds when the buyers possess several different utility functions. These functions are typical in competitive equilibrium literature and are widely studied. Firstly, we show that the incentive ratio in a Fisher market is at most 2 if goods are *gross substitutes* from each buyer's standpoint. The *Weak Gross Substitute* (WGS) condition captures the case that the substitution parameter ρ, of the *Constant Elasticity of Substitution* (CES) functions, is non-negative. Our proofs in this scenario leverage the properties of WGS. Secondly, when buyers have Cobb-Douglas utility functions that correspond to the case that ρ approaches 0, we show that the incentive ratio is decreased to $e^{1/e} \approx 1.445$. Thirdly, when buyers have Leontief utility functions that correspond to the case that ρ approaches $-\infty$, we show that the incentive ratio is also at most 2. En route to prove this result, we establish reductions between large markets and small markets, in terms of capturing buyers' best response and dominant strategies. This equivalence of markets at different sizes conform to the fact that worst-case metrics rest with a single instance rather than the size of the inputs. This way, we narrow down the analysis to 2-buyer 2-item markets. All of our constant bounds are tight.

A small constant incentive ratio, in particular, 2 and 1.445 in our results, indicate that although truthfully revealing their private information is not a dominant strategy in Fisher markets, the agents' incentive to misreport is relatively small. There are two points to support this speculation. On the one hand, the definition of incentive ratio is on a worst-case sense. That is, the incentive ratio bounds are the strongest approximation guarantee, and they do not take into consideration the likelihood that the extreme cases happen. By looking at our tight bound instances, it is obvious that these extreme cases would rarely occur. On the other hand, our analysis is built upon the assumption that a buyer obtains complete information about other buyers' utility functions and budgets, and can correctly figure out its best strategy to manipulate the market. A buyer without complete information about the market and perfect rationality would not be able to achieve these maximum utility increments. In contrast, we illustrate an example where a buyer's incentive ratio could be unbounded if the utility function is piecewise linear.

We acknowledge that people may not completely agree with us on the above justification to introduce the solution concept of incentive ratio. But our proof reveals something mathematically true, and we believe it will come to have a role to play in practice in the economy. Facing the fact an agent may deviate to benefit its own economic interest, there is a future need to develop new theories of Economics for its cure.

Another approach in the field is examing a system's efficiency loss at equilibrium due to its agents' selfish behavior. For example, the Price of Anarchy (respectively, the Price of Stability) is very successful in measuring the ratio of the efficiency of the worst (the best) equilibrium versus the optimal centralized solution. However, the Price of Anarchy bounds could be large in some problems unless additional constraints are assumed in place. For example, Brânzei et al. [9] showed that the Price of Anarchy of the Fisher market game is bound by $\sqrt{n}$ or n for buyers with typical Constant Elasticity of Substitution utility functions. While it is impossible to root out selfish behavior in many scenarios, we propose approbating systems in which the users' gain by manipulation is bound by a reasonably small value. Owing to inherent costs for an individual to figure out a beneficial manipulation, rational individuals will behave truthfully, and hence the systems are accountable.

1.1. Related work

Eisenberg and Gale [22] introduced a convex program to capture market equilibria of Fisher markets with linear utilities. Their convex program can be solved in polynomial time using the ellipsoid algorithm [25] and interior point algorithms [34]. Devanur et al. [20] gave the first combinatorial polynomial time algorithm for computing a Fisher market equilibrium with linear utility functions. The first strongly polynomial time algorithm for this problem was recently given by Orlin [28]. Codenotti and Varadarajan [16] modeled the Leontief market equilibrium problem as a concave maximization problem; Gary et al. [23] gave a polynomial time algorithm to compute an equilibrium for Leontief utilities. For Cobb-Douglas market, Eaves [21] gave the necessary and sufficient conditions for existence of the market equilibrium, and gave an algorithm to compute a market equilibrium in polynomial time. Other computational studies on different market equilibrium models and utilities can be found in, e.g., [18,16,10,25,19] and the references within.

Agents' incentives for deviating from price-taking behavior are shown to be decreasing as the size of a market grows [30,24,29,5]. Saraiva [31] presents evidence for vanishing manipulability in large stable mechanisms in two folds. On the one hand, convergence toward truth-telling in stable mechanisms can be achieved much faster if colleges' preferences are independently drawn from a uniform distribution. On the other, the results can be applied to competitive environments in which virtually all vacancies end up being filled. In contrast with considering the deviation incentive in the scope of large economies, we bound the incentive under a worst-case analysis framework. Our analysis, therefore, provides a theoretical guarantee on the incentive to manipulate in a market from a different angle. Recently, incentive ratio has been adapted to characterize the extent to which strategic agents can increase their utility in the Probabilistic Serial mechanism [33], the proportional sharing mechanism [15], and bandwidth sharing [14].

A related concept, *price of anarchy* [26], together with several variants (e.g., *price of stability* [2]), models the loss of social efficiency due to self-interested behavior. Cole and Tao [17] show that under suitable conditions, the PoA tends to 1 as the market size increases in Large Fisher markets and Large Walrasian auctions. In [9], the authors proved Price of Anarchy bounds for the *Fisher market game* when the utility functions of buyers are linear, Leontief, and Cobb-Douglas functions, respectively. We note, though, that the incentive ratio does not deal with social welfare but that of individual optimality. It is close to the approximate market equilibrium price introduced by [18], in which every individual achieves a solution within a constant factor from its optimum under the given price. In both concepts, individuals do not achieve their optimum but are bounded by a constant factor away. However, Deng et al. [18] focuses on computational complexity while we consider individual manipulations by strategic plays.

Organization. In Section 2, we introduce the setting, the incentive ratio, and the notations that will be used throughout the paper. In Section 3, 4, and 5, we consider WGS, Cobb-Douglas, and Leontief utility markets and derive the matching incentive ratios, respectively. We conclude in Section 7.

2. Preliminaries

In a Fisher market, there are n buyers and m divisible goods (items, interchangeably) of unit quantity each. We use $[n] = \{1, 2, \dots, n\}$ and $[m] = \{1, 2, \dots, m\}$ to denote the set of buyers and items, respectively. Each buyer has an initial cash endowment e_i, which is normalized to be $\sum_{i\in[n]} e_i = 1$ and a utility function $u_i : [0, 1]^m \to \mathbb{R}$. Note that budget normalization does not affect market equilibrium computation but facilitates our proofs for buyers' utility gain as a ratio. In an allocation $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{im}) \in [0, 1]^m$, x_{ij} is the allocation of item j to buyer i and $u_i(\mathbf{x}_i)$ denotes the utility of buyer i.

An outcome of the market is represented by a tuple $(\mathbf{p}, \mathbf{x})$, where $\mathbf{p} = (p_1, p_2, \dots, p_m)$ is a price vector of all items and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ is an allocation vector of all buyers. We say that $\mathbf{x}_i^*$ is an optimal allocation for buyer i with respect to a price vector $\mathbf{p}$ if $\mathbf{x}_i^*$ maximizes his utility function $u_i(\mathbf{x})$ subject to the endowment constraint $\mathbf{p} \cdot \mathbf{x} \le e_i$. An outcome is a *market equilibrium* if the following two conditions hold.

(1) Market clearance: All items are sold out and all cash endowments are spent, i.e., $\sum_{i\in[n]} x_{ij} = 1$, $\forall j$, and $\sum_{j\in[m]} p_j = \sum_{i\in[n]} e_i = 1$.
(2) Individual optimality: The market allocation $\mathbf{x}_i$ is an optimal allocation for each buyer i with respect to the price vector $\mathbf{p}$.

2.1. Utility functions

We now review some standard utility functions and properties.

Definition 2.1 *(Concavity).* A utility function $u(\cdot)$ is *concave* if for any $\mathbf{x}, \mathbf{x}' \in [0, 1]^m$ and for any $t \in [0, 1]$,

$$u\big(t \cdot \mathbf{x} + (1 - t) \cdot \mathbf{x}'\big) \ge t \cdot u(\mathbf{x}) + (1 - t) \cdot u(\mathbf{x}')$$

Concavity is one of the most natural assumptions for utility functions, on which the seminal existence result of a market equilibrium is established [3]. In the following, we focus on concave utility functions.

Definition 2.2 *(Demand set).* For a given price vector $\mathbf{p}$, the demand set of buyer i under its initial cash endowment e_i, denoted by $D_i(\mathbf{p}, e_i)$, is the set of all optimal allocations of the following program:

$$\begin{aligned} \max \quad & u_i(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{p} \cdot \mathbf{x} \le e_i \end{aligned} \tag{1}$$

A nice characterization of the demand set $D_i(\mathbf{p}, e_i)$ states that the marginal bang-per-buck ratio of all purchased items (which measures the per-buck utility of the items) is the same in an optimal allocation. The following proposition relates different allocations to their respective utilities.

Proposition 2.1. *Let $\mathbf{y}_i = (y_{ij})_j \in D_i(\mathbf{p}, e_i)$ be an optimal allocation of buyer i with respect to a price vector $\mathbf{p}$. Then there exists a constant $c_i \ge 0$ such that for any other allocation $\mathbf{y}'_i$ for buyer i, we have*

$$\sum_{j \in [m]} c_i p_j (y_{ij} - y'_{ij}) \le u_i(\mathbf{y}_i) - u_i(\mathbf{y}'_i)$$

In particular, this implies that $c_i e_i \le u_i(\mathbf{y}_i)$.

Proof. Let $L(\mathbf{x}_i, \lambda) = -u_i(\mathbf{x}_i) + \lambda(\mathbf{p} \cdot \mathbf{x}_i - e_i)$ be the Lagrangian associated with the problem (1) and

$$g(\lambda) = \inf_{\mathbf{x}_i \in [0,1]^m} -u_i(\mathbf{x}_i) + \lambda(\mathbf{p} \cdot \mathbf{x}_i - e_i)$$

be the dual function. Since $u_i(\cdot)$ is concave and the constraint is affine, it is not hard to see that the problem (1) satisfies strong duality (see Chapter 5.2 in [7] for more details). Let c_i be the optimal dual solution of $g(\lambda)$ (note that c_i is positive). Then

$$-u_i(\mathbf{y}_i) = g(c_i) \le L(\mathbf{y}_i, c_i) \le -u_i(\mathbf{y}_i)$$

where the first equality follows from strong duality. Hence

$$L(\mathbf{y}_i, c_i) = g(c_i) \le L(\mathbf{y}'_i, c_i)$$

for any other allocation vector $\mathbf{y}'_i \in [0, 1]^m$. The proposition follows from rearranging the terms. □

In the above claim, c_i provides a lower bound on the bang-per-buck ratio of buyer i's optimal consumption, in which the *bang-per-buck* ratio is given by $u_i(\mathbf{y}_i)/e_i$.

One extensively studied class of utility functions is the Constant Elasticity of Substitution (CES) functions [32].

Definition 2.3 *(CES).* For each agent i, its CES utility function is represented as $u_i(x_i) = \left(\sum_{j=1}^m a_{ij} x_{ij}^\rho\right)^{\frac{1}{\rho}}$, where $-\infty < \rho < 1$ and $\rho \ne 0$, and $a_i = (a_{i1}, \ldots, a_{im}) \ge 0$ is a given vector associated with buyer i.

The CES utility functions allow us to model a wide range of realistic preferences of buyers, and have been shown to derive, in the limit, a number of special classes. In this paper, we will consider the following typical functions.

- Linear utilities (derived when $\rho \to 1$): $u_i(x_i) = \sum_{j \in [m]} a_{ij} x_{ij}$. Linear functions are arguably the most natural and well-studied utilities.
- Cobb-Douglas utilities (derived when $\rho \to 0$): $u_i(x_i) = \prod_{j \in [m]} x_{ij}^{a_{ij}}$, where $\sum_{j=1}^m a_{ij} = 1$, for all $i \in [n]$.
- Leontief utilities (derived when $\rho \to -\infty$): $u_i(x_i) = \min_j \left\{\frac{x_{ij}}{a_{ij}}\right\}$, where $a_{ij} > 0$. Leontief functions indicate perfect complementarity between different items.

For CES functions, an equilibrium allocation can be captured by the seminal Eisenberg-Gale convex program [22].

$$\begin{aligned} \max \quad & \sum_{i=1}^n e_i \log u_i \\ \text{s.t.} \quad & u_i = \left(\sum_{j=1}^m a_{ij} x_{ij}^\rho\right)^{\frac{1}{\rho}}, \quad \forall\, i \in [n] \\ & \sum_{i=1}^n x_{ij} \le 1, \quad \forall\, j \in [m] \\ & x_{ij} \ge 0, \quad \forall\, i \in [n], j \in [m] \end{aligned} \tag{2}$$

A class of utility functions that we are interested in this paper is that when they satisfy the following (weak) gross substitute property (note that there are several equivalent definitions; here we adopt the one in [27], page 70).

Definition 2.4 *(Weak) gross substitute.* A utility function $u_i(\cdot)$ is said to have the *weak gross substitute (*$\mathtt{WGS}$*)* property if the demand set $D_i(\mathbf{p}, e_i)$ of buyer i contains a unique allocation, that is, if the maximization of $u_i(\mathbf{x})$ subject to the endowment constraint, as a function of prices, yields demand functions $d_{ij}(\mathbf{p})$ from each buyer i to item j that is differentiable, and

$$\frac{\partial d_{ij}}{\partial p_k} \geq 0, \ \forall k \neq j$$

If the inequality is strict, $u(\cdot)$ is said to have the *gross substitute* property.

The WGS property ensures that when increasing the price of some items, buyers will not reduce their consumption of items whose prices are not changed. It is also not hard to see that equilibrium prices are unique (e.g. by applying Proposition 3.1). When the substitution parameter $0 < \rho < 1$, the CES utility functions satisfy this property.

2.2. Incentive ratio

As discussed in the Introduction, a buyer can manipulate his private utility function and endowment in order to obtain a larger utility in a market equilibrium. A natural question is then to what extend such benefits can be obtained from manipulations. To this end, we adopt the notion of incentive ratio introduced by [13,12].

In a market M, we consider a buyer $i \in [n]$ who attempts to improve his market allocation through misreporting his utility function and endowment. Specifically, in each of the following sections, a buyer can misreport a utility function within the class of WGS, Cobb-Douglas, and Leontief functions, respectively. For endowment, a buyer is bound by its real budget, so it may misreport a smaller budget, but it is not possible to overbid.[1] Let U_i denote the class of utility functions that i can feasibly report. Given a reported profile P, which consists of a vector of utility functions and a vector of endowments of all buyers, let $\mathbf{x}_i(P)$ denote the equilibrium allocation of buyer i. If $u_i(\cdot) \in U_i$ and e_i are buyer i's private true utility function and endowment, respectively, then the *incentive ratio* of buyer i in the market M is defined to be

$$\zeta_i^M = \sup_{u_{-i};\ e_{-i}} \ \sup_{u_i' \in U_i;\ e_i' \leq e_i} \frac{u_i\big(\mathbf{x}_i(u_i', e_i'; u_{-i}, e_{-i})\big)}{u_i\big(\mathbf{x}_i(u_i, e_i; u_{-i}, e_{-i})\big)},$$

in which we assume that one cannot report a budget beyond his true endowment, i.e., $e_i' \leq e_i$. In the above definition, the denominator is the utility of buyer i when he bids u_i and e_i truthfully, and the numerator is the largest possible utility of buyer i when he unilaterally changes his utility function and budget given all other buyers' profiles unchanged. The incentive ratio of buyer i is defined as the maximum ratio over all possible utilities and endowments of other buyers; it thus implies an upper bound on utility gain from manipulation.[2]

The incentive ratio of the market M with respect to a given class of utility functions is then defined as

$$\zeta^M = \max_{i \in [n]} \zeta_i^M.$$

3. Weak Gross substitute utility functions

In this section, we analyze the incentive ratio of Fisher markets when agents' utility functions satisfy the WGS condition. We first prove the following proposition, which says that if prices are changed, a buyer will spend no less money on those items whose prices are decreased.

Proposition 3.1. *Given a utility function $u(\cdot)$ that satisfies the WGS condition, let $\mathbf{p}$ and $\mathbf{p}'$ be two price vectors and $\mathbf{x} \in D(\mathbf{p}, e)$ and $\mathbf{x}' \in D(\mathbf{p}', e)$ be the corresponding optimal demands. Let $S = \{j \in [m] \mid p_j > p_j'\}$ denote the set of items whose prices are decreased from $\mathbf{p}$ to $\mathbf{p}'$. Then*

$$\sum_{j \in S} x_j p_j \leq \sum_{j \in S} x_j' p_j'$$

[1] In Section 3, our proofs for the case that a buyer has WGS utility function do not depend on the budget normalization. In Sections 4 and 5, we prove that a buyer would not benefit from reporting a smaller budget when it has Cobb-Douglas and Leontief utility functions, respectively.

[2] For some utility functions, equilibrium allocations and even obtained utilities may not be unique. This may lead to different true utilities for a given bid. The definition of incentive ratio here is the strongest in the sense that it bounds the largest possible utility gains in all possible equilibrium allocations at manipulation, which include, of course, a most favorable one.

and

$$\sum_{j\notin S} x'_j p'_j \le \sum_{j\notin S} x_j p_j.$$

Proof. Define a price vector $\mathbf{p}^* = (p^*_j)_j$ as $p^*_j = \min\{p_j, p'_j\}$ and consider an optimal demand $\mathbf{x}^* \in D(\mathbf{p}^*, e)$. Note here that $p^*_j = p'_j < p_j$, for all $j \in S$; otherwise $p^*_j = p_j$. That is, the prices of all item $j \in S$ decrease from p to p^* while others' prices remain the same. For all $j \in [m] \setminus S$, applying the WGS property on $u(\cdot)$ with prices $\mathbf{p}$ and $\mathbf{p}^*$, we have $x^*_j \le x_j$ and $p^*_j x^*_j = p_j x^*_j \le p_j x_j$. Due to the fact that $\sum_{j\in[m]} x_j p_j = e = \sum_{j\in[m]} x^*_j p^*_j$, we have

$$\sum_{j\in S} x^*_j p'_j = \sum_{j\in S} x^*_j p^*_j \ge \sum_{j\in S} x_j p_j.$$

Using the WGS condition again with prices $\mathbf{p}'$ and $\mathbf{p}^*$, we get $x^*_j \le x'_j$ for all $j \in S$. Combining these two inequalities, we have that

$$\sum_{j\in S} x_j p_j \le \sum_{j\in S} x'_j p'_j.$$

This completes the proof. □

Our main result of this section is the following.

Theorem 3.1. *For any market M with utility functions that satisfy the WGS condition, the incentive ratio of the market is at most 2, i.e.,*

$$\zeta^{WGS} \le 2.$$

Proof. Without loss of generality, we will consider buyer 1 and show that $\zeta^M_1 \le 2$. For any fixed bids of other buyers, let $(\mathbf{p}, \mathbf{x})$ be an equilibrium when buyer 1 bids truthfully and $(\mathbf{p}', \mathbf{x}')$ be an equilibrium when he bids strategically. It suffices to prove that $u_1(\mathbf{x}'_1) \le 2u_1(\mathbf{x}_1)$. Let S denote the set of items whose price are decreased from $\mathbf{p}$ to $\mathbf{p}'$, i.e., $S = \left\{ j \in [m] \mid p_j > p'_j \right\}$, and let $T = [m] \setminus S$ denote the set of items whose prices are not decreased.

By Proposition 2.1, there exists a constant c such that for buyer 1

$$\begin{aligned} u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1) &\le \sum_{j\in[m]} c \cdot p_j \cdot (x'_{1j} - x_{1j}) \\ &= \sum_{j\in T} c \cdot p_j \cdot (x'_{1j} - x_{1j}) + c \cdot \left(\sum_{j\in S} p_j (x'_{1j} - x_{1j}) \right) \end{aligned} \tag{3}$$

Note that $\mathbf{x}_i \in D_i(\mathbf{p}, e_i)$ and $\mathbf{x}'_i \in D_i(\mathbf{p}', e_i)$, for any buyer $i \ne 1$. Hence by Proposition 3.1, they spend more money on the items in S as their prices are decreased. Hence (aggregately),

$$\begin{aligned} \sum_{j\in S} p_j - \sum_{j\in S} x_{1j} p_j &= \sum_{i\ne 1} \sum_{j\in S} x_{ij} p_j \\ &\le \sum_{i\ne 1} \sum_{j\in S} x'_{ij} p'_j = \sum_{j\in S} p'_j - \sum_{j\in S} x'_{1j} p'_j \end{aligned}$$

which implies that

$$\sum_{j\in S} x_{1j} p_j \ge \sum_{j\in S} x'_{1j} p'_j + \sum_{j\in S} (p_j - p'_j).$$

Therefore,

$$\begin{aligned} \sum_{j\in S} x'_{1j} p_j - \sum_{j\in S} x_{1j} p_j &\le \sum_{j\in S} x'_{1j} p_j - \sum_{j\in S} x'_{1j} p'_j - \left(\sum_{j\in S} p_j - \sum_{j\in S} p'_j \right) \\ &\le \sum_{j\in S} (x'_{1j} - 1)(p_j - p'_j) \\ &\le 0 \end{aligned}$$

In addition, for any $j \in T$, since $p_j < p'_j$,

$$\begin{aligned}\sum_{j\in T} c\cdot p_j\cdot(x'_{1j}-x_{1j}) &\le \sum_{j\in T} c\cdot p_j x'_{1j} \le \sum_{j\in T} c\cdot p'_j x'_{1j}\\ &\le c\cdot e'_1 \le c\cdot e_1\\ &\le u_1(\mathbf{x}_1) \quad \text{(by Proposition 2.1)}\end{aligned}$$

Substituting the above inequalities to Equation (3) yields $u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1) \le u_1(\mathbf{x}_1)$. This completes the proof. □

Note that the theorem applies to all utility functions that satisfy the WGS condition. In addition, it holds even if different buyers have different utility functions. The following example shows that the ratio of 2 is tight.

Example 1 *(Tight incentive ratio).* There are two items and two buyers in the market. Each buyer has a linear utility function, given by the following table. It is not hard to see that the utility functions satisfy the WGS condition.

	u_{i1}	u_{i2}	endowment
buyer 1	εx_{11}	$(1-\varepsilon)x_{12}$	ε
buyer 2	$\varepsilon^2 x_{21}$	$1-\varepsilon^2 x_{22}$	$1-\varepsilon$

If both buyers bid truthfully, the equilibrium price is $\mathbf{p} = (\varepsilon, 1-\varepsilon)$ and buyer 1's utility is $u_1(\mathbf{x}_1) = \varepsilon$. If buyer 1 misreports his utility as $u'_1(\cdot) = u_2(\cdot)$ (and endowment $e'_1 = e_1$), the equilibrium price becomes $\mathbf{p}' = \left(\varepsilon^2, 1-\varepsilon^2\right)$, and the best equilibrium allocation for buyer 1 is $x'_1 = (1, \frac{\varepsilon}{1+\varepsilon})$ with utility $u'_1(\mathbf{x_1}') = \varepsilon + \frac{\varepsilon(1-\varepsilon)}{1+\varepsilon}$. Thus, the utility gain approaches 2 when ε is arbitrarily small.

3.1. WGS and homogeneous utility functions

In addition to the WGS condition, if the utility functions are also homogeneous, then we can obtain a more refined analysis and bound the incentive ratio by the manipulator's money endowment. We first introduce homogeneous utility functions.

Definition 3.1 *(Homogeneity).* A utility function $u(\mathbf{x})$ is homogeneous of degree k if for all $c \in \mathbb{R}$,

$$u(c\cdot\mathbf{x}) = c^k \cdot u(\mathbf{x}).$$

With the homogeneity and the WGS property, we have a more precise characterization of a buyer's consumption after price changes, as the following generalization of Proposition 3.1 demonstrates.

Proposition 3.2. *Assume that $u(\cdot)$ is homogeneous*[3] *and satisfies the WGS condition. Let $\mathbf{p}$ and $\mathbf{p}'$ be two price vectors and $\mathbf{x} \in D(\mathbf{p}, e)$ and $\mathbf{x}' \in D(\mathbf{p}', e)$ be the corresponding optimal demands. For any $q > 0$, let $S(q) = \left\{ j \in [m] \mid q \cdot p_j > p'_j \right\}$. Then*

$$\sum_{j\in S(q)} x_j p_j \le \sum_{j\in S(q)} x'_j p'_j.$$

Equivalently, if we denote $T(q) = [m] \setminus S(q) = \left\{ j \in [m] \mid q\cdot p_j \le p'_j \right\}$, then

$$\sum_{j\in T(q)} x'_j p'_j \le \sum_{j\in T(q)} x_j p_j.$$

We show a negative correlation between a buyer's incentive ratio and his money endowment. The following theorem suggests that the more endowment a buyer has, the less incentive he has to manipulate. The intuition behind the theorem is that a buyer who has a large endowment has already adequate allocation, and thus, the space to improve his utility from manipulations is quite small.

[3] Note that the proposition can be generalized to the family of homothetic preferences that satisfy the WGS condition. Here we simplify our presentation based on the family of homogeneous functions.

Theorem 3.2. *For any market M with homogeneous utilities that satisfy the WGS condition, consider any buyer i with an initial endowment of e_i, we have*

$$\zeta_i^M \le 2 - e_i.$$

We defer the proof of the theorem to the Appendix.

4. Cobb-Douglas utility functions

In a market with Cobb-Douglas utility functions,[4] a buyer's utility is given by $u_i(\mathbf{x}_i) = \prod_{j\in[m]} x_{ij}^{\alpha_{ij}}$, where $\sum_{j=1}^m \alpha_{ij} = 1$, for all $i \in [n]$, and buyer i has a money endowment of e_i. For convenience, it is usually assumed that each item is desired by at least one buyer, i.e., $\alpha_{ij} > 0$ for some i. By solving the Eisenberg-Gale convex program, one can obtain that the market equilibrium prices and allocations are unique and are given by the following equations:

$$p_j = \sum_{i=1}^n e_i\alpha_{ij} \tag{4}$$

$$x_{ij} = \frac{e_i\alpha_{ij}}{\sum_{i=1}^n e_i\alpha_{ij}} \tag{5}$$

Based on these equations, we analyze the incentive ratio of Cobb-Douglas markets.

Theorem 4.1. *The incentive ratio of Cobb-Douglas markets is*

$$\zeta^{\text{Cobb-Douglas}} \le e^{1/e} \approx 1.445.$$

Proof. Note that Equation (5) implies that x_{ij} is monotonically increasing with respect to e_i. Therefore, a buyer will not report a budget that is smaller than his actual budget. Hence, together with $e_i' \le e_i, \forall i$, a buyer will not misreport his budget. We consider two scenarios: For the fixed bid vector $(\alpha_{ij})_{j\in[m]}$ of buyer $i \neq 1$, buyer 1 bids $(\alpha_{1j})_{j\in[m]}$ and $(\alpha'_{1j})_{j\in[m]}$, respectively, with resulting equilibrium allocations $\mathbf{x}_1 = (x_{1j})_{j\in[m]}$ and $\mathbf{x}'_1 = (x'_{1j})_{j\in[m]}$. Then,

$$\begin{aligned}\zeta &= \frac{u_1(\mathbf{x}'_1)}{u_1(\mathbf{x}_1)} = \frac{\prod_{j=1}^m {x'_{1j}}^{\alpha_{1j}}}{\prod_{j=1}^m x_{1j}^{\alpha_{1j}}} = \frac{\prod_{j=1}^m \left(\frac{e_1\alpha'_{1j}}{e_1\alpha'_{1j}+\sum_{i\neq 1} e_i\alpha_{ij}}\right)^{\alpha_{1j}}}{\prod_{j=1}^m \left(\frac{e_1\alpha_{1j}}{e_1\alpha_{1j}+\sum_{i\neq 1} e_i\alpha_{ij}}\right)^{\alpha_{1j}}} \\ &= \prod_{j=1}^m \left(\frac{e_1\alpha'_{1j}\alpha_{1j} + \alpha'_{1j}\sum_{i\neq 1} e_i\alpha_{ij}}{e_1\alpha'_{1j}\alpha_{1j} + \alpha_{1j}\sum_{i\neq 1} e_i\alpha_{ij}}\right)^{\alpha_{1j}} \triangleq \prod_{j=1}^m R_j\end{aligned}$$

where R_j, $j = 1, \ldots, m$, is the j-th term of the above formula.

For each item j, it is easy to see that $R_j > 1$ if and only if $\alpha'_{1j} > \alpha_{1j}$. Let $S = \{j \mid R_j > 1\}$, and $r_j = R_j^{1/\alpha_{1j}}$ for $j \in S$ (note that r_j is well defined for those items in S as $\alpha_{1j} > 0$). Therefore, $r_j > 1$ if and only if $R_j > 1$. Further, when $r_j > 1$, one can see that

$$(\alpha'_{1j} - r_j\alpha_{1j})\sum_{i\neq 1} e_i\alpha_{1j} = (r_j - 1)e_1\alpha'_{1j}\alpha_{1j} > 0 \Longrightarrow \alpha'_{1j} \ge r_j\alpha_{1j}$$

This implies that $\sum_{j\in S} r_j\alpha_{1j} \le \sum_{j\in S} \alpha'_{1j} \le 1$. Hence,

$$\zeta \le \prod_{j\in S} r_j^{\alpha_{1j}} \le \prod_{j\in S} e^{\frac{r_j\alpha_{1j}}{e}} \le e^{\frac{\sum_{j\in S} r_j\alpha_{1j}}{e}} \le e^{1/e}.$$

Note that the above second inequality follows from the fact that for any $x, y \ge 0$, $x^y \le e^{xy/e}$, which can be verified easily. Therefore, the theorem follows. □

The upper bound established in the above claim is tight, which can be seen from the following example.[5]

[4] The Cobb-Douglas utility functions stands somewhere in between gross substitute and gross complement utility functions, because its cross elasticity of demand is zero.

[5] The endowments in this example are not normalized for the ease of presentation.

Example 2 *(Tight incentive ratio).* There are 2 items and 2 buyers with profiles as follows:

	α_{i1}	α_{i2}	endowment
buyer 1	$\frac{1}{e}$	$1-\frac{1}{e}$	e_1
buyer 2	$1-\epsilon_2$	ϵ_2	e_2

where we define $\epsilon_1 = \frac{1}{n}$, $\epsilon_2 = \frac{1}{n^3}$ and $n = \frac{e_2}{e_1}$. When buyer 1 bids $u_1' = (1-\epsilon_1, \epsilon_1)$ instead of u_1, the incentive ratio is given by the following formula:

$$\left(\frac{\frac{1-\epsilon_1}{e} + n(1-\epsilon_2)(1-\epsilon_1)}{\frac{1-\epsilon_1}{e} + n(1-\epsilon_2)\frac{1}{e}}\right)^{\frac{1}{e}} \cdot \left(\frac{(1-\frac{1}{e}) + \frac{1}{n^2}}{(1-\frac{1}{e}) + \frac{1}{n}\cdot(1-\frac{1}{e})}\right)^{1-\frac{1}{e}}$$

When n goes to infinity, the limit of the left factor is $e^{1/e}$, and the limit of the right factor is 1. Therefore, the ratio approaches $e^{1/e}$.

Note that when buyer 1 changes his bids $(\frac{1}{e}, 1-\frac{1}{e}) \to (1-\epsilon, 1-\frac{1}{e})$, he still gets most $(\frac{n^2}{n^2+1})$ of this item while paying much less $((1-\frac{1}{e})e_1 \to \frac{e_1}{n})$.

5. Leontief utility functions

In this section, we analyze the incentive ratio of the Fisher markets when buyers have Leontief utility functions. Note that Leontief functions do not satisfy the WGS condition. We will show that given any n-buyer m-item market, there exists a 2-buyer 2-item market such that its incentive ratio is equal to the incentive ratio of the larger market. The reduction from a large market to a smaller market is possible as the incentive ratio notion is a worst-case metric. Essentially, the incentive ratio is determined by the worst-case inputs, even though these worst-case inputs are more likely to occur in a small-size market than that of a large market. This circumstance is not uncommon as it happens to other literature that considers an agent's manipulation in a mechanism. For example, Bouveret and Lang ([6], Proposition 7) show that finding a manipulation for an n-agent picking sequence can be reduced to finding a manipulation for a 2-agent picking sequence. Given that, without loss of generality, they consider the scenario that there are only two agents in which one of them is the manipulating agent.

Codenotti and Varadarajan [16] prove that buyers' utilities can be computed by solving the following convex program:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} e_i \log u_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} a_{ij} u_i \le 1, \ \ \forall\, j = 1, \ldots, m \\ & u_i \ge 0, \ \ \forall\, i = 1, \ldots, n. \end{aligned} \tag{6}$$

By applying the KKT conditions, we get that $p_j\left(\sum_{i=1}^{n} a_{ij}u_i - 1\right) = 0$ and $u_i = \frac{e_i}{\sum_{j=1}^{m} p_j a_{ij}}$, where the Lagrangian variables p_j are actually the market equilibrium prices.

We first establish some properties regarding buyers' best response and dominant strategies in the market. Some of the proofs are deferred to the Appendix.

Proposition 5.1. *In Leontief markets, buyers do not benefit from misreporting their budgets.*

Since reporting their budgets truthfully is a dominant strategy for the buyers, from now on, we only consider the scenario that a buyers misreports its utility function. In the following, we denote $b_i^{\max} = \max_j\{b_{ij}\}$, $b_i^{\min} = \min_j\{b_{ij}\}$, $\forall i$.

Lemma 5.1. *Given any 2-buyer market and any strategy $\mathbf{b_2} = (b_{2j})_{j\in[m]}$ of buyer 2, the best response strategy for buyer 1 is $\mathbf{b_1} = (b_{1j})_{j\in[m]}$, where $b_{1j} = 1 - \frac{b_{2j}}{b_2^{\max}} e_2, \forall j$. The utility of buyer 1 is $u_1(\mathbf{b_1}, \mathbf{b_2}) = \min_j \left\{ \frac{1-\frac{b_{2j}}{b_2^{\max}}e_2}{a_{1j}} \right\}$.*

Since $\sum_j p_j = 1$, $u_2 = \frac{e_2}{\sum_j p_j b_{2j}} \in \left[\frac{e_2}{b_2^{\max}}, \frac{e_2}{b_2^{\min}}\right]$. Therefore, the best response strategy of buyer 1 is the strategy such that the utility of buyer 2 is the minimum possible value $\frac{e_2}{b_2^{\max}}$ and buyer 2 receives the minimum possible allocation which is $x_{2j} = b_{2j}\frac{e_2}{b_2^{\max}}, \forall j$.

Lemma 5.2. *Given the utility function $\mathbf{a_1}$ of buyer 1 and any strategies $\mathbf{b_{-1}}$ of other buyers, let $\mathcal{G}_1 = \{j | \operatorname{argmax}\{a_{1j}\}\}$ and $\mathcal{G}_i = \{j | \operatorname{argmax}\{b_{ij}\}\}$, $i = 2, \ldots, n$. If $\mathcal{G} = \mathcal{G}_1 \bigcap \mathcal{G}_2 \bigcap \cdots \bigcap \mathcal{G}_n \neq \emptyset$, then truth-telling is a dominant strategy for buyer* 1.

The above lemma implies that if there is a common item that other buyers desire most, and it is also most preferable to an individual buyer, then the allocation of this item determines buyers' utilities and allocation, and there is no space for the individual buyer to improve its utility.

We now establish the connection between incentive ratios of markets with different number of buyers and items. Specifically, we use two reductions to show that the incentive ratio of any n-buyer m-item market is no larger than the incentive ratio of a 2-buyer 2-item market. These results significantly simplify the proof of the main theorem in this section. Intuitively, we can characterize the best response strategy in 2-buyer markets by Lemma 5.1, but it would be difficult to characterize the best response strategy in n-buyer markets.

Lemma 5.3. *Given any n-buyer market, there exists a 2-buyer market such that its incentive ratio is no less than the incentive ratio of the n-buyer market.*

Given any n-buyer market, by "unifying" $n-1$ buyers, we show that there exists a 2-buyer market such that its incentive ratio is not smaller than that of the n-buyer market. The detail of the proof is in the Appendix.

Similarly, we have the following lemma with respect to the number of items in the markets.

Lemma 5.4. *Given any 2-buyer m-item market, its incentive ratio is no more than the incentive ratio of a 2-buyer 2-item market.*

Our main result in this section is the following.

Theorem 5.5. *The incentive ratio of Leontief markets is*

$$\zeta^{Leontief} < 2.$$

Proof. Given the properties of the incentive ratio established before, it suffices to bound the incentive ratio for 2-buyer 2-item markets.

Let M be a market with 2 buyers and 2 items. Without loss of generality, assume buyer 1 achieves a bigger incentive ratio than buyer 2. We will only analyze the incentive ratio of buyer 1 as the same argument applies to buyer 2. Denote the utility function of buyer 1 by $\mathbf{a_1} = (a_{11}, a_{12})$ and the report of buyer 2 by $\mathbf{b_2} = (b_{21}, b_{22})$. According to Lemma 5.2, assume without loss of generality that $a_{11} > a_{12}$ and $b_{21} < b_{22}$, otherwise the truthful strategy is a dominant strategy for buyer 1. Let $u_1 = u_1(\mathbf{a_1}, \mathbf{b_2})$, $u_2 = u_2(\mathbf{a_1}, \mathbf{b_2})$. The incentive ratio of buyer 1 is

$$\zeta = \frac{\min\limits_j \left\{ \frac{1 - \frac{b_{2j}}{b_2^{\max}} e_2}{a_{1j}} \right\}}{u_1} = \frac{\min \left\{ \frac{1 - \frac{b_{21}}{b_{22}} e_2}{a_{11}}, \frac{e_1}{a_{12}} \right\}}{u_1}.$$

According to the equilibrium price when they report $(\mathbf{a_1}, \mathbf{b_2})$, there are three cases to discuss.

- $0 < p_1 < 1, 0 < p_2 < 1$.
  In this case, according to the complementary slackness of the KKT conditions, we have

$$\begin{cases} a_{11}u_1 + b_{21}u_2 - 1 = 0 \\ a_{12}u_1 + b_{22}u_2 - 1 = 0 \end{cases}.$$

  Thus

$$u_1 = \frac{b_{22} - b_{21}}{a_{11}b_{22} - a_{12}b_{21}}, \quad u_2 = \frac{a_{11} - a_{12}}{a_{11}b_{22} - a_{12}b_{21}}.$$

  Since $u_i = \frac{e_i}{\sum_j p_j a_{ij}}$ and $0 < p_i < 1, i = 1, 2$, we know that $\frac{e_1}{a_{11}} < u_1 < \frac{e_1}{a_{12}}$, $\frac{e_2}{b_{22}} < u_2 < \frac{e_2}{b_{21}}$. Therefore,

$$\frac{e_1}{a_{11}} < \frac{b_{22} - b_{21}}{a_{11}b_{22} - a_{12}b_{21}} < \frac{e_1}{a_{12}}, \quad \frac{e_2}{b_{22}} < \frac{a_{11} - a_{12}}{a_{11}b_{22} - a_{12}b_{21}} < \frac{e_2}{b_{21}}.$$

  So,

$$\frac{a_{12}(b_{22} - b_{21})}{a_{11}b_{22} - a_{12}b_{21}} < e_1 < \frac{a_{11}(b_{22} - b_{21})}{a_{11}b_{22} - a_{12}b_{21}}.$$

Thus,

$$\frac{1-\frac{b_{21}}{b_{22}}e_2}{a_{11}} - \frac{e_1}{a_{12}} = \frac{a_{12}(b_{22}-b_{21})-(a_{11}b_{22}-a_{12}b_{21})e_1}{a_{11}b_{22}a_{12}} < \frac{a_{12}(b_{22}-b_{21})-a_{12}(b_{22}-b_{21})}{a_{11}b_{22}a_{12}} = 0.$$

Therefore,

$$\begin{aligned}\zeta &= \frac{\min_j \left\{ \frac{1-\frac{b_{2j}}{b_2^{\max}}e_2}{a_{1j}} \right\}}{u_1} = \frac{\frac{1-\frac{b_{21}}{b_{22}}e_2}{a_{11}}}{\frac{b_{22}-b_{21}}{a_{11}b_{22}-a_{12}b_{21}}} \\ &= \frac{b_{22}-b_{21}+b_{21}e_1}{a_{11}b_{22}} \cdot \frac{a_{11}b_{22}-a_{12}b_{21}}{b_{22}-b_{21}} \\ &< 1 + \frac{b_{21}}{b_{22}}\left(1-\frac{a_{12}}{a_{11}}\right) < 2,\end{aligned}$$

where the last inequality is due to $0 < b_{22} < b_{21} + \varepsilon$ and $0 < a_{12} < \varepsilon$, as ε approaches 0.

- $p_1 = 1, p_2 = 0$.
  In this case, $u_1 = \frac{e_1}{\sum_j p_j a_{1j}} = \frac{e_1}{a_{11}}, u_2 = \frac{e_2}{\sum_j p_j b_{2j}} = \frac{e_2}{b_{21}}$. According to the constraint of (6),

$$a_{12}u_1 + b_{22}u_2 - 1 = a_{12}\frac{e_1}{a_{11}} + b_{22}\frac{e_2}{b_{21}} - 1 \le 0,$$

so $e_1 \ge \frac{a_{11}(b_{22}-b_{21})}{a_{11}b_{22}-a_{12}b_{21}}$. Therefore,

$$\frac{1-\frac{b_{21}}{b_{22}}e_2}{a_{11}} - \frac{e_1}{a_{12}} = \frac{a_{12}(b_{22}-b_{21})-(a_{11}b_{22}-a_{12}b_{21})e_1}{a_{11}b_{22}a_{12}} \le \frac{a_{12}(b_{22}-b_{21})-a_{11}(b_{22}-b_{21})}{a_{11}b_{22}a_{12}} < 0.$$

Hence,

$$\begin{aligned}\zeta &= \frac{\min_j \left\{ \frac{1-\frac{b_{2j}}{b_2^{\max}}e_2}{a_{1j}} \right\}}{u_1} = \frac{\frac{1-\frac{b_{21}}{b_{22}}e_2}{a_{11}}}{\frac{e_1}{a_{11}}} = \frac{b_{22}-b_{21}+b_{21}e_1}{b_{22}e_1} \\ &\le \frac{b_{21}}{b_{22}} + \frac{a_{11}b_{22}-a_{12}b_{21}}{a_{11}b_{22}} = 1 + \frac{b_{21}}{b_{22}}\left(1-\frac{a_{12}}{a_{11}}\right) < 2.\end{aligned}$$

- $p_1 = 0, p_2 = 1$.
  In this case buyer 2's utility is the minimum possible. By a similar argument as in the proof of Lemma 5.1, buyer 1 gets the largest possible utility hence its incentive ratio is 1. □

Moreover, the ratio given by the theorem is tight as the following example shows. The example can be extended to markets with arbitrary number of buyers and items by splitting a buyer into a number of homogeneous buyers and making copies of items.

Example 3 *(Tight incentive ratio).* There are two buyers and two items in the market. Their Leontief utility functions and budgets are given in the following table.

	a_{i1}	a_{i2}	endowment
buyer 1	$1-\varepsilon$	ε	$4\varepsilon - 4\varepsilon^2 + \varepsilon^3$
buyer 2	$\frac{1}{2}-\varepsilon$	$\frac{1}{2}+\varepsilon$	$1 - 4\varepsilon + 4\varepsilon^2 - \varepsilon^3$

When both buyers report truthfully, buyer 1's utility is $u_1 = \frac{4\varepsilon-4\varepsilon^2+\varepsilon^3}{1-\varepsilon}$. If buyer 1 strategically reports

$$\mathbf{b_1} = \left(\frac{8\varepsilon - 12\varepsilon^2 + 9\varepsilon^3 - 2\varepsilon^4}{1+2\varepsilon}, 4\varepsilon - 4\varepsilon^2 + \varepsilon^3\right),$$

then its utility becomes

$$u_1' = \frac{8\varepsilon - 12\varepsilon^2 + 9\varepsilon^3 - 2\varepsilon^4}{(1+2\varepsilon)(1-\varepsilon)},$$

and the incentive ratio is

$$\frac{8 - 12\varepsilon + 9\varepsilon^2 - 2\varepsilon^3}{4 + 4\varepsilon - 7\varepsilon^2 + 2\varepsilon^3},$$

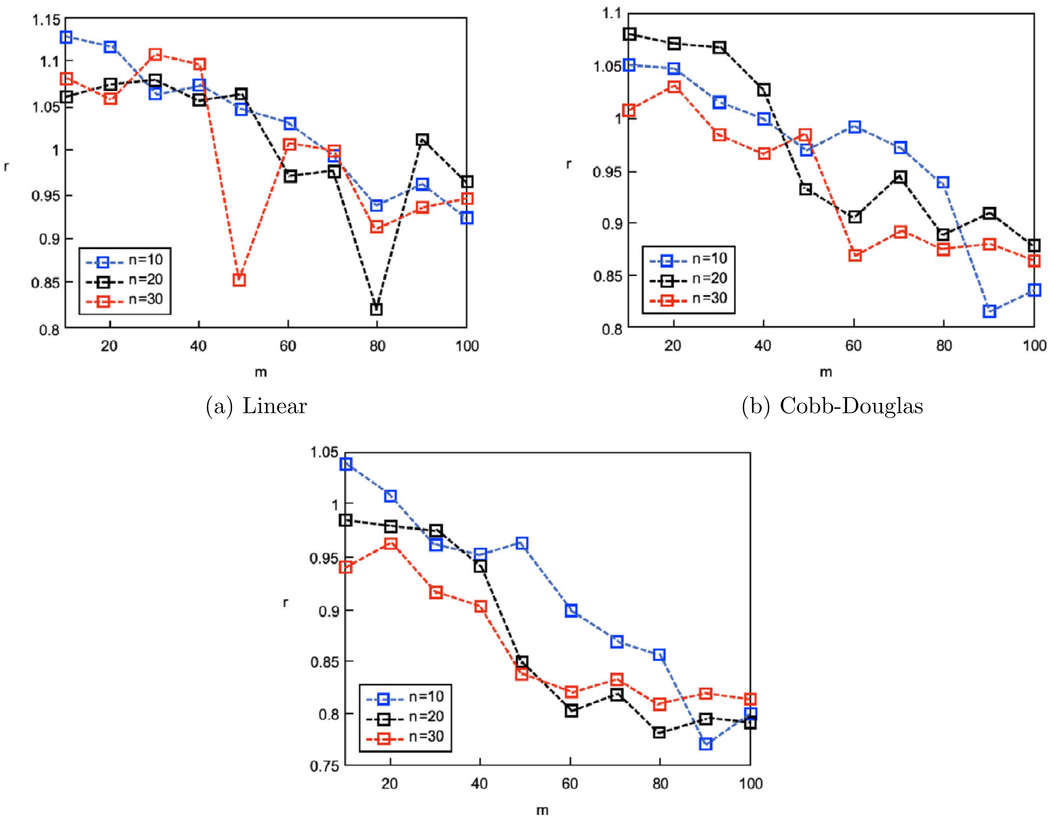
which converges to 2 as ε approaches 0.

6. Simulation

While the worst-case incentive ratios are revealing, the tight bound examples are customized. In this section, we present simulation results on the extent to which an agent's utility may change by unilateral misreporting and examine how much an average instance differs from these worst-case instances. Hence, going forward, we complement theoretical results with simulations showing what the utility gain looks like for a more typical instance. To this end, we use synthetically generated data to examine the ratio in three typical CES utility functions.

For each class of utility functions (linear, Cobb-Douglas, and Leontief, respectively), we draw a_{ij} and e_i from the uniform distribution $\mathbb{U}(0,1)$. By computing the market equilibrium, we obtain Agent 1's utility u_1 under the truthful profile. Next, we uniformly at random generate a different utility profile $\mathbf{a}_1' = (a_{1j}')_{j\in[m]}$ of Agent 1, and compute its utility u_1' in this case. We generate k times misreporting profile $\mathbf{a}_1'$, and output the largest ratio $r = \frac{u_1'}{u_1}$ amongst these k iterations. Last, we repeat this simulation 100 times, and examine the ratio when n and m vary. In Fig. 1, we illustrate the simulation results, in which $k = 1000$, m starts from 10 to 100 with an increment of 10 each time, and $n = 10, 20, 30$, respectively. Therefore, for each node in the figure, it represents the largest ratio of $100 \times 1000 = 100,000$ random instances, given a fixed n and m.

Our first finding is that an agent may not necessarily find a beneficial misreport, even if we generate 100 instances and for each of them, we generate 1000 misreports. Secondly, in these scenarios that Agent 1 finds a beneficial misreport, the



(a) Linear

(b) Cobb-Douglas

(c) Leontief

Fig. 1. The ratio $r = \frac{u_1'}{u_1}$ when inputs are independently and identically drawn from the Uniform distribution $\mathbb{U}(0,1)$, where m is the number of goods.

ratio $r = \frac{u_1'}{u_1}$ is at most 1.15, which is much smaller than the worst-case incentive ratio. Lastly, we repeat the simulation by drawing the random inputs from a Gaussian distribution and find similar results to Fig. 1.

7. Conclusions

In this paper, we quantified the extent to which an agent can improve its utility by strategically misreports the utility functions, in the form of incentive ratio. We presented the incentive ratio upper bounds of Fisher markets when the agents are equipped with several different utility functions. We also established a negative correlation between incentive ratio and endowment. It turned out that these ratios are upper bounded by small constants, which indicates that the incentives to manipulate the Fisher markets are relatively small. In contrast, we note that the incentive ratio could be very large and even unbounded if we generalize the investigated utility functions. The following example demonstrates an unbounded incentive ratio when the utility functions are piecewise linear.

Example 4 *(Unbounded incentive ratio).* We consider a market with two items and two buyers where both buyers have linear utilities. Their utilities and endowments are as follows.

	u_{i1}	u_{i2}	endowment
buyer 1	$\frac{x_{11}}{2}$	$\frac{x_{12}}{2}$	ϵ
buyer 2	x_{21}	$h(x_{22})$	$1-\epsilon$

Here $h(x)$ is a piecewise linear and concave function defined as below:

$$h(x) = \begin{cases} kx & \text{if } x \le t \\ -\frac{(k-1)x^2}{2\delta} + (\frac{k-1}{\delta}t + k)x - \frac{(k-1)t^2}{2\delta} & \text{if } t < x \le t+\delta \\ x + (k-1)t + \frac{(k-1)\delta}{2} & \text{if } x > t+\delta \end{cases}$$

where $k = \frac{1-\frac{\epsilon}{2}}{\frac{\epsilon}{2}}$ and $t = \frac{1-\epsilon}{1-\frac{\epsilon}{2}}$, and δ is a sufficiently small number. Fig. 2 shows an example of $h(x)$ when $\epsilon = 0.2$ (thus, $k = 9$ and $t = \frac{8}{9}$) and $\delta = \frac{t}{100} = \frac{8}{900}$.

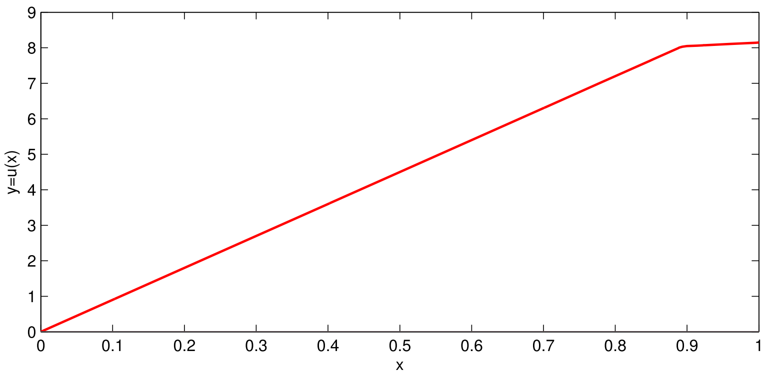


Fig. 2. An illustration of $h(x)$: the marginal utility diminishes quickly after the threshold t.

Note that the utility function of the second buyer does not satisfy the WGS condition. Indeed, when the first item's price decreases from $\frac{1}{2}$ to $\frac{\epsilon}{2}$ and the second item's price increases from $\frac{1}{2}$ to $1-\frac{\epsilon}{2}$, the second buyer's demand for the first item decreases, contradicting the WGS condition.

If the first buyer bids truthfully, the equilibrium price vector is $(\frac{1}{2}, \frac{1}{2})$ and his utility is ϵ. But if he misreports his utility function to be $\left(\frac{\epsilon}{2}x_{11}, (1-\frac{\epsilon}{2})x_{12}\right)$, the equilibrium price vector becomes $(\frac{\epsilon}{2}, 1-\frac{\epsilon}{2})$ and he gets the first item with a utility of at least $\frac{1}{2}$. Thus, his incentive ratio is at least $\frac{1}{2\epsilon}$, which is unbounded when ϵ approaches 0.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

Proof of Proposition 3.2. Let $\mathbf{p}'' = \mathbf{p}' \setminus q$ and $\mathbf{x}'' = q\mathbf{x}'$. Since $u(\cdot)$ is homogeneous, $u(\mathbf{x}'') = q^k \cdot u(\mathbf{x}')$. Since $\mathbf{x}' \in D(\mathbf{p}', e)$ and $\mathbf{x}''$ is a feasible allocation given price $\mathbf{p}''$, we know that $\mathbf{x}'' \in D(\mathbf{p}'', e)$.

By the definition of $\mathbf{p}''$, for all item $j \in S(q)$, $p''_j = p'_j/q < p_j$; and for all item $j \notin S(q)$, $p''_j \geq p_j$. Applying Proposition 3.1 with prices $\mathbf{p}$ and $\mathbf{p}''$ and the corresponding optimal allocations $\mathbf{x}$ and $\mathbf{x}''$ gives

$$\sum_{j\in S} x''_j p''_j \geq \sum_{j\in S} x_j p_j$$

where $S = \left\{ j \in [m] \mid p_j > p''_j \right\} = S(q)$. Substituting $x''_j = qx'_j$ and $p''_j = \frac{p'_j}{q}$ to the above inequality yields

$$\sum_{j\in S(q)} x'_j p'_j \geq \sum_{j\in S(q)} x_j p_j.$$

Hence, the claim follows. □

Proof of Theorem 3.2. Without loss of generality, we will only consider buyer 1 and prove that $\zeta_1 \leq 2 - e_1$. Let $(\mathbf{p}, \mathbf{x})$ and $(\mathbf{p}', \mathbf{x}')$ be market equilibria when buyer 1 bids truthfully and strategically, respectively. We will show that

$$u_1(\mathbf{x}'_1) \leq (2 - e_1) \cdot u_1(\mathbf{x}_1).$$

Let $R(q_k) = \left\{ j \in [m] \mid p'_j = q_k \cdot p_j \right\}$. Divide all items into a collection of subsets such that:

$$[m] = \bigcup_{k=1}^{t} R(q_k)$$

where $q_1 > q_2 > \cdots > q_t$. Let

$$\gamma_k = \sum_{j\in R(q_k)} p_j \text{ and } \gamma'_k = \sum_{j\in R(q_k)} p'_j$$

be the sum of the prices of the items in $R(q_k)$ with respect to p and p', respectively. Further, define

$$\beta_k = \sum_{j\in R(q_k)} p_j x_{1j} \text{ and } \beta'_k = \sum_{j\in R(q_k)} p'_j x'_{1j}$$

to be the amount of money that buyer 1 spends on the set of items $R(q_k)$ in the consumptions $\mathbf{x}_1$ and $\mathbf{x}'_1$, respectively. It follows that $\sum_{k=1}^t \beta_k = e_1$ and $\sum_{k=1}^t \beta'_k = e'_1$, where e'_1 is the amount of endowment that buyer 1 spends in $\mathbf{x}'$.

Next, by Proposition 2.1, there exists a constant c such that

$$\begin{aligned} u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1) &\leq \sum_{j\in[m]} c \cdot p_j(x'_{1j} - x_{1j}) \\ &= \sum_{k=1}^{t} \sum_{j\in R(q_k)} c \cdot \left(\frac{p'_j x'_{1j}}{q_k} - p_j x_{1j} \right) \\ &= \sum_{k=1}^{t} c \cdot \left(\frac{\beta'_k}{q_k} - \beta_k \right) \triangleq \Delta \end{aligned} \tag{A.1}$$

Thus, to have an upper bound on the utility gain $u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1)$, it suffices to bound Δ. Particularly we will try to identify the constraints for and optimize over the sequence $\left\{\beta'_k\right\}$, while assuming that other parameters are already fixed.

For any $k = 1, \ldots, t-1$, note that $T(q_k) = \{j \in [m] \mid q_k \cdot p_j \leq p'_j\} = \bigcup_{\ell=1}^k R(q_\ell)$. By Proposition 3.2, for all buyers $i \neq 1$, they spend less money on the items in $T(q_k)$ after prices are changed from p to p'. We therefore have

$$\sum_{\ell=1}^{k}(\gamma'_\ell - \beta'_\ell) = \sum_{j\in T(q_k)} p'_j - \sum_{j\in T(q_k)} p'_j x'_{1j} \leq \sum_{j\in T(q_k)} p_j - \sum_{j\in T(q_k)} p_j x_{1j} = \sum_{\ell=1}^{k}(\gamma_\ell - \beta_\ell)$$

This implies that

$$\sum_{\ell=1}^{k}\beta'_\ell \geq \sum_{\ell=1}^{k}(\beta_\ell + \gamma'_\ell - \gamma_\ell), \tag{A.2}$$

for any $k = 1, \ldots, t-1$. Note as well that the inequality holds in (A.2) when $k = t$.

Now we are ready to estimate Δ. Since $\{q_k\}$ is a decreasing sequence, Δ can be bounded by the case when the vector $(\beta'_1, \beta'_2, \ldots, \beta'_t)$ is lexicographically minimized, subject to the set of constraints in (A.2). In other words, the utility gain is maximized when less money is spent on those items whose price are increased more. However, under the set of constraints in (A.2), $(\beta'_1, \beta'_2, \ldots, \beta'_t)$ is lexicographically minimized when

$$\beta'_k = \beta_k + \gamma'_k - \gamma_k, \ \forall\, k = 1, \ldots, t$$

And this assignment also satisfies $\sum_{k=1}^{t}\beta'_k = \sum_{k=1}^{t}(\beta_k + \gamma'_k - \gamma_k)$.

Summing over the above arguments, Δ/c in Formula (A.1) becomes:

$$\begin{aligned}
\sum_{k=1}^{t}\left(\frac{\beta'_k}{q_k} - \beta_k\right) &\leq \sum_{k=1}^{t}\left(\frac{\beta_k + \gamma'_k - \gamma_k}{q_k} - \beta_k\right) \\
&= \sum_{k=1}^{t}\frac{\gamma_k(\beta_k + \gamma'_k - \gamma_k) - \beta_k q_k \gamma_k}{q_k\gamma_k} \\
&= \sum_{k=1}^{t}\frac{(\gamma_k - \beta_k)(\gamma'_k - \gamma_k)}{\gamma'_k} \quad (\text{since } q_k\gamma_k = \gamma'_k) \\
&\leq \sum_{k=1}^{t}\frac{(\gamma_k - \beta_k)(\beta_k + \gamma'_k - \gamma_k)}{\gamma'_k} \quad (\text{since } \gamma_k - \beta_k \geq 0) \\
&\leq \frac{\sum_{k=1}^{t}(\gamma_k - \beta_k)\cdot\sum_{k=1}^{t}(\beta_k + \gamma'_k - \gamma_k)}{\sum_{k=1}^{t}\gamma'_k} \\
&= \frac{\left(\sum_{k=1}^{t}\gamma_k - \sum_{k=1}^{t}\beta_k\right)\cdot\sum_{k=1}^{t}\beta'_k}{1} \\
&\leq (1 - e_1)\cdot e'_1
\end{aligned}$$

The second last inequality follows from repeatedly applying the following Fact A.1.

Therefore, Formula (A.1) becomes

$$u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1) \leq ce'_1\cdot(1-e_1) \leq ce_1\cdot(1-e_1) \leq u_1(\mathbf{x}_1)\cdot(1-e_1),$$

where the last inequality is by Proposition 2.1. Thus, we have

$$u_1(\mathbf{x}'_1) \leq (2 - e_1)\cdot u_1(\mathbf{x}_1).$$

This completes the proof. □

The following fact is used in the above proof.[6]

[6] It can be verified that the inequality is equivalent to $(a_1b_2 - a_2b_1)^2 \geq 0$.

Lemma A.1. *Assume that $a_1 + b_1$ and $a_2 + b_2$ are positive, then*

$$\frac{a_1 b_1}{a_1 + b_1} + \frac{a_2 b_2}{a_2 + b_2} \leq \frac{(a_1 + a_2)(b_1 + b_2)}{a_1 + b_1 + a_2 + b_2}$$

Proof of Proposition 5.1. Denote buyers' budgets by e_i and utility functions by $\mathbf{a_i} = (a_{ij})_{j\in[m]}, i = 1, \ldots, n$. Denote their strategies by $\mathbf{s_i} = (e'_i, \mathbf{b_i})$, where $e'_i \leq e_i$ and $\mathbf{b_i} = (b_{ij})_{j\in[m]}, i = 1, \ldots, n$. Without loss of generality we consider buyer 1. When buyer 1 uses strategy $(e_1, \mathbf{b_1})$, its utility, evaluated by the reported utility function, is captured by the solution of the convex program

$$\begin{aligned} \max \quad & f(u_1, u_2, \ldots, u_n) = e_1 \log u_1 + \sum_{i=2}^{n} e_i \log u_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} b_{ij} u_i \leq 1, \quad \forall\, j = 1, \ldots, m \\ & u_i \geq 0, \quad \forall\, i = 1, \ldots, n. \end{aligned} \tag{A.3}$$

When buyer 1 uses strategy $(e'_1, \mathbf{b_1})$, its utility, evaluated by the reported utility function, is captured by the solution of the convex program

$$\begin{aligned} \max \quad & g(u_1, u_2, \ldots, u_n) = e'_1 \log u_1 + \sum_{i=2}^{n} e_i \log u_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} b_{ij} u_i \leq 1, \quad \forall\, j = 1, \ldots, m \\ & u_i \geq 0, \quad \forall\, i = 1, \ldots, n. \end{aligned} \tag{A.4}$$

Let $(x_1, x_2, \ldots, x_n)$ denotes the optimal solution of (A.3), and $(y_1, y_2, \ldots, y_n)$ denotes the optimal solution of (A.4). Since (A.3) and (A.4) have the same feasible region, $(x_1, x_2, \ldots, x_n)$ is a feasible solution of (A.4), and $(y_1, y_2, \ldots, y_n)$ is a feasible solution of (A.3). Moreover, as the objective functions of (A.3) and (A.4) are strictly concave, we have

$$\begin{aligned} & f(x_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, y_n) > 0, \\ & g(y_1, y_2, \ldots, y_n) - g(x_1, x_2, \ldots, x_n) > 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} & g(y_1, y_2, \ldots, y_n) - g(x_1, x_2, \ldots, x_n) \\ = & \left[e'_1 \log y_1 + \sum_{i=2}^{n} e_i \log y_i\right] - \left[e'_1 \log x_1 + \sum_{i=2}^{n} e_i \log x_i\right] \\ = & \left[e_1 \log y_1 + \sum_{i=2}^{n} e_i \log y_i\right] - \left[e_1 \log x_1 + \sum_{i=2}^{n} e_i \log x_i\right] + (e_1 - e'_1)(\log x_1 - \delta \log y_1) \\ = & f(y_1, y_2, \ldots, y_n) - f(x_1, x_2, \ldots, x_n) + (e_1 - e'_1) \log \frac{x_1}{y_1}. \end{aligned}$$

Therefore $(e_1 - e'_1) \log \frac{x_1}{y_1} > 0$. Hence $x_1 > y_1$ which implies that buyer 1 gets a strictly larger utility, evaluated by the reported utility function, by reporting true budget than that of by reporting a smaller budget. Hence buyer 1 gets strictly more allocation on every item so its utility, evaluated by its true utility function, is also larger. □

Proof of Lemma 5.1. Since $\sum_j p_j = 1$, the utility of buyer i, evaluated using the reported utility function, is bounded by

$$\frac{e_i}{b_i^{\max}} = \frac{e_i}{(\sum_j p_j) b_i^{\max}} \leq u_i = \frac{e_i}{\sum_j p_j b_{ij}} \leq \frac{e_i}{(\sum_j p_j) b_i^{\min}} = \frac{e_i}{b_i^{\min}}.$$

According to the convex program (6), obviously buyer 1's utility is maximized when $u_2 = \frac{e_2}{b_2^{\max}}$ and buyer 2's allocation is minimum possible, that is, $x_{2j} = b_{2j} u_2 = \frac{b_{2j}}{b_2^{\max}} e_2, \forall j$.

Now we just need to verify that when buyers use strategies $(\mathbf{b_1}, \mathbf{b_2})$, buyer 2's utility is $u_2 = \frac{e_2}{b_2^{\max}}$ and its allocation is $x_{2j} = b_{2j}u_2 = \frac{b_{2j}}{b_2^{\max}}e_2, \forall j$. To verify that, we claim that the market equilibrium price when buyers use the strategy pair $(\mathbf{b_1}, \mathbf{b_2})$ is

$$p_j = \begin{cases} 1 & \text{if } j \in \arg\max\{b_{2j}\} \\ 0 & \text{otherwise} \end{cases}$$

and the utilities of the buyers, evaluated by their reported utility function, are $u_1 = 1, u_2 = \frac{e_2}{b_2^{\max}}$.

It's easy to check that these values satisfy the KKT conditions of the convex program (6), and the first constraint of (6) is tight. Because the objective function of the convex program is strictly concave, the market equilibrium prices and utilities are unique. Note that buyer 1's utility, evaluated by its true utility function, is $u_1(\mathbf{b_1}, \mathbf{b_2}) = \min_j \left\{ \frac{1 - \frac{b_{2j}}{b_2^{\max}} e_2}{a_{1j}} \right\}$. □

Proof of Lemma 5.2. When buyer 1 reports truthfully, $\forall k \in \mathcal{G}$,

$$a_{1k}u_1 + \sum_{i \neq 1} b_{ik}u_i = a_1^{\max}u_1 + \sum_{i \neq 1} b_i^{\max}u_i \geq a_{1j}u_1 + \sum_{i \neq 1} b_{ij}u_i, \forall j.$$

Therefore to maximize the objective function of (6), the k-th constraint must be tight and the corresponding Lagrangian variables satisfy $\sum_{k \in \mathcal{G}} p_k = 1$. So, $u_1 = \frac{e_1}{a_1^{\max}}, u_i = \frac{e_i}{b_i^{\max}}, \forall i \neq 1$. Note that in this case the utilities of buyers $i, i \neq 1$ already have the minimum possible values and buyer 1 cannot increase its utility by obtaining more allocation of the zero-priced items. Hence, truth-telling is a dominant strategy for buyer 1. □

Proof of Lemma 5.3. We prove the lemma by a reduction from any n-buyer market M_n to a 2-buyer market M_2. Denote the incentive ratio of M_n and M_2 by ζ^{M_n} and ζ^{M_2}, respectively. W.l.o.g., assume $\zeta^{M_n} > 1$. We will construct a 2-buyer market and prove $\zeta^{M_n} \leq \zeta^{M_2}$.

Without loss of generality assume that buyer 1 has the maximal incentive ratio amongst all the buyers, i.e., $\zeta^{M_n} = \zeta_1^{M_n}$. Buyers' endowments are $e_i, i = 1, \ldots, n$. Their utilities are $\mathbf{a_i} = (a_{ij})$ while they report $\mathbf{b_i} = (b_{ij})$, where $a_{ij} > 0$ and $b_{ij} > 0$. Since the incentive ratio is defined as the factor of the largest possible utility gain that a participant can achieve by behaving strategically, given that all other participants have their strategies unchanged, for simplicity, assume $\mathbf{b_i} = \mathbf{a_i}, i = 2, \ldots, n$ where $\mathbf{a_i}$ could be any feasible utility function. Let $\mathbf{b_1}$ be buyer 1's best response strategy to $\mathbf{b_i}, i = 2, \ldots, n$. When buyer 1 reports truthfully, denote its utility by $u_1 = u_1(\mathbf{a_1}, \mathbf{a_{-1}}) = \min_{j \in [m]} \left\{ \frac{x_{1j}}{a_{1j}} \right\}$; when buyer 1 uses best response strategy $\mathbf{b_1}$, denote its utility by $u'_1 = u'_1(\mathbf{b_1}, \mathbf{a_{-1}}) = \min_{j \in [m]} \left\{ \frac{1 - \sum_{i=2}^n a_{ij}u'_i}{a_{1j}} \right\}$, where $u'_i, i = 2, \ldots, n$ are other buyers' utility when buyer 1 uses best response strategy. Denote the two buyers in market M_2 by 1^* and 2^*. Their endowments are $e_{i^*}, i = 1, 2$. Their utilities are $\mathbf{a_{i^*}} = (a_{i^*,j})$ while they report $\mathbf{b_{i^*}} = (b_{i^*,j})$, where $a_{i^*,j} > 0$ and $b_{i^*,j} > 0$. Again, for simplification of notations, assume $\mathbf{b_{2^*}} = \mathbf{a_{2^*}}$. Let $\mathbf{b_{1^*}}$ be the best response strategy of buyer 1^*. When buyer 1^* reports truthfully, its utility is denoted by $u_{1^*} = u_{1^*}(\mathbf{a_{1^*}}, \mathbf{a_{2^*}}) = \min_{j \in [m]} \left\{ \frac{x_{1^*,j}}{a_{1^*,j}} \right\}$; when it uses best response strategy $\mathbf{b_{1^*}}$, its utility is $u'_{1^*} = u'_{1^*}(\mathbf{b_{1^*}}, \mathbf{a_{2^*}}) = \min_{j \in [m]} \left\{ \frac{1 - a_{2^*,j}u'_{2^*}}{a_{1^*,j}} \right\}$, where u'_{2^*} is the utility of buyer 2^*. Let $a_i^{\max} = \max_j\{a_{ij}\}, i = 1, \ldots, n$, $a_{i^*}^{\max} = \max_j\{a_{i^*,j}\}, i = 1, 2$. According to Lemma 5.1, $u'_{1^*} = \min_{j \in [m]} \left\{ \frac{1 - \frac{a_{2^*,j}}{a_{2^*}^{\max}} e_{2^*}}{a_{1j}} \right\}$.

We will prove $\zeta_1^{M_n} \leq \zeta_{1^*}^{M_2}$, together with $\zeta^{M_n} = \zeta_1^{M_n}$, we conclude $\zeta^{M_n} = \zeta_1^{M_n} \leq \zeta_{1^*}^{M_2} \leq \zeta^{M_2}$.

Before introducing the reduction, we need some notations for easy of reading. Let $\mathcal{A} = \left\{ j \mid a_{1j} = a_1^{\max} \right\}$; $\mathcal{B} = \left\{ j \mid 1 - e_1 \leq \sum_{i=2}^n a_{ij}u'_i \right\}$, and $z_1 = \max_{j \in \mathcal{B}}\{a_{1j}\}$, $\mathcal{B}_1 = \left\{ j \mid j \in \mathcal{B}, a_{1j} = z_1 \right\}$, $\mathcal{B}_2 = \left\{ j \mid j \in \mathcal{B}, a_{1j} < z_1 \right\}$, hence $\mathcal{B} = \mathcal{B}_1 \bigcup \mathcal{B}_2$; $\mathcal{C} = [m] \setminus (\mathcal{A} \bigcup \mathcal{B})$.

We claim that the set of items is well partitioned. Because $\mathcal{A} \neq \emptyset$, $\mathcal{A} \bigcap \mathcal{C} = \emptyset$, $\mathcal{B} \bigcap \mathcal{C} = \emptyset$, and $\mathcal{A} \bigcup \mathcal{B} \bigcup \mathcal{C} = [m]$ are trivial by definition. Moreover, $\mathcal{B} \neq \emptyset$ since if $\mathcal{B} = \emptyset$, then $1 - e_1 > \sum_{i=2}^n a_{ij}u'_i, \forall j \in [m]$, and

$$1 - e_1 = \sum_j p'_j(1 - e_1) > \sum_j p'_j \sum_{i=2}^n a_{ij}u'_i = 1 - e_1,$$

so a contradiction occurs, where p'_j is the equilibrium price when i_1 uses best response strategy. Specifically, $\mathcal{B}_1 \neq \emptyset$ as there always exists a maximum element in a compact nonempty set. Also, $\mathcal{A} \bigcap \mathcal{B} = \emptyset$. To prove this, assume $\mathcal{A} \bigcap \mathcal{B} \neq \emptyset$, then $\exists j_0 \in \mathcal{A} \bigcap \mathcal{B}$ such that $a_{1j_0} = a_1^{\max}$ and $1 - e_1 \leq \sum_{i=2}^n a_{ij_0}u'_i$, which implies

$$u'_1 = \min_j \left\{ \frac{1 - \sum_{i=2}^n a_{ij} u'_i}{a_{1j}} \right\} \le \frac{1 - \sum_{i=2}^n a_{ij_0} u'_i}{a_{1j_0}} \le \frac{e_1}{a_1^{\max}}.$$

However $u_1 \ge \frac{e_1}{a_1^{\max}}$; hence $\zeta^{M_n} = \zeta_1^{M_n} = \frac{u'_1}{u_1} \le 1$. A contradiction occurs.

The reduction is given below.

- Input of M_n: buyers' endowment e_i, utility $u_i = \min_j \left\{ \frac{x_{ij}}{a_{ij}} \right\}$, $i \in [n]$, $j \in [m]$.
- Construction of M_2: there are two buyers $1^*, 2^*$.
  - for 1^*, endowment $e_{1^*} = \max \left\{ \max_{j \in \mathcal{B}} \left\{ 1 - \sum_{i=2}^n a_{ij} u'_i \right\}, \frac{z_1}{a_1^{\max}} \right\}$, utility $u_{1^*} = \min_j \left\{ \frac{x_{1^*,j}}{a_{1^*,j}} \right\}$, where $a_{1^*,j} = a_{1j}$;
  - for 2^*, endowment $e_{2^*} = 1 - e_{1^*}$, utility $u_{2^*} = \min_j \left\{ \frac{x_{2^*,j}}{a_{2^*,j}} \right\}$,
    - when $j \in \mathcal{A}$,
      $$a_{2^*,j} = z_2 = \min \left\{ \min_{j \in \mathcal{A}} \left\{ K \cdot \frac{\sum_{i=2}^n a_{ij} u'_i}{1 - e_{1^*}} \right\}, \frac{1 - e_{1^*}}{1 - \frac{z_1}{a_1^{\max}} e_{1^*}} - \varepsilon \right\};$$
    - when $j \in \mathcal{B}$, $a_{2^*,j} = 1$;
    - when $j \in \mathcal{C}$, $a_{2^*,j} = \varepsilon$,

  where ε is arbitrarily small positive quantity and $K = \min_{j \in \mathcal{A}} \left\{ 1, \frac{1 - e_{1^*}}{\sum_{i=2}^n a_{ij} u'_i} \right\}$ is constant.

We claim that the utility function of buyer 2^* is well defined, since all of $a_{2^*,j} > 0$. In addition, $a_{2^*}^{\max} = 1$. That is, $a_{2^*,j}$ is maximized when $j \in \mathcal{B}$.

In the following, we are going to prove $\zeta_1^{M_n} = \frac{u'_1}{u_1} \le \frac{u'_{1^*}}{u_{1^*}} = \zeta_{1^*}^{M_2}$. This is done by proving $u'_1 \le u'_{1^*}$ and $u_1 \ge u_{1^*}$.

- When buyers 1 and 1^* use their best response strategies in markets M_n and M_2 respectively, we prove $u'_1 \le u'_{1^*}$. Note that in M_n, $u'_1 = \min_{j \in [m]} \left\{ \frac{1 - \sum_{i=2}^n a_{ij} u'_i}{a_{1j}} \right\}$, and in M_2, according to Lemma 5.1, $u'_{1^*} = \min_{j \in [m]} \left\{ \frac{1 - \frac{a_{2^*,j}}{a_{2^*}^{\max}}(1 - e_{1^*})}{a_{1j}} \right\}$. Therefore we can prove

  $$u'_1 = \min_{j \in [m]} \left\{ \frac{1 - \sum_{i=2}^n a_{ij} u'_i}{a_{1j}} \right\} \le \min_{j \in [m]} \left\{ \frac{1 - \frac{a_{2^*,j}}{a_{2^*}^{\max}}(1 - e_{1^*})}{a_{1j}} \right\} = u'_{1^*},$$

  by proving a stronger version

  $$1 - \sum_{i=2}^n a_{ij} u'_i \le 1 - \frac{a_{2^*,j}}{a_{2^*}^{\max}}(1 - e_{1^*}), \quad \forall j \in [m],$$

  or equivalently,

  $$\sum_{i=2}^n a_{ij} u'_i \ge \frac{a_{2^*,j}}{a_{2^*}^{\max}}(1 - e_{1^*}), \quad \forall j \in [m].$$

  We check the above inequality for each set of items.
  - when $j \in \mathcal{A}$, the inequality takes the form

    $$\sum_{i=2}^n a_{ij} u'_i \ge z_2(1 - e_{1^*}), \quad \forall j \in \mathcal{A}$$

    According to the definition of z_2, $z_2 \le K \cdot \frac{\sum_{i=2}^n a_{ij} u'_i}{1 - e_{1^*}} \le \frac{\sum_{i=2}^n a_{ij} u'_i}{1 - e_{1^*}}, \forall j \in \mathcal{A}$, where $K \le 1$ by definition. Hence the inequality holds.
  - when $j \in \mathcal{B}$, the inequality takes the form

    $$\sum_{i=2}^n a_{ij} u'_i \ge 1 - e_{1^*}, \quad \forall j \in \mathcal{B}$$

    According to the definition of e_{1^*}, $e_{1^*} \ge \max_{j \in \mathcal{B}} \left\{ 1 - \sum_{i=2}^n a_{ij} u'_i \right\}$. Hence the inequality holds.

– when $j \in \mathcal{C}$, the inequality takes the form

$$\sum_{i=2}^{n} a_{ij} u_i' \geq \epsilon(1 - e_{1^*}), \quad \forall j \in \mathcal{C}$$

This is true as ϵ is arbitrarily small positive quantity.

Therefore $u_1' \leq u_{1^*}'$.

- When buyers 1 and 1^* use their truthful strategies in markets M_n and M_2 respectively, we prove $u_1 \geq u_{1^*}$.
First, we claim that when buyer 1^* reports truthfully, the market equilibrium of M_2 is,

$$u_{1^*} = \frac{1 - z_2}{a_1^{\max} - z_1 z_2}, \quad u_{2^*} = \frac{a_1^{\max} - z_1}{a_1^{\max} - z_1 z_2},$$

$$\sum_j p_j^* = \begin{cases} \frac{a_1^{\max} e_{1^*} - z_1 + (1 - e_{1^*}) z_1 z_2}{(1 - z_2)(a_1^{\max} - z_1)} & \text{if } j \in \mathcal{A} \\ \frac{(1 - e_{1^*}) a_1^{\max} + z_1 z_2 e_{1^*} - z_2 a_1^{\max}}{(1 - z_2)(a_1^{\max} - z_1)} & \text{if } j \in \mathcal{B}_1 \\ 0 & \text{otherwise} \end{cases}$$

where p_j^* is the equilibrium price. Note that $\sum_{j=1}^{m} p_j^* = 1$ and $p_j^* > 0, \forall j \in \mathcal{A} \bigcup \mathcal{B}_1$. These values can be verified by the KKT conditions. To verify the linear complementarity,

– when $j \in \mathcal{A}$,

$$a_{1j} u_{1^*} + a_{2^*,j} u_{2^*} - 1 = a_1^{\max} \frac{1 - z_2}{a_1^{\max} - z_1 z_2} + z_2 \frac{a_1^{\max} - z_1}{a_1^{\max} - z_1 z_2} - 1 = 0$$

This is coincide with $p_j^* > 0, \forall j \in \mathcal{A}$.

– when $j \in \mathcal{B}_1$,

$$a_{1j} u_{1^*} + a_{2^*,j} u_{2^*} - 1 = z_1 \frac{1 - z_2}{a_1^{\max} - z_1 z_2} + \frac{a_1^{\max} - z_1}{a_1^{\max} - z_1 z_2} - 1 = 0$$

This is coincide with $p_j^* > 0, \forall j \in \mathcal{B}_1$.

– when $j \in \mathcal{B}_2$,

$$a_{1j} u_{1^*} + a_{2^*,j} u_{2^*} - 1 < z_1 \frac{1 - z_2}{a_1^{\max} - z_1 z_2} + \frac{a_1^{\max} - z_1}{a_1^{\max} - z_1 z_2} - 1 = 0$$

This is coincide with $p_j^* = 0, \forall j \in \mathcal{B}_2$.

– when $j \in \mathcal{C}$,

$$a_{1j} u_{1^*} + a_{2^*,j} u_{2^*} - 1 = a_{1j} \frac{1 - z_2}{a_1^{\max} - z_1 z_2} + \epsilon \cdot \frac{a_1^{\max} - z_1}{a_1^{\max} - z_1 z_2} - 1 < 0$$

This is coincide with $p_j^* = 0, \forall j \in \mathcal{C}$.

Hence $u_{1^*} = \frac{1 - z_2}{a_1^{\max} - z_1 z_2}$.

As $1 > z_2$ and $a_1^{\max} > z_1$, we have

$$a_1^{\max} u_1 - 1 > z_2(z_1 u_1 - 1),$$

so

$$u_1 > \frac{1 - z_2}{a_1^{\max} - z_1 z_2} = u_{1^*}.$$

Combining these two cases together, we conclude

$$\zeta_1^{M_n} = \frac{u_1'}{u_1} \leq \frac{u_{1^*}'}{u_{1^*}} = \zeta_{1^*}^{M_2}.$$

Therefore

$$\zeta^{M_n} = \zeta_1^{M_n} \leq \zeta_{1^*}^{M_2} \leq \zeta^{M_2}. \quad \square$$

Proof of Lemma 5.4. Given any 2-buyer m-item market M_2, we will construct a 2-buyer 2-item market M_2^* without decreasing the incentive ratio. Let the two buyers in market M_2 be buyer 1 and buyer 2. Their endowments are e_1 and e_2; their utilities are $\mathbf{a_1} = (a_{1j})$ and $\mathbf{a_2} = (a_{2j})$, $j = 1, \dots, m$. W.l.o.g., assume $\mathbf{b_2} = \mathbf{a_2}$ and $\mathbf{b_1}$ is buyer 1's best response strategy to $\mathbf{b_2}$. Let $u_i = u_i(\mathbf{a_1}, \mathbf{a_2}), u_i' = u_i'(\mathbf{b_1}, \mathbf{a_2}), i = 1, 2$. W.l.o.g., assume buyer 1 obtains a larger incentive ratio than buyer 2. All the notations in market M_2^* are defined accordingly.

Let $\mathcal{A} = \{j \mid a_{1j}u_1 + a_{2j}u_2 = 1\}$, $\mathcal{B} = \{j \mid a_{2j} = a_2^{\max}\}$. In other words, $\mathcal{A}$ is the set of items in M_2 such that the corresponding j-th constraints of (6) are tight, and $\mathcal{B}$ is the set of items in M_2 such that buyer 2 desires most. Note that $\mathcal{A} \neq \emptyset$ otherwise none of the constraints of the convex program is tight so the value of the objective function can be increased; $\mathcal{B} \neq \emptyset$ since there always exists a maximum element in a compact nonempty set. The construction of 2-item market M_2^* is given by the table below.

- Input of M_2: endowment e_i, utility $u_i = \min_j \left\{\frac{x_{ij}}{a_{ij}}\right\}$, $i = 1, 2$, $j = 1, \dots, m$.
- Construction of M_2^*: randomly choose $j_{1^*} \in \mathcal{A}$ and $j_{2^*} \in \mathcal{B}$, delete all other items. W.l.o.g., denote $j_{1^*} = j_1$, $j_{2^*} = j_2$.
  - for buyer 1^*, endowment $e_{1^*} = e_1$,
    utility $u_{1^*} = \min\left\{\frac{x_{1^*,1}}{a_{1^*,1}}, \frac{x_{1^*,2}}{a_{1^*,2}}\right\}$, where $a_{1^*,1} = a_{11}$, and
    * $a_{1^*,2} = a_{12}$, if $j_2 \in \mathcal{A}$ (i.e., $a_{12}u_1 + a_2^{\max}u_2 = 1$)
    * $a_{1^*,2} = \frac{1-a_2^{\max}u_2}{u_1}$, if $j_2 \notin \mathcal{A}$ (i.e., $a_{12}u_1 + a_2^{\max}u_2 < 1$)
  - for buyer 2^*, endowment $e_{2^*} = e_2$,
    utility $u_{2^*} = \min\left\{\frac{x_{2^*,1}}{a_{2^*,1}}, \frac{x_{2^*,2}}{a_{2^*,2}}\right\}$, where $a_{2^*,j} = a_{2j}$, $j = 1, 2$.

We will prove $\zeta_1^{M_2} \le \zeta_{1^*}^{M_2^*}$ and conclude that $\zeta^{M_2} = \zeta_1^{M_2} \le \zeta_{1^*}^{M_2^*} \le \zeta^{M_2^*}$. To prove $\zeta_1^{M_2} = \frac{u_1'}{u_1} \le \frac{u_{1^*}'}{u_{1^*}} = \zeta_{1^*}^{\bar{M}_2}$, we should verify that $u_1' \le u_{1^*}'$ and $u_1 = u_{1^*}$.

- $u_1 = u_{1^*}$.
  When both of buyers 1^* and 2^* report truthfully in M_2^*, their market equilibrium utilities are captured by the following convex program.

$$\begin{aligned} \max \quad & e_1 \log u_{1^*} + e_2 \log u_{2^*} \\ s.t. \quad & a_{11}u_{1^*} + a_{21}u_{2^*} \le 1 \\ & a_{1^*,2}u_{1^*} + a_2^{\max}u_{2^*} \le 1 \\ & u_{1^*}, u_{2^*} \ge 0. \end{aligned} \tag{A.5}$$

  For the case $j_2 \in \mathcal{A}$, $a_{1^*,2} = a_{12}$, we claim $u_{i^*} = u_i, i = 1, 2$. Obviously u_1, u_2 is a feasible solution of (A.5). If it is not the optimal solution of (A.5), denote the optimal solution of (A.5) by y_1, y_2. Then y_1, y_2 is also a feasible solution of the convex program to solve the market equilibrium for market M_2, and it improves the value of the objective function, which contradicts with the fact that u_1, u_2 is the market equilibrium solution.
  For the case $j_2 \notin \mathcal{A}$, $a_{1^*,2} = \frac{1-a_2^{\max}u_2}{u_1}$, we claim $u_{i^*} = u_i, i = 1, 2$, $p_{1^*} = \frac{a_2^{\max} - \frac{e_2}{u_2}}{a_2^{\max} - a_{21}}$, $p_{2^*} = \frac{\frac{e_2}{u_2} - a_{21}}{a_2^{\max} - a_{21}}$. It is easy to verify these values satisfy the KKT conditions of the convex program, so u_1, u_2 is the optimal solution.
- $u_1' \le u_{1^*}'$.
  When $j_2 \in A$, according to Lemma 5.1, the maximum utility of buyer 1 in M_2 by playing best response strategy is $u_1' = \min_{j\in[m]} \left\{\frac{1-\frac{a_{2j}}{a_2^{\max}}e_2}{a_{1j}}\right\}$; the maximum utility of buyer 1^* in M_2^* is $u_{1^*}' = \min_{j\in\{j_1,j_2\}} \left\{\frac{1-\frac{a_{2^*,j}}{a_{2^*}^{\max}}e_{2^*}}{a_{1^*j}}\right\} = \min_{j\in\{j_1,j_2\}} \left\{\frac{1-\frac{a_{2j}}{a_2^{\max}}e_2}{a_{1j}}\right\}$. Since $\{j_1, j_2\} \subset [m]$, $u_1' \le u_{1^*}'$.

  When $j_2 \notin \mathcal{A}$, the maximum utility of buyer 1 in M_2 is $u_1' = \min_{j\in[m]} \left\{\frac{1-\frac{a_{2j}}{a_2^{\max}}e_2}{a_{1j}}\right\}$; the maximum utility of buyer 1^* in M_2^* is

$$u_{1^*}' = \min_{j\in\{j_1,j_2\}} \left\{\frac{1-\frac{a_{2^*,j}}{a_{2^*}^{\max}}e_{2^*}}{a_{1^*,j}}\right\} = \min\left\{\frac{1-\frac{a_{21}}{a_2^{\max}}e_2}{a_{11}}, \frac{e_1}{a_{1^*,2}}\right\}$$

  We will prove $\frac{1-\frac{a_{21}}{a_2^{\max}}e_2}{a_{11}} \le \frac{e_1}{a_{1^*,2}}$; hence $u_{1^*}' = \frac{1-\frac{a_{21}}{a_2^{\max}}e_2}{a_{11}}$. Together with $\frac{1-\frac{a_{21}}{a_2^{\max}}e_2}{a_{11}} \ge \min_{j\in[m]}\left\{\frac{1-\frac{a_{2j}}{a_2^{\max}}e_2}{a_{1j}}\right\}$, we get $u_1' \le u_{1^*}'$.

As we have shown $u_{i^*} = u_i, i = 1, 2$, according to the construction of market M_2^*, it is easy to see that when the objective function of (A.5) is maximized, both of its constraints are tight. Hence

$$\begin{cases} a_{11}u_{1^*} + a_{21}u_{2^*} - 1 = 0 \\ a_{1^*,2}u_{1^*} + a_2^{\max}u_{2^*} - 1 = 0. \end{cases}$$

Therefore,

$$u_{2^*} = \frac{a_{11} - a_{1^*,2}}{a_{11}a_2^{\max} - a_{1^*,2}a_{21}}.$$

Recall that $u_{2^*} \geq \frac{e_{2^*}}{a_{2^*}^{\max}} = \frac{e_2}{a_2^{\max}}$, we have

$$\frac{a_{11} - a_{1^*,2}}{a_{11}a_2^{\max} - a_{1^*,2}a_{21}} \geq \frac{e_2}{a_2^{\max}}.$$

Thus,

$$\frac{1 - \frac{a_{21}}{a_2^{\max}}e_2}{a_{11}} \leq \frac{e_1}{a_{1^*,2}}.$$

Therefore, $u_1' \leq u_{1^*}'$. In conclusion, $\zeta^{M_2} = \zeta_1^{M_2} \leq \zeta_{1^*}^{M_2^*} \leq \zeta^{M_2^*}$. □

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