THE MOD-p HOMOLOGY OF THE CLASSIFYING SPACES OF CERTAIN GAUGE GROUPS

DAISUKE KISHIMOTO AND STEPHEN THERIAULT

Abstract. Let G be a compact connected simple Lie group of type $(n_1, \dots, n_l)$, where $n_1 < \cdots < n_l$. Let $\mathcal{G}_k$ be the gauge group of the principal G-bundle over S^4 corresponding to $k \in \pi_3(G) \cong \mathbb{Z}$. We calculate the mod-$p$ homology of the classifying space $B\mathcal{G}_k$ provided that $n_l < p - 1$.

1. Introduction

Let G be a Lie group and $P \to X$ be a principal G-bundle over a manifold X. Automorphisms of P are by definition G-equivariant self-maps of P covering the identity map of X. The topological group of automorphisms of P is called the *gauge group* of P; we denote this group by $\mathcal{G}(P)$. The classifying space of $\mathcal{G}(P)$ is denoted by $B\mathcal{G}(P)$.

Gauge groups are fundamental in modern physics and geometry. Since the classifying space $B\mathcal{G}(P)$ is homotopy equivalent to the moduli space of connections on P as in [AB], the topology of gauge groups over 4-manifolds and their classifying spaces has proved to be of immense value in studying diffeomorphism structures on 4-manifolds [D], Yang-Mills theory [AJ], and invariants of 3-manifolds [F]. Donaldson famously used the rational cohomology of $B\mathcal{G}(P)$ in the case when $G = SU(2)$ and X is a simply-connected 4-manifold in order to construct polynomial invariants to distinguish diffeomorphism types. Ever since, an important problem has been to calculate the mod-p (co)-homology of $B\mathcal{G}(P)$ when $G = SU(2)$ and X is a simply-connected 4-manifold for a prime p, in the hope of finding new polynomial invariants of diffeomorphism types.

A certain subring of the mod-2 cohomology of $B\mathcal{G}(P)$ was studied by Masbaum [Ma] when $G = SU(2)$ and X is a simply-connected closed 4-manifold, but otherwise nothing else is known. In terms of other Lie groups, Choi [C] has some partial results on the mod-2 homology for $G = Sp(n)$ and $X = S^4$. In this paper we make significant progress, completely calculating the mod-p homology of $B\mathcal{G}(P)$ for a family of Lie groups G when $X = S^4$. In particular, this includes the pivotal case of $G = SU(2)$ for $p \geq 5$.

2010 *Mathematics Subject Classification.* Primary 55R40, Secondary 81T13.
Key words and phrases. gauge group, classifying space, homology.

To state our results, we need some notation. Let G be a compact connected simple Lie group. Principal G-bundles over S^4 are classified by $\pi_3(G)$, where $\pi_3(G) \cong \mathbb{Z}$ since G is simple. Let $\mathcal{G}_k$ denote the gauge group of a principal G-bundle over S^4 corresponding to $k \in \mathbb{Z} \cong \pi_3(G)$. Let $\mathrm{Map}_k(S^4, BG)$ be the component of the space of continuous (not necessarily pointed) maps from S^4 to BG which are of degree k, and similarly define $\mathrm{Map}_k^*(S^4, BG)$ with respect to pointed maps. There is a fibration

$$\mathrm{Map}_k^*(S^4, BG) \to \mathrm{Map}_k(S^4, BG) \xrightarrow{ev} BG \tag{1.1}$$

where ev evaluates a map at the basepoint of S^4. Let $G\langle 3\rangle$ be the three-connected cover of G. For each $k \in \mathbb{Z}$, the space $\mathrm{Map}_k^*(S^4, BG)$ is homotopy equivalent to $\Omega^3 G\langle 3\rangle$. By [AB, G], there is a homotopy equivalence $B\mathcal{G}_k \simeq \mathrm{Map}_k(S^4, BG)$. Thus there is a homotopy fibration

$$\Omega^3 G\langle 3\rangle \to B\mathcal{G}_k \xrightarrow{ev} BG. \tag{1.2}$$

The Lie group G has *type* $(n_1, \dots, n_l)$ if the rational cohomology of G is generated by elements in degrees $2n_1 - 1, \dots, 2n_l - 1$, where $n_1 < \cdots < n_l$. Unless otherwise indicated, homology is assumed to be with mod-p coefficients.

Theorem 1.1. *Let G be a compact connected simple Lie group of type $(n_1, \dots, n_l)$ and let p be a prime. If $n_l < p - 1$ then there is an isomorphisms of $\mathbb{Z}/p\mathbb{Z}$-vector spaces*

$$H_*(B\mathcal{G}_k) \cong H_*(BG) \otimes H_*(\Omega^3 G\langle 3\rangle).$$

Under the assumption of Theorem 1.1 the Lie group G is p-locally homotopy equivalent to the product $\prod_{i=1}^{l} S^{2n_i - 1}$, so we can also calculate the Poincaré series of $H_*(B\mathcal{G}_k)$ by Theorem 1.1 (Corollary 4.5). Remarkably, Theorem 1.1 implies that the mod-p homology of $B\mathcal{G}_k$ is independent of k for p large, whereas there is more than one p-local homotopy type in the family $\{B\mathcal{G}_k\}_{k \in \mathbb{Z}}$ for p large as was proved in [KT1]. The approach to Theorem 1.1 is to consider the Serre spectral sequence applied to the fiberwise coproduct of (1.2). Control is obtained over the differentials by showing that the first nontrivial differential is a transgression on a certain element, which is not obvious, and then atomicity-style arguments (cf. [S]) are used to show the spectral sequence must collapse at the E^2-term.

Of key interest is when $G = SU(2)$. Theorem 1.1 holds if $p \geq 5$ in this case. We also obtain partial results for the prime 3, which are of a different flavour than those in Theorem 1.1. Let (m, n) denote the gcd of integers m and n. For $(k, 3) = 1$, it is the Serre spectral sequence for the homotopy fibration $SU(2) \to \Omega^3 SU(2)\langle 3\rangle \to B\mathcal{G}_k$ induced from (1.2) that collapses at the E^2-term.

Theorem 1.2. *Let $G = SU(2)$ and $p = 3$. If $(k, 3) = 1$ then there is an isomorphism of $\mathbb{Z}/3\mathbb{Z}$-vector spaces*

$$H_*(B\mathcal{G}_k) \cong H_*(\Omega^3 S^3\langle 3\rangle)/(x_3)$$

where x_3 is a generator of $H_3(\Omega^3 S^3\langle 3\rangle) \cong \mathbb{Z}/3\mathbb{Z}$ and (x_3) is the ideal generated by x_3.

The case $(k,3)=3$ is still open. The difference of the mod-p homology of $B\mathcal{G}_k$ for $G=SU(2)$ in Theorems 1.1 and 1.2 comes from the homotopy commutativity of $SU(2)$; it is homotopy commutative if and only if $p\ge 5$ as in [Mc]. This is notable because the product decomposition $\mathcal{G}_k\simeq G\times\Omega^4G\langle 3\rangle$ as A_n-spaces is guaranteed by the higher homotopy commutativity of G as in [KiKo, KT1], whereas Theorem 1.1 shows the homological product decomposition as A_∞-spaces.

Acknowledgement: The authors would like to thank the referee for the helpful comments. The first author was partly supported by JSPS KAKENHI (No. 17K05248).

2. Serre spectral sequence

Consider a homotopy fibration

$$F\to E\to B \tag{2.1}$$

over a path-connected base B such that F is an H-space and there is a fiberwise action of F on E which restricts to the multiplication of F. We assume that $\pi_1(B)$ acts trivially on $H_*(F)$. Let (E^r,d^r) denote the associated homology Serre spectral sequence.

Lemma 2.1. *There is a coalgebra map*

$$\mu\colon E^r_{p,q}\otimes H_{q'}(F)\to E^r_{p,q+q'}$$

having the following properties.

(1) *The map*
$$\mu\colon H_p(B)\otimes H_q(F)=E^2_{p,0}\otimes H_q(F)\to E^2_{p,q}$$
coincides with the canonical isomorphism;

(2) *For $x\in E^r$ and $y\in H_*(F)$,*
$$d^r(\mu(x\otimes y))=\mu(d^r(x)\otimes y).$$

Proof. Since the action of F on E is fiberwise, there is a homotopy commutative diagram

$$\begin{tikzcd} F\times F \arrow[r]\arrow[d] & F\times E \arrow[r]\arrow[d] & B \arrow[d, equal] \\ F \arrow[r] & E \arrow[r] & B. \end{tikzcd}$$

Since rows of this diagram are homotopy fibrations, there is a map between associated homology Serre spectral sequences. Since the homology Serre spectral sequences of the top row is isomorphic with $(E^r\otimes H_*(F),d^r\otimes 1)$, we obtain the map μ. By definition, μ is a coalgebra map. The second statement holds because the differential on $H_*(F)$ in $E^r\otimes H_*(F)$ is trivial. Let $p\colon E\to B$ denote the projection. For each $y\in E$, $\pi^{-1}(\pi(y))$ is homotopy equivalent to the orbit space $y\cdot F$ including y. Therefore, by the construction of the Serre spectral sequence, the first statement holds. $\square$

Corollary 2.2. *If $d^2 = \cdots = d^{r-1} = 0$, then the map $\mu\colon E^r_{p,0} \otimes H_q(F) \to E^r_{p,q}$ coincides with the canonical isomorphism*

$$E^r_{p,q} \cong H_p(B) \otimes H_q(F).$$

Let $\overline{F} \to \overline{E} \to B$ be a homotopy fibration which is a homotopy retract of (2.1). Let $(\overline{E}^r, \overline{d}^r)$ denote the associated homology Serre spectral sequence.

Lemma 2.3. *If $\overline{d}^2 = \cdots = \overline{d}^{r-1} = 0$ and $\overline{d}^r \neq 0$, then the following statements hold:*

1. *$\overline{d}^r x \neq 0$ for some $x \in H_*(B) = \overline{E}^r_{*,0}$;*
2. *If y is an element of least degree in $H_*(B) = \overline{E}^r_{*,0}$ with $\overline{d}^r y \neq 0$, then y is transgressive and $\overline{d}^r(y)$ is a primitive element of $H_{*-1}(F) = \widehat{E}^r_{0,*-1}$.*

Proof. Let $i\colon \overline{E}^2 \to E^2$ and $q\colon E^2 \to \overline{E}^2$ denote the inclusion and the retraction, respectively. Since

$$d^2(x) = d^2(i_*(x)) = i_*(\overline{d}^2(x)) = 0$$

the hypothesis that $\overline{d}^2 = 0$ implies that $d^2 = 0$. Therefore $\overline{E}^3 = \overline{E}^2$ is a retract of $E^3 = E^2$. Iterating this argument, since $\overline{d}_3 = \cdots = \overline{d}^{r-1} = 0$, we also obtain $d^2 = \cdots = d^{r-1} = 0$ and therefore $\overline{E}^r$ is a retract of E^r. The inclusion and retraction at the r^{th}-stage may still be denoted by $i\colon \overline{E}^2 \to E^2$ and $q\colon E^2 \to \overline{E}^2$. Suppose that $\overline{d}^r(x) = 0$ for all $x \in H_p(B) = \widehat{E}^r_{p,0}$. Then

$$d^r(x) = d^r(i_*(x)) = i_*(\overline{d}^r(x)) = 0,$$

implying $d^r = 0$ by Lemma 2.1 and Corollary

Let y be an element of least degree element in $H_*(B) = \overline{E}^r_{*,0}$ such that $\overline{d}^r y \neq 0$. Let Δ denote comultiplication, and set

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \sum_i y'_i \otimes y''_i$$

where $|y'_i| < |y|$ and $|y''_i| < |y|$ for each i. Since y is an element of least degree in $H_*(B) = E^r_{*,0}$ with $d^r y \neq 0$, we have $d^r y'_i = 0$ and $d^r y''_i = 0$ for each i. Therefore

$$\Delta(d^r(y)) = d^r(\Delta(y)) = d^r\left(y \otimes 1 + 1 \otimes y + \sum_i y'_i \otimes y''_i\right) = d^r(y) \otimes 1 + 1 \otimes d^r(y)$$

so $d^r(y)$ is primitive.

If $r < |y|$, then by Corollary 2.2, $d^r(y) = \mu(a \otimes b)$ for some non-trivial $a \in H_{|y|-r}(B)$ and $b \in H_{r-1}(F)$. This is impossible because $\mu(a \otimes b)$ is not primitive by Lemma 2.1. Thus $r \geq |y|$. Clearly, $r \leq |y|$ since $r > |y|$ implies $d^r(y)$ lands in the second quadrant of the spectral sequence, which is zero. Therefore $r = |y|$, implying that y is transgressive. Moreover, since

$$\overline{d}^r(y) = \overline{d}^r(q_*(y)) = q_*(d^r(y))$$

and $d^r y$ is primitive, $\overline{d}^r y$ is also primitive. Therefore the second statement is proved. $\square$

Next, we consider a family of homotopy fibrations $F_n \to E_n \to B$ with a common base B for $n \in \mathbb{Z}$. Let (E_n^r, d_n^r) denote the associated homology Serre spectral sequence. We can form a homotopy fibration

$$\coprod_{n\in\mathbb{Z}} F_n \to \coprod_{n\in\mathbb{Z}} E_n \to B.$$

Let $(\widehat{E}^r, \widehat{d}^r)$ denote the associated homology spectral sequence.

Lemma 2.4. *If $(\widehat{E}^r, \widehat{d}^r)$ collapses at the second term, then so does (E_n^r, d_n^r) for each $n \in \mathbb{Z}$.*

Proof. Since there are isomorphisms

$$(E_n^2)_{p,q} \cong H_p(B) \otimes H_q(F_n) \quad \text{and} \quad \widehat{E}^2_{p,q} \cong H_p(B) \otimes \left(\bigoplus_{n\in\mathbb{Z}} H_q(F_n) \right),$$

the inclusion $E_n \to E$ induces an injection $E_n^2 \to \widehat{E}^2$. Then the statement is proved by induction on r. $\square$

3. The mod-p homology of $\Omega^3 G\langle 3\rangle$ when G is p-regular

Localize at an odd prime p and take homology with mod-p coefficients. If G is p-regular of type $(n_1, \ldots, n_l)$ then there is a homotopy equivalence $G \simeq \prod_{i=1}^{l} S^{2n_i-1}$. Therefore

$$\Omega^3 G\langle 3\rangle \simeq \Omega^3 S^3\langle 3\rangle \times \prod_{i=2}^{l} \Omega^3 S^{2n_i-1}. \tag{3.1}$$

This is an equivalence of H-spaces and so induces an isomorphism of Hopf algebras in homology. In this section we record a property of $\Omega^3 G\langle 3\rangle$ which will be important later. This begins with a general definition.

In general, for a path-connected space X of finite type, let $PH_*(X)$ be the subspace of primitive elements in $H_*(X)$. Let

$$\mathcal{P}^n_*\colon H_q(X) \to H_{q-2(p-1)n}(X)$$

be the dual of the Steenrod operation $\mathcal{P}^n$. For $r \geq 1$, let β^r be the r^{th}-Bockstein. Let $MH_*(X)$ be the subspace of $PH_*(X)$ defined by

$$MH_*(X) = \{x \in PH_*(X) \mid \mathcal{P}^n_*(x) = 0 \text{ for all } n > 0 \text{ and } \beta^r(x) = 0 \text{ for all } r \geq 1\}.$$

Since $MH_*(X)$ records information about the primitive elements in $H_*(X)$, if $X \simeq A \times B$ then

$$MH_*(X) = MH_*(A \times B) = MH_*(A) \oplus MH_*(B).$$

Now consider $MH_*(\Omega^3 G\langle 3\rangle)$. The product decomposition (3.1) implies that

$$MH_*(\Omega^3\langle G\rangle) = MH_*(\Omega^3 S^3\langle 3\rangle) \oplus \left(\bigoplus_{i=2}^{l} MH_*(\Omega^3 S^{2n_i-1})\right). \tag{3.2}$$

By [CLM] there is an isomorphism of Hopf algebras

$$H_*(\Omega^3 S^{2n+1}) \cong \bigotimes_{k\geq 1, j\geq 0} \Lambda(a_{2(np^k-1)p^j-1}) \otimes \bigotimes_{k\geq 1, j\geq 1} \mathbb{Z}/p\mathbb{Z}[b_{2(np^k-1)p^j-2}] \otimes \bigotimes_{k\geq 0} \mathbb{Z}/p\mathbb{Z}[c_{2n^k-2}] \tag{3.3}$$

such that $|a_i| = |b_i| = i$. Here, the generators are primitive and many are related by the action of the dual Steenrod algebra. Selick [S] determined $MH_*(\Omega^3 S^{2n+1})$ in full, we record only the subset of elements of odd degree.

Lemma 3.1. *If $n > 1$ then $MH_{odd}(\Omega^3 S^{2n+1}) = \mathbb{Z}/p\mathbb{Z}\{a_{2np-3}\}$.*

In the references that follow for $\Omega^3 S^3\langle 3\rangle$, the statements in [Th] are in terms of Anick spaces, but by [GT] the space $\Omega S^3\langle 3\rangle$ is homotopy equivalent to the Anick space $T^{2p+1}(p)$ for $p \geq 3$. By [Th, Proposition 4.1], for p odd there is an isomorphism of Hopf algebras

$$H_*(\Omega^3 S^3\langle 3\rangle) \cong H_*(\Omega^2 S^{2p-1}) \otimes H_*(\Omega^3 S^{2p+1}) \tag{3.4}$$

which respects the action of the dual Steenrod operations and the Bockstein operations. This can be phrased in terms of a generating set using the isomorphism of Hopf algebras

$$H_*(\Omega^2 S^{2p-1}) \cong \bigotimes_{k=0}^{\infty} \Lambda(\bar{a}_{2(p-1)p^k-1}) \otimes \bigotimes_{k=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[\bar{b}_{2(p-1)p^k-2}] \tag{3.5}$$

proved in [CLM], where $|\bar{a}_i| = |\bar{b}_i| = i$, and the $n = p$ case of (3.3). Again, the generators are primitive and many are related by the action of the dual Steenrod algebra. A description of $MH_*(\Omega^3\langle 3\rangle)$ in full was given in [Th, Lemma 4.2], but again we need only record the subset of elements of odd degree.

Lemma 3.2. $MH_{odd}(\Omega^3 S^3\langle 3\rangle) = \mathbb{Z}/p\mathbb{Z}\{\bar{a}_{2p-3}\}$.

Combining (3.2), Lemmas 3.1 and 3.2 we obtain the following.

Lemma 3.3. *If G is p-regular of type $(n_1, \ldots, n_l)$ then*

$$MH_{odd}(\Omega^3 G\langle 3\rangle) = \mathbb{Z}/p\mathbb{Z}\{\bar{a}_{2p-3}, a_{2n_2p-3}, \ldots, a_{2n_lp-3}\}.$$

4. The proof of Theorem 1.1

We continue to localize at an odd prime p and take homology with mod-p coefficients. Let G be a compact connected simple Lie group of type $(n_1, \ldots, n_l)$. Consider the homotopy fibration sequence

$$G \xrightarrow{\partial_k} \Omega^3 G\langle 3\rangle \xrightarrow{g_k} B\mathcal{G}_k \xrightarrow{ev} BG.$$

First, we show properties of $\partial_k \colon G \to \Omega^3 G\langle 3\rangle$.

Lemma 4.1. *Suppose that G is p-regular. Then ∂_k are null homotopic for all k if and only if $n_l < p-1$.*

Proof. Let $\epsilon_i \colon S^{2n_i-1} \to G$ be the inclusion for $i = 1, \ldots, l$, where $G \simeq \prod_{i=1}^{l} S^{2n_i-1}$. By [L] ∂_k corresponds to the Samelson product $\langle k\epsilon_1, 1_G\rangle$ through the adjoint congruence $[G, \Omega^3 G\langle 3\rangle] \cong [S^3 \wedge G, G]$, where $\Omega^3 G\langle 3\rangle$ is homotopy equivalent to the component of $\Omega^3 G$ containing the basepoint. By the linearity of Samelson products, $\langle k\epsilon_1, 1_G\rangle = k\langle \epsilon_1, 1_G\rangle$. Thus we aim to get a condition that guarantees the triviality of $\langle \epsilon_1, 1_G\rangle$. Arguing as in [KaKi], we can see that $\langle \epsilon_1, 1_G\rangle$ is trivial if and only if $\langle \epsilon_1, \epsilon_i\rangle$ is trivial for all i. It is shown that $\langle \epsilon_1, \epsilon_i\rangle$ is trivial for all i if and only if $n_l < p-1$ in [KT2] when G is a classical group and in [HKO] when G is an exceptional group. Thus the proof is complete. □

Since $\mathcal{G}_k$ is homotopy equivalent to the homotopy fiber of ∂_k, the following is immediate from Proposition 4.1.

Corollary 4.2. *If $n_l < p-1$ then there is a homotopy equivalence $\mathcal{G}_k \simeq G \times \Omega^4 G\langle 3\rangle$.*

Next, we consider the map $g_k \colon \Omega^3 G\langle 3\rangle \to B\mathcal{G}_k$.

Lemma 4.3. *If $n_l < p-1$ then the restriction of $(g_k)_*$ to $MH_{odd}(\Omega^3 G\langle 3\rangle)$ is an injection.*

Proof. First, for $n \geq 2$, consider the homotopy fibration $\Omega^4 S^{2n+1} \to * \to \Omega^3 S^{2n+1}$. We claim that the element $a_{2np-3} \in H_*(\Omega^3 S^{2n+1})$ transgresses to a nonzero element in $H_*(\Omega^4 S^{2n+1})$. To see this, let $E^2 \colon S^{2n-1} \to \Omega^2 S^{2n+1}$ be the double suspension. Let W_n be the homotopy fibre of E^2. Then there is a homotopy fibration $\Omega S^{2n+1} \xrightarrow{\Omega E^2} \Omega^3 S^{2n+1} \to W_n$. By [CLM] this fibration induces an isomorphism of Hopf algebras

$$H_*(\Omega^3 S^{2n+1}) \cong H_*(\Omega S^{2n+1}) \otimes H_*(W_n).$$

In particular, since W_n is $(2np-4)$-connected, the element $a_{2np-3} \in H_*(\Omega^3 S^{2n+1})$ corresponds to an element $c \in H_{2np-3}(W_n)$. On the other hand, W_n has a single cell in dimension

$2np-3$, so c represents the inclusion of the bottom cell. Now consider the homotopy fibration diagram

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \arrow[r, "\Omega^2 E^2"] \arrow[d] & \Omega^4 S^{2n+1} \arrow[r] \arrow[d] & \Omega W_n \arrow[d] \\
* \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega S^{2n+1} \arrow[r, "\Omega E^2"] & \Omega^3 S^{2n+1} \arrow[r] & W_n.
\end{tikzcd}$$

As c represents the bottom cell in $H_{2np-3}(W_n)$, it transgresses nontrivially to a class $d \in H_{2np-4}(\Omega W_n)$. Since a_{2np-3} maps to c, the naturality of the transgression implies that a_{2np-3} must transgress to a nontrivial class in $H_{2np-4}(\Omega^4 S^{2n+1})$.

Next, consider the homotopy fibration diagram

$$\begin{tikzcd}
\Omega^4 G\langle 3\rangle \arrow[r, "\Omega g_k"] \arrow[d] & \mathcal{G}_k \arrow[r, "\Omega ev"] \arrow[d] & G \arrow[d] \\
* \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega^3 G\langle 3\rangle \arrow[r, "g_k"] & B\mathcal{G}_k \arrow[r, "ev"] & BG.
\end{tikzcd} \tag{4.1}$$

By Corollary 4.2, the map Ωg_k has a left homotopy inverse. In particular, $(\Omega g_k)_*$ is an injection. Since G is p-regular for $n_l < p$, the left column in the fibration diagram is a product of the homotopy fibrations $\Omega^4 S^3\langle 3\rangle \to * \to \Omega^3 S^3\langle 3\rangle$ and, for $2 \le i \le n_l$, $\Omega^4 S^{2n_i-1} \to * \to \Omega^3 S^{2n_i-1}$. Therefore, by the argument in the first paragraph of the proof, the element $a_{2n_i-1} \in MH_*(\Omega^3 G\langle 3\rangle)$ transgresses to a nontrivial element in $H_{2n_ip-4}(\Omega^4 G\langle 3\rangle)$. As this element injects into $H_{2n_ip-4}(\mathcal{G}_k)$, the naturality of the transgression in (4.1) implies that $(g_k)_*(a_{2n_ip-3})$ must be nontrivial.

Finally, consider the element $\bar{a}_{2p-3} \in H_*(\Omega^3 G\langle 3\rangle)$. It comes from an element $x \in H_{2p-3}(\Omega^3 S^3\langle 3\rangle)$. The description of $H_*(\Omega^3 S^3\langle 3\rangle)$ in (3.4) implies that $\Omega^3 S^3\langle 3\rangle$ is $(2p-4)$-connected and has a single cell in dimension $2p-3$. Thus x represents the inclusion of the bottom cell. Therefore, x transgresses to a nontrivial element in $H_{2p-4}(\Omega^4 S^3\langle 3\rangle)$, and so $\bar{a}_{2p-3}$ transgresses to a nontrivial element in $H_{2p-4}(\Omega^4 G\langle 3\rangle)$. Since $(\Omega g_k)_*$ is an injection, the naturality of the transgression in (4.1) implies that $(g_k)_*(a_{2p-3})$ is nontrivial. □

Consider the fibration

$$\mathrm{Map}^*(S^4, BG) \to \mathrm{Map}(S^4, BG) \xrightarrow{ev} BG \tag{4.2}$$

where ev is the evaluation map at the basepoint of S^4. Since G is p-regular and we are localizing at the prime p, S^3 is a homotopy retract of G. Therefore (4.2) is a homotopy retract of the fibration

$$\mathrm{Map}^*(\Sigma G, BG) \to \mathrm{Map}(\Sigma G, BG) \xrightarrow{ev} BG. \tag{4.3}$$

We may identify $\mathrm{Map}^*(\Sigma G, BG)$ with $\mathrm{Map}^*(G,G)$ by the adjoint congruence. $\mathrm{Map}^*(G,G)$ is an H-space by the composite of maps, and there is a fiberwise action of $\mathrm{Map}^*(G,G)$ on $\mathrm{Map}(\Sigma G, BG)$ given by

$$\mathrm{Map}(\Sigma G, BG) \times \mathrm{Map}^*(G,G) \to \mathrm{Map}(\Sigma G, BG), \quad (f,g) \mapsto f \circ \Sigma g.$$

Thus Lemma 2.3 applies to the fibration (4.2). The inclusion of the fibre in (4.2) may be identified with the coproduct of the maps $\mathrm{Map}^*_k(S^4, BG) \to \mathrm{Map}_k(S^4, BG)$ for all $k \in \mathbb{Z}$. Equivalently, this is the coproduct of the maps $g_k \colon \Omega^3 G\langle 3\rangle \to B\mathcal{G}_k$ for all $k \in \mathbb{Z}$. Each $(g_k)_*$ is injective on $MH_{odd}(\mathrm{Map}^*(S^4, BG))$ by Lemma 4.3, so the coproduct is as well. This leads to the mod-p homology Serre spectral sequence for (4.2) collapsing at the E^2-term.

Proposition 4.4. *Let G be a compact connected simple Lie group of type $(n_1, \dots, n_l)$, let p be a prime, and suppose that $n_l < p-1$. Then the mod-p homology Serre spectral sequence for (4.2) is totally nonhomologous to zero.*

Proof. Let (E^r, d^r) denote the mod-p homology Serre spectral sequence for (4.2). Let $z \in H_m(\mathrm{Map}^*(S^4, BG))$ be an element in the kernel of g_* of least dimension, where $g\colon \mathrm{Map}^*(S^4, BG) \to \mathrm{Map}(S^4, BG)$ is the fiber inclusion of (4.2). We begin by establishing some properties of z.

Property 1: z is primitive. If not, then $\overline{\Delta}(z) = \Sigma_{\alpha \in A} z'_\alpha \otimes z''_\alpha$ for some elements z'_α, z''_α of degrees $< m$ such that $\{z'_\alpha \otimes z''_\alpha\}_{\alpha \in A}$ is linearly independent, where $\overline{\Delta}$ is the reduced diagonal. The reduced diagonal is natural for any map of spaces, so $(g_* \otimes g_*) \circ \overline{\Delta} = \overline{\Delta} \circ g_*$. Now

$$(g_* \otimes g_*) \circ \overline{\Delta}(z) = (g_* \otimes g_*)(\Sigma_\alpha z''_\alpha \otimes z''_\alpha) = \Sigma_\alpha g_*(z'_\alpha) \otimes g_*(z''_\alpha)$$

is a sum of linearly independent elements since z'_α, z''_α are of degrees $< m$ while the element of least degree in $\operatorname{Ker} g_*$ is of degree m. On the other hand,

$$\overline{\Delta} \circ g_*(z) = \overline{\Delta}(0) = 0.$$

This contradiction implies that $\overline{\Delta}(z)$ must be 0; that is, z is primitive.

Property 2: $\mathcal{P}^n_*(z) = 0$ for every $n \geq 1$. Suppose $\mathcal{P}^n_*(z) = y$ for some $n \geq 1$, where y is nonzero. The (dual) Steenrod operations are natural for any map of spaces, so $g_* \mathcal{P}^n_* = \mathcal{P}^n_* g_*$. Now

$$g_* \mathcal{P}^n_*(z) = g_*(y)$$

is nonzero, since $|y| < m$ while the element of least degree in $\operatorname{Ker} g_*$ is of degree m. On the other hand,

$$\mathcal{P}^n_* g_*(z) = \mathcal{P}^n_*(0) = 0.$$

This contradiction implies that $\mathcal{P}^n_*(z) = 0$ for every $n \geq 1$.

Property 3: $\beta^r(z) = 0$ for every $r \geq 1$. The reasoning is exactly as in the proof of Property 2.

Property 4: z is in the image of the transgression for the mod-p homology Serre spectral sequence for the homotopy fibration (4.2). By assumption, the first nontrivial differential d^r in the mod-p homology Serre spectral sequence for (4.2) satisfies the properties of Lemma 2.3. In particular, d^r is determined by how it acts on the elements in $H_*(BG)$. The E^2-term of the spectral sequence is $H_*(BG) \otimes H_*(\mathrm{Map}^*(S^4, BG))$. Since z is an element of least degree in the kernel of $(g_k)_*$ and is of degree m, the map $(g_k)_*$ is an injection in degrees $< m$. Thus the spectral sequence is totally non-homologous to zero in this degree range, implying that it collapses at E^2. In particular, the differential d^r on an element of $H_*(BG)$ of degree $\leq m$ is zero, and so as d^r satisfies the properties of Lemma 2.3, $d^r(x \otimes y) = 0$ for $x \in H_t(BG)$ with $t \leq m$ and $y \in H_*(\mathrm{Map}^*(S^4, BG))$. Hence, for degree reasons, if z is in the image of d^r then the only possibility is that $z = d^r(x)$ and $r = m+1$ for some $x \in H_{m+1}(BG)$. That is, z is in the image of the transgression. On the other hand, as $z \in \operatorname{Ker} g_*$, it cannot survive the spectral sequence and so must be hit by some differential.

Property 5: z has odd degree. By Property 4, in the mod-p homology Serre spectral sequence for (4.2), we have $z = d^{m+1}(x)$ for some $x \in H_{m+1}(BG)$. As $H_*(BG)$ is concentrated in even degrees, the degree of z must be odd.

Let's examine the consequences of Properties 1 to 5. Collectively, Properties 1 to 3 imply that $z \in MH_*(\mathrm{Map}^*(S^4, BG))$. Property 5 then implies that $z \in MH_{odd}(\mathrm{Map}^*(S^4, BG))$. But g_* is an injection on $MH_{odd}(\mathrm{Map}^*(S^4, BG))$ as we saw above, implying in turn that $z \notin \operatorname{Ker} g_*$, a contradiction. Thus g_* is an injection, and this completes the proof. $\square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.4, the mod-p homology Serre spectral sequence for (4.2) collapses at the E^2-term. Thus, as (4.2) is the fiberwise coproduct of the fibrations $\mathrm{Map}_k^*(S^4, BG) \to \mathrm{Map}_k(S^4, BG) \xrightarrow{ev} BG$ in (1.1) for all $k \in \mathbb{Z}$, it follows from Lemma 2.4 that the mod-p homology Serre spectral sequence for each fibration collapses at the E^2-term. Equivalently, the mod-p homology Serre spectral sequence for $\Omega^3 G\langle 3\rangle \to B\mathcal{G}_k \xrightarrow{ev} BG$ collapses at the E^2-term for all $k \in \mathbb{Z}$. $\square$

We calculate the Poincaré series of the mod-p homology of $B\mathcal{G}_k$. Let $P_t(X)$ denote the Poincaré series of the mod-p homology of a space X. Let

$$P(t) = \prod_{k\geq 0}(1 + t^{2(p-1)p^k-1}) \prod_{k\geq 1} \frac{1}{1 - t^{2(p-1)p^k-2}}$$

$$Q_n(t) = \prod_{k\geq 1, j\geq 0}(1 + t^{2(np^k-1)p^j-1}) \prod_{k\geq 1, j\geq 1} \frac{1}{1 - t^{2(np^k-1)p^j-2}} \prod_{k\geq 0} \frac{1}{1 - t^{2n^k-2}}.$$

Corollary 4.5. *Under the same hypothesis of Theorem 1.1, the Poincaré series of the mod-p homology of $B\mathcal{G}_k$ is given by*

$$P_t(B\mathcal{G}_k) = P(t)Q_{p-1}(t)\prod_{i=2}^{l} Q_{n_i-1}(t)\prod_{i=1}^{l}\frac{1}{1-t^{2n_i}}.$$

Proof. By Theorem 1.1, $P_t(B\mathcal{G}_k) = P_t(BG)P_t(\Omega^3 G\langle 3\rangle)$. Since $n_l < p-1$, the mod-p cohomology of BG is a polynomial algebra with generators in dimension $2n_1, \ldots, 2n_l$, and so $P_t(BG) = \prod_{i=1}^{l}\frac{1}{1-t^{2n_i}}$. By (3.1), $P_t(\Omega^3 G\langle 3\rangle) = P_t(\Omega^3 S^3\langle 3\rangle)\prod_{i=2}^{l} P_t(\Omega^3 S^{2n_i-1})$. By (3.3), (3.4) and (3.5), $P_t(\Omega^3 S^3\langle 3\rangle) = P(t)Q_{p-1}(t)$ and $P_t(\Omega^3 S^{2n_i-1}) = P_{n_i-1}(t)$, completing the proof. □

5. The mod-3 homology of $SU(2)$-gauge groups

The mod-p homology of $B\mathcal{G}_k$ for $G = SU(2)$ and $p \geq 5$ is given by Theorem 1.1. Since $SU(2)$-gauge groups are pivotal in Donaldson Theory, in this section we expand beyond the statement of Theorem 1.1 by also calculating the mod-3 homology.

Since there is a homeomorphism $SU(2) \cong S^3$, we phrase what follows in terms of S^3. By (1.2) there is a homotopy fibration sequence

$$S^3 \xrightarrow{\partial_k} \Omega^3 S^3\langle 3\rangle \xrightarrow{g_k} B\mathcal{G}_k \xrightarrow{ev} BS^3.$$

By [K] we have the following.

Lemma 5.1. *The map $\partial_1 \colon S^3 \to \Omega^3 S^3\langle 3\rangle$ has order 12.*

In the proof of Lemma 4.1 it is shown that $\partial_k = k \circ \partial_1$. In particular, if we localize at $p = 3$ then ∂_k has order $3/(k,3)$. Observe that for $p = 3$, the description of $H_*(\Omega^3 S^3\langle 3\rangle)$ in (3.4) implies that $\Omega^3 S^3\langle 3\rangle$ is 2-connected with a single cell in dimension 3.

Lemma 5.2. *If $p = 3$ and $(k,3) = 1$ then the map $\partial_k \colon S^3 \to \Omega^3 S^3\langle 3\rangle$ induces an injection in mod-3 homology.*

Proof. By hypothesis, $(k,3) = 1$ so the above discussion implies that ∂_k is nontrivial. Thus it represents a nontrivial element in $\pi_3(\Omega^3 S^3\langle 3\rangle) \cong \pi_6(S^3\langle 3\rangle) \cong \pi_6(S^3)$. On the other hand, by [T], the 3-component of $\pi_6(S^3)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Therefore ∂_k represents a generator of the 3-component of $\pi_3(\Omega^3 S^3\langle 3\rangle)$. Next, since $\Omega^3 S^3\langle 3\rangle$ is 2-connected, the Hurewicz map $\pi_3(\Omega^3 S^3\langle 3\rangle) \otimes \mathbb{Z}/3\mathbb{Z} \to H_*(\Omega^3 S^3\langle 3\rangle)$ is an isomorphism, where we take homology with mod-3 coefficients. By the naturality of Hurewicz maps there is a commutative diagram

$$\begin{tikzcd}
\pi_3(S^3) \otimes \mathbb{Z}/3\mathbb{Z} \arrow[r, "(\partial_k)_*"] \arrow[d] & \pi_3(\Omega^3 S^3\langle 3\rangle) \otimes \mathbb{Z}/3\mathbb{Z} \arrow[d] \\
H_3(S^3) \arrow[r, "(\partial_k)_*"] & H_3(\Omega^3 S^3\langle 3\rangle)
\end{tikzcd}$$

where the vertical maps are the Hurewicz maps. We saw that the top and the right maps are isomorphisms. The left arrow is an isomorphism by the Hurewicz theorem. Thus the bottom arrow is an isomorphism, completing the proof. □

Next, by (3.4) together with (3.3) and (3.5), $H_3(\Omega^3 S^3\langle 3\rangle)\cong\mathbb{Z}/3\mathbb{Z}$ and if x_3 is its generator, then $\Lambda(x_3)$ is a tensor product factor of $H_*(\Omega^3 S^3\langle 3\rangle)$.

Proof of Theorem 1.2. Consider the homotopy fibration $S^3\xrightarrow{\partial_k}\Omega^3 S^3\langle 3\rangle\xrightarrow{g_k}B\mathcal{G}_k$. By Lemma 5.2, $(\partial_k)_*$ is an injection. Therefore the homology Serre spectral sequence for this homotopy fibration is totally non-homologous to zero, implying that it collapses at the E^2-term, so $H_*(\Omega^3 S^3\langle 3\rangle)\cong H_*(S^3)\otimes H_*(B\mathcal{G}_k)$. Since $H_*(S^3)\cong\Lambda(y_3)$ for $|y_3|=3$ such that y_3 corresponds to x_3 and as observed above, $\Lambda(x_3)$ is a tensor product factor of $H_*(\Omega^3 S^3\langle 3\rangle)$, we obtain the isomorphism asserted in the statement of the Theorem. □

References

[AB] M.F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, *Philos. Trans. Roy. Soc. London Ser. A* **308** (1983), 523-615.

[AJ] M.F. Atiyah and J.D.S. Jones, Topological aspects of Yang-Mills theory, *Commun. Math. Phys.* **61** (1978), 97-118.

[C] Y. Choi, Homology of the classifying space of $Sp(n)$ gauge groups, *Israel J. Math.* **151** (2006), 167-177.

[CLM] F.R. Cohen, T.J. Lada, and J.P. May, *The homology of iterated loop spaces*, Lecture Notes in Math. **533**, Springer-Verlag, 1976.

[CS] M. C. Crabb and W. A. Sutherland, Counting homotopy types of gauge groups, *Proc. London Math. Soc.* **81** (2000), 747-768.

[D] S.K. Donaldson, Connections, cohomology and the intersection forms on 4-manifolds, *J. Differential Geom.* **24** (1986), 275-341.

[F] A. Floer, An instanton invariant for 3-manifolds, *Commun. Math. Phys.* **118** (1988), 215-240.

[G] D.H. Gottlieb, Applications of bundle map theory, *Trans. Amer. Math. Soc.* **171** (1972), 23-50.

[GT] B. Gray and S. Theriault, An elementary construction of Anick's fibration, *Geom. Topol.* **14** (2010), 243-276.

[HKO] S. Hasui, D. Kishimoto and A. Ohsita, Samelson products in p-regular exceptional Lie groups, *Topology Appl.* **178** (2014), 17-29.

[KaKi] S. Kaji and D. Kishimoto, Homotopy nilpotency in p-regular loop spaces, *Math. Z.* **264** (2010), no. 1, 209-224.

[KiKo] D. Kishimoto and A. Kono, Splitting of gauge groups, *Trans. Amer. Math. Soc.* **362** (2010), 6715-6731.

[KT1] D. Kishimoto and M. Tsutaya, Infiniteness of A_∞-types of gauge groups, *J. Topol.* **9** (2016), no. 1, 181-191.

[KT2] D.Kishimoto and M. Tsutaya, Samelson products in p-regular $\mathrm{SO}(2n)$ and its homotopy normality, *Glasg. Math. J.* **60** (2018), no. 1, 165-174.

[K] A. Kono, A note on the homotopy type of certain gauge groups, *Proc. Roy. Soc. Edinburgh Sect. A* **117** (1991), 295-297.

[L] G.E. Lang, The evaluation map and EHP sequences, *Pacific J. Math.* **44** (1973), 201-210.

[Ma] G. Masbaum, On the cohomology of the classifying space of the gauge group over some 4-complexes, *Bull. Soc. Math. France* **119** (1991), 1-31.

[Mc] C.A. McGibbon, Homotopy commutativity in localized groups, *Amer. J. Math.* **106** (1984), 665-687.

[MT] M. Mimura and H. Toda, *Topology of Lie groups, I and II*, Translations of Mathematical Monographs vol. **91**, American Mathematical Society, 1991.
[S] P.S. Selick, A reformulation of the Arf invariant one mod-p problem and applications to atomic spaces, *Pacific J. Math.* **108** (1983), 431-450.
[Th] S.D. Theriault, Atomicity for Anick's spaces, *J. Pure Appl. Alg.* **219** (2015), 2346-2358.
[T] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Math. Studies No. **49**, Princeton University Press, Princeton NJ, 1962.

Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan
E-mail address: kishimoto@math.kyushu-u.ac.jp

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom
E-mail address: s.d.theriault@soton.ac.uk