University of Southampton

Faculty of Social Sciences
School of Mathematical Sciences

Equivariant bounded cohomology

by

Kevin Li

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Abstract

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This research paper thesis consists of the following articles:

[1] K. Li. Bounded cohomology of classifying spaces for families of subgroups. To appear in *Algebr. Geom. Topol.*, arXiv:2105.05223, 2021.

[2] K. Li, C. Löh, and M. Moraschini. Bounded acyclicity and relative simplicial volume. Preprint, arXiv:2202.05606, 2022.

[3] K. Li. Amenable covers of right-angled Artin groups. Preprint, arXiv:2204.01162, 2022.

[4] K. Li. On the topological complexity of toral relatively hyperbolic groups. *Proc. Amer. Math. Soc.*, 150(3):967–974, 2022.

In [1], we introduce a bounded version of Bredon cohomology for groups relative to a family of subgroups. We obtain cohomological characterisations of relative amenability and relative hyperbolicity, analogous to the results of Johnson and Mineyev for bounded cohomology.

In [2], joint with Clara Löh and Marco Moraschini, we provide new vanishing and glueing results for relative simplicial volume. We consider equivariant nerve pairs and relative classifying spaces for families of subgroups. Our methods also lead to vanishing results for ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable category.

In [3], we consider the right-angled Artin group A_L associated to a finite flag complex L. We show that the amenable category of A_L equals the virtual cohomological dimension of the right-angled Coxeter group W_L.

In [4], we prove that the topological complexity $\mathsf{TC}(\pi)$ equals $\mathrm{cd}(\pi \times \pi)$ for certain toral relatively hyperbolic groups π.

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

[1] K. Li. Bounded cohomology of classifying spaces for families of subgroups. To appear in *Algebr. Geom. Topol.*, arXiv:2105.05223, 2021.

[2] K. Li, C. Löh, and M. Moraschini. Bounded acyclicity and relative simplicial volume. Preprint, arXiv:2202.05606, 2022.

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Signed:.. Date:..................

Declaration of Joint Authorship

I declare that Chapter 3 of this thesis, which is the article [LLM], is joint work with Clara Löh and Marco Moraschini.

I confirm that all authors have contributed equally to the project.

Signed:... Date:..................

Contents

Chapter 1

Introduction

The present research paper thesis consists of the articles [Lib, LLM, Lia, Li22] given by the Chapters 2, 3, 4, and 5, respectively. In this introduction we explain the context in which the four articles form a coherent body of work and highlight some of our main results. These include cohomological characterisations of relative amenability (Theorem 1.29) and relative hyperbolicity (Theorem 1.32) and a vanishing theorem for relative simplicial volume (Theorem 1.37). Furthermore, we compute the amenable category of right-angled Artin groups (Theorem 1.38) and Farber's topological complexity of certain relatively hyperbolic groups (Theorem 1.45).

1.1 Classifying spaces for families of subgroups

The connecting theme of this thesis is classifying spaces for families of subgroups. These are important objects in equivariant algebraic topology and geometric group theory that were introduced by tom Dieck. The equivariant cohomology of classifying spaces for families is a generalisation of group cohomology and provides lower bounds for the generalised Lusternik–Schnirelmann category. Throughout the entire section, let G be a discrete group.

1.1.1 Basics on classifying spaces for families

We will mostly work in the equivariant setting of G-CW-complexes and G-maps. For background on G-CW-complexes and a survey on classifying spaces for families we refer to [Lüc05].

A *G-CW-complex* Y is a G-space together with a G-invariant filtration

$$\emptyset = Y^{(-1)} \subset Y^{(0)} \subset Y^{(1)} \subset \cdots \subset Y^{(n)} \subset \cdots \subset Y \tag{1.1}$$

such that the following hold:

- $Y = \bigcup_{n=0}^{\infty} Y^{(n)}$;
- Y carries the weak topology with respect to the filtration (1.1);
- $Y^{(n)}$ is obtained from $Y^{(n-1)}$ as a G-pushout of the form

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \longrightarrow & Y^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^{n} & \longrightarrow & Y^{(n)}\,. \end{array}$$

The subgroups H_i of G and their conjugates are called *isotropy groups* of Y.

For example, given a path-connected CW-complex X with fundamental group G, the universal covering space $\widetilde{X}$ carries the structure of a G-CW-complex with trivial isotropy groups. For an aspherical space BG, the universal covering space EG is characterised (up to G-homotopy) as a contractible space with a free G-action. Equivalently, EG is terminal (up to G-homotopy) among G-CW-complexes with trivial isotropy groups.

Classifying spaces for families of subgroups are generalisations of EG to the setting of G-CW-complexes with (not necessarily trivial) isotropy groups that are contained in a specified family of subgroups.

Definition 1.1 (Family of subgroups)**.** A *family $\mathcal{F}$ of subgroups of G* is a non-empty set of subgroups of G that is closed under conjugation by elements of G and under taking subgroups.

Important examples of families are the following:

$$\begin{aligned} \mathcal{TR} &= \{\text{the trivial subgroup}\}; \\ \mathcal{FIN} &= \{\text{all finite subgroups}\}; \\ \mathcal{VCY} &= \{\text{all virtually cyclic subgroups}\}; \\ \mathcal{AME} &= \{\text{all amenable subgroups}\}; \\ \mathcal{ALL} &= \{\text{all subgroups}\}. \end{aligned}$$

A definition of amenable groups is given later (Definition 1.25). Amenable groups may be considered as "small" for many purposes in geometry, topology, and dynamics and will play a key role throughout this thesis.

The above examples of families are defined by group-theoretic properties that are invariant under conjugation and under taking subgroups. Another source of examples is determined by any set of subgroups.

Example 1.2 (Family generated by a set of subgroups)**.** Let $\mathcal{H}$ be a set of subgroups of G. The *family* $\mathcal{F}\langle\mathcal{H}\rangle$ *generated by* $\mathcal{H}$ is defined as the smallest (with respect to inclusion) family of subgroups of G containing $\mathcal{H}$. Explicitly,

$$\mathcal{F}\langle\mathcal{H}\rangle = \{F \subset G \mid F \subset gHg^{-1} \text{ for some } H \in \mathcal{H}, g \in G\}\,.$$

We say that a G-CW-complex *has isotropy in a family* $\mathcal{F}$ if all its isotropy groups are contained in $\mathcal{F}$.

Definition 1.3 (Classifying space for a family of subgroups)**.** Let $\mathcal{F}$ be a family of subgroups of G. A *classifying space* $E_{\mathcal{F}}G$ *of* G *with respect to* $\mathcal{F}$ is a terminal object in the G-homotopy category of G-CW-complexes with isotropy in $\mathcal{F}$.

The usual constructions of Segal and Milnor for EG generalise to give generic constructions for $E_{\mathcal{F}}G$, proving the existence of such classifying spaces. An equivalent characterisation of $E_{\mathcal{F}}G$ can be given in terms of fixed-point sets.

Theorem 1.4 ([Lüc05, Theorem 1.9])**.** *A G-CW-complex Y is a model for $E_{\mathcal{F}}G$ if and only if Y has isotropy in $\mathcal{F}$ and for all $H \in \mathcal{F}$ the fixed-point set Y^H is contractible (and in particular non-empty).*

As mentioned above, a model for $E_{\mathcal{TR}}G$ is the same as a model for EG. A model for $E_{\mathcal{ALL}}G$ is given by the point G/G. Since every family $\mathcal{F}$ contains the trivial family $\mathcal{TR}$, by the universal property of $E_{\mathcal{F}}G$ there exists a G-map $EG \to E_{\mathcal{F}}G$ that is unique up to G-homotopy. We illustrate this map in an example.

Example 1.5 (The infinite dihedral group)**.** Consider the infinite dihedral group

$$D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle s, t \mid s^2 = t^2 = 1\rangle\,.$$

A model for $E(D_\infty)$ is given by the D_∞-pushout

$$\begin{array}{ccc} D_\infty \times S^0 & \longrightarrow & D_\infty \times_{\langle s\rangle} S^\infty \coprod D_\infty \times_{\langle t\rangle} S^\infty \\ \downarrow & & \downarrow \\ D_\infty \times D^1 & \longrightarrow & E(D_\infty)\,, \end{array}$$

where the maps are the obvious ones. Here S^∞ is the infinite dimensional sphere viewed as a $\mathbb{Z}/2\mathbb{Z}$-CW-complex via the antipodal action and $D_\infty \times_{\langle s\rangle} S^\infty$ is the induced D_∞-CW-complex.

A model for $E_{\mathcal{FIN}}(D_\infty)$ is given by the real line $\mathbb{R}$ on which s and t act by reflection in 0 and 1, respectively. This action endows $\mathbb{R}$ with the structure of a D_∞-CW-complex given by the following D_∞-pushout

$$\begin{array}{ccc} D_\infty \times S^0 & \longrightarrow & D_\infty/\langle s\rangle \coprod D_\infty/\langle t\rangle \\ \downarrow & & \downarrow \\ D_\infty \times D^1 & \longrightarrow & \mathbb{R}\,. \end{array}$$

Then the canonical D_∞-map $E(D_\infty) \to E_{\mathcal{FIN}}(D_\infty)$ is induced by collapsing each copy of S^∞ to a point.

Since D_∞ is virtually cyclic and hence amenable, a model for $E_{\mathcal{VCY}}(D_\infty)$ and $E_{\mathcal{AME}}(D_\infty)$ is given by the point D_∞/D_∞. Thus the canonical D_∞-maps $E(D_\infty) \to E_{\mathcal{VCY}}(D_\infty)$ and $E(D_\infty) \to E_{\mathcal{AME}}(D_\infty)$ are the constant maps.

Especially for the family $\mathcal{FIN}$ of finite subgroups, where we are considering proper actions, many interesting constructions of $E_{\mathcal{FIN}}G$ for various classes of groups are known (Table 1.1). For more details see [Lüc05, Section 4]. Note that if G is torsion-free, we have $\mathcal{FIN} = \mathcal{TR}$ and hence $E_{\mathcal{FIN}}G = EG$.

Group G	Model for $E_{\mathcal{FIN}}G$
graph of finite groups	Bass–Serre tree
right-angled Coxeter group	Davis complex
hyperbolic group	Rips complex
mapping class group	Teichmüller space
$\mathrm{Out}(F_n)$	Culler–Vogtmann outer space

Table 1.1: Constructions of $E_{\mathcal{FIN}}G$.

For concrete computations it is desirable to have models for $E_{\mathcal{F}}G$ of small dimension or with a small number of (equivariant) cells. For two nested families $\mathcal{E} \subset \mathcal{F}$ satisfying a certain maximality condition, there is a construction of Lück–Weiermann [LW12] that produces a "small" model for $E_{\mathcal{F}}G$ from $E_{\mathcal{E}}G$ (Theorem 5.3).

Remark 1.6 (Isomorphism conjectures in K-theory). The classifying spaces for the families $\mathcal{FIN}$ and $\mathcal{VCY}$ make a prominent appearance in the isomorphism conjectures of Farrell–Jones and Baum–Connes. For a comprehensive introduction to the topic, see [Lüc]. One version of the Farrell–Jones conjecture predicts that the assembly map

$$H_n^G(E_{\mathcal{VCY}}G; \mathbf{K}) \to K_n(\mathbb{Z}G) \tag{1.2}$$

is an isomorphism. Here $K_n(\mathbb{Z}G)$ is the algebraic K-theory of the group ring $\mathbb{Z}G$ and $H_n^G(-;\mathbf{K})$ is the G-homology theory associated to the K-theory spectrum $\mathbf{K}$. Many powerful obstructions, such as the Wall finiteness obstruction or the Whitehead torsion, live in (quotients of) the algebraic K-groups. While the right hand side of (1.2) is extremely difficult to compute in general, the left hand side is accessible via standard tools from equivariant algebraic topology. In principle, the left hand side of (1.2) can be computed using the equivariant Atiyah–Hirzebruch spectral sequence from the Bredon homology of $E_{\mathcal{VCY}}G$ with appropriate coefficients. This is one reason why the Bredon (co)homology (Section 1.1.2) of classifying spaces for families is of interest.

Remark 1.7 (Convention on families). We point out that in Chapter 3 a different convention for families of subgroups is used. In Chapter 3 a family is only assumed to be closed under conjugation (and not necessarily under taking subgroups). In all other chapters including this introduction the term "family" is used as in Definition 1.1.

1.1.2 Bredon cohomology

The equivariant version of cellular cohomology for G-CW-complexes is given by Bredon cohomology [Bre67]. A nice introduction to Bredon cohomology is provided in [Flu].

Fix a ground ring R. Let $\mathcal{F}$ be a family of subgroups of G. The *($\mathcal{F}$-restricted) orbit category* $\mathcal{O}_{\mathcal{F}}G$ has as objects G-sets G/H with $H \in \mathcal{F}$ and as morphisms G-maps. An *$\mathcal{O}_{\mathcal{F}}G$-module* is a contravariant functor $M\colon \mathcal{O}_{\mathcal{F}}G \to R\text{-}\mathbf{Mod}$ to the category of R-modules. Morphisms of $\mathcal{O}_{\mathcal{F}}G$-modules are natural transformations.

For example, any RG-module V gives rise to an $\mathcal{O}_{\mathcal{F}}G$-module $V^?\colon \mathcal{O}_{\mathcal{F}}G \to R\text{-}\mathbf{Mod}$ by taking invariants $V^?(G/H) = V^H$. For a G-CW-complex Y, the cellular chains of its fixed-point sets form an $\mathcal{O}_{\mathcal{F}}G$-module $C_n^{\mathrm{cell}}(Y^?)\colon \mathcal{O}_{\mathcal{F}}G \to R\text{-}\mathbf{Mod}$,

$$C_n^{\mathrm{cell}}\big(Y^?\big)(G/H) = C_n^{\mathrm{cell}}\big(Y^H\big)\,.$$

Taking cohomology of the $\mathcal{O}_{\mathcal{F}}G$-chain complex $C_*^{\mathrm{cell}}(-^?)$ defines a G-cohomology theory. Equivalently, one can also use the singular chain complex instead.

Definition 1.8 (Bredon cohomology)**.** Let $\mathcal{F}$ be a family of subgroups of G and let M be an $\mathcal{O}_{\mathcal{F}}G$-module. Let Y be a G-CW-complex with isotropy in $\mathcal{F}$. The *(G-equivariant) Bredon cohomology of* Y with coefficients in M is defined as

$$H_G^*(Y;M) \coloneqq H^*\big(\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}G}(C_*^{\mathrm{cell}}(Y^?),M)\big)\,.$$

For an RG-module V, we denote $H_G^*(Y;V) \coloneqq H_G^*(Y;V^?)$.

The *Bredon cohomology of the group G with respect to $\mathcal{F}$* and with coefficients in M is defined to be

$$H_{\mathcal{F}}^*(G;M) \coloneqq H_G^*(E_{\mathcal{F}}G;M)\,. \tag{1.3}$$

For the trivial family $\mathcal{TR}$, the orbit category $\mathcal{O}_{\mathcal{TR}}G$ is isomorphic to the groupoid G, which shows that an $\mathcal{O}_{\mathcal{TR}}G$-module is the same as an RG-module. Since $E_{\mathcal{TR}}G = EG$, in this case the Bredon cohomology of G recovers ordinary group cohomology [Bro82]. The notions of geometric dimension $\mathrm{gd}(G)$ and cohomological dimension $\mathrm{cd}(G)$ admit generalisations to the setting of families.

Definition 1.9 (Bredon geometric and cohomological dimension)**.** Let $\mathcal{F}$ be a family of subgroups of G. The *geometric dimension* $\mathrm{gd}_{\mathcal{F}}(G)$ *of* G *with respect to* $\mathcal{F}$ is the infimum of $n \in \mathbb{N}_{\geq 0}$ for which there exists a model for $E_{\mathcal{F}}G$ of dimension n. (In particular, if no finite dimensional model for $E_{\mathcal{F}}G$ exists, then $\mathrm{gd}_{\mathcal{F}}(G) = \infty$.) For $R = \mathbb{Z}$, the *cohomological dimension* $\mathrm{cd}_{\mathcal{F}}(G)$ *of* G *with respect to* $\mathcal{F}$ is the supremum of $k \in \mathbb{N}_{\geq 0}$ for which $H^k_{\mathcal{F}}(G; M)$ is non-trivial for some $\mathcal{O}_{\mathcal{F}}G$-module M.

Clearly, we have $\mathrm{cd}_{\mathcal{F}}(G) \leq \mathrm{gd}_{\mathcal{F}}(G)$. The classical result $\mathrm{gd}(G) \leq \max\{\mathrm{cd}(G), 3\}$ of Eilenberg–Ganea [EG57] holds more generally with respect to every family.

Theorem 1.10 (Lück–Meintrup [LM00])**.** *For every family* $\mathcal{F}$ *of subgroups of* G*, we have*

$$\mathrm{gd}_{\mathcal{F}}(G) \leq \max\{\mathrm{cd}_{\mathcal{F}}(G), 3\}\,.$$

Thus, if $\mathrm{cd}_{\mathcal{F}}(G) \geq 3$, then the Bredon geometric and cohomological dimensions coincide.

Remark 1.11 (Algebraic definition of Bredon group cohomology)*.* Recall that ordinary group cohomology can be defined purely algebraically via derived functors and also concretely via a standard resolution [Bro82]. While our definition of (1.3) is topological, the Bredon cohomology $H^*_{\mathcal{F}}(G; M)$ may alternatively be defined algebraically as $\mathrm{Ext}^*_{\mathcal{O}_{\mathcal{F}}G}(\underline{R}, M)$, where $\underline{R}$ is the constant $\mathcal{O}_{\mathcal{F}}G$-module with value R. There is a standard resolution of the $\mathcal{O}_{\mathcal{F}}G$-module $\underline{R}$ which can be used to compute $H^*_{\mathcal{F}}(G; M)$. The usual computational tools for group cohomology, such as the long exact coefficient sequence, Shapiro's lemma, and the Lyndon–Hochschild–Serre spectral sequence, all have versions for Bredon cohomology [Flu, MP02]. More details on this algebraic point of view are given in Chapter 2.

1.1.3 Generalised Lusternik–Schnirelmann category

A basic approach to study a topological space X is to find a cover $\mathcal{U}$ of X by open subsets that are simple or small in an appropriate sense and to analyse how these overlap. More precisely, one wants to understand how the topology of X is reflected in the nerve $N(\mathcal{U})$ of $\mathcal{U}$. By a classical result of Leray, if all open subsets in $\mathcal{U}$ and all their non-empty intersections are contractible, then X is in fact homotopy equivalent to $N(\mathcal{U})$.

The minimal number of contractible open subsets needed to cover X (not requiring conditions on intersections) yields a naive "measure of complexity" for X, the Lusternik–Schnirelmann category. It is a well-studied homotopy invariant originating from critical point theory [CLOT03].

Definition 1.12 (Lusternik–Schnirelmann category)**.** Let X be a topological space. The *Lusternik–Schnirelmann category* (*LS-category* for short) $\text{LS-cat}(X)$ is the infimum of $n \in \mathbb{N}_{\geq 0}$ for which there exists a cover $X = \bigcup_{i=0}^{n} U_i$ by (not necessarily path-connected) open subsets U_i that are contractible in X.

The usual strategy to determine the precise value of the LS-category and similar invariants below is twofold: First, to construct an explicit open cover as in Definition 1.12 and second, to show that no smaller cover can exist using cohomological obstructions.

The LS-category of a simplicial complex X is bounded above by its dimension. Indeed, we have $X = \bigcup_{i=0}^{\dim(X)} U_i$, where U_i is the disjoint union of open stars corresponding to i-simplices of X. For any space X, an obstruction to admitting open covers by contractible sets is given by cup products on cohomology. If there exists a non-trivial cup product of k cohomology classes in positive degrees, then $\text{LS-cat}(X) \geq k$.

Since the LS-category is homotopy invariant, we obtain an invariant of discrete groups by defining the *LS-category of* G to be $\text{LS-cat}(G) := \text{LS-cat}(BG)$. This corresponds to restricting ourselves to aspherical spaces, which we will do similarly for other invariants below. The LS-category of an aspherical space can be identified with the cohomological dimension of its fundamental group. For aspherical spaces this provides an algebraic characterisation of the topologically defined LS-category.

Theorem 1.13 (Eilenberg–Ganea [EG57], Stallings [Sta68], Swan [Swa69])**.** *We have*

$$\text{LS-cat}(G) = \text{cd}(G)\,.$$

Note that for an aspherical space X, the requirement on U_i to be contractible in X is equivalent to being simply-connected in X. From this point of view, it is natural to relax the condition on U_i to having fundamental group in a prescribed family of subgroups of $\pi_1(X)$.

Definition 1.14 (Generalised LS-category)**.** Let $\mathcal{F}$ be a family of subgroups of G. Let X be a path-connected topological space with fundamental group G. The *generalised LS-category with respect to* $\mathcal{F}$ (also $\mathcal{F}$*-category*) $\text{cat}_{\mathcal{F}}(X)$ is the infimum of $n \in \mathbb{N}_{\geq 0}$ for which there exists a cover $X = \bigcup_{i=0}^{n} U_i$ by (not necessarily path-connected) open subsets U_i satisfying

$$\text{im}\big(\pi_1(U_i \hookrightarrow X, x)\big) \in \mathcal{F}$$

for all $x \in U_i$.

The $\mathcal{F}$*-category of the group* G is defined to be $\text{cat}_{\mathcal{F}}(G) := \text{cat}_{\mathcal{F}}(BG)$.

This general setup encompasses the following interesting special cases:

- $\text{cat}_{\mathcal{TR}}(G) = \text{LS-cat}(G) = \text{cd}(G)$ the cohomological dimension;
- $\text{cat}_{\mathcal{AME}}(G)$ the amenable category;
- $\text{cat}_{\mathcal{D}}(\pi \times \pi) = \mathsf{TC}(\pi)$ Farber's topological complexity, where π is a group and $\mathcal{D}$ is the family of subgroups of $\pi \times \pi$ generated by the diagonal subgroup.

The amenable category will be discussed further in Section 1.2. It was systematically studied as an invariant of 3-manifolds in [GLGAH13, GLGAH14] and for arbitrary spaces recently in [CLM, LM]. Lower bounds for the amenable category are provided by several important invariants from algebraic topology, such as bounded cohomology, ℓ^2-Betti numbers, and mod p homology growth (Section 1.2.5). Farber's topological complexity is motivated by robotics and it is the subject of Section 1.3.

The following theorem characterises the $\mathcal{F}$-category of G in terms of classifying spaces for families. In the author's opinion, it is a beautiful result because it relates two important concepts which facilitates the transfer of methods. As we shall see in Chapter 3, it allows to prove new results and re-prove existing results for the amenable category in a very conceptual way. In special cases the next theorem was previously obtained in [FGLO19, BCVEB22].

Theorem 1.15 (Capovilla–Löh–Moraschini [CLM, Proposition 7.5])**.** *Let $\mathcal{F}$ be a family of subgroups of G. The $\mathcal{F}$-category $\operatorname{cat}_{\mathcal{F}}(G)$ coincides with the infimum of $n \in \mathbb{N}_{\geq 0}$ for which the canonical G-map $EG \to E_{\mathcal{F}}G$ is equivariantly homotopic to a map with values in the n-skeleton of $E_{\mathcal{F}}G$.*

Roughly speaking, Theorem 1.15 states that the $\mathcal{F}$-category $\operatorname{cat}_{\mathcal{F}}(G)$ coincides with the "geometric dimension of the G-map $EG \to E_{\mathcal{F}}G$". This interpretation offers an obvious lower bound given by the "cohomological dimension of the G-map $EG \to E_{\mathcal{F}}G$" in terms of Bredon cohomology.

Definition 1.16. Let $\mathcal{F}$ be a family of subgroups of G. We define $\operatorname{cd}_{\mathcal{TR}\subset\mathcal{F}}(G)$ to be the supremum of $k \in \mathbb{N}_{\geq 0}$ for which the induced map on Bredon cohomology

$$H^k_G(E_{\mathcal{F}}G; M) \to H^k_G\big(EG; M(G/1)\big)$$

is non-trivial for some $\mathcal{O}_{\mathcal{F}}G$-module M.

Clearly, we have

$$\operatorname{cd}_{\mathcal{TR}\subset\mathcal{F}}(G) \leq \operatorname{cat}_{\mathcal{F}}(G) \leq \min\{\operatorname{gd}(G), \operatorname{gd}_{\mathcal{F}}(G)\}\,. \tag{1.4}$$

The question when the equality $\operatorname{cd}_{\mathcal{TR}\subset\mathcal{F}}(G) = \operatorname{cat}_{\mathcal{F}}(G)$ holds is posed and addressed briefly in Section 1.3.4.

Remark 1.17 (Convention on normalisation). We point out that in Chapter 3 a different normalisation for $\operatorname{cat}_{\mathcal{F}}(X)$ is used, which produces values that are by 1 larger than in Definition 1.14.

1.2 Bounded cohomology and simplicial volume

Bounded cohomology is a homotopy invariant defined in terms of bounded singular cochains. It is often described as a "functional analytic variant" of singular cohomology, though it is not a cohomology theory. Originally, Gromov [Gro82] studied the simplicial volume of manifolds using the duality between the ℓ^1-seminorm on singular homology and the ℓ^∞-seminorm on bounded cohomology. Today bounded cohomology has become an object of interest in its own right due to plentiful connections to topics in geometric group theory [Mon06]. For instance, bounded cohomology characterises both amenable groups and hyperbolic groups. The standard references are [Fri17, Mon01]. In this section, the ground ring is $\mathbb{R}$.

1.2.1 Basics on bounded cohomology

We recall the definition of simplicial volume which is a homotopy invariant of oriented closed connected manifolds. At the same time, the simplicial volume of Riemannian manifolds encodes geometric information about the Riemannian volume. It was introduced by Gromov in his proof of Mostow rigidity.

Singular homology is naturally equipped with an ℓ^1-seminorm as follows. For a topological space X, the singular chain module $C_n(X;\mathbb{R})$ is equipped with an ℓ^1-norm, for a singular chain $c = \sum_i c_i\sigma_i \in C_n(X;\mathbb{R})$ with singular simplices σ_i and $c_i \in \mathbb{R}$ given by

$$|c|_1 \coloneqq \sum_i |c_i| \,.$$

This ℓ^1-norm on $C_n(X;\mathbb{R})$ descends to an ℓ^1-seminorm on singular homology $H_n(X;\mathbb{R})$, for a singular homology class $\alpha \in H_n(X;\mathbb{R})$ given by

$$\|\alpha\|_1 \coloneqq \inf\bigl\{|c|_1 \bigm| c \in C_n(X;\mathbb{R}), \partial c = 0, [c] = \alpha\bigr\} \,.$$

Definition 1.18 (Simplicial volume)**.** Let M be an oriented closed connected manifold of dimension n. The real fundamental class $[M]_\mathbb{R} \in H_n(M;\mathbb{R})$ is the image of the fundamental class $[M]_\mathbb{Z} \in H_n(M;\mathbb{Z})$ under the change of coefficients homomorphism $H_n(M;\mathbb{Z}) \to H_n(M;\mathbb{R})$. The *simplicial volume of* M is defined as

$$\|M\| \coloneqq \bigl\|[M]_\mathbb{R}\bigr\|_1 \in \mathbb{R}_{\geq 0} \,.$$

It is easy to see from the definition that the circle S^1 has vanishing simplicial volume. More generally, the simplicial volume of manifolds with amenable fundamental group vanishes (Corollary 1.27). On the other hand, the simplicial volume of hyperbolic manifolds is proportional to the Riemannian volume, in particular it is positive. The value of simplicial volume and in particular its (non-)vanishing behaviour is detected by the dual theory of bounded cohomology. The definition is analogous to that of singular cohomology, however, using only *bounded* cochains.

Definition 1.19 (Bounded cohomology)**.** Let X be a topological space. Consider the singular chain module $C_n(X;\mathbb{R})$ equipped with the ℓ^1-norm. Taking topological duals yields the bounded cochain complex

$$C_b^*(X;\mathbb{R}) \coloneqq B\big(C_*(X;\mathbb{R}),\mathbb{R}\big)\,.$$

The *bounded cohomology of* X is defined as

$$H_b^n(X;\mathbb{R}) \coloneqq H^n\big(C_b^*(X;\mathbb{R})\big)\,.$$

The inclusion of bounded cochains into all (not necessarily bounded) cochains induces the so-called *comparison map*

$$\operatorname{comp}^*\colon H_b^*(X;\mathbb{R}) \to H^*(X;\mathbb{R})\,.$$

We mention that the comparison map is neither injective nor surjective in general.

The operator-norm on $C_b^n(X;\mathbb{R})$ descends to an ℓ^∞-seminorm on $H_b^n(X;\mathbb{R})$, for a class $\varphi \in H_b^n(X;\mathbb{R})$ given by

$$\|\varphi\|_\infty \coloneqq \inf\big\{|f|_\infty \;\big|\; f \in C_b^n(X;\mathbb{R}), \delta f = 0, [f] = \varphi\big\}\,.$$

Proposition 1.20 (Duality principle [Fri17, Lemma 6.1])**.** *Let X be a topological space and let $\alpha \in H_n(X;\mathbb{R})$ be a homology class. We have*

$$\|\alpha\|_1 = \max\big\{\langle \operatorname{comp}^n(\varphi), \alpha\rangle \;\big|\; \varphi \in H_b^n(X;\mathbb{R}), \|\varphi\|_\infty \leq 1\big\}\,.$$

In particular, for an oriented closed connected manifold M of dimension n, the comparison map $\operatorname{comp}^n\colon H_b^n(M;\mathbb{R}) \to H^n(M;\mathbb{R}) \cong \mathbb{R}$ *is surjective if and only if* $\|M\| > 0$.

We warn the reader that bounded cohomology is *not* a cohomology theory. While it is functorial and homotopy invariant, it does not satisfy the Mayer–Vietoris or excision axiom. This makes explicit computations very difficult, which is why the focus is often on qualitative (non-)vanishing results. Another feature of bounded cohomology is that it is insensitive to abelian groups (and more generally, to amenable groups (Theorem 1.26)) and hence to higher homotopy groups.

Theorem 1.21 (Mapping theorem [Fri17, Theorem 5.9])**.** *Let X be a path-connected topological space with fundamental group G. The classifying map $X \to BG$ induces an isometric isomorphism*

$$H_b^*(BG;\mathbb{R}) \xrightarrow{\cong} H_b^*(X;\mathbb{R})\,.$$

Theorem 1.21 essentially reduces the study of bounded cohomology of arbitrary spaces to that of aspherical spaces, in other words, to that of groups.

Equivariant bounded cohomology. The definition of bounded cohomology can be extended to G-spaces and to allow for twisted coefficients.

A *normed G-module V* is a G-module equipped with a G-invariant norm. For two normed G-modules U and V, we denote by $B_G(U, V)$ the set of G-equivariant bounded linear maps.

Definition 1.22 (Equivariant bounded cohomology)**.** Let G be a group and let V be a normed G-module. Let Y be a G-space. The singular chain module $C_n(Y; \mathbb{R})$ equipped with the ℓ^1-norm is a normed G-module. The cochain complex of G-equivariant bounded cochains with coefficients in V is

$$C^*_{G,b}(Y; V) \coloneqq B_G\bigl(C_*(Y; \mathbb{R}), V\bigr) .$$

The *(G-equivariant) bounded cohomology of Y* with coefficients in V is defined as

$$H^n_{G,b}(Y; V) \coloneqq H^n\bigl(C^*_{G,b}(Y; V)\bigr) .$$

Forgetting boundedness of the cochains induces the *comparison map*

$$\mathrm{comp}^* \colon H^*_{G,b}(Y; V) \to H^*_G(Y; V) .$$

The *bounded cohomology of the group G* with coefficients in V is defined to be

$$H^*_b(G; V) \coloneqq H^*_{G,b}(EG; V) . \tag{1.5}$$

On the one hand, bounded cohomology (with dual coefficients) vanishes for amenable groups (Theorem 1.26), such as the integers $\mathbb{Z}$. On the other hand, bounded cohomology is non-trivial in the presence of negative curvature, e.g., for hyperbolic groups (Theorem 1.30). As a matter of fact, for the free group F_2 on two generators, $H^2_b(F_2; \mathbb{R})$ is infinite dimensional. This example shows that bounded cohomology does not satisfy the Mayer–Vietoris axiom nor the dimension axiom.

Remark 1.23 (Alternative definition of bounded group cohomology)*.* There is also a concrete definition of bounded group cohomology via a standard resolution and an algebraic framework of relative homological algebra due to Ivanov [Iva85]. However, an easy algebraic description via derived functors does not seem to be available, having to do with the fact that the category of normed G-modules is not abelian. Nevertheless, some cohomological tools are still available [Mon01].

The main novelty of Chapter 2 is a bounded version of Bredon group cohomology.

Definition 1.24 (Bounded Bredon cohomology for groups)**.** Let G be a group and let $\mathcal{F}$ be a family of subgroups. For a normed G-module V, we define

$$H^*_{\mathcal{F},b}(G;V) := H^*_{G,b}(E_{\mathcal{F}}G;V)\,.$$

Clearly, Definition 1.24 is a generalisation of ordinary bounded group cohomology (1.5) which is recovered for the trivial family $\mathcal{TR}$. In Chapter 2 we also give an alternative definition of bounded Bredon cohomology via a standard resolution. We generalise two essential theorems on bounded cohomology to the setting of families: characterisations of amenable groups and of hyperbolic groups.

1.2.2 Amenability

The class of amenable groups is fundamental in geometric group theory and beyond. Amenable groups can be considered as small from the point of view of large scale geometry.

Let G be a group and S be a G-set. The $\mathbb{R}$-module $\ell^\infty(S,\mathbb{R})$ of bounded functions $S \to \mathbb{R}$ admits the structure of an $\mathbb{R}G$-module, given by $(g \cdot f)(s) = f(g^{-1}s)$ for $f \in \ell^\infty(S,\mathbb{R})$, $g \in G$, and $s \in S$.

A *G-invariant mean on S* is an $\mathbb{R}$-linear function $m\colon \ell^\infty(S,\mathbb{R}) \to \mathbb{R}$ satisfying

- $m(\mathrm{const}_1) = 1$, where const_1 is the constant function with value 1;
- $m(f) \geq 0$ for all $f \in \ell^\infty(S,\mathbb{R})$ with $f \geq 0$;
- $m(g \cdot f) = m(f)$ for all $f \in \ell^\infty(S,\mathbb{R})$ and $g \in G$.

Definition 1.25 (Amenable group)**.** A group G is *amenable* if the G-set G admits a G-invariant mean.

For example, all finite groups and all abelian groups are amenable. The class of amenable groups is closed under taking subgroups, quotients, extensions, and directed unions. The free group F_2 is not amenable, a fact that lies at the core of the Banach–Tarski paradox.

Amenability is characterised by the vanishing of bounded cohomology with dual coefficients. For a normed $\mathbb{R}G$-module V, we denote its topological dual by $V^\# := B(V,\mathbb{R})$ which is again a normed $\mathbb{R}G$-module.

Theorem 1.26 (Johnson [Joh72])**.** *Let G be a group. The following are equivalent:*

(i) *G is amenable;*

(ii) *$H^n_b(G;V^\#) = 0$ for every dual normed $\mathbb{R}G$-module $V^\#$ and all $n \geq 1$;*

(iii) $H^1_b(G; V^\#) = 0$ *for every dual normed* $\mathbb{R}G$*-module* $V^\#$.

Combining the implication (i) ⇒ (ii) of Theorem 1.26 with the mapping theorem (Theorem 1.21) and the duality principle (Proposition 1.20), we deduce the following.

Corollary 1.27. *Let* M *be an oriented closed connected manifold of positive dimension. If the fundamental group of* M *is amenable, then we have* $\|M\| = 0$.

Relative amenability. There is a generalisation of amenability to the relative setting, see e.g., [MP03, JOR12].

Definition 1.28. Let G be a group and let $\mathcal{H}$ be a set of subgroups of G. We say that G is *amenable relative to* $\mathcal{H}$ if the G-set $\coprod_{H\in\mathcal{H}} G/H$ admits a G-invariant mean. If $\mathcal{H}$ consists of a single subgroup H, we also say that G is amenable relative to H. (In the literature it is also common to say that H is *co-amenable in* G.)

We collect some examples: Obviously, a group G is amenable if and only if it is amenable relative to the trivial subgroup. Given a normal subgroup N of G, we have that G is amenable relative to N if and only if the quotient group G/N is amenable. An ascending HNN-extension $H*_\varphi$ is amenable relative to H [MP03]. If $\mathcal{H}$ contains a subgroup that has finite index in G or that contains the commutator subgroup of G, then G is amenable relative to $\mathcal{H}$.

We consider the bounded Bredon cohomology (Definition 1.24) with respect to the family $\mathcal{F}\langle\mathcal{H}\rangle$ generated by $\mathcal{H}$ (Example 1.2).

Theorem 1.29 (Characterisation of relative amenability, Theorem 2.3)**.** *Let* G *be a group and let* $\mathcal{H}$ *be a set of subgroups of* G. *The following are equivalent:*

(i) G *is amenable relative to* $\mathcal{H}$*;*

(ii) $H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G; V^\#) = 0$ *for every dual normed* $\mathbb{R}G$*-module* $V^\#$ *and all* $n \geq 1$*;*

(iii) $H^1_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G; V^\#) = 0$ *for every dual normed* $\mathbb{R}G$*-module* $V^\#$.

The proof of Theorem 1.29 is analogous to the classical proof of Theorem 1.26 in the non-relative case. It uses the algebraic definition of bounded (Bredon) group cohomology and relies on a transfer map and the long exact coefficient sequence.

1.2.3 Hyperbolicity

Another class of groups that is fundamental in geometric group theory and geometric topology is that of hyperbolic groups. In a certain probabilistic sense, most finitely presented groups are hyperbolic.

A finitely generated group G is *hyperbolic* if its Cayley graph with respect to some finite generating set is hyperbolic as a metric space.

Examples of hyperbolic groups are finite groups, free groups, fundamental groups of surfaces of genus ≥ 2, and more generally, fundamental groups of closed hyperbolic manifolds. Hyperbolic groups are always finitely presented and cannot contain $\mathbb{Z}^2$ as a subgroup.

Hyperbolicity is characterised by surjectivity of the comparison map for all coefficients.

Theorem 1.30 (Mineyev [Min01, Min02])**.** *Let G be a finitely presented group. The following are equivalent:*

(i) *G is hyperbolic;*

(ii) *$\mathrm{comp}^n \colon H_b^n(G; V) \to H^n(G; V)$ is surjective for every normed $\mathbb{R}G$-module V and all $n \geq 2$;*

(iii) *$\mathrm{comp}^2 \colon H_b^2(G; V) \to H^2(G; V)$ is surjective for every normed $\mathbb{R}G$-module V.*

Theorem 1.30 was motivated by the fact that hyperbolic manifolds have positive simplicial volume. The implication (i) $\Rightarrow$ (ii) together with the duality principle (Proposition 1.20) yields the following generalisation.

Corollary 1.31. *Let M be an aspherical oriented closed connected manifold of dimension $n \geq 2$. If the fundamental group of M is hyperbolic, then we have $\|M\| > 0$.*

Relative hyperbolicity. Various generalisations of hyperbolicity have been at the forefront of mathematical research in recent years. A key notion is that of relatively hyperbolic groups, see e.g., [Hru10].

Let G be a finitely generated group and let $\mathcal{P}$ be a finite set of subgroups. We say that G is *hyperbolic relative to* $\mathcal{P}$ if the coned-off Cayley graph of G with respect to $\mathcal{P}$ is hyperbolic and fine.

For example, a free product $G_1 * G_2$ is hyperbolic relative to its factors $\{G_1, G_2\}$. The fundamental group of a finite volume hyperbolic manifold is hyperbolic relative to the cusp subgroups. If G is torsion-free hyperbolic relative to $\mathcal{P}$, then the set $\mathcal{P}$ is a *malnormal collection*, i.e., for all $P_i, P_j \in \mathcal{P}$ and $g \in G$, we have $P_i \cap gP_jg^{-1} = \{1\}$ unless $i = j$ and $g \in P_j$.

We consider the comparison map for bounded Bredon cohomology (Definition 1.24) with respect to the family $\mathcal{F}\langle\mathcal{P}\rangle$ generated by $\mathcal{P}$ (Example 1.2).

Theorem 1.32 (Characterisation of relative hyperbolicity, Theorem 2.31)**.** *Let G be a finitely presented torsion-free group and let $\mathcal{P}$ be a finite malnormal collection of subgroups consisting of finitely presented groups. The following are equivalent:*

(i) *G is hyperbolic relative to $\mathcal{P}$;*

(ii) $\mathrm{comp}^n \colon H^n_{\mathcal{F}\langle\mathcal{P}\rangle,b}(G;V) \to H^n_{\mathcal{F}\langle\mathcal{P}\rangle}(G;V)$ *is surjective for every normed $\mathbb{R}G$-module V and all $n \geq 2$;*

(iii) $\mathrm{comp}^2 \colon H^2_{\mathcal{F}\langle\mathcal{P}\rangle,b}(G;V) \to H^2_{\mathcal{F}\langle\mathcal{P}\rangle}(G;V)$ *is surjective for every normed $\mathbb{R}G$-module V.*

The proof strategy of Theorem 1.32 follows that of Theorem 1.30 in the non-relative case. The key ingredients are bicombing techniques developed by Mineyev–Yaman [MY] and a criterion for relative hyperbolicity in terms of linear homological isoperimetric inequalities due to Martínez-Pedroza [MP16] and Franceschini [Fra18].

1.2.4 A question of Gromov

One observes that, among aspherical manifolds, many conditions ensuring the vanishing of simplicial volume also imply the vanishing of the Euler characteristic. For example, both invariants vanish for aspherical manifold that have amenable fundamental group, that are the total space of a fibre bundle whose fibre has amenable fundamental group, that admit a non-trivial self-covering, or that admit a smooth non-trivial S^1-action. This raises the following question, which is one of the main open problems in the field of simplicial volume.

Question 1.33 (Gromov)**.** Let M be an aspherical oriented closed connected manifold. Does vanishing of the simplicial volume $\|M\|$ imply vanishing of the Euler characteristic $\chi(M)$?

For more motivation and a recent status report on Question 1.33 we refer to [LMR]. At the time of writing, it is completely open. The main source of examples satisfying Question 1.33 is the topic of the next section.

1.2.5 Vanishing results for amenable covers

We have discussed that the simplicial volume of manifolds with amenable fundamental group vanishes (Corollary 1.27). The following theorem is a generalisation to the case when the manifold has small amenable category (Definition 1.14).

For a topological space X, we have $\mathrm{cat}_{\mathcal{AME}}(X) \leq n$ if there exists a cover $X = \bigcup_{i=0}^n U_i$ by (not necessarily path-connected) open subsets U_i satisfying

$$\mathrm{im}\big(\pi_1(U_i \hookrightarrow X, x)\big) \in \mathcal{AME}$$

for all basepoints $x \in U_i$.

Theorem 1.34 (Vanishing theorem [Gro82, Iva85]). *Let X be a path-connected topological space. Then the comparison map* $\mathrm{comp}^k\colon H_b^k(X;\mathbb{R}) \to H^k(X;\mathbb{R})$ *is trivial for all* $k > \mathrm{cat}_{\mathcal{AME}}(X)$.

In particular, if M is an oriented closed connected manifold with $\mathrm{cat}_{\mathcal{AME}}(M) < \dim(M)$, *then* $\|M\| = 0$.

Theorem 1.34 is one of the strongest known vanishing results for simplicial volume. Apart from the original proofs of Gromov [Gro82] and Ivanov [Iva85], recently several new accounts have been given [FM, LS20, Iva, Rap].

There is an analogous vanishing result for the ℓ^2-Betti numbers of aspherical spaces in degrees larger than the amenable category. Moreover, in the case of aspherical manifolds one even obtains vanishing in *all* degrees.

Theorem 1.35 (Sauer [Sau09]). *Let X be an aspherical path-connected CW-complex. We have* $b_k^{(2)}(X) = 0$ *for all* $k > \mathrm{cat}_{\mathcal{AME}}(X)$.

Moreover, let M be an aspherical oriented closed connected manifold. If M satisfies $\mathrm{cat}_{\mathcal{AME}}(M) < \dim(M)$, *then we have* $b_k^{(2)}(M) = 0$ *for all* $k \geq 0$ *and in particular,* $\chi(M) = 0$.

Together, Theorem 1.34 and Theorem 1.35 show that aspherical manifolds M with $\mathrm{cat}_{\mathcal{AME}}(M) < \dim(M)$ satisfy Question 1.33.

Remark 1.36 (Vanishing theorems for acyclic covers). Both Theorem 1.34 and Theorem 1.35 involve the amenable category because amenable groups are both *boundedly acyclic* as well as *ℓ^2-acyclic*. In fact, Theorem 1.34 holds more generally for uniformly boundedly acyclic covers (Corollary 3.78) and Theorem 1.35 for ℓ^2-acyclic covers (Theorem 3.86).

A vanishing theorem for relative simplicial volume. There are notions of relative simplicial volume for manifolds with boundary and of relative bounded cohomology for pairs of topological spaces. It is indispensable to consider these relative versions, e.g., when studying manifolds that are constructed by glueing manifolds with boundary (Section 3.8).

Joint with Clara Löh and Marco Moraschini, we introduce the *relative amenable category* (Definition 3.43) and obtain a vanishing theorem analogous to Theorem 1.34.

Theorem 1.37 (Relative vanishing theorem, Corollary 3.80). *Let (X, A) be a CW-pair with path-connected ambient space X. Suppose that A has only finitely many connected components, each of which is π_1-injective in X. Then the comparison map*

$$\mathrm{comp}^k\colon H_b^k(X, A;\mathbb{R}) \to H^k(X, A;\mathbb{R})$$

is trivial for all $k > \mathrm{cat}_{\mathcal{AME}}(X, A)$.

In particular, if $(M, \partial M)$ *is an oriented compact connected manifold with* π_1*-injective boundary components and* $\mathrm{cat}_{\mathcal{AME}}(M, \partial M) < \dim(M)$*, then* $\|M, \partial M\| = 0$.

To prove Theorem 1.37, we adapt Löh–Sauer's recent proof [LS20] of Theorem 1.34 to the relative setting. To this end we introduce classifying spaces for group pairs and equivariant nerve pairs. Using recent results of Moraschini–Raptis [MR], we are able to allow for more general uniformly boundedly acyclic covers (Corollary 3.78). Furthermore, we obtain new estimates on the simplicial volume of manifolds constructed by glueing manifolds with boundedly acyclic boundary (Section 3.8).

Chapter 3 also contains several other results that use classifying spaces for group pairs, most notably a vanishing theorem for relative ℓ^2-Betti numbers (Theorem 3.86) and a relative version of a result of Dranishnikov–Rudyak [DR09] for maps of degree one (Corollary 3.29).

Amenable category of aspherical spaces. In view of the vanishing results for the comparison map (Theorem 1.34) and for ℓ^2-Betti numbers (Theorem 1.35), the amenable category is an interesting and meaningful invariant. As usual, by restricting ourselves to aspherical spaces, we obtain an invariant of discrete groups. We list some examples for which the precise value of the amenable category can be computed. We have

- $\mathrm{cat}_{\mathcal{AME}}(G) = 0$ if and only if G is amenable (Definition 1.14);
- $\mathrm{cat}_{\mathcal{AME}}(G) = 1$ if and only if G is a non-amenable fundamental group of a graph of amenable groups [CLM, Corollary 5.4];
- $\mathrm{cat}_{\mathcal{AME}}(G) = \mathrm{cd}(G)$ if G is torsion-free non-elementary hyperbolic (Theorem 1.30 and Theorem 1.34).

In Chapter 4 we compute the amenable category for all right-angled Artin groups. These groups interpolate between free groups and free abelian groups, and are an important source of (counter-)examples in geometric group theory.

Let L be a finite flag complex, i.e., a simplicial complex in which every clique spans a simplex. The *right-angled Artin group* A_L has as generators vertices of L subject to the relations that two generators commute if the corresponding vertices span an edge in L. The *right-angled Coxeter group* W_L is the quotient of A_L obtained by adding the relations that each generator is of order two. Since W_L is virtually torsion-free, the *virtual cohomological dimension* $\mathrm{vcd}(W_L)$ is well-defined as the cohomological dimension of a finite index torsion-free subgroup of W_L.

Theorem 1.38 (Amenable category of right-angled Artin groups, Corollary 4.17)**.** *Let A_L be the right-angled Artin group associated to a finite flag complex L. We have*

$$\mathrm{cat}_{\mathcal{AME}}(A_L) = \mathrm{vcd}(W_L)\,.$$

The proof of Theorem 1.38 relies on combining existing results from the literature. The upper bound follows from a construction of classifying spaces that are of small dimension by Petrosyan–Prytuła [PP]. The lower bound uses a vanishing theorem analogous to Theorem 1.35 for mod p homology growth by Sauer [Sau16] and a recent computation of this growth for right-angled Artin groups by Avramidi–Okun–Schreve [AOS21].

Our methods also yield a complete characterisation of right-angled Artin groups with (non-)vanishing minimal volume entropy (Theorem 4.20). This resolves the remaining cases left open by recent work of Haulmark–Schreve [HS] (see also [BC21]).

1.3 Farber's topological complexity

The notion of topological complexity was introduced by Farber [Far03] motivated by the motion planning problem in robotics. For example, let us consider a robot moving on the floor of a warehouse. A motion planning algorithm takes as input an initial and final location of the robot and produces as output a trajectory connecting the two that avoids obstacles in the floor plan. In theory the robot can then move autonomously along the trajectory. More generally, the initial and final states can be viewed as points in a configuration space encoding various parameters (e.g., the angle determining the position of a robot arm). Then the trajectory is a path in this configuration space. The idea of topological complexity is to measure the "complexity" of a motion planning algorithm in a given space. This concept has attracted the interest of both applied and pure mathematicians.

1.3.1 Basics on topological complexity

Finding a path connecting prescribed start and end points can be formalised mathematically via sections of the path fibration.

For a path-connected topological space X, we have the path fibration

$$p_X \colon X^{[0,1]} \to X \times X, \quad p_X(\gamma) = (\gamma(0), \gamma(1))\,.$$

It is easy to observe that p_X admits a global section (i.e., a map $s \colon X \times X \to X^{[0,1]}$ such that $p_X \circ s = \mathrm{id}_{X \times X}$) if and only if X is contractible. In the spirit of the Lusternik–Schnirelmann category (Section 1.1.3), open subsets of $X \times X$ on which p_X admits a *local* section may be regarded as small, and we consider the minimal number of such open subsets needed to cover $X \times X$. This number yields a homotopy invariant measuring how many continuous rules are required to implement a motion planning algorithm in X.

Definition 1.39 (Topological complexity)**.** Let X be a path-connected topological space. The *topological complexity* $\mathsf{TC}(X)$ is the infimum of $n \in \mathbb{N}_{\geq 0}$ for which there exists an open cover $X \times X = \bigcup_{i=0}^{n} U_i$ such that p_X admits a local section over each U_i.

The *topological complexity of a group* π is defined to be $\mathsf{TC}(\pi) \coloneqq \mathsf{TC}(B\pi)$.

The general strategy to computing the precise value of $\mathsf{TC}(X)$ is as for the Lusternik–Schnirelmann category. One constructs an explicit open cover together with local sections of p_X as in Definition 1.39, and then one needs to show that this cover is minimal. A basic obstruction to the existence of such covers is given by non-trivial cup-products of cohomology classes in the kernel of the map $H^*(X \times X) \to H^*(X)$ induced by the diagonal map.

For example, the topological complexity of spheres S^n for $n > 0$ is given by

$$\mathsf{TC}(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

The topological complexity of real projective space $\mathbb{R}P^k$ is closely related to the smallest dimension n for which $\mathbb{R}P^k$ can be immersed into $\mathbb{R}^n$ [FTY03]. The latter example gives evidence that the topological complexity encodes interesting geometric information.

Remark 1.40 (Convention on normalisation)*.* In the literature also a different normalisation for TC is used, producing values that are by 1 larger than ours. Therefore, Definition 1.39 is sometimes called the *reduced* topological complexity.

Topological complexity of aspherical spaces. The connection between generalised LS-category and classifying spaces for families (Theorem 1.15) was first observed in the context of topological complexity by Farber–Grant–Lupton–Oprea in their remarkable work [FGLO19].

For a group π, we consider the product $\pi \times \pi$ and denote by $\mathcal{D} \coloneqq \mathcal{F}\langle\Delta\rangle$ the family of subgroups of $\pi \times \pi$ generated by the diagonal subgroup $\Delta \subset \pi \times \pi$ (Example 1.2).

Theorem 1.41 (Farber–Grant–Lupton–Oprea [FGLO19])**.** *Let π be a group. The topological complexity $\mathsf{TC}(\pi)$ coincides with the infimum of $n \in \mathbb{N}_{\geq 0}$ for which the canonical $(\pi \times \pi)$-map $E(\pi \times \pi) \to E_{\mathcal{D}}(\pi \times \pi)$ is equivariantly homotopic to a map with values in the n-skeleton of $E_{\mathcal{D}}(\pi \times \pi)$.*

The topological complexity $\mathsf{TC}(\pi)$ satisfies the inequalities

$$\mathrm{cd}(\pi) \leq \mathsf{TC}(\pi) \leq \mathrm{cd}(\pi \times \pi)\,, \tag{1.6}$$

which can easily be deduced from Theorem 1.41. Since the topological complexity of groups containing torsion is infinite, one reduces to the study of torsion-free groups. The precise value of $\mathsf{TC}(\pi)$ has been computed for several classes of groups, for an account see e.g., [FGLO19, FM20, Dra20]. For torsion-free abelian groups π, we have $\mathsf{TC}(\pi) = \mathrm{cd}(\pi)$ because in this case a model for $E_{\mathcal{D}}(\pi \times \pi)$ is given by $E\pi$. For right-angled Artin groups in general, the topological complexity provides a new invariant.

Theorem 1.42 (González et al. [GGG+15], Cohen–Pruidze [CP08])**.** *Let A_L be the right-angled Artin group associated to a finite flag complex L with vertex set V. Then*

$$\mathsf{TC}(A_L) = \max\bigl\{|V_1 \cup V_2| \bigm| V_1, V_2 \subset V \text{ each spanning a simplex in } L\bigr\}.$$

In particular, for $n, k \in \mathbb{N}$ with $n \leq k \leq 2n$, the right-angled Artin group $\mathbb{Z}^n * \mathbb{Z}^{k-n}$ satisfies $\mathrm{cd}(\mathbb{Z}^n * \mathbb{Z}^{k-n}) = n$ and $\mathsf{TC}(\mathbb{Z}^n * \mathbb{Z}^{k-n}) = k$ [Rud16]. This shows that $\mathsf{TC}(\pi)$ can attain any value between the lower bound $\mathrm{cd}(\pi)$ and the upper bound $\mathrm{cd}(\pi \times \pi)$ in (1.6).

1.3.2 A question of Farber

The topological complexity is defined topologically and similar in spirit to the Lusternik–Schnirelmann category (Definition 1.12). We have seen that the LS-category of a group coincides with its cohomological dimension (Theorem 1.13). This raises the following question which is one of the main open problems in the field of topological complexity.

Question 1.43 (Farber)**.** Let π be a group. What is an algebraic characterisation of $\mathsf{TC}(\pi)$?

We do not specify what is meant precisely by an "algebraic characterisation" in Question 1.43. Certainly a characterisation in terms of (Bredon) group cohomology qualifies as such.

Recall from Definition 1.16 that $\mathrm{cd}_{\mathcal{TR}\subset\mathcal{D}}(\pi \times \pi)$ denotes the supremum of degrees k for which the map

$$H^k_{\pi\times\pi}\bigl(E_{\mathcal{D}}(\pi \times \pi); M\bigr) \to H^k_{\pi\times\pi}\bigl(E(\pi \times \pi); M(\pi \times \pi/1)\bigr)$$

is non-trivial for some $\mathcal{O}_{\mathcal{D}}(\pi \times \pi)$-module M. By Theorem 1.41, it provides a lower bound

$$\mathrm{cd}_{\mathcal{TR}\subset\mathcal{D}}(\pi \times \pi) \leq \mathsf{TC}(\pi). \tag{1.7}$$

To the author's knowledge, there is no known example of a group π for which the inequality (1.7) is strict. This makes $\mathrm{cd}_{\mathcal{TR}\subset\mathcal{D}}(\pi \times \pi)$ an obvious candidate to answer Question 1.43. In the next section, we discuss that the lower bound (1.7) can be used to compute the topological complexity of hyperbolic groups. In particular, for hyperbolic groups the inequality (1.7) is in fact an equality.

1.3.3 Computations for hyperbolic groups

To compute the topological complexity of a group π, knowledge about centralisers in π is essential. In torsion-free hyperbolic groups, the centraliser of any non-trivial element is cyclic. This property is sufficient to show that $\mathsf{TC}(\pi)$ takes the maximal value $\mathrm{cd}(\pi \times \pi)$.

Theorem 1.44 (Dranishnikov [Dra20]). *Let π be a torsion-free group with $\mathrm{cd}(\pi) \geq 2$. Assume that the centraliser of every non-trivial element in π is cyclic. Then we have $\mathsf{TC}(\pi) = \mathrm{cd}(\pi \times \pi)$.*

For groups π as in Theorem 1.44, it was previously shown by Farber–Mescher [FM20] that $\mathsf{TC}(\pi) \in \{\mathrm{cd}(\pi \times \pi) - 1, \mathrm{cd}(\pi \times \pi)\}$ using a purely algebraic approach. On the other hand, Dranishnikov's proof of Theorem 1.44 uses the characterisation of $\mathsf{TC}(\pi)$ via classifying spaces for families (Theorem 1.41), an original construction for $E_{\mathcal{D}}(\pi \times \pi)$, and cohomology with compact support.

We were able to generalise Theorem 1.44 to a relative setting using different methods. More precisely, we show that the lower bound (1.7) in terms of Bredon cohomology is maximal by building a model for $E_{\mathcal{D}}(\pi \times \pi)$ in *two* steps via a construction of Lück–Weiermann (Theorem 5.3).

Theorem 1.45 (Theorem 5.11). *Let π be a torsion-free group with $\mathrm{cd}(\pi) \geq 2$. Suppose that π admits a malnormal collection of abelian groups $\mathcal{P} = \{P_i \mid i \in I\}$ satisfying $\mathrm{cd}(P_i \times P_i) < \mathrm{cd}(\pi \times \pi)$. Assume that the centraliser of every element of π that is not conjugate into any of the P_i is cyclic. Then we have*

$$\mathsf{TC}(\pi) = \mathrm{cd}(\pi \times \pi)\,.$$

Our main examples of groups satisfying the assumptions of Theorem 1.45 are certain toral relatively hyperbolic groups.

Theorem 1.45 was generalised by Sam Hughes and the author in [HL] to Rudyak's higher topological complexities [Rud10]. We deduce that hyperbolic groups satisfy a conjecture by Farber–Oprea [FO19] on the rationality of the TC-generating function [HL, Corollary 1.3]. The article [HL] was also produced during the author's candidature. It is not included in this thesis due to its similarity with Chapter 5.

1.3.4 Questions

We conclude this introduction by posing two questions that are inspired by Farber's Question 1.43. We ask the same question more generally for the $\mathcal{F}$-category (Definition 1.14), where $\mathcal{F}$ is an arbitrary family of subgroups.

Question 1.46. Let G be a group and let $\mathcal{F}$ be a family of subgroups of G. What is an algebraic characterisation of $\mathrm{cat}_{\mathcal{F}}(G)$?

Question 1.46 contains Question 1.43 a special case since $\mathsf{TC}(\pi) = \mathrm{cat}_{\mathcal{D}}(\pi \times \pi)$. In Chapter 4 we apply established methods from topological complexity, most notably from [FGLO19], to other families. One might hope that, vice versa, finding answers to Question 1.46 for other families can provide new insights to the case of topological complexity.

An obvious candidate to answer Question 1.46 is the lower bound in terms of Bredon cohomology (Definition 1.16)

$$\mathrm{cd}_{\mathcal{TR}\subset\mathcal{F}}(G) \leq \mathrm{cat}_{\mathcal{F}}(G)\,. \tag{1.8}$$

However, we note that the inequality (1.8) can be strict. Namely, when $\mathcal{F}$ is generated by a single normal subgroup N, then the invariants in (1.8) reduce to geometric and cohomological invariants of the group epimorphism $G \twoheadrightarrow G/N$. Goodwillie [Gra] exhibited an example satisfying $\mathrm{cd}_{\mathcal{TR}\subset\mathcal{F}\langle N\rangle}(G) < \mathrm{cat}_{\mathcal{F}\langle N\rangle}(G)$. In a slightly different language, the question when the equality $\mathrm{cd}_{\mathcal{TR}\subset\mathcal{F}\langle N\rangle}(G) = \mathrm{cat}_{\mathcal{F}\langle N\rangle}(G)$ holds was recently investigated in [Sco, DK]. It is a special case of our following general question.

Question 1.47. For which groups G and families $\mathcal{F}$ does the equality

$$\mathrm{cat}_{\mathcal{F}}(G) = \mathrm{cd}_{\mathcal{TR}\subset\mathcal{F}}(G)$$

hold?

Goodwillie's example shows that Question 1.47 does not hold in general. On the other hand, in this thesis we will encounter several interesting examples that satisfy Question 1.47, some of which are listed in Table 1.2.

Group G	Family $\mathcal{F}$	$\mathrm{cat}_{\mathcal{F}}(G)$	Reference
every group G	$\mathcal{TR}$	$\mathrm{cd}(G)$	Theorem 1.13
right-angled Coxeter group W_L	$\mathcal{FIN}$	$\mathrm{vcd}(W_L)$	Corollary 4.15
right-angled Artin group A_L	$\mathcal{F}\langle\mathcal{S}\rangle$	$\mathrm{vcd}(W_L)$	Proposition 4.13
torsion-free non-elementary hyperbolic group G	$\mathcal{AME}$	$\mathrm{cd}(G)$	[CLM, Example 7.8]
$\pi \times \pi$, where π is a torsion-free non-elementary hyperbolic group	$\mathcal{D}$	$\mathrm{cd}(\pi \times \pi)$	Theorem 5.11

TABLE 1.2: Examples satisfying $\mathrm{cat}_{\mathcal{F}}(G) = \mathrm{cd}_{\mathcal{TR}\subset\mathcal{F}}(G)$.

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Chapter 2

Bounded cohomology of classifying spaces for families of subgroups

This chapter is the article [Lib] which has been accepted for publication in the journal *Algebraic and Geometric Topology*.

Abstract. We introduce a bounded version of Bredon cohomology for groups relative to a family of subgroups. Our theory generalises bounded cohomology and differs from Mineyev–Yaman's relative bounded cohomology for pairs. We obtain cohomological characterisations of relative amenability and relative hyperbolicity, analogous to the results of Johnson and Mineyev for bounded cohomology.

2.1 Introduction

Bounded cohomology is a homotopy invariant of topological spaces with deep connections to Riemannian geometry via the simplicial volume of manifolds [Gro82]. An astonishing phenomenon known as Gromov's Mapping Theorem is that for every CW-complex X, the classifying map $X \to B\pi_1(X)$ induces an isometric isomorphism on bounded cohomology. This emphasises the importance of the corresponding theory of bounded cohomology for groups, which is also of independent interest due to its plentiful applications in geometric group theory [Mon01, Mon06, Fri17]. The bounded cohomology $H_b^n(G;V)$ of a (discrete) group G with coefficients in a normed G-module V is the cohomology of the cochain complex of bounded G-maps $G^{n+1} \to V$. The inclusion of bounded G-maps into (not necessarily bounded) G-maps induces the so-called *comparison map* $H_b^n(G;V) \to H^n(G;V)$. On the one hand, the bounded cohomology groups are very difficult to compute in general. On the other hand, they characterise interesting group-theoretic properties such as amenability [Joh72] and hyperbolicity [Min01, Min02].

Theorem 2.1 (Johnson)**.** *Let G be a group. The following are equivalent:*

(i) *G is amenable;*

(ii) $H_b^n(G; V^\#) = 0$ *for all dual normed* $\mathbb{R}G$*-modules* $V^\#$ *and all* $n \geq 1$*;*

(iii) $H_b^1(G; V^\#) = 0$ *for all dual normed* $\mathbb{R}G$*-modules* $V^\#$*.*

Theorem 2.2 (Mineyev)**.** *Let G be a finitely presented group. The following are equivalent:*

(i) *G is hyperbolic;*

(ii) *The comparison map* $H_b^n(G; V) \to H^n(G; V)$ *is surjective for all normed* $\mathbb{Q}G$*-modules V and all* $n \geq 2$*;*

(iii) *The comparison map* $H_b^2(G; V) \to H^2(G; V)$ *is surjective for all normed* $\mathbb{R}G$*-modules V.*

There are well-studied notions of relative amenability and relative hyperbolicity in the literature [JOR12, Hru10]. In the present article we introduce a new "relative bounded cohomology theory" characterising these relative group-theoretic properties as a bounded version of Bredon cohomology. For a group G, a family of subgroups $\mathcal{F}$ is a non-empty set of subgroups which is closed under conjugation and taking subgroups. For a set of subgroups $\mathcal{H}$ of G, we denote by $\mathcal{F}\langle\mathcal{H}\rangle$ the smallest family containing $\mathcal{H}$. The Bredon cohomology $H^n_{\mathcal{F}}(G; V)$ with coefficients in a G-module V (or more general coefficient systems) is a generalisation of group cohomology, which is recovered when $\mathcal{F}$ consists only of the trivial subgroup. A fundamental feature of Bredon cohomology is that for a normal subgroup N of G there is an isomorphism $H^n_{\mathcal{F}\langle N\rangle}(G; V) \cong H^n(G/N; V^N)$. From a topological point of view, the Bredon cohomology of G can be identified with the equivariant cohomology of the classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$, which is a terminal object in the G-homotopy category of G-CW-complexes with stabilisers in $\mathcal{F}$. Especially the classifying spaces $E_{\mathcal{FIN}}G$ and $E_{\mathcal{VCY}}G$ for the family of finite groups and virtually cyclic groups have received a lot of attention in recent years due to their prominent role in the Isomorphism Conjectures of Baum–Connes and Farrell–Jones, respectively.

We introduce the *bounded Bredon cohomology* $H^n_{\mathcal{F},b}(G; V)$ of G with coefficients in a normed G-module V, which generalises bounded cohomology (Definition 2.7). Our theory still is well-behaved with respect to normal subgroups (Corollary 2.17) and admits a topological interpretation in terms of classifying spaces for families (Theorem 2.16). We obtain the following generalisations of Theorems 2.1 and 2.2. A group G is called amenable relative to a set of subgroups $\mathcal{H}$ if there exists a G-invariant mean on the G-set $\coprod_{H\in\mathcal{H}} G/H$.

Theorem 2.3. *Let G be a group and* $\mathcal{H}$ *be a set of subgroups. The following are equivalent:*

(i) *G is amenable relative to $\mathcal{H}$;*

(ii) *$H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V^\#) = 0$ for all dual normed $\mathbb{R}G$-modules $V^\#$ and all $n \geq 1$;*

(iii) *$H^1_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V^\#) = 0$ for all dual normed $\mathbb{R}G$-modules $V^\#$.*

Theorem 2.3 is a special case of the more general Theorem 2.23. We also provide a characterisation of relative amenability in terms of relatively injective modules (Proposition 2.26). Recall that a finite set of subgroups $\mathcal{H}$ is called a malnormal (resp. almost malnormal) collection if for all $H_i, H_j \in \mathcal{H}$ and $g \in G$ we have $H_i \cap gH_jg^{-1}$ is trivial (resp. finite), unless $i = j$ and $g \in H_i$. A group G is said to be of type $F_{n,\mathcal{F}}$ for a family of subgroups $\mathcal{F}$, if there exists a model for the classifying space $E_{\mathcal{F}}G$ with cocompact n-skeleton.

Theorem 2.4 (Theorem 2.31)**.** *Let G be a finitely generated torsion-free group and $\mathcal{H}$ be a finite malnormal collection of subgroups. Suppose that G is of type $F_{2,\mathcal{F}\langle\mathcal{H}\rangle}$ (e.g., G and all subgroups in $\mathcal{H}$ are finitely presented). Then the following are equivalent:*

(i) *G is hyperbolic relative to $\mathcal{H}$;*

(ii) *The comparison map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V) \to H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V)$ is surjective for all normed $\mathbb{Q}G$-modules V and all $n \geq 2$;*

(iii) *The comparison map $H^2_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V) \to H^2_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V)$ is surjective for all normed $\mathbb{R}G$-modules V.*

In Theorem 2.4 the equivalence of (i) and (iii) still holds if the group G contains torsion and $\mathcal{H}$ is almost malnormal, see Remark 2.32. Note that condition (iii) is trivially satisfied for groups of Bredon cohomological dimension $\mathrm{cd}_{\mathcal{F}\langle\mathcal{H}\rangle}$ equal to 1.

The topological interpretation of bounded Bredon cohomology via classifying spaces for families was used by Löh–Sauer [LS20] to give a new proof of the Nerve Theorem and Vanishing Theorem for amenable covers. We prove a converse of [LS20, Proposition 5.2], generalising a recent result of [MR, Theorem 3.1.3] where the case of a normal subgroup is treated.

Theorem 2.5. *Let G be a group and $\mathcal{F}$ be a family of subgroups. The following are equivalent:*

(i) *All subgroups in $\mathcal{F}$ are amenable;*

(ii) *The canonical map $H^n_{\mathcal{F},b}(G;V^\#) \to H^n_b(G;V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$-modules $V^\#$ and all $n \geq 0$;*

(iii) *The canonical map $H^1_{\mathcal{F},b}(G;V^\#) \to H^1_b(G;V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$-modules $V^\#$.*

Theorems 2.5 is a special case of the more general Theorem 2.23. As an application of Theorem 2.5, the comparison map vanishes for groups which admit a "small" model for $E_{\mathcal{F}}G$, where $\mathcal{F}$ is any family consisting of amenable subgroups (Corollary 2.24). Examples are graph products of amenable groups (e.g., right-angled Artin groups) and fundamental groups of graphs of amenable groups.

There is another natural relative cohomology theory given by the relative cohomology of a pair of spaces. For a set of subgroups $\mathcal{H}$, it gives rise to the cohomology $H^n(G,\mathcal{H};V)$ of the group pair $(G,\mathcal{H})$ introduced by Bieri–Eckmann [BE78]. A bounded version $H^n_b(G,\mathcal{H};V)$ was defined by Mineyev–Yaman [MY] to give a characterisation of relative hyperbolicity (see also [Fra18]). A characterisation of relative amenability in terms of this relative theory was obtained in [JOR12]. There is a canonical map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V) \to H^n(G,\mathcal{H};V)$ for $n \geq 2$ which is an isomorphism if $\mathcal{H}$ is malnormal (see Remark 2.6). Similarly, there is a map for the bounded versions but we do not know when it is an isomorphism due to the failure of the excision axiom for bounded cohomology (see Remark 2.18). We also mention that Mineyev–Yaman's relative bounded cohomology was extended to pairs of groupoids in [Bla16].

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2.2 Preliminaries on Bredon cohomology and classifying spaces

In this section we briefly recall the notion of Bredon cohomology for groups and its topological interpretation as the equivariant cohomology of classifying spaces for families of subgroups. For an introduction to Bredon cohomology we refer to [Flu] and for a survey on classifying spaces to [Lüc05].

Let G be a group, which shall always mean a discrete group. A *family of subgroups* $\mathcal{F}$ is a non-empty set of subgroups of G that is closed under conjugation by elements of G and under taking subgroups. Typical examples are $\mathcal{TR} = \{1\}$, $\mathcal{FIN} = \{\text{finite subgroups}\}$, $\mathcal{VCY} = \{\text{virtually cyclic subgroups}\}$, and $\mathcal{ALL} = \{\text{all subgroups}\}$. We will moreover be interested in $\mathcal{AME} = \{\text{amenable subgroups}\}$. For a subgroup H of G, we denote by $\mathcal{F}|_H$ the family $\{L \cap H \mid L \in \mathcal{F}\}$ of subgroups of H. (In the literature this family is sometimes denoted by $\mathcal{F}\cap H$ instead.) For a set of subgroups $\mathcal{H}$, one can consider the smallest family containing $\mathcal{H}$ which is $\mathcal{F}\langle\mathcal{H}\rangle = \{\text{conjugates of elements in } \mathcal{H} \text{ and their subgroups}\}$ and called the *family generated by* $\mathcal{H}$. When $\mathcal{H}$ consists of a single subgroup H, we denote $\mathcal{F}\langle\mathcal{H}\rangle$ instead by $\mathcal{F}\langle H\rangle$ and call it the *family generated by* H. We denote by $G/\mathcal{H}$ the G-set $\coprod_{H\in\mathcal{H}} G/H$.

Let R be a ring and $\mathbf{Mod}_R$ denote the category of R-modules. We will often suppress the ring R, so that G-modules are understood to be RG-modules. The *($\mathcal{F}$-restricted) orbit category* $\mathcal{O}_\mathcal{F}G$ has as objects G-sets of the form G/H with $H \in \mathcal{F}$ and as morphisms G-maps. An *$\mathcal{O}_\mathcal{F}G$-module* is a contravariant functor $M\colon \mathcal{O}_\mathcal{F}G \to \mathbf{Mod}_R$, the category of which is denoted by $\mathcal{O}_\mathcal{F}G$-$\mathbf{Mod}_R$. Note that $\mathcal{O}_{\mathcal{TR}}G$-$\mathbf{Mod}_R$ can be identified with the category of G-modules (see e.g., [Flu, Chapter 1, Section 4]). For a G-module V, there is a coinduced $\mathcal{O}_\mathcal{F}G$-module $V^?$ given by $V^?(G/H) = V^H$. (In the literature this is sometimes called a *fixed-point functor*.) Observe that $(-)^?$ is right-adjoint to the restriction $\mathcal{O}_\mathcal{F}G$-$\mathbf{Mod}_R \to \mathcal{O}_{\mathcal{TR}}G$-$\mathbf{Mod}_R$, $M \mapsto M(G/1)$ (see e.g., [Flu, Proposition 1.31]). That is, for every $\mathcal{O}_\mathcal{F}G$-module M and G-module V there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{F}G\text{-}\mathbf{Mod}_R}(M, V^?) \cong \mathrm{Hom}_{RG}(M(G/1), V)\,. \tag{2.1}$$

For a G-space X and a G-CW-complex Y with stabilisers in $\mathcal{F}$, there are singular and cellular $\mathcal{O}_\mathcal{F}G$-chain complexes $C_*(X^?)(G/H) = C_*(X^H)$ and $C_*^{\mathrm{cell}}(Y^?)(G/H) = C_*^{\mathrm{cell}}(Y^H)$, where $C_*(X^H)$ and $C_*^{\mathrm{cell}}(Y^H)$ denote the usual singular and cellular chain complexes, respectively.

The *Bredon cohomology* of G with coefficients in an $\mathcal{O}_\mathcal{F}G$-module M is defined as the R-module

$$H^n_\mathcal{F}(G; M) \coloneqq \mathrm{Ext}^n_{\mathcal{O}_\mathcal{F}G\text{-}\mathbf{Mod}_R}(R, M)$$

for $n \geq 0$, where R is regarded as a constant $\mathcal{O}_\mathcal{F}G$-module. It can be computed as the cohomology of the cochain complex $\mathrm{Hom}_{\mathcal{O}_\mathcal{F}G\text{-}\mathbf{Mod}_R}(R[((G/\mathcal{F})^{*+1})^?], M)$ (see e.g., [Flu, Proposition 3.5]). We define the G-chain complex $C_*^\mathcal{F}(G)$ given by G-modules

$$C_n^\mathcal{F}(G) \coloneqq R[(G/\mathcal{F})^{n+1}]$$

with the diagonal G-action and differentials $\partial_n\colon C_n^\mathcal{F}(G) \to C_{n-1}^\mathcal{F}(G)$,

$$\partial_n(g_0H_0, \dots, g_nH_n) = \sum_{i=0}^{n} (-1)^i (g_0H_0, \dots, \widehat{g_iH_i}, \dots, g_nH_n)\,.$$

For a G-module V, the G-cochain complex $C^*_\mathcal{F}(G; V)$ is given by

$$C^n_\mathcal{F}(G; V) \coloneqq \mathrm{Hom}_R(C_n^\mathcal{F}(G), V)$$

so that by adjunction (2.1)

$$H^n_{\mathcal{F}}(G;V) \coloneqq H^n_{\mathcal{F}}(G;V^?) \cong H^n(C^*_{\mathcal{F}}(G;V)^G)\,.$$

For a G-space X with stabilisers in $\mathcal{F}$, the *Bredon cohomology* of X with coefficients in an $\mathcal{O}_{\mathcal{F}}G$-module M is defined as

$$H^n_G(X;M) \coloneqq H^n(\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{-}\mathbf{Mod}_R}(C_*(X^?), M))$$

for $n \geq 0$. If X is a G-CW-complex, then $H^n_G(X;M)$ can be computed using $C^{\mathrm{cell}}_*(X^?)$ instead of $C_*(X^?)$.

A *classifying space* $E_{\mathcal{F}}G$ *for the family* $\mathcal{F}$ is a terminal object in the G-homotopy category of G-CW-complexes with stabilisers in $\mathcal{F}$. It can be shown that a G-CW-complex X is a model for $E_{\mathcal{F}}G$ if and only if the fixed-point set X^H is contractible for $H \in \mathcal{F}$ and empty otherwise (see e.g., [Lüc05, Theorem 1.9]). An explicit model is given by the geometric realisation Y of the semi-simplicial set $\{(G/\mathcal{F})^{n+1} \mid n \geq 0\}$ with the usual face maps. Then Y has (non-equivariant) n-cells corresponding to $(G/\mathcal{F})^{n+1}$ and we refer to Y as the *simplicial model* for $E_{\mathcal{F}}G$. Note that a model for $E_{\mathcal{TR}}G$ is given by EG and a model for $E_{\mathcal{ALL}}G$ is the point G/G. The cellular $\mathcal{O}_{\mathcal{F}}G$-chain complex of any model for $E_{\mathcal{F}}G$ is a projective resolution of the constant $\mathcal{O}_{\mathcal{F}}G$-module R (see e.g., [Flu, Proposition 2.9]) and thus we have

$$H^n_{\mathcal{F}}(G;M) \cong H^n_G(E_{\mathcal{F}}G;M) \tag{2.2}$$

for all $\mathcal{O}_{\mathcal{F}}G$-modules M. If N is a normal subgroup of G, then a model for $E_{\mathcal{F}\langle N\rangle}G$ is given by $E(G/N)$ regarded as a G-CW-complex and we find

$$H^n_{\mathcal{F}\langle N\rangle}(G;M) \cong H^n(G/N; M(G/N)) \tag{2.3}$$

(see e.g., [ANCMSS21, Corollary 4.11]).

For a subgroup H of G, when viewed as an H-space $E_{\mathcal{F}}G$ is a model for $E_{\mathcal{F}|_H}H$ which induces the restriction map

$$\mathrm{res}^n_{H\subset G}\colon H^n_{\mathcal{F}}(G;M) \to H^n_{\mathcal{F}|_H}(H;M) \tag{2.4}$$

for all $\mathcal{O}_{\mathcal{F}}G$-modules M. For two families of subgroups $\mathcal{F} \subset \mathcal{G}$, the up to G-homotopy unique G-map $E_{\mathcal{F}}G \to E_{\mathcal{G}}G$ induces the canonical map

$$\mathrm{can}^n_{\mathcal{F}\subset\mathcal{G}}\colon H^n_{\mathcal{G}}(G;M) \to H^n_{\mathcal{F}}(G;M) \tag{2.5}$$

for all $\mathcal{O}_{\mathcal{G}}G$-modules M.

Remark 2.6 (Bieri–Eckmann's relative cohomology). For a group G and a set of subgroups $\mathcal{H}$, Bieri–Eckmann [BE78] have introduced the *relative cohomology* $H^n(G,\mathcal{H};V)$ *of the pair* $(G,\mathcal{H})$ with coefficients in a G-module V. It can be identified with the relative cohomology $H^n_G(EG, \coprod_{H\in\mathcal{H}} G\times_H EH; V)$ of the pair of G-spaces $(EG, \coprod_{H\in\mathcal{H}} G\times_H EH)$. Here a model for EG is chosen that contains $\coprod_{H\in\mathcal{H}} G\times_H EH$ as a subcomplex by taking mapping cylinders. Hence there is a long exact sequence

$$\cdots H^n(G,\mathcal{H};V) \to H^n(G;V) \to \prod_{H\in\mathcal{H}} H^n(H;V) \to \cdots,$$

which is one of the main features of the relative cohomology groups.

There is a relation between Bredon cohomology and Bieri–Eckmann's relative cohomology as follows. Consider the G-space X obtained as the G-pushout

$$\begin{array}{ccc} \coprod_{H\in\mathcal{H}} G\times_H EH & \longrightarrow & EG \\ \downarrow & & \downarrow \\ \coprod_{H\in\mathcal{H}} G/H & \longrightarrow & X, \end{array}$$

where the left vertical map is induced by collapsing each EH to a point. Then the G-space X has stabilisers in $\mathcal{F}\langle\mathcal{H}\rangle$ and hence admits a G-map $X \to E_{\mathcal{F}\langle\mathcal{H}\rangle}G$. For an $\mathcal{O}_{\mathcal{F}}G$-module M, we have maps

$$\begin{array}{ccc} H^n_G(X;M) & \longleftarrow & H^n_G(X, \coprod_{H\in\mathcal{H}} G/H; M) \\ \uparrow & & \downarrow\cong \\ H^n_G(E_{\mathcal{F}\langle\mathcal{H}\rangle}G;M) & & H^n_G(EG, \coprod_{H\in\mathcal{H}} G\times_H EH; M), \end{array}$$

where the right vertical map is an isomorphism by excision. Now, if $\mathcal{H}$ is a malnormal collection, then X is a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ and we have

$$H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G;M) \cong H^n(G,\mathcal{H};M(G/1))$$

for $n \geq 2$. This was shown in [ANCMSS21, Theorem 4.16] for the special case when $\mathcal{H}$ consists of a single subgroup.

2.3 Bounded Bredon cohomology

In this section we introduce a bounded version of Bredon cohomology and develop some of its basic properties. We follow the exposition in [Fri17] for bounded cohomology. Throughout, let G be a group and $\mathcal{F}$ be a family of subgroups.

From now on, let the ring R be one of $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. A *normed G-module V* is a G-module equipped with a G-invariant norm $\|\cdot\|\colon V \to \mathbb{R}$. (That is, for all $v,u \in V$, $r \in R$, and $g \in G$ we have $\|v\| = 0$ if and only if $v = 0$, $\|rv\| \leq |r|\cdot\|v\|$, $\|v+u\| \leq \|v\| + \|u\|$, and $\|g\cdot v\| = \|v\|$.) A morphism $f\colon V \to W$ of normed G-modules is a morphism of G-modules with finite operator-norm $\|f\|_\infty$. By $\mathrm{bHom}_R(V,W)$ we denote the G-module of R-linear maps $f\colon V \to W$ with finite operator-norm, where the G-action is given by $(g \cdot f)(v) = g \cdot f(g^{-1}v)$. We denote the topological dual $\mathrm{bHom}_R(V, \mathbb{R})$ of V by $V^{\#}$. For a set S and a normed module V, we denote by $\mathrm{bMap}(S, V)$ the module of functions $S \to V$ with bounded image. Instead of $\mathrm{bMap}(S, \mathbb{R})$ we also write $\ell^\infty(S)$.

The following is our key definition. Recall the notation $G/\mathcal{F} = \coprod_{H \in \mathcal{F}} G/H$ and consider $C_n^{\mathcal{F}}(G) = R[(G/\mathcal{F})^{n+1}]$ as a normed G-module equipped with the ℓ^1-norm with respect to the R-basis $(G/\mathcal{F})^{n+1}$. For a normed G-module V, we define the cochain complex $C^*_{\mathcal{F},b}(G; V)$ of normed G-modules by

$$C^n_{\mathcal{F},b}(G; V) \coloneqq \mathrm{bHom}_R(C_n^{\mathcal{F}}(G), V)$$

together with the differentials $\delta^n\colon C^n_{\mathcal{F},b}(G; V) \to C^{n+1}_{\mathcal{F},b}(G; V)$,

$$\delta^n(f)(g_0H_0, \ldots, g_{n+1}H_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0H_0, \ldots, \widehat{g_iHi}, \ldots, g_{n+1}H_{n+1})\,.$$

Definition 2.7 (Bounded Bredon cohomology of groups)**.** The *bounded Bredon cohomology* of G with coefficients in a normed G-module V is defined as

$$H^n_{\mathcal{F},b}(G; V) \coloneqq H^n(C^*_{\mathcal{F},b}(G; V)^G)$$

for $n \geq 0$. The inclusion $C^n_{\mathcal{F},b}(G; V) \subset C^n_{\mathcal{F}}(G; V)$ induces a map

$$c^n_{\mathcal{F}}\colon H^n_{\mathcal{F},b}(G; V) \to H^n_{\mathcal{F}}(G; V)$$

called the *comparison map*.

Note that for $\mathcal{F} = \mathcal{TR}$, Definition 2.7 recovers the usual definition of bounded cohomology.

Remark 2.8 (Coefficient modules). We only consider normed G-modules as coefficients, rather than more general $\mathcal{O}_{\mathcal{F}}G$-modules equipped with a "compatible norm". Hence strictly speaking our theory is a bounded version of Nucinkis' cohomology relative to the G-set $G/\mathcal{F}$ [Nuc99], rather than a bounded version of Bredon cohomology.

Remark 2.9 (Canonical semi-norm). The ℓ^∞-norm on $C^n_{\mathcal{F},b}(G; V)$ descends to a *canonical semi-norm* on $H^n_{\mathcal{F},b}(G; V)$. However, we do not consider semi-norms anywhere in this article and regard $H^n_{\mathcal{F},b}(G; V)$ merely as an R-module.

Bounded Bredon cohomology satisfies the following basic properties.

Lemma 2.10. *The following hold:*

(i) *Let* $0 \to V_0 \to V_1 \to V_2 \to 0$ *be a short exact sequence of normed* G*-modules such that* $0 \to V_0^H \to V_1^H \to V_2^H \to 0$ *is exact for each* $H \in \mathcal{F}$*. Then there exists a long* *exact sequence*

$$0 \to H^0_{\mathcal{F},b}(G;V_0) \to H^0_{\mathcal{F},b}(G;V_1) \to H^0_{\mathcal{F},b}(G;V_2) \to H^1_{\mathcal{F},b}(G;V_0) \to \cdots ;$$

(ii) $H^0_{\mathcal{F},b}(G;V) \cong V^G$ *for all normed* G*-modules* V*;*

(iii) $H^1_{\mathcal{F},b}(G;\mathbb{R}) = 0$.

Proof. (i) For a G-set $S = \coprod_{i\in I} G/H_i$ and a normed G-module V, we can identify the module $\mathrm{bMap}_G(S,V)$ with the submodule of $\prod_{i\in I} V^{H_i}$ consisting of the elements $(v_i)_{i\in I}$ satisfying $\sup_{i\in I}\|v_i\| < \infty$. It follows that for a G-set S with stabilisers in $\mathcal{F}$, the sequence of modules

$$0 \to \mathrm{bMap}_G(S,V_0) \to \mathrm{bMap}_G(S,V_1) \to \mathrm{bMap}_G(S,V_2) \to 0$$

is exact. Applying the above to the G-sets $(G/\mathcal{F})^{n+1}$ for $n \geq 0$, we obtain that the sequence of cochain complexes

$$0 \to C^*_{\mathcal{F},b}(G;V_0)^G \to C^*_{\mathcal{F},b}(G;V_1)^G \to C^*_{\mathcal{F},b}(G;V_2)^G \to 0$$

is exact. Then the associated long exact sequence on cohomology is as desired.

(ii) We have $H^0_{\mathcal{F},b}(G;V) = \ker(\delta^0)$, where

$$\delta^0\colon \mathrm{bHom}_{RG}(R[G/\mathcal{F}],V) \to \mathrm{bHom}_{RG}(R[(G/\mathcal{F})^2],V)$$

is given by $\delta^0(f)(g_0H_0,g_1H_1) = f(g_1H_1) - f(g_0H_0)$. Hence $\ker(\delta^0)$ consists precisely of the constant G-maps $G/\mathcal{F} \to V$, which are in correspondence to V^G.

(iii) We identify

$$C^n_{\mathcal{F},b}(G;\mathbb{R})^G \cong \mathrm{bMap}\Big(\coprod_{H_0,\dots,H_n\in\mathcal{F}} H_0\backslash(G/H_1\times\cdots\times G/H_n),\mathbb{R}\Big)$$

for $n \geq 1$ and $C^0_{\mathcal{F},b}(G;\mathbb{R})^G \cong \mathrm{bMap}\big(\coprod_{H_0\in\mathcal{F}} *_{H_0},\mathbb{R}\big)$. The differentials of this "inhomogeneous" complex in low degrees are given by

$$\begin{aligned}\delta^0(f)(H_0g_1H_1) &= f(*_{H_1}) - f(*_{H_0})\\ \delta^1(\varphi)(H_0(g_1H_1,g_2H_2)) &= \varphi(H_1g_1^{-1}g_2H_2) - \varphi(H_0g_2H_2) + \varphi(H_0g_1H_1)\,.\end{aligned}$$

Then it is not difficult to check that $\ker(\delta^1) = \mathrm{im}(\delta^0)$. □

We also define the bounded cohomology of a G-space X as follows. Denote by $S_n(X)$ the set of singular n-simplices in X and consider $C_n(X) = R[S_n(X)]$ equipped with the ℓ^1-norm as a normed G-module. For a normed G-module V, we define the cochain complex $C_b^*(X;V)$ of normed G-modules by

$$C_b^n(X;V) \coloneqq \mathrm{bHom}_R(C_n(X),V)$$

together with the usual differentials.

Definition 2.11 (Bounded cohomology of G-spaces)**.** The *(G-equivariant) bounded cohomology* of a G-space X with coefficients in a normed G-module V is defined as

$$H^n_{G,b}(X;V) \coloneqq H^n(C_b^*(X;V)^G)$$

for $n \geq 0$. The inclusion $C_b^n(X;V) \subset C^n(X;V)$ induces a map

$$c_X^n \colon H^n_{G,b}(X;V) \to H^n_G(X;V)$$

called the *comparison map*.

Note that the functors $H^*_{G,b}$ are G-homotopy invariant and that $H^n_{G,b}(G/H;V)$ is isomorphic to V^H for $n=0$ and trivial otherwise. However, beware that $H^*_{G,b}$ is neither a G-cohomology theory, nor can it be computed cellularly for G-CW-complexes, as is the case already when G is the trivial group (see e.g., [Fri17, Remark 5.6]).

Relative homological algebra. We develop the relative homological algebra that will allow us to compute bounded Bredon cohomology via resolutions, analogous to Ivanov's approach for bounded cohomology [Iva85].

A map $p\colon A \to B$ of G-modules is called *$\mathcal{F}$-strongly surjective* if for each $H \in \mathcal{F}$ there exists a map $\tau_H \colon B \to A$ of H-modules such that $p \circ \tau_H = \mathrm{id}_B$. A G-module P is called *relatively $\mathcal{F}$-projective* if for every $\mathcal{F}$-strongly surjective G-map $p\colon A \to B$ and every G-map $\phi\colon P \to B$, there exists a G-map $\Phi\colon P \to A$ such that $p \circ \Phi = \phi$. A chain complex of G-modules is called *relatively $\mathcal{F}$-projective* if each chain module is relatively $\mathcal{F}$-projective. A resolution (C_*, ∂_*) of G-modules is called *$\mathcal{F}$-strong* if it is contractible as a resolution of H-modules for each $H \in \mathcal{F}$. (That is, there exist H-maps $k_*^H \colon C_* \to C_{*+1}$ such that $\partial_{n+1} \circ k_n^H + k_{n-1}^H \circ \partial_n = \mathrm{id}_{C_n}$.)

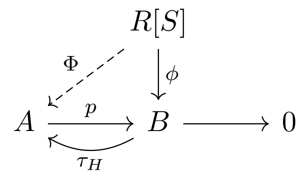
Lemma 2.12. *The following hold:*

(i) *If S is a G-set with stabilisers in $\mathcal{F}$, then the G-module $R[S]$ is relatively $\mathcal{F}$-projective;*

(ii) *If S is a G-set with $S^H \neq \emptyset$ for all $H \in \mathcal{F}$, then the resolution $R[S^{*+1}] \to R$ of G-modules is $\mathcal{F}$-strong;*

(iii) *If X is a G-space with contractible fixed-point set X^H for each $H \in \mathcal{F}$, then the resolution $C_*(X) \to R$ of G-modules is $\mathcal{F}$-strong.*

Proof. (i) Given a lifting problem as in the definition of relative $\mathcal{F}$-projectivity,



we construct a lift Φ as follows. Let T be a set of representatives of $G\backslash S$ and denote the stabiliser of an element $t \in T$ by G_t. Then for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1}s = t_s$. Define $\Phi\colon R[S] \to A$ on generators by

$$\Phi(s) = g_s \cdot \tau_{G_{t_s}}(\phi(g_s^{-1}s))$$

which is independent of the choice of g_s, since the map $\tau_{G_{t_s}}$ is G_{t_s}-equivariant. Then Φ is a G-equivariant lift of ϕ.

(ii) For $H \in \mathcal{F}$, fix an element $s_H \in S^H$ and define $k_*^H : R[S^{*+1}] \to R[S^{*+2}]$ on generators by

$$k_n^H(s_0, \ldots, s_n) = (s_H, s_0, \ldots, s_n)\,.$$

Then k_*^H is an H-equivariant contraction.

(iii) For $H \in \mathcal{F}$, fix a point $x_H \in X^H$ and define a contraction $k_*^H\colon C_*(X) \to C_{*+1}(X)$ of H-chain complexes inductively as follows. Starting with $k_{-1}^H\colon R \to C_0(X)$, $r \mapsto r \cdot x_H$, we may assume that k_{n-1}^H has been constructed. Let s be a singular n-simplex in X and denote its stabiliser by H_s. Then there exists a singular $(n+1)$-simplex s' with 0-th vertex x_H and opposite face s, satisfying $\partial_{n+1}(s') + k_{n-1}^H(\partial_n(s)) = s$. Moreover, since X^{H_s} is contractible we may choose s' such that its image is contained in X^{H_s}. Now, for each H-orbit of singular n-simplices in X choose a representative s, define $k_n^H(s)$ to be s' and then extend H-equivariantly. □

The proof of the following proposition is standard and omitted.

Proposition 2.13. *Let $f\colon V \to W$ be a map of G-modules, $P_* \to V$ be a G-chain complex with P_n relatively $\mathcal{F}$-projective for all $n \geq 0$, and $C_* \to W$ be an $\mathcal{F}$-strong resolution of G-modules. Then there exists a G-chain map $f_*\colon P_* \to C_*$ extending f, which is unique up to G-chain homotopy.*

While relatively $\mathcal{F}$-projective $\mathcal{F}$-strong resolutions are useful to compute Bredon homology, the following dual approach will compute bounded Bredon cohomology.

A map $i\colon A \to B$ of normed G-modules is called *$\mathcal{F}$-strongly injective* if for each $H \in \mathcal{F}$ there exists a map $\sigma_H\colon B \to A$ of normed H-modules with $\|\sigma_H\|_\infty \le K$ such that $\sigma_H \circ i = \mathrm{id}_A$, for a uniform constant $K \ge 0$. A normed G-module I is called *relatively $\mathcal{F}$-injective* if for every $\mathcal{F}$-strongly injective G-map $i\colon A \to B$ and every map $\psi\colon A \to I$ of normed G-modules, there exists a map $\Psi\colon B \to I$ of normed G-modules such that $\Psi \circ i = \psi$. A chain complex of normed G-modules is called *relatively $\mathcal{F}$-injective* if each chain module is relatively $\mathcal{F}$-injective. A resolution of normed G-modules is called *$\mathcal{F}$-strong* if it is contractible as a resolution of normed H-modules for each $H \in \mathcal{F}$.

Dually to Lemma 2.12 and Proposition 2.13 we obtain the following.

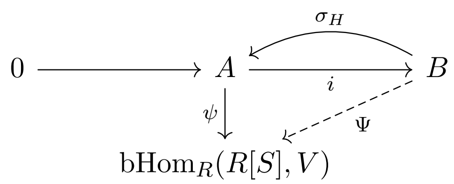
Lemma 2.14. *Let V be a normed G-module. The following hold:*

(i) *If S is a G-set with stabilisers in $\mathcal{F}$, then $\mathrm{bHom}_R(R[S], V)$ is a relatively $\mathcal{F}$-injective normed G-module;*

(ii) *If S is a G-set with $S^H \neq \emptyset$ for all $H \in \mathcal{F}$, then the resolution of normed G-modules $V \to \mathrm{bHom}_R(R[S^{*+1}], V)$ is $\mathcal{F}$-strong;*

(iii) *If X is a G-space with contractible fixed-point set X^H for each $H \in \mathcal{F}$, then the resolution $V \to C_b^*(X; V)$ of normed G-modules is $\mathcal{F}$-strong.*

Proof. (i) Given an extension problem as in the definition of relative $\mathcal{F}$-injectivity,



we construct an extension Ψ as follows. Let T be a set of representatives of $G\backslash S$ and denote the stabiliser of an element $t \in T$ by G_t. Then for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1} s = t_s$. Define the map $\Psi\colon B \to \mathrm{bHom}_R(R[S], V)$ for $b \in B$ and $s \in S$ by

$$\Psi(b)(s) = \psi(g_s \cdot \sigma_{G_{t_s}}(g_s^{-1} b))(s)\,.$$

One checks that Ψ is a well-defined map of normed RG-modules extending ψ.

The proofs of (ii) and (iii) are dual to those of Lemma 2.12 (ii) and (iii), respectively, and are left to the reader. □

Proposition 2.15. *Let $f\colon V \to W$ be a map of normed G-modules, $V \to C^*$ be an $\mathcal{F}$-strong resolution of normed G-modules, and $W \to I^*$ be a G-chain complex with I^n relatively $\mathcal{F}$-injective for all $n \ge 0$. Then there exists a G-chain map $f^*\colon C^* \to I^*$ extending f, which is unique up to G-chain homotopy.*

As a consequence of Proposition 2.15, we may use any relatively $\mathcal{F}$-injective $\mathcal{F}$-strong resolution to compute bounded Bredon cohomology. We obtain the isomorphisms analogous to (2.2) and (2.3) for Bredon cohomology.

Theorem 2.16. *Let G be a group, $\mathcal{F}$ be a family of subgroups, and V be a normed G-module. For all $n \geq 0$ there is an isomorphism*

$$H^n_{\mathcal{F},b}(G;V) \cong H^n_{G,b}(E_{\mathcal{F}}G;V)\,.$$

Proof. Both $C^*_{\mathcal{F},b}(G;V)$ and $C^*_b(E_{\mathcal{F}}G;V)$ are relatively $\mathcal{F}$-injective $\mathcal{F}$-strong resolutions of V by Lemma 2.14 and hence G-chain homotopy equivalent by Proposition 2.15. □

Corollary 2.17. *Let G be a group, N be a normal subgroup of G, and V be a normed G-module. For all $n \geq 0$ there is an isomorphism*

$$H^n_{\mathcal{F}\langle N\rangle,b}(G;V) \cong H^n_b(G/N;V^N)\,.$$

Proof. As a model for $E_{\mathcal{F}\langle N\rangle}G$ we take $E(G/N)$ regarded as a G-space. Then it suffices to observe that

$$\mathrm{bHom}_{RG}(R[S_n(E(G/N))],V) \cong \mathrm{bHom}_{R[G/N]}(R[S_n(E(G/N))],V^N)$$

and to apply Theorem 2.16 twice. □

Analogous to (2.4) and (2.5) for Bredon cohomology, for a subgroup H of G and two families of subgroups $\mathcal{F} \subset \mathcal{G}$, we have the maps

$$\begin{aligned} \mathrm{res}^n_{H\subset G,b}\colon H^n_{\mathcal{F},b}(G;V) &\to H^n_{\mathcal{F}|_H,b}(H;V)\,;\\ \mathrm{can}^n_{\mathcal{F}\subset\mathcal{G},b}\colon H^n_{\mathcal{G},b}(G;V) &\to H^n_{\mathcal{F},b}(G;V) \end{aligned}$$

for all normed G-modules V.

Remark 2.18 (Mineyev–Yaman's relative bounded cohomology)*.* Mineyev–Yaman have introduced the bounded analogue of Bieri–Eckmann's relative cohomology for pairs (Remark 2.6) in [MY]. For a group G, a finite set of subgroups $\mathcal{H}$, and a normed G-module V, their relative bounded cohomology groups $H^n_b(G,\mathcal{H};V)$ can be identified with $H^n_{G,b}(EG,\coprod_{H\in\mathcal{H}} G\times_H EH;V)$ and therefore fit in a long exact sequence

$$\cdots \to H^n_b(G,\mathcal{H};V) \to H^n_b(G;V) \to \prod_{H\in\mathcal{H}} H^n_b(H;V) \to \cdots\,.$$

As in Remark 2.6, we denote by X the G-space obtained as a G-pushout from EG by collapsing $G \times_H EH$ to G/H for each $H \in \mathcal{H}$. Then we have maps

$$\begin{array}{ccc} H^n_{G,b}(X;V) & \longleftarrow & H^n_{G,b}(X, \coprod_{H\in\mathcal{H}} G/H;V) \\ \uparrow & & \downarrow \\ H^n_{G,b}(E_{\mathcal{F}\langle\mathcal{H}\rangle}G;V) & & H^n_{G,b}(EG, \coprod_{H\in\mathcal{H}} G\times_H EH;V)\,. \end{array}$$

For $n \geq 2$, the horizontal map is an isomorphism by the long exact sequence of a pair, using the fact that $H^*_{G,b}(G/H;V) = 0$ for $* \geq 1$. Hence for $n \geq 2$ we obtain a map

$$H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V) \to H^n_b(G,\mathcal{H};V)\,.$$

However, even if $\mathcal{H}$ is a malnormal collection in which case X is a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$, this map need not be an isomorphism due to the failure of the excision axiom for bounded cohomology.

2.4 Characterisation of relative amenability

In this section we prove a characterisation of relatively amenable groups in terms of bounded Bredon cohomology analogous to Theorem 2.1.

Recall that a *G-invariant mean* on a G-set S is an $\mathbb{R}$-linear map $m\colon \ell^\infty(S) \to \mathbb{R}$ which is normalised, non-negative, and G-invariant. (That is, for the constant function $1 \in \ell^\infty(S)$, $f \in \ell^\infty(S)$, and $g \in G$ we have $m(1) = 1$, $m(f) \geq 0$ if $f \geq 0$, and $m(g\cdot f) = m(f)$.) Note that for a G-map $S_1 \to S_2$ of G-sets, a G-invariant mean on S_1 is pushed forward to a G-invariant mean on S_2.

Definition 2.19 (Relative amenability)**.** A group G is *amenable relative* to a set of subgroups $\mathcal{H}$ if the G-set $G/\mathcal{H}$ admits a G-invariant mean. When G is amenable relative to $\mathcal{H}$ consisting of a single subgroup H, we also say that H is *co-amenable* in G.

When $\mathcal{H}$ is a finite set of subgroups, we recover the notion of relative amenability studied in [JOR12] (see also [MP03]).

Example 2.20. Let G be a group, H be a subgroup, and $\mathcal{H}$ be a set of subgroups.

(i) If G is amenable, then G is amenable relative to $\mathcal{H}$;

(ii) If H is a normal subgroup, then H is co-amenable in G if and only if the quotient group G/H is amenable;

(iii) If H has finite index in G or contains the commutator subgroup $[G,G]$, then H is co-amenable in G;

(iv) If $\mathcal{H}$ is finite and G is amenable relative to $\mathcal{H}$, then $\mathcal{H}$ contains an element that is co-amenable in G;

(v) G is amenable relative to $\mathcal{H}$ if and only if G is amenable relative to $\mathcal{F}\langle\mathcal{H}\rangle$.

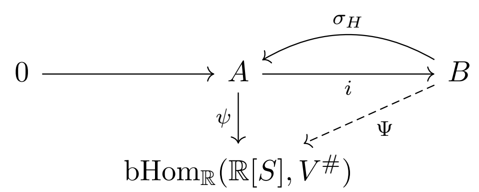
The following lemma is proved analogously to [Fri17, Lemma 3.2] (see also [Mon01, Corollary 5.3.8]).

Lemma 2.21. *Let G be a group and $\mathcal{H}$ be a set of subgroups. Then G is amenable relative to $\mathcal{H}$ if and only if there exists a non-trivial G-invariant element in $\ell^\infty(G/\mathcal{H})^\#$.*

By Proposition 2.15 bounded Bredon cohomology can be computed using relatively $\mathcal{F}$-injective $\mathcal{F}$-strong resolutions. If one considers coefficients in dual normed $\mathbb{R}G$-modules, then such resolutions can be obtained from G-sets whose stabilisers are amenable relative to $\mathcal{F}$.

Lemma 2.22. *Let G be a group, $\mathcal{F}$ be a family of subgroups, and $V^\#$ be a dual normed $\mathbb{R}G$-module. If S is a G-set such that every stabiliser G_s is amenable relative to $\mathcal{F}|_{G_s}$, then the normed $\mathbb{R}G$-module* $\mathrm{bHom}_\mathbb{R}(\mathbb{R}[S^{n+1}], V^\#)$ *is relatively $\mathcal{F}$-injective for all $n \geq 0$.*

Proof. Since the stabilisers of S^{n+1} are intersections of stabilisers of S, and relative amenability passes to subgroups, it is enough to consider the case $n = 0$. Let an extension problem as in the definition of relative $\mathcal{F}$-injectivity be given.



Let T be a set of representatives of $G\backslash S$. We denote the stabiliser of an element $t \in T$ by G_t and by assumption there exists a G_t-invariant mean m_t on $G_t/\mathcal{F}|_{G_t}$. Note that any subgroup $L \in \mathcal{F}|_{G_t}$ can also be viewed as an element in $\mathcal{F}$. Now, for every $s \in S$ there exist unique elements $t_s \in T$ and $g_sG_{t_s} \in G/G_{t_s}$ such that $g_s^{-1}s = t_s$. Define $\Psi\colon B \to \mathrm{bHom}_\mathbb{R}(\mathbb{R}[S], V^\#)$ for $b \in B$, $s \in S$, and $v \in V$ by

$$\Psi(b)(s)(v) = m_{t_s}\big(gL \mapsto (g_sg \cdot \psi(\sigma_L(g^{-1}g_s^{-1}b)))(s)(v)\big)\,.$$

One checks that Ψ is a well-defined map of normed $\mathbb{R}G$-modules extending ψ. □

For a family of subgroups $\mathcal{F}$, consider the short exact sequence of normed $\mathbb{R}G$-modules

$$0 \to \mathbb{R} \to \ell^\infty(G/\mathcal{F}) \to \ell^\infty(G/\mathcal{F})/\mathbb{R} \to 0\,,$$

where $\mathbb{R}$ is regarded as the constant functions. Then the sequence of topological duals

$$0 \to (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\# \to \ell^\infty(G/\mathcal{F})^\# \to \mathbb{R} \to 0$$

is exact, since an $\mathbb{R}$-linear split $\mathbb{R} \to \ell^\infty(G/\mathcal{F})^\#$ is given by evaluation at the trivial coset of the trivial subgroup in $G/\mathcal{F}$. We define the *relative Johnson class*

$$[J_\mathcal{F}] \in H^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\#)$$

as the cohomology class of the 1-cocycle $J_\mathcal{F} \in C^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\#)$ given by

$$J_\mathcal{F}(g_0H_0, g_1H_1) = \epsilon_{g_1H_1} - \epsilon_{g_0H_0}\,,$$

where $\epsilon_{g_iH_i}$ is the evaluation map at g_iH_i for $i = 0, 1$.

Theorem 2.23. *Let G be a group and $\mathcal{F} \subset \mathcal{G}$ be two families of subgroups. The following are equivalent:*

(i) *Every subgroup $H \in \mathcal{G}$ is amenable relative to $\mathcal{F}|_H$;*

(ii) *The canonical map $H^n_{\mathcal{G},b}(G; V^\#) \to H^n_{\mathcal{F},b}(G; V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$-modules $V^\#$ and all $n \geq 0$;*

(iii) *The canonical map $H^1_{\mathcal{G},b}(G; V^\#) \to H^1_{\mathcal{F},b}(G; V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$-modules $V^\#$;*

(iv) *The relative Johnson class $[J_\mathcal{F}] \in H^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\#)$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$.*

Proof. Suppose that every subgroup $H \in \mathcal{G}$ is amenable relative to $\mathcal{F}|_H$. Then the resolution of normed $\mathbb{R}G$-modules $V^\# \to C^*_\mathcal{G}(G; V^\#)$ is $\mathcal{F}$-strong and relatively $\mathcal{F}$-injective by Lemmas 2.14 (ii) and 2.22 applied to the G-set $G/\mathcal{G}$. Hence the canonical map $\mathrm{can}^n_{\mathcal{F}\subset\mathcal{G},b}$ is an isomorphism for all $n \geq 0$ by Proposition 2.15.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. Suppose that the relative Johnson class $[J_\mathcal{F}]$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$ and denote $V := \ell^\infty(G/\mathcal{F})/\mathbb{R}$. We claim that for every subgroup $H \in \mathcal{G}$, the image of $[J_\mathcal{F}]$ under the restriction map

$$\mathrm{res}^1_{H\subset G,b}\colon H^1_{\mathcal{F},b}(G; V^\#) \to H^1_{\mathcal{F}|_H,b}(H; V^\#)$$

is trivial. Indeed, there is a commutative diagram

$$\begin{array}{ccccc}
H^1_{G,b}(E_\mathcal{G}G; V^\#) & \xrightarrow{\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}} & H^1_{G,b}(E_\mathcal{F}G; V^\#) & & \\
\downarrow & & \downarrow & \searrow^{\mathrm{res}^1_{H\subset G,b}} & \\
H^1_{H,b}(E_\mathcal{G}G; V^\#) & \longrightarrow & H^1_{H,b}(E_\mathcal{F}G; V^\#) & \xrightarrow{\cong} & H^1_{H,b}(E_{\mathcal{F}|_H}H; V^\#)\,,
\end{array}$$

where the vertical maps are induced by viewing a G-space as an H-space. Observe that the lower left corner $H^1_{H,b}(E_{\mathcal{G}}G; V^\#)$ is trivial, since when viewed as an H-space $E_{\mathcal{G}}G$ is a model for $E_{\mathcal{ALL}|_H}H$ and hence H-equivariantly contractible. This proves the claim.

Now, fix a subgroup $H \in \mathcal{G}$ and denote $W := \ell^\infty(H/\mathcal{F}|_H)/\mathbb{R}$. Consider the commutative diagram of normed $\mathbb{R}H$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^\# & \longrightarrow & \ell^\infty(G/\mathcal{F})^\# & \longrightarrow & \mathbb{R} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & W^\# & \longrightarrow & \ell^\infty(H/\mathcal{F}|_H)^\# & \longrightarrow & \mathbb{R} & \longrightarrow & 0\,, \end{array}$$

where the rows are exact, and remain exact when restricted to L-fixed-points for every $L \in \mathcal{F}|_H$. By Lemma 2.10 there are associated long exact sequences on bounded cohomology

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & (V^\#)^H & \longrightarrow & (\ell^\infty(G/\mathcal{F})^\#)^H & \longrightarrow & \mathbb{R} & \xrightarrow{\partial^0_{V^\#}} & H^1_{\mathcal{F}|_H,b}(H; V^\#) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & (W^\#)^H & \longrightarrow & (\ell^\infty(H/\mathcal{F}|_H)^\#)^H & \longrightarrow & \mathbb{R} & \xrightarrow{\partial^0_{W^\#}} & H^1_{\mathcal{F}|_H,b}(H; W^\#) & \longrightarrow & \cdots\,. \end{array}$$

Observe that the image of $\partial^0_{V^\#}$ is precisely $\mathbb{R}\cdot \mathrm{res}^1_{H\subset G,b}[J_{\mathcal{F}}]$ and hence trivial by the claim above. This implies that the map $\partial^0_{W^\#}$ is trivial and hence there exists a non-trivial H-invariant element in $\ell^\infty(H/\mathcal{F}|_H)^\#$. Thus H is amenable relative to $\mathcal{F}|_H$ by Lemma 2.21. This finishes the proof. □

As special cases of Theorem 2.23 we obtain Theorem 2.3 by taking $\mathcal{G} = \mathcal{ALL}$ and Theorem 2.5 by taking $\mathcal{F} = \mathcal{TR}$. The case when $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G} = \mathcal{ALL}$ recovers Theorem 2.1.

Corollary 2.24. *Let X be a CW-complex with fundamental group G and $\mathcal{F}$ be a family consisting of amenable subgroups of G. Suppose that there exists a model for $E_{\mathcal{F}}G$ whose orbit space $G\backslash E_{\mathcal{F}}G$ is homotopy equivalent to a k-dimensional CW-complex. Then the comparison map $c^n_X \colon H^n_b(X;\mathbb{R}) \to H^n(X;\mathbb{R})$ vanishes for all $n > k$.*

Proof. By Gromov's Mapping Theorem [Gro82, page 40] (see also [Fri17, Theorem 5.9]), the comparison map c^n_X vanishes if the map $c^n_{EG} \colon H^n_{G,b}(EG;\mathbb{R}) \to H^n_G(EG;\mathbb{R})$ vanishes. The G-map $EG \to E_{\mathcal{F}}G$ induces a commutative square

$$\begin{array}{ccc} H^n_{G,b}(EG;\mathbb{R}) & \xrightarrow{c^n_{EG}} & H^n_G(EG;\mathbb{R}) \\ {\scriptstyle \mathrm{can}^n_{\mathcal{TR}\subset\mathcal{F},b}}\uparrow\cong & & \uparrow{\scriptstyle \mathrm{can}^n_{\mathcal{TR}\subset\mathcal{F}}} \\ H^n_{G,b}(E_{\mathcal{F}}G;\mathbb{R}) & \xrightarrow{c^n_{E_{\mathcal{F}}G}} & H^n_G(E_{\mathcal{F}}G;\mathbb{R})\,, \end{array}$$

where the canonical map $\mathrm{can}^n_{\mathcal{TR}\subset\mathcal{F},b}$ is an isomorphism by Theorem 2.5. Since we are considering trivial coefficients, the lower right corner can be identified with the (non-equivariant) cohomology of the orbit space

$$H^n_G(E_{\mathcal{F}}G;\mathbb{R}) \cong H^n(G\backslash E_{\mathcal{F}}G;\mathbb{R})$$

(see e.g., [Flu, Theorem 4.2]). □

As an application of Corollary 2.24 we obtain the following well-known examples.

Example 2.25. The comparison map vanishes in all positive degrees for CW-complexes with the following fundamental groups:

(i) Graph products of amenable groups (e.g., right-angled Artin groups);

(ii) Fundamental groups of graphs of groups with amenable vertex groups.

Indeed, if G_Γ is a graph product of amenable groups, we consider the family $\mathcal{F}$ generated by the vertex groups and direct products of vertex groups whenever the corresponding vertices form a clique in the underlying graph Γ. We claim that there exists a model for $E_{\mathcal{F}}(G_\Gamma)$ with contractible orbit space. If Γ is a complete graph, then a model for $E_{\mathcal{F}}(G_\Gamma)$ is given by the point. Otherwise, Γ can be written as $\Gamma_1 \cup_{\Gamma_0} \Gamma_2$, where Γ_i are proper full subgraphs of Γ for $i = 0, 1, 2$, and we have $G_\Gamma \cong G_{\Gamma_1} *_{G_{\Gamma_0}} G_{\Gamma_2}$. Let $\mathcal{F}_i$ be the corresponding family of subgroups of G_{Γ_i} for $i = 0, 1, 2$. Then a model for the classifying space $E_{\mathcal{F}}(G_\Gamma)$ can be constructed as the following G-pushout

$$\begin{array}{ccc} G_\Gamma \times_{G_{\Gamma_0}} E_{\mathcal{F}_0}(G_{\Gamma_0}) \times S^0 & \longrightarrow & G_\Gamma \times_{G_{\Gamma_1}} E_{\mathcal{F}_1}(G_{\Gamma_1}) \coprod G_\Gamma \times_{G_{\Gamma_2}} E_{\mathcal{F}_2}(G_{\Gamma_2}) \\ \downarrow & & \downarrow \\ G_\Gamma \times_{G_{\Gamma_0}} E_{\mathcal{F}_0}(G_{\Gamma_0}) \times D^1 & \longrightarrow & E_{\mathcal{F}}(G_\Gamma)\,. \end{array}$$

By induction on the number of vertices of Γ, the classifying spaces $E_{\mathcal{F}_i}(G_{\Gamma_i})$ have contractible orbit spaces for $i = 0, 1, 2$, and hence so does $E_{\mathcal{F}}(G_\Gamma)$.

If G is the fundamental group of a graph of groups with amenable vertex groups, we consider the family $\mathcal{F}$ generated by the vertex groups. Then the Bass–Serre tree is a 1-dimensional model for $E_{\mathcal{F}}G$. Recall that the comparison map always vanishes in degree 1, since $H^1_b(G;\mathbb{R})$ is trivial for every group G.

We also obtain a characterisation of relative amenability via relatively $\mathcal{F}$-injective modules, analogous to [Fri17, Proposition 4.18] (see also [Mon01, Theorem 5.7.1]).

Proposition 2.26. *Let G be a group and $\mathcal{F}$ be a family of subgroups. The following are equivalent:*

(i) *G is amenable relative to $\mathcal{F}$;*

(ii) *Every dual normed $\mathbb{R}G$-module $V^\#$ is relatively $\mathcal{F}$-injective;*

(iii) *The trivial normed $\mathbb{R}G$-module $\mathbb{R}$ is relatively $\mathcal{F}$-injective.*

Proof. Suppose that G is amenable relative to $\mathcal{F}$ and let $m_\mathcal{F}$ be a G-invariant mean on $G/\mathcal{F}$. The inclusion $V^\# \to C^0_{\mathcal{F},b}(G; V^\#)$ of normed G-modules admits a right inverse r given by

$$r(f)(v) = m_\mathcal{F}\big(gH \mapsto f(gH)(v)\big)$$

for $f \in C^0_{\mathcal{F},b}(G; V^\#)$ and $v \in V$. Then the relative $\mathcal{F}$-injectivity of $V^\#$ follows from the relative $\mathcal{F}$-injectivity of $C^0_{\mathcal{F},b}(G; V^\#)$.

Clearly, condition (ii) implies (iii). Suppose that $\mathbb{R}$ is relatively $\mathcal{F}$-injective. Consider the strongly $\mathcal{F}$-injective map $i\colon \mathbb{R} \to \ell^\infty(G/\mathcal{F})$ of normed G-modules that has an H-section τ_H given by $\tau_H(f) = f(eH)$ for each $H \in \mathcal{F}$. Then the identity $\mathrm{id}_\mathbb{R}$ admits an extension along i which yields a non-trivial G-invariant element in $\ell^\infty(G/\mathcal{F})^\#$. By Lemma 2.21 this finishes the proof. □

Characterisation of relative finiteness. Analogously to Theorem 2.23, when instead considering all (not necessarily dual) normed $\mathbb{R}G$-modules, one obtains the theorem below. Let G be a group and $\mathcal{F}$ be a family of subgroups.

Let $\ell^1(G/\mathcal{F})$ denote the normed $\mathbb{R}G$-module of summable functions $f\colon G/\mathcal{F} \to \mathbb{R}$ with norm $\|f\|_1 = \sum_{gH\in G/\mathcal{F}} |f(gH)|$. Let $\ell^1_0(G/\mathcal{F})$ be the kernel of the map $\ell^1(G/\mathcal{F}) \to \mathbb{R}$, $f \mapsto \sum_{gH\in G/\mathcal{F}} f(gH)$. We define the class $[K_\mathcal{F}] \in H^1_{\mathcal{F},b}(G; \ell^1_0(G/\mathcal{F}))$ as the cohomology class of the 1-cocycle $K_\mathcal{F} \in C^1_{\mathcal{F},b}(G; \ell^1_0(G/\mathcal{F}))$ given by

$$K_\mathcal{F}(g_0H_0, g_1H_1) = \chi_{g_1H_1} - \chi_{g_0H_0}\,,$$

where $\chi_{g_iH_i}$ is the characteristic function supported at g_iH_i for $i = 0, 1$.

We say that G is *finite relative* to $\mathcal{F}$, if $\mathcal{F}$ contains a finite index subgroup of G.

Theorem 2.27. *Let G be a group and $\mathcal{F} \subset \mathcal{G}$ be two families of subgroups. The following are equivalent:*

(i) *Every subgroup $H \in \mathcal{G}$ is finite relative to $\mathcal{F}|_H$;*

(ii) *The canonical map $H^n_{\mathcal{G},b}(G; V) \to H^n_{\mathcal{F},b}(G; V)$ is an isomorphism for all normed $\mathbb{R}G$-modules V and all $n \geq 0$;*

(iii) *The canonical map $H^1_{\mathcal{G},b}(G; V) \to H^1_{\mathcal{F},b}(G; V)$ is an isomorphism for all normed $\mathbb{R}G$-modules V;*

(iv) *The class* $[K_{\mathcal{F}}] \in H^1_{\mathcal{F},b}(G;\ell^1_0(G/\mathcal{F}))$ *lies in the image of the canonical map* $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$.

Proof. We only give a sketch of the proof which is entirely analogous to that of Theorem 2.23. Suppose that every subgroup $H \in \mathcal{G}$ is finite relative to $\mathcal{F}|_H$. One shows that the resolution of normed $\mathbb{R}G$-modules $V \to C^*_{\mathcal{G}}(G;V)$ is relatively $\mathcal{F}$-injective by taking averages over finite sets of cosets. Moreover, the resolution is $\mathcal{F}$-strong by Lemma 2.14 (ii) and hence the canonical map $\mathrm{can}^n_{\mathcal{F}\subset\mathcal{G},b}$ is an isomorphism for all $n \geq 0$ by Proposition 2.15.

The implications (ii) ⇒ (iii) ⇒ (iv) are obvious. Suppose that the class $[K_{\mathcal{F}}]$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$. Fix a subgroup $H \in \mathcal{G}$ and consider the diagram of normed $\mathbb{R}H$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell^1_0(G/\mathcal{F}) & \longrightarrow & \ell^1(G/\mathcal{F}) & \longrightarrow & \mathbb{R} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & \ell^1_0(H/\mathcal{F}|_H) & \longrightarrow & \ell^1(H/\mathcal{F}|_H) & \longrightarrow & \mathbb{R} & \longrightarrow & 0\,. \end{array}$$

Following the proof of Theorem 2.23, one obtains a non-trivial H-invariant element $f \in \ell^1(H/\mathcal{F}|_H)$. Since f is constant on H-orbits, non-trivial, and summable, there exists a finite H-orbit in $H/\mathcal{F}|_H$. Thus H is finite relative to $\mathcal{F}|_H$. □

Theorem 2.27 has the following interesting special cases. If $\mathcal{F}$ is arbitrary and $\mathcal{G} = \mathcal{ALL}$, we characterise that $\mathcal{F}$ contains a finite index subgroup of G. If $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G}$ is arbitrary, we characterise that all subgroups in $\mathcal{G}$ are finite, generalising [MR, Theorem B]. We recover the characterisation of finite groups ([Fri17, Theorem 3.12]) for $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G} = \mathcal{ALL}$.

2.5 Characterisation of relative hyperbolicity

In this section we prove a characterisation of relatively hyperbolic groups in terms of bounded Bredon cohomology analogous to Theorem 2.2.

Let G be a finitely generated group and $\mathcal{H}$ be a finite set of subgroups. Recall that G is *hyperbolic relative to* $\mathcal{H}$ if the coned–off Cayley graph is hyperbolic and fine (see e.g., [Hru10]). For example, hyperbolic groups are hyperbolic relative to the trivial subgroup, free products $G_1 * G_2$ are hyperbolic relative to $\{G_1, G_2\}$, and fundamental groups of finite volume hyperbolic manifolds are hyperbolic relative to the cusp subgroups. If G is hyperbolic relative to $\mathcal{H}$, then $\mathcal{H}$ is almost malnormal and hence malnormal if G is torsion-free.

From now on, let the ring R be either $\mathbb{Q}$ or $\mathbb{R}$. A map $f\colon C \to B$ of normed RG-modules is called *undistorted* if there exists a constant $K \geq 0$ such that for all $b \in \mathrm{im}(f)$ there exists $c \in C$ with $f(c) = b$ such that $\|c\|_C \leq K \cdot \|b\|_B$. A normed RG-module P is called *boundedly projective* if for every undistorted epimorphism $f\colon C \to B$ and every map $\phi\colon P \to B$ of normed RG-modules, there exists a map $\Phi\colon P \to C$ of normed RG-modules such that $f \circ \Phi = \phi$.

The following lemma [MY, Lemma 52] is useful to construct G-equivariant maps.

Lemma 2.28 (Mineyev–Yaman)**.** *Let G be a group and S be a G-set with finite stabilisers. Then $\mathbb{Q}[S]$ is projective as a $\mathbb{Q}G$-module and boundedly projective as a normed $\mathbb{Q}G$-module when equipped with the ℓ^1-norm.*

Let X be a G-CW-complex with cocompact $(n+1)$-skeleton and consider for $k \geq 0$ the cellular chains $C_k^{\mathrm{cell}}(X;R)$ as a normed RG-module equipped with the ℓ^1-norm. We say that X satisfies a *linear homological isoperimetric inequality over R in degree n* if the boundary map

$$\partial_{n+1}\colon C_{n+1}^{\mathrm{cell}}(X;R) \to C_n^{\mathrm{cell}}(X;R)$$

is undistorted. Equivalently, there exists a constant $K \geq 0$ such that for every cellular n-boundary $b \in B_n^{\mathrm{cell}}(X;R)$ we have $\|b\|_\partial \leq K \cdot \|b\|_1$, where

$$\|b\|_\partial \coloneqq \inf\{\|c\|_1 \mid c \in C_{n+1}^{\mathrm{cell}}(X;R),\, \partial_{n+1}(c) = b\}$$

(which is sometimes called the *filling norm*). (In [MM85], the terminology of the *uniform boundary condition* is used for a linear homological isoperimetric inequality.)

If G is hyperbolic relative to $\mathcal{H}$, Mineyev–Yaman [MY, Theorem 41] have constructed the so-called "ideal complex" X. It is in particular a cocompact G-CW-complex with precisely one equivariant 0-cell G/H for each each $H \in \mathcal{H}$ and finite edge-stabilisers. Moreover, X is (non-equivariantly) contractible and hence a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ provided that G is torsion-free. We summarise some of its properties [MY, Theorem 47 and 51].

Theorem 2.29 (Mineyev–Yaman)**.** *Let G be a finitely generated torsion-free group and $\mathcal{H}$ be a finite set of subgroups. If G is hyperbolic relative to $\mathcal{H}$, then there exists a cocompact model X for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ satisfying the following:*

(i) *X satisfies linear homological isoperimetric inequalities over $\mathbb{Q}$ in degree n for all $n \geq 1$;*

(ii) *There exists a map $q\colon X^{(0)} \times X^{(0)} \to C_1^{\mathrm{cell}}(X;\mathbb{Q})$ with $\partial_1(q(a,b)) = b - a$, called a homological $\mathbb{Q}$-bicombing, that is G-equivariant and satisfies*

$$\|q(a,b) + q(b,c) - q(a,c)\|_1 \leq K$$

for all $a, b, c \in X^{(0)}$ and a uniform constant $K \geq 0$.

The following criterion for relative hyperbolicity is a combination of [Fra18, Proposition 8.3 and Theorem 8.5] (see also [MP16, Theorems 1.6 and 1.10]).

Theorem 2.30 (Franceschini, Martínez-Pedroza)**.** *Let G be a group and $\mathcal{H}$ be a finite set of subgroups. Then G is hyperbolic relative to $\mathcal{H}$ if there exists a G-CW-complex Z satisfying the following:*

(i) *Z is simply-connected;*

(ii) *The 2-skeleton $Z^{(2)}$ is cocompact;*

(iii) *$\mathcal{H}$ is a set of representatives of distinct conjugacy classes of vertex-stabilisers such that each infinite stabiliser is represented;*

(iv) *The edge-stabilisers of Z are finite;*

(v) *Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1.*

We prove a characterisation of relative hyperbolicity closely following Mineyev's original proof of Theorem 2.2 ([Min01, Theorem 11] and [Min02, Theorem 9]).

Theorem 2.31. *Let G be a finitely generated torsion-free group and $\mathcal{H}$ be a finite malnormal collection of subgroups. Let $\mathcal{F}$ be the family $\mathcal{F}\langle\mathcal{H}\rangle$ and suppose that G is of type $F_{2,\mathcal{F}}$. Then the following are equivalent:*

(i) *G is hyperbolic relative to $\mathcal{H}$;*

(ii) *The comparison map $H^n_{\mathcal{F},b}(G;V) \to H^n_{\mathcal{F}}(G;V)$ is surjective for all normed $\mathbb{Q}G$-modules V and all $n \geq 2$;*

(iii) *The comparison map $H^2_{\mathcal{F},b}(G;V) \to H^2_{\mathcal{F}}(G;V)$ is surjective for all normed $\mathbb{R}G$-modules V.*

Proof. Suppose that G is hyperbolic relative to $\mathcal{H}$. Let X be the model for $E_{\mathcal{F}}G$ that is given by Mineyev–Yaman's ideal complex (Theorem 2.29) and Y be the simplicial model for $E_{\mathcal{F}}G$ with (non-equivariant) n-cells corresponding to $(G/\mathcal{F})^{n+1}$ for all $n \geq 0$. We claim that there is a G-chain map

$$\varphi_*\colon C^{\mathrm{cell}}_*(Y;\mathbb{Q}) \to C^{\mathrm{cell}}_*(X;\mathbb{Q})$$

with φ_n bounded for all $n \geq 2$, admitting a G-homotopy left inverse. We construct φ_* inductively as follows. In degree 0, we define

$$\varphi_0\colon C^{\mathrm{cell}}_0(Y;\mathbb{Q}) = \mathbb{Q}[G/\mathcal{F}] \to C^{\mathrm{cell}}_0(X;\mathbb{Q})$$

to map a generator of the form eH to the vertex of X with stabiliser containing H. Then extend G-equivariantly and $\mathbb{Q}$-linearly to all of $\mathbb{Q}[G/\mathcal{F}]$. In degree 1, we define $\varphi_1\colon C_1^{\mathrm{cell}}(Y;\mathbb{Q}) \to C_1^{\mathrm{cell}}(X;\mathbb{Q})$ on generators by

$$\varphi_1(g_0H_0, g_1H_1) = q(\varphi_0(g_0H_0), \varphi_0(g_1H_1))\,,$$

where q is the homological $\mathbb{Q}$-bicombing on X from Theorem 2.29 (ii). Since both q and φ_0 are G-equivariant, so is φ_1. In degree 2, we consider the maps

$$\begin{array}{ccc} C_2^{\mathrm{cell}}(Y;\mathbb{Q}) & \xrightarrow{\partial_2^Y} & C_1^{\mathrm{cell}}(Y;\mathbb{Q}) \\ & & \downarrow{\scriptstyle\varphi_1} \\ C_2^{\mathrm{cell}}(X;\mathbb{Q}) & \xrightarrow{\partial_2^X} & C_1^{\mathrm{cell}}(X;\mathbb{Q}) \end{array} \tag{2.6}$$

and observe that the composition $\varphi_1 \circ \partial_2^Y$ is bounded by properties of q and that ∂_2^X is undistorted by Theorem 2.29 (i). There is a G-invariant decomposition

$$C_2^{\mathrm{cell}}(Y;\mathbb{Q}) \cong \mathbb{Q}[S_1] \oplus \mathbb{Q}[S_2]\,,$$

where S_1 and S_2 denote the sets of 2-cells of Y with trivial resp. non-trivial stabilisers. We obtain a bounded G-map $\varphi_2\colon C_2^{\mathrm{cell}}(Y;\mathbb{Q}) \to C_2^{\mathrm{cell}}(X;\mathbb{Q})$ by using the bounded projectivity of $\mathbb{Q}[S_1]$ (Lemma 2.28) and by setting φ_2 to be zero on $\mathbb{Q}[S_2]$. This renders the square (2.6) commutative because the edge-stabilisers of X are trivial.

Assuming that φ_n has been constructed, one analogously defines a bounded G-map φ_{n+1} using that ∂_{n+1}^X is undistorted by Theorem 2.29 (i). Thus one obtains a G-chain map φ_* with φ_n bounded for $n \geq 2$. To conclude the claim, we note that $C_*^{\mathrm{cell}}(Y;\mathbb{Q})$ is a relatively $\mathcal{F}$-projective $\mathcal{F}$-strong resolution of $\mathbb{Q}$ by Lemma 2.12. Hence by Proposition 2.13 any G-chain map $\psi_*\colon C_*^{\mathrm{cell}}(X;\mathbb{Q}) \to C_*^{\mathrm{cell}}(Y;\mathbb{Q})$ extending $\mathrm{id}_{\mathbb{Q}}$ is a G-homotopy left inverse of φ_*.

Now, let V be a normed $\mathbb{Q}G$-module. Applying $\mathrm{Hom}_{\mathbb{Q}G}(-,V)$ yields a cochain map

$$\varphi^*\colon C^*_{\mathrm{cell}}(X;V)^G \to C^*_{\mathrm{cell}}(Y;V)^G$$

with homotopy right inverse ψ^*. In particular, the composition $\varphi^* \circ \psi^*$ induces the identity on $H^*(C^*_{\mathrm{cell}}(Y;V)^G) \cong H^*_{\mathcal{F}}(G;V)$. Finally, for $n \geq 2$ let $c \in C^n_{\mathrm{cell}}(Y;V)^G$ be a cocycle. Then $\varphi^n(\psi^n(c))$ and c represent the same cohomology class in $H^n_{\mathcal{F}}(G;V)$. We have

$$\|\varphi^n(\psi^n(c))\|_\infty = \|\psi^n(c) \circ \varphi_n\|_\infty \leq \|\psi^n(c)\|_\infty \cdot \|\varphi_n\|_\infty\,,$$

where φ_n is bounded by construction and so is $\psi^n(c) \in C^n_{\mathrm{cell}}(X;V)^G$ because X has only finitely many orbits of n-cells. Thus we have shown that for $n \geq 2$ every cohomology class in $H^n_{\mathcal{F}}(G;V)$ admits a bounded representative.

Obviously condition (ii) implies (iii). Suppose that the comparison map is surjective in degree 2 for coefficients in every normed $\mathbb{R}G$-module. Let Z be a model for $E_{\mathcal{F}}G$ with cocompact 2-skeleton. Since $\mathcal{H}$ is malnormal, by collapsing fixed-point sets of Z we may assume that for every non-trivial subgroup $H \in \mathcal{F}$ the fixed-point set Z^H consists of precisely one point.[1] In other words, Z has one equivariant 0-cell of the form G/H for each $H \in \mathcal{H}$ and all other cells have trivial stabilisers. In order to apply Theorem 2.30 and conclude that G is hyperbolic relative to $\mathcal{H}$, it remains to verify that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1.

We take as coefficients the cellular 1-boundaries $V := B_1^{\mathrm{cell}}(Z;\mathbb{R})$ equipped with the norm $\|\cdot\|_\partial$. Let Y be the simplicial model for $E_{\mathcal{F}}G$. Then there is a G-chain homotopy equivalence

$$\psi_*\colon C_*^{\mathrm{cell}}(Z;\mathbb{R}) \to C_*^{\mathrm{cell}}(Y;\mathbb{R})$$

with G-homotopy inverse φ_*. Applying $\mathrm{Hom}_{\mathbb{R}G}(-,V)$ yields a cochain homotopy equivalence

$$\psi^*\colon C^*_{\mathrm{cell}}(Y;V)^G \to C^*_{\mathrm{cell}}(Z;V)^G$$

with homotopy inverse φ^*. In particular, the composition $\psi^* \circ \varphi^*$ induces the identity on $H^*(C^*_{\mathrm{cell}}(Z;V)^G) \cong H^*_{\mathcal{F}}(G;V)$. Consider the 2-cocycle $u \in C^2_{\mathrm{cell}}(Z;V)^G$ given by the boundary map

$$u = \partial_2\colon C_2^{\mathrm{cell}}(Z;\mathbb{R}) \to B_1^{\mathrm{cell}}(Z;\mathbb{R}) = V\,.$$

Then we can write

$$u = (\psi^2 \circ \varphi^2)(u) + \delta_Z^1(v) \tag{2.7}$$

for some $v \in C^1_{\mathrm{cell}}(Z;V)^G$. Since the comparison map $H^2_{\mathcal{F},b}(G;V) \to H^2_{\mathcal{F}}(G;V)$ is surjective by hypothesis, we can write

$$\varphi^2(u) = u' + \delta_Y^1(v') \tag{2.8}$$

for a *bounded* 2-cocycle $u' \in C^2_{\mathrm{cell}}(Y;V)^G$ and some $v' \in C^1_{\mathrm{cell}}(Y;V)^G$. For a fixed vertex $y \in Y^{(0)} = G/\mathcal{F}$, let $\mathrm{Cone}_y\colon C_1^{\mathrm{cell}}(Y;\mathbb{R}) \to C_2^{\mathrm{cell}}(Y;\mathbb{R})$ be defined on generators by

$$\mathrm{Cone}_y\left((g_0H_0, g_1H_1)\right) = (y, g_0H_0, g_1, H_1)\,.$$

Obviously Cone_y preserves the ℓ^1-norms. For a G-CW-complex W, we denote the evaluation pairing by

$$\langle\cdot,\cdot\rangle_W\colon C^*_{\mathrm{cell}}(W;V)^G \times C_*^{\mathrm{cell}}(W;\mathbb{R}) \to V\,.$$

[1] A detailed proof of this fact was communicated to us by Sam Hughes.

Now, for $b \in C_1^{\mathrm{cell}}(Z;\mathbb{R})$ and $c \in C_2^{\mathrm{cell}}(Z;\mathbb{R})$ with $\partial_2(c) = b$, we find by (2.7) that

$$\begin{aligned} b = \langle u, c\rangle_Z &= \langle (\psi^2 \circ \varphi^2)(u) + \delta_Z^1(v), c\rangle_Z \\ &= \langle (\psi^2 \circ \varphi^2)(u), c\rangle_Z + \langle v, \partial_2^Y(c)\rangle_Z \\ &= \langle \varphi^2(u), \psi_2(c)\rangle_Y + \langle v, b\rangle_Z\,. \end{aligned}$$

Since $\varphi^2(u)$ is a cocycle and $\psi_2(c) - \mathrm{Cone}_y(\partial_2^Y(\psi_2(c)))$ is a cycle and hence a boundary, we have

$$\begin{aligned} \langle \varphi^2(u), \psi_2(c)\rangle_Y &= \langle \varphi^2(u), \mathrm{Cone}_y(\partial_2^Y(\psi_2(c)))\rangle_Y \\ &= \langle \varphi^2(u), \mathrm{Cone}_y(\psi_1(b))\rangle_Y \\ &= \langle u' + \delta_Y^1(v'), \mathrm{Cone}_y(\psi_1(b))\rangle_Y \\ &= \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle v', \partial_2^Y(\mathrm{Cone}_y(\psi_1(b)))\rangle_Y \\ &= \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle v', \psi_1(b)\rangle_Y \\ &= \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle \psi^1(v'), b\rangle_Z\,, \end{aligned}$$

where we used (2.8). Together, we have

$$b = \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle \psi^1(v') + v, b\rangle_Z\,.$$

We claim that $\|\langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y\|_\partial \le \|u'\|_\infty \cdot \|\mathrm{Cone}_y(\psi_1(b))\|_1$. Indeed, consider the map induced by ∂_2 on coefficients

$$(\partial_2)_*\colon C^2_{\mathrm{cell}}(Y; C_2^{\mathrm{cell}}(Z;\mathbb{R}))^G \to C^2_{\mathrm{cell}}(Y;V)^G\,.$$

Since ∂_2 is surjective, there exists a preimage $\widetilde{u'} \in C^2_{\mathrm{cell}}(Y; C_2^{\mathrm{cell}}(Z;\mathbb{R}))^G$ of u' under $(\partial_2)_*$ with $\|\widetilde{u'}\|_\infty \le \|u'\|_\infty$. Then $\langle \widetilde{u'}, \mathrm{Cone}_y(\psi_1(b))\rangle_Y \in C_2^{\mathrm{cell}}(Z;\mathbb{R})$ is a preimage of $\langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y$ under ∂_2 witnessing the desired inequality. Similarly, one shows that $\|\langle \psi^1(v') + v, b\rangle_Z\|_\partial \le \|\psi^1(v') + v\|_\infty \cdot \|b\|_1$. It follows that

$$\begin{aligned} \|b\|_\partial &\le \|\langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y\|_\partial + \|\langle \psi^1(v') + v, b\rangle_Z\|_\partial \\ &\le \|u'\|_\infty \cdot \|\mathrm{Cone}_y(\psi_1(b))\|_1 + \|\psi^1(v') + v\|_\infty \cdot \|b\|_1 \\ &= \|u'\|_\infty \cdot \|\psi_1(b)\|_1 + \|\psi^1(v') + v\|_\infty \cdot \|b\|_1 \\ &\le (\|u'\|_\infty \cdot \|\psi_1\|_\infty + \|\psi^1(v') + v\|_\infty) \cdot \|b\|_1\,. \end{aligned}$$

Finally, u' is bounded by construction and so are ψ_1 and $\psi^1(v') + v$ because they are G-maps with domain $C_1^{\mathrm{cell}}(Z;\mathbb{R})$ and Z has only finitely many orbits of 1-cells. Thus we have shown that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1. This finishes the proof. □

Remark 2.32 (Groups with torsion). In Theorem 2.31, if the group G is not assumed to be torsion-free and $\mathcal{H}$ is instead assumed to be almost malnormal, one can still prove the equivalence of (i) and (iii). However, a few modifications are necessary which we shall only outline.

Assuming that G is hyperbolic relative to $\mathcal{H}$, Mineyev–Yaman's ideal complex has to be replaced by a Rips type construction X due to Martínez-Pedroza–Przytycki that is a model for $E_{\mathcal{F}\cup\mathcal{FIN}}G$. This complex X satisfies a linear homological isoperimetric inequality over $\mathbb{Z}$ in degree 1 ([MPP19, Corollary 1.5]). It is part of a hyperbolic tuple in the sense of [MY, Definition 38] and hence admits a homological $\mathbb{Q}$-bicombing by [MY, Theorem 47]. Then one can construct a G-chain map φ_* with φ_2 bounded similarly as before and conclude surjectivity of the comparison map in degree 2 for the family $\mathcal{F}\cup\mathcal{FIN}$. This implies the same for the family $\mathcal{F}$ over the ring $\mathbb{R}$.

For the converse implication, since $\mathcal{H}$ is almost malnormal, there exists a model Z for $E_{\mathcal{F}}G$ such that for every infinite subgroup $H \in \mathcal{F}$ the fixed-point set Z^H consists of precisely one point. Then one shows as before that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1 and concludes by Theorem 2.30.

We do not know whether condition (ii) is equivalent to (i) and (iii) in this case.

Chapter 3

Bounded acyclicity and relative simplicial volume

This chapter is the article [LLM] which is joint with Clara Löh and Marco Moraschini.

Abstract. We provide new vanishing and glueing results for relative simplicial volume, following up on two current themes in bounded cohomology: The passage from amenable groups to boundedly acyclic groups and the use of equivariant topology.

More precisely, we consider equivariant nerve pairs and relative classifying spaces for families of subgroups. Typically, we apply this to uniformly boundedly acyclic families of subgroups. Our methods also lead to vanishing results for ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable category and to a relative version of a result by Dranishnikov and Rudyak concerning mapping degrees and the inheritance of freeness of fundamental groups.

3.1 Introduction

Bounded cohomology is defined as the cohomology of the bounded dual of the singular or bar chain complex [Gro82] and it has many applications in group theory and geometry of manifolds. A fundamental phenomenon is that bounded cohomology of amenable groups is trivial (i.e., amenable groups are boundedly acyclic). On the other hand, the bounded cohomology of negatively curved groups surjects onto ordinary cohomology. In manifold topology, the simplicial volume of an oriented compact manifold is a homotopy invariant defined as the ℓ^1-seminorm of the $\mathbb{R}$-fundamental class [Gro82]. Using a duality argument, the simplicial volume can be expressed in terms of bounded cohomology.

We provide new vanishing results for relative simplicial volume, following up on two current themes in bounded cohomology:

- The passage from amenable groups to boundedly acyclic groups;

- The use of equivariant topology, most notably of classifying spaces for families of subgroups.

A technical difficulty in the passage from amenable to boundedly acyclic groups is that the class of amenable groups possesses a large degree of uniformity when it comes to bounded cohomology. This includes the fact that the class of amenable groups is closed under subgroups and quotients and the fact that amenable groups are not only boundedly acyclic, but uniformly boundedly acyclic. Therefore, in the setting of boundedly acyclic groups, generalised vanishing results for simplicial volume come with additional uniformity and closure hypotheses.

As we aim at results for relative bounded cohomology and relative simplicial volume, we adapt tools from equivariant topology to this relative setting.

Uniform bounded acyclicity. Group actions with amenable stabilisers have proved to be a valuable tool to compute bounded cohomology [Mon01, BM02, BI09]. Similarly, also *uniformly boundedly acyclic actions* allow us to compute bounded cohomology, where the uniformity refers to a uniform bound on the norms of primitives. Recently, uniformly boundedly acyclic actions have been used to compute the bounded cohomology of geometrically relevant groups [FFLMa, MN].

Let X be a path-connected space. We say that a set of path-connected subspaces $\mathcal{A}$ of X is *uniformly boundedly acyclic* [*of order* n] *in* X if the collection of all finite [resp. n-fold] intersections of conjugates of the subgroups

$$\bigl(\operatorname{im}(\pi_1(A \hookrightarrow X))\bigr)_{A \in \mathcal{A}}$$

in $\pi_1(X)$ is uniformly boundedly acyclic (Definition 3.58). In the special case when the above groups are amenable, we also speak of an *amenable* set of subspaces. The issue of basepoints is addressed in Section 3.1. We have two geometric situations in which uniformly boundedly acyclic sets of subspaces lead to interesting uniformly boundedly acyclic actions: Open covers and glueing loci of manifolds obtained by glueing manifolds with boundary.

Vanishing via relative open covers. Gromov [Gro82] and Ivanov [Iva85] established vanishing results for the comparison map (and thus for simplicial volume) in the presence of amenable open covers with small multiplicity.

Following the approach by Löh and Sauer [LS20] through equivariant nerves and classifying spaces for families, we generalise these vanishing results in two directions. First, we allow more general covers: A cover $\mathcal{U}$ of X by path-connected open subsets is *uniformly boundedly acyclic* if the underlying set of subsets of X is uniformly boundedly acyclic in X. Second, we adapt the setting to pairs of CW-complexes (X, A), where A is π_1-injective in X (Theorem 3.76). To this end, we introduce the notion of [*weakly convex*] *relative covers* (Definition 3.41). Using equivariant nerve pairs and classifying spaces of group pairs for families, we obtain:

Theorem 3.1 (Corollary 3.78)**.** *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A has only finitely many connected components, each of which is π_1-injective in X. Let $\mathcal{U}$ be a relative cover of (X, A) that is uniformly boundedly acyclic.*

(i) *If $\mathcal{U}$ is weakly convex, then the comparison map*

$$\operatorname{comp}^k \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \operatorname{mult}_A(\mathcal{U})$.

(ii) *Let $\nu \colon (X, A) \to (|N(\mathcal{U})|, |N_A(\mathcal{U})|)$ be a nerve map. If $\mathcal{U}$ is convex, then the comparison map* comp^* *factors through ν:*

$$\begin{array}{ccc} H_b^*(X, A; \mathbb{R}) & \xrightarrow{\operatorname{comp}^*} & H^*(X, A; \mathbb{R}) \\ & \searrow & \uparrow {\scriptstyle H^*(\nu;\mathbb{R})} \\ & & H^*(|N(\mathcal{U})|, |N_A(\mathcal{U})|; \mathbb{R}) \, . \end{array}$$

Here $\operatorname{mult}_A(\mathcal{U})$ denotes the relative multiplicity of $\mathcal{U}$ with respect to A (Definition 3.35) and the simplicial complex $N_A(\mathcal{U})$ is a suitable subcomplex of the nerve $N(\mathcal{U})$ (Definition 3.36).

In the absolute case, Ivanov proved a similar vanishing theorem for *weakly boundedly acyclic covers* using spectral sequences [Iva]. Our notion of uniformly boundedly acyclic covers is similar, but the relation between the two is unclear (Remark 3.69).

Theorem 3.1 applies in particular to relative covers that are amenable. We introduce the *relative amenable multiplicity* $\operatorname{mult}_{\mathcal{AME}}(X, A)$ (Definition 3.45) as the minimal relative multiplicity of weakly convex relative amenable covers of (X, A) by path-connected open subsets.

Theorem 3.2 (Corollary 3.80)**.** *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A consists of finitely many connected components, each of which is π_1-injective in X. Then the comparison map*

$$\operatorname{comp}^k \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \operatorname{mult}_{\mathcal{AME}}(X, A)$.

In particular, if $(M, \partial M)$ is an oriented compact connected triangulable manifold with π_1-injective boundary components and $\operatorname{mult}_{\mathcal{AME}}(M, \partial M) \leq \dim(M)$, then the relative simplicial volume $\|M, \partial M\|$ vanishes.

In the absolute case, every cover is a weakly convex relative cover, hence $\mathrm{mult}_{\mathcal{AME}}(X, \emptyset)$ is the minimal multiplicity of amenable covers of X. For a CW-complex X, this coincides with the minimal *cardinality* of amenable covers of X by not necessarily path-connected subsets [CLM, Remark 3.13]. The latter quantity is called the *amenable category* $\mathrm{cat}_{\mathcal{AME}}(X)$ (Remark 3.47), a notion that is modelled on the classical LS-category [CLOT03].

As an application of Theorem 3.2, we give an alternative proof of a relative vanishing theorem, which is a consequence of Gromov's vanishing theorem for non-compact manifolds (Theorem 3.82).

Our methods for equivariant nerve pairs and relative classifying spaces also lead to vanishing results for ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable multiplicity (Theorem 3.86). In the absolute case (Corollary 3.87), this recovers a result by Sauer [Sau09, Theorem C].

Glueings. We adapt the additivity of relative simplicial volume for glueings along amenable boundaries [Gro82, BBF$^+$14, Kue15] to situations with boundedly acyclic boundaries. As we move away from amenability, we lose control on the norm, and thus only retain control on the vanishing behaviour.

Theorem 3.3 (Theorem 3.88)**.** *Let $n \geq 3$ and $(M_i, \partial M_i)_{i \in I}$ be a finite collection of oriented compact connected n-manifolds. Assume that every connected component of every boundary component ∂M_i has boundedly acyclic fundamental group. Let $\mathcal{N}$ be a set of π_1-injective boundary components of the $(M_i)_{i\in I}$ and let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i\in I}$ by a pairwise glueing of the boundary components in $\mathcal{N}$.*

If $\mathcal{N}$, viewed as a set of subsets of M, is uniformly boundedly acyclic of order n in M, then the following are equivalent:

(i) *We have $\|M, \partial M\| = 0$;*

(ii) *For all $i \in I$, we have $\|M_i, \partial M_i\| = 0$.*

Mapping degrees. One of the classical applications of simplicial volume is an a priori estimate on mapping degrees [Gro82, Thu, FM21]. In contrast, the exact relation between mapping degrees and monotonicity of (generalised) LS-category invariants is still wide open [Rud17, CLM].

In the absolute case, Eilenberg and Ganea showed that the LS-category invariant for the family containing only the trivial subgroup of an aspherical space recovers the cohomological dimension of its fundamental group [EG57]. Moreover, cohomological dimension one can be characterised in terms of freeness. Thus, the monotonicity problem for (generalised) LS-category leads to inheritance properties of fundamental groups under maps of non-zero degree.

We use equivariant and group cohomological methods to establish the following relative version (and a simplified proof) of a monotonicity result by Dranishnikov and Rudyak for closed manifolds [DR09]:

Theorem 3.4 (Corollary 3.29)**.** *Let $f\colon (M, \partial M) \to (N, \partial N)$ be a map between oriented compact connected manifolds of the same dimension with π_1-injective boundary components. Let $\partial M = \coprod_{i=1}^m M_i$ and $\partial N = \coprod_{i=1}^n N_i$ be decompositions into connected components. If $\deg(f) = \pm 1$ and there exists a free group F_M such that*

$$\pi_1(M) \cong F_M * *_{i=1}^m \pi_1(M_i)\,,$$

then there exists a free group F_N such that $\pi_1(N) \cong F_N * *_{i=1}^n \pi_1(N_i)$.

For closed manifolds our approach also yields inheritance properties for virtual freeness:

Theorem 3.5 (Corollary 3.31)**.** *Let $f\colon M \to N$ be a map between oriented closed connected manifolds of the same dimension. If $\deg(f) \neq 0$ and $\pi_1(M)$ is virtually free, then also $\pi_1(N)$ is virtually free.*

Conventions. In this article, we adhere to the following conventions:

Instead of the usual notion of families of sets, groups, modules, ..., we will speak of *collections*; this is to avoid confusion with the term "families of subgroups". I.e., a collection $(H_i)_{i\in I}$ of groups [or sets, ...] is a map $I \to \text{Group}$, $i \mapsto H_i$ from a set I to the class of all groups [or sets, ...]. In particular, collections can contain repetitions.

Families of subgroups will only be closed under conjugation but not necessarily under finite intersections or taking subgroups (Definition 3.9).

All groups will be discrete groups; in particular, we consider bounded cohomology of discrete groups and G-CW-complexes for discrete groups G. The geometric realisation of G-simplicial complexes will always be equipped with the G-CW-structure coming from the barycentric subdivision (Example 3.7).

Given a compact manifold M with non-empty boundary, we say that M has π_1-*injective boundary* if every connected component of ∂M is π_1-injective in M.

We usually refrain from spelling out basepoints for fundamental groups. Strictly speaking, fixing basepoints is necessary to make the notion of the image of "the" fundamental group of a subspace in "the" fundamental group of an ambient space precise. However, we will always deal with situations concerning conjugation-invariant properties or concerning collections of all conjugates of such subgroups. Therefore, all choices of basepoints would lead to the same outcome.

We always work with open covers consisting of path-connected sets. We explain in Remark 3.47 why this condition is not restrictive in our setting.

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3.2 Classifying spaces of group pairs for families of subgroups

The goal of this section is to introduce classifying spaces for families of subgroups in a relative setting. We first recall classifying spaces for families of subgroups and then explain the extension to the relative setting.

This is motivated by our geometric situations of topological pairs (X, A) (e.g., manifolds with boundary), where two classes of subgroups of the fundamental group $G \coloneqq \pi_1(X)$ will be involved:

- A family $\mathcal{F}$ of subgroups of G, describing the allowed fundamental groups of sets in open covers of X;
- A collection $\mathcal{H}$ of subgroups of G, coming from the fundamental groups of the components of A.

3.2.1 G-CW-complexes

We briefly recall the definitions of G-CW-complexes and the induction functor. For more background on G-CW-complexes we refer the reader to the literature [Lüc89, Lüc05].

Definition 3.6 (G-CW-complex)**.** A *G-CW-complex* Y is a G-space equipped with a G-invariant filtration

$$\emptyset = Y^{(-1)} \subset Y^{(0)} \subset Y^{(1)} \subset \cdots \subset Y^{(n)} \subset \cdots \subset Y \tag{3.1}$$

such that the following hold:

- $Y = \bigcup_{n\geq 0} Y^{(n)}$;
- Y carries the weak topology with respect to the filtration (3.1);
- $Y^{(n)}$ is obtained from $Y^{(n-1)}$ as a G-pushout of the form

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \longrightarrow & Y^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \longrightarrow & Y^{(n)}\,. \end{array}$$

The subgroups H_i of G and their conjugates are called *isotropy groups* of Y. If all isotropy groups of Y are trivial, we also say that Y is a *free* G-CW-complex.

A morphism of G-CW-complexes is a G-map.

For example, the universal covering $\widetilde{X}$ of a path-connected CW-complex X is a free $\pi_1(X)$-CW-complex with respect to the CW-structure inherited from X.

Example 3.7 (Barycentric subdivision of G-simplicial complexes)**.** Let N be an (abstract) simplicial complex and let G be a group acting on N via simplicial automorphisms. Then the geometric realisation $|N'|$ of the barycentric subdivision is a G-CW-complex, while $|N|$ need not be a G-CW-complex in general. The standard homeomorphism between the geometric realisations $|N| \to |N'|$ is a G-homeomorphism. Therefore, $|N|$ admits a canonical structure as a G-CW-complex and we will always use this G-CW-structure.

Given a subgroup $H \subset G$, there is a natural way to associate to an H-CW-complex a G-CW-complex.

Definition 3.8 (Induction)**.** Let H be a subgroup of G. The *induction (along the inclusion $H \subset G$)* is the functor

$$G \times_H (-)\colon H\text{-CW-complexes} \to G\text{-CW-complexes}$$

that assigns to an H-CW-complex B the G-CW-complex $G \times_H B$, that is the quotient of $G \times B$ by the (right) H-action $(g, b) \cdot h = (gh, h^{-1}b)$. Here G acts on $G \times_H B$ by left multiplication. We denote elements of $G \times_H B$ by $[g, b]$.

For an H-map $f\colon B \to C$ between H-CW-complexes, the induced G-map

$$G \times_H f\colon G \times_H B \to G \times_H C$$

is given by $G \times_H f([g, b]) = [g, f(b)]$.

The induction functor is left-adjoint to the restriction functor, which associates to a G-CW-complex the same space viewed as an H-CW-complex.

3.2.2 Classifying spaces for families of subgroups

We use the following (non-standard) convention for families of subgroups:

Definition 3.9 (Family of subgroups)**.** Let G be a group and $\mathcal{F}$ be a non-empty set of subgroups of G. We say that $\mathcal{F}$ is a *family of subgroups* (or *conjugation-closed family of subgroups*) of G if it is closed under conjugation. We say that $\mathcal{F}$ is an *intersection-closed family of subgroups* of G if it is closed under conjugation and taking finite intersections.

Example 3.10. The following are examples of families of subgroups:

(i) The set of isotropy groups of a G-CW-complex;

(ii) The family $\mathcal{TR}$ consisting only of the trivial subgroup;

(iii) The family $\mathcal{FIN}$ consisting of all finite subgroups;

(iv) The family $\mathcal{AME}$ consisting of all amenable subgroups;

(v) Let H be a subgroup of G and let $\mathcal{F}$ be a family of subgroups of G. Then the set $\mathcal{F}|_H = \{L \subset H \mid L \in \mathcal{F}\}$ is a family of subgroups of H.

Example 3.11 (Families generated by a set of subgroups)**.** Let G be a group and let $\mathcal{G}$ be a non-empty set of subgroups. The *intersection-closed family* $\mathcal{F}\langle\mathcal{G}\rangle$ *generated by* $\mathcal{G}$ is defined to be the smallest (with respect to inclusion) intersection-closed family containing $\mathcal{G}$, that is

$$\mathcal{F}\langle\mathcal{G}\rangle = \left\{ \bigcap_{i=1}^{n} g_i H_i g_i^{-1} \;\middle|\; n \in \mathbb{N}, H_i \in \mathcal{G}, g_i \in G \right\}.$$

For $n \in \mathbb{N}$ we define the (conjugation-closed) family

$$\mathcal{F}_n\langle\mathcal{G}\rangle := \left\{ \bigcap_{i=1}^{n} g_i H_i g_i^{-1} \;\middle|\; H_i \in \mathcal{G}, g_i \in G \right\}.$$

Recall that EG, the universal covering of an Eilenberg–MacLane space BG, is a terminal object in the G-homotopy category of free G-CW-complexes. The following is a generalisation to G-CW-complexes with not necessarily trivial isotropy groups.

Definition 3.12 (Classifying space for a family of subgroups)**.** Let G be a group and let $\mathcal{F}$ be a (conjugation-closed) family of subgroups of G. A *classifying space for G with respect to $\mathcal{F}$* is a G-CW-complex E with the following universal property:

- All isotropy groups of E lie in $\mathcal{F}$;
- For each G-CW-complex Y whose isotropy groups all lie in $\mathcal{F}$, there is up to G-homotopy exactly one G-map $Y \to E$.

We usually denote such classifying spaces by $E_{\mathcal{F}}G$ (even though they are only unique up to G-homotopy equivalence).

If $\mathcal{F}$ contains the trivial group, then the universal property of $E_{\mathcal{F}}G$ ensures that there exists a G-map $EG \to E_{\mathcal{F}}G$, which is unique up to G-homotopy.

Theorem 3.13 (Existence of classifying spaces for families of subgroups)**.** *Let G be a group and let $\mathcal{F}$ be a (conjugation-closed) family of subgroups.*

(i) *A G-CW-complex Y is a classifying space for G with respect to $\mathcal{F}$ if and only if the following conditions are satisfied:*
  - *All isotropy groups of Y lie in $\mathcal{F}$;*
  - *For all $H \in \mathcal{F}$, the fixed point set Y^H is contractible.*

(ii) *There exists a classifying space for G with respect to $\mathcal{F}$.*

Proof. (i) This follows from an equivariant version of the Whitehead theorem [Lüc05, Theorem 1.6] applied to the map $Y \to G/G$.

(ii) In view of the first part, such a classifying space can be constructed by inductively attaching cells to kill homotopy groups of the fixed point sets [Lüc89, Proposition 2.3]. □

Remark 3.14. We point out that the construction of classifying spaces in Theorem 3.13 (ii) indeed works for (conjugation-closed) families of subgroups with no additional closure properties. This level of generality is usually not considered in the literature, where classifying spaces are often defined only for families that are intersection-closed or closed under taking arbitrary subgroups.

Many interesting constructions of classifying spaces for intersection-closed families, most notably for $\mathcal{FIN}$, are known [Lüc05, Section 4]. For a (conjugation-closed) family $\mathcal{G}$ of subgroups of G, we can consider the intersection-closed family $\mathcal{F}\langle\mathcal{G}\rangle$ generated by $\mathcal{G}$ (Example 3.11). Then a model for $E_{\mathcal{G}}G$ is given by the G-CW-subcomplex of $E_{\mathcal{F}\langle\mathcal{G}\rangle}G$ consisting of all cells with isotropy in $\mathcal{G}$.

Example 3.15. Let $D_\infty = \langle s, t \mid s^2 = t^2 = e\rangle$ be the infinite dihedral group. A model for $E_{\mathcal{FIN}}D_\infty$ is given by the real line $\mathbb{R}$ on which s and t act via reflection at 0 and 1, respectively. Considering the (conjugation-closed) family $\mathcal{FIN} \setminus \mathcal{TR}$, a model for $E_{\mathcal{FIN}\setminus\mathcal{TR}}D_\infty$ is given by the subcomplex $\mathbb{Z} \subset \mathbb{R}$, that is $D_\infty/\langle s\rangle \sqcup D_\infty/\langle t\rangle$.

3.2.3 $(G, \mathcal{H})$-CW-pairs

In this section we introduce a notion of pairs of equivariant CW-complexes adapted to a collection of subgroups.

Definition 3.16 (Group pair)**.** A *group pair* is a pair $(G, \mathcal{H})$, consisting of a group G and a collection $\mathcal{H}$ of subgroups of G (see Section 3.1 for the term "collection").

For our purposes, the most important examples of group pairs will arise as pairs of fundamental groups.

Example 3.17 (Fundamental group pair)**.** Let (X, A) be a CW-pair, where X is path-connected, and let $x_0 \in X$. Moreover, we assume that each connected component of A is π_1-injective in X. A group pair $(G, \mathcal{H})$ is a *fundamental group pair* of (X, A) (at the basepoint x_0) if:

- $G = \pi_1(X, x_0)$ and
- $\mathcal{H} = (H_i)_{i \in I}$, where $A = \coprod_{i \in I} A_i$ is a decomposition of A into connected components and for each $i \in I$, there exists a basepoint $x_i \in A_i$ and a path γ_i in X from x_0 to x_i, such that H_i is the subgroup of $\pi_1(X, x_0)$ isomorphic to $\pi_1(A_i, x_i)$ via the homomorphism induced by γ_i.

In this situation, we will also abuse notation and just write $\pi_1(A_i)$ for this subgroup H_i in G. It should be noted that there is always an implicit (fixed) choice of basepoints and paths involved. In many cases, these choices will be of no consequence; when these choices would matter, we will be in situations, where we include all G-conjugates of the $(H_i)_{i \in I}$ in the collection of subgroups in question, and thus avoid ambiguities.

Definition 3.18 ($(G, \mathcal{H})$-CW-pair)**.** Let $(G, \mathcal{H})$ be a group pair with $\mathcal{H} = (H_i)_{i \in I}$. A $(G, \mathcal{H})$*-CW-pair* is a G-CW-pair (Y, B) together with a decomposition

$$B = \coprod_{i \in I} G \times_{H_i} B_i \,,$$

where B_i is an H_i-CW-complex.

Let $\mathcal{F}$ be a family of subgroups of G. We say that (Y, B) *has isotropy in* $\mathcal{F}$ if all isotropy groups of Y lie in $\mathcal{F}$.

The *relative dimension* $\dim(Y, B) \in \mathbb{N} \cup \{\infty\}$ is the dimension of the relative G-CW-complex (Y, B).

A map of $(G, \mathcal{H})$-CW-pairs $f \colon (Y, B) \to (Z, C)$ is a G-map of pairs such that the restriction $f|_B$ is of the form $\coprod_{i \in I} G \times_{H_i} f_i$, where $f_i \colon B_i \to C_i$ is an H_i-map.

If a $(G, \mathcal{H})$-CW-pair (Y, B) as above has isotropy in $\mathcal{F}$, then the isotropy groups of the H_i-CW-complex B_i lie in $\mathcal{F}|_{H_i}$.

In the situation of Definition 3.18, replacing a subgroup H_i by a conjugate gH_ig^{-1} with $g \in G$ corresponds to changing the reference point in the description of the induced space $G \times_{H_i} B_i$. Thus, the collection $\mathcal{H}$ can also be viewed as a collection of representatives of conjugacy classes of subgroups of G.

Example 3.19 (Universal covering pair)**.** Let (X, A) be a CW-pair with fundamental group pair $(G, \mathcal{H})$, where $A = \coprod_{i \in I} A_i$ and $\mathcal{H} = (H_i)_{i \in I} = (\pi_1(A_i))_{i \in I}$ are as in Example 3.17. Denote by $p\colon \widetilde{X} \to X$ the universal covering map. Then there is a G-homeomorphism $p^{-1}(A) \cong \coprod_{i \in I} G \times_{H_i} \widetilde{A_i}$, where $\widetilde{A_i}$ is the universal covering of A_i. This shows that $(\widetilde{X}, p^{-1}(A))$ is a $(G, \mathcal{H})$-CW-pair with isotropy in the trivial family $\mathcal{TR}$.

3.2.4 Classifying spaces of group pairs with respect to a family

We now let families of subgroups and an additional collection of subgroups interact:

Definition 3.20 (Classifying space for group pairs)**.** Let G be a group, $\mathcal{H}$ be a collection of subgroups of G, and $\mathcal{F}$ be a family of subgroups of G. A *classifying space for the group pair* $(G, \mathcal{H})$ *with respect to* $\mathcal{F}$ is a $(G, \mathcal{H})$-CW-pair (E, D) with the following universal property:

- The pair (E, D) has isotropy in $\mathcal{F}$;
- For each $(G, \mathcal{H})$-CW-pair (Y, B) with isotropy in $\mathcal{F}$, there is up to G-homotopy exactly one G-map $(Y, B) \to (E, D)$ of $(G, \mathcal{H})$-CW-pairs.

We usually denote such classifying spaces by $E_{\mathcal{F}}(G, \mathcal{H})$ (even though they are only unique up to G-homotopy equivalence of pairs).

Models for $E_{\mathcal{F}}(G, \mathcal{H})$ can be constructed as mapping cylinders:

Lemma 3.21 (Existence of classifying spaces for group pairs)**.** *Let* $(G, \mathcal{H})$ *be a group pair and let* $\mathcal{F}$ *be family of subgroups of* G*. Then there exists a classifying space for the group pair* $(G, \mathcal{H})$ *with respect to* $\mathcal{F}$*.*

Proof. We write the collection $\mathcal{H}$ as $(H_i)_{i \in I}$. For every $i \in I$, let $E_{\mathcal{F}|_{H_i}} H_i$ be a classifying space for H_i with respect to the family $\mathcal{F}|_{H_i}$. The induced G-CW-complex $G \times_{H_i} (E_{\mathcal{F}|_{H_i}} H_i)$ has isotropy in the family $\mathcal{F}$ and hence, by the universal property of $E_{\mathcal{F}} G$, there exists a G-map $G \times_{H_i} (E_{\mathcal{F}|_{H_i}} H_i) \to E_{\mathcal{F}} G$ (that is unique up to G-homotopy). Then the mapping cylinder of the G-map

$$\coprod_{i \in I} G \times_{H_i} (E_{\mathcal{F}|_{H_i}} H_i) \to E_{\mathcal{F}} G$$

is a model for $E_{\mathcal{F}}(G, \mathcal{H})$. This follows from the universal properties of the classifying spaces $E_{\mathcal{F}|_{H_i}} H_i$ and $E_{\mathcal{F}} G$. □

3.3 Relative cohomological dimension and mapping degrees

We discuss an application of classifying spaces for group pairs and relative group cohomology to maps between manifolds. We obtain inheritance results for the freeness of fundamental groups of manifolds in terms of mapping degrees. The results of this section are independent from the rest of the paper.

3.3.1 Relative group cohomology

We recall the definition of relative cohomology of a group pair and a characterisation of groups pairs with relative cohomological dimension one.

Definition 3.22 (Relative cohomology of group pairs)**.** Let $(G, \mathcal{H})$ be a group pair and R be a commutative ring. We define the *relative cohomology* $H^*(G, \mathcal{H}; V)$ with coefficients in an RG-module V to be the R-module

$$H^*(G, \mathcal{H}; V) \coloneqq H^*_G(E_{\mathcal{TR}}(G, \mathcal{H}); V)\,,$$

where $E_{\mathcal{TR}}(G, \mathcal{H})$ is a classifying space for $(G, \mathcal{H})$ with respect to the trivial family. Here the equivariant cohomology $H^*_G(E_{\mathcal{TR}}(G, \mathcal{H}); V)$ is by definition the cohomology $H^*(BG, \coprod_{H \in \mathcal{H}} BH; V)$ with twisted coefficients.

Definition 3.23 (Relative cohomological dimension)**.** The *relative cohomological dimension* $\operatorname{cd}_R(G, \mathcal{H})$ of the group pair $(G, \mathcal{H})$ over the ring R is defined as follows:

$$\operatorname{cd}_R(G, \mathcal{H}) \coloneqq \sup\{n \in \mathbb{N} \mid H^n(G, \mathcal{H}; V) \ncong 0 \text{ for some } RG\text{-module } V\}.$$

For simplicity we denote $\operatorname{cd}_{\mathbb{Z}}(G, \mathcal{H})$ by $\operatorname{cd}(G, \mathcal{H})$.

To illustrate these definitions, we mention that for $\mathcal{H} = (H_i)_{i \in I}$ the long exact sequence for the pair $E_{\mathcal{TR}}(G, \mathcal{H})$ takes the following form:

$$\cdots \to \prod_{i \in I} H^{n-1}(H_i; V) \to H^n(G, \mathcal{H}; V) \to H^n(G; V) \to \prod_{i \in I} H^n(H_i; V) \to \cdots\,.$$

This shows that $\operatorname{cd}_R(G, \mathcal{H}) \leq n$ if and only if for all RG-modules V the restriction map $H^k(G; V) \to \prod_{i \in I} H^k(H_i; V)$ is an isomorphism for $k > n$ and an epimorphism for $k = n$.

Remark 3.24. Let $(G, \mathcal{H})$ be a group pair with $\mathcal{H} = (H_i)_{i \in I}$. While our definition of $H^*(G, \mathcal{H}; V)$ is purely topological, one may also define it algebraically via derived functors. More precisely, consider the augmentation RG-map $R[\coprod_{i \in I} G/H_i] \to R$ and let Δ denote its kernel. Then there exists a natural isomorphism

$$H^*(G, \mathcal{H}; V) \cong \operatorname{Ext}^{*-1}_{RG}(\Delta, V)\,.$$

In this situation we have that $\mathrm{cd}_R(G, \mathcal{H}) = \mathrm{pd}_{RG}(\Delta) + 1$, where pd_{RG} denotes the projective dimension.

Many of the usual cohomological tools for group cohomology have been developed for the relative case as well [Tak59, BE78, Alo91].

In the case of a single subgroup H, the relation between $H^*(G, H; V)$ and the Bredon cohomology $H^*_G(E_{\mathcal{F}\langle H\rangle}G; V)$ has recently been investigated [ANCM17, ANCMSS21].

By the work of Stallings [Sta68] and Swan [Swa69] groups of cohomological dimension one are precisely the free groups. The following is a generalisation to the relative setting.

Theorem 3.25 (Group pairs of relative cohomological dimension one [Dic80, Alo91])**.** *Let $(G, \mathcal{H})$ be a group pair with $\mathcal{H} = (H_i)_{i \in I}$. Then the following are equivalent:*

(i) $\mathrm{cd}(G, \mathcal{H}) = 1$;

(ii) *There exists a free group F such that $G \cong F * *_{i \in I} H_i$.*

3.3.2 Mapping degrees and monotonicity

For maps between manifolds with π_1-injective boundary components, we prove monotonicity results on the cohomological dimension of the fundamental group pairs.

First, we introduce some notation. Let (X, A) be a CW-pair with X path-connected. Let $A = \coprod_{i \in I} A_i$ be a decomposition into connected components and assume that each A_i is π_1-injective in X. We denote by $(\pi_1(X), \pi_1(A))$ a fundamental group pair of (X, A) (Example 3.17) and by $p \colon \widetilde{X} \to X$ the universal covering map. Let $H^*(X, A; V)$ be the cohomology with twisted coefficients in a $\pi_1(X)$-module V. Then the classifying map $\varphi_{(X,A)} \colon (\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{TR}}(\pi_1(X), \pi_1(A))$ induces a map on cohomology:

$$H^*(\varphi_{(X,A)}) \colon H^*(\pi_1(X), \pi_1(A); V) \to H^*(X, A; V)\,.$$

Lemma 3.26. *Let (X, A) be a CW-pair with fundamental group pair $(\pi_1(X), \pi_1(A))$ as above. The map $H^2(\varphi_{(X,A)}) \colon H^2(\pi_1(X), \pi_1(A); V) \to H^2(X, A; V)$ is injective for every $\pi_1(X)$-module V.*

Proof. We argue via the four lemma for monomorphisms. We consider the following commutative diagram, where the coefficient module is omitted:

$$\begin{array}{ccccccc} H^1(\pi_1(X)) & \longrightarrow & \prod_{i \in I} H^1(\pi_1(A_i)) & \longrightarrow & H^2(\pi_1(X), \pi_1(A)) & \longrightarrow & H^2(\pi_1(X)) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle H^2(\varphi_{(X,A)})} & & \downarrow \\ H^1(X) & \longrightarrow & H^1(A) & \longrightarrow & H^2(X, A) & \longrightarrow & H^2(X)\,. \end{array}$$

Here all vertical maps are induced by the respective classifying maps. Since a model for $E_{\mathcal{TR}}(\pi_1(X))$ [resp. for $E_{\mathcal{TR}}(\pi_1(A_i))$] can be built from $\widetilde{X}$ [resp. from $\widetilde{A_i}$] by attaching cells of dimension greater than or equal to 3, the first and second vertical arrows are isomorphisms, while the last vertical arrow is injective. Applying the four lemma for monomorphisms, we conclude that $H^2(\varphi_{(X,A)})$ is injective. □

We use the convention (Section 3.1) to say that a manifold M has π_1-*injective boundary* ∂M if every component of ∂M is π_1-injective in M.

Theorem 3.27 (Mapping degree and relative cohomological dimension one)**.** *Let $f\colon (M,\partial M) \to (N,\partial N)$ be a map between oriented compact connected manifolds of the same dimension with π_1-injective boundary. Then the following hold:*

(i) *If* $\deg(f) = \pm 1$ *and* $\operatorname{cd}(\pi_1(M),\pi_1(\partial M)) \leq 1$*, then* $\operatorname{cd}(\pi_1(N),\pi_1(\partial N)) \leq 1$*;*

(ii) *If* $\deg(f) \neq 0$ *and* $\operatorname{cd}_{\mathbb{Q}}(\pi_1(M),\pi_1(\partial M)) \leq 1$*, then* $\operatorname{cd}_{\mathbb{Q}}(\pi_1(N),\pi_1(\partial N)) \leq 1$*.*

Proof. We proceed by contraposition. Let $R = \mathbb{Z}$ [resp. $R = \mathbb{Q}$] and suppose that $\operatorname{cd}_R(\pi_1(N),\pi_1(\partial N)) > 1$. Then by a dimension shifting argument, there exists an $R\pi_1(N)$-module V such that $H^2(\pi_1(N),\pi_1(\partial N);V)$ is non-trivial. We denote by $f^{-1}V$ the $R\pi_1(M)$-module that is obtained from V by restriction along $\pi_1(f)$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^2(N,\partial N;V) & \xrightarrow{H^2(f)} & H^2(M,\partial M;f^{-1}V) \\ {\scriptstyle H^2(\varphi_{(N,\partial N)})}\uparrow & & {\scriptstyle H^2(\varphi_{(M,\partial M)})}\uparrow \\ H^2(\pi_1(N),\pi_1(\partial N);V) & \xrightarrow{H^2(\pi_1(f))} & H^2(\pi_1(M),\pi_1(\partial M);f^{-1}V)\,. \end{array}$$

Here the vertical maps, which are induced by the respective classifying maps, are injective in degree 2 (Lemma 3.26). By Poincaré–Lefschetz duality with twisted coefficients, there exists an Umkehr map

$$f_!\colon H^2(M,\partial M;f^{-1}V) \to H^2(N,\partial N;V)$$

such that the composition $f_! \circ H^2(f)\colon H^2(N,\partial N;V) \to H^2(N,\partial N;V)$ is given by multiplication with $\deg(f)$. Hence the map $H^2(f)$ is injective in each of the following cases:

(i) If $\deg(f) = \pm 1$ and $R = \mathbb{Z}$;

(ii) If $\deg(f) \neq 0$ and $R = \mathbb{Q}$.

This shows that in the situations (i) and (ii) the composition

$$H^2(f) \circ H^2(\varphi_{(N,\partial N)}) = H^2(\varphi_{(M,\partial M)}) \circ H^2(\pi_1(f))$$

is injective, whence $H^2(\pi_1(f))$ is injective. Therefore the relative cohomology group $H^2(\pi_1(M), \pi_1(\partial M); f^{-1}V)$ is non-trivial and we have $\mathrm{cd}_R(\pi_1(M), \pi_1(\partial M)) > 1$. □

Remark 3.28. If the universal coverings of N and ∂N are k-connected, then the map $H^{k+1}(\varphi_{(N,\partial N)})$ is injective and hence similar monotonicity results hold for cohomological dimension at most k.

Theorem 3.27 readily implies the following:

Corollary 3.29. *Let $f\colon (M, \partial M) \to (N, \partial N)$ be a map between oriented compact connected manifolds of the same dimension with π_1-injective boundary. Let $\partial M = \coprod_{i=1}^m M_i$ and $\partial N = \coprod_{i=1}^n N_i$ be decompositions into connected components. If $\deg(f) = \pm 1$ and there exists a free group F_M such that*

$$\pi_1(M) \cong F_M * *_{i=1}^m \pi_1(M_i)\,,$$

then there exists a free group F_N such that $\pi_1(N) \cong F_N * *_{i=1}^n \pi_1(N_i)$.

Proof. This follows from Theorem 3.27 (i) and the group-theoretic characterisation of relative cohomological dimension one (Theorem 3.25). □

Examples of manifolds satisfying the assumptions of Corollary 3.29 are the following:

Example 3.30. Let F_k be a free group of rank k, and let $\mathcal{H} = (H_1, \dots, H_m)$ be a finite collection of finitely presented groups. Then for every $n \geq 7$, there exists a compact connected n-dimensional manifold $(M, \partial M)$ with fundamental group pair $(G, \mathcal{H})$ such that $G \cong F_k * *_{i=1}^m H_i$. Indeed, let L_i be an oriented closed connected 4-manifold with $\pi_1(L_i) \cong H_i$ and consider

$$(M, \partial M) \coloneqq (\#_{i=1}^k S^1 \times S^{n-1}) \# (\#_{i=1}^m L_i \times D^{n-4})\,.$$

Then $\partial M \cong \coprod_{i=1}^m L_i \times S^{n-5}$ and $\pi_1(M) \cong F_k * *_{i=1}^m \pi_1(L_i)$.

In the case of closed manifolds, Corollary 3.29 provides a simplified proof of a result by Dranishnikov and Rudyak [DR09, Theorem 5.2], without making use of the Berstein class. We also obtain the following analogue for maps of non-zero degree:

Corollary 3.31. *Let $f\colon M \to N$ be a map between oriented closed connected manifolds of the same dimension with $\deg(f) \neq 0$. If $\pi_1(M)$ is the fundamental group of a graph of finite groups, then so is $\pi_1(N)$. In particular, if $\pi_1(M)$ is virtually free, then $\pi_1(N)$ is virtually free.*

Proof. This follows from Theorem 3.27 (ii) and Dunwoody's characterisation [Dun79] of groups of cohomological dimension one over arbitrary rings. As fundamental groups of closed manifolds are finitely generated, this characterisation can also be expressed in terms of virtual freeness [Dun79, Corollary 1.2]. □

We are not aware of a relative version of Dunwoody's result that would characterise group pairs of relative cohomological dimension one over arbitrary rings. The results in this section motivate the following:

Question 3.32. For which other classes $\mathcal{C}$ of groups does the following hold? Whenever $f\colon M \to N$ is a map between oriented closed connected manifolds of the same dimension with $\deg(f) \neq 0$ [or $\deg(f) = 1$] and $\pi_1(M)$ is the fundamental group of a graph of groups from $\mathcal{C}$, also $\pi_1(N)$ must be the fundamental group of a graph of groups from $\mathcal{C}$.

Positive answers to Question 3.32 lead to corresponding monotonicity results for the generalised LS-category ≤ 2 associated with the class $\mathcal{C}$, provided that $\mathcal{C}$ is closed under isomorphisms, subgroups, and quotients [CLM, Corollary 5.4 and the following paragraph].

3.4 Relative open covers and equivariant nerve pairs

In this section, we study equivariant nerves of open covers in a relative setting. Given an open cover of a space, the nerve of the cover gives an approximation of the space. Considering actions on spaces and compatible covers leads to equivariant nerves, studied by Löh and Sauer [LS20] for universal covering spaces with the action by the fundamental group. We adapt this approach to pairs of spaces.

3.4.1 Open covers and nerve pairs

We fix some notation and terminology on open covers and their nerves.

Let Y be a space and $\mathcal{V}$ be a cover of Y by path-connected open subsets. We regard $\mathcal{V}$ as a set of subsets of Y (and not as a collection of subsets).

Definition 3.33 ($\mathcal{F}$-Cover)**.** Let $\mathcal{F}$ be a family of subgroups of $\pi_1(Y)$. We say that $\mathcal{V}$ is an *$\mathcal{F}$-cover* of Y if $\operatorname{im}(\pi_1(V,x) \to \pi_1(Y,x)) \in \mathcal{F}$ for all $V \in \mathcal{V}$ and all $x \in V$.

Definition 3.34 (Convex cover)**.** The cover $\mathcal{V}$ of Y is said to be *convex* if every intersection of finitely many elements of $\mathcal{V}$ is path-connected or empty.

The *multiplicity* $\operatorname{mult}(\mathcal{V})$ of $\mathcal{V}$ is defined as follows:

$$\operatorname{mult}(\mathcal{V}) := \sup\Big\{n \in \mathbb{N} \;\Big|\; \bigcap_{i=1}^{n} V_i \neq \emptyset \text{ for some pairwise different } V_1, \dots, V_n \in \mathcal{V}\Big\}.$$

The *nerve* $N(\mathcal{V})$ of $\mathcal{V}$ is the (abstract) simplicial complex with vertex set $\mathcal{V}$; pairwise different $V_0, \dots, V_n \in \mathcal{V}$ span an n-simplex in $N(\mathcal{V})$ if $V_0 \cap \dots \cap V_n \neq \emptyset$. By definition, $\dim(N(\mathcal{V})) = \mathrm{mult}(\mathcal{V}) - 1$.

Let $|N(\mathcal{V})|$ be the geometric realisation of the nerve $N(\mathcal{V})$. Given a partition of unity $(\psi_V)_{V \in \mathcal{V}}$ on Y subordinate to $\mathcal{V}$, there is an associated *nerve map*

$$\mu \colon Y \to |N(\mathcal{V})| , \quad y \mapsto \mu(y) := \sum_{V \in \mathcal{V}} \psi_V(y) \cdot V . \tag{3.2}$$

The nerve map is unique up to homotopy, since different choices of partitions of unity lead to homotopic nerve maps.

Definition 3.35 (Relative multiplicity)**.** For a subspace B of Y, we define the *relative multiplicity* $\mathrm{mult}_B(\mathcal{V})$ of $\mathcal{V}$ (with respect to B) as follows:

$$\begin{aligned}\mathrm{mult}_B(\mathcal{V}) := \sup\Big\{ n \in \mathbb{N} \;\Big|\; \bigcap_{i=1}^{n} V_i \neq \emptyset \text{ and } B \cap \Big(\bigcap_{i=1}^{n} V_i\Big) = \emptyset \\ \text{for some pairwise different } V_1, \dots, V_n \in \mathcal{V} \Big\} .\end{aligned}$$

Definition 3.36 (Nerve pair)**.** For a subspace B of Y, we denote by $N_B(\mathcal{V})$ the simplicial subcomplex of $N(\mathcal{V})$ with vertex set $\mathcal{V}_B := \{V \in \mathcal{V} \mid V \cap B \neq \emptyset\}$, and pairwise different $V_0, \dots, V_n \in \mathcal{V}_B$ span a simplex in $N_B(\mathcal{V})$ if $V_0 \cap \dots \cap V_n \cap B \neq \emptyset$. By construction we have $N(\mathcal{V}) = N_Y(\mathcal{V})$.

The nerve map $\mu \colon Y \to |N(\mathcal{V})|$ induces a map of pairs

$$\mu \colon (Y, B) \to (|N(\mathcal{V})|, |N_B(\mathcal{V})|) .$$

The *relative dimension* $\dim(N(\mathcal{V}), N_B(\mathcal{V}))$ is the dimension of the relative simplicial complex $(N(\mathcal{V}), N_B(\mathcal{V}))$. By definition, $\dim(N(\mathcal{V}), N_B(\mathcal{V})) = \mathrm{mult}_B(\mathcal{V}) - 1$.

3.4.2 Equivariant nerve pairs

We now consider open covers that are compatible with a group action, giving rise to an action on their nerve.

Definition 3.37 (Invariant cover and partition of unity)**.** Let Y be a G-CW-complex and $\mathcal{V}$ be a cover of Y by path-connected open subsets. We say that $\mathcal{V}$ is *G-invariant* if for all $g \in G$ and $V \in \mathcal{V}$, we have $g \cdot V \in \mathcal{V}$. For $V \in \mathcal{V}$, we write

$$\mathrm{Stab}_G(V) := \{ g \in G \mid g \cdot V = V \} .$$

For a family of subgroups $\mathcal{F}$ of G, we say that the cover $\mathcal{V}$ *has isotropy in $\mathcal{F}$* if for all $V \in \mathcal{V}$, we have $\mathrm{Stab}_G(V) \in \mathcal{F}$.

A partition of unity $(\psi_V)_{V\in\mathcal{V}}$ on Y subordinate to a G-invariant cover $\mathcal{V}$ is said to be *G-invariant* if for all $g \in G$ and $y \in Y$, we have

$$\psi_V(y) = \psi_{g\cdot V}(g \cdot y)\,.$$

The key examples will come from covers of a space X giving rise to $\pi_1(X)$-invariant covers of the universal covering space $\widetilde{X}$ (Example 3.40).

We recall basic properties of equivariant nerves [LS20, Lemmas 4.8 and 4.11] and their proofs for completeness.

Lemma 3.38 (Equivariant nerve)**.** *Let Y be a G-CW-complex and $\mathcal{V}$ be a G-invariant cover of Y. Then the following hold:*

(i) *Let B be an H-invariant subcomplex of Y for a subgroup H of G. Then $N_B(\mathcal{V})$ is an H-simplicial complex and its geometric realisation $|N_B(\mathcal{V})|$ is an H-CW-complex. In particular, $N(\mathcal{V})$ is a G-simplicial complex;*

(ii) *Suppose that $g \cdot V \cap V \neq \emptyset$ implies $g \cdot V = V$ for all $g \in G, V \in \mathcal{V}$. Let $\mathcal{F}$ be an intersection-closed family of subgroups of G. If the cover $\mathcal{V}$ has isotropy in $\mathcal{F}$, then the G-CW-complex $|N(\mathcal{V})|$ has isotropy in $\mathcal{F}$;*

(iii) *Let $(\psi_V)_{V\in\mathcal{V}}$ be a G-invariant partition of unity on Y subordinate to $\mathcal{V}$. Then the induced nerve map $\mu\colon Y \to |N(\mathcal{V})|$ is G-equivariant.*

Proof. (i) To show that $N_B(\mathcal{V})$ is an H-simplicial complex, it suffices to prove that the H-action sends simplices of $N_B(\mathcal{V})$ to simplices of $N_B(\mathcal{V})$. Let v be a vertex of $N_B(\mathcal{V})$ corresponding to $V \in \mathcal{V}$ with $V \cap B \neq \emptyset$. Then for every $h \in H$, we have

$$\emptyset \neq h \cdot (V \cap B) = (h \cdot V) \cap (h \cdot B) \subset (h \cdot V) \cap B\,.$$

This shows that the vertex $h \cdot v$ of $N(\mathcal{V})$ corresponding to $h \cdot V \in \mathcal{V}$ lies in $N_B(\mathcal{V})$. The same argument also extends to higher-dimensional simplices, which proves the claim. Then the geometric realisation $|N_B(\mathcal{V})|$ is an H-CW-complex (Example 3.7).

(ii) We show that the isotropy groups of the vertices of the barycentric subdivision of $N(\mathcal{V})$ lie in $\mathcal{F}$. This is indeed sufficient; since the action is simplicial the stabiliser of every interior point of a k-simplex in the barycentric subdivision of $N(\mathcal{V})$ is the intersection of the stabilisers of its $k+1$ vertices. Then the fact that $\mathcal{F}$ is closed under finite intersections yields the thesis.

Let v be a vertex in the barycentric subdivision of $N(\mathcal{V})$, associated to a k-simplex corresponding to $V_0, \dots, V_k \in \mathcal{V}$ with $V_0 \cap \dots \cap V_k \neq \emptyset$. It remains to show that the subgroup

$$G_v = \bigl\{g \in G \bigm| \{g \cdot V_0, \dots, g \cdot V_k\} = \{V_0, \dots, V_k\}\bigr\}$$

of G lies in the family $\mathcal{F}$. For $k = 0$, we know that $G_v = \mathrm{Stab}_G(V_0) \in \mathcal{F}$ by the assumption that $\mathcal{V}$ has isotropy in $\mathcal{F}$. On the other hand, for $k > 0$, by the assumption that $g \cdot V \cap V \neq \emptyset$ implies $g \cdot V = V$, we have that $g \cdot V_i = V_j$ for $i, j \in \{0, \dots, k\}$ implies $i = j$. Hence $G_v = \mathrm{Stab}_G(V_0) \cap \dots \cap \mathrm{Stab}_G(V_k)$. This group lies in $\mathcal{F}$ since $\mathcal{F}$ is closed under finite intersections.

(iii) The G-invariance of the partition of unity implies the G-equivariance of the nerve map (3.2) as follows: For all $g \in G$ and all $y \in Y$, we have

$$\begin{aligned}\mu(g \cdot y) &= \sum_{V \in \mathcal{V}} \psi_V(g \cdot y) \cdot V = \sum_{V \in \mathcal{V}} \psi_{g^{-1} \cdot V}(y) \cdot V \\ &= \sum_{V \in \mathcal{V}} \psi_V(y) \cdot (g \cdot V) = g \cdot \mu(y)\,.\end{aligned}$$

This finishes the proof. □

We extend the previous results to the relative situation:

Lemma 3.39 (Equivariant nerve pair)**.** *Let $(G, \mathcal{H})$ be a group pair and let (Y, B) be a $(G, \mathcal{H})$-CW-pair. Let $\mathcal{F}$ be an intersection-closed family of subgroups of G and $\mathcal{V}$ be a G-invariant cover of Y with isotropy in $\mathcal{F}$. Suppose that the following hold:*

(i) *For all $V \in \mathcal{V}, g \in G$ with $g \cdot V \cap V \neq \emptyset$, we have $g \cdot V = V$;*

(ii) *There exists a G-invariant partition of unity on Y subordinate to $\mathcal{V}$;*

(iii) *For all $V \in \mathcal{V}$ with $V \cap B \neq \emptyset$, the intersection $V \cap B$ is connected.*

Then $(|N(\mathcal{V})|, |N_B(\mathcal{V})|)$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$. Moreover, the nerve map $\mu\colon Y \to |N(\mathcal{V})|$ induces a map of $(G, \mathcal{H})$-CW-pairs:

$$\mu\colon (Y, B) \to \big(|N(\mathcal{V})|, |N_B(\mathcal{V})|\big)\,.$$

Proof. By Lemma 3.38, assumption (i) implies that $|N(\mathcal{V})|$ is a G-CW-complex with isotropy in $\mathcal{F}$, and assumption (ii) implies that the nerve map $\mu\colon Y \to |N(\mathcal{V})|$ is G-equivariant.

We write the collection $\mathcal{H}$ as $(H_i)_{i \in I}$. Since (Y, B) is a $(G, \mathcal{H})$-CW-pair, we have a decomposition $B = \coprod_{i \in I} G \times_{H_i} B_i$, where B_i is an H_i-CW-complex. We identify B_i with the subset $[e, B_i] \subset B \subset Y$, where $e \in G$ denotes the neutral element. We also identify the set of vertices of $|N(\mathcal{V})|$ with $\mathcal{V}$. There is a G-map

$$\Phi\colon \coprod_{i \in I} G \times_{H_i} |N_{B_i}(\mathcal{V})| \to |N_B(\mathcal{V})|\,,$$

mapping a vertex $[g, V]$ of $G \times_{H_i} |N_{B_i}(\mathcal{V})|$ to the vertex $g \cdot V$ of $|N_B(\mathcal{V})|$, and that is defined by affine extension. The affine extension is well-defined, since the images of vertices spanning a simplex in $|N_{B_i}(\mathcal{V})|$ also span a simplex in $|N_B(\mathcal{V})|$. We claim that assumption (iii) implies that Φ is a G-homeomorphism.

Indeed, the inverse map Φ^{-1} is given as follows: For a vertex V of $|N_B(\mathcal{V})|$, we have $V \cap B \neq \emptyset$ and hence the intersection $V \cap B$ is connected by assumption (iii). Thus, there exists a unique element $i \in I$ and a unique coset $gH_i \in G/H_i$ such that $V \cap [g, B_i] \neq \emptyset$. Since $V \cap [g, B_i] = g(g^{-1} \cdot V \cap B_i)$, we may define Φ^{-1} to map the vertex V to the vertex $[g, g^{-1} \cdot V]$. This assignment is G-equivariant: Indeed, for every $g' \in G$, we have $\emptyset \neq g'(V \cap [g, B_i]) = g'g(g^{-1} \cdot V \cap B_i)$. Hence under Φ^{-1} the vertex $g' \cdot V$ is mapped to the vertex $[g'g, g^{-1} \cdot V]$.

Then Φ^{-1} is determined by affine extension. This is well-defined because the images of vertices spanning a simplex in $|N_B(\mathcal{V})|$ also span a simplex in the corresponding $|N_{B_i}(\mathcal{V})|$. Thus Φ is a G-homeomorphism, showing that $(|N(\mathcal{V})|, |N_B(\mathcal{V})|)$ is a $(G, \mathcal{H})$-CW-pair.

The G-map $\mu\colon (Y, B) \to (|N(\mathcal{V})|, |N_B(\mathcal{V})|)$ is a map of $(G, \mathcal{H})$-CW-pairs, since we have $\mu(B_i) \subset |N_{B_i}(\mathcal{V})|$. □

3.4.3 Relative open covers

We study our main example of equivariant nerve pairs coming from lifted covers of CW-pairs.

Example 3.40 (Lifted cover)**.** Let X be a connected CW-complex, let $G := \pi_1(X)$, and let $p\colon \widetilde{X} \to X$ denote the universal covering. For a cover $\mathcal{U}$ of X by path-connected open subsets, we consider the *lifted cover* $\widetilde{\mathcal{U}}$ of $\widetilde{X}$:

$$\widetilde{\mathcal{U}} := \bigl\{ V \subset \widetilde{X} \bigm| V \text{ is a path-connected component of } p^{-1}(U) \text{ for some } U \in \mathcal{U} \bigr\}.$$

Clearly, $\widetilde{\mathcal{U}}$ is a G-invariant cover of $\widetilde{X}$. Note that for every $g \in G$, $V \in \widetilde{\mathcal{U}}$, the condition $g \cdot V \cap V \neq \emptyset$ implies $g \cdot V = V$. Moreover, for every $V \in \widetilde{\mathcal{U}}$ we have that $\operatorname{Stab}_G(V)$ is conjugate to $\operatorname{im}(\pi_1(p(V)) \to \pi_1(X))$. This shows that if $\mathcal{U}$ is an $\mathcal{F}$-cover of X, then $\widetilde{\mathcal{U}}$ has isotropy in $\mathcal{F}$.

Every given partition of unity $(\varphi_U)_{U \in \mathcal{U}}$ on X subordinate to $\mathcal{U}$ lifts to a G-invariant partition of unity $(\widetilde{\varphi}_V)_{V \in \widetilde{\mathcal{U}}}$ on $\widetilde{X}$ subordinate to $\widetilde{\mathcal{U}}$ as follows: For $V \in \widetilde{\mathcal{U}}$, define

$$\widetilde{\varphi}_V := \chi_V \cdot (\varphi_{p(V)} \circ p)\colon \widetilde{X} \to [0,1],$$

where $\chi_V\colon \widetilde{X} \to [0,1]$ denotes the characteristic function on $V \subset \widetilde{X}$. Let ν and $\widetilde{\nu}$ be the nerve maps associated to $(\varphi_U)_{U \in \mathcal{U}}$ and $(\widetilde{\varphi}_V)_{V \in \widetilde{\mathcal{U}}}$, respectively.

The simplicial map $N(p)\colon N(\widetilde{\mathcal{U}}) \to N(\mathcal{U})$, which maps a simplex of $N(\widetilde{\mathcal{U}})$ corresponding to $V_0, \dots, V_k \in \widetilde{\mathcal{U}}$ to the simplex of $N(\mathcal{U})$ corresponding to $p(V_0), \dots, p(V_k) \in \mathcal{U}$, makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\nu}} & |N(\widetilde{\mathcal{U}})| \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle |N(p)|} \\ X & \xrightarrow{\nu} & |N(\mathcal{U})| \,. \end{array}$$

Given a pair of spaces (X, A), we introduce conditions on covers of X requiring a certain regularity near the subspace A. These conditions will give rise to desirable properties of the lifted covers.

Definition 3.41 (Relative cover)**.** Let (X, A) be a pair of spaces with path-connected ambient space X. A *relative cover* of (X, A) is a cover $\mathcal{U}$ of X by path-connected open subsets such that for all $U \in \mathcal{U}$ the following hold:

(RC1) If $U \cap A \neq \emptyset$, then $U \cap A$ is path-connected;

(RC2) If $U \cap A \neq \emptyset$, then the inclusion

$$\operatorname{im}\big(\pi_1(U \cap A, x) \to \pi_1(X, x)\big) \hookrightarrow \operatorname{im}\big(\pi_1(U, x) \to \pi_1(X, x)\big)$$

is an isomorphism for some (whence every) $x \in U \cap A$.

A relative open cover $\mathcal{U}$ is *weakly convex* if for every $k \in \mathbb{N}$ and all $U_1, \dots, U_k \in \mathcal{U}$ with $U_1 \cap \dots \cap U_k \cap A \neq \emptyset$, each path-connected component of $U_1 \cap \dots \cap U_k$ intersects A.

We also say that a relative open cover $\mathcal{U}$ of (X, A) is *convex* if the underlying cover $\mathcal{U}$ of X is convex. Clearly, every convex relative cover is in particular weakly convex.

Given a family $\mathcal{F}$ of subgroups of $\pi_1(X)$, a relative cover $\mathcal{U}$ of (X, A) is a *relative $\mathcal{F}$-cover* if the cover $\mathcal{U}$ of X is an $\mathcal{F}$-cover.

Keeping the same notation as in Example 3.40, we have the following:

Proposition 3.42 (Equivariant nerve pair of lifted covers)**.** *Let (X, A) be a CW-pair with fundamental group pair $(G, \mathcal{H})$. Let $\mathcal{F}$ be an intersection-closed family of subgroups of G and $\mathcal{U}$ be a relative $\mathcal{F}$-cover of X. Let ν and $\widetilde{\nu}$ be the above nerve maps of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, respectively.*

Then $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$ and the map $\widetilde{\nu}$ induces a map of $(G, \mathcal{H})$-CW-pairs

$$\widetilde{\nu}\colon \big(\widetilde{X}, p^{-1}(A)\big) \to \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big)$$

that makes the following diagram commute:

$$\begin{array}{ccc} \big(\widetilde{X}, p^{-1}(A)\big) & \xrightarrow{\ \widetilde{\nu}\ } & \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle |N(p)|} \\ (X, A) & \xrightarrow{\ \nu\ } & \big(|N(\mathcal{U})|, |N_A(\mathcal{U})|\big) . \end{array}$$

Moreover, we have the following:

(i) *If $\mathcal{U}$ is weakly convex, then*

$$\dim\big(N(\widetilde{\mathcal{U}}), N_{p^{-1}(A)}(\widetilde{\mathcal{U}})\big) = \operatorname{mult}_A(\mathcal{U}) - 1 ;$$

(ii) *If $\mathcal{U}$ is convex, then the map $N(p)$ induces isomorphisms of simplicial complexes:*

$$G\backslash N(\widetilde{\mathcal{U}}) \cong N(\mathcal{U}) ;$$
$$G\backslash N_{p^{-1}(A)}(\widetilde{\mathcal{U}}) \cong N_A(\mathcal{U}) .$$

Proof. To show that $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$, we verify that the lifted G-invariant cover $\widetilde{\mathcal{U}}$ of $\widetilde{X}$ satisfies all assumptions of Lemma 3.39. By Example 3.40, we know that $\widetilde{\mathcal{U}}$ has isotropy in $\mathcal{F}$, that there exists a G-invariant partition of unity on $\widetilde{X}$ subordinate to $\widetilde{\mathcal{U}}$, and that $g \cdot V \cap V \neq \emptyset$ implies $g \cdot V = V$ for all $g \in G, V \in \widetilde{\mathcal{U}}$. Hence, we are left to show that if $V \in \widetilde{\mathcal{U}}$ with $V \cap p^{-1}(A) \neq \emptyset$, then $V \cap p^{-1}(A)$ is connected.

Assume for a contradiction that $V \cap p^{-1}(A)$ is disconnected. Let us set $U := p(V)$. By condition (RC1) we know that $U \cap A$ is connected. This shows that there exists a point $a \in U \cap A$ with two lifts $\widetilde{a}_1, \widetilde{a}_2$ contained in different components of $V \cap p^{-1}(A)$. Since V is path-connected, there exists a path γ in V connecting $\widetilde{a}_1$ to $\widetilde{a}_2$. By construction, the image of γ under p is a loop $p_*\gamma$ in U based at a. Then, by condition (RC2), the homotopy class $[p_*\gamma] \in \pi_1(X, a)$ admits a representative whose support is contained in $U \cap A$. Thus, there exists a lifted homotopy in $\widetilde{X}$ relative to the endpoints from γ to a path in $V \cap p^{-1}(A)$. This contradicts the fact that $\widetilde{a}_1$ and $\widetilde{a}_2$ lie in different components of $V \cap p^{-1}(A)$. Hence Lemma 3.39 applies and yields the claim.

(i) We show that $\operatorname{mult}_{p^{-1}(A)}(\widetilde{\mathcal{U}}) = \operatorname{mult}_A(\mathcal{U})$, which immediately implies the claim. The inequality $\operatorname{mult}_{p^{-1}(A)}(\widetilde{\mathcal{U}}) \geq \operatorname{mult}_A(\mathcal{U})$ is clear. To show the opposite inequality, let $V_1, \dots, V_k \in \widetilde{\mathcal{U}}$ with $\bigcap_{i=1}^k V_i \neq \emptyset$ and $\bigcap_{i=1}^k V_i \cap p^{-1}(A) = \emptyset$. We claim that we have $\bigcap_{i=1}^k p(V_i) \cap A = \emptyset$, whence $k \leq \operatorname{mult}_A(\mathcal{U})$ because the $(p(V_i))_i$ are pairwise different. Indeed, assume for a contradiction that $\bigcap_{i=1}^k p(V_i) \cap A \neq \emptyset$. Take a point $\widetilde{x} \in \bigcap_{i=1}^k V_i$ and consider $p(\widetilde{x}) \in \bigcap_{i=1}^k p(V_i)$. Then the component of $\bigcap_{i=1}^k p(V_i)$ containing $p(\widetilde{x})$ intersects A by weak convexity of $\mathcal{U}$. Hence, we can choose a path τ in $\bigcap_{i=1}^k p(V_i)$ connecting $p(\widetilde{x})$ to some point in A. Then the lifted path $\widetilde{\tau}$ of τ in $\widetilde{X}$ with starting point $\widetilde{x}$

has endpoint in $p^{-1}(A)$ and is supported in $\bigcap_{i=1}^k V_i$. This shows $\bigcap_{i=1}^k V_i \cap p^{-1}(A) \neq \emptyset$, which is a contradiction.

(ii) If $\mathcal{U}$ is convex, then the map $N(p)\colon N(\widetilde{\mathcal{U}}) \to N(\mathcal{U})$ induces the first isomorphism $G\backslash N(\widetilde{\mathcal{U}}) \cong N(\mathcal{U})$ by [LS20, Lemma 4.5 (3)]. Hence, to deduce the second isomorphism, it suffices to show that $N_{p^{-1}(A)}(\widetilde{\mathcal{U}}) = N(p)^{-1}(N_A(\mathcal{U}))$. The inclusion of $N_{p^{-1}(A)}(\widetilde{\mathcal{U}})$ into $N(p)^{-1}(N_A(\mathcal{U}))$ is clear. To prove the opposite inclusion, let $V_1, \ldots, V_k \in \widetilde{\mathcal{U}}$ span a simplex in $N(p)^{-1}(N_A(\mathcal{U}))$. This means that $\bigcap_{i=1}^k V_i \neq \emptyset$ and $\bigcap_{i=1}^k p(V_i) \cap A \neq \emptyset$, and we need to show that $\bigcap_{i=1}^k V_i \cap p^{-1}(A) \neq \emptyset$. Take a point $\widetilde{x} \in \bigcap_{i=1}^k V_i$ and consider $p(\widetilde{x}) \in \bigcap_{i=1}^k p(V_i)$. Since $\mathcal{U}$ is convex and $\bigcap_{i=1}^k p(V_i) \cap A \neq \emptyset$, we can choose a path τ in $\bigcap_{i=1}^k p(V_i)$ with starting point $p(\widetilde{x})$ and endpoint in A. As before, the lifted path $\widetilde{\tau}$ of τ in $\widetilde{X}$ starting at $\widetilde{x}$ shows that $\bigcap_{i=1}^k V_i \cap p^{-1}(A) \neq \emptyset$. □

3.4.4 Relative generalised LS-category

We introduce a relative version of the generalised Lusternik–Schnirelmann category for families of subgroups [CLM, Definition 2.16].

Definition 3.43 (Relative $\mathcal{F}$-category)**.** Let (X, A) be a CW-pair with fundamental group pair $(G, \mathcal{H})$ and let $p\colon \widetilde{X} \to X$ denote the universal covering. Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup. The *relative $\mathcal{F}$-category* of (X, A), denoted by $\mathrm{cat}_{\mathcal{F}}(X, A)$, is the minimal $n \in \mathbb{N}$ such that there exists a $(G, \mathcal{H})$-CW-pair (Y, B) with isotropy in $\mathcal{F}$ of relative dimension $n-1$ and a map of $(G, \mathcal{H})$-CW-pairs $(\widetilde{X}, p^{-1}(A)) \to (Y, B)$. If no such integer n exists, we set $\mathrm{cat}_{\mathcal{F}}(X, A) := +\infty$.

We will refer to $\mathrm{cat}_{\mathcal{AME}}(X, A)$ also as the *relative amenable category* of (X, A).

Remark 3.44. In the situation of Definition 3.43, let $E_{\mathcal{F}}(G, \mathcal{H})$ be a model for the classifying space of the group pair $(G, \mathcal{H})$ with respect to the family $\mathcal{F}$. Consider the (up to G-homotopy unique) map of $(G, \mathcal{H})$-CW-pairs

$$f\colon \big(\widetilde{X}, p^{-1}(A)\big) \to E_{\mathcal{F}}(G, \mathcal{H}).$$

Let $n \in \mathbb{N}$. Then the following are equivalent:

(i) We have $\mathrm{cat}_{\mathcal{F}}(X, A) \leq n$;

(ii) The map f factors (up to G-homotpy) through a $(G, \mathcal{H})$-CW-pair (Y, B) with isotropy in $\mathcal{F}$ of relative dimension $n-1$;

(iii) The map f is G-homotopic to a map of $(G, \mathcal{H})$-CW-pairs with values in the relative $(n-1)$-skeleton of $E_{\mathcal{F}}(G, \mathcal{H})$.

Indeed, the equivalence of these conditions follows from the universal property of $E_{\mathcal{F}}(G, \mathcal{H})$ and the equivariant cellular approximation theorem [Lüc89, Theorem 2.1].

By definition, the relative $\mathcal{F}$-category satisfies $\mathrm{cat}_{\mathcal{F}}(X,A) \leq \dim(X,A)+1$. A more efficient upper bound for the relative category is provided by the existence of weakly convex relative covers:

Definition 3.45 (Relative $\mathcal{F}$-multiplicity)**.** Let (X,A) be a pair of spaces and let $\mathcal{F}$ be a family of subgroups of $\pi_1(X)$. The *relative $\mathcal{F}$-multiplicity* of (X,A), denoted by $\mathrm{mult}_{\mathcal{F}}(X,A)$, is the minimal $n \in \mathbb{N}$ such that there exists a weakly convex relative $\mathcal{F}$-cover $\mathcal{U}$ of (X,A) with $\mathrm{mult}_A(\mathcal{U}) = n$. If no such integer n exists, we set $\mathrm{mult}_{\mathcal{F}}(X,A) \coloneqq +\infty$.

We will refer to $\mathrm{mult}_{\mathcal{AME}}(X,A)$ also as the *relative amenable multiplicity* of (X,A).

Lemma 3.46. *Let (X,A) be a CW-pair with fundamental group pair $(G,\mathcal{H})$. Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup. Then we have*

$$\mathrm{cat}_{\mathcal{F}}(X,A) \leq \mathrm{mult}_{\mathcal{F}}(X,A)\,.$$

Proof. We may assume that $n \coloneqq \mathrm{mult}_{\mathcal{F}}(X,A)$ is finite. Let $\mathcal{U}$ be a weakly convex relative $\mathcal{F}$-cover of (X,A) with $\mathrm{mult}_A(\mathcal{U}) = n$. By Proposition 3.42, the equivariant nerve pair $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ of the lifted cover $\widetilde{\mathcal{U}}$ of $\widetilde{X}$ is a $(G,\mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$ and of relative dimension $n-1$. Hence the nerve map

$$\widetilde{\nu}\colon \bigl(\widetilde{X}, p^{-1}(A)\bigr) \to \bigl(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\bigr)$$

exhibits the desired inequality. □

Remark 3.47 (Category and multiplicity, absolute case)*.* Let X be a path-connected CW-complex with fundamental group G and let $\mathcal{F}$ be a family of subgroups of G. In the absolute case, the generalised Lusternik–Schnirelmann category $\mathrm{cat}_{\mathcal{F}}(X)$ is defined as the minimal n for which there exists an open $\mathcal{F}$-cover of X by n many *not necessarily path-connected* subsets. If the family $\mathcal{F}$ is closed under taking subgroups, this is compatible with Definition 3.43 in the sense that $\mathrm{cat}_{\mathcal{F}}(X,\emptyset) = \mathrm{cat}_{\mathcal{F}}(X)$ by [CLM, Lemma 7.6].

In particular, also the converse estimate of Lemma 3.46 holds in the absolute case: Indeed, taking path-connected components of open $\mathcal{F}$-covers with n not necessarily path-connected members produces an $\mathcal{F}$-cover of multiplicity at most n. Therefore, we obtain $\mathrm{cat}_{\mathcal{F}}(X,\emptyset) = \mathrm{mult}_{\mathcal{F}}(X,\emptyset)$.

If $f\colon (Z,C) \to (X,A)$ is a homotopy equivalence of CW-pairs, then pulling back fundamental group pairs and families of subgroups along the induced map $\pi_1(f)$ shows that $\mathrm{cat}_{\pi_1(f)^*\mathcal{F}}(Z,C) = \mathrm{cat}_{\mathcal{F}}(X,A)$. In contrast, it is not clear whether $\mathrm{mult}_{\mathcal{F}}$ is also a relative homotopy invariant.

3.5 Simplicial volume, bounded cohomology, and acyclicity

In this section we recall the notions of simplicial volume and bounded cohomology. We also recall bounded acyclicity and we introduce a uniform version of bounded acyclicity. In particular, we explain how uniformly boundedly acyclic actions lead to computations of bounded cohomology. This is an adaptation of standard techniques in bounded cohomology [Mon01, MR]; similar results are also discussed in recent computations of bounded cohomology groups [FFLMa, MN].

3.5.1 Simplicial volume

We recall the definition of simplicial volume [Gro82]. Let (X, A) be a pair of spaces. For every singular n-chain $c = \sum_{i=1}^{k} a_i \sigma_i \in C_n(X, A; \mathbb{R})$, written in reduced form, we define the ℓ^1-*norm* as follows:

$$|c|_1 \coloneqq \sum_{i=1}^{k} |a_i| \,.$$

The restriction of the ℓ^1-norm to the subspace of relative cycles induces a quotient ℓ^1-seminorm (denoted by $\|\cdot\|_1$) on the homology group $H_n(X, A; \mathbb{R})$.

Definition 3.48 (Relative simplicial volume)**.** Let M be an oriented connected compact n-manifold with (possibly non-empty) boundary. Then the *relative simplicial volume* of M is

$$\|M, \partial M\| \coloneqq \big\|[M, \partial M]\big\|_1 \,,$$

where $[M, \partial M] \in H_n(M, \partial M; \mathbb{R}) \cong \mathbb{R}$ denotes the relative fundamental class of M.

Example 3.49. Let M be an oriented compact connected n-manifold.

(i) If the interior M° admits a complete finite-volume hyperbolic metric, then we have $\|M, \partial M\| = \mathrm{vol}(M)/v_n$ [Gro82, FM11];

(ii) If M is a handlebody of genus $g \geq 2$, then $\|M, \partial M\| = 3 \cdot (g - 1)$ [BFP15];

(iii) If $M = \Sigma_g \times I$, where Σ_g is a surface of genus $g \geq 2$, then $\|M, \partial M\| = \frac{5}{4} \cdot \|\partial M\|$ [BFP15];

(iv) The simplicial volume of graph manifolds is zero [Som81, Gro82].

(v) If M admits a self-map f with $|\deg(f)| \geq 2$, then $\|M, \partial M\| = 0$ [Gro82].

(vi) If M is closed and admits an open cover by amenable subsets of multiplicity at most $\dim(M)$, then $\|M\| = 0$ [Gro82]. Many examples are known to satisfy this condition [LMS, Section 1.1].

Further computations of simplicial volume are surveyed in the literature [LMR].

We also recall the locally finite version of simplicial volume for non-compact manifolds [Gro82, Löh08, FM]. Given a topological space X, a (possibly infinite) real singular n-chain $c = \sum_{\sigma\in\operatorname{map}(\Delta^n,X)} a_\sigma\sigma$ is *locally finite* if every compact subset of X intersects only finitely many simplices with non-trivial coefficient. We define $C_*^{\mathrm{lf}}(X;\mathbb{R})$ as the $\mathbb{R}$-module of locally finite chains on X. The usual boundary operator for finite chains admits a canonical extension to $C_*^{\mathrm{lf}}(X;\mathbb{R})$. The *locally finite homology* $H_*^{\mathrm{lf}}(X;\mathbb{R})$ of X is the homology of the complex $C_*^{\mathrm{lf}}(X;\mathbb{R})$.

As in the finite case, the ℓ^1-norm of a locally finite chain $c = \sum_{\sigma\in\operatorname{map}(\Delta^n,X)} a_\sigma\sigma$ in $C_n^{\mathrm{lf}}(X;\mathbb{R})$ is given by

$$|c|_1 \coloneqq \sum_{\sigma\in\operatorname{map}(\Delta^n,X)} |a_\sigma| \in [0,+\infty]\,.$$

As before, this norm induces an ℓ^1-seminorm $\|\cdot\|_1$ on $H_n^{\mathrm{lf}}(X;\mathbb{R})$.

Definition 3.50 (Locally finite simplicial volume)**.** Let M be an oriented (possibly non-compact) connected n-manifold without boundary. The *locally finite simplicial volume* of M is defined by

$$\|M\|_{\mathrm{lf}} \coloneqq \big\|[M]_{\mathrm{lf}}\big\|_1\,,$$

where $[M]_{\mathrm{lf}} \in H_n^{\mathrm{lf}}(M;\mathbb{R}) \cong \mathbb{R}$ denotes the locally finite fundamental class.

The [locally finite] simplicial volume can be defined for every normed ring R. In this case, we will consider the ℓ^1-seminorm on $H_*^{[\mathrm{lf}]}(-;R)$ and we will talk about [*locally finite*] *R-simplicial volume* $\|\cdot\|_{R,[\mathrm{lf}]}$.

3.5.2 Bounded cohomology

We recall the definition of bounded cohomology of groups and spaces [Gro82, Iva85, Mon01, Fri17] as well as its equivariant version [LS20, Lib].

For a group G and a normed $\mathbb{R}G$-module V, we write

$$C_b^*(G;V) \coloneqq \ell^\infty(G^{*+1},V)^G$$

(equipped with the simplicial coboundary operator) for the *bounded cochain complex of G with coefficients in V*.

Definition 3.51 (Bounded cohomology of groups)**.** The *bounded cohomology of G with coefficients in V* is defined by

$$H_b^*(G;V) \coloneqq H^*\big(C_b^*(G;V)\big)\,.$$

Similarly, if (X,A) is a topological pair, we can consider the singular cochain complex

$$C^*(X,A;\mathbb{R}) \coloneqq \big\{f\in C^*(X;\mathbb{R}) \bigm| f(\sigma)=0 \text{ for all } \sigma \text{ supported in } A\big\}\,,$$

where a singular n-simplex σ is *supported in* A if $\sigma(\Delta^n) \subset A$. We can restrict to the subcomplex of bounded cochains:

$$C_b^*(X,A;\mathbb{R}) \coloneqq \left\{ f \in C^*(X,A;\mathbb{R}) \;\middle|\; \sup_{\sigma\in \operatorname{map}(\Delta^*,X)} |f(\sigma)| < \infty \right\}.$$

Definition 3.52 (Bounded cohomology of spaces)**.** Let (X, A) be a pair of spaces. The *bounded cohomology of* (X, A) (with real coefficients) is defined by

$$H_b^*(X,A;\mathbb{R}) \coloneqq H^*\big(C_b^*(X,A;\mathbb{R})\big).$$

The inclusion of complexes $C_b^*(X,A;\mathbb{R}) \hookrightarrow C^*(X,A;\mathbb{R})$ induces a natural map from bounded cohomology to ordinary cohomology, the *comparison map*:

$$\operatorname{comp}^*_{(X,A)} \colon H_b^*(X,A;\mathbb{R}) \to H^*(X,A;\mathbb{R}).$$

The connection between bounded cohomology and simplicial volume is encoded in the following classical result:

Proposition 3.53 (Duality principle, qualitative version [Gro82, Fri17])**.** *Let M be an oriented connected compact n-manifold with (possibly empty) boundary. Then the following are equivalent:*

(i) $\|M, \partial M\| > 0$*;*

(ii) *The comparison map* $\operatorname{comp}^n_{(M,\partial M)}$ *is surjective.*

We also recall the equivariant version of bounded cohomology [LS20, Definition 5.1]:

Definition 3.54 (Equivariant [bounded] cohomology)**.** Let Y be a G-space and $C_*(Y;\mathbb{R})$ denote the singular chain complex. For coefficients in a [normed] $\mathbb{R}G$-module V, we define the cochain complex

$$C_G^*(Y;V) \coloneqq \operatorname{Hom}_{\mathbb{R}G}\big(C_*(Y;\mathbb{R}), V\big)$$

and the subcomplex $C^*_{G,b}(Y;V) \subset C^*_G(Y;V)$ consisting of [bounded] $\mathbb{R}G$-homomorphisms. Then we set

$$H^n_G(Y;V) \coloneqq H^n\big(C^*_G(Y;V)\big);$$
$$H^n_{G,b}(Y;V) \coloneqq H^n\big(C^*_{G,b}(Y;V)\big).$$

For a pair of G-spaces (Y, B) one similarly defines $H^n_G(Y,B;V)$ and $H^n_{G,b}(Y,B;V)$. As in the absolute case, there is a *comparison map*

$$\operatorname{comp}^n_{G,(Y,B)} \colon H^n_{G,b}(Y,B;V) \to H^n_G(Y,B;V).$$

We have the following induction isomorphisms:

Lemma 3.55. *Let H be a subgroup of G, let B be an H-space, and let $n \in \mathbb{N}$. Then there are natural isomorphisms of $\mathbb{R}$-vector spaces:*

$$H^n_G(G \times_H B; \mathbb{R}) \xrightarrow{\cong} H^n_H(B; \mathbb{R})\,;$$
$$H^n_{G,b}(G \times_H B; \mathbb{R}) \xrightarrow{\cong} H^n_{H,b}(B; \mathbb{R})\,.$$

Proof. Let $C_*(-)$ denote the singular chain complexes (with real coefficients). Since the induced G-space $G \times_H B$ consists of disjoint copies of B, the image of a singular simplex in $G \times_H B$ is contained in a single copy. Hence we have a natural isomorphism of $\mathbb{R}G$-chain complexes $C_*(G \times_H B) \cong \mathbb{R}G \otimes_{\mathbb{R}H} C_*(B)$ and thus an adjunction isomorphism

$$\Phi \colon C^*_G(G \times_H B; \mathbb{R}) \xrightarrow{\cong} C^*_H(B; \mathbb{R})\,.$$

This yields the isomorphism on equivariant cohomology. The claim on equivariant bounded cohomology follows by observing that Φ restricts to an isomorphism on the subcomplexes of bounded cochains $C^*_{G,b}(G \times_H B; \mathbb{R}) \to C^*_{H,b}(B; \mathbb{R})$. □

3.5.3 Bounded acyclicity

We recall the definition of bounded acyclicity for modules and groups:

Definition 3.56 (Boundedly acyclic group)**.** Let G be a group and let $n \in \mathbb{N}$. A normed $\mathbb{R}G$-module V is *boundedly n-acyclic* if $H^k_b(G; V) \cong 0$ for all $k \in \{1, \dots, n\}$. A normed $\mathbb{R}G$-module V is *boundedly acyclic* if $H^k_b(G; V) \cong 0$ for all $k \in \mathbb{N}_{\geq 1}$.

The group G is *boundedly n-acyclic* [resp. *boundedly acyclic*] if the trivial $\mathbb{R}G$-module $\mathbb{R}$ is boundedly n-acyclic [resp. boundedly acyclic].

Amenable groups are boundedly acyclic; by now, there is a wide range of known examples of non-amenable boundedly acyclic groups, including finitely presented examples [MM85, Löh17, FFLMb, FFLMa, MN, Mon].

Resolutions by boundedly acyclic modules can be used to compute bounded cohomology [MR, Proposition 2.5.4 and Remark 2.5.5]:

Theorem 3.57 (Fundamental lemma for boundedly acyclic resolutions [MR])**.** *Let G be a group and let $n \in \mathbb{N}$. Let $0 \to V \to V^*$ be a resolution of normed $\mathbb{R}G$-modules such that V^j is a dual normed $\mathbb{R}G$-module and boundedly $(n-j)$-acyclic for every $j \in \{0, \dots, n-1\}$. Then there is a canonical isomorphism (of $\mathbb{R}$-vector spaces)*

$$H^k(V^{*G}) \xrightarrow{\cong} H^k_b(G; V)$$

for all $k \in \{0, \dots, n\}$ and a canonical injective map

$$H^{n+1}(V^{*G}) \hookrightarrow H_b^{n+1}(G; V).$$

Moreover, if the given resolution is strong, then these maps are the ones induced by the canonical G-cochain homotopy class $V^* \to \ell^\infty(G^{*+1}, V)$.

3.5.4 Uniform bounded acyclicity

To formulate uniform bounded acyclicity for collections of groups, we need additional control on the norms of primitives. This can be expressed in terms of the uniform boundary condition (see Appendix 3.A for a definition and some properties). More precisely, we use the *uniform uniform boundary condition* denoted by UUBC (Definition 3.108).

Definition 3.58 (Uniformly boundedly acyclic collection of groups)**.** A collection $\mathcal{G}$ of groups is *uniformly boundedly acyclic* if

- all members of $\mathcal{G}$ are boundedly acyclic and
- the collection $(C_b^*(H; \mathbb{R}))_{H \in \mathcal{G}}$ satisfies UUBC^k for all $k \in \mathbb{N}$.

Similarly, for $n \in \mathbb{N}$, we define *uniformly boundedly n-acyclic* collections of groups if the previous conditions are satisfied up to degree n. Moreover, we extend these definitions to sets of groups.

For example, all collections consisting of amenable groups are uniformly boundedly acyclic (Example 3.109). Also, all *finite* collections of boundedly acyclic groups are uniformly boundedly acyclic (Example 3.110).

Proposition 3.59. *Let $n \in \mathbb{N}$, let G be a group, let $(H_i)_{i \in I}$ be a uniformly boundedly n-acyclic collection of subgroups of G, and let $k \in \{1, \dots, n\}$. Then*

$$H_b^k\big(G; \ell^\infty(G/H_i, \mathbb{R})\big) \cong 0$$

for all $i \in I$ and the collection $(C_b^*(G; \ell^\infty(G/H_i, \mathbb{R})))_{i \in I}$ satisfies UUBC^k.

Proof. This is a boundedly controlled version of the Shapiro lemma: By the Shapiro lemma in bounded cohomology [Mon01, Proposition 10.1.3], we have

$$H_b^k\big(G; \ell^\infty(G/H_i, \mathbb{R})\big) \cong H_b^k(H_i; \mathbb{R}) \cong 0$$

for all $i \in I$ and all $k \in \{1, \dots, n\}$. In order to conclude that $(C_b^*(G; \ell^\infty(G/H_i, \mathbb{R})))_{i \in I}$ satisfies UUBC^k, we make the proof of the Shapiro lemma more explicit:

Let $H \subset G$ be a subgroup of G. Then there is a non-empty set J such that G, as an H-space, is isomorphic to $J \times H$ (with the translation action on the H-factor). Therefore, on the one hand, we obtain isometric isomorphisms

$$\begin{aligned} C_b^*\big(G; \ell^\infty(G/H,\mathbb{R})\big) &\cong \ell^\infty\big(G^{*+1}, \ell^\infty(G/H,\mathbb{R})\big)^G \cong \ell^\infty\big(\operatorname{res}_H^G G^{*+1}, \mathbb{R}\big)^H \\ &\cong \ell^\infty\big((J \times H)^{*+1}, \mathbb{R}\big)^H \end{aligned}$$

of cochain complexes (each equipped with the simplicial coboundary operator). On the other hand, $C_b^*(H;\mathbb{R}) = \ell^\infty(H^{*+1},\mathbb{R})^H$. Both sides are connected through mutually homotopy inverse cochain homotopy equivalences

$$\ell^\infty\big(H^{*+1}, \mathbb{R}\big)^H \leftrightarrow \ell^\infty\big((J \times H)^{*+1}, \mathbb{R}\big)^H$$

given by (where $0 \in J$ is a chosen basepoint)

$$\begin{aligned} \varphi^* \colon \ell^\infty\big(H^{*+1}, \mathbb{R}\big)^H &\to \ell^\infty\big((J \times H)^{*+1}, \mathbb{R}\big)^H \\ f &\mapsto \big(((i_0,h_0),\dots,(i_k,h_k)) \mapsto f(h_0,\dots,h_k)\big) \\ \psi^* \colon \ell^\infty\big((J \times H)^{*+1}, \mathbb{R}\big)^H &\to \ell^\infty\big(H^{*+1}, \mathbb{R}\big)^H \\ f &\mapsto \big((h_0,\dots,h_k) \mapsto f\big((0,h_0),\dots,(0,h_k)\big)\big)\,; \end{aligned}$$

these cochain maps have norm 1 in each degree. Indeed, $\psi^* \circ \varphi^*$ is the identity on $\ell^\infty(H^{*+1},\mathbb{R})^H$ and the standard map

$$\begin{aligned} \ell^\infty\big((J \times H)^{*+1}, \mathbb{R}\big)^H &\to \ell^\infty\big((J \times H)^{*}, \mathbb{R}\big)^H \\ f &\mapsto \Big(((i_0,h_0),\dots,(i_{k-1},h_{k-1})) \mapsto \\ &\qquad \sum_{j=0}^{k-1} (-1)^j \cdot f\big((i_0,h_0),\dots,(i_j,h_j),(0,h_j),\dots,(0,h_{k-1})\big)\Big) \end{aligned}$$

is a cochain homotopy between $\varphi^* \circ \psi^*$ and the identity on $\ell^\infty((J\times H)^{*+1},\mathbb{R})^H$, with norm k in degree k. In particular, all these norms are independent of the subgroup H of G.

Hence, the claim follows by applying these considerations and homotopy inheritance of UBC (Proposition 3.102) to the subgroups $(H_i)_{i\in I}$ of G. □

3.5.5 Uniformly boundedly acyclic actions

Group actions with amenable stabilisers, so-called amenable actions, have proved to be a valuable tool to compute bounded cohomology in specific cases [Mon01, BM02, BI09]. Similarly, also uniformly boundedly acyclic actions allow us to compute bounded cohomology. This is an easy application of the fact that bounded cohomology can be computed via acyclic resolutions (Theorem 3.57). However, usually, in this approach we cannot compute the seminorm on bounded cohomology.

Definition 3.60 (Uniformly boundedly acyclic action)**.** A group action on a set is *uniformly boundedly acyclic* if the collection of all stabilisers forms a uniformly boundedly acyclic collection of groups. Similarly, for $n \in \mathbb{N}$, we introduce the notion of *uniformly boundedly n-acyclic actions.*

Uniformly boundedly acyclic actions lead to boundedly acyclic modules:

Proposition 3.61. *Let G be a group, let $n \in \mathbb{N}$, and let $G \curvearrowright S$ be a uniformly boundedly n-acyclic action on a set S. Then, for all $k \in \{1, \dots, n\}$, we have*

$$H_b^k\bigl(G; \ell^\infty(S, \mathbb{R})\bigr) \cong 0\,.$$

Proof. Without loss of generality, we may assume that $S = \coprod_{i \in I} G/H_i$ with the left translation action on each summand. Using the uniform version of the Shapiro lemma (Proposition 3.59) and the compatibility with bounded products (Theorem 3.114), we obtain for every $k \in \{1, \dots, n\}$

$$\begin{aligned}
0 &\cong \prod_{i \in I}^{b} H_b^k\bigl(G; \ell^\infty(G/H_i, \mathbb{R})\bigr) && \text{(Proposition 3.59)}\\
&\cong H^k\Bigl(\prod_{i \in I}^{b} C_b^*\bigl(G; \ell^\infty(G/H_i, \mathbb{R})\bigr)\Bigr) && \text{(Theorem 3.114)}\\
&\cong H^k\Bigl(C_b^*\Bigl(G; \prod_{i \in I}^{b} \ell^\infty(G/H_i, \mathbb{R})\Bigr)\Bigr) && \text{(direct computation)}\\
&\cong H^k\Bigl(C_b^*\Bigl(G; \ell^\infty\Bigl(\coprod_{i \in I} G/H_i, \mathbb{R}\Bigr)\Bigr) && \text{(Example 3.112)}\\
&= H_b^k\bigl(G; \ell^\infty(S, \mathbb{R})\bigr)\,,
\end{aligned}$$

as claimed. □

Corollary 3.62 (Bounded cohomology via uniformly boundedly acyclic actions)**.** *Let G be a group, let $G \curvearrowright S$ be an action on a non-empty set S. Let $n \in \mathbb{N}_{>0}$ and suppose that the diagonal action $G \curvearrowright S^n$ is uniformly boundedly n-acyclic. Then the cohomology of the simplicial cochain complex $\ell^\infty(S^{*+1}, \mathbb{R})^G$ is canonically isomorphic to $H_b^*(G; \mathbb{R})$ in all degrees $\leq n$ and there exists a canonical injective map*

$$H^{n+1}\bigl(\ell^\infty(S^{*+1}, \mathbb{R})^G\bigr) \hookrightarrow H^{n+1}(G; \mathbb{R})\,.$$

More precisely, every G-cochain map $\ell^\infty(S^{*+1}, \mathbb{R}) \to \ell^\infty(G^{*+1}, \mathbb{R})$ that is degree-wise bounded and extends $\mathrm{id}_\mathbb{R} \colon \mathbb{R} \to \mathbb{R}$ induces an isomorphism [resp. injection]

$$H^k(\ell^\infty(S^{*+1}, \mathbb{R})^G) \to H_b^k(G; \mathbb{R})$$

in the corresponding range for k.

Proof. Bounded cohomology can be computed through boundedly acyclic resolutions (Theorem 3.57). As S is non-empty, $\ell^\infty(S^{*+1}, \mathbb{R})$ is a strong resolution of $\mathbb{R}$ by normed $\mathbb{R}G$-modules. Therefore, it suffices to notice that in the present situation the $\mathbb{R}G$-modules $\ell^\infty(S^{k+1}, \mathbb{R})$ are boundedly acyclic (Definition 3.56) for every $k \in \{0, \dots, n-1\}$ by Proposition 3.61. □

Remark 3.63 (Amenable actions). Proposition 3.61 and Corollary 3.62 are analogous to the corresponding results for amenable actions [Mon01][Fri17, Section 4.9]: If the action of G on a set S is amenable, then the normed $\mathbb{R}G$-module $\ell^\infty(S, \mathbb{R})$ is relatively injective and hence the cochain complex $\ell^\infty(S^{*+1}, \mathbb{R})^G$ computes $H^*_b(G; \mathbb{R})$.

Remark 3.64 (Bounded cohomology via alternating cochains). Let G be a group and let $G \curvearrowright S$ be an action on a non-empty set S. A bounded function $f \in \ell^\infty(S^k, \mathbb{R})$ is *alternating* if

$$f\big(s_{\sigma(1)}, \dots, s_{\sigma(k)}\big) = \operatorname{sign}(\sigma) \cdot f(s_1, \dots, s_k)$$

holds for every permutation $\sigma \in \Sigma_k$ and all $(s_1, \dots, s_k) \in S^k$. We write

$$\ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R}) \subset \ell^\infty(S^{*+1}, \mathbb{R})$$

for the subcomplex of alternating functions, which is well-defined since the coboundary operator preserves being alternating.

Let $n \in \mathbb{N}_{>0}$ and suppose that the diagonal action $G \curvearrowright S^n$ is uniformly boundedly n-acyclic. Then also the cohomology of the simplicial cochain complex

$$\ell^\infty_{\mathrm{alt}}(S, \mathbb{R})^G \to \ell^\infty_{\mathrm{alt}}(S^2, \mathbb{R})^G \to \ell^\infty_{\mathrm{alt}}(S^3, \mathbb{R})^G \to \cdots$$

is canonically isomorphic to $H^*_b(G; \mathbb{R})$ in all degrees $\leq n$ and the canonical map

$$H^{n+1}(\ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G) \to H^{n+1}(G; \mathbb{R})$$

is injective. Indeed, by Corollary 3.62, we already know that the previous result holds for the non-alternating complex. Moreover, the inclusion $\ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R}) \hookrightarrow \ell^\infty(S^{*+1}, \mathbb{R})$ induces an isomorphism on cohomology; this can be seen from the same computation as in the case of the complex $\ell^\infty(G^{*+1}, \mathbb{R})$ [Fri17, Proposition 4.26].

We conclude this section by showing that the computation of bounded cohomology via alternating cochains of boundedly acyclic actions is natural in the following sense. This is analogous to the case of amenable actions [BBF+14, Lemma 2.2].

Lemma 3.65. *Let $i\colon H \to G$ be a group homomorphism. Let $H \curvearrowright S_H$ and $G \curvearrowright S_G$ be actions on non-empty sets S_H and S_G, respectively. Let $\varphi\colon S_H \to S_G$ be an i-equivariant map.*

(i) *Then the following diagram commutes, where the horizontal arrows are the canonical maps (induced by restriction to a single orbit):*

$$\begin{array}{ccc} H^*\big(\ell^\infty_{\mathrm{alt}}(S_G^{*+1},\mathbb{R})^G\big) & \longrightarrow & H_b^*(G;\mathbb{R}) \\ \big\downarrow {\scriptstyle H^*(\varphi^{*+1})} & & \big\downarrow {\scriptstyle H_b^*(i)} \\ H^*\big(\ell^\infty_{\mathrm{alt}}(S_H^{*+1},\mathbb{R})^H\big) & \longrightarrow & H_b^*(H;\mathbb{R})\,. \end{array}$$

(ii) *Let $n \in \mathbb{N}_{>0}$ and suppose that the diagonal actions $H \curvearrowright S_H^n$ and $G \curvearrowright S_G^n$ are uniformly boundedly n-acyclic. Then the horizontal arrows are isomorphisms in all degrees $\leq n$ and injective in degree $n+1$.*

Proof. Part (ii) is shown in Remark 3.64.

Part (i) is a straightforward computation: As S_H and S_G are non-empty, we can choose a point $x_H \in S_H$ and set $x_G := \varphi(x_H)$. The orbit maps $\psi_H\colon H \to S_H$ for x_H and $\psi_G\colon G \to S_G$ for x_G induce cochain maps extending $\mathrm{id}_\mathbb{R}\colon \mathbb{R} \to \mathbb{R}$ and therefore induce the canonical maps $H^*(\ell^\infty_{\mathrm{alt}}(S_H^{*+1},\mathbb{R})^H) \to H_b^*(H;\mathbb{R})$ and $H^*(\ell^\infty_{\mathrm{alt}}(S_G^{*+1},\mathbb{R})^G) \to H_b^*(G;\mathbb{R})$, respectively. Because φ is i-equivariant and because $\varphi(x_H) = x_G$, the diagram

$$\begin{array}{ccc} \ell^\infty_{\mathrm{alt}}(S_G^{*+1},\mathbb{R})^G & \xrightarrow{\psi_G^{*+1}} & C_b^*(G;\mathbb{R}) \\ {\scriptstyle \varphi^{*+1}}\big\downarrow & & \big\downarrow {\scriptstyle C_b^*(i;\mathbb{R})} \\ \ell^\infty_{\mathrm{alt}}(S_H^{*+1},\mathbb{R})^H & \xrightarrow[\psi_H^{*+1}]{} & C_b^*(H;\mathbb{R})\,, \end{array}$$

commutes. Taking cohomology proves part (i). □

3.6 A vanishing theorem for relative simplicial volume

We prove vanishing theorems for the comparison map and for relative simplicial volume in the presence of uniformly boundedly acyclic open covers with small multiplicity, as outlined in Section 3.1. The proof uses equivariant nerve pairs and equivariant bounded cohomology with respect to families of boundedly acyclic subgroups.

3.6.1 Uniformly boundedly acyclic open covers

We introduce the notion of uniformly boundedly acyclic open covers.

Definition 3.66 (Families associated to a set of subspaces)**.** Let X be a path-connected space with $\pi_1(X) = G$ and $\mathcal{A}$ be a set of path-connected subspaces of X. Consider the set

$$\mathcal{G} := \big\{\mathrm{im}\big(\pi_1(A \hookrightarrow X)\big) \;\big|\; A \in \mathcal{A}\big\}$$

of subgroups of G; again, we implicitly use the convention on basepoints (Section 3.1). Then the *intersection-closed family of subgroups of G associated to $\mathcal{A}$* is defined as

$$\mathcal{F}\langle\mathcal{A}\rangle := \mathcal{F}\langle\mathcal{G}\rangle = \left\{ \bigcap_{i=1}^{n} g_i H_i g_i^{-1} \;\middle|\; n \in \mathbb{N}, H_i \in \mathcal{G}, g_i \in G \right\}.$$

For fixed $n \in \mathbb{N}$, we define the (conjugation-closed) family of subgroups of G associated to $\mathcal{A}$ as

$$\mathcal{F}_n\langle\mathcal{A}\rangle := \mathcal{F}_n\langle\mathcal{G}\rangle = \left\{ \bigcap_{i=1}^{n} g_i H_i g_i^{-1} \;\middle|\; H_i \in \mathcal{G}, g_i \in G \right\}.$$

Using the notion of uniformly boundedly acyclic collections of groups (Definition 3.58), we define the following:

Definition 3.67 (Uniformly boundedly acyclic set of subspaces)**.** Let X be a path-connected space and $\mathcal{A}$ be a set of path-connected subspaces of X. We say that $\mathcal{A}$ is *uniformly boundedly acylic* [*of order n*] *in X* if the associated family $\mathcal{F}\langle\mathcal{A}\rangle$ [resp. $\mathcal{F}_n\langle\mathcal{A}\rangle$] is uniformly boundedly acyclic.

Definition 3.68 (Uniformly boundedly acyclic open cover)**.** Let X be a path-connected space and $\mathcal{U}$ be a cover of X by path-connected open subsets. We say that $\mathcal{U}$ is *uniformly boundedly acyclic* if it is uniformly boundedly acyclic in X when viewed as a set of subspaces of X.

By Example 3.109, every amenable cover is uniformly boundedly acyclic.

Remark 3.69. The above notion of uniformly boundedly acyclic open covers is similar to Ivanov's notion of weakly boundedly acyclic open covers [Iva, Section 4]. The difference is that we consider intersections of subgroups of the fundamental group whereas Ivanov considers fundamental groups of intersections of subspaces. The key steps of our arguments happen on the level of bounded cohomology. In contrast, Ivanov's arguments via spectral sequences target the comparison map more directly. It is not clear to us whether one of the concepts contains the other.

3.6.2 Strong H_b^*-admissibility

We show that classifying spaces with respect to uniformly boundedly acyclic families can be used to compute bounded cohomology. This is phrased in terms of equivariant bounded cohomology (Definition 3.54).

Definition 3.70 (Strongly H_b^*-admissible family)**.** Let G be a group and let $\mathcal{F}$ be a (conjugation-closed) family of subgroups of G that contains the trivial subgroup 1. Since $1 \in \mathcal{F}$, there is a canonical (up to G-homotopy) G-map $f\colon EG \to E_{\mathcal{F}}G$. The family $\mathcal{F}$ is *strongly H_b^*-admissible* if the induced map

$$H_{G,b}^*(f;\mathbb{R})\colon H_{G,b}^*(E_{\mathcal{F}}G;\mathbb{R}) \to H_{G,b}^*(EG;\mathbb{R}) \cong H_b^*(G;\mathbb{R})$$

is bijective.

Definition 3.70 is a slight generalisation of the original definition [LS20, Definition 5.1], where only families are considered that are closed under taking subgroups.

Proposition 3.71. *Let G be a group and let $\mathcal{F}$ be an intersection-closed family of subgroups of G that is uniformly boundedly acyclic and contains the trivial subgroup. Then $\mathcal{F}$ is strongly H_b^*-admissible.*

Proof. We use Proposition 3.61 and the fact that bounded cohomology can be computed using acyclic resolutions (Theorem 3.57). Let $E_{\mathcal{F}}G$ be a model for the classifying space of G with respect to the family $\mathcal{F}$. We show that the chain complex $C_b^*(E_{\mathcal{F}}G;\mathbb{R})$ together with the canonical augmentation $\mathbb{R} \to C_b^0(E_{\mathcal{F}}G;\mathbb{R})$ is a boundedly acylic resolution of $\mathbb{R}$ over G: As $1 \in \mathcal{F}$, the space $E_{\mathcal{F}}G$ is contractible (Theorem 3.13). Therefore, $C_b^*(E_{\mathcal{F}}G;\mathbb{R})$ is a resolution of $\mathbb{R}$.

Moreover, for each $n \in \mathbb{N}$, the Banach G-module $C_b^n(E_{\mathcal{F}}G;\mathbb{R})$ is boundedly acyclic: By definition, $C_b^n(E_{\mathcal{F}}G;\mathbb{R}) = \ell^\infty(S,\mathbb{R})$ with $S := \operatorname{map}(\Delta^n, E_{\mathcal{F}}G)$. In view of Proposition 3.61, it thus suffices to show that the stabilisers of the G-action on S lie in the uniformly boundedly acyclic collection $\mathcal{F}$. If $\sigma\colon \Delta^n \to E_{\mathcal{F}}G$ is a singular simplex, then $\sigma(\Delta^n)$ meets only finitely many cells of $E_{\mathcal{F}}G$. Therefore, the stabiliser of σ is an intersection of finitely many elements of $\mathcal{F}$ and thus lies in $\mathcal{F}$.

Let $f\colon EG \to E_{\mathcal{F}}G$ be the canonical (up to G-homotopy) G-map. Then the fundamental lemma for boundedly acyclic resolutions (Theorem 3.57) shows that the cochain map

$$C_b^*(f;\mathbb{R})^G\colon C_b^*(E_{\mathcal{F}}G;\mathbb{R})^G \to C_b^*(EG;\mathbb{R})^G$$

induces an isomorphism in bounded cohomology. □

Corollary 3.72 ([LS20, Proposition 5.2])**.** *Every intersection-closed family $\mathcal{F}$ that consists of amenable groups and contains the trivial subgroup is strongly H_b^*-admissible.* □

3.6.3 A relative vanishing theorem

In this section, we prove the relative vanishing theorem by making use of the relative equivariant setting developed in the previous sections. This result extends the classical vanishing theorem by Ivanov [Iva85, Iva] to the relative setting. By now there are several alternative proofs of Ivanov's result [FM, LS20, Rap]; moreover, in the case of aspherical manifolds, the vanishing of simplicial volume can also be obtained directly through the amenable reduction lemma [AK16, FM, LMS]. We will follow the approach via classifying spaces by Löh and Sauer [LS20].

First, we prove a vanishing result for the comparison map in terms of the relative generalised LS-category (Definition 3.43).

Setup 3.73 (CW-pair with a family of subgroups)**.** Let (X, A) be a CW-pair with X path-connected and let A have only finitely many connected components.

- We suppose that the inclusion of every component of A into X is π_1-injective and we let $(G, \mathcal{H} = (H_i)_{i\in I})$ be a fundamental group pair for (X, A) (Example 3.17);
- Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup;
- For every $i \in I$, let $\mathcal{F}|_{H_i}$ be the restricted family of subgroups of H_i (Example 3.10 (v));
- Let $p\colon \widetilde{X} \to X$ be the universal covering of X.

Lemma 3.74. *In the situation of Setup 3.73, suppose that the family $\mathcal{F}$ of subgroups of G is strongly H_b^*-admissible and that for every $i \in I$ the family $\mathcal{F}|_{H_i}$ of subgroups of H_i is strongly H_b^*-admissible.*

Then the canonical map $f\colon (\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{F}}(G, \mathcal{H})$ of $(G, \mathcal{H})$-CW-pairs induces an isomorphism in equivariant bounded cohomology:

$$H^*_{G,b}(f)\colon H^*_{G,b}\bigl(E_{\mathcal{F}}(G,\mathcal{H});\mathbb{R}\bigr) \xrightarrow{\cong} H^*_{G,b}\bigl(\widetilde{X}, p^{-1}(A);\mathbb{R}\bigr)\,.$$

In particular, the comparison map $\operatorname{comp}^k_{(X,A)}\colon H_b^k(X, A;\mathbb{R}) \to H^k(X, A;\mathbb{R})$ vanishes in all degrees $k \geq \operatorname{cat}_{\mathcal{F}}(X, A)$.

Proof. By Example 3.19, $(\widetilde{X}, p^{-1}(A))$ is a $(G, \mathcal{H})$-CW-pair with isotropy in the trivial family $\mathcal{TR}$. Thus, since $\mathcal{F}$ contains the trivial subgroup, the G-map f can be factored (up to G-homotopy) as

$$\bigl(\widetilde{X}, p^{-1}(A)\bigr) \xrightarrow{f_1} E_{\mathcal{TR}}(G,\mathcal{H}) \xrightarrow{f_2} E_{\mathcal{F}}(G,\mathcal{H})\,.$$

Using the long exact sequence for pairs in equivariant bounded cohomology together with the induction isomorphism (Lemma 3.55), we have the following: The induced map $H^*_{G,b}(f_1)$ is a (not necessarily isometric) isomorphism by the mapping theorem [Gro82] and the five lemma. The map $H^*_{G,b}(f_2)$ is an isomorphism by the five lemma and the assumption that the families $\mathcal{F}$ and $\mathcal{F}|_{H_i}$ are strongly H_b^*-admissible. Together, this yields the desired isomorphism.

Moreover, by naturality of the comparison map we have a commutative diagram, where we omit the trivial $\mathbb{R}$-coefficients:

$$\begin{array}{ccccc} H^k_{G,b}\big(E_{\mathcal{F}}(G,\mathcal{H})\big) & \xrightarrow[\cong]{H^k_{G,b}(f)} & H^k_{G,b}\big(\widetilde{X},p^{-1}(A)\big) & \xleftarrow[\cong]{H^*_b(p)} & H^k_b(X,A) \\ \Big\downarrow{\scriptstyle \mathrm{comp}^k_{G,E_{\mathcal{F}}(G,\mathcal{H})}} & & \Big\downarrow{\scriptstyle \mathrm{comp}^k_{G,(\widetilde{X},p^{-1}(A))}} & & \Big\downarrow{\scriptstyle \mathrm{comp}^k_{(X,A)}} \\ H^k_G\big(E_{\mathcal{F}}(G,\mathcal{H})\big) & \xrightarrow[H^k_G(f)]{} & H^k_G\big(\widetilde{X},p^{-1}(A)\big) & \xleftarrow[H^*(p)]{\cong} & H^k(X,A)\,. \end{array}$$

The claim follows, since the map $H^k_G(f)$ is trivial in all degrees $k \geq \mathrm{cat}_{\mathcal{F}}(X,A)$ (Remark 3.44). □

Second, we use the relation between relative category and relative multiplicity to derive a vanishing theorem via relative open covers:

Setup 3.75 (Relative $\mathcal{F}$-cover and its nerves)**.** In the situation of Setup 3.73, we additionally consider a relative $\mathcal{F}$-cover $\mathcal{U}$ of (X,A) (Definition 3.41). We will use the notions of relative multiplicity (Definition 3.35) and weak convexity (Definition 3.41) of relative covers. Moreover, we fix the following notation:

- Let $\widetilde{\mathcal{U}}$ be the lifted G-invariant cover of $\widetilde{X}$ (Example 3.40);
- Let $\nu\colon \big(X,A\big) \to \big(|N(\mathcal{U})|, |N_A(\mathcal{U})|\big)$ be a nerve map;
- Let $\widetilde{\nu}\colon \big(\widetilde{X},p^{-1}(A)\big) \to \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big)$ be a corresponding nerve map of $(G,\mathcal{H})$-CW-pairs (Proposition 3.42).

The following theorem is a relative version of the vanishing theorem for strongly H^*_b-admissible families [LS20, Theorem 5.3].

Theorem 3.76 (Relative vanishing theorem)**.** *In the situation of Setups 3.73 and 3.75, suppose that the family $\mathcal{F}$ of subgroups of G is strongly H^*_b-admissible and that for all $i \in I$ the family $\mathcal{F}|_{H_i}$ of subgroups of H_i is strongly H^*_b-admissible.*

Then the comparison map comp^*_G factors through the equivariant nerve map $\widetilde{\nu}$:

$$\begin{array}{ccc} H^*_{G,b}(\widetilde{X},p^{-1}(A);\mathbb{R}) & \xrightarrow{\mathrm{comp}^*_G} & H^*_G(\widetilde{X},p^{-1}(A);\mathbb{R}) \\ & \searrow & \Big\uparrow{\scriptstyle H^*_G(\widetilde{\nu};\mathbb{R})} \\ & & H^*_G(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|;\mathbb{R})\,. \end{array}$$

In particular, the following hold:

(i) *If $\mathcal{U}$ is weakly convex, then the comparison map*

$$\mathrm{comp}^k\colon H^k_b(X,A;\mathbb{R}) \to H^k(X,A;\mathbb{R})$$

vanishes in all degrees $k \geq \operatorname{mult}_A(\mathcal{U})$*;*

(ii) *If* $\mathcal{U}$ *is convex, then the comparison map* comp^* *factors through the nerve map* ν*:*

$$\begin{array}{ccc} H_b^*(X,A;\mathbb{R}) & \xrightarrow{\operatorname{comp}^*} & H^*(X,A;\mathbb{R}) \\ & \searrow & \uparrow{\scriptstyle H^*(\nu;\mathbb{R})} \\ & & H^*(|N(\mathcal{U})|,|N_A(\mathcal{U})|;\mathbb{R}) \,. \end{array}$$

Proof. Let $E_{\mathcal{F}}(G,\mathcal{H})$ be a model for the classifying space of $(G,\mathcal{H})$ with respect to the family $\mathcal{F}$ (Lemma 3.21). By Proposition 3.42, $(|N(\widetilde{\mathcal{U}})|,|N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G,\mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$. By Example 3.19, $(\widetilde{X},p^{-1}(A))$ is a $(G,\mathcal{H})$-CW-pair with isotropy in the trivial family $\mathcal{TR}$. The universal property of $E_{\mathcal{F}}(G,\mathcal{H})$ yields that the canonical map $f\colon (\widetilde{X},p^{-1}(A)) \to E_{\mathcal{F}}(G,\mathcal{H})$ of $(G,\mathcal{H})$-CW-pairs factors (up to G-homotopy) through the equivariant nerve map $\widetilde{\nu}$:

$$\big(\widetilde{X},p^{-1}(A)\big) \xrightarrow{\widetilde{\nu}} \big(|N(\widetilde{\mathcal{U}})|,|N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \xrightarrow{\varphi} E_{\mathcal{F}}(G,\mathcal{H})\,.$$

Hence, we have the following commutative diagram, where the trivial coefficient module $\mathbb{R}$ is omitted from the notation.

$$\begin{array}{ccccc} H_b^*(X,A) & \xrightarrow{\operatorname{comp}^*} & H^*(X,A) & \xleftarrow{H^*(\nu)} & H^*\big(|N(\mathcal{U})|,|N_A(\mathcal{U})|\big) \\ {\scriptstyle H_b^*(p)}\downarrow\cong & & {\scriptstyle H^*(p)}\downarrow\cong & & \downarrow{\scriptstyle H^*(|N(p)|)} \\ H_{G,b}^*\big(\widetilde{X},p^{-1}(A)\big) & \xrightarrow{\operatorname{comp}_G^*} & H_G^*\big(\widetilde{X},p^{-1}(A)\big) & \xleftarrow{H_G^*(\widetilde{\nu})} & H_G^*\big(|N(\widetilde{\mathcal{U}})|,|N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \\ {\scriptstyle H_{G,b}^*(f)}\uparrow\cong & & {\scriptstyle H_G^*(f)}\uparrow & \nearrow{\scriptstyle H_G^*(\varphi)} & \\ H_{G,b}^*\big(E_{\mathcal{F}}(G,\mathcal{H})\big) & \xrightarrow{\operatorname{comp}_{G,E_{\mathcal{F}}(G,\mathcal{H})}^*} & H_G^*\big(E_{\mathcal{F}}(G,\mathcal{H})\big) & & \end{array}$$

The map $H_{G,b}^*(f)$ is an isomorphism by Lemma 3.74. Therefore, the desired factorisation of the comparison map comp_G^* is given by

$$\operatorname{comp}_G^* = H_G^*(\widetilde{\nu}) \circ H_G^*(\varphi) \circ \operatorname{comp}_{G,E_{\mathcal{F}}(G,\mathcal{H})}^* \circ H_{G,b}^*(f)^{-1}\,.$$

(i) If $\mathcal{U}$ is weakly convex, then we have $\operatorname{cat}_{\mathcal{F}}(X,A) \leq \operatorname{mult}_A(\mathcal{U})$ (Lemma 3.46) and we conclude by Lemma 3.74. More explicitly, let $n = \operatorname{mult}_A(\mathcal{U})$. Then, by Proposition 3.42 (i), we have $\dim(|N(\widetilde{\mathcal{U}})|,|N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|) = n-1$ and hence $H_G^k(|N(\widetilde{\mathcal{U}})|,|N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|) = 0$ for $k \geq n$. This shows that comp^k vanishes in every degree $k \geq n$.

(ii) If $\mathcal{U}$ is convex, then the map $H^*(|N(p)|)$ is an isomorphism by Proposition 3.42 (ii). Hence the desired factorisation of the comparison map comp^* is given by

$$\operatorname{comp}^* = H^*(\nu) \circ H^*(|N(p)|)^{-1} \circ H_G^*(\varphi) \circ \operatorname{comp}_{G,E_{\mathcal{F}}(G,\mathcal{H})}^* \circ H_{G,b}^*(f)^{-1} \circ H_b^*(p)\,.$$

This proves the statement. $\square$

Remark 3.77 (H_b^*-admissible families). A family $\mathcal{F}$ of subgroups of G containing the trivial subgroup is called (not necessarily strongly) *H_b^*-admissible* if the canonical G-map $EG \to E_{\mathcal{F}}G$ induces a surjective map in equivariant bounded cohomology in all degrees [LS20, Definition 5.1].

The conclusions of Theorem 3.76 hold more generally, if in the situation of Setup 3.75 the family $\mathcal{F}$ of subgroups of G is only assumed to be (not necessarily strongly) H_b^*-admissible, while the families $\mathcal{F}|_{H_i}$ of subgroups of H_i are still assumed to be strongly H_b^*-admissible. Indeed, similarly to Lemma 3.74, in this case it follows from the four lemma for epimorphisms that the canonical G-map $(\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{F}}(G, \mathcal{H})$ induces a surjective map in equivariant bounded cohomology in all degrees. This suffices to carry out the above proof of Theorem 3.76.

As an application of the previous theorem, we can deduce the vanishing theorem for uniformly boundedly acyclic open covers (Definition 3.68), which complements a recent result by Ivanov [Iva].

Corollary 3.78 (Relative vanishing theorem for uniformly boundedly acyclic covers)**.** *In the situation of Setup 3.73 and Setup 3.75, suppose that the relative cover $\mathcal{U}$ of (X, A) is uniformly boundedly acyclic, viewed as an open cover of X.*

Then all statements in Theorem 3.76 hold.

Proof. We take $\mathcal{F}$ to be the family $\mathcal{F}\langle\mathcal{U}\rangle \cup \{1\}$ of subgroups of G. Since the cover $\mathcal{U}$ is uniformly boundedly acyclic, the family $\mathcal{F}$ is uniformly boundedly acyclic and hence strongly H_b^*-admissible (Proposition 3.71). As subsets of uniformly boundedly acyclic sets of groups are again uniformly boundedly acyclic, the restricted families $\mathcal{F}|_{H_i}$ of subgroups of H_i are strongly H_b^*-admissible for every $i \in I$. Thus, Theorem 3.76 applies and yields the thesis. □

Remark 3.79 (ℓ^1-Homology). In view of *strong* H_b^*-admissibility, the analogues of Theorem 3.76 and Corollary 3.78 for ℓ^1-homology also hold. One can simply argue through duality as in the absolute case [LS20, Section 6].

In particular, Theorem 3.76 gives a relative vanishing theorem in the presence of "small" relative amenable multiplicity (Definition 3.45).

Corollary 3.80 (Relative vanishing theorem for amenable covers)**.** *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A consists of finitely many connected components, each of which is π_1-injective in X. Then the comparison map*

$$\operatorname{comp}^k_{(X,A)} \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \operatorname{mult}_{\mathcal{AME}}(X, A)$.

In particular, if $(M, \partial M)$ *is an oriented compact connected triangulable manifold with (possibly empty)* π_1*-injective boundary and* $\operatorname{mult}_{\mathcal{AME}}(M, \partial M) \leq \dim(M)$*, then the relative simplicial volume* $\|M, \partial M\|$ *vanishes.*

Proof. The claim on the comparison map follows from the strong H_b^*-admissibility of the family $\mathcal{AME}$ (Corollary 3.72).

Because $(M, \partial M)$ is triangulable, both M and ∂M admit compatible triangulations; in particular, we can view $(M, \partial M)$ as a CW-pair. Therefore, the statement on relative simplicial volume follows from the duality principle (Proposition 3.53). □

Remark 3.81 (Optimality of assumptions). Let Σ denote the surface of genus one with one boundary component. Since the interior of Σ admits a complete hyperbolic metric with finite volume, we know that $\|\Sigma, \partial\Sigma\| > 0$ (Example 3.49 (i)). This shows that $\operatorname{mult}_{\mathcal{AME}}(\Sigma, \partial\Sigma) = 3$. However, when either one of the conditions (RC1), (RC2), or weak convexity on the open cover is dropped, then it is not difficult to construct amenable covers of $(\Sigma, \partial\Sigma)$ with relative multiplicity at most 2. In this sense, our set of assumptions in Corollary 3.80 is optimal.

3.6.4 Amenable covers with small multiplicity on the boundary

We compare Corollary 3.80 with existing results in the literature [LMR, Section 3.3]. The main available relative vanishing result is the following, which is based on Gromov's vanishing theorem for non-compact manifolds [Gro82][FM, Corollary 11]:

Theorem 3.82 ([LMR, Theorem 3.13]). *Let* $(M, \partial M)$ *be an oriented compact connected* n*-manifold with non-empty boundary. Let* $\mathcal{U}$ *be an open cover of* M *and let* $\mathcal{U}|_{\partial M}$ *denote the restriction of* $\mathcal{U}$ *to* ∂M*, i.e.,* $\mathcal{U}|_{\partial M} := \{U \cap \partial M \mid U \in \mathcal{U}\}$*. Suppose that the following hold:*

(i) $\operatorname{mult}(\mathcal{U}) \leq n$*;*

(ii) $\operatorname{mult}(\mathcal{U}|_{\partial M}) \leq n - 1$*;*

(iii) *The open covers* $\mathcal{U}$ *of* M *and* $\mathcal{U}|_{\partial M}$ *of* ∂M *are amenable.*

Then $\|M, \partial M\| = 0$.

We emphasise that (contrary to our convention) here the restricted cover $\mathcal{U}|_{\partial M}$ of ∂M may consist of disconnected subsets.

Since this result neither assumes that the manifold M is triangulable, nor that the boundary inclusion is π_1-injective (in the sense of Section 3.1), while our Corollary 3.80 does so, we cannot recover Theorem 3.82 in full generality. However, in the special situation of triangulable manifolds with π_1-injective boundary, we can provide a simplified proof that does not make use of Gromov's theory of diffusion of chains:

Proof of Theorem 3.82 for triangulable manifolds with π_1-injective boundary.
We show that there exists a weakly convex relative $\mathcal{AME}$-cover $\mathcal{V}$ of $(M, \partial M)$ such that $\operatorname{mult}_{\partial M}(\mathcal{V}) \leq n = \dim(M)$. This implies that $\operatorname{mult}_{\mathcal{AME}}(M, \partial M) \leq n$ and thus by Corollary 3.80 the thesis.

Let $m \coloneqq \{\operatorname{mult}(\mathcal{U}), \operatorname{mult}(\mathcal{U}|_{\partial M})+1\}$. By conditions (i) and (ii), we have $m \leq n$. From $\mathcal{U}$ we will construct a new cover $\mathcal{V}$ of $(M, \partial M)$ that is a weakly convex relative $\mathcal{AME}$-cover with $\operatorname{mult}_{\partial M}(\mathcal{V}) \leq m$. To this end, we follow a classical strategy for modifying open covers (while controlling the multiplicity) in the case of compact manifolds with boundary [FM, proof of Theorem 11.2.3][LS09a, proof of Theorem 5.3].

By compactness, we may assume that $\mathcal{U}$ is finite, say $\mathcal{U} = \{U_1, \dots, U_k\}$. Since ∂M is collared in M, we have the following identification:

$$(M, \partial M) \cong (M', \partial M') \coloneqq \big(M \cup_{\partial M \cong (\partial M \times \{0\})} (\partial M \times [0,1]), \partial M \times \{1\}\big)\,.$$

Let $\varepsilon \coloneqq 1/3(k+1)$ and $t_i \coloneqq i/(k+1)$. Moreover, for $i \in \{1, \dots, k\}$, let $\mathcal{U}(i)$ denote the set of connected components of $U_i \cap \partial M$. We set for every $i \in \{1, \dots, k\}$ such that $U_i \cap \partial M \neq \emptyset$

$$U_i' \coloneqq U_i \cup \big((U_i \cap \partial M) \times [0, t_i + \varepsilon)\big) \subset M'$$

and

$$\mathcal{U}'(i) \coloneqq \big\{U \times (t_i - \varepsilon, 1] \bigm| U \in \mathcal{U}(i)\big\}\,.$$

When $U_i \cap \partial M = \emptyset$, we just set $U_i' \coloneqq U_i$ and $\mathcal{U}'(i) \coloneqq \emptyset$. This produces a new open amenable cover

$$\mathcal{U}' \coloneqq \big\{U_1', \dots, U_k'\big\} \cup \mathcal{U}'(1) \cup \dots \cup \mathcal{U}'(k)$$

of M'.

Finally, we obtain $\mathcal{V}$ from $\mathcal{U}'$ by discarding all sets of the form $U \times (t_i - \varepsilon, 1]$ with $U \in \mathcal{U}(j)$ for some $j > i$. Then $\mathcal{V}$ is an amenable open cover of M' and a straightforward case analysis shows that $\operatorname{mult}(\mathcal{V}) \leq m$. In particular, this implies that $\mathcal{V}$ satisfies $\operatorname{mult}_{\partial M'}(\mathcal{V}) \leq \operatorname{mult}(\mathcal{V}) \leq m \leq n$ by assumptions (i) and (ii).

We are left to show that $\mathcal{V}$ satisfies all the conditions in Definition 3.41. The key observation is the following: Only sets $V \in \mathcal{V}$ of the form $V = U \times (t_i - \varepsilon, 1]$ with $U \in \mathcal{U}(i)$ intersect $\partial M'$. In particular, if $V \in \mathcal{V}$ satisfies $V \cap \partial M' \neq \emptyset$, then $V \cap \partial M'$ is connected. Thus, (RC1) is satisfied. Moreover, for the same reason every $V \in \mathcal{V}$ with $V \cap \partial M' \neq \emptyset$ deformation retracts onto $V \cap \partial M'$, whence (RC2) holds. It remains to show that the relative cover $\mathcal{V}$ is weakly convex. If $V_1, \dots, V_j \in \mathcal{V}$ satisfy $V_1 \cap \dots \cap V_j \cap \partial M' \neq \emptyset$, then

$V_1 \cap \dots \cap V_j$ is of the form $(V_1 \cap \dots \cap V_j \cap \partial M') \times (r, 1]$ for some $r \in (0, 1)$. In particular, each component of $V_1 \cap \dots \cap V_j$ intersects $\partial M'$. Hence, $\mathcal{V}$ is weakly convex.

We conclude that $\mathcal{V}$ is a weakly convex relative $\mathcal{AME}$-cover of $(M', \partial M')$ satisfying $\operatorname{mult}_{\partial M'}(\mathcal{V}) \leq n$. Using the identification of $(M', \partial M')$ with $(M, \partial M)$, we get the thesis. □

We conclude this section by showing that for general CW-pairs (X, A) such that the inclusion of A into X is π_1-injective the hypotheses of Corollary 3.80 are weaker than the ones of Theorem 3.82:

Example 3.83. Let $n \geq 3$. Let M be an oriented closed connected hyperbolic $(n-1)$-manifold and denote $I = [0, 1]$. We consider the CW-pair $(M \times I, M \times \{1\})$. Since $M \times \{1\} \cong M$ is hyperbolic, we have $\|M \times \{1\}\| > 0$ [Thu, Gro82]. Hence, $M \times \{1\}$ cannot admit an open amenable cover with multiplicity at most $n - 1$. This shows that there is no open amenable cover of M whose restriction to $M \times \{1\}$ is both amenable and with multiplicity at most $n - 1$.

On the other hand, it is easy to construct a weakly convex relative $\mathcal{AME}$-cover $\mathcal{V}$ of $(M \times I, M \times \{1\})$ with $\operatorname{mult}_{M \times \{1\}}(\mathcal{V}) \leq n$. Let $\mathcal{U}$ be the open star cover of $M \times \{1\}$ and let $\mathcal{V}$ be the cover of $M \times I$ defined as follows:

$$\mathcal{V} \coloneqq \{U \times I \mid U \in \mathcal{U}\} .$$

Since by construction $\mathcal{V}$ consists of contractible sets, $\mathcal{V}$ is an amenable open cover. Moreover, each member of $\mathcal{V}$ intersects $M \times \{1\}$ in a contractible, whence connected, set. The same argument also applies to multiple intersections. Finally, since each element V in $\mathcal{V}$ is a product $U \times I$ with $U \in \mathcal{U}$, it follows that V retracts by deformation onto $V \cap (M \times \{1\})$. This shows that $\mathcal{V}$ is a weakly convex relative $\mathcal{AME}$-cover of $(M \times I, M \times \{1\})$.

Since by construction the relative multiplicity of $\mathcal{V}$ is zero, we have obtained our desired cover.

We do not know whether the previous example can be improved to the situation of compact manifolds with π_1-injective boundary.

3.7 A vanishing theorem for relative ℓ^2-Betti numbers

We prove a vanishing theorem for the relative ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable multiplicity using equivariant nerve pairs. In the absolute case for ℓ^2-Betti numbers, more sophisticated arguments involving nerves have previously been used by Sauer [Sau09]. For further background on ℓ^2-Betti numbers we refer to the literature [Lüc02, Kam19]. The results of this section are not used in the rest of the article.

We use the following (non-standard) notation:

Definition 3.84 (ℓ^2-Homology and ℓ^2-Betti numbers)**.** Let Y be a G-space. The *ℓ^2-homology* $H_*^{(2)}(G \curvearrowright Y)$ is defined as the singular homology of Y with twisted coefficients in the group von Neumann algebra $\mathcal{N}G$, that is the $\mathcal{N}G$-module

$$H_*^{(2)}(G \curvearrowright Y) \coloneqq H_*^G(Y; \mathcal{N}G) .$$

The *n-th ℓ^2-Betti number* $b_n^{(2)}(G \curvearrowright Y) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ is

$$b_n^{(2)}(G \curvearrowright Y) \coloneqq \dim_{\mathcal{N}G}\bigl(H_n^{(2)}(G \curvearrowright Y)\bigr) ,$$

where $\dim_{\mathcal{N}G}$ is the von Neumann dimension function. For a pair (Y, B) of G-spaces, one similarly defines $H_*^{(2)}(G \curvearrowright (Y, B))$ and $b_*^{(2)}(G \curvearrowright (Y, B))$.

The ℓ^2-Betti numbers $b_*^{(2)}(G)$ of a group G are defined as

$$b_*^{(2)}(G) \coloneqq b_*^{(2)}(G \curvearrowright EG) .$$

We say that G is *ℓ^2-acyclic* if $b_k^{(2)}(G) = 0$ for all $k > 0$.

For example, amenable groups are ℓ^2-acyclic [Lüc02, Corollary 6.75]. The following shows that ℓ^2-Betti numbers can be computed using classifying spaces for families consisting of ℓ^2-acyclic subgroups.

Proposition 3.85 ([Kam19, Theorem 4.14])**.** *Let G be a group and $\mathcal{F}$ be a (conjugation-closed) family of subgroups of G that consists of ℓ^2-acyclic groups and contains the trivial subgroup. Then the canonical G-map $EG \to E_{\mathcal{F}}G$ induces a dimension-isomorphism in ℓ^2-homology*

$$H_*^{(2)}(G \curvearrowright EG) \xrightarrow{\cong_{\dim}} H_*^{(2)}(G \curvearrowright E_{\mathcal{F}}G) .$$

In particular, $b_*^{(2)}(G) = b_*^{(2)}(G \curvearrowright E_{\mathcal{F}}G)$.

Proof. Since $\mathcal{F}$ contains the trivial subgroup, $EG \times E_{\mathcal{F}}G$ equipped with the diagonal G-action is a model for EG. The projection $EG \times E_{\mathcal{F}}G \to E_{\mathcal{F}}G$ onto the second factor induces a dimension-isomorphism

$$H_*^{(2)}(G \curvearrowright EG) \cong H_*^{(2)}(G \curvearrowright EG \times E_{\mathcal{F}}G) \xrightarrow{\cong_{\dim}} H_*^{(2)}(G \curvearrowright E_{\mathcal{F}}G) ,$$

because all members of $\mathcal{F}$ are ℓ^2-acyclic [Lüc02, proof of Theorem 6.54 (2)]. □

For the vanishing theorem, we consider the following situation: Let (X, A) be a CW-pair with X path-connected. Suppose that A has only finitely many connected components and let $A = \coprod_{i\in I} A_i$ be a decomposition into connected components. Assume that each A_i is π_1-injective in X and let $(G,\mathcal{H})$ be a fundamental group pair for (X,A) (Example 3.17). Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup. Denote by $p\colon \widetilde{X} \to X$ the universal covering.

Moreover, let $\mathcal{U}$ be a relative $\mathcal{F}$-cover of (X,A) (Definition 3.41) with relative multiplicity $\operatorname{mult}_A(\mathcal{U})$ (Definition 3.35). Let $\widetilde{\mathcal{U}}$ be the lifted G-invariant cover of $\widetilde{X}$ (Example 3.40) and $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ be its equivariant nerve pair (Proposition 3.42). We will also use the notion of weakly convex relative covers (Definition 3.41).

Theorem 3.86 (Relative vanishing theorem for ℓ^2-Betti numbers)**.** *In the above situation, if X and all the $(A_i)_{i\in I}$ are aspherical and $\mathcal{F}$ consists of ℓ^2-acyclic groups, then we have*

$$b_*^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) \le b_*^{(2)}\big(G \curvearrowright (|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)\big)\,.$$

In particular, if $\mathcal{U}$ is weakly convex, then

$$b_k^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) = 0$$

for all $k \ge \operatorname{mult}_A(\mathcal{U})$.

Proof. By Proposition 3.42, the equivariant nerve pair $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G,\mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$. By the universal property of the classifying space $E_{\mathcal{F}}(G,\mathcal{H})$, the canonical G-map $f\colon (\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{F}}(G,\mathcal{H})$ factors (up to G-homotopy) through the equivariant nerve map $\widetilde{\nu}$:

$$\big(\widetilde{X}, p^{-1}(A)\big) \xrightarrow{\widetilde{\nu}} \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \to E_{\mathcal{F}}(G,\mathcal{H})\,.$$

Since X and all the $(A_i)_{i\in I}$ are aspherical by assumption, Proposition 3.85 and a five lemma for dimension-isomorphisms [Sau05, Section 2] imply that f induces a dimension-isomorphism

$$H_*^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) \xrightarrow{\cong_{\dim}} H_*^{(2)}\big(G \curvearrowright E_{\mathcal{F}}(G,\mathcal{H})\big)\,.$$

Thus, the above factorisation of the G-map f shows

$$b_*^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) \le b_*^{(2)}\big(G \curvearrowright (|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)\big)\,,$$

as claimed.

To conclude the vanishing result, suppose that the relative cover $\mathcal{U}$ is weakly convex with $\operatorname{mult}_A(\mathcal{U}) = n$. Then we have $\dim(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|) = n-1$ by Proposition 3.42 (i) and hence $b_k^{(2)}(G \curvearrowright (|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)) = 0$ for all degrees $k \ge n$. □

In the absolute case for the family $\mathcal{AME}$, we recover a result of Sauer [Sau09, Theorem C]:

Corollary 3.87. *Let X be a path-connected aspherical CW-complex. Then we have $b_k^{(2)}(\pi_1(X) \curvearrowright \widetilde{X}) = 0$ for all $k \geq \operatorname{cat}_{\mathcal{AME}}(X)$.* □

3.8 Glueing estimates for relative simplicial volume

The classical glueing estimates for simplicial volume [Gro82, BBF+14, Kue15] require that the boundary components used in the glueing have amenable fundamental groups. Replacing amenability with bounded acyclicity and the uniform boundary condition also leads to (albeit weaker) glueing estimates for simplicial volume (Theorem 3.88). The uniform boundary condition also has been used for versions of simplicial volume where the strong glueing formulae are unknown and no suitable dual theory is available [FL21, FFL19].

Theorem 3.88 (Vanishing inheritance for boundedly acyclic glueings)**.** *Let $n \geq 3$ and $(M_i, \partial M_i)_{i \in I}$ be a finite collection of oriented compact connected n-manifolds. Assume that every connected component of every boundary component ∂M_i has boundedly acyclic fundamental group. Let $\mathcal{N}$ be a set of π_1-injective boundary components of the $(M_i)_{i \in I}$ and let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i \in I}$ by a pairwise glueing of the boundary components in $\mathcal{N}$ (along orientation-reversing homeomorphisms).*

If $\mathcal{N}$, viewed as a set of subsets of M, is uniformly boundedly acyclic of order n in M (Definition 3.67) then the following are equivalent:

(i) *We have $\|M, \partial M\| = 0$;*

(ii) *For all $i \in I$, we have $\|M_i, \partial M_i\| = 0$.*

The implication (ii) ⇒ (i) is proved in Section 3.8.1. The implication (i) ⇒ (ii) is established in Section 3.8.2.

Remark 3.89. More generally, the conclusion of Theorem 3.88 holds for *compatible* glueings [BBF+14], where the boundary components in $\mathcal{N}$ need not be π_1-injective. On the other hand, it remains unclear whether the assumption of bounded acyclicity on the boundary components that are not in $\mathcal{N}$ can be dropped.

Remark 3.90. In the situation of Theorem 3.88, if the collection of fundamental groups of all members of $\mathcal{N}$ is *malnormal* in $\pi_1(M)$, then the uniform bounded acyclicity condition is automatically satisfied as soon as all members of $\mathcal{N}$ have boundedly acyclic fundamental group.

3.8.1 Upper glueing estimates via the uniform boundary condition

We begin with the, easier, upper glueing estimate; this estimate works over all normed rings and also gives rough estimates in the non-vanishing case:

Proposition 3.91. *Let R be a normed ring, let $K \in \mathbb{R}_{>0}$, let I be a finite set, and let $(M_i, \partial M_i)_{i\in I}$ be a finite collection of oriented compact connected manifolds of the same dimension n. Moreover, let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i\in I}$ by a pairwise glueing (along orientation-reversing homeomorphisms) of a set $\mathcal{N}$ of boundary components such that K is a UBC_{n-1}-constant of $C_*(\bigcup \mathcal{N}; R)$. Then*

$$\|M, \partial M\|_R \leq \bigl(1 + K \cdot (n+1)\bigr) \cdot \sum_{i\in I} \|M_i, \partial M_i\|_R .$$

In particular, if $\|M_i, \partial M_i\|_R = 0$ for all $i \in I$, then $\|M, \partial M\|_R = 0$.

Proof. This is the standard filling argument [Löh, Example 6.18][BBF+14, Remark 6.2], adapted to the general UBC-setting; for the sake of completeness, we give the argument:

For notational simplicity, we view the $(M_i)_{i\in I}$ as subspaces of the glued manifold M. Moreover, we write $N \subset M$ for the (disjoint) union of the glueing loci. Hence, K is a UBC_{n-1}-constant for N. Let $(z_i \in C_n(M_i; R))_{i\in I}$ be a collection of relative fundamental cycles of $(M_i, \partial M_i)_{i\in I}$. As we glue along orientation-reversing homeomorphisms, the chain

$$b \coloneqq \sum_{i\in I} \partial z_i|_N \in C_{n-1}(N; R)$$

is null-homologous. By UBC_{n-1} for $C_*(N; R)$, there exists a chain $c \in C_n(N; R)$ with

$$\partial c = b \quad \text{and} \quad |c|_1 \leq K \cdot |b|_1 \leq K \cdot \sum_{i\in I} |\partial z_i|_1 \leq K \cdot (n+1) \cdot \sum_{i\in I} |z_i|_1 .$$

Then $z \coloneqq \sum_{i\in I} z_i - c \in C_n(M; R)$ is a relative cycle on $(M, \partial M)$; checking the local contributions on the components $(M_i, \partial M_i)$ shows that z is a relative R-fundamental cycle of $(M, \partial M)$. Therefore, we obtain

$$\begin{aligned}\|M, \partial M\|_R \leq |z|_1 &\leq \sum_{i\in I} |z_i|_1 + |c|_1 \\ &\leq \sum_{i\in I} |z_i|_1 + K \cdot (n+1) \cdot \sum_{i\in I} |z_i|_1 .\end{aligned}$$

Taking the infimum over all relative fundamental cycles $(z_i)_{i\in I}$ proves the claim. □

Proof of Theorem 3.88, (ii) ⇒ (i).
All boundedly acyclic groups satisfy the uniform boundary condition in all degrees (Theorem 3.105). As only finitely many components are involved, we also find a joint UBC_{n-1}-constant for $C_*(\bigcup \mathcal{N}; \mathbb{R})$. Applying Proposition 3.91 thus proves the implication (ii) ⇒ (i). □

In the same way, we also obtain the following estimate for the locally finite simplicial volume [Gro82, Löh] for interiors of compact manifolds with UBC-boundary; similar results have been obtained previously for amenable boundaries (or other more restrictive conditions on the boundary) via the uniform boundary condition [LS09b, Löh] or bounded cohomology [KK15].

Proposition 3.92. *Let R be a normed ring and let M be an oriented connected compact n-manifold with boundary satisfying the following properties:*

- *We have* $\|\partial M\|_R = 0$*;*
- *The boundary* ∂M *satisfies* UBC_{n-1} *over* R*. Let* K *be a* UBC_{n-1}*-constant for* $C_*(\partial M; R)$*.*

Then

$$\|M^\circ\|_{R,\mathrm{lf}} \leq \bigl(K \cdot (n+1) + 1\bigr) \cdot \|M, \partial M\|_R\,.$$

Proof. We will follow the previously known UBC-arguments: Let $c \in C_n(M; R)$ be a relative fundamental cycle of $(M, \partial M)$ and let $\varepsilon \in \mathbb{R}_{>0}$.

Because of $\|\partial M\|_R = 0$, there exists a sequence $(z_k)_{k\in\mathbb{N}}$ in $C_{n-1}(\partial M; R)$ of fundamental cycles of ∂M with $|z_k|_1 \leq \varepsilon \cdot 1/2^k$ for all $k \in \mathbb{N}$. From this sequence, we can construct a locally finite relative fundamental cycle $z \in C_n^{\mathrm{lf}}(\partial M \times \mathbb{R}_{\geq 0}; R)$ of the half-open cylinder $\partial M \times \mathbb{R}_{\geq 0}$ with $\partial z = -\partial c$ and

$$\begin{aligned}
|z|_1 &\leq K \cdot \bigl(|\partial c|_1 + |z_0|_1\bigr) + n \cdot |z_0|_1 + \sum_{k=0}^{\infty} \Bigl(K \cdot \bigl(|z_k|_1 + |z_{k+1}|_1\bigr) + n \cdot |z_{k+1}|_1\Bigr) \\
&\leq K \cdot |\partial c|_1 + n \cdot \varepsilon \cdot 2 + K \cdot 2 \cdot \varepsilon \cdot 2 \\
&\leq K \cdot (n+1) \cdot |c|_1 + 2 \cdot \varepsilon \cdot (n + 2 \cdot K)
\end{aligned}$$

by UBC-filling the differences between subsequent z_k with "small" chains and then spreading out the result over the half-open cylinder [Löh, proof of Theorem 6.1]. Here we filled the "cylinders" by using the canonical triangulation of $\Delta^{n-1} \times [0,1]$ into n simplices of dimension n.

Moreover, $c + z$ is a locally finite fundamental cycle of $M \cup_{\partial M} (\partial M \times \mathbb{R}_{\geq 0}) \cong M^\circ$ (via the topological collar theorem) and so

$$\|M^\circ\|_{R,\mathrm{lf}} \leq |c + z|_1 \leq |c|_1 + K \cdot (n+1) \cdot |c|_1 + 2 \cdot \varepsilon \cdot (n + 2 \cdot K)\,.$$

Taking the infimum over $\varepsilon \to 0$ and then over all relative fundamental cycles c thus shows that

$$\|M^\circ\|_{R,\mathrm{lf}} \leq \|M, \partial M\|_R + K \cdot (n+1) \cdot \|M, \partial M\|_R\,,$$

as claimed. ☐

Corollary 3.93. *Let M be an oriented connected compact n-manifold with boundary satisfying the following properties:*

- *We have $\|M, \partial M\| = 0$;*
- *The boundary ∂M satisfies UBC_{n-1}.*

Then $\|M^\circ\|_{\mathrm{lf}} = 0$.

Proof. Since $\|\partial M\| \leq (n+1) \cdot \|M, \partial M\| = 0$, we can apply Proposition 3.92. □

In particular, the conditions on the boundary in Proposition 3.92 (over $\mathbb{R}$) and Corollary 3.93 are satisfied if the boundary has vanishing bounded cohomology in positive degrees.

Remark 3.94 (Group-theoretic Dehn fillings). A classical application of upper glueing estimates are (generalised) Dehn fillings of manifolds [FM11, BBF$^+$14]. The simplicial volume of *group-theoretic* Dehn fillings was recently investigated [PS]. In particular, the simplicial volume does not increase when performing a group-theoretic Dehn filling with resulting peripheral subgroups that are amenable and of small cohomological dimension [PS, Theorem 6.3]. One obtains an analogous result for the vanishing behaviour of simplicial volume if amenability is replaced by bounded acyclicity.

3.8.2 Lower glueing estimates via bounded acyclicity

We prove the lower glueing estimate (i) ⇒ (ii) of Theorem 3.88, adapting the argument in the amenable case by Bucher, Burger, Frigerio, Iozzi, Pagliantini, and Pozzetti [BBF$^+$14].

In this section, all [bounded] cohomology groups are taken with trivial coefficients in $\mathbb{R}$.

Proof of Theorem 3.88, (i) ⇒ (ii). We proceed by contraposition, i.e., we assume that one of the building blocks satisfies $\|M_i, \partial M_i\| > 0$ and show that $\|M, \partial M\| > 0$. By the duality principle (Proposition 3.53), it suffices to show that the comparison map $H^n_b(M, \partial M) \to H^n(M, \partial M)$ is non-zero.

Let $N \subset M$ be the union of the glueing loci. We consider the following diagram. Here, all unlabelled arrows are induced by inclusions (and direct sums) and the maps labelled by ev are given by evaluation on the fundamental class.

$$\begin{array}{ccccccc}
H_b^n(M) & \xleftarrow{\ \cong\ } & H_b^n(M,\partial M) & \xrightarrow[\mathrm{comp}^n_{(M,\partial M)}]{} & H^n(M,\partial M) & \xrightarrow[\mathrm{ev}]{\ \cong\ } & \mathbb{R} \\
 & & \cong\big\uparrow & & \big\uparrow & & \\
\big\downarrow{\scriptstyle (**)} & & H_b^n(M,N\cup\partial M) & \xrightarrow[\mathrm{comp}^n_{(M,N\cup\partial M)}]{} & H^n(M,N\cup\partial M) & & \big\uparrow{\scriptstyle \mathrm{sum}} \\
 & & {\scriptstyle (***)}\big\downarrow & & \big\downarrow & & \\
\bigoplus_{i\in I} H_b^n(M_i) & \xleftarrow{\ \cong\ } & \bigoplus_{i\in I} H_b^n(M_i,\partial M_i) & \xrightarrow[\bigoplus_{i\in I}\mathrm{comp}^n_{(M_i,\partial M_i)}]{(*)} & \bigoplus_{i\in I} H^n(M_i,\partial M_i) & \xrightarrow[\mathrm{ev}]{\ \cong\ } & \bigoplus_I \mathbb{R}
\end{array}$$

This diagram is commutative: The leftmost and middle squares are commutative by functoriality of bounded cohomology and naturality of the comparison map. The rightmost square is commutative because one can construct a relative fundamental cycle of $(M, \partial M)$ out of relative fundamental cycles of the $(M_i, \partial M_i)$ plus a chain on N (see proof of Proposition 3.91).

By the duality principle and the hypothesis that one of the $\|M_i, \partial M_i\|$ is non-zero, the arrow $(*)$ is non-zero; isolating the corresponding index shows that also the composition $\mathrm{sum} \circ \mathrm{ev} \circ (*)$ is non-zero.

In the leftmost square, both horizontal arrows and the upper right vertical arrow are isomorphisms by bounded acyclicity of all boundary components (and the long exact sequence of pairs in bounded cohomology). In Section 3.8.3, using graphs of groups (Theorem 3.98 and Example 3.99), we will show that the left vertical arrow $(**)$ induced by the inclusions $M_i \hookrightarrow M$ is surjective.

Therefore, the leftmost square shows that $(***)$ is surjective. Together with the non-triviality of the composition $\mathrm{sum} \circ \mathrm{ev} \circ (*)$, we thus obtain that the comparison map $H_b^n(M, \partial M) \to H^n(M, \partial M)$ must be non-zero, as desired. $\square$

3.8.3 Graphs of groups with boundedly acyclic edge groups

We consider the bounded cohomology of finite graphs of groups with boundedly acyclic edge groups in relation to the bounded cohomology of the vertex groups. We adapt the proof in the amenable case by Bucher et al. [BBF+14] to the boundedly acyclic situation by using uniformly boundedly acyclic actions instead of amenable actions.

We first fix basic notation.

Definition 3.95 (Graph)**.** A *graph* is a tuple $\Gamma = (V, E, o, t, \overline{\cdot})$, consisting of a set V, a set E, a map $(o, t)\colon E \to V^2$, and a fixed point-free involution $\overline{\cdot}\colon E \to E$ with

$$o(e) = t(\overline{e})$$

for all $e \in E$. The elements of V are called *vertices*, the elements of E are called *edges*. The set of *geometric edges* is defined by

$$\overline{E} \coloneqq \bigl\{(e, \overline{e}) \bigm| e \in E\bigr\}.$$

Definition 3.96 (Graph of groups)**.** Let $\Gamma = (V, E, o, t, \overline{\cdot})$ be a finite graph (i.e., V and E are finite). A *graph of groups* $\mathbb{G}$ *over* Γ is a Γ-shaped diagram in the category of groups and injective group homomorphisms, i.e., $\mathbb{G}$ consists of the following data:

- A map that associates a group G_v to each $v \in V$;
- A map that associates a group G_e to each $e \in E$ such that $G_e = G_{\overline{e}}$;
- A map that associates to each edge $e \in E$ an injective group homomorphism $h_e \colon G_e \to G_{t(e)}$.

If G is the fundamental group of a graph of groups, then the vertex and edge groups admit canonical inclusions into G [Ser03, Chapter 5] and we will identify these groups with their image inside of G.

We consider finite graphs of groups with boundedly acyclic edge groups, in analogy with the amenable case [BBF+14, Theorem 1.1]; more precisely:

Setup 3.97.

- Let $n \geq 1$;
- Let $\Gamma = (V, E, o, t, \overline{\cdot})$ be a finite graph;
- Let $\mathbb{G}$ be a graph of groups over Γ;
- Let G be the fundamental group of $\mathbb{G}$; for $v \in V$, we denote the corresponding inclusion by $i_v \colon G_v \hookrightarrow G$;
- The edge groups $(G_e)_{e \in E}$ are *uniformly boundedly acyclic of order n in G*, i.e., the set
$$\Biggl\{\bigcap_{i=1}^{n} g_i G_{e_i} g_i^{-1} \Biggm| g_1, \dots, g_n \in G,\ e_1, \dots, e_n \in E\Biggr\}$$
of subgroups of G is uniformly boundedly acyclic.

Theorem 3.98. *In the situation of Setup 3.97, let $n \geq 3$ and $k \in \{3, \dots, n\}$. Then the map*

$$\bigoplus_{v \in V} H_b^k(i_v) \colon H_b^k(G) \to \bigoplus_{v \in V} H_b^k(G_v)$$

induced by the inclusions is surjective.

For the proof, we describe the bounded cohomology of G and the vertex groups via suitable uniformly boundedly acyclic actions. In the situation of Setup 3.97, we consider the set

$$S \coloneqq (G \times V) \sqcup \coprod_{e \in \overline{E}} G/G_e$$

with the G-action

- given on $G \times V$ by left translation on the first factor;
- given on each G/G_e by left translation of cosets.

In particular, for $k \in \{0, \dots, n-1\}$, the diagonal action of G on S^{k+1} is uniformly boundedly acyclic, since we assumed uniform bounded acyclicity of order n for the collection of edge groups. By Remark 3.64, the bounded cohomology of G is canonically isomorphic to the cohomology of the complex $\ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G$ in degrees $\leq n$.

Similarly, for each vertex $v \in V$, we consider the G_v-set

$$S_v \coloneqq G_v \sqcup \coprod_{e \in E, t(e) = v} G_v/G_e \,.$$

with the left translation action. In the situation of Setup 3.97, the diagonal action on S_v^{k+1} is uniformly boundedly acyclic for $k \in \{0, \dots, n-1\}$ and thus the bounded cohomology of G_v is canonically isomorphic to the cohomology of the complex $\ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}$ in degrees $\leq n$ (Remark 3.64).

For each vertex $v \in V$, there is a canonical inclusion $\varphi_v \colon S_v \to S$; on the first summand, this is given by $\varphi_v(g) \coloneqq (g, v)$ for all $g \in G_v$, on the other summands, we use the canonical maps induced by the canonical inclusions $i_v \colon G_v \hookrightarrow G$. By construction, φ_v is G_v-equivariant with respect to the inclusion i_v.

With this preparation, we give the proof of Theorem 3.98:

Proof of Theorem 3.98. We write

$$\varphi^* \coloneqq \bigoplus_{v \in V} \varphi_v^* \colon \ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G \to \bigoplus_{v \in V} \ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}$$

for the combination of the φ_v^*. Bucher et al. [BBF+14, Theorem 4.1] provide a construction of a cochain map $\psi^* \colon \bigoplus_{v \in V} \ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v} \to \ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G$ in degrees ≥ 2 that is right-inverse to φ^*. Then also $H^k(\varphi^*)$ has a right inverse in degrees ≥ 2.

Let $k \in \{3, \dots, n\}$. By Lemma 3.65 (applied to i_v and φ_v for each vertex $v \in V$) and using that V is finite, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
H^k\big(\ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G\big) & & \longrightarrow & & H^k_b(G; \mathbb{R}) \\
{\scriptstyle H^k(\varphi^*)}\big\downarrow & & & & \big\downarrow{\scriptstyle \bigoplus_{v \in V} H^k_b(i_v)} \\
H^k\big(\bigoplus_{v \in V} \ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}\big) & \overset{\cong}{\longleftrightarrow} & \bigoplus_{v \in V} H^k\big(\ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}\big) & \longrightarrow & \bigoplus_{v \in V} H^k_b(G_v; \mathbb{R}) \,.
\end{array}$$

Here the horizontal maps are the canonical maps. As $k \in \{3, \dots, n\}$, these horizontal maps are isomorphism and the left vertical arrow admits a right inverse (given by ψ^*). Therefore, also the right vertical arrow has a right inverse. In particular, the right vertical arrow $\bigoplus_{v \in V} H^k_b(i_v)$ is surjective. □

In particular, Theorem 3.98 applies to the glueing situation of Theorem 3.88:

Example 3.99. In the situation of Theorem 3.98, the fundamental group $\pi_1(M)$ is isomorphic to the fundamental group of a finite graph of groups that satisfies the conditions of Setup 3.97. More specifically, the vertex groups are isomorphic to the fundamental groups of the $(M_i)_{i \in I}$ and the edge groups are isomorphic to the fundamental groups of the boundary components in $\mathcal{N}$ along which we glue.

This concludes the proof of Theorem 3.88.

3.A Appendix A. The uniform boundary condition

We recall the uniform boundary condition and its basic properties and consequences in the context of bounded cohomology. Moreover, we introduce the uniform uniform boundary condition and use it to compute the bounded cohomology of bounded products.

3.A.1 Normed chain complexes

We begin with basic terminology for normed chain complexes. A *normed chain complex* over a normed ring R is a chain complex in the category of normed R-modules with bounded linear maps; i.e., the boundary operators in normed chain complexes are degree-wise bounded linear operators. A *Banach chain complex* is a normed chain complex over $\mathbb{R}$ consisting of Banach spaces. Similarly, one has *normed* [*resp. Banach*] *cochain complexes.*

Let C_* be a normed chain complex over a normed ring R with boundary operator $\partial_* \colon C_* \to C_{*-1}$.

- We write $B(C_*, R)$ for the normed cochain complex whose cochain modules are the bounded duals of the chain modules of C_* and whose coboundary operators are the duals of the boundary operators ∂_*.
- We write $\overline{C}_*$ for the degree-wise completion of C_* with the degree-wise continuous extension $\overline{\partial}_*$ of the boundary operator ∂_*.

Over $\mathbb{R}$, both $B(C_*, \mathbb{R})$ and $\overline{C}_*$ are Banach [co]chain complexes.

If C_* is a normed chain complex, then we obtain an induced seminorm on $H_*(C_*)$ via

$$\|\alpha\| := \inf\bigl\{|c| \bigm| c \in C_* \text{ is a cycle representing } \alpha\bigr\}$$

for all $\alpha \in H_*(C_*)$. Similarly, this also works for normed cochain complexes.

3.A.2 The uniform boundary condition

The uniform boundary condition asks for uniform control on fillings of null-homologous [co]cycles in normed [co]chain complexes. In some contexts, similar properties are encoded in the language of isoperimetric inequalities. In the following, we will stick to the terminology of Matsumoto and Morita [MM85].

Definition 3.100 (Uniform boundary condition)**.** Let C_* be a normed chain complex over a normed ring and let $k \in \mathbb{N}$. Then C_* satisfies the *uniform boundary condition in degree* k if there exists a constant $K \in \mathbb{R}_{>0}$ with

$$\forall_{b \in \operatorname{im} \partial_{k+1} \subset C_k} \quad \exists_{c \in C_{k+1}} \quad \partial_{k+1}(c) = b \quad \text{and} \quad |c| \leq K \cdot |b| \,.$$

We abbreviate the uniform boundary condition in degree k by UBC_k.

Remark 3.101. Through re-indexing, we can translate the notion of uniform boundary condition also to normed cochain complexes. In this case, we use the notation UBC^k for the uniform boundary condition in degree k. Moreover, all of the results below apply both to chain and cochain complexes (with appropriate re-indexing).

We recall basic inheritance properties of UBC:

Proposition 3.102 (Homotopy inheritance of UBC)**.** *Let $k \in \mathbb{N}$ and let C_*, D_* be normed chain complexes over a normed ring R that are chain homotopic in the category of normed R-chain complexes. Then C_* satisfies UBC_k if and only if D_* satisfies UBC_k.*

More precisely, let $f_* \colon C_* \to D_*$ and $g_* \colon D_* \to C_*$ be chain maps that are bounded in each degree and let $h_* \colon D_* \to D_{*+1}$ be a corresponding degree-wise bounded chain homotopy between $f_* \circ g_*$ and id_{D_*}. If $K \in \mathbb{R}_{>0}$ is a UBC_k-constant for C_*, then

$$\|f_{k+1}\| \cdot \|g_k\| \cdot K + \|h_k\|$$

is a UBC_k*-constant for* D_**.*

Proof. Let $x \in \operatorname{im} \partial^D_{k+1}$. Then $g_k(x) \in \operatorname{im} \partial^C_{k+1}$. As K is a UBC_k-constant for C_*, there exists $\widetilde{y} \in C_{k+1}$ with $\partial^C_{k+1}(\widetilde{y}) = g_k(x)$ and $|\widetilde{y}| \leq K \cdot \bigl|g_k(x)\bigr|$. Then

$$y \coloneqq f_{k+1}(\widetilde{y}) - h_k(x)$$

satisfies $\partial^D_{k+1}(y) = x$ and $|y| \leq \bigl(\|f_{k+1}\| \cdot \|g_k\| \cdot K + \|h_k\|\bigr) \cdot |x|$, as desired. □

Proposition 3.103 (Dense subcomplexes and UBC)**.** *Let* R *be a normed ring, let* D_* *be a normed chain complex over* R*, and let* $C_* \subset D_*$ *be a dense subcomplex. Let* $k \in \mathbb{N}$*. Then the following are equivalent:*

(i) C_* *satisfies* UBC_k*;*

(ii) $\overline{D}_*$ *satisfies* UBC_k *and* $\ker \partial^C_{k+1}$ *is dense in* $\ker \partial^{\overline{D}}_{k+1}$*;*

(iii) D_* *satisfies* UBC_k *and* $\ker \partial^C_{k+1}$ *is dense in* $\ker \partial^D_{k+1}$*.*

Proof. Because C_* is dense in D_*, the completion $\overline{D}_*$ of D_* is also a completion of C_*. The argument by Matsumoto and Morita [MM85, Theorem 2.8] applies to all normed chain complexes and their completions; this shows the equivalence of (i) and (ii).

We may apply this to D_* and its completion $\overline{D}_*$, since with $\ker \partial^C_{k+1}$ also $\ker \partial^D_{k+1}$ is dense in $\ker \partial^{\overline{D}}_{k+1}$. Hence (ii) implies that D_* satisfies UBC_k; whence (ii) implies (iii), because $\ker \partial^C_{k+1}$ being dense in $\ker \partial^{\overline{D}}_{k+1}$ also implies density of $\ker \partial^C_{k+1}$ in $\ker \partial^D_{k+1}$.

Also, the argument by Matsumoto and Morita shows that (iii) implies (i), as this implication does not rely on completeness of the ambient complex. □

The following characterisations of UBC apply to normed [resp. Banach] chain complexes over $\mathbb{R}$.

Theorem 3.104 ([MM85, Theorem 2.8])**.** *Let* C_* *be a normed chain complex over* $\mathbb{R}$ *and let* $k \in \mathbb{N}$*. Then the following are equivalent:*

(i) C_* *satisfies* UBC_k*;*

(ii) $\overline{C}_*$ *satisfies* UBC_k *and* $\ker \partial_{k+1}$ *is dense in* $\ker \overline{\partial}_{k+1}$*;*

(iii) *The comparison map* $H^{k+1}(B(C_*, \mathbb{R})) \to H^{k+1}(\operatorname{Hom}_{\mathbb{R}}(C_*, \mathbb{R}))$ *is injective.*

Proof. The equivalence of the first two items is contained in Proposition 3.103. The argument by Matsumoto and Morita [MM85, Theorem 2.8] for the remaining implications applies to all normed chain complexes over $\mathbb{R}$. □

Theorem 3.105 ([MM85, Theorem 2.3])**.** *Let C_* be a Banach chain complex over $\mathbb{R}$ and let $k \in \mathbb{N}$. Then the following are equivalent:*

(i) *C_* satisfies* UBC_k*;*

(ii) $\operatorname{im} \partial_{k+1}$ *is closed in C_k;*

(iii) *$H_k(C_*)$ is a Banach space with respect to the induced seminorm;*

(iv) *$H^{k+1}(B(C_*, \mathbb{R}))$ is a Banach space with respect to the induced seminorm.*

Proof. The argument by Matsumoto and Morita [MM85, Theorem 2.3] applies to all Banach chain complexes. □

Example 3.106. Let X be a space or a group with $H_b^*(X;\mathbb{R}) \cong H_b^*(1;\mathbb{R})$. In particular, this bounded cohomology is Banach in all degrees. Then the cochain complex version of Theorem 3.105 (Remark 3.101) shows that $C_b^*(X;\mathbb{R})$ satisfies UBC^k for all $k \in \mathbb{N}$. Moreover, Theorem 3.104 shows that $C_*(X;\mathbb{R})$ satisfies UBC_k for all $k \in \mathbb{N}$.

This applies to all path-connected spaces with amenable fundamental group, to all amenable groups, and to the known boundedly acyclic groups [MM85, Löh17, FFLMb, FFLMa, MN, Mon]. In particular, there exist finitely presented non-amenable groups G such that $C_b^*(G;\mathbb{R})$ satisfies UBC^k for all $k \in \mathbb{N}$ [FFLMb, Corollary 5.2][Mon].

Remark 3.107. For the free group F_2 of rank 2 it is well known that $H_b^2(F_2;\mathbb{R})$ and $H_b^3(F_2;\mathbb{R})$ are infinite-dimensional. But it is unknown whether the higher bounded cohomology of F_2 is trivial or not. We outline an approach through the uniform boundary condition: Let $k \in \mathbb{N}_{\geq 4}$. Then the following are equivalent:

(i) $H_b^k(F_2;\mathbb{R}) \cong 0$;

(ii) $C_*(F_2;\mathbb{R})$ satisfies UBC_{k-1};

(iii) $C_*(F_2;\mathbb{Q})$ satisfies UBC_{k-1};

Indeed, the first two items are equivalent by Theorem 3.104 and the fact that $H^k(F_2;\mathbb{R})$ is trivial. The equivalence of the last two items follows from Proposition 3.103, the fact that $C_*(F_2;\mathbb{Q}) \hookrightarrow C_*(F_2;\mathbb{R})$ induces an isometric embedding on the level of homology [Löh, Proposition 1.7], and a small computation using the universal coefficient theorem.

Moreover, the last condition can be reformulated as follows:

$$\exists_{K \in \mathbb{R}_{>0}} \quad \forall_{c \in C_k(F_2;\mathbb{Z})} \quad \exists_{c' \in C_k(F_2;\mathbb{Q})} \quad \partial_k(c') = \partial_k(c) \quad \text{and} \quad |c'|_1 \leq K \cdot |\partial_k(c)|_1 .$$

In principle, this allows for experimental testing whether $H_b^k(F_2;\mathbb{R})$ is trivial or not [Lan]. Of course, the main challenge is to efficiently generate large amounts of "interesting" chains in $C_k(F_2;\mathbb{Z})$.

The uniform boundary condition is useful, e.g., in glueing estimates for simplicial volume (Section 3.8.1), in vanishing results for bounded cohomology of certain groups [MM85, Löh17, FFLMb, FFLMa], and in vanishing results for ℓ^1-homology [MM85, FL21].

3.A.3 The uniform uniform boundary condition

We introduce a uniform version of the uniform boundary condition for collections of normed cochain complexes.

Definition 3.108 (Uniform uniform boundary condition)**.** Let R be a normed ring and let $k \in \mathbb{N}$. A collection $(C_i^*)_{i\in I}$ of normed cochain complexes over R satisfies the *uniform uniform boundary condition in degree k* (UUBC^k for short) if there exists $K \in \mathbb{R}_{>0}$ that is a UBC^k-constant for all C_i^* with $i \in I$.

Example 3.109 ([FFLMa, Example 4.11])**.** We consider a collection $(H_i)_{i\in I}$ of amenable groups. Then $(C_b^*(H_i;\mathbb{R}))_{i\in I}$ satisfies UUBC^k for all $k \in \mathbb{N}$. Indeed, if H is amenable, then 1 is a UBC^k-constant for $C_b^*(H;\mathbb{R})$: There exists a contracting cochain homotopy s for $C_b^*(H;\mathbb{R})$ with $\|s\| \leq 1$ [Fri17, Theorem 3.6]; thus, for every cocycle b, the cochain $c := s(b)$ satisfies

$$b = \delta \circ s(b) + s \circ \delta(b) = \delta(c) \quad \text{and} \quad |c|_\infty \leq \|s\| \cdot |b|_\infty \leq |b|_\infty\,.$$

Example 3.110. Every finite collection of cochain complexes whose members all satisfy UBC^k satisfies UUBC^k.

In particular, from Example 3.106, we obtain: If $(H_i)_{i\in I}$ is a *finite* collection of boundedly acyclic groups, then $(C_b^*(H_i;\mathbb{R}))_{i\in I}$ satisfies UUBC^k for all $k \in \mathbb{N}$.

It is unknown whether all collections of boundedly acyclic groups satisfy UUBC in all degrees (as the open mapping theorem used in the proof of Theorem 3.105 does not give a priori estimates on the norms of the partial inverses). Every collection of boundedly acyclic groups satisfies UUBC in degree 2 [FFLMa, Proposition 4.15].

3.A.4 Bounded products

Finite degree-wise products are compatible with taking cohomology of bounded cochain complexes. For infinite degree-wise products, in general, one needs to impose boundedness conditions in a uniform way. To this end, we introduce bounded products and prove a compatibility statement for cohomology of certain degree-wise bounded products.

Definition 3.111 (Bounded product)**.** Let R be a normed ring. Let $(V_i)_{i\in I}$ be a collection of normed modules over R. The *bounded product* of $(V_i)_{i\in I}$ is the normed R-module

$$\prod_{i\in I}^b V_i := \Big\{ x \in \prod_{i\in I} V_i \;\Big|\; \sup_{i\in I} |x_i| < \infty \Big\} \subset \prod_{i\in I} V_i$$

with respect to the supremum norm $|\cdot|_\infty$.

Example 3.112. Let $(S_i)_{i\in I}$ be a collection of sets and let R be a normed ring. Then the canonical inclusions $(S_j \hookrightarrow \coprod_{i\in I} S_i)_{j\in I}$ induce a natural isometry

$$\ell^\infty\Big(\coprod_{i\in I} S_i, R\Big) \to \prod_{i\in I}^b \ell^\infty(S_i, R)$$

of normed R-modules.

Remark 3.113 (Bounded product of normed cochain complexes). Let R be a normed ring. A collection $(C_i^*)_{i\in I}$ of normed cochain complexes over R is called *uniform* if for each $k \in \mathbb{N}$, the supremum $\sup_{i\in I} \|\delta_i^k\|$ is finite. For example, all collections of normed cochain complexes built using simplicial coboundary operators are uniform (such as bounded cochain complexes of groups or spaces).

If $(C_i^*)_{i\in I}$ is a uniform collection of normed cochain complexes over R, then the degree-wise bounded product $(\prod_{i\in I}^b C_i^k)_{k\in\mathbb{N}}$ is a normed cochain complex over R with respect to the supremum norm and the degree-wise product coboundary operator

$$\begin{aligned}\prod_{i\in I}^b C_i^* &\to \prod_{i\in I}^b C_i^{*+1}\\ (x_i)_{i\in I} &\mapsto (\delta_i^*(x_i))_{i\in I}\,.\end{aligned}$$

Theorem 3.114 (Cohomology of bounded products)**.** *Let $k \in \mathbb{N}$. Let $(C_i^*)_{i\in I}$ be a uniform collection of normed cochain complexes over a normed ring R that satisfies* UUBC^k*. Then the map*

$$\Phi\colon H^k\Big(\prod_{i\in I}^b C_i^*\Big) \to \prod_{i\in I}^b H^k(C_i^*)$$

induced by the canonical projections is a continuous isomorphism of R-modules with continuous inverse. Here, we equip $H^k(C_i^*)$ with the seminorm induced by the given norm on C_i^*.

Proof. Clearly, the map Φ is well-defined and continuous.

We construct an explicit inverse: Let $\varepsilon \in \mathbb{R}_{>0}$. Let $(\varphi_i)_{i\in I} \in \prod_{i\in I}^b H^k(C_i^*)$; for each $i \in I$, there exists a cocycle $f_i \in C_i^k$ representing φ_i in $H^k(C_i^*)$ with

$$|f_i| \le \|\varphi_i\| + \varepsilon\,.$$

Because $(\varphi_i)_{i\in I}$ lies in the bounded product, the norms $(|f_i|)_{i\in I}$ are a bounded collection, and so $f := (f_i)_{i\in I} \in \prod_{i\in I}^b C_i^k$; moreover, $\delta(f) = (\delta_i(f_i))_{i\in I} = 0$. Therefore, we obtain a cohomology class

$$\varphi := [f] \in H^k\Big(\prod_{i\in I}^b C_i^*\Big)\,.$$

By construction, $\Phi(\varphi) = (\varphi_i)_{i\in I}$.

If this construction is independent of the chosen collection $(f_i)_{i\in I}$, then it provides an R-linear inverse of Φ; moreover, as we can take $\varepsilon \to 0$, we also see that this inverse is bounded.

Thus, it remains to show that φ is independent of the choice of the collection $(f_i)_{i\in I}$. To show this, we use the uniform uniform boundary condition: Let $(f_i')_{i\in I} \in \prod_{i\in I} C_i^k$ be a collection of cocycles with

$$[f_i'] = \varphi_i \in H^k(C_i^*) \quad \text{and} \quad |f_i'| \leq \|\varphi_i\| + 1$$

for all $i \in I$. Let K be a UUBC^k-constant for $(C_i^*)_{i\in I}$. Then, for each $i \in I$, there is a cochain $c_i \in C_i^{k-1}$ with

$$\delta_i(c_i) = f_i - f_i' \quad \text{and} \quad |c_i| \leq K \cdot |f_i - f_i'| \leq K \cdot 2 \cdot \bigl(\|\varphi_i\| + 1\bigr)\,.$$

Thus, $c := (c_i)_{i\in I}$ lies in $\prod_{i\in I}^b C_i^{k-1}$ and $\delta(c) = f - f'$. □

Chapter 4

Amenable covers of right-angled Artin groups

This chapter is the article [Lia].

ABSTRACT. Let A_L be the right-angled Artin group associated to a finite flag complex L. We show that the amenable category of A_L equals the virtual cohomological dimension of the right-angled Coxeter group W_L. In particular, right-angled Artin groups satisfy a question of Capovilla–Löh–Moraschini proposing an inequality between the amenable category and Farber's topological complexity.

4.1 Introduction

A classical approach to study a topological space X is to cover it by open subsets $U_0, \ldots, U_n$ that are simpler or small in an appropriate sense and to analyse how these overlap. The minimal possible cardinality n of such a cover yields a measure of complexity of the space X. When the subsets $(U_i)_i$ are required to be contractible in X, we obtain the Lusternik–Schnirelmann category (LS-category for short) LS-cat(X) which is a well-studied homotopy invariant originating from critical point theory [CLOT03]. We will relax the contractibility assumption and instead require the subsets $(U_i)_i$ to be *amenable in* X, in the sense that the group

$$\operatorname{im}\bigl(\pi_1(U_i \hookrightarrow X, x)\bigr)$$

is amenable for every basepoint $x \in U_i$. Then the *amenable category* $\operatorname{cat}_{\mathcal{AME}}(X)$ of X is the minimal $n \in \mathbb{N}_{\geq 0}$ for which there exists an open cover $X = \bigcup_{i=0}^{n} U_i$ by $n+1$ many amenable subsets. Clearly, we have $\operatorname{cat}_{\mathcal{AME}}(X) \leq \text{LS-cat}(X)$.

Amenable groups (such as finite or abelian groups) and hence amenable subsets can be considered as small for many purposes in geometry, topology, and dynamics. Therefore the amenable category is a meaningful threshold, especially for aspherical spaces. For instance, there are vanishing results in all degrees larger than the amenable category for the comparison map from bounded cohomology to singular cohomology [Gro82, Iva85], for ℓ^2-Betti numbers [Sau09], and for homology growth [Sau16, HS]. The amenable category was systematically studied as an invariant of 3-manifolds in [GLGAH13, GLGAH14] and for arbitrary spaces recently in [CLM, LM].

The focus of this note is on the amenable category of aspherical spaces. Since the amenable category is a homotopy invariant, it yields an invariant of discrete groups G by setting $\mathrm{cat}_{\mathcal{AME}}(G) := \mathrm{cat}_{\mathcal{AME}}(BG)$. Here BG is an Eilenberg–MacLane space. By the classical work of [EG57, Sta68, Swa69], the LS-category LS-cat(BG) coincides with the cohomological dimension $\mathrm{cd}(G)$. In particular, we always have $\mathrm{cat}_{\mathcal{AME}}(G) \leq \mathrm{cd}(G)$. The amenable category is difficult to compute in general, the usual strategy being to exhibit an explicit open cover by amenable subsets and to prove its minimality using (co)homological obstructions. The precise value of $\mathrm{cat}_{\mathcal{AME}}(G)$ is known, e.g., for the following classes of groups:

- $\mathrm{cat}_{\mathcal{AME}}(G) = 0$ if and only if G is amenable;
- $\mathrm{cat}_{\mathcal{AME}}(G) = 1$ if and only if G is a non-amenable fundamental group of a graph of amenable groups [CLM, Corollary 5.4];
- $\mathrm{cat}_{\mathcal{AME}}(G) = \mathrm{cd}(G)$ if G is torsion-free non-elementary hyperbolic [Min01][CLM, Example 7.8].

The main result of the present note is a computation of the amenable category for all right-angled Artin groups. These form an important class of groups in geometric group theory, interpolating between free groups and free abelian groups. Let L be a finite flag complex (i.e., a simplicial complex in which every clique spans a simplex) with vertex set V. The right-angled Artin group A_L has as generators vertices $v \in V$, subject to the relation that v_1 and v_2 commute if and only if they are connected by an edge in L. The right-angled Coxeter group W_L is the quotient of A_L obtained by adding the relations that each generator $v \in V$ is of order 2. Since W_L is virtually torsion-free, its virtual cohomological dimension $\mathrm{vcd}(W_L)$ is well-defined as the cohomological dimension of a finite index torsion-free subgroup.

Theorem 4.1 (Corollary 4.17). *Let A_L be the right-angled Artin group associated to a finite flag complex L. Then we have*

$$\mathrm{cat}_{\mathcal{AME}}(A_L) = \mathrm{vcd}(W_L)\,.$$

Theorem 4.1 provides many examples of groups for which the amenable category is not extremal, in the sense that $1 < \mathrm{cat}_{\mathcal{AME}}(G) < \mathrm{cd}(G)$. Furthermore, it follows from Theorem 4.1 and [Dra97] that there are right-angled Artin groups A_{L_1} and A_{L_2} satisfying $\mathrm{cat}_{\mathcal{AME}}(A_{L_1} \times A_{L_2}) < \mathrm{cat}_{\mathcal{AME}}(A_{L_1}) + \mathrm{cat}_{\mathcal{AME}}(A_{L_2})$.

Another invariant of a similar spirit is Farber's topological complexity TC which is motivated by the motion planning problem in robotics [Far03]. In [CLM, Question 8.1] it is asked for which topological spaces X the following inequality holds:

$$\mathrm{cat}_{\mathcal{AME}}(X \times X) \leq \mathsf{TC}(X)\,.$$

Examples of spaces and groups satisfying this inequality can be found in [CLM, Section 8], and no counter-example seems to be known at the time of writing. We show that all right-angled Artin groups are positive examples.

Theorem 4.2 (Proposition 4.18)**.** *Let A_L be the right-angled Artin group associated to a finite flag complex L. Then we have* $\mathrm{cat}_{\mathcal{AME}}(A_L \times A_L) \leq \mathsf{TC}(A_L)$.

We also obtain a complete characterisation of right-angled Artin groups with (non-)vanishing minimal volume entropy (Theorem 4.20), resolving the cases that were not covered by recent work in [HS, BC21].

Our proofs rely on combining upper and lower bounds (Lemma 4.4) with existing results on generalised LS-category, classifying spaces for families of subgroups, and homology growth from [CLM, HS, LM, OS, PP, Sau16].

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4.2 Preliminaries

4.2.1 Generalised LS-category

Let G be a group. A *family $\mathcal{F}$ of subgroups* of G is a non-empty set of subgroups of G that is closed under conjugation and under taking subgroups. Important examples are the families $\mathcal{TR}$ consisting only of the trivial subgroup, $\mathcal{FIN}$ consisting of all finite subgroups, and $\mathcal{AME}$ consisting of all amenable subgroups. For a set $\mathcal{H}$ of subgroups of G, the *family $\mathcal{F}\langle\mathcal{H}\rangle$ generated by $\mathcal{H}$* is defined as the smallest family containing $\mathcal{H}$. For a family $\mathcal{F}$ and a subgroup H of G, we can form the family $\mathcal{F}|_H = \{F \subset H \mid F \in \mathcal{F}\}$ of subgroups of H.

Definition 4.3 ([CLM, Definition 2.16]). Let X be a path-connected space with fundamental group G and let $\mathcal{F}$ be a family of subgroups of G. A (not necessarily path-connected) open subset U of X is an *$\mathcal{F}$-set* if

$$\operatorname{im}\bigl(\pi_1(U \hookrightarrow X, x)\bigr) \in \mathcal{F}$$

for all $x \in U$. The *generalised LS-category with respect to $\mathcal{F}$* (also *$\mathcal{F}$-category*) $\operatorname{cat}_{\mathcal{F}}(X)$ is the minimal $n \in \mathbb{N}_{\geq 0}$ for which there exists an open cover $X = \bigcup_{i=0}^{n} U_i$ by $n+1$ many $\mathcal{F}$-sets. If no such finite cover of X exists, we set $\operatorname{cat}_{\mathcal{F}}(X) = \infty$.

The $\mathcal{F}$-category of the group G is defined as $\operatorname{cat}_{\mathcal{F}}(G) \coloneqq \operatorname{cat}_{\mathcal{F}}(BG)$.

We point out that we use a different normalisation than in [CLM], our value for $\operatorname{cat}_{\mathcal{F}}(X)$ is smaller by 1. In the literature similar invariants are sometimes defined in terms of the multiplicity of open covers rather than the cardinality. However, for CW-complexes there is no difference [CLM, Remark 3.13]. From here onwards, we will study the generalised LS-category for groups, that is for aspherical spaces (even though some results hold more generally for not necessarily aspherical spaces).

It is a classical result [EG57, Sta68, Swa69] that the $\mathcal{TR}$-category $\operatorname{cat}_{\mathcal{TR}}(G)$ coincides with the cohomological dimension $\operatorname{cd}(G)$. The following upper and lower bounds for the $\mathcal{F}$-category are immediate.

Lemma 4.4. *Let G be a group and let $\mathcal{F}$ be a family of subgroups of G.*

(i) *For a subfamily $\mathcal{E} \subset \mathcal{F}$, we have $\operatorname{cat}_{\mathcal{F}}(G) \leq \operatorname{cat}_{\mathcal{E}}(G)$.*
In particular, $\operatorname{cat}_{\mathcal{F}}(G) \leq \operatorname{cd}(G)$;

(ii) *For a subgroup $H \subset G$, we have $\operatorname{cat}_{\mathcal{F}|_H}(H) \leq \operatorname{cat}_{\mathcal{F}}(G)$.*
In particular, if $\mathcal{F}|_H = \mathcal{TR}$ then $\operatorname{cd}(H) \leq \operatorname{cat}_{\mathcal{F}}(G)$.

Our main object of interest is the $\mathcal{AME}$-category (also *amenable category*) $\operatorname{cat}_{\mathcal{AME}}(G)$. A lower bound for the amenable category is given by homology growth. Recall that a group G is *of type F* if there exists a finite model for BG. A group G is *residually finite* if it admits a *residual chain* $(\Gamma_i)_{i\in\mathbb{N}}$, i.e., a nested sequence $G = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that each Γ_i is a finite index normal subgroup of G and $\bigcap_{i\in\mathbb{N}} \Gamma_i = \{1\}$. We denote by $b_k(\Gamma_i; \mathbb{F}_p)$ the k-th Betti number of $B\Gamma_i$ with coefficients in $\mathbb{F}_p$.

Theorem 4.5 ([HS, Theorem 3.2][Sau16, Theorem 1.6]). *Let G be a residually finite group of type F and let $(\Gamma_i)_i$ be a residual chain. Then we have*

$$\lim_{i\to\infty} \frac{b_k(\Gamma_i; \mathbb{F}_p)}{[G : \Gamma_i]} = 0$$

for all $k > \operatorname{cat}_{\mathcal{AME}}(G)$ and all primes p.

4.2.2 Classifying spaces for families of subgroups

Let G be a group and let $\mathcal{F}$ be a family of subgroups of G. A *classifying space* $E_{\mathcal{F}}G$ for G with respect to the family $\mathcal{F}$ is a terminal object in the G-homotopy category of G-CW-complexes whose isotropy groups lie in $\mathcal{F}$ [Lüc05]. For the trivial family $\mathcal{TR}$, a model for $E_{\mathcal{TR}}G$ is given by the universal covering space EG of BG. In particular, for every family $\mathcal{F}$ there is a unique (up to G-homotopy) G-map $EG \to E_{\mathcal{F}}G$.

The $\mathcal{F}$-category of groups can be characterised via classifying spaces for families.

Theorem 4.6 ([CLM, Proposition 7.5])**.** *Let G be a group and let $\mathcal{F}$ be a family of subgroups of G. Then $\operatorname{cat}_{\mathcal{F}}(G)$ equals the infimum of $n \in \mathbb{N}_{\geq 0}$ for which the canonical G-map $EG \to E_{\mathcal{F}}G$ is G-homotopic to a G-map with values in the n-skeleton of $E_{\mathcal{F}}G$.*

The usual notions of geometric and cohomological dimension of groups admit generalisations to the setting of families. The *geometric dimension* $\operatorname{gd}_{\mathcal{F}}(G)$ of G with respect to $\mathcal{F}$ is the smallest possible dimension of a model for $E_{\mathcal{F}}G$. The *cohomological dimension* $\operatorname{cd}_{\mathcal{F}}(G)$ of G with respect to $\mathcal{F}$ is the supremum of degrees in which the G-equivariant Bredon cohomology of $E_{\mathcal{F}}G$ is non-trivial for some Bredon coefficient module [Bre67].

Corollary 4.7. *Let G be a group and let $\mathcal{F}$ be a family of subgroups of G. Then we have $\operatorname{cat}_{\mathcal{F}}(G) \leq \operatorname{gd}_{\mathcal{F}}(G)$.*

Moreover, if $\operatorname{cd}_{\mathcal{F}}(G) = 1$ implies $\operatorname{gd}_{\mathcal{F}}(G) = 1$, then $\operatorname{cat}_{\mathcal{F}}(G) \leq \operatorname{cd}_{\mathcal{F}}(G)$.

Proof. The inequality $\operatorname{cat}_{\mathcal{F}}(G) \leq \operatorname{gd}_{\mathcal{F}}(G)$ is an immediate consequence of Theorem 4.6. Since $\operatorname{gd}_{\mathcal{F}}(G) \leq \max\{\operatorname{cd}_{\mathcal{F}}(G), 3\}$ [LM00], it remains to treat the case that $\operatorname{cd}_{\mathcal{F}}(G) = 2$ and $\operatorname{gd}_{\mathcal{F}}(G) = 3$. We follow a standard argument using equivariant obstruction theory (see e.g., [GMP, Theorem 3.6]). Let $E_{\mathcal{F}}G$ be a 3-dimensional model and consider the identity map $\operatorname{id}_2\colon (E_{\mathcal{F}}G)_2 \to (E_{\mathcal{F}}G)_2$ on its 2-skeleton. The obstruction to extending the restriction $\operatorname{id}_2|_{(E_{\mathcal{F}}G)_1}$ to a G-map $E_{\mathcal{F}}G \to (E_{\mathcal{F}}G)_2$ lies in the Bredon cohomology of $E_{\mathcal{F}}G$ in degree 3. This cohomology group is trivial by the assumption that $\operatorname{cd}_{\mathcal{F}}(G) = 2$ and hence there exists a G-map $\varphi\colon E_{\mathcal{F}}G \to (E_{\mathcal{F}}G)_2$. By considering the composition

$$EG \to E_{\mathcal{F}}G \xrightarrow{\varphi} (E_{\mathcal{F}}G)_2 \hookrightarrow E_{\mathcal{F}}G\,,$$

it follows from Theorem 4.6 that $\operatorname{cat}_{\mathcal{F}}(G) \leq 2$. □

It is conjectured that $\operatorname{cd}_{\mathcal{F}}(G) = 1$ implies $\operatorname{gd}_{\mathcal{F}}(G) = 1$ for every family $\mathcal{F}$, see [GMP] for a recent account. While the conjecture is open in general, it is known to hold, e.g., for the family $\mathcal{FIN}$ [Dun79].

4.2.3 Graph products of groups

Let L be a flag complex, which shall always mean a finite flag complex, with vertex set V. Let G be a group and for all $v \in V$ let $G_v = G$. The *graph product* G_L [Gre] is the group

$$G_L = *_{v \in V} G_v / \langle [G_{v_1}, G_{v_2}] \text{ for } v_1, v_2 \in V \text{ spanning an edge in } L \rangle \,.$$

The *right-angled Artin group* (RAAG for short) associated to L is $A_L = \mathbb{Z}_L$. The *right-angled Coxeter group* (RACG for short) associated to L is $W_L = (\mathbb{Z}/2\mathbb{Z})_L$.

Remark 4.8. The results of this note hold, when suitably modified, also for graph products with varying vertex groups $(G_v)_{v \in V}$. However, we restrict ourselves to the case of identical vertex groups for ease of notation.

For every full subcomplex K of L, the graph product G_L retracts onto G_K by mapping the factors $(G_v)_v$ corresponding to vertices in $L \setminus K$ to the trivial element in G_K.

Consider the obvious projection $q\colon G_L \to \prod_{v \in V} G_v$. If G is abelian, then the kernel of q is the commutator subgroup G'_L. Moreover, since the restriction of q to G_σ is injective for every simplex $\sigma \subset L$, the intersection of G'_L with conjugates of G_σ in G_L is trivial.

By functoriality of the graph product construction $(-)_L$ in the group variable, the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ induces a map $p\colon A_L \to W_L$ which restricts to the commutator subgroups $p'\colon A'_L \to W'_L$. Since $W'_L \subset W_L$ is of finite index and torsion-free, the *virtual cohomological dimension* $\operatorname{vcd}(W_L)$ equals $\operatorname{cd}(W'_L)$.

Lemma 4.9. *Let L be a flag complex. The group homomorphism $p'\colon A'_L \to W'_L$ admits a right-inverse. In particular,* $\operatorname{vcd}(W_L) \leq \operatorname{cd}(A'_L)$.

Proof. We argue on the level of topological spaces using the polyhedral product construction (see e.g., [PV16]). Models for $B(A'_L)$ and $B(W'_L)$ are given by the polyhedral products $(\mathbb{R}, \mathbb{Z})^L$ and $([0,1], \{0,1\})^L$, respectively. Consider the map

$$f\colon (\mathbb{R}, \mathbb{Z}) \to ([0,1], \{0,1\}) \,, \quad x \mapsto \begin{cases} x - \lfloor x \rfloor & \text{if } \lfloor x \rfloor \text{ is even;} \\ 1 - (x - \lfloor x \rfloor) & \text{if } \lfloor x \rfloor \text{ is odd} \end{cases}$$

that "folds" the real line onto the unit interval. The map f induces a map on polyhedral products and on their fundamental groups the map $p'\colon A'_L \to W'_L$. A right-inverse to f is given by the inclusion $([0,1], \{0,1\}) \hookrightarrow (\mathbb{R}, \mathbb{Z})$. It follows that W'_L is a retract of A'_L and in particular $\operatorname{cd}(W'_L) \leq \operatorname{cd}(A'_L)$. □

We recall an explicit formula for the virtual cohomological dimension of RACGs [Dav08, Corollary 8.5.5]. For the right-angled Coxeter group W_L associated to a flag complex L, we have

$$\mathrm{vcd}(W_L) = \max\{n \mid \widetilde{H}^{n-1}(L \setminus \sigma; \mathbb{Z}) \neq 0 \text{ for some simplex } \sigma \subset L \text{ or } \sigma = \emptyset\}\,. \tag{4.1}$$

Here $\widetilde{H}^*$ denotes reduced cohomology. The virtual cohomological dimension of RACGs will play a key role due to its interpretation as the cohomological dimension of graph products with respect to the following family.

Let G_L be a graph product and let $\mathcal{F}\langle\mathcal{S}\rangle$ be the family of subgroups of G_L that is generated by the set of *spherical subgroups*

$$\mathcal{S} = \{G_\sigma \subset G_L \mid \sigma \subset L \text{ simplex}\}\,.$$

In the case of RACGs, we have $\mathcal{F}\langle\mathcal{S}\rangle = \mathcal{FIN}$. In the case of RAAGs, the family $\mathcal{F}\langle\mathcal{S}\rangle$ consists of free abelian groups and in particular, $\mathcal{F}\langle\mathcal{S}\rangle \subset \mathcal{AME}$.

Theorem 4.10 ([PP, Corollaries 8.3 and 1.10])**.** *Let G_L be the graph product associated to a non-trivial group G and a flag complex L. Then $\mathrm{cd}_{\mathcal{F}\langle\mathcal{S}\rangle}(G_L) = \mathrm{vcd}(W_L)$.*

Moreover, $\mathrm{cd}_{\mathcal{F}\langle\mathcal{S}\rangle}(G_L) = 1$ implies $\mathrm{gd}_{\mathcal{F}\langle\mathcal{S}\rangle}(G_L) = 1$.

In view of Theorem 4.5, we recall a computation of homology growth for graph products (using that residually finite amenable groups of type F are $\mathbb{F}_p$-ℓ^2-*acyclic* by Theorem 4.5). Graph products of residually finite groups are residually finite [Gre].

Theorem 4.11 ([OS, Theorem 5.1])**.** *Let G be a non-trivial residually finite amenable group of type F and let G_L be the graph product associated to a flag complex L. Then, for any residual chain $(\Gamma_i)_i$ in G_L, we have*

$$\lim_{i\to\infty} \frac{b_k(\Gamma_i; \mathbb{F}_p)}{[G_L : \Gamma_i]} = \widetilde{b}_{k-1}(L; \mathbb{F}_p)$$

for all $k > 0$ and all primes p.

Here $\widetilde{b}_{k-1}(L; \mathbb{F}_p)$ denotes the reduced Betti number of L with coefficients in $\mathbb{F}_p$. Our formulations of Theorem 4.10 and Theorem 4.11 for graph products are special cases of the results in [PP, OS] which apply to the more general context of group actions with a strict fundamental domain.

4.3 Generalised LS-category of right-angled Artin groups

We investigate the generalised LS-category of RAAGs with respect to several interesting families. Throughout, let L be a finite flag complex. The $\mathcal{TR}$-category $\mathrm{cat}_{\mathcal{TR}}(A_L)$ equals the cohomological dimension $\mathrm{cd}(A_L)$ which is $\dim(L) + 1$. The following lemma provides an upper bound for various families and will be used frequently.

Lemma 4.12. *Let G_L be the graph product associated to a group G and a flag complex L. Let $\mathcal{F}$ be a family of subgroups of G_L satisfying $\mathcal{F}\langle\mathcal{S}\rangle \subset \mathcal{F}$. Then* $\operatorname{cat}_{\mathcal{F}}(G_L) \leq \operatorname{vcd}(W_L)$*.*

Proof. Combining Lemma 4.4 (i), Corollary 4.7, and Theorem 4.10 yields the claim. □

4.3.1 Spherical category

We compute the generalised LS-category with respect to the family $\mathcal{F}\langle\mathcal{S}\rangle$ for RAAGs and RACGs.

Proposition 4.13 (Spherical category of RAAGs)**.** *Let A_L be the right-angled Artin group associated to a flag complex L. Then we have* $\operatorname{cat}_{\mathcal{F}\langle\mathcal{S}\rangle}(A_L) = \operatorname{vcd}(W_L)$*.*

In particular, $\operatorname{cd}(A'_L) = \operatorname{vcd}(W_L)$*.*

Proof. Lemma 4.12 provides the upper bound $\operatorname{cat}_{\mathcal{F}\langle\mathcal{S}\rangle}(A_L) \leq \operatorname{vcd}(W_L)$. For the lower bound, consider the commutator subgroup A'_L which satisfies $\mathcal{F}\langle\mathcal{S}\rangle|_{A'_L} = \mathcal{TR}$. Then Lemma 4.4 (ii) applied to A'_L together with Lemma 4.9 yields

$$\operatorname{cat}_{\mathcal{F}\langle\mathcal{S}\rangle}(A_L) \geq \operatorname{cat}_{\mathcal{F}\langle\mathcal{S}\rangle|_{A'_L}}(A'_L) = \operatorname{cd}(A'_L) \geq \operatorname{vcd}(W_L)\,,$$

concluding the proof. □

An alternative proof for the lower bound $\operatorname{cat}_{\mathcal{F}\langle\mathcal{S}\rangle}(A_L) \geq \operatorname{vcd}(W_L)$ will be provided by Theorem 4.16 below.

A virtually torsion-free group G satisfies $\operatorname{cd}_{\mathcal{FIN}}(G) \geq \operatorname{vcd}(G)$, which follows from the Shapiro lemma for Bredon cohomology. This inequality can be strict, but it is in fact an equality for right-angled Coxeter groups (Theorem 4.10), as well as for many other examples [DMP16].

Proposition 4.14. *Let G be a virtually torsion-free group satisfying* $\operatorname{cd}_{\mathcal{FIN}}(G) = \operatorname{vcd}(G)$*. Then we have* $\operatorname{cat}_{\mathcal{FIN}}(G) = \operatorname{vcd}(G)$*.*

Proof. By Corollary 4.7, we have $\operatorname{cat}_{\mathcal{FIN}}(G) \leq \operatorname{cd}_{\mathcal{FIN}}(G) = \operatorname{vcd}(G)$. The opposite inequality $\operatorname{vcd}(G) \leq \operatorname{cat}_{\mathcal{FIN}}(G)$ follows from Lemma 4.4 (ii) by restricting to a finite index torsion-free subgroup of G. □

Corollary 4.15 (Finite category of RACGs)**.** *Let W_L be the right-angled Coxeter group associated to a flag complex L. Then we have* $\operatorname{cat}_{\mathcal{FIN}}(W_L) = \operatorname{vcd}(W_L)$*.*

4.3.2 Amenable category

We prove the main result of this note.

Theorem 4.16. *Let G be a non-trivial residually finite amenable group of type F and let G_L be the graph product associated to a flag complex L. Let $\mathcal{F}$ be a family of subgroups of G_L satisfying $\mathcal{F}\langle\mathcal{S}\rangle \subset \mathcal{F} \subset \mathcal{AME}$. Then we have*

$$\operatorname{cat}_{\mathcal{F}}(G_L) = \operatorname{vcd}(W_L)\,.$$

Proof. On the one hand, since $\mathcal{F}\langle\mathcal{S}\rangle \subset \mathcal{F}$ we have $\operatorname{cat}_{\mathcal{F}}(G_L) \leq \operatorname{vcd}(W_L)$ by Lemma 4.12. On the other hand, since $\mathcal{F} \subset \mathcal{AME}$ we have $\operatorname{cat}_{\mathcal{F}}(G_L) \geq \operatorname{cat}_{\mathcal{AME}}(G_L)$ by Lemma 4.4 (i) and it remains to prove that $\operatorname{cat}_{\mathcal{AME}}(G_L) \geq \operatorname{vcd}(W_L)$.

Let $n = \operatorname{vcd}(W_L)$. By (4.1) there exists a (possibly empty) simplex $\sigma \subset L$ such that $\widetilde{H}^{n-1}(L \setminus \sigma; \mathbb{Z}) \neq 0$. Let K be the full subcomplex of L spanned by the vertices in $L \setminus \sigma$. Then the graph product G_K associated to K is a subgroup of G_L. Hence $\operatorname{cat}_{\mathcal{AME}}(G_L) \geq \operatorname{cat}_{\mathcal{AME}}(G_K)$ by Lemma 4.4 (ii) and it suffices to prove that $\operatorname{cat}_{\mathcal{AME}}(G_K) \geq n$.

Indeed, since K is homotopy equivalent to $L \setminus \sigma$, we have $\widetilde{H}^{n-1}(K; \mathbb{Z}) \neq 0$. The universal coefficient theorem implies that $\widetilde{H}_{n-1}(K; \mathbb{F}_p) \neq 0$ for some prime p. By Theorem 4.11 we have

$$\lim_{i\to\infty} \frac{b_n(\Gamma_i; \mathbb{F}_p)}{[G_K : \Gamma_i]} = \widetilde{b}_{n-1}(K; \mathbb{F}_p) \neq 0\,,$$

where $(\Gamma_i)_i$ is any residual chain in G_K. Thus we conclude from Theorem 4.5 that $\operatorname{cat}_{\mathcal{AME}}(G_K) \geq n$. This finishes the proof. □

Applying Theorem 4.16 to $G = \mathbb{Z}$ and $\mathcal{F} = \mathcal{AME}$ yields the following.

Corollary 4.17 (Amenable category of RAAGs)**.** *Let A_L be the right-angled Artin group associated to a flag complex L. Then we have* $\operatorname{cat}_{\mathcal{AME}}(A_L) = \operatorname{vcd}(W_L)$.

4.3.3 Topological complexity

Another important generalised LS-category is Farber's topological complexity [Far03]. Let G be a torsion-free group and consider the product $G \times G$. Let $\mathcal{F}\langle\Delta\rangle$ be the family of subgroups of $G \times G$ generated by the diagonal subgroup $\Delta \subset G \times G$. The *topological complexity* $\mathsf{TC}(G)$ coincides with the $\mathcal{F}\langle\Delta\rangle$-category $\operatorname{cat}_{\mathcal{F}\langle\Delta\rangle}(G \times G)$ of $G \times G$ by [FGLO19], which might as well be taken as the definition of $\mathsf{TC}(G)$.

The topological complexity of RAAGs has been computed [CP08, GGG+15]. We recall the precise result for completeness, even though we will not need it in the sequel. For the right-angled Artin group A_L associated to a flag complex L with vertex set V, we have

$$\mathsf{TC}(A_L) = \max\{|V_1 \cup V_2| \mid V_1, V_2 \subset V \text{ each spanning a simplex in } L\}\,.$$

The topological complexity and amenable category of RAAGs are related by the following inequality, providing positive examples to [CLM, Question 8.1].

Proposition 4.18. *Let A_L be the right-angled Artin group associated to a flag complex L. Then we have* $\mathrm{cat}_{\mathcal{AME}}(A_L \times A_L) \leq \mathsf{TC}(A_L)$.

Proof. We prove the inequalities $\mathrm{cat}_{\mathcal{AME}}(A_L \times A_L) \leq 2\,\mathrm{vcd}(W_L) \leq \mathsf{TC}(A_L)$. Since the product $A_L \times A_L$ is a right-angled Artin group (associated to the join $L * L$), by Lemma 4.12 we have $\mathrm{cat}_{\mathcal{AME}}(A_L \times A_L) \leq \mathrm{vcd}(W_L \times W_L) \leq 2\,\mathrm{vcd}(W_L)$.

To prove the remaining inequality, let $\sigma \subset L$ be a simplex of maximal dimension. Consider the subgroup $A_\sigma \times A'_L$ of $A_L \times A_L$ which satisfies $\mathcal{F}\langle\Delta\rangle|_{A_\sigma \times A'_L} = \mathcal{TR}$. It follows from Lemma 4.4 (ii) that $\mathsf{TC}(A_L) \geq \mathrm{cd}(A_\sigma \times A'_L)$. Since A_σ is free abelian and using Lemma 4.9, we obtain

$$\mathrm{cd}(A_\sigma \times A'_L) = \mathrm{cd}(A_\sigma) + \mathrm{cd}(A'_L) \geq 2\,\mathrm{vcd}(W_L)\,.$$

This concludes the proof. □

Remark 4.19. The analogous inequality holds for all higher topological complexities [Rud10], i.e., we have $\mathrm{cat}_{\mathcal{AME}}((A_L)^r) \leq \mathsf{TC}_r(A_L)$ for all $r \in \mathbb{N}_{\geq 2}$.

4.3.4 Minimal volume entropy

The generalised LS-category with respect to families of subgroups with controlled growth is closely related to the (non-)vanishing of minimal volume entropy. The arguments in this section follow [HS] to which we refer for the precise definitions.

Let G be a group of type F with $\mathrm{gd}(G) = n$. The *minimal volume entropy* $\omega(G)$ is defined as the minimal exponential growth rate of balls in cocompact models for EG of dimension n. There are sufficient conditions for the (non-)vanishing of minimal volume entropy, called the fiber (non-)collapsing assumption (F(N)CA for short) [BS]. More precisely,

- if there exists a finite model for BG of dimension n satisfying FCA, then $\omega(G) = 0$;
- if every finite model for BG of dimension n satisfies FNCA and G has uniform uniform exponential growth, then $\omega(G) > 0$.

The conditions FCA and FNCA are not complementary in general.

We will use a reformulation of condition FCA in the language of generalised LS-category [BS, LM]. Let G be a group as above. For $\delta \in \mathbb{R}_{>0}$, let $\mathsf{Subexp}_{<\delta}$ denote the family of subgroups H of G such that every finitely generated subgroup of H has subexponential growth with subexponential growth rate $< \delta$. It follows from [LM, Corollary 5.9] that there exists a finite model for BG of dimension n satisfying FCA if and only if

$$\mathrm{cat}_{\mathsf{Subexp}_{<(n-k)/n}}(G) < k+1$$

for some $k \in \{0, \ldots, n-1\}$. (Our values for the generalised LS-category are smaller by 1 than in [LM] because we use a different normalisation.)

The following is a complete characterisation of RAAGs with (non-)vanishing minimal volume entropy.

Theorem 4.20 (Minimal volume entropy of RAAGs)**.** *Let A_L be the right-angled Artin group associated to a flag complex L of dimension d. Then $\widetilde{H}^d(L;\mathbb{Z}) \neq 0$ if and only if $\omega(A_L) > 0$.*

Proof. Using (4.1) we observe that $\widetilde{H}^d(L;\mathbb{Z}) \neq 0$ is equivalent to $\mathrm{vcd}(W_L) = d+1$. For all $\delta > 0$ we have $\mathcal{F}\langle\mathcal{S}\rangle \subset \mathsf{Subexp}_{<\delta} \subset \mathcal{AME}$ and hence Theorem 4.16 implies

$$\mathrm{cat}_{\mathsf{Subexp}_{<\delta}}(A_L) = \mathrm{vcd}(W_L)\,.$$

By the above, we have $\mathrm{vcd}(W_L) < d+1$ if and only if there exists a finite model for $B(A_L)$ of dimension $d+1$ satisfying FCA. In this case $\omega(A_L) = 0$. On the other hand, the conditions FCA and FNCA are in fact complementary for $B(A_L)$ [BC21]. Thus, we have $\mathrm{vcd}(W_L) = d+1$ if and only if every finite model for $B(A_L)$ of dimension $d+1$ satisfies FNCA. In this case $\omega(A_L) > 0$, using that RAAGs have uniform uniform exponential growth. □

Most cases of Theorem 4.20 appeared in [HS, Theorem 1.1], which however left open if $\widetilde{H}^d(L;\mathbb{Z}) = 0$ implies $\omega(A_L) = 0$ in the case when $d = 2$. Our resolution of this case goes back to the obstruction theoretical argument in the proof of Corollary 4.7.

4.4 Graph products of hyperbolic groups

We provide examples of graph products whose amenable category is maximal, i.e., it equals the cohomological dimension. For a group G and a flag complex L of dimension d, we have $\mathrm{cd}(G_L) = \mathrm{cd}(G^{d+1}) \leq (d+1) \cdot \mathrm{cd}(G)$.

Lemma 4.21. *Let G be a group and let L be a flag complex of dimension d. Let $\mathcal{F}$ be a family of subgroups of the graph product G_L. If there is a simplex $\sigma \subset L$ of dimension d such that $\mathrm{cat}_{\mathcal{F}|_{G_\sigma}}(G_\sigma) = \mathrm{cd}(G_\sigma)$, then $\mathrm{cat}_{\mathcal{F}}(G_L) = \mathrm{cd}(G^{d+1})$.*

Proof. The claim follows at once from Lemma 4.4 (ii) by restricting to the subgroup G_σ of G_L. □

In the following proof we use the notion of simplicial volume of manifolds and some of its standard properties [Gro82, Fri17].

Proposition 4.22. *Let G be the fundamental group of an oriented closed connected hyperbolic manifold and let G_L be the graph product associated to a flag complex L. Let $\mathcal{F}$ be a family of subgroups of G_L satisfying $\mathcal{F} \subset \mathcal{AME}$. Then we have*

$$\mathrm{cat}_{\mathcal{F}}(G_L) = (\dim(L) + 1) \cdot \mathrm{cd}(G)\,.$$

Proof. Let $d = \dim(L)$ and let M be an oriented closed connected hyperbolic manifold with $\pi_1(M) \cong G$. Since $\mathcal{F} \subset \mathcal{AME}$, by Lemma 4.4 (i) we have

$$\mathrm{cat}_{\mathcal{AME}}(G_L) \le \mathrm{cat}_{\mathcal{F}}(G_L) \le \mathrm{cd}(G_L) \le (d+1) \cdot \mathrm{cd}(G)\,.$$

Since M is a model for BG, the product M^{d+1} is a model for BG^{d+1} and we have $\mathrm{cd}(G^{d+1}) = (d+1) \cdot \mathrm{cd}(G)$. Hence by Lemma 4.21 it suffices to show that we have $\mathrm{cat}_{\mathcal{AME}}(M^{d+1}) = \dim(M^{d+1})$. Indeed, the hyperbolic manifold M has positive simplicial volume and by the product inequality for simplicial volume so does M^{d+1}. Finally, the amenable category of manifolds with positive simplicial volume is maximal, i.e., it equals the dimension of the manifold. □

Remark 4.23. More generally, Proposition 4.22 holds by the same proof for every group G that is the fundamental group of an oriented closed connected aspherical manifold with positive simplicial volume. For a recent list of manifolds known to have positive simplicial volume, see e.g., [LMR, Example 3.1].

Chapter 5

On the topological complexity of toral relatively hyperbolic groups

This chapter is the article [Li22] which has been published in the journal *Proceedings of the American Mathematical Society*.

Abstract. We prove that the topological complexity $\mathsf{TC}(\pi)$ equals $\mathrm{cd}(\pi \times \pi)$ for certain toral relatively hyperbolic groups π.

5.1 Introduction

The (reduced) topological complexity $\mathsf{TC}(X)$ of a space X is defined as the minimal integer n for which there exists a cover of $X \times X$ by $n+1$ open subsets $U_0, \ldots, U_n$ such that the path fibration $X^{[0,1]} \to X \times X$ admits a local section over each U_i. This quantity, which is similar in spirit to the classical Lusternik–Schnirelmann category, was introduced by Farber [Far03] in the context of robot motion planning. In fact, $\mathsf{TC}(-)$ is a homotopy invariant and hence one can define the topological complexity $\mathsf{TC}(\pi)$ of a group π to be $\mathsf{TC}(B\pi)$, where $B\pi$ is a classifying space for π. There are bounds (see e.g., [FGLO19, FM20, Dra20])

$$\mathrm{cd}(\pi) \leq \mathsf{TC}(\pi) \leq \mathrm{cd}(\pi \times \pi)\,, \tag{5.1}$$

where $\mathrm{cd}(-)$ denotes the cohomological dimension. However, the precise value of $\mathsf{TC}(\pi)$ is known only for a small class of groups, which contains for instance the abelian groups, hyperbolic groups, free products of the form $H*H$ for H geometrically finite, right-angled Artin groups, and certain subgroups of braid groups. We refer to [FM20] and [Dra20] for a more thorough account on this topic.

It is the decisive insight of [FGLO19] that the topological complexity of groups can be expressed in terms of classifying spaces for families of subgroups, which are well-studied objects in equivariant topology. For a family $\mathcal{F}$ of subgroups of a group G, a classifying space $E_{\mathcal{F}}G$ is a terminal object in the G-homotopy category of G-CW-complexes with stabilisers in $\mathcal{F}$. Farber, Grant, Lupton, and Oprea showed that $\mathsf{TC}(\pi)$ equals the minimal integer n for which the canonical $(\pi\times\pi)$-map $E(\pi\times\pi)\to E_{\mathcal{D}}(\pi\times\pi)$ is equivariantly homotopic to a map with values in the n-skeleton $E_{\mathcal{D}}(\pi\times\pi)^{(n)}$. Here $\mathcal{D}$ is the family of subgroups of $\pi\times\pi$ consisting of all conjugates of the diagonal subgroup $\Delta(\pi)$ and their subgroups. Using this characterisation of $\mathsf{TC}(\pi)$, in a recent breakthrough Dranishnikov [Dra20] has computed the topological complexity of torsion-free hyperbolic groups and more generally, of geometrically finite groups with cyclic centralisers.

Theorem 5.1 (Dranishnikov)**.** *Let π be a geometrically finite group with $\mathrm{cd}(\pi)\geq 2$ such that the centraliser $Z_\pi(b)$ is cyclic for every $b\in\pi\setminus\{e\}$. Then we have $\mathsf{TC}(\pi)=\mathrm{cd}(\pi\times\pi)$.*

Recall that a group π is called *geometrically finite* if it admits a finite model for $B\pi$. Note that for geometrically finite groups π we have $\mathrm{cd}(\pi\times\pi)=2\,\mathrm{cd}(\pi)$, see [Dra19]. Previously, Farber and Mescher [FM20] had shown for groups π as in Theorem 5.1 that $\mathsf{TC}(\pi)$ equals either $\mathrm{cd}(\pi\times\pi)$ or $\mathrm{cd}(\pi\times\pi)-1$. The main contribution of the present note is the following generalisation of Theorem 5.1.

Theorem 5.2. *Let π be a torsion-free group with $\mathrm{cd}(\pi)\geq 2$ admitting a malnormal collection of abelian subgroups $\mathcal{P}=\{P_i\mid i\in I\}$ that satisfy $\mathrm{cd}(P_i\times P_i)<\mathrm{cd}(\pi\times\pi)$. Suppose that the centraliser $Z_\pi(b)$ is cyclic for every $b\in\pi$ that is not conjugate into any of the P_i. Then $\mathsf{TC}(\pi)=\mathrm{cd}(\pi\times\pi)$.*

Recall that a set $\mathcal{P}=\{P_i\mid i\in I\}$ of subgroups of π is called a *malnormal collection* if for all $P_i,P_j\in\mathcal{P}$ and $g\in\pi$ we have $gP_ig^{-1}\cap P_j=\{e\}$, unless $i=j$ and $g\in P_i$. Our main examples of groups satisfying the assumptions of Theorem 5.2 are torsion-free relatively hyperbolic groups π with $\mathrm{cd}(\pi)\geq 2$ and finitely generated abelian peripheral subgroups $P_1,\ldots,P_k$ satisfying $\mathrm{cd}(P_i)<\mathrm{cd}(\pi)$. Note that Theorem 5.2 recovers Theorem 5.1 as a special case when $\mathcal{P}$ consists only of the trivial subgroup and that the assumption of geometric finiteness has been dropped.

In light of the upper bound $\mathsf{TC}(\pi)\leq\mathrm{cd}(\pi\times\pi)$, Theorem 5.1 and Theorem 5.2 are statements about the maximality of topological complexity. They share a common strategy of proof based on the characterisation of $\mathsf{TC}(\pi)$ in terms of classifying spaces from [FGLO19]. Namely, we construct a "small" model for $E_{\mathcal{D}}(\pi\times\pi)$ from $E(\pi\times\pi)$ allowing us to show that the map $E(\pi\times\pi)\to E_{\mathcal{D}}(\pi\times\pi)$ induces a non-trivial map on cohomology in degree $\mathrm{cd}(\pi\times\pi)$. Hence one has equality $\mathsf{TC}(\pi)=\mathrm{cd}(\pi\times\pi)$. Nevertheless, even for the case when $\mathcal{P}$ consists only of the trivial subgroup, our proof is different from Dranishnikov's. He constructed a specific model for $E_{\mathcal{D}}(\pi\times\pi)$ and used cohomology with compact support, while we employ a general construction due to Lück and Weiermann and use equivariant Bredon cohomology. Lück and Weiermann's construction (Theorem 5.3) is a general recipe to efficiently construct $E_{\mathcal{F}}G$ from $E_{\mathcal{E}}G$ for two families of subgroups $\mathcal{E} \subset \mathcal{F}$ of a group G satisfying a certain maximality condition. While for the group $\pi \times \pi$ this condition is not satisfied for the families $\{\{e\}\} \subset \mathcal{D}$, we define an intermediate family $\{\{e\}\} \subset \mathcal{F}_1 \subset \mathcal{D}$ such that we can apply two iterations of the construction.

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5.2 Preliminaries on classifying spaces for families

We briefly review the notion of classifying spaces for families of subgroups due to tom Dieck and their equivariant Bredon cohomology. For a survey on classifying spaces for families we refer to [Lüc05] and for an introduction to Bredon cohomology to [Flu]. Let G be a group, which shall always mean a discrete group.

A *family of subgroups* $\mathcal{F}$ is a non-empty set of subgroups of G that is closed under conjugation by elements of G and under taking subgroups. Typical examples are $\mathcal{TR} = \{\{e\}\}$, $\mathcal{FIN} = \{\text{finite subgroups}\}$, $\mathcal{VCY} = \{\text{virtually cyclic subgroups}\}$, and $\mathcal{ALL} = \{\text{all subgroups}\}$. For a set $\mathcal{H}$ of subgroups of G, one can consider the family $\mathcal{F}\langle\mathcal{H}\rangle = \{\text{conjugates of subgroups in } \mathcal{H} \text{ and their subgroups}\}$ which is the smallest family containing $\mathcal{H}$ and called the *family generated by* $\mathcal{H}$. When $\mathcal{H} = \{H\}$ consists of a single subgroup, we denote $\mathcal{F}\langle\{H\}\rangle$ instead by $\mathcal{F}\langle H\rangle$ and call it the *family generated by* H. Note that for two families $\mathcal{E}$ and $\mathcal{F}$ of subgroups of G, the union $\mathcal{E} \cup \mathcal{F}$ is again a family. For a family $\mathcal{F}$ of subgroups of G and any subgroup $H \subset G$, we denote by $\mathcal{F}|_H$ the family $\{K \cap H \mid K \in \mathcal{F}\}$ of subgroups of H. (In the literature this family is sometimes denoted by $\mathcal{F} \cap H$ instead.)

A *classifying space* $E_{\mathcal{F}}G$ *for the family* $\mathcal{F}$ is a terminal object in the G-homotopy category of G-CW-complexes with stabilisers in $\mathcal{F}$. It can be shown that $E_{\mathcal{F}}G$ always exists and that a G-CW-complex X is a model for $E_{\mathcal{F}}G$ if and only if the fixed-point set X^H is contractible for $H \in \mathcal{F}$ and empty otherwise. In particular, there exists a G-map $EG \to E_{\mathcal{F}}G$ which is unique up to G-homotopy.

The *orbit category* $\mathcal{O}_{\mathcal{F}}G$ has as objects G-sets of the form G/H for $H \in \mathcal{F}$ and as morphisms G-maps. Let $\mathcal{O}_{\mathcal{F}}G$-**Mod** denote the category of contravariant functors

$M: \mathcal{O}_{\mathcal{F}}G \to \mathbb{Z}$-**Mod** with values in the category of $\mathbb{Z}$-modules, which are called *$\mathcal{O}_{\mathcal{F}}G$-modules*. For a G-CW-complex X with stabilisers in $\mathcal{F}$, the *G-equivariant Bredon cohomology* $H^*_G(X;M)$ with coefficients in an $\mathcal{O}_{\mathcal{F}}G$-module M is the cohomology of the cochain complex $\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{-}\mathbf{Mod}}(C_*(X^?), M)$, where $C_*(X^?)(G/H) = C_*(X^H)$ is the cellular chain complex.

Passage to larger families. Let G be a group and $\mathcal{E} \subset \mathcal{F}$ be two families of subgroups.

We say that G satisfies *condition* $(M_{\mathcal{E}\subset\mathcal{F}})$ if every element $H \in \mathcal{F} \setminus \mathcal{E}$ is contained in a unique element $M \in \mathcal{F} \setminus \mathcal{E}$ which is maximal in $\mathcal{F} \setminus \mathcal{E}$ (with respect to inclusion). We say that G satisfies *condition* $(NM_{\mathcal{E}\subset\mathcal{F}})$ if every maximal element $M \in \mathcal{F} \setminus \mathcal{E}$ is self-normalising, i.e., M equals its normaliser N_GM in G. Let $\mathcal{M} = \{M_i \mid i \in I\}$ be a complete set of representatives for the conjugacy classes of maximal elements in $\mathcal{F} \setminus \mathcal{E}$, i.e., each M_i is maximal in $\mathcal{F} \setminus \mathcal{E}$ and every maximal element in $\mathcal{F} \setminus \mathcal{E}$ is conjugate to precisely one of the M_i. The following [LW12, Corollary 2.8] is a special case of a more general construction due to Lück and Weiermann.

Theorem 5.3 (Lück–Weiermann)**.** *Let G be a group satisfying condition $(M_{\mathcal{E}\subset\mathcal{F}})$ for two families of subgroups $\mathcal{E} \subset \mathcal{F}$. Consider a cellular G-pushout of the form*

$$\begin{array}{ccc} \coprod_{i\in I} G \times_{N_GM_i} E_{\mathcal{E}|_{N_GM_i}}(N_GM_i) & \xrightarrow{\varphi} & E_{\mathcal{E}}G \\ \downarrow{\scriptstyle \coprod_{i\in I} \mathrm{id}_G \times_{N_GM_i} f_i} & & \downarrow \\ \coprod_{i\in I} G \times_{N_GM_i} E_{\mathcal{ALL}|_{M_i}\cup\mathcal{E}|_{N_GM_i}}(N_GM_i) & \longrightarrow & X \end{array}$$

such that each f_i is a cellular N_GM_i-map and φ is an inclusion of G-CW-complexes, or such that each f_i is an inclusion of N_GM_i-CW-complexes and φ is a cellular G-map. Then X is a model for $E_{\mathcal{F}}G$.

Note that a G-pushout as in Theorem 5.3 with maps f_i and φ as required always exists by using equivariant cellular approximation and mapping cylinders.

Corollary 5.4. *Let G be a group and $\mathcal{E} \subset \mathcal{F}$ be two families of subgroups.*

(i) *If G satisfies condition $(M_{\mathcal{TR}\subset\mathcal{F}})$, then a model for $E_{\mathcal{F}}G$ can be constructed as a G-pushout of the form*

$$\begin{array}{ccc} \coprod_{i\in I} G \times_{N_GM_i} E(N_GM_i) & \longrightarrow & EG \\ \downarrow & & \downarrow \\ \coprod_{i\in I} G \times_{N_GM_i} E(N_GM_i/M_i) & \longrightarrow & E_{\mathcal{F}}G\,; \end{array}$$

(ii) *If G satisfies conditions $(M_{\mathcal{E}\subset\mathcal{F}})$ and $(NM_{\mathcal{E}\subset\mathcal{F}})$, then a model for $E_{\mathcal{F}}G$ can be constructed as a G-pushout of the form*

$$\begin{array}{ccc} \coprod_{i\in I} G\times_{M_i} E_{\mathcal{E}|_{M_i}}M_i & \longrightarrow & E_{\mathcal{E}}G \\ \downarrow & & \downarrow \\ \coprod_{i\in I} G/M_i & \longrightarrow & E_{\mathcal{F}}G\,. \end{array}$$

Proof. This follows from Theorem 5.3 by observing that if $\mathcal{E}|_{N_GM_i}\subset\mathcal{ALL}|_{M_i}$, then a model for $E_{\mathcal{ALL}|_{M_i}\cup\mathcal{E}|_{N_GM_i}}(N_GM_i)$ is given by $E(N_GM_i/M_i)$ regarded as a N_GM_i-CW-complex. □

Topological complexity via classifying spaces for families. Let G be a group and $\mathcal{E}\subset\mathcal{F}$ be two families of subgroups. The following notation is not standard.

We denote by $\mathrm{hdim}_{\mathcal{E}\subset\mathcal{F}}(G)$ the infimum of integers n for which the canonical G-map $E_{\mathcal{E}}G\to E_{\mathcal{F}}G$ is G-homotopic to a G-map with values in the n-skeleton $(E_{\mathcal{F}}G)^{(n)}$. We denote by $\mathrm{cd}_{\mathcal{E}\subset\mathcal{F}}(G)$ the supremum of integers k for which the induced map on Bredon cohomology $H^k_G(E_{\mathcal{F}}G;M)\to H^k_G(E_{\mathcal{E}}G;M)$ is non-trivial for some $\mathcal{O}_{\mathcal{F}}G$-module M. One clearly has the inequality

$$\mathrm{cd}_{\mathcal{E}\subset\mathcal{F}}(G)\le\mathrm{hdim}_{\mathcal{E}\subset\mathcal{F}}(G)\,. \tag{5.2}$$

Let π be a group and $\Delta(\pi)\subset\pi\times\pi$ be the diagonal subgroup. Consider the family $\mathcal{D}:=\mathcal{F}\langle\Delta(\pi)\rangle$ of subgroups of $\pi\times\pi$ that is generated by $\Delta(\pi)$. The following is the main result of [FGLO19, Theorem 3.3].

Theorem 5.5 (Farber–Grant–Lupton–Oprea)**.** *Let π be a group. Then we have*

$$\mathsf{TC}(\pi)=\mathrm{hdim}_{\mathcal{TR}\subset\mathcal{D}}(\pi\times\pi)\,.$$

Theorem 5.5 was recently generalised to families generated by a single subgroup in [BCVEB22, Theorem 1.1] and to arbitrary families in [CLM, Proposition 7.5].

5.3 Structure of the diagonal family of $\pi\times\pi$

Let π be a group and $\Delta\colon\pi\to\pi\times\pi$ be the diagonal map. For a subset $S\subset\pi$, denote by $Z_\pi(S)$ the centraliser of S in π. The following notation is adopted from [FGLO19] and [Dra20].

For $\gamma\in\pi$ and a subset $S\subset\pi$, define the subgroup $H_{\gamma,S}$ of $\pi\times\pi$ to be

$$H_{\gamma,S}:=(\gamma,e)\cdot\Delta(Z_\pi(S))\cdot(\gamma^{-1},e)\,.$$

When S is a singleton set $\{b\}$, we write $H_{\gamma,b}$ instead of $H_{\gamma,\{b\}}$. Note that we have $H_{e,e} = \Delta(\pi)$. The proof of the following identities is elementary and left to the reader.

Lemma 5.6. *Let $\gamma, \delta \in \pi$ and $S, T \subset \pi$ be subsets. Then the following hold:*

(i) $(g,h) \cdot H_{\gamma,S} \cdot (g^{-1}, h^{-1}) = H_{g\gamma h^{-1}, hSh^{-1}}$ *for all* $(g,h) \in \pi \times \pi$*;*

(ii) $H_{\gamma,S} \cap H_{\delta,T} = H_{\gamma, S\cup T\cup\{\delta^{-1}\gamma\}}$*;*

(iii) $N_{\pi\times\pi} H_{\gamma,S} = \{(\gamma k h \gamma^{-1}, h) \in \pi \times \pi \mid h \in N_\pi(Z_\pi(S)), k \in Z_\pi(Z_\pi(S))\}$*.*

We define the families $\mathcal{F}_1 \subset \mathcal{D}$ of subgroups of $\pi \times \pi$ to be

$$\begin{aligned} \mathcal{D} &\coloneqq \mathcal{F}\langle \Delta(\pi) \rangle\,; \\ \mathcal{F}_1 &\coloneqq \mathcal{F}\langle \{H_{\gamma,b} \mid \gamma \in \pi, b \in \pi \setminus \{e\}\} \rangle\,. \end{aligned} \tag{5.3}$$

In view of Lemma 5.6 (i) and (ii), the family $\mathcal{F}_1$ is generated by the intersections of conjugates of the diagonal subgroup $\Delta(\pi)$.

Lemma 5.7. *Let π be a group. Then condition $(M_{\mathcal{F}_1\subset\mathcal{D}})$ holds for the group $\pi \times \pi$. Moreover, if the centre $Z_\pi(\pi)$ of π is trivial, then condition $(NM_{\mathcal{F}_1\subset\mathcal{D}})$ holds.*

Proof. If $\mathcal{F}_1$ equals $\mathcal{D}$, then the statement is vacuous, so we may assume that $\mathcal{F}_1$ is strictly contained in $\mathcal{D}$. For $\gamma \in \pi$, conjugates of $H_{\gamma,e}$ are of the form $H_{\delta,e}$ for some $\delta \in \pi$ by Lemma 5.6 (i). If $\gamma \neq \delta$, then $H_{\gamma,e} \cap H_{\delta,e} \in \mathcal{F}_1$ by Lemma 5.6 (ii). Hence the $\{H_{\gamma,e} \mid \gamma \in \pi\}$ are precisely the maximal elements in $\mathcal{D}\setminus\mathcal{F}_1$ and condition $(M_{\mathcal{F}_1\subset\mathcal{D}})$ holds. Moreover, given that $Z_\pi(\pi)$ is trivial, we have $N_{\pi\times\pi}(H_{\gamma,e}) = H_{\gamma,e}$ by Lemma 5.6 (iii). □

From now on and for the remainder of this note, we specialise to the following situation.

Setup 5.8. Let π be a torsion-free group admitting a malnormal collection of abelian subgroups $\mathcal{P} = \{P_i \mid i \in I\}$ such that the centraliser $Z_\pi(b)$ is cyclic for every $b \in \pi$ that is not conjugate into any of the P_i.

Note that in the situation of Setup 5.8, we have $N_\pi(Z_\pi(P_i)) = Z_\pi(P_i) = P_i$ for every $P_i \in \mathcal{P}$. Our main examples are torsion-free relatively hyperbolic groups with finitely generated abelian peripheral subgroups, so-called *toral* relatively hyperbolic groups, which satisfy Setup 5.8 by [Osi06, Theorem 1.14].

The following Lemma 5.9 for the case when $\mathcal{P} = \{\{e\}\}$ can be found in [FGLO19, Lemma 8.0.4] from where the first part of the proof is recalled.

Lemma 5.9. *Let π be a group as in Setup 5.8. Then for $b, c \in \pi \setminus \{e\}$, we have either $Z_\pi(b) = Z_\pi(c)$ or $Z_\pi(b) \cap Z_\pi(c) = \{e\}$.*

Proof. Let $b, c \in \pi \setminus \{e\}$ be two elements. Suppose neither b nor c are conjugate into any of the P_i and that $Z_\pi(b) \cap Z_\pi(c)$ is non-trivial. Let $Z_\pi(b)$, $Z_\pi(c)$ and $Z_\pi(b) \cap Z_\pi(c)$ be generated by x, y and z, respectively. Then $x^n = z = y^m$ for some $n, m \in \mathbb{Z}$. Observe that z is not conjugate into any of the P_i. Thus its centraliser $Z_\pi(z)$ is infinite cyclic and contains both x and y. Therefore, x and y commute and it follows that $Z_\pi(b) = Z_\pi(c)$.

Suppose $b \in \pi \setminus \{e\}$ and $c \in gP_ig^{-1}$ for some $g \in \pi$, $P_i \in \mathcal{P}$. Note that $Z_\pi(c) = gP_ig^{-1}$. If $Z_\pi(b) \cap gP_ig^{-1}$ is non-trivial, then $b \in gP_ig^{-1}$ by malnormality of $\mathcal{P}$ and hence we have $Z_\pi(b) = Z_\pi(c)$. □

Lemma 5.10. *Let π be a group as in Setup 5.8. Then we have the following:*

(i) *Condition $(M_{\mathcal{TR}\subset\mathcal{F}_1})$ holds for the group $\pi\times\pi$. Moreover, for $\gamma \in \pi$ and $b \in \pi\setminus\{e\}$ there is an isomorphism $N_{\pi\times\pi}H_{\gamma,b} \cong Z_\pi(b) \times Z_\pi(b)$;*

(ii) *Conditions $(M_{\mathcal{TR}\subset\mathcal{F}_1|_{H_{e,e}}})$ and $(NM_{\mathcal{TR}\subset\mathcal{F}_1|_{H_{e,e}}})$ hold for the group $H_{e,e}$.*

Proof. (i) For $\gamma \in \pi$ and $b \in \pi\setminus\{e\}$, conjugates of $H_{\gamma,b}$ are of the form $H_{\delta,c}$ for some $\delta \in \pi$, $c \in \pi \setminus \{e\}$ by Lemma 5.6 (i). We have either $H_{\gamma,b} = H_{\delta,c}$ or $H_{\gamma,b} \cap H_{\delta,c} = \{(e,e)\}$ by Lemma 5.6 (ii) and Lemma 5.9. Hence the $\{H_{\gamma,b} \mid \gamma \in \pi, b \in \pi \setminus \{e\}\}$ are precisely the maximal elements in $\mathcal{F}_1 \setminus \mathcal{TR}$ and condition $(M_{\mathcal{TR}\subset\mathcal{F}_1})$ holds. Moreover, for $b \in \pi$ that is not conjugate into any of the P_i, observe that $N_\pi(Z_\pi(b))$ is torsion-free virtually cyclic and hence infinite cyclic. It follows that $N_\pi(Z_\pi(b)) = Z_\pi(b) \cong \mathbb{Z}$. If $b \in gP_ig^{-1}$ for some $g \in \pi$ and $P_i \in \mathcal{P}$, we have $N_\pi(Z_\pi(b)) = gP_ig^{-1}$ which is abelian and coincides with $Z_\pi(b)$. Thus, for every $b \in \pi \setminus \{e\}$ we have

$$N_{\pi\times\pi}H_{\gamma,b} = \{(\gamma kh\gamma^{-1}, h) \mid h, k \in Z_\pi(b)\} \cong Z_\pi(b) \times Z_\pi(b)$$

by Lemma 5.6 (iii).

(ii) Under the isomorphism $H_{e,e} \cong \pi$, the family $\mathcal{F}_1|_{H_{e,e}}$ is identified with the family $\mathcal{F}\langle\{Z_\pi(b) \mid b \in \pi \setminus \{e\}\}\rangle$. The claim follows as before by Lemma 5.9 and the observation that $Z_\pi(b)$ is self-normalising for every $b \in \pi \setminus \{e\}$. □

5.4 Maximality of topological complexity

The following is the main technical result of this note and will immediately imply Theorem 5.2.

Theorem 5.11. *Let π be a torsion-free group with $\mathrm{cd}(\pi) \geq 2$ admitting a malnormal collection of abelian subgroups $\mathcal{P} = \{P_i \mid i \in I\}$ that satisfy $\mathrm{cd}(P_i \times P_i) < \mathrm{cd}(\pi \times \pi)$. Suppose that the centraliser $Z_\pi(b)$ is cyclic for every $b \in \pi$ that is not conjugate into any of the P_i. Then $\mathrm{cd}_{\mathcal{TR}\subset\mathcal{D}}(\pi \times \pi) = \mathrm{cd}(\pi \times \pi)$.*

Proof. If $\mathrm{cd}(\pi \times \pi)$ is infinite, then so is $\mathrm{cd}(\pi)$. One observes that $\mathrm{cd}(\pi)$ coincides with $\mathrm{cd}_{\mathcal{TR}\subset\mathcal{D}|_{\pi\times\{e\}}}(\pi \times \{e\})$ which is a lower bound for $\mathrm{cd}_{\mathcal{TR}\subset\mathcal{D}}(\pi \times \pi)$ by Shapiro's Lemma for Bredon cohomology (see e.g., [Flu, Proposition 3.31]). We may assume that $\mathrm{cd}(\pi\times\pi)$ is finite and denote it by n. Note that $n \geq \mathrm{cd}(\pi \times \mathbb{Z}) = \mathrm{cd}(\pi) + 1 \geq 3$. Consider the families $\mathcal{TR} \subset \mathcal{F}_1 \subset \mathcal{D}$ of subgroups of $\pi \times \pi$ as defined in (5.3).

First, condition $(M_{\mathcal{TR}\subset\mathcal{F}_1})$ holds by Lemma 5.10 (i) and hence Corollary 5.4 (i) yields a $(\pi \times \pi)$-pushout

$$\begin{array}{ccc} \coprod_{H_{\gamma,b}\in\mathcal{M}}(\pi\times\pi)\times_{N_{\pi\times\pi}H_{\gamma,b}} E(N_{\pi\times\pi}H_{\gamma,b}) & \longrightarrow & E(\pi\times\pi) \\ \downarrow & & \downarrow \\ \coprod_{H_{\gamma,b}\in\mathcal{M}}(\pi\times\pi)\times_{N_{\pi\times\pi}H_{\gamma,b}} E(N_{\pi\times\pi}H_{\gamma,b}/H_{\gamma,b}) & \longrightarrow & E_{\mathcal{F}_1}(\pi\times\pi)\,, \end{array} \tag{5.4}$$

where $\mathcal{M}$ is a complete set of representatives of conjugacy classes of maximal elements in $\mathcal{F}_1 \setminus \mathcal{TR}$. Moreover, in Lemma 5.10 (i) we identified $N_{\pi\times\pi}H_{\gamma,b}$ with $Z_\pi(b) \times Z_\pi(b)$ which is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ or $P_i \times P_i$ for some $P_i \in \mathcal{P}$ and hence has cohomological dimension strictly less than n. Thus, for every $\mathcal{O}_{\mathcal{D}}(\pi \times \pi)$-module M, we have

$$H^n_{\pi\times\pi}((\pi\times\pi)\times_{N_{\pi\times\pi}H_{\gamma,b}} E(N_{\pi\times\pi}H_{\gamma,b}); M) = 0\,.$$

Applying the Mayer–Vietoris sequence for $H^*_{\pi\times\pi}(-;M)$ to the pushout (5.4) yields that the map

$$H^n_{\pi\times\pi}(E_{\mathcal{F}_1}(\pi\times\pi); M) \to H^n_{\pi\times\pi}(E(\pi\times\pi); M)$$

is surjective.

Second, conditions $(M_{\mathcal{F}_1\subset\mathcal{D}})$ and $(NM_{\mathcal{F}_1\subset\mathcal{D}})$ hold by Lemma 5.7 and hence Corollary 5.4 (ii) yields a $(\pi \times \pi)$-pushout

$$\begin{array}{ccc} (\pi\times\pi)\times_{H_{e,e}} E_{\mathcal{F}_1|_{H_{e,e}}}(H_{e,e}) & \longrightarrow & E_{\mathcal{F}_1}(\pi\times\pi) \\ \downarrow & & \downarrow \\ (\pi\times\pi)/H_{e,e} & \longrightarrow & E_{\mathcal{D}}(\pi\times\pi)\,. \end{array} \tag{5.5}$$

Applying the Mayer–Vietoris sequence for $H^*_{\pi\times\pi}(-;M)$ to the pushout (5.5) yields that the map

$$H^n_{\pi\times\pi}(E_{\mathcal{D}}(\pi\times\pi); M) \to H^n_{\pi\times\pi}(E_{\mathcal{F}_1}(\pi\times\pi); M)$$

is surjective provided that

$$H^n_{\pi\times\pi}((\pi\times\pi)\times_{H_{e,e}} E_{\mathcal{F}_1|_{H_{e,e}}}(H_{e,e}); M) = 0\,. \tag{5.6}$$

The latter is true by another application of Corollary 5.4 (ii) using that conditions $(M_{\mathcal{TR}\subset\mathcal{F}_1|_{H_{e,e}}})$ and $(NM_{\mathcal{TR}\subset\mathcal{F}_1|_{H_{e,e}}})$ hold for the group $H_{e,e}$ by Lemma 5.10 (ii). It yields an $H_{e,e}$-pushout

$$\begin{array}{ccc} \coprod_{H_{e,b}\in\mathcal{M}'} H_{e,e}\times_{H_{e,b}} E(H_{e,b}) & \longrightarrow & E(H_{e,e}) \\ \downarrow & & \downarrow \\ \coprod_{H_{e,b}\in\mathcal{M}'} H_{e,e}/H_{e,b} & \longrightarrow & E_{\mathcal{F}1|_{H_{e,e}}}(H_{e,e}) \, , \end{array} \tag{5.7}$$

where $\mathcal{M}'$ is a complete set of representatives of conjugacy classes of maximal elements in $\mathcal{F}1|_{H_{e,e}} \setminus \mathcal{TR}$. The Mayer–Vietoris sequence for $H^*_{H_{e,e}}(-;M)$ applied to the pushout (5.7) shows that (5.6) indeed holds, using that $\mathrm{cd}(H_{e,e}) < n$ and $\mathrm{cd}(H_{e,b}) < n-1$ for $b \in \pi \setminus \{e\}$.

Together, the map

$$H^n_{\pi\times\pi}(E_{\mathcal{D}}(\pi\times\pi);M) \to H^n_{\pi\times\pi}(E(\pi\times\pi);M)$$

is surjective for every $\mathcal{O}_{\mathcal{D}}(\pi\times\pi)$-module M. Finally, the coefficients M can be chosen such that $H^n_{\pi\times\pi}(E(\pi\times\pi);M)$ is non-trivial. This concludes the proof. □

Proof of Theorem 5.2. It follows from Theorem 5.11 that the inequalities

$$\mathrm{cd}_{\mathcal{TR}\subset\mathcal{D}}(\pi\times\pi) \le \mathsf{TC}(\pi) \le \mathrm{cd}(\pi\times\pi)$$

given by (5.1) and (5.2) are in fact equalities. □

Chapter 6

Conclusion

The mathematical part of this thesis has ended with Chapter 5.

We draw a conclusion by explaining how our articles demonstrate an original contribution to the subject and by examining their strengths and weaknesses. As is customary in pure mathematics, these aspects are addressed in the introductions of our papers. For the convenience of the reader, we repeat here shortened versions of the introductions from Sections 2.1, 3.1, 4.1, and 5.1. Some questions for future directions of research were posed in Section 1.3.4.

Bounded cohomology of classifying spaces for families

In Chapter 2, we introduce and study a bounded version of Bredon cohomology for groups with respect to a family of subgroups. The bounded cohomology $H_b^n(G; V)$ of a (discrete) group G with coefficients in a normed G-module V is the cohomology of the cochain complex of bounded G-maps $G^{n+1} \to V$. The inclusion of bounded G-maps into (not necessarily bounded) G-maps induces the so-called *comparison map* $H_b^n(G; V) \to H^n(G; V)$. On the one hand, the bounded cohomology groups are very difficult to compute in general. On the other hand, they characterise interesting group-theoretic properties such as amenability [Joh72] and hyperbolicity [Min01, Min02].

Theorem 6.1 (Johnson)**.** *Let G be a group. The following are equivalent:*

(i) *G is amenable;*

(ii) *$H_b^n(G; V^\#) = 0$ for all dual normed $\mathbb{R}G$-modules $V^\#$ and all $n \geq 1$;*

(iii) *$H_b^1(G; V^\#) = 0$ for all dual normed $\mathbb{R}G$-modules $V^\#$.*

Theorem 6.2 (Mineyev)**.** *Let G be a finitely presented group. The following are equivalent:*

(i) *G is hyperbolic;*

(ii) *The comparison map $H^n_b(G;V) \to H^n(G;V)$ is surjective for all normed $\mathbb{Q}G$-modules V and all $n \geq 2$;*

(iii) *The comparison map $H^2_b(G;V) \to H^2(G;V)$ is surjective for all normed $\mathbb{R}G$-modules V.*

There are well-studied notions of relative amenability and relative hyperbolicity in the literature [JOR12, Hru10]. In Chapter 2 we introduce a new "relative bounded cohomology theory" characterising these relative group-theoretic properties as a bounded version of Bredon cohomology. For a group G, a family of subgroups $\mathcal{F}$ is a non-empty set of subgroups which is closed under conjugation and taking subgroups. For a set of subgroups $\mathcal{H}$ of G, we denote by $\mathcal{F}\langle\mathcal{H}\rangle$ the smallest family containing $\mathcal{H}$. The Bredon cohomology $H^n_{\mathcal{F}}(G;V)$ with coefficients in a G-module V (or more general coefficient systems) is a generalisation of group cohomology, which is recovered when $\mathcal{F}$ consists only of the trivial subgroup. A fundamental feature of Bredon cohomology is that for a normal subgroup N of G there is an isomorphism $H^n_{\mathcal{F}\langle N\rangle}(G;V) \cong H^n(G/N;V^N)$. From a topological point of view, the Bredon cohomology of G can be identified with the equivariant cohomology of the classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$, which is a terminal object in the G-homotopy category of G-CW-complexes with stabilisers in $\mathcal{F}$.

We introduce the *bounded Bredon cohomology* $H^n_{\mathcal{F},b}(G;V)$ of G with coefficients in a normed G-module V, which generalises bounded cohomology (Definition 2.7). Our theory still is well-behaved with respect to normal subgroups (Corollary 2.17) and admits a topological interpretation in terms of classifying spaces for families (Theorem 2.16). We obtain the following generalisations of Theorems 6.1 and 6.2. A group G is called amenable relative to a set of subgroups $\mathcal{H}$ if there exists a G-invariant mean on the G-set $\coprod_{H\in\mathcal{H}} G/H$.

Theorem 6.3. *Let G be a group and $\mathcal{H}$ be a set of subgroups. The following are equivalent:*

(i) *G is amenable relative to $\mathcal{H}$;*

(ii) *$H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V^\#) = 0$ for all dual normed $\mathbb{R}G$-modules $V^\#$ and all $n \geq 1$;*

(iii) *$H^1_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V^\#) = 0$ for all dual normed $\mathbb{R}G$-modules $V^\#$.*

Theorem 6.3 is a special case of the more general Theorem 2.23. We also provide a characterisation of relative amenability in terms of relatively injective modules (Proposition 2.26). Recall that a finite set of subgroups $\mathcal{H}$ is called a malnormal (resp. almost malnormal) collection if for all $H_i, H_j \in \mathcal{H}$ and $g \in G$ we have $H_i \cap gH_jg^{-1}$ is trivial (resp. finite), unless $i = j$ and $g \in H_i$. A group G is said to be of type $F_{n,\mathcal{F}}$ for a family of subgroups $\mathcal{F}$, if there exists a model for the classifying space $E_{\mathcal{F}}G$ with cocompact n-skeleton.

Theorem 6.4 (Theorem 2.31)**.** *Let G be a finitely generated torsion-free group and $\mathcal{H}$ be a finite malnormal collection of subgroups. Suppose that G is of type $F_{2,\mathcal{F}\langle\mathcal{H}\rangle}$ (e.g., G and all subgroups in $\mathcal{H}$ are finitely presented). Then the following are equivalent:*

(i) *G is hyperbolic relative to $\mathcal{H}$;*

(ii) *The comparison map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V) \to H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V)$ is surjective for all normed $\mathbb{Q}G$-modules V and all $n \geq 2$;*

(iii) *The comparison map $H^2_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V) \to H^2_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V)$ is surjective for all normed $\mathbb{R}G$-modules V.*

In Theorem 6.4 the equivalence of (i) and (iii) still holds if the group G contains torsion and $\mathcal{H}$ is almost malnormal, see Remark 2.32. Note that condition (iii) is trivially satisfied for groups of Bredon cohomological dimension $\mathrm{cd}_{\mathcal{F}\langle\mathcal{H}\rangle}$ equal to 1.

The topological interpretation of bounded Bredon cohomology via classifying spaces for families was used by Löh–Sauer [LS20] to give a new proof of the Nerve Theorem and Vanishing Theorem for amenable covers. We prove a converse of [LS20, Proposition 5.2], generalising a recent result of [MR, Theorem 3.1.3] where the case of a normal subgroup is treated.

Theorem 6.5. *Let G be a group and $\mathcal{F}$ be a family of subgroups. The following are equivalent:*

(i) *All subgroups in $\mathcal{F}$ are amenable;*

(ii) *The canonical map $H^n_{\mathcal{F},b}(G;V^\#) \to H^n_b(G;V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$-modules $V^\#$ and all $n \geq 0$;*

(iii) *The canonical map $H^1_{\mathcal{F},b}(G;V^\#) \to H^1_b(G;V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$-modules $V^\#$.*

Theorems 6.5 is a special case of the more general Theorem 2.23. As an application of Theorem 6.5, the comparison map vanishes for groups which admit a "small" model for $E_{\mathcal{F}}G$, where $\mathcal{F}$ is any family consisting of amenable subgroups (Corollary 2.24). Examples are graph products of amenable groups (e.g., right-angled Artin groups) and fundamental groups of graphs of amenable groups.

There is another natural relative cohomology theory given by the relative cohomology of a pair of spaces. For a set of subgroups $\mathcal{H}$, it gives rise to the cohomology $H^n(G,\mathcal{H};V)$ of the group pair $(G,\mathcal{H})$ introduced by Bieri–Eckmann [BE78]. A bounded version $H_b^n(G,\mathcal{H};V)$ was defined by Mineyev–Yaman [MY] to give a characterisation of relative hyperbolicity (see also [Fra18]). A characterisation of relative amenability in terms of this relative theory was obtained in [JOR12]. There is a canonical map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V) \to H^n(G,\mathcal{H};V)$ for $n \geq 2$ which is an isomorphism if $\mathcal{H}$ is malnormal (see Remark 2.6). Similarly, there is a map for the bounded versions but we do not know when it is an isomorphism due to the failure of the excision axiom for bounded cohomology (see Remark 2.18). We also mention that Mineyev–Yaman's relative bounded cohomology was extended to pairs of groupoids in [Bla16].

Bounded acyclicity and relative simplicial volume

In Chapter 3, joint with Clara Löh and Marco Moraschini, we provide new vanishing results for relative simplicial volume, following up on two current themes in bounded cohomology:

- The passage from amenable groups to boundedly acyclic groups;
- The use of equivariant topology, most notably of classifying spaces for families of subgroups.

A technical difficulty in the passage from amenable to boundedly acyclic groups is that the class of amenable groups possesses a large degree of uniformity when it comes to bounded cohomology. This includes the fact that the class of amenable groups is closed under subgroups and quotients and the fact that amenable groups are not only boundedly acyclic, but uniformly boundedly acyclic. Therefore, in the setting of boundedly acyclic groups, generalised vanishing results for simplicial volume come with additional uniformity and closure hypotheses.

As we aim at results for relative bounded cohomology and relative simplicial volume, we adapt tools from equivariant topology to this relative setting.

Uniform bounded acyclicity. Group actions with amenable stabilisers have proved to be a valuable tool to compute bounded cohomology [Mon01, BM02, BI09]. Similarly, also *uniformly boundedly acyclic actions* allow us to compute bounded cohomology, where the uniformity refers to a uniform bound on the norms of primitives. Recently, uniformly boundedly acyclic actions have been used to compute the bounded cohomology of geometrically relevant groups [FFLMa, MN].

Let X be a path-connected space. We say that a set of path-connected subspaces $\mathcal{A}$ of X is *uniformly boundedly acyclic* [*of order* n] *in* X if the collection of all finite [resp. n-fold] intersections of conjugates of the subgroups

$$\big(\operatorname{im}(\pi_1(A \hookrightarrow X))\big)_{A\in\mathcal{A}}$$

in $\pi_1(X)$ is uniformly boundedly acyclic (Definition 3.58). In the special case when the above groups are amenable, we also speak of an *amenable* set of subspaces. We have two geometric situations in which uniformly boundedly acyclic sets of subspaces lead to interesting uniformly boundedly acyclic actions: Open covers and glueing loci of manifolds obtained by glueing manifolds with boundary.

Vanishing via relative open covers. Gromov [Gro82] and Ivanov [Iva85] established vanishing results for the comparison map (and thus for simplicial volume) in the presence of amenable open covers with small multiplicity.

Following the approach by Löh and Sauer [LS20] through equivariant nerves and classifying spaces for families, we generalise these vanishing results in two directions. First, we allow more general covers: A cover $\mathcal{U}$ of X by path-connected open subsets is *uniformly boundedly acyclic* if the underlying set of subsets of X is uniformly boundedly acyclic in X. Second, we adapt the setting to pairs of CW-complexes (X, A), where A is π_1-injective in X (Theorem 3.76). To this end, we introduce the notion of [*weakly convex*] *relative covers* (Definition 3.41). Using equivariant nerve pairs and classifying spaces of group pairs for families, we obtain:

Theorem 6.6 (Corollary 3.78)**.** *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A has only finitely many connected components, each of which is π_1-injective in X. Let $\mathcal{U}$ be a relative cover of (X, A) that is uniformly boundedly acyclic.*

(i) *If $\mathcal{U}$ is weakly convex, then the comparison map*

$$\operatorname{comp}^k \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \operatorname{mult}_A(\mathcal{U})$.

(ii) *Let $\nu \colon (X, A) \to (|N(\mathcal{U})|, |N_A(\mathcal{U})|)$ be a nerve map. If $\mathcal{U}$ is convex, then the comparison map* comp^* *factors through ν:*

$$\begin{array}{ccc} H_b^*(X, A; \mathbb{R}) & \xrightarrow{\operatorname{comp}^*} & H^*(X, A; \mathbb{R}) \\ & \searrow & \uparrow {\scriptstyle H^*(\nu; \mathbb{R})} \\ & & H^*(|N(\mathcal{U})|, |N_A(\mathcal{U})|; \mathbb{R}) . \end{array}$$

Here $\operatorname{mult}_A(\mathcal{U})$ denotes the relative multiplicity of $\mathcal{U}$ with respect to A (Definition 3.35) and the simplicial complex $N_A(\mathcal{U})$ is a suitable subcomplex of the nerve $N(\mathcal{U})$ (Definition 3.36).

In the absolute case, Ivanov proved a similar vanishing theorem for *weakly boundedly acyclic covers* using spectral sequences [Iva]. Our notion of uniformly boundedly acyclic covers is similar, but the relation between the two is unclear (Remark 3.69).

Theorem 6.6 applies in particular to relative covers that are amenable. We introduce the *relative amenable multiplicity* $\operatorname{mult}_{\mathcal{AME}}(X, A)$ (Definition 3.45) as the minimal relative multiplicity of weakly convex relative amenable covers of (X, A) by path-connected open subsets.

Theorem 6.7 (Corollary 3.80). *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A consists of finitely many connected components, each of which is π_1-injective in X. Then the comparison map*

$$\operatorname{comp}^k \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \operatorname{mult}_{\mathcal{AME}}(X, A)$.

In particular, if $(M, \partial M)$ is an oriented compact connected triangulable manifold with π_1-injective boundary components and $\operatorname{mult}_{\mathcal{AME}}(M, \partial M) \leq \dim(M)$, then the relative simplicial volume $\|M, \partial M\|$ vanishes.

As an application of Theorem 6.7, we give an alternative proof of a relative vanishing theorem, which is a consequence of Gromov's vanishing theorem for non-compact manifolds (Theorem 3.82).

Our methods for equivariant nerve pairs and relative classifying spaces also lead to vanishing results for ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable multiplicity (Theorem 3.86). In the absolute case (Corollary 3.87), this recovers a result by Sauer [Sau09, Theorem C].

Glueings. We adapt the additivity of relative simplicial volume for glueings along amenable boundaries [Gro82, BBF$^+$14, Kue15] to situations with boundedly acyclic boundaries. As we move away from amenability, we lose control on the norm, and thus only retain control on the vanishing behaviour.

Theorem 6.8 (Theorem 3.88). *Let $n \geq 3$ and $(M_i, \partial M_i)_{i \in I}$ be a finite collection of oriented compact connected n-manifolds. Assume that every connected component of every boundary component ∂M_i has boundedly acyclic fundamental group. Let $\mathcal{N}$ be a set of π_1-injective boundary components of the $(M_i)_{i \in I}$ and let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i \in I}$ by a pairwise glueing of the boundary components in $\mathcal{N}$.*

If $\mathcal{N}$, viewed as a set of subsets of M, is uniformly boundedly acyclic of order n in M, then the following are equivalent:

(i) *We have $\|M, \partial M\| = 0$;*

(ii) *For all $i \in I$, we have $\|M_i, \partial M_i\| = 0$.*

Mapping degrees. We use equivariant and group cohomological methods to establish the following relative version (and a simplified proof) of a monotonicity result by Dranishnikov and Rudyak for closed manifolds [DR09]:

Theorem 6.9 (Corollary 3.29)**.** *Let $f\colon (M, \partial M) \to (N, \partial N)$ be a map between oriented compact connected manifolds of the same dimension with π_1-injective boundary components. Let $\partial M = \coprod_{i=1}^{m} M_i$ and $\partial N = \coprod_{i=1}^{n} N_i$ be decompositions into connected components. If $\deg(f) = \pm 1$ and there exists a free group F_M such that*

$$\pi_1(M) \cong F_M * *_{i=1}^{m} \pi_1(M_i)\,,$$

then there exists a free group F_N such that $\pi_1(N) \cong F_N * *_{i=1}^{n} \pi_1(N_i)$.

For closed manifolds our approach also yields inheritance properties for virtual freeness:

Theorem 6.10 (Corollary 3.31)**.** *Let $f\colon M \to N$ be a map between oriented closed connected manifolds of the same dimension. If $\deg(f) \neq 0$ and $\pi_1(M)$ is virtually free, then also $\pi_1(N)$ is virtually free.*

Amenable covers of right-angled Artin groups

In Chapter 4, we compute the amenable category for all right-angled Artin groups. A subset U of a topological space X is said to be *amenable in X* if the group

$$\operatorname{im}\bigl(\pi_1(U \hookrightarrow X, x)\bigr)$$

is amenable for every basepoint $x \in U$. The *amenable category* $\operatorname{cat}_{\mathcal{AME}}(X)$ of X is the minimal $n \in \mathbb{N}_{\geq 0}$ for which there exists an open cover $X = \bigcup_{i=0}^{n} U_i$ by $n+1$ many amenable subsets. Clearly, we have $\operatorname{cat}_{\mathcal{AME}}(X) \leq \text{LS-cat}(X)$.

The focus of Chapter 4 is on the amenable category of aspherical spaces. Since the amenable category is a homotopy invariant, it yields an invariant of discrete groups G by setting $\operatorname{cat}_{\mathcal{AME}}(G) := \operatorname{cat}_{\mathcal{AME}}(BG)$. Here BG is an Eilenberg–MacLane space. By the classical work of [EG57, Sta68, Swa69], the LS-category LS-cat(BG) coincides with the cohomological dimension $\operatorname{cd}(G)$. In particular, we always have $\operatorname{cat}_{\mathcal{AME}}(G) \leq \operatorname{cd}(G)$. The amenable category is difficult to compute in general, the usual strategy being to exhibit an explicit open cover by amenable subsets and to prove its minimality using (co)homological obstructions. The precise value of $\operatorname{cat}_{\mathcal{AME}}(G)$ is known, e.g., for the following classes of groups:

- $\operatorname{cat}_{\mathcal{AME}}(G) = 0$ if and only if G is amenable;
- $\operatorname{cat}_{\mathcal{AME}}(G) = 1$ if and only if G is a non-amenable fundamental group of a graph of amenable groups [CLM, Corollary 5.4];

- $\mathrm{cat}_{\mathcal{AME}}(G) = \mathrm{cd}(G)$ if G is torsion-free non-elementary hyperbolic [Min01][CLM, Example 7.8].

The main result of Chapter 4 is a computation of the amenable category for all right-angled Artin groups. These form an important class of groups in geometric group theory, interpolating between free groups and free abelian groups. Let L be a finite flag complex (i.e., a simplicial complex in which every clique spans a simplex) with vertex set V. The right-angled Artin group A_L has as generators vertices $v \in V$, subject to the relation that v_1 and v_2 commute if and only if they are connected by an edge in L. The right-angled Coxeter group W_L is the quotient of A_L obtained by adding the relations that each generator $v \in V$ is of order 2. Since W_L is virtually torsion-free, its virtual cohomological dimension $\mathrm{vcd}(W_L)$ is well-defined as the cohomological dimension of a finite index torsion-free subgroup.

Theorem 6.11 (Corollary 4.17)**.** *Let A_L be the right-angled Artin group associated to a finite flag complex L. Then we have*

$$\mathrm{cat}_{\mathcal{AME}}(A_L) = \mathrm{vcd}(W_L)\,.$$

Theorem 6.11 provides many examples of groups for which the amenable category is not extremal, in the sense that $1 < \mathrm{cat}_{\mathcal{AME}}(G) < \mathrm{cd}(G)$. Furthermore, it follows from Theorem 6.11 and [Dra97] that there are right-angled Artin groups A_{L_1} and A_{L_2} satisfying $\mathrm{cat}_{\mathcal{AME}}(A_{L_1} \times A_{L_2}) < \mathrm{cat}_{\mathcal{AME}}(A_{L_1}) + \mathrm{cat}_{\mathcal{AME}}(A_{L_2})$.

Another invariant of a similar spirit is Farber's topological complexity TC which is motivated by the motion planning problem in robotics [Far03]. In [CLM, Question 8.1] it is asked for which topological spaces X the following inequality holds:

$$\mathrm{cat}_{\mathcal{AME}}(X \times X) \leq \mathsf{TC}(X)\,.$$

Examples of spaces and groups satisfying this inequality can be found in [CLM, Section 8], and no counter-example seems to be known at the time of writing. We show that all right-angled Artin groups are positive examples.

Theorem 6.12 (Proposition 4.18)**.** *Let A_L be the right-angled Artin group associated to a finite flag complex L. Then we have* $\mathrm{cat}_{\mathcal{AME}}(A_L \times A_L) \leq \mathsf{TC}(A_L)$.

We also obtain a complete characterisation of right-angled Artin groups with (non-)vanishing minimal volume entropy (Theorem 4.20), resolving the cases that were not covered by recent work in [HS, BC21].

Our proofs rely on combining upper and lower bounds (Lemma 4.4) with existing results on generalised LS-category, classifying spaces for families of subgroups, and homology growth from [CLM, HS, LM, OS, PP, Sau16].

Topological complexity of relatively hyperbolic groups

In Chapter 5, we compute the topological complexity for certain toral relatively hyperbolic groups. The (reduced) topological complexity $\mathsf{TC}(X)$ of a space X is defined as the minimal integer n for which there exists a cover of $X \times X$ by $n+1$ open subsets $U_0, \ldots, U_n$ such that the path fibration $X^{[0,1]} \to X \times X$ admits a local section over each U_i. This quantity, which is similar in spirit to the classical Lusternik–Schnirelmann category, was introduced by Farber [Far03] in the context of robot motion planning. In fact, $\mathsf{TC}(-)$ is a homotopy invariant and hence one can define the topological complexity $\mathsf{TC}(\pi)$ of a group π to be $\mathsf{TC}(B\pi)$, where $B\pi$ is a classifying space for π. There are bounds (see e.g., [FGLO19, FM20, Dra20])

$$\mathrm{cd}(\pi) \leq \mathsf{TC}(\pi) \leq \mathrm{cd}(\pi \times \pi)\,, \tag{6.1}$$

where $\mathrm{cd}(-)$ denotes the cohomological dimension. However, the precise value of $\mathsf{TC}(\pi)$ is known only for a relatively small class of groups.

Farber, Grant, Lupton, and Oprea [FGLO19] showed that $\mathsf{TC}(\pi)$ equals the minimal integer n for which the canonical $(\pi \times \pi)$-map $E(\pi \times \pi) \to E_{\mathcal{D}}(\pi \times \pi)$ is equivariantly homotopic to a map with values in the n-skeleton $E_{\mathcal{D}}(\pi \times \pi)^{(n)}$. Here $\mathcal{D}$ is the family of subgroups of $\pi \times \pi$ consisting of all conjugates of the diagonal subgroup $\Delta(\pi)$ and their subgroups. Using this characterisation of $\mathsf{TC}(\pi)$, in a recent breakthrough Dranishnikov [Dra20] has computed the topological complexity of torsion-free hyperbolic groups and more generally, of geometrically finite groups with cyclic centralisers.

Theorem 6.13 (Dranishnikov)**.** *Let π be a geometrically finite group with $\mathrm{cd}(\pi) \geq 2$ such that the centraliser $Z_\pi(b)$ is cyclic for every $b \in \pi \setminus \{e\}$. Then we have $\mathsf{TC}(\pi) = \mathrm{cd}(\pi \times \pi)$.*

Recall that a group π is called *geometrically finite* if it admits a finite model for $B\pi$. Note that for geometrically finite groups π we have $\mathrm{cd}(\pi \times \pi) = 2\,\mathrm{cd}(\pi)$, see [Dra19]. Previously, Farber and Mescher [FM20] had shown for groups π as in Theorem 6.13 that $\mathsf{TC}(\pi)$ equals either $\mathrm{cd}(\pi \times \pi)$ or $\mathrm{cd}(\pi \times \pi) - 1$. The main contribution of Chapter 5 is the following generalisation of Theorem 6.13.

Theorem 6.14. *Let π be a torsion-free group with $\mathrm{cd}(\pi) \geq 2$ admitting a malnormal collection of abelian subgroups $\mathcal{P} = \{P_i \mid i \in I\}$ that satisfy $\mathrm{cd}(P_i \times P_i) < \mathrm{cd}(\pi \times \pi)$. Suppose that the centraliser $Z_\pi(b)$ is cyclic for every $b \in \pi$ that is not conjugate into any of the P_i. Then $\mathsf{TC}(\pi) = \mathrm{cd}(\pi \times \pi)$.*

Our main examples of groups satisfying the assumptions of Theorem 6.14 are torsion-free relatively hyperbolic groups π with $\mathrm{cd}(\pi) \geq 2$ and finitely generated abelian peripheral subgroups $P_1, \ldots, P_k$ satisfying $\mathrm{cd}(P_i) < \mathrm{cd}(\pi)$. Note that Theorem 6.14 recovers Theorem 6.13 as a special case when $\mathcal{P}$ consists only of the trivial subgroup and that the assumption of geometric finiteness has been dropped.

In light of the upper bound $\mathsf{TC}(\pi) \leq \mathrm{cd}(\pi \times \pi)$, Theorem 6.13 and Theorem 6.14 are statements about the maximality of topological complexity. They share a common strategy of proof based on the characterisation of $\mathsf{TC}(\pi)$ in terms of classifying spaces from [FGLO19]. Namely, we construct a "small" model for $E_{\mathcal{D}}(\pi \times \pi)$ from $E(\pi \times \pi)$ allowing us to show that the map $E(\pi \times \pi) \to E_{\mathcal{D}}(\pi \times \pi)$ induces a non-trivial map on cohomology in degree $\mathrm{cd}(\pi \times \pi)$. Hence one has equality $\mathsf{TC}(\pi) = \mathrm{cd}(\pi \times \pi)$. Nevertheless, even for the case when $\mathcal{P}$ consists only of the trivial subgroup, our proof is different from Dranishnikov's. He constructed a specific model for $E_{\mathcal{D}}(\pi \times \pi)$ and used cohomology with compact support, while we employ a general construction due to Lück and Weiermann and use equivariant Bredon cohomology.

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