UNIVERSITY OF SOUTHAMPTON

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Mathematical Sciences

Realisation of Quantum Entanglement and Chaos in Gravity

by

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ABSTRACT

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Originating from string theory, the holographic correspondence provides a dictionary to convert a quantum theory on the boundary of the Anti-deSitter space (AdS) into a theory of gravity in the bulk AdS space. In this thesis we will study the intersection of quantum information and quantum gravity, focusing on methods of quantifying quantum entanglement and chaos in gravity via the AdS/CFT holographic correspondence.

Entanglement entropy of a bipartite quantum system on the boundary is equivalent to the area of a minimal surface in the bulk. In both the boundary and bulk pictures, entanglement entropy is divergent, meaning it equals to infinity. Hence we need to renormalise the entanglement entropy to obtain a finite quantity. The variation of the entanglement entropy is related to the dynamics of the bulk spacetime via the first law of entanglement entropy. We will first present a way to express the renormalised entanglement entropy in terms of the Euler invariant of the bulk entangling surface and other renormalised curvature invariants. Then we use this expression and independently derived the renormalised version of the first law of entanglement entropy. In particular, we use the Hamiltonian formalism of holographic renormalisation to derive the integral form of the first law of entanglement entropy.

Quantum chaos is characterised by the scrambling of information that increases exponentially in time. The rate of the exponential growth, known as the Lyapunov exponent, can be measured via the out-of-time-ordered correlation function (OTOC). In holography, the OTOC becomes the gravitational scattering amplitude of high energy particles. We investigate a possible correction to the Lyapunov exponent by considering the classical stringy effect in the bulk gravitational scattering. Following the semi-classical shock wave calculation of gravitational eikonal scattering, we obtain the classical string transverse oscillation contribution to the eikonal phase. We conclude such correction is negligible in the high energy eikonal limit, hence satisfying the chaos bound.

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Declaration of Authorship

I, Linus Ho Yi Too, declare that the thesis entitled *Realisation of Quantum Entanglement and Chaos in Gravity* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
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- where I have consulted the published work of others, this is always clearly attributed;
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- I have acknowledged all main sources of help;
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- parts of this work have been published as:
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CHAPTER 1

Introduction

A theory in Physics is a way of quantifying information in physical systems. The two foundational theories in mordern physics are, the theory of general relativity that describes gravity in large scale in terms of geometrical quantities, and the theory of quantum mechanics that describe particles dynamics in small scale in terms of probabilistic quantities. The quest of quantum gravity is to find a microscopic description of gravity that is consistent with quantum mechanics. One of the theory of quantum gravity is the holographic correspondence that provides a duality between gravitational systems and quantum mechanical systems.

But before going into the details of holography and quantum information, let us see an illustrative example of equivalent descriptions between the macroscopic theory of thermodynamics and the microscopic theory of statistical physics. The laws of thermodynamics are so general that it withstands the bizarre development the quantum mechanics, and even reincarnate itself in the theory of gravity. Classical thermodynamics is a powerful framework allowing us to describe the physics of systems with quantities like energy, temperature and entropy. These thermodynamics quantities give a macroscopic description of the systems in the sense that the nature of its constituents is neglected. Credit to Boltzmann and others, thermodynamics can be explained via statistical behaviour of microscopic theories. Most notably, the Boltzmann entropy S_B formula relates the thermal entropy to the size of the ensemble of microscopic configurations Ω,

$$S_B = k_B \log \Omega \tag{1.0.1}$$

where k_B is the Boltzmann constant. This later led to the discovery of Shannon entropy in information theory and entanglement entropy in quantum information which we will review in section 2.3. Often entropy is called the measure of disorder because large entropy means many microscopic configurations are allowed.

Our focus will be on the equivalent descriptions between the macroscopic theory of gravity and the microscopic quantum theory. This duality is called gauge/gravity duality or holography for reason explained below. The first evidence of holography is the discovery of black holes thermodynamics that is only characterised by the black hole's energy, temperature and entropy [3]. Particularly insightful is the Bekenstein-Hawking black hole entropy formula [4],

$$S_{BH} = \frac{A(\mathcal{H})}{4G_N}, \tag{1.0.2}$$

where A is the area of the black hole horizon $\mathcal{H}$ and G_N is the Netwon's constant. The fact that black hole has finite entropy suggests there are finite number of possible microscopic configurations given a fixed horizon area. This means information of the whole black hole is stored on the horizon, hence the black hole interior can be viewed as the hologram of the horizon. The current holographic correspondence states a gravitational theory of a spacetime is equivalent to a quantum theory on the boundary of that spacetime, which we will review in section 2.2. Hence via holography one can relate gravity in the bulk to quantum mechanical system living on the boundary. If this duality holds in general, this strongly suggests gravity in spacetime is an emergent property of some quantum system. So all gravitational description in the bulk is simply the average description of the quantum ensemble.

As information theory was inspired by the formulation of statistical physics and thermodynamics. The interest in the study of quantum information theory within holography is growing rapidly. The realisation of quantum information quantities and phenomena in the bulk theory of gravity not only provides us additional methods of computation, it also gives us great insights into the quantum nature of spacetime itself. There have been holographic realisation of entanglement entropy [5, 6, 7, 8], quantum chaos [9, 10], quantum complexity [11, 12, 13, 14, 15] and quantum computing code [16]. We are going to investigate methods of utilising this duality to describe entanglement entropy and quantum chaos in the bulk.

Quantum mechanics is intrinsically probabilistic and a key enhancement from classical statistical system is the property of entanglement. Entanglement makes a system not separable, so an individual subsystem cannot be fully described as a pure quantum state. Heuristically entangled quantum system means from the sole perspective of an individual subsystem, it forms a classical ensemble of quantum states instead of one single quantum state. This property of entanglement allow us to quantify the amount of entanglement using entropy. In holographic system, the entanglement entropy, S_{EE}, is equal to the area of a surface, Σ, partitioning the bulk systems which given by the Ryu-Takayangi formula [5, 6]

$$S_{EE} = \frac{A(\Sigma)}{4G_N}, \tag{1.0.3}$$

where S_{EE} is the entanglement entropy, $A(\Sigma)$ is the area of the entangling surface Σ and G_N is the gravitational constant. This should remind you of the Bekenstein-Hawking black hole entropy formula. The Ryu-Takayangi prescription of entanglement entropy in gravity teaches us that spacetime itself is related to entanglement. Entanglement entropy plays an important role in our quest to understand quantum gravity, that includes recent development on reconstruction of spacetime from entanglement [17, 18, 19] and the black hole information paradox [20, 21, 22].

Quantum chaos is quantified by the growth of certain correlation function of operators in the quantum system that measures how the interference between operators increases with time. On the gravity side of the holographic duality, this correlation function is realised in the form of high energy scattering amplitude of two particles [23]. A more chaotic system will induce more interference between the two particles in the scattering process. Since the gauge/gravity duality originated from string theory, we can probe the correction to quantum chaos effect coming from stringy scattering i.e. replacing the particles with strings. We will show addition correction will only make the system less chaotic. There is a proposed bound on chaos for quantum system dual to gravity [24]. Analogous to the principle of maximum entropy which states the most likely macroscopic state has maximum entropy. The fact classical gravity or general relativity is the most chaotic theory of gravity may indicate why our universe is governed by general relativity instead of other theory of gravity in the low energy limit.

Renormalisation is an essential procedure if we want our quantity of interest to be finite. In essense, renormalisation in quantum field theory is a procedure that removes the divergences coming from higher order interactions. One introduces counterterms to cancel the divegences, the famous infinity minus infinity "trick", $\infty - \infty$. However, it is important to know counterterm renormalisation does have its root in the elegant Wilsonian renormalisation. The major obstacle to quantising gravity is due to its non-renormalisability. In the direct quantization of gravity, after applying the perturbative renormalisation procedure, one finds the coupling increases as the energy scale increases. This means one would need infinite amount of counterterms to remove all the divergences. However, there are renormalisable theory of gravity.

We have motivated the topics of interest of the thesis, namely, holography, entanglement entropy, chaos and renormalisation. Now we will go through the developments in past few decades leading up to the current stage of research. After the success of the standard model, the quantum field theory description of all interactions except gravity, people tried to incorporate quantum mechanics into quantum gravity. The early attempts of semi-classical gravity that introduced relativistic quantum mechanics into curved spacetime produced extraordinary results such as Hawking radiation [25], black holes thermodynamics [3] and Regge behaviour in high energy gravitational scattering [26, 27]. From the former results, Hawking developed the black holes information paradox that until now still intrigues physicists. The form of Regge behaviour in quantum field theory and gravity set the point of reference for the full theory of quantum gravity. Remarkably, string theory was able to reproduce the Regge behaviour of the high energy gravitational scattering amplitude [28, 29]. By going to the two dimensional worldsheet, string theory is renormlisable. Not only that, the vanishing of beta functions give the equations of motion to classical gravity. Subsequently, with the formulation of D-brane, string theory is now able to describe both gauge theory and black hole physics. In a specific set up, a string theory with D-branes related gauge theory and gravity in curved background, this was the birth of the holographic correspondence [30, 31]. On top of opening the door for us to investigate the quantum nature of gravity. Holographic principle, began from the black holes thermodynamics and through the success of string theory, transforms into a powerful technique that can be applied to many physical systems. The applications of holography range from the fundamentals in high energy physics like the form of entanglement entropy in CFT to beyond high energy physics like cosmology [32], condensed matter systems and hydrodynamics [33]. The field of holography is growing rapidly, by combining techniques from quantum information and computing, many fascinating holographic models were developed. Among most models, entanglement entropy and chaotic behaviour are key quantities and features to study. For example, entanglement entropy plays an important role in the new proposals on the resolution of black holes information paradox [20, 21, 22].

In this thesis, we are going to highlight the recent development of renormalised entanglement entropy, its application to the dynamics of gravity and the chaotic behaviour of holographic system in the present of classical string. As a starter, in chapter 2, we are going to review the background materials. We will show the origin of holography and elaborate on the holographic renormalisation procedure that will be used extensively in chapter 3 and 4. We will learn about entanglement entropy, from the definition to the calculation in quantum field theory and eventually in holography. Since correlation function and scattering amplitude play important roles in the way of quantifying chaotic behaviour in quantum system and gravity, we will review the key results and techniques in quantum and gravitational scatterings. Finally, we will describe how to quantify chaos using the out-of-time-order correlation function in quantum system and in holography.

In chapter 3, we present a new way of expressing the renormalised entanglement entropy in even spacetime dimension by writing the renormalised area of the bulk entangling surface in terms of Euler characteristic of the bulk entangling surface and other curvature invariants defined with respect to the bulk entangling surface. In chapter 4, we will discuss about the first law entanglement entropy and its application in gravity. Then we will present three methods of obtaining the renormalised version of the first law of entanglement entropy, first by direct renormalisation of the infinitesimal first law, second by applying the renormalised entanglement entropy formula developed in chapter 3 and lastly by applying holographic renormalisation in the Hamiltonian formalism to renormlise the integral first law in terms of charges. In chapter 5, we will be investigating a possible correction to the holographic chaos by introducing classical string dynamics in the bulk. We will proceed to calculate subleading and next-to-subleading contributions to the Lyapunov exponent induced by the classical string oscillation. In chapter 6, we will end with the conclusion and outlook from the methodologies and results developed in the previous chapters.

CHAPTER 2

Background

In this chapter we are going to review the background on holographic entanglement entropy and holographic quantum chaos. Since these two subjects relate concepts in quantum information to dynamics in gravity, for each subject we will establish the relevant fundamentals on the quantum side then move to the related topics on the gravity side and finally present the holographic version. But first let us review the basics of Anti deSitter space and holographic duality.

2.1 Anti deSitter space

The following three chapters will mainly be in the Anti deSitter space. So in this section we will provide some background on the techniques used in the later chapters. Anti deSitter space is a Lorentzian space with constant negative curvature; it is a solution to the Einstein equation with negative cosmological constant. In particular, the $d+1$ dimensional AdS geometry is a submanifold of $\mathbb{R}^{2,d}$ with coordinates X^m satisfying the equation

$$-\left(X^0\right)^2-\left(X^1\right)^2+\cdots+\left(X^{d+1}\right)^2=-l_{AdS}^2. \tag{2.1.1}$$

where l_{AdS} is the AdS radius. From (2.1.1) we can immediately deduced by setting a timelike coordinate to constant, this spacelike hypersurface is the hyperbolic space. Both AdS and hyperbolic space can be thought of as spheres with different signatures. The important fact is the volume diverges as one approach the boundary. The intuition is as the timelike coordinates increase, the spacelike coordinates need to increase so that the radius in (2.1.1) remains the same. It is apparent, in the embedded coordinates, that $SO(2,d)$ and $SO(1,d)$ are isometries in AdS_{d+1} and $\mathbf{H}_d$ respectively.

The metric $G_{\mu\nu}$ in the global coordinates can be expressed as

$$ds^2 = -\frac{l_{AdS}^2 + r^2}{l_{AdS}^2}dt^2 + \frac{l_{AdS}^2}{l_{AdS}^2 + r^2}dr^2 + r^2 d\Omega_{d-1}. \tag{2.1.2}$$

where r is the radial coordinate and $r = \infty$ hypersurface is the boundary. In the Poincare coordinates, we can express the metric in the Poincare patch of AdS as

$$ds^2 = \frac{l_{AdS}^2}{z^2}(dz^2 + g_{\mu\nu}dx^\mu dx^\nu), \tag{2.1.3}$$

where $g_{\mu\nu} = \eta_{\mu\nu}$. We will be calling z as the radial coordinate and $z = 0$ hypersurface is the boundary of the AdS space in the Poincare patch. These metrics satisfy the Einstein equation. Setting the cosmological constant $\Lambda = -\frac{d(d+1)}{2}$ the Einstein equation becomes

$$R_{ab} - \frac{R}{2}G_{ab} = \frac{d(d+1)}{2}G_{ab}, \tag{2.1.4}$$

where R_{ab} is the Ricci tensor with respect to the $d+1$ dimensional spacetime metric G_{ab}. Then we can obtain the Ricci tensor and Ricci scalar as

$$R_{ab} = -d\,G_{ab}, \qquad R = -d(d-1). \tag{2.1.5}$$

For asymptotically locally AdS, $g_{\mu\nu}$ can differ from the Minkowski metric. In general the Ricci tensor for metric G_{ab} in (2.1.3) is

$$\begin{aligned} R_{\mu\nu} &= -\frac{d}{z^2}g_{\mu\nu} + \overline{R}_{\mu\nu} - \frac{1}{2}g''_{\mu\nu} + \frac{1}{2}(g'g^{-1}g')_{\mu\nu} \\ &\quad - \frac{1}{4}Tr(g^{-1}g')g'_{\mu\nu} + \frac{1}{2z}\Big[Tr(g^{-1}g')g_{\mu\nu} + (d-1)g'_{\mu\nu}\Big] && (2.1.6) \\ R_{\mu z} &= -\frac{1}{2}\nabla_\mu Tr(g^{-1}g') + \frac{1}{2}\nabla^\sigma g'_{\mu\sigma} && (2.1.7) \\ R_{zz} &= -\frac{d}{z^2} - \frac{1}{2}Tr(g^{-1}g'') + \frac{1}{4}Tr(g^{-1}g'g^{-1}g') + \frac{1}{2z}Tr(g^{-1}g'). && (2.1.8) \end{aligned}$$

where $' = \partial_z$, $\overline{R}_{\mu\nu}$ and ∇_μ is the Ricci tensor and covariant derivative for $g_{\mu\nu}$. Explicitly, the Einstein equations are

$$\overline{R}_{\mu\nu} + \frac{1}{2}\left[-g'' + (g'g^{-1}g') - \frac{1}{2}Tr(g^{-1}g')g' + \frac{1}{z}Tr(g^{-1}g')g + \frac{(d-1)}{z}g'\right]_{\mu\nu} = 0 \tag{2.1.9}$$

$$-\frac{1}{2}\nabla_\mu Tr(g^{-1}g') + \frac{1}{2}\nabla^\sigma g'_{\mu\sigma} = 0 \tag{2.1.10}$$

$$-\frac{1}{2}Tr(g^{-1}g'') + \frac{1}{4}Tr(g^{-1}g'g^{-1}g') + \frac{1}{2z}Tr(g^{-1}g') = 0. \tag{2.1.11}$$

One can use a series expansion of $g_{\mu\nu}(z, x^\sigma)$ in z to solve the above Einstein equations. We use the Fefferman-Graham expansion of $g_{\mu\nu}(z, x^\sigma)$ [34],

$$g_{\mu\nu}(z,x) = g^{(0)}_{\mu\nu}(x) + z^2 g^{(2)}_{\mu\nu}(x) + \cdots + z^d g^{(d)}_{\mu\nu}(x) + z^2 \log z^2 \tilde{g}^{(d)}_{\mu\nu}(x) + \cdots \tag{2.1.12}$$

where the logarithm is present for even d. $g^{(0)}_{\mu\nu}$ and $g^{(d)}_{\mu\nu}$ are the only independent terms. The higher order terms like $g^{(k)}_{\mu\nu}$ for $0 < k < n$ and $\tilde{g}^{(d)}_{\mu\nu}$ are functions of $g^{(0)}_{\mu\nu}$. This near boundary expansion can be derived from solving the Einstein equation order by order in z.

Since the AdS space has a boundary, the gravitational action needs to include the Gibbon-Hawking boundary term for the validity of the variational problem with Dirichlet boundary condition. The full action is

$$I = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^{d+1}x\sqrt{G}(R - 2\Lambda) - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^d x\sqrt{\gamma}K \tag{2.1.13}$$

where K is the trace of the extrinsic curvature of the boundary,

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu}. \tag{2.1.14}$$

where $\mathcal{L}_n$ is the Lie derivative with respect to the unit normal n and $h_{\mu\nu}$ is the boundary induced metric. The onshell action then becomes a multiple of the volume of the spacetime. Since we know the volume diverges, even within just the Poincare patch, the onshell action is divergent too. We will discuss about the treatment of this infinity in section 2.2.3 on holographic renormalisation. In fact, as the metric diverges as one approach the boundary, we can deduce geodesics or more generally minimal surfaces have to be perpendicular to boundary. Only by moving away from the boundary perpendicularly can one minimise its length or area.

2.2 Holographic duality

In this section we are going to state the holographic duality and do a lightning review on the most well know example of holographic duality, AdS/CFT. The holographic correspondence states that a gravitational theory in $d+1$ dimensional spacetime is equivalent to the a quantum theory without gravity in d dimensional spacetime. As mentioned in the introduction, the earlier hint of holographic principle was the Bekenstein-Hawking black hole entropy formula. The laws of black hole thermodynamic, including the area law of black hole entropy, were derived by introducing quantum fluctuation to a black hole. The entropy would tell us the number of quantum microstates of the black hole interior. Hence suggesting the information lives on the horizon. Holography is not limited to black holes or string theory which we will discuss shortly; there are alternate theories that display the property of holographic duality.

2.2.1 Maldacena Conjecture

The famous Maldacena conjecture states a theory of quantum gravity, the full Type IIB string theory in 10 dimensional spacetime asymptotic to $AdS_5 \times S^5$ with 5-form field strength F_5 sourced by N coincident D3-branes, is equivalent to a quantum field theory, the $\mathcal{N} = 4$ super-Yang-Mill theory with gauge group $SU(N)$ in 4 dimensional Minkowski spacetime [30]. Although we will only be interested in the low energy limit of this duality, it is insightful to see the top-down string theory construction of the original correspondence. Detail explanations and further discussion can be found in [35, 36]

The 10 dimensional Type IIB superstring theory contains closed oriented strings with left and right movers matching chirality. The Ramond sector (R-sector) fermions have periodic boundary condition and are tachyon free; R-sector ground states are therefore spacetime fermions that transform under $SO(8)$ spinor representation $\mathbf{8_s}$ or $\mathbf{8_c}$ depending on their chirality. In IIB, the massless modes in R-R sector are spacetime bosons with matching spinor representation that decompose into

$$\mathbf{8_c} \otimes \mathbf{8_c} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+, \tag{2.2.1}$$

$C^{(0)}$ 0-form, $C^{(2)}$ 2-form and $C^{(4)}$ 4-form bosonic potentials in 10 dimensions. The 5-form field strength $F_5 = dC^{(4)}$ is self dual and is sourced by the 4 dimension D3-brane. From self duality and Dirac quantisation, N coincident D3-branes have charge

$$N\sqrt{2\pi} = \int_{S^5} F_5. \tag{2.2.2}$$

The tension of a D3-brane, T_3, is the mass per volume of the brane and it determines how strongly the brane is couple to gravity. For N coincident D3-branes, the tension is

$$T_3 = \frac{N}{8\pi^2 g_s \alpha'}, \tag{2.2.3}$$

where g_s is the closed string coupling and α' is related to the reciprocal of the string tension. Since D3-branes have both mass and charge, it will backreact with the 10 dimensional target space and effectively curving the spacetime. However the full string theoretic backreaction is hard to compute. Therefore we go down in energy scale to look for hint of the duality.

Considering the following limits, first $\alpha' \to 0$, the string tension becomes large and suppressing massive excitation, second $g_s \to 0$, coupling decreases means string loop effect is suppressed. Hence the low energy effective field is governed by the massless modes. Since the 2D wordsheet theory is conformal, the beta functions will vanish. In the $\alpha' \to 0$ limit, the beta functions equate to the equations of motion in IIB supergravity. Hence IIB supergravity is the low energy effective field theory for Type IIB superstring theory. One can equally obtain the IIB supergravity effective action through integrating out modes in the string path integral after neglecting subleading terms in the expansion of massive excitation and worldsheet topology.

Similar to Reissner-Nordstrom black hole created by a point source of mass and charge, the geometry created by the massive and charged N coincident D3-branes are black 3-branes. Black p-brane are higher dimensional generalisation of black hole or black 0-brane where there are p longitudinal directions. One can write an ansatz for the supergravity solution with N D3-branes as source by separating the metric into parts respecting the symmetry group $Poincare(1,3) \times SO(6)$. This is the unbroken subgroup of the original symmetry $Poincare(1,9)$; the symmetry is broken by the D3-branes as a 4D defect. The backreacted metric is

$$ds^2 = H(r)^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5) \tag{2.2.4}$$

where

$$H(r) = 1 + \frac{R^4}{r^4} \qquad\qquad R^4 = 4\pi g_s N. \tag{2.2.5}$$

The location of the D-branes is at $r = 0$. Note that the N D3-branes preserve 16 out of 32 supercharges, which is called half BPS. Using black hole language, N D3-branes are extremal, saturating the mass and charge bound. More details can be found in [37, 38, 39, 40, 41].

In the $r \gg R$ region, the metric is flat 10D Minkowski spacetime, same as RN black hole which is also asymptotically flat. The intuition is very far from the source the spacetime should have minimal deformation. In the $r \ll R$ region, the metric becomes $AdS_5 \times S^5$,

$$ds^2 = \frac{R^2}{r^2}dr^2 + \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + R^2 d\Omega_5. \tag{2.2.6}$$

Using coordinate $z = \frac{R}{r}$ we can see it is the product of the Poincare patch of the AdS_5 and S^5 both having radius R,

$$ds^2 = \frac{R^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) + R^2 d\Omega_5. \tag{2.2.7}$$

Note the radius of the S^5 is constant in this region and the proper radial distance diverges as $z \to \infty$ or $r \to 0$ approaching the N D3-branes. This is a geometry of infinite throat with constant radius. The N D3-branes are nowhere to be seen, only the F_5 flux reminds. The constant negative curvature or cosmological constant pulls fields inwards, outgoing fields will be redshifted. We can define the horizon to be location at which the redshift is infinite i.e. $z = \infty$ or $r = 0$. In the near horizon limit, we can consider the quantum theory to be the full Type IIB superstring theory with F_5 field strength. In fact the asymptotically flat region will decouple from the strings in the throat due to the redshift. Now we see z or r is acting as the energy scale, this will become important later.

In the low energy/near horizon limit $z \to \infty$ or $r \to 0$, we can neglect the asymptotically flat region. The target space for the Type IIB superstring theory then becomes $AdS_5 \times S^5$. Also since the position of the N D3-branes is at $z = \infty$ or $r = 0$, it is infinite proper distance away from even the near horizon region. Hence we do not have open strings dynamics in this region, the only effect is the charge of the N D3-branes is still carried by the F_5 flux. The isometry of AdS_5 enhances another 16 conformal supercharges. Giving 32 total supercharges, maximal in 10D with spin less than or equal to two.

Our focus is mainly on the gravity side therefore here we will only layout some key aspects on the quantum field theory side of the duality. Massless excitation of open strings ending on N coincident D3-branes can be separated into brane's world-volume vector gauge fields and scalar fields. Both fields carry the Chan-Paton factors and transform under adjoint representation of $U(N)$. Since $U(N) = SU(N) \times U(1)$, we can separate out the $U(1)$ gauge field. The adjoint representation of $U(1)$ is trivial; the $U(1)$ gauge field does not interact and is decoupled. The final gauge group is $SU(N)$. D-branes are half BPS object which brings the total number of supercharges of the Type IIB superstring from 32 to 16. But in the conformal phase, when the scalar field have vanishing vacuum expectation value, there are extra 16 enhanced conformal supersymmetry. Hence the number of supercharges is 32, in 4D they are from the maximal massless non-gravitational representation $\mathcal{N} = 4$ vector multiplet plus conformal supercharges. The low energy limit of the world volume effective field theory then becomes 4D $\mathcal{N} = 4$ super-Yang-Mill with gauge group $SU(N)$. Note we are taking the large N limit so string coupling g_s is small for fix t'Hooft coupling then taking the large t'Hooft coupling limit so α' is small. With these limit, we arrives at strongly coupled CFT limit and the weakly coupled gravity limit.

One of the evidence of duality is the matching of symmetries between the two side of the duality. Also one observes the super-Yang-Mill theory lives on the 4D flat Minokowski spacetime which is also the boundary of AdS_5, the non-compact part of the quantum gravity theory. Hence this led to the holographic correspondence; the conformal super-Yang-Mill theory living on the boundary of the bulk AdS is described by a quantum gravity theory in the bulk AdS. In particular, there is a UV-IR duality between the two theories; $z \to 0$ or $r \to \infty$ represents the UV behaviour of the quantum field theory and IR behaviour of the gravity theory.

The duality needs to match the fields and operator between the two theories. The partition function of the conformal super-Yang-Mill theory with sources ϕ for operator $\mathcal{O}$ is equal to the partition function of the Type IIB superstring theory containing fields with boundary condition $\Phi_{\partial\mathcal{M}} = \phi$,

$$Z_{SYM}[\phi] = Z_{string}|_{\Phi_{\partial\mathcal{M}}=\phi}, \tag{2.2.8}$$

the source ϕ is sometime called the external field coupled to operator $\mathcal{O}$.

Since S^5 is a compacted space, the propagating gravitons and other massless fields will receive a mass from the quantized momentum along the compact direction. In the weak gravity limit, the zero mode in S^5 dominates and the Type IIB superstring on $AdS_5 \times S^5$ can be effectively written as a classical gravity theory on AdS_5 with a dual conformal field theory living on the boundary of AdS_5.

2.2.2 AdS/CFT

With the evidence and motivation given above, we can state the duality as the equality of partition functions of the bulk classical/onshell gravity theory and the boundary conformal field theory,

$$Z_{CFT} = Z_{grav}, \tag{2.2.9}$$

so any expectation values in the CFT side can be calculated using the gravity partition function. More precisely, we can express the partition function as Euclidean path integral with source,

$$Z_{CFT}[\phi] = \int \mathcal{D}\mathcal{O} e^{-I_{CFT}(\mathcal{O}) - \int \phi\mathcal{O}} \tag{2.2.10}$$

where ϕ is the source that couples to operator $\mathcal{O}$. By the external field method, we can take functional derivative with respect to ϕ to generator expectation value of $\mathcal{O}$,

$$-\frac{1}{Z_{CFT}[\phi]} \frac{\delta Z_{CFT}[\phi]}{\delta\phi} = \langle\mathcal{O}\rangle_\phi \, . \tag{2.2.11}$$

In the weak coupling limit, the Newton's constant on the gravity side is small. Hence, similar to the classical limit of quantum theory where $\hbar \to 0$, the gravitation path integral with $G_N \to 0$ is given by the saddle point approximation. The classical/onshell fields satisfying the equations of motion are extrema of the action which dominate the path integral. The onshell gravity partition function is simply the exponential of the onshell action,

$$\begin{aligned} Z_{grav} &= \int_{onshell} \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi e^{-I_E(g,\Phi)} \\ &\approx e^{-I_E^{onshell}(g,\Phi)}. \end{aligned} \tag{2.2.12}$$

Hence the $-I_E^{onshell}$ is equal to the generating functional of connected diagrams in the conformal field theory,

$$\langle \mathcal{O} \rangle_\phi = \frac{\delta I_E^{onshell}(g, \Phi)}{\delta \Phi}|_{\Phi \to \phi}. \tag{2.2.13}$$

The equations of motion of bulk fields Φ are obtained via the usual variational problem with Dirichlet boundary condition $\Phi|_{\partial\mathcal{M}} = \phi$. The bulk fields are onshell hence satisfy the equations of motion of the bulk classical theory with gravity. In fact, there are two independent solutions for each bulk field's equation of motion called, normalisable and non-normalisable modes. The two modes have different scaling behaviour with respect to the radial coordinate z. The origin of their names came from the normalisable condition in some relativistic inner product of the Hilbert space. Similar to the Fefferman-Graham expansion of the boundary metric $g_{\mu\nu}$ that solves the Einsteins equation order by order, in general bulk fields can be expanded in z in the near boundary region [42],

$$\Phi(z, x) = z^m \left(\phi_0(x) + z^2\phi_2(x) + \cdots + z^n\phi_n(x) + z^n \log z^2 \tilde{\phi}_n(x) + \cdots\right) \tag{2.2.14}$$

where $n, m > 0$ are constant related to the dimension of the spacetime and the scaling dimension of the dual oprerator, ϕ_0 is the non-normalisable mode and ϕ_n is the normalisable mode. The non-normalisable modes are divergent in the Klein-Gordon inner production defined by,

$$(\Phi_1, \Phi_2) = -i \int_{\mathcal{C}} \boldsymbol{\epsilon}^\mu (\Phi_1^* \partial_\mu \Phi_2 - \Phi_2^* \partial_\mu \Phi_1), \tag{2.2.15}$$

where $\mathcal{C}$ is a spacelike hypersurface. The higher order terms like ϕ_k for $0 < k < n$ and $\tilde{\phi}_n$ are functions of ϕ_0 as they need to solve Φ equation of motion order by order.

This is a remarkable result, the duality relates a quantum many-body system, uncountably many as we are in the continuum, to a deterministic gravitation system governs by geometry. Analogous to relationship of statistical mechanics and thermodynamics, the expectation values of the microscopic quantum system are given by functions of parametrised by the macroscopic gravitation system. Since quantum mechanics is intrinsically probabilistic, the generating functional of connected diagrams is the functional form of moment generating function in classical probability theory.

To our interest, the AdS/CFT holographic duality is stated as the duality between the onshell fields of the classical gravity in the Poincare patch of AdS_{d+1} and operators of conformal field theory lives on the d dimensional boundary ∂AdS_{d+1}. We will take the holographic duality to be true or treat it as our ansatz for the remaining part of the thesis.

2.2.3 Holographic renormalisation

We now know that the partition function of the boundary CFT is equivalent to the saddle point approximation of the gravitational path integral. To evaluate the classical partition function, we need to compute the onshell action of the AdS spacetime. As mentioned before, since the onshell Ricci scalar is a constant, the onshell action is proportional to the volume of the AdS. However we have seen the volume of AdS diverges, so the action and the expectation of the operators on the CFT are also divergent, for example the expectation value of the CFT stress tensor

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{\gamma}} \frac{\delta I_{grav}}{\delta \gamma^{\mu\nu}}, \tag{2.2.16}$$

where $\gamma_{\mu\nu}$ is the induced metric of boundary. It is easy to see why the CFT stress tensor is sourced by the boundary metric by considering the variation of CFT action,

$$I_{CFT}[\gamma + \delta\gamma] = I_{CFT}[\gamma] + \int_{\partial\mathcal{M}} \delta\gamma_{\mu\nu} T^{\mu\nu}. \tag{2.2.17}$$

Then the last term is essentially the source term. Therefore we need to renormalise the onshell action by adding covariant counterterms. This is analogous to the counterterms renormalisation in quantum field theory that removes UV divergences coming from higher loops Feynman diagrams by introducing counterterm interactions with the same divergence behaviour.

In conventional QFT renormalisation, we would regulate the divergent diagrams by setting a UV cutoff. Just as we have seen in the Maldacena conjecture, the radial coordinate acts as the energy scale of the bulk gravity theory. Through the IR-UV duality, we can see as we approach the boundary, $z \to 0$, the IR volume divergence in the bulk is related to the UV divergence in the boundary quantum theory. So to regulate the bulk theory, we set a radial cutoff $z \geq \epsilon$ such that metric at the regulated boundary $z = \epsilon$ is finite. By analysing the dependence on ϵ in the near boundary region, one can construct covariant boundary counterterms on the regulated boundary to cancel out all the divergences. With all the divergences cancelled out, the limit of $\epsilon \to 0$ pushes the regulated boundary to the real boundary of the spacetime. Finally one can define renormalised quantity in this limit. In fact, we only need to obtain the renormalised action; other quantities like n-points function and entanglement entropy are derivatives and functions of the renormalised action.

The outline of the holographic renormalisation procedure to obtain the renormalised action is as follow [42],

1. Find the equations of motion of the gravity theory with Dirichlet boundary condition.

2. Solve the equations of motion by series expansion of the bulk fields in the radial

coordinate z.

3. Identify the normalisable and non-normalisable modes of the bulk fields.

4. Substitute the bulk fields series expansion into the regularised action with radial cutoff .

5. Separate the divergences as a function of non-normalisable modes and radial cutoff.

6. Invert the bulk fields radial expansion to write the non-normalisable modes in terms of the bulk fields.

7. Construct the covariant counterterms as the divergences of the regularised action in terms of bulk fields but with an overall opposite sign.

8. Equate the renormalised action as the limit of vanishing radial cutoff of the sum of regularised action and the covariant counterterms.

The emphasis on covariant counterterms is to ensure the general covariance of the renormalised action. The finite quantities like one-point function will be a function of the normalisable and non-normalisable modes. The part related to the non-normalisable modes is called scheme dependent as this can be altered by finite counterterms. The part related to the normalisable modes is called scheme independent as it is universal in all renormalisation scheme and it captures the dynamics of the system as it is dual to the operator in the CFT.

As an example, we will demonstrate the holographic renormalisation in bulk theory with pure gravity. The regularised action is the Einstein-Hilbert action with negative cosmological constant and Hawking-Gibbon boundary term for $z > \epsilon$,

$$I_{reg} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}_\epsilon} d^{d+1}x\sqrt{G}(R[G]-\Lambda) - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}_\epsilon} d^d x\sqrt{\gamma}K \tag{2.2.18}$$

where we can set the cosmological constant to $\Lambda = \frac{-d(d-1)}{2l_{AdS}^2}$. The Hawking-Gibbon boundary term is necessary for the action to remain invariant under infinitesimal variation around the classical solution with Dirichlet boundary condition,

$$\lim_{z\to 0} z^2\gamma_{\mu\nu} = g^{(0)}_{\mu\nu} \tag{2.2.19}$$

where $g^{(0)}_{\mu\nu}$ is the metric of the boundary CFT. Shown in section 2.1, we can express the solution to the Einstein equation in the Fefferman-Graham gauge [34] where we only need two independent terms, namely $g^{(0)}_{\mu\nu}$ and $g^{(d)}_{\mu\nu}$. The other terms in the Fefferman-Graham expansion with order lower than $O(z^{d-2})$ can be expressed as a function of the non-normalisable mode $g^{(0)}_{\mu\nu}$. To obtain a covariant expression, one can perturbatively invert the Fefferman-Graham expansion to express $g^{(0)}_{\mu\nu}$ and it's function in terms of $\gamma_{\mu\nu}$. In particular, from (2.1.9) one can write $g^{(n<d)}_{\mu\nu}$ and $\tilde{g}^{(d)}_{\mu\nu}$ as contractions, combinations and derivatives of the Riemann curvature tensor $\bar{R}_{\mu\nu\rho\sigma}$ with respect to the boundary metric $\gamma_{\mu\nu}$.

The divergences of the regularised action in the expansion of ϵ is [43],

$$I^{div}_{reg} = \frac{1}{16\pi G_N}\int_{\partial\mathcal{M}_\epsilon} d^d x\sqrt{g^{(0)}}\left(\epsilon^{-d}a_{(0)} + \cdots + \epsilon^{-2}a_{(d-2)} - \log\epsilon^2 a_{(d)}\right) \tag{2.2.20}$$

where the logarithm term is only present in even d. By construction the counterterms action is equal but opposite to the regularised divergences,

$$I_{ct} = -I^{div}_{reg} \tag{2.2.21}$$

But the covariant expression in terms of $\gamma_{\mu\nu}$ is [43],

$$I_{ct} = -\frac{1}{16\pi G_N}\int_{\partial\mathcal{M}_\epsilon} d^d x\sqrt{\gamma}\left[2(1-d) - \frac{R}{d-2} - \frac{R^{\mu\nu}R_{\mu\nu} - \frac{d}{4(d-1)}R^2}{(d-4)(d-2)} - \log\epsilon^2 a_{(d)}\right]. \tag{2.2.22}$$

Finally the renormalised action is defined to be the limit of $\epsilon \to 0$ the sum of regularised and counterterms actions.

$$I_{ren} = \lim_{\epsilon\to 0}\left[I_{reg} + I_{ct}\right]. \tag{2.2.23}$$

The renormalised stress tensor can be obtain via the variation of the onshell renormalised action. Hence the it is separable into the regularised and counterterms parts,

$$T^{ren}_{\mu\nu} = \lim_{\epsilon\to 0}\left[T^{reg}_{\mu\nu} + T^{ct}_{\mu\nu}\right]. \tag{2.2.24}$$

Since the action is onshell, the variation of the action with respect to the metric should vanish under Dirichlet boundary condition. However, when we take the functional derivative with respect to the boundary metric, this will pick up the boundary piece that would have vanished due to the Dirichlet boundary condition. The regularised part is actually the Brown-York stress tensor of the regularised spacetime,

$$\begin{aligned} T^{reg}_{\mu\nu} &= \frac{2}{\sqrt{\gamma}}\frac{\delta I_{reg}}{\delta\gamma^{\mu\nu}} \\ &= \frac{2}{\sqrt{\gamma}}\frac{\delta I_{\mathcal{M}_\epsilon}}{\delta\gamma^{\mu\nu}} + \frac{2}{\sqrt{\gamma}}\frac{\delta I_{\partial\mathcal{M}_\epsilon}}{\delta\gamma^{\mu\nu}} \\ &= -\frac{1}{8\pi G_N}(K_{\mu\nu} - K\gamma_{\mu\nu})|_{z=\epsilon}. \end{aligned} \tag{2.2.25}$$

Taking the functional derivative of the counterterms action in (2.2.22), one finds the divergence of $T_{\mu\nu}^{reg}$ is cancelled out. Finally we have the renormalised stress tensor of the boundary CFT as

$$T_{\mu\nu}^{ren} = \frac{d}{16\pi G_N} g_{\mu\nu}^{(d)} + \mathcal{A}_{\mu\nu} + \tilde{f}_{\mu\nu} \tag{2.2.26}$$

where $\mathcal{A}_{\mu\nu}$ is the trace/Weyl anomaly component and $\tilde{f}_{\mu\nu}$ is the traceless scheme dependent compnent of the stress tensor. The last two terms are only present in even d. By trace/Weyl anomaly, it means the contribution of the stress tensor that violate the tracelessness condition from the Ward identity of the Weyl transformation's conserved current,

$$\nabla_\nu \langle x_\mu T^{\mu\nu} \rangle = \mathcal{A} \tag{2.2.27}$$

$$\left\langle T^\mu_\mu \right\rangle = \mathcal{A} \tag{2.2.28}$$

where $\mathcal{A}$ is the Weyl anomaly in the CFT. Scheme dependent terms are contribution from finite counterterm introduced in a particular renormalisation scheme. For example in $d = 2$, the Weyl anomaly is proportional to the central charge of the CFT. From [44, 45, 46], we also know that $\mathcal{A}$ takes the form,

$$\mathcal{A} \sim \mathcal{E} + \mathcal{I} + \nabla \cdot \mathcal{J}, \tag{2.2.29}$$

where $\mathcal{E}$ is proportional to the Euler density, $\mathcal{I}$ is a conformal invariant, ∇ is the covariant derivative of $g^(0)_{\mu\nu}$ and $\mathcal{J}$ is a tensor constructed from curvature tensors. Using similar technique in section 3, these terms can be written as combination of Weyl tensors and extrinsic curvatures.

Now we see that $g_{\mu\nu}^{(d)}$ provides the information for the one point function of the CFT stress tensor. Hence bulk field $g_{\mu\nu}^{(d)}$ is dual to boundary operator $T_{\mu\nu}$. Also $g_{\mu\nu}^{(0)}$ provides the Dirichlet boundary condition for the metric $g_{\mu\nu}$. In the absence of Weyl anomaly, the Dirichlet boundary condition can be lifted to conformal Dirichlet boundary condition. So any boundary metric in the conformal class $[g_{\mu\nu}^{(0)}]$ is valid. These two facts extend beyond pure gravity theory, so the non-normalisable modes define the boundary value of the bulk fields and the normalisable modes govern the one point function of the dual operators.

Note this is the holographic renormalisation in Lagrangian formalism. There is a more direct method of doing holographic renormalisation that avoids inversion of the radial expansion. The holographic renormalisation in the Hamiltonian formalism utilises the scaling behaviour of the bulk fields to construct appropriate counter terms. With our previous emphasis on covariant counterterms, we need to use covariant phase space formalism to define the conserved currents and charges. We will review and apply both formalisms extensively in chapter 4.

2.2.4 Holographic correlation function

Although we will only see a limited use of holographic propagators in the following sections, we briefly introduce the two propagators in holography. The bulk-to-bulk propagator $G(z,x;z',x')$ or sometimes expressed as

$$G(z,x;z',x') = \langle \Phi(z,x)\Phi(z',x') \rangle \tag{2.2.30}$$

they satisfy the differential equation

$$\mathcal{D}^2 G(z,x;z',x') = \delta(z,x;z',x') \tag{2.2.31}$$

where $\mathcal{D}^2$ is a second order linear differential operator in the equation of motion of bulk field derived from the action. So the $G(z,x;z',x')$ is the Green's function of the differential operator $\mathcal{D}^2$. The bulk field $\Phi(z,x)$ is then

$$\Phi(z,x) = \int dz d^d x' G(z,x;z',x') J(z',x') \tag{2.2.32}$$

where $J(z,x)$ is the bulk source of bulk field $\Phi(z,x)$ such that the bulk field satisfies the bulk equation of motion. Similarly, the boundary-to-bulk propagator $K(z,x;x')$ or sometimes expressed as

$$K(z,x;x') = \langle \Phi(z,x)\mathcal{O}(x') \rangle \tag{2.2.33}$$

where $\mathcal{O}(x)$ is the dual operator of bulk field $\Phi(z,x)$. The boundary-to-bulk propagator satisfies the differential equation

$$\mathcal{D}^2 K(z,x;x') = 0 \tag{2.2.34}$$

So $K(z,x;x')$ is the homogeneous solution to differential operator $\mathcal{D}^2$. The bulk field $\Phi(z,x)$ is related to the boundary source $\phi(x)$ by

$$\Phi(z,x) = \int d^d x' K(z,x;x')\phi(x'). \tag{2.2.35}$$

2.3 Entanglement entropy

To begin, let us understand what is quantum entanglement and what entanglement entropy measures. Entanglement is the quantum correlation between systems. Contrasting to classical correlation, measurement on a quantum state will change the complementary quantum state in the entangled system. For entangled system, the overall quantum state cannot be written as a product state,

$$|\psi\rangle_{AB} = \sum_{n,m} C_{nm} |n\rangle_A |m\rangle_B \neq |\phi\rangle_A |\phi\rangle_B \, , \tag{2.3.1}$$

because for product state, measurement on system A will not change the quantum state of system B. An entangle state viewed from the perspective of system B, where no information about system A is given, it forms a classical statistical ensemble of quantum state. In terms of density matrix, the reduced density matrix of system B ρ_B, the partial trace of system A Tr_A of the total density matrix ρ,

$$\begin{aligned}\rho_B &= Tr_A(\rho) \\ &= \sum_i p_i |i\rangle_B \langle i|_B \, ,\end{aligned} \tag{2.3.2}$$

is mixed with p_m is the probability of the system B being in quantum state $|i\rangle_B$. An example is the thermofield double (TFD) state,

$$|TFD\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta E_n}{2}} |n\rangle |n\rangle \, , \tag{2.3.3}$$

the states of the subsystems are entangled such that the reduced density matrix,

$$\rho_r = \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle \langle n| \, , \tag{2.3.4}$$

is in a thermal canonical ensemble with inverse temperature β and normalised by the partition function Z. The statistical ensemble of the reduced density matrix represents our ignorance on the system B alone. Hence, the more entangled the overall system is, the more system B is dependent on system A and the less we know about system B alone. Therefore we can measure entanglement by the entropy for this ensemble, similar to Boltzmann entropy that counts the microstates or Shannon entropy that measure the average uncertainty on the possible outcomes.

In a quantum system, the quantum entanglement between a bipartite region can be measured by the entanglement entropy which is the von Neumann entropy of the reduced density matrix,

$$S_B = -Tr(\rho_B \log \rho_B). \tag{2.3.5}$$

The modular Hamiltonian is defined as the exponent of the density matrix; for the reduced density matrix of region B it is,

$$\rho_B = e^{-H_B}. \tag{2.3.6}$$

In general the modular Hamiltonian is not local and hard to compute, the exception is when the density matrix is thermal. As the terminologies hinted, the entanglement entropy and modular Hamiltonian has similar structure as thermal entropy and thermal Hamiltonian. Since the thermal entropy of a thermal state is also given by the von Neumann entropy, if the reduced density matrix is thermally mixed, the entanglement entropy is identical to the thermal entropy. Similarly the thermal Hamiltonian is the exponent of the thermal density matrix in the rest frame,

$$\rho_{th} = \frac{1}{Z} e^{-\beta H_{th}} \tag{2.3.7}$$

where β is the inverse temperature and Z is the partition function. Then the modular Hamiltonian can be expressed simply as

$$H_B = \beta H_{th} - \log Z = \beta(H_{th} - F) \tag{2.3.8}$$

where the second equality follows from the definition of the free energy. For conformal field theory in flat space and B is a ball region, the modular Hamiltonian can be calculated directly from the energy momentum tensor, see section 4.2.3.

From now on we will be studying relativistic continuous quantum systems, namely quantum field theory. Since the Hilbert space is continuous we need to replace the sum of states to the path integral. The wavefunction can be expressed in path integral as

$$\Psi(\psi_n) \left|\psi_n\right\rangle = \sqrt{N} \int_{\psi(x,-\infty)}^{\psi(x,0^-)=\psi_n} \mathcal{D}\psi \; e^{-S[\psi]} \left|\psi_n\right\rangle \tag{2.3.9}$$

Similarly, entanglement entropy can be calculated in quantum field theory using path integral to generate the density matrix.

$$\begin{aligned}\rho_{nm} &= N \int_{\psi'(x,-\infty)}^{\psi'(x,0^-)=\psi'_n} \mathcal{D}\psi' \; e^{-S[\psi']} \int_{\psi(x,0^+)=\psi_m}^{\psi(x,+\infty)} \mathcal{D}\psi \; e^{-S[\psi]} & (2.3.10)\\ &= N \int \mathcal{D}\psi \; \delta(\psi(x,0^-) - \psi_n) \; \delta(\psi(x,0^+) - \psi_m) e^{-S[\psi]} & (2.3.11)\end{aligned}$$

Bipartition the fields into $\psi = \{\psi^A, \psi^B\}$ then the reduce density matrix is

$$\rho_{Bij} = \int \mathcal{D}\psi_A \; \left\langle \psi^A, \psi^B_i \middle| \rho \middle| \psi^A, \psi^B_j \right\rangle. \tag{2.3.12}$$

In general, the logarithm of the reduced density matrix is hard to compute. We can use another entropy measure, called Renyi Entropy, defined by

$$S_n(B) = -\frac{1}{n-1} \log Tr \rho_B^n. \tag{2.3.13}$$

Using L'Hopital's rule, we see Renyi entropy tends to Von Neumann entropy in the limit

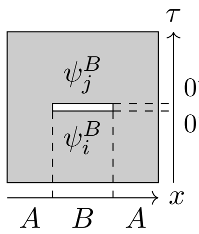


Figure 2.3.1: This is a diagram of the path integral representation of reduced density matrix with the shaded region being integrated over. The whole space is integrated except a thin slit in region B with infinitesimally width. The ψ_i^B and ψ_j^B mark the boundary value of the field at the two sides of the slit.

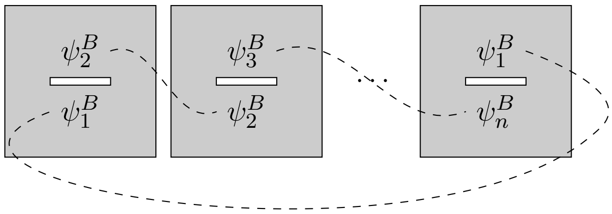


Figure 2.3.2: The n replica density partition function is made up of n identical density matrix. Each boxes represent a reduced density matrix in the product ρ_B^n. The dashed line indicate the identification of boundary value between slits. Since all insertions are identified in pairs, this represent the overall trace $Tr_B(\rho_B^n)$.

of $n \to 1$ where we assume analyticity in n. Hence to calculate the entanglement entropy we need to work out the trace of reduced density matrix to the n^{th} power and analytically continue $n \to 1$.

2.3.1 Replica trick

We now introduce the replica trick in quantum field theory to help us to obtain the Renyi entropy. We need to generate the trace of the n^{th} power of the reduced density matrix in the terms of path integral,

$$Tr_B(\rho_B^n) = \int \prod_i^n \mathcal{D}\psi_i^B \left\langle \psi_i^B \middle| \rho_B \middle| \psi_{i+1}^B \right\rangle , \tag{2.3.14}$$

where $\psi_{n+1}^B = \psi_1^B$. Diagrammatically, the QFT reduced density matrix of B where A is traced out is shown in figure 2.3.1 and the replica trick in quantum field theory is shown in figure 2.3.2.

Since the partition function equals to the trace of the density matrix,

$$Z = Tr\rho, \tag{2.3.15}$$

equally we can define a partition function associates with the n^{th} power of the density matrix

$$Z(n) = Tr\rho^n \tag{2.3.16}$$

From the definition of Renyi entropy in (2.3.13) and its relation to the von Neumann entropy we can deduced the entanglement entropy is the limit of Renyi entropy as follow,

$$S = -n\partial_n[\log Z(n) - n\log Z(1)]_{n=1}. \tag{2.3.17}$$

Similar to the thermal entropy, the entanglement can be obtained purely from the partition function. This expression will be helpful in the later sections.

In quantum field theory, fields are continuous in space so one would expect there are entanglement between neighbouring fields. In bipartite system, no matter how small, there will be uncountably many neighbouring fields separated by the boundary shared between the two subregions, called entangling surface. This suggests there are infinite amount of entanglement between any bipartite system in quantum field theory. We can regularised the entanglement entropy by putting the field theory on lattice and using the lattice spacing, ϵ, as UV cutoff. Then the divergence behaviour of the entanglement entropy can be capture in series expansion of the UV cutoff,

$$S = S_{div}[\epsilon] + S_{finite} \tag{2.3.18}$$

In general, the entanglement entropy also follows an area law where the coefficient of the leading divergent term is equal to the area of the entangling surface [47]. For CFT in even dimensions, the divergent term is a logarithm of the length scale of the region over the UV cutoff [48]. Some coefficients in the ϵ expansion of the entanglement can have physical interpretations, like the coefficient of the universal logarithm term is related to the central charge. There are interesting theorem relating the coefficients in different energy scale, from UV to IR, like the c-theorem [49]. In this thesis, we will mainly be focusing on the finite term of the entanglement entropy which can be obtained via systematic renormalisation procedure.

2.4 Holographic entanglement entropy

As we have showed above, entanglement entropy measures the information shared between the entanglement pair. This is an purely quantum mechanical quantity as there is no classical analogue of entanglement. However, in the holographic correspondence, we are able to represent the entanglement entropy of the boundary quantum theory as a geometrical quantity in the bulk. From (2.3.13) we see can obtain the entanglement entropy via the replica partition function. In the field theory calculation, one would need to perform the path integral in order to obtain the partition function and hence the entanglement entropy. For free theory, one can still perform Gaussian path integral. But the calculation get very non-trivial for complicated theory. However, as we will discover, if we use the gravity partition function in (2.3.13), we can see the area of a bulk minimal surface is the dual of the entanglement entropy.

2.4.1 Ryu-Takayanagi prescription

The Ryu-Takayanagi prescription is the bulk holographic description of entanglement entropy on the boundary quantum field theory [6, 5]. Since the bulk interior is dual to the boundary theory, the bulk holographic entanglement entropy also measures the entanglement between two bulk systems separated by a bulk entangling surface. The entanglement entropy S of the d dimensional boundary quantum field theory with boundary entangling surface $\partial\Sigma$ is related to a minimal surface extending into the bulk spacetime $\mathcal{M}$ with the boundary entangling surface as its boundary. The bulk extending co-dimension 2 minimal surface is defined to be the bulk entangling surface Σ or the Ryu-Takayangi surface. The entanglement entropy is then equal to the area of the bulk entangling surface,

$$S = \frac{A(\Sigma)}{4G_N} \tag{2.4.1}$$

where $A(\Sigma)$ is the area of Σ and G_N is the gravitational constant. Since the AdS metric diverges near the boundary, the area of the minimal surface Σ suspending from the boundary also diverges. But we will save the discussion of renormalisation in section 2.4.3

This minimal surface description can be extended in multiple entangling surfaces where the combination of bulk extending surfaces that have the minimal overall area are chosen to be the bulk entangling surface. For thermal boundary CFT dual to gravity in AdS black hole background, certain configurations of entanglement entropy captures the entropy of the black hole. In those configurations, the set of entangling surface contains a surface wrapping around the black hole horizon. This analysis is a way to study the black hole microstates in holography [50].

2.4.2 Replica trick in gravity

The Ryu-Takayangi formula for holographic entanglement entropy can be derive using the replica trick on the gravity partition function as in (2.3.17). To apply the replica trick on the gravity side we first define the n^{th} replica partition function by replacing the manifold $\mathcal{M}$ with the n^{th} replica geometry $\mathcal{M}(n)$ such that $\mathcal{M}(1) = \mathcal{M}$. There are many methods of deriving the Ryu-Takayangi formula [5, 51, 52], we will be following the more straight forward approach in [53]. All the methods utilise the conical singularity in the replica geometry to relate the entanglement entropy to the area of a minimal surface.

The replica geometry is constructed by gluing the n identical copy of manifold $\mathcal{M}$, similar to figure 2.3.2. First, for each copy of $\mathcal{M}$ a co-dimension one cut is made along the bulk entangling region enclosed by the co-dimension two bulk entangling surface Σ and spatial boundary $\partial\mathcal{M}|_{t=const}$. The geometry $\mathcal{M}(n)$ is the overall geometry after identification or gluing along the cuts of n replica together in the Euclidean time direction.

The gravity partition function is given by the exponential of the gravity action. Hence we need to obtain the gravity action on the n replica geometry $\mathcal{M}(n)$

$$-\log Z(n) = I_{grav}[\mathcal{M}(n)]. \tag{2.4.2}$$

The gluing procedure of more than one replica creates a conical singularity in the boundary of the cut, i.e. Σ, in $\mathcal{M}(n)$. The terms in our gravitational action that are sensible to the conical singularity are terms related to the Ricci tensor.

We now sketch out the procedure for obtaining the conical singularity terms in curvature invariants as distribution, i.e. delta function. From the original metric $g_{\mu\nu}$, here assumed to be static,

$$ds^2 = B(x)dt^2 + h_{ij}(x)dx^i dx^j \tag{2.4.3}$$

to construction the replica geometry we need to separate the orthogonal directions of the bulk entangling surface Σ. Since we are interested in the tip of the cone, near the bulk entangling surface Σ we can do a coordinates transformation to a Rinder frame that locally looks like,

$$ds^2 = -r^2 d\tau^2 + dr^2 + (\gamma_{ab} + 2r\cosh\tau K_{ab})dy^a dy^b \tag{2.4.4}$$

followed by a Wick rotation to,

$$ds^2 = r^2 d\tau^2 + dr^2 + (\gamma_{ab} + 2r\cos\tau K_{ab})dy^a dy^b \tag{2.4.5}$$

where γ_{ab} is the metric for Σ, τ and r are the orthogonal coordinates to Σ and K_{ab} is the extrinsic curvature. The metric is periodic in τ hence we define the n^{th} replica geometry by extending the domain of from $0 \leq \tau < 2\pi$ to $0 \leq \tau < 2\pi n$. For $n > 1$ there are n sheets of identical geometries. Since the period exceeds 2π there is a conical singularity at $r = 0$. To capture the behaviour of the curvatures near the singularity, we first regularized the replica metric by smoothening the conical singularity or squashing the tip of the conical replica geometry and look at the limit as $n \to 1$. For example, the metric of the regularized replica geometry for S^{d-1} entangling surface is [53],

$$ds^2 = r^2 d\tau^2 + \sqrt{\frac{r^2 + n^2\beta^2}{r^2 + \beta^2}}dr^2 + (\lambda + r^n A^{1-n}\cos\tau)^2 d\Omega_{d-2}^2 \tag{2.4.6}$$

where β is the regularization parameter and A is an arbitrary constant. As $r \to 0$, the extra factor of n^2 in the radial component of the metric resolves the conical singularity.

In the limit of $\beta \to 0$ and $n \to 1$ one recovers the original geometry.

To see the distributional property we need to analyse the replica curvatures invariants integrals,

$$\int_{\mathcal{M}^{(n)}_{reg}} d^{d+1}x\sqrt{g}\, O(\mathcal{R}^{(n)}_{\mu\nu\rho\sigma}) \tag{2.4.7}$$

where $O(\mathcal{R}^{(n)}_{\mu\nu\rho\sigma})$ are integrands like $\mathcal{R}^{(n)}, \mathcal{R}^{(n)^2}, \mathcal{R}^{(n)}_{\mu\nu}\mathcal{R}^{(n)^{\mu\nu}}$ etc. The limiting behaviour of the replica curvature integral must depend only on the intrinsic curvature of the original manifold $\mathcal{M}$, extrinsic curvature of the extended entangling surface Σ and their derivatives. For a quadratic curvature integral, by dimensional and parity reason it can only depend on the extrinsic curvature in this form,

$$(n-1)\int_{\Sigma} d^{d-1}x\sqrt{\gamma}[c_1(TrK)^2 + c_2 Tr(K\cdot K)]. \tag{2.4.8}$$

Since this combination of extrinsic curvatures is general and independent of the choice of Σ, one can deduce the coefficient c_1 and c_2 by doing explicit calculation with two different Σ and solving the simultaneous equations for c_1 and c_2.

The relevant replica curvature integrals up to first order in $1-n$ are the following [53],

$$\int_{\mathcal{M}_n} d^{d+1}x\sqrt{g}\mathcal{R}^{(n)} = n\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\mathcal{R} + 4\pi(1-n)\int_{\Sigma} d^{d-1}x\sqrt{\gamma} \tag{2.4.9}$$

$$\int_{\mathcal{M}_n} d^{d+1}x\sqrt{g}\mathcal{R}^{(n)2} = n\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\mathcal{R}^2 + 8\pi(1-n)\int_{\Sigma} d^{d-1}x\sqrt{\gamma}\mathcal{R} \tag{2.4.10}$$

$$\begin{aligned}\int_{\mathcal{M}_n} d^{d+1}x\sqrt{g}\mathcal{R}^{(n)}{}_{\mu\nu}\mathcal{R}^{(n)\mu\nu} &= n\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \\ &+ 4\pi(1-n)\int_{\Sigma} d^{d-1}x\sqrt{\gamma}(\mathcal{R}_{\mu\nu}n^{\mu}\cdot n^{\nu} - \frac{1}{2}(TrK)^2)\end{aligned} \tag{2.4.11}$$

Note the integrals on the left hand side diverge as the regulator $\beta \to 0$. From these integral we can write the replica curvature invariants as distribution

$$\mathcal{R}^{(n)} = n\mathcal{R} + 4\pi(1-n)\delta_{\Sigma} \tag{2.4.12}$$

$$\mathcal{R}^{(n)2} = n\mathcal{R}^2 + 8\pi(1-n)\delta_{\Sigma}\mathcal{R} \tag{2.4.13}$$

$$\mathcal{R}^{(n)}{}_{\mu\nu}\mathcal{R}^{(n)\mu\nu} = n\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + 4\pi(1-n)\delta_{\Sigma}(\mathcal{R}_{\mu\nu}n^{\mu}\cdot n^{\nu} - \frac{1}{2}(TrK)^2). \tag{2.4.14}$$

where δ_{Σ} is the delta function in Σ, indicating the conical singularity is at Σ. Now we have all the ingredients for the derivation of the holographic entanglement entropy and its counterterms

2.4.3 Holographic renormalised entanglement entropy

Having established the holographic renormalisation procedure and replica trick in gravity, we can proceed to derive the holographic renormalised entanglement entropy directly from the renormalised action. This is a systematic way of renormalising the entanglement entropy which is conceptually different from the direct removal of the divergences. Although in some dimensions the end results of the two methods are identical, we still need to use proper renormalisation procedure if we want to relate the renormalised entanglement entropy to other renormalised quantity.

The renormalised action can be written as [54],

$$I_{ren} = \lim_{\epsilon \to 0} [I_{reg} - I_{ct}] \tag{2.4.15}$$

and explicitly,

$$\begin{aligned} I_{ren} = & \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} (\mathcal{R} + \Lambda) \\ & - \frac{1}{16\pi G_N} \int_{\partial \mathcal{M}} d^d x \sqrt{\gamma} \Big[2(1-d)\mathcal{R} + \frac{1}{2-d} \mathcal{R} \\ & \qquad - \frac{1}{(d-4)(d-2)^2} (\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} - \frac{d}{4(d-1)} \mathcal{R}^2) \\ & \qquad - \log \epsilon a_{(d)} + \cdots \Big]. \end{aligned} \tag{2.4.16}$$

The explicit expression of the renormalised entanglement entropy as the limit of Renyi entropy is

$$\begin{aligned} S_{ren} &= -n\partial_n [\log Z(n) - n \log Z(1)]_{n=1} \\ S_{ren} &= n\partial_n [I_{ren}(n) - n I_{ren}(1)]_{n=1}. \end{aligned} \tag{2.4.17}$$

Applying the replica curvature integrals and distribution from (2.4.9 − 2.4.14) we can obtain the holographic renormalised entanglement entropy [54],

$$\begin{aligned} S_{ren} = & \frac{1}{4G_N} \int_{\Sigma} d^{d-1}x \sqrt{h} - \frac{1}{4(d-2)G_N} \int_{\partial\Sigma} d^{d-2}x \sqrt{\tilde{h}} \\ & - \frac{1}{4(d-2)(d-4)G_N} \int_{\partial\Sigma} d^{d-2}x \sqrt{\tilde{h}} \Big(\mathcal{R}_{\mu\nu} n^{\mu} \cdot n^{\nu} - \frac{1}{2}(TrK)^2 - \frac{d}{2(d-1)} \mathcal{R} \Big) \end{aligned} \tag{2.4.18}$$

where the third integral only arises for $d \geq 4$ and at $d = 4$ the coefficient changes to logarithm of the regulator. Cubic and higher order replica curvature integrals are relevant for higher dimensional entanglement entropy and can be calculated following the same procedure as above. As in the renormalised action, we obtained the covariant boundary counterterms for the entanglement entropy. In chapter 3 we will express the renormalised entanglement entropy in another set of curvature invariant that allow us to understand property of this finite quantity.

Holographic entanglement entropy is a rich topic to study and has proved to be a useful tool in quantum gravity. The area law of bulk entanglement entropy is directly related to the black hole thermal entropy. There are rapid development on the application of holographic entanglement entropy to the black hole information paradox [20, 21, 22]. Another fundamental application of holographic entanglement entropy is on the emergent property of dynamical spacetime or gravity [55, 56, 57]. In chapter 4 we will be investigating the first law of entanglement entropy. There we will review how the linearised Einstein equation is related to the variation of entanglement entropy.

2.5 Quantum scattering

In this section, we will be reviewing the past development in high energy scattering, further details can be found in [58, 59]. To begin, let us briefly go over the basics of potential scattering in $3+1D$ quantum mechnaics. In the traditional non-relativistic scattering theory of quantum mechanics, incoming plane wavefunction is scattered by a spherically symmetric potential and becomes an outgoing wavefunction. The Schrodinger equation with potential $V(\mathbf{r})$ can be rearranged as

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = U(\mathbf{r})\psi(\mathbf{r}) \tag{2.5.1}$$

where $k^2 = 2mE$ and $U(\mathbf{r}) = 2mV(\mathbf{r})$. At a steady state, the overall wavefunction is a superposition of the incoming and outgoing wave. The coefficient of the outgoing spherical wave at infinity is known as the scattering amplitude,

$$A(\mathbf{k}, \mathbf{k}') = -\frac{1}{4\pi}\int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}} U(\mathbf{r})\psi(\mathbf{r}) \tag{2.5.2}$$

and the scattering matrix or S-matrix is defined by

$$S(\mathbf{k}, \mathbf{k}') = 1 + 2ikA(\mathbf{k}, \mathbf{k}'), \tag{2.5.3}$$

where k is the non-relativistic momentum vector. Since the wavefunction satisfies the Schrodinger equation with spherically symmetric potential, one can separate the radial and angular solutions. They can be express as partial waves

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty} \frac{\phi_l(r)}{r} P_l(\cos\theta) \tag{2.5.4}$$

where $P_l(\cos\theta)$ are the Legendre polynomials and l is the angular momentum quantum number. Similarly, the radial wavefunction can be separated into incoming and outgoing parts and the coefficient of the outgoing part is the partial wave S-matrix S_l and is related to the partial wave scattering amplitude by

$$A_l(k) = \frac{S_l(k) - 1}{2ik}. \tag{2.5.5}$$

We can also define the S_l as a phase shift by

$$S_l(k) = e^{2i\delta_l(k)}. \tag{2.5.6}$$

For elastic scattering $\delta_l(k)$ is real. Partial wave analysis is useful in low energy regime when the high angular momentum modes are suppressed. Although the application of partial waves in high energy scattering is limited, we will use it to define the Regge behaviour in QFT.

At high energy forward scattering regime, where the energy is large compare to the potential and backscattering is low, we can use the eikonal approximation for the scattering amplitude

$$A(\mathbf{k}, \mathbf{k}') = \frac{ik}{2\pi} \int d^2b' e^{i\mathbf{k}\cdot\mathbf{b}'}(1 - e^{i\chi(\mathbf{b})}). \tag{2.5.7}$$

where $\mathbf{b}$ is the impact parameter that is the fourier transform of the transverse momentum. So the S-matrix in the impact parameter space is given by the eikonal phase factor $e^{i\chi(\mathbf{b})}$. A more insightful expression of the eikonal phase is to write it as the Fourier transform of the Born amplitude

$$\chi(\mathbf{b}) = \frac{1}{2\pi k'} \int d^2k e^{i\mathbf{k}\cdot\mathbf{b}'} A^B(\mathbf{k}, \mathbf{k}') \tag{2.5.8}$$

where Born amplitude is

$$A^B(\mathbf{k}, \mathbf{k}') = -\frac{1}{4\pi} \int d^3r e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} U(\mathbf{r}). \tag{2.5.9}$$

The Born approximation is valid for weak potential.

The interpretation of the eikonal approximation give us insight into how high energy scattering works. Since the eikonal amplitude is related to the exponential of the Born amplitude, we can think of the eikonal amplitude of as the infinite sum of Born amplitudes. Also the form of the eikonal S-matrix in parameter space can be interpreted as picking up the contribution from partial waves with largest angular momentum. In the relativistic case, there are normalisation factor difference but the conceptual framework is the same. In fact using Feynman diagrams, the above interpretation of eikonal scattering is more apparent.

2.5.1 Regge scattering

In the partial waves expansion (2.5.4), the wavefunction is separated into the radial and angular parts. Since the each angular part is an eigenfunction of the angular Laplacian, this modifies potential by the eigenvalue which is related to angular momentum. For each partial wave labelled by its angular momentum, there are bound states or resonances

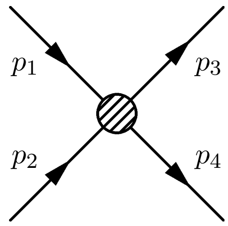


Figure 2.5.1: The 2-2 scattering with the external legs labelled by the incoming momenta p_1, p_2 and outgoing momenta p_3, p_4.

associate with the modified potential. We know that by analytical continuation of the linear momentum, complex poles in the scattering amplitude represent bound states or resonances. Regge showed it is also possible to write angular momentum as a complex variable and physical solutions have poles at real integer angular momentum. More importantly, bound states or resonances made from composite particles will also have poles in the complex angular momentum plane as well. Poles of the partial waves are called Regge poles representing Reggeons which can be composite particles.

For relativistic 2-2 scattering, the standard kinematic parameters are the Mandelstam variables,

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^2 \qquad u = (p_1 - p_4)^2. \tag{2.5.10}$$

We will mainly be focusing on the scattering process where s denotes the square of centre of mass energy and momentum transfer is in the t-channel. At the centre of mass frame, the partial waves expansion of the scattering amplitude takes similar form as the non-relativistic quantum mechanics case,

$$A(s,t) = \sum_l (2l+1) A_l(s) P_l(\cos\theta). \tag{2.5.11}$$

Then one notices the sum of integer l can be reformulated as a contour integral in the complex l plane as

$$A(s,t) = \frac{1}{2i} \oint_C dl (2l+1) \frac{A(l,t)}{\sin \pi l} P(l, 1 + 2s/t) \tag{2.5.12}$$

where the contour C encircles the infinitesimal region above and below the positive real axis. In analytical continuation, it is very important to keep track of the poles within the contour. Restricting our contour to C, the contour integral (2.5.12) by itself reproduces the standard partial waves expansion. However, in contour C the domain for l is restricted to essentially the real line. To fully explore the complex plane, we would like to extend the contour to larger part of the complex l plane. Now one can deform the contour to C' which encircles the $Re(l) > -\frac{1}{2}$ region. Poles and branch cuts within this region will need to be subtracted from main contour integral. For simplicity we will assume there is no

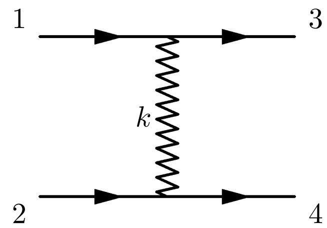


Figure 2.5.2: The $1, 2 \to 3, 4$ scattering is represented by the tree level diagram with Reggeon transfer in the t-channel carrying momentum k.

branch cut and the scattering amplitude becomes

$$\begin{aligned} A(s,t) =& \frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl(2l+1)\frac{A(l,t)}{\sin \pi l}P(l,1+2s/t) \\ &+\sum_i (2\alpha_i(t)+1)\beta_i(t)\frac{P(\alpha_i(t),1+2s/t)}{\sin \pi\alpha_i(t)} \end{aligned} \tag{2.5.13}$$

where $\beta_i(t)$ is the residue of $A(l,t)$ at $l = \alpha_i(t)$ and $\alpha_i(t)$ are called the Regge poles. In the limit $s \gg |t|$ the analytic continuation of the Legendre polynomial behaves like

$$\lim_{s\gg|t|} P(l,1+2s/t) \to \frac{\Gamma(2l+1)}{\Gamma^2(l+1)}\left(\frac{s}{2l}\right)^l. \tag{2.5.14}$$

Hence the first term in (2.5.13) vanishes by construction as the real part of l is negative. The Regge pole with the largest real part will dominate the sum. We are then left with the famous Regge power law behaviour

$$A(s,t) \sim s^{\alpha(t)}. \tag{2.5.15}$$

This can be thought of as an exchange of a Reggeon with angular momentum $\alpha(t)$ which is also called the Regge trajectory. This is diagrammatically represented in figure 2.5.2. An example of Regge behaviour is in the Veneziano amplitude, in string theory it is the tree level four open strings tachyon amplitude, with linear Regge trajectory

$$\alpha(t) = 1 + \alpha' t. \tag{2.5.16}$$

So the Regge intercept is 1 and the Regge slope is α'.

2.6 Gravitational scattering

The behaviour of eikonal 2-2 scattering in gravity can be derived in various methods [26, 27, 29, 60]. We will consider the semi-classical shock wave interpretation where the highly energetic particles backreact on the spacetime, creating shock waves and altering the trajectory of the other particles [27, 29, 61, 62]. This is an analogue of the potential scattering in quantum mechanics with gravitational potential induced by the particles. The other interpretation is the perturbative linear quantum gravity where the sum of ladder diagrams from linearized gravity interaction reproduces the eikonal amplitude [63].

2.6.1 Shock wave

The shock wave geometry is a solution to the Einstein equation with the energy momentum tensor being sourced by the high energy particle [62, 61, 64]. To begin, we write the metric as

$$ds^2 = -a(u,v')dudv' + r^2(u,v')h_{ij}(x^k)dx^i dx^j. \tag{2.6.1}$$

Since we expect the shock wave to shift the position of the spacetime after the high energy particle passes by, we introduce the shift in coordinate by a Heaviside function

$$v' \mapsto v = v' + \theta(u) f(x^k). \tag{2.6.2}$$

After the coordinate transformation, one arrives at the shock wave metric,

$$ds^2 = -a(u,v)dudv + r^2(u,v)h_{ij}(x^k)dx^i dx^j + a(u,v)f(x^k)\delta(u)du^2. \tag{2.6.3}$$

Shock wave geometry was also introduced to joining of two Einstein solutions, so each sides of the shock wave satisfy the Einstein equation individually.

An alternate approach to deriving a shock wave metric is by considering the massless limit of Schwarzchild metric has a discontinuity at $u = 0$. Then the 2-2 scattering becomes just the scattering of one moving particle by the gravitational potential created by the stationary particle. In the massless limit and taking the centre of mass frame, this arrives at the shock wave graviational scattering.

In general, the total energy momentum tensor is of two parts, one associates with the background geometry in the absent of shock wave and the other associates to the source of the shock wave induced by the high energy particle,

$$T_{\mu\nu} = T^{(b)}_{\mu\nu} + T^{(p)}_{\mu\nu}. \tag{2.6.4}$$

For high energy particle located at the $u = 0$ null surface, the energy momentum tensor takes the form,

$$T^{(p)}_{uu} = J(x^k)\delta(u). \tag{2.6.5}$$

Then the Einstein equations give following conditions at $u = 0$

$$a_{,v} = 0 \qquad 2rr_{,v} = 0, \qquad \left(a\nabla^2_{(D-2)} - ar^2\Lambda - 2rr_{,uv}\right) f = J. \tag{2.6.6}$$

In the semi-classical picture, the high energy 2-2 scattering process is considered as the wavefunction of the one particle being scattered off a shock wave geometry created by the other ultra-energetic particle.

For simplicity, let us focus on the flat background example. In the flat space coordinates, the plane wavefunction is

$$\psi = e^{ip\cdot x'}. \tag{2.6.7}$$

The wavefunctions before and after the shock wave are different by a shift in the null coordinate,

$$\psi_- = e^{ip_i x^i + ip_u u + ip_v v} \tag{2.6.8}$$

$$\psi_+ = e^{ip_i x^i + ip_u u + ip_v (v-f)} \tag{2.6.9}$$

where shift function for flat space shock wave in $D = 4$ is

$$f(x^k) = -2G_N p_1^u \log\left(\mu x^i x_i\right). \tag{2.6.10}$$

We can expand the wavefunction after the shock wave in momentum eigenstates with momentum conservation condition

$$\psi_+(p,x) = \int d^D k \delta\left(k^u - \frac{k_i k^i - m^2}{k^v}\right)(1 - S(p,k))e^{ik\cdot x} \tag{2.6.11}$$

where $S(p,k)$ is the S-matrix element. So by doing the inverse Fourier transform one arrives at the scattering amplitude as

$$A(s,t) = G_N \frac{s}{t}\frac{\Gamma(1-iG_N s)}{\Gamma(1+iG_N s)}(-t)^{iG_N s} \tag{2.6.12}$$

One obtain the same result if consider Klein-Gordon equation of ultra-energetic particle in a static spherically symmetric background generated by the other massive particle. This would give Schrodinger equation with gravitation potential that has the same form as Coulomb potential.

For the holographic quantum chaos, we will mainly being using the shock wave approach to high energy gravitational scattering.

2.6.2 Perturbative quantum gravity

To get a better understand and to see direct relation with quantum gravity, we will demonstrate the perturbative quantum gravity approach to high energy graviational scattering.

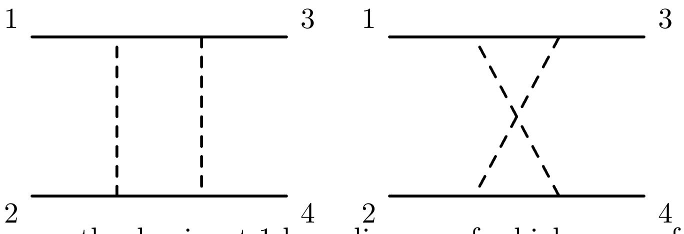


Figure 2.6.1: These are the dominant 1-loop diagrams for high energy forward scattering. The solid lines are the incoming and outgoing particles which are the scalars in our case. The dash lines represents the higher spin particles which are the gravitons in our case. Note that there is no four point gravitons vertex in the diagram on the right.

First, let us write the action for linearized gravity with massless scalar

$$I = \frac{1}{16\pi G_N}\int d^Dx\, h\cdot\mathcal{D}^2(h) + \frac{1}{2}\phi\nabla^2\phi + \frac{1}{2}h(\partial\phi,\partial\phi) - \frac{1}{4}h\partial\phi\cdot\partial\phi \tag{2.6.13}$$

where kinetic term for the graviton or metric perturbation is expressed by the differential operator $\mathcal{D}^2$ defined by

$$h\cdot\mathcal{D}^2(h) = \frac{1}{8}h_{\mu\nu}\left(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\sigma}\right)\nabla^2 h_{\lambda\sigma}. \tag{2.6.14}$$

With this action, we have directly apply the Feynman rule to Feynman diagrams and obtain the amplitudes. The ingredients for the Feynman rules are scalar propagator

$$i\Delta(p) = -\frac{i}{p^2 - i\epsilon}, \tag{2.6.15}$$

the graviton propagator

$$iD^{\mu\nu\rho\sigma}(k) = -\frac{16\pi G_N}{k^2 - i\epsilon}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right) \tag{2.6.16}$$

and the scalar-graviton vertex

$$V_{\mu\nu}(p,p') = \frac{i}{2}\left(p_\mu p'_\nu + p_\nu p'_\mu - \eta_{\mu\nu}p\cdot p'\right) \tag{2.6.17}$$

where p and p' are the scalar momenta and k is graviton momentum. At large energy, the tree level diagram then gives the amplitude analogous to the Born amplitude

$$A_0(s,t) \sim -8\pi G_N\frac{s^2}{t} \tag{2.6.18}$$

where the s^2 originated from the fact the intermediate particle is spin-2. In general the power of s is determined by the spin of the intermediate particle because of the momentum dependence of the vertex. For large s and at each loop order, the forward scattering is dominated by diagrams with intermediated by graviton only. So in one loop diagrams like figure 2.6.1 dominate. We approximate the intermediate scalar propagator as

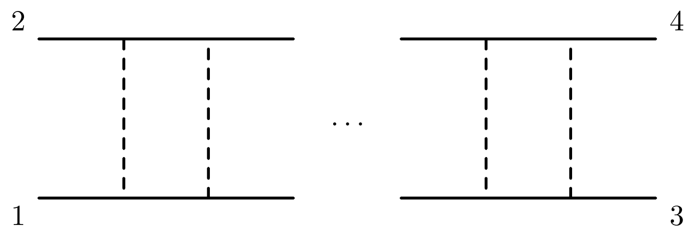


Figure 2.6.2: This is a schematic representation of the ladder diagrams with only gravitons intermediating between the large momentum scalar lines. Crossing of the gravitons are permitted.

$$\begin{aligned} i\Delta(p+k) &= -\frac{i}{(p+k)^2 - i\epsilon} \\ i\Delta(p+k) &\sim -\frac{i}{2p\cdot k - i\epsilon}. \end{aligned} \tag{2.6.19}$$

then we obtain the 1-loop amplitude as

$$A_1(s,t) = 2s \int d^{D-2}b e^{-i\mathbf{q}\cdot\mathbf{b}} \frac{(i\chi(\mathbf{b},s))^2}{2!} \tag{2.6.20}$$

where

$$\chi(\mathbf{b},s) = \frac{1}{2s}\int \frac{d^{D-2}q}{(2\pi)^{D-2}} e^{i\mathbf{q}\cdot\mathbf{b}} A_0(s,-q^2). \tag{2.6.21}$$

The factor of s infront is due to the normalisation factor when changing basis between transverse momentum space to impact parameter space [65]. This can be interpreted as the products of the tree level amplitudes. Also we scaled the IR regulator/(graviton mass) μ to 1. One can show, see [66], the infinite sum of ladder diagrams like figure 2.6.2 reproduces the eikonal scattering amplitude,

$$iA_{eik}(s,t) = 2s\int d^{D-2}b e^{-i\mathbf{q}\cdot\mathbf{b}} \left(e^{i\chi(\mathbf{b},s)} - 1\right). \tag{2.6.22}$$

We can evaluating the integral and get

$$A_{eik}(s,t) = 8\pi G_N \frac{s^2}{-t}\frac{\Gamma(1-iG_N s)}{\Gamma(1+iG_N s)}\left(\frac{4}{-t}\right)^{-iG_N s} \tag{2.6.23}$$

which is equal to (2.6.12) up to some kinetic normalisation factor. From the form of (2.6.22), the S-matrix in impact parameter space is again a phase shift,

$$S(s,\mathbf{b}) = e^{i\chi(s,\mathbf{b})}. \tag{2.6.24}$$

In fact the eikonal phase can be represented as the classical action,

$$\chi(s,\mathbf{b}) = I_{cl}(s,\mathbf{b}). \tag{2.6.25}$$

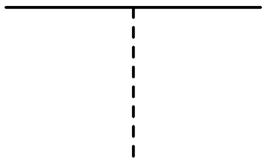


Figure 2.6.3: This is a diagram representing the scalar sourcing linearised graviton as it propagates.

Here is a sketch of the argument, we can calculate the connected amputated Green's function for the scalar in space with linear metric perturbation, representing all the connected diagrams in the 2-2 scattering between four scalars. The path integral of the Green's function is

$$G(x_1, x_2; x_3, x_4) = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\phi_1\mathcal{D}\phi_2\phi_1(x_1)\phi_1(x_3)\phi_2(x_2)\phi_2(x_4)e^{i\left(I_{grav}+I_\phi+I_{int}\right)}. \tag{2.6.26}$$

We then integrate out the scalar fields

$$G(x_1, x_2; x_3, x_4) = \int \mathcal{D}h_{\mu\nu}G_1^c(x_1, x_3|h_{\mu\nu})G_2^c(x_2, x_4|h_{\mu\nu})e^{iI^{(2)}_{grav}} \tag{2.6.27}$$

where $G_1^c(x_1, x_3|h_{\mu\nu})$ and $G_2^c(x_2, x_4|h_{\mu\nu})$ are connected amputated Green's function in linearly perturbed metric, see figure 2.6.3. These two point Green's functions are the source term for the linearised graviton, i.e.

$$G^c_{eik}(p_1^u, \mathbf{x}_1; p_3^u, \mathbf{x}_3|h_{\mu\nu}) \sim exp\left[\frac{i}{2}\int d^Dx\sqrt{-G}T_1\cdot h\right]. \tag{2.6.28}$$

Combining everything together we get the S-matrix to be

$$S(s, \mathbf{b}) \sim \int \mathcal{D}h_{\mu\nu} \exp\left[i\int d^Dx\sqrt{-G}\left(h\cdot\mathcal{D}^2(h) + \frac{1}{2}T_1\cdot h + \frac{1}{2}T_2\cdot h\right)\right] \tag{2.6.29}$$

$$S(s, \mathbf{b}) \sim \exp\left[iI^{(2)}_{grav} + iI_{source}\right] \tag{2.6.30}$$

To go to second line, one can compute the Gaussian path integral by change of variable or use saddle point approximation.

2.7 Quantum chaos

In this section we will review Quantum Chaos base on [9, 10, 23, 67, 68, 69]. Classical chaos is the phenomenon of classical system being very sensitive to perturbation of initial condition. More precisely, the distance between the perturbed phase space trajectory and the unperturbed trajectory grow exponentially in time

$$\delta x \sim e^{\lambda_L t} \tag{2.7.1}$$

where λ_L is the Lyapunov exponent that governs the rate of growth. A system with larger λ_L is more chaotic. There are quantum analogues of chaos in terms of phase space trajectory governs by the Schrodinger equation. However, we will focus on the formalism of quantum chaos measured by the expectation value of the commutator between two operators at different time [68],

$$C(t) = \left\langle - [W(t,x), V(0,0)]^2 \right\rangle_\beta . \tag{2.7.2}$$

where β is the inverse temperature of the system. Generically, operators $W(0,x)$ and $V(0,0)$ at time equal to zero should commute with each other as they act on different position and the size of the operator is not large. As time increases, in chaotic system the operator size of $W(t,x)$ increases, eventually covering the site that $V(0,0)$ acts on. As a result the commutator increases, the exponential grows in the onset of chaos is also governed by the Lyapunov exponent,

$$C(t) \sim e^{\lambda_L t}, \tag{2.7.3}$$

for t be greater than the thermal dissipation time but smaller than the scrambling time when the size of the operator $W(t,x)$ reach the size of the system.

2.7.1 Out-of-time-order correlation function

Let W and V be unitary and Hermitian operators. The commutator in $C(t)$ can be expanded, the only term that behaves differently in chaotic system is the out of time order correlator (OTOC) [9, 10, 23, 67, 68, 69],

$$\langle W(t,x)V(0,0)W(t,x)V(0,0)\rangle_\beta . \tag{2.7.4}$$

The only requirement on these operators are they have vanishing thermal expectation value and their time ordered two point function decay normally,

$$\langle W \rangle = \langle V \rangle = 0 \tag{2.7.5}$$

$$\langle W(t,x)W(0,0)\rangle,\ \langle V(t,x)V(0,0)\rangle \sim e^{-t/t_d} \tag{2.7.6}$$

where t_d is the thermal dissipation time, the time scale that the operator evolves to be significantly different from the initial operator.

To explore the operator evolution with time, we use the BCH formula to write the operator as

$$W(t,x) = e^{iHt}W(0,x)e^{-iHt}$$
$$W(t,x) = Ad[e^{iHt}]W(0,x)$$
$$W(t,x) = \sum_{n=0} \frac{(-it)^n}{n!} Ad[H]^n W(0,x) \tag{2.7.7}$$

where H is the Hamiltonian of the theory and $Ad[\quad]$ is the adjoint representation, $Ad[H] = [H, \quad]$. The growth of the operator $W(0,x)$ can be thought of as the scramble of W in the system by commuting with the Hamiltonian. $Ad[H]^n W(0,x)$ can be written as a product of operators and typically this operator product increases in size with respect to n. The relationship between operator growth and scrambling can be demonstrated in simple discrete one dimensional quantum system with qubits and Hamiltonian that include nearest neighbour interaction, see [68].

For integrable system the time evolution can be solved analytically hence late time behaviour can be determined. The operator decreases in size and 'unscramble' at later time or rather it was never properly scramble in the system. In chaotic systems the operator becomes thoroughly scrambled in the system after passing the scrambling time and remains large.

The late time behaviour of the OTOC depends on the Hamiltonian of the system, if the system is integrable then OTOC tends to,

$$\langle W(t,x)W(t,x)\rangle_\beta \langle V(0,0)V(0,0)\rangle_\beta \tag{2.7.8}$$

but, for our interest, in chaotic systems the normalised OTOC vanishes like [9, 10, 23, 67, 68, 69],

$$\frac{\langle W(t,x)V(0,0)W(t,x)V(0,0)\rangle_\beta}{\langle W(t,x)W(t,x)\rangle_\beta \langle V(0,0)V(0,0)\rangle_\beta} \sim 1 - exp\left[\lambda_L\left(t - t_* - \frac{|x|}{v_B}\right)\right] \tag{2.7.9}$$

where λ_L is the Lyapunov exponent, t_* is the scrambling time, v_B is the butterfly velocity and the coefficient are schematically hidden. As mentioned before the physical interpretation of the scrambling time is the time needed for a whole system to become properly scrambled. And the butterfly velocity is the speed for which the scrambling is spread through the system. The Lyapunov exponent is the decay rate of the OTOC after the system is properly scrambled.

This manifests the same chaotic growth behaviour of the expectation value of the commutator,

$$C(t) \sim exp\left[\lambda_L\left(t - t_* - \frac{|x|}{v_B}\right)\right]. \tag{2.7.10}$$

To better understand the relationship of OTOC and chaos, consider the OTOC as an overlap of these two states [10],

$$|in\rangle = W(t,x)V(0,0)\,|\beta\rangle\,, \quad |out\rangle = V(0,0)W(t,x)\,|\beta\rangle \tag{2.7.11}$$

The construction of the *out* state is to evolve the thermal state $|\beta\rangle$ in time then perturb by the operator W, $We^{-iHt}\,|\beta\rangle$. The state is evolved back in time such that the system thermalized and feature of the W perturbation is dissipated out. Following by a V perturbation to the state, $Ve^{iHt}We^{-iHt}\,|\beta\rangle$, the state now only has feature of the V perturbation.

Similarly, the construction of the in state is to first perturb the thermal state $|\beta\rangle$ by the operator V then evolve the state in time, $e^{-iHt}V\,|\beta\rangle$, such that the system is thermalized and feature of the V perturbation is dissipated out. Followed by a W perturbation to the state and evolve back in time, $e^{iHt}We^{-iHt}V\,|\beta\rangle$.

In ordinary non-chaotic cases, the information about the W and V operators acting on the state does not scramble or mix with each other. So the perturbations can be independent evolve forward and backward in time. So for large enough t the *in* state will appear to have only the V perturbation and the information about W is thermalised. The *out* state only sees the V perturbation as the W perturbation is thermalised. Hence the *in* and *out* overlap and OTOC is non-vanishing even at large t.

A chaotic system would scramble the information of both operators even when acted on different time. The *in* state would lose the feature of the V perturbation when evolved back in time due to the thermalisation of the scrambled information. Again the *out* state only show features of the V perturbation. Therefore the *in* and *out* overlap, $\langle out|in\rangle$, and OTOC is small. The size of the overlap decay with respect to the time passed proper scrambling because the smearing of the V perturbation intensifies.

2.8 Holographic chaos

Holographic conformal field theory that possesses chaotic behaviour [10] and can be describe by black hole physics in the bulk [9]. The holographic dual to a d dimensional conformal field theory with inverse temperature β is the theory of Einstein gravity in AdS_{d+1} black hole geometry with the same temperature.

We have seen the thermofield double state in section 2.3 as an example of entangled state. But let us further elaborate the thermal property of this state. Analogous to the use of Boltzmann factor in statistical physics to describe the proportion of microstate being in a certain energy level, we can use the same logic to express the thermal quantum state.

The density matrix of a thermal mixed state at inverse temperature β is,

$$\rho(\beta) = \frac{1}{Z}\sum_n e^{-\beta E_n} |n\rangle\langle n| \tag{2.8.1}$$

which is a ensemble of energy eigenstates with probability proportional to the Boltzmann factor. The purification of this state is called thermofield double state,

$$|TFD\rangle = \frac{1}{Z^{\frac{1}{2}}}\sum_n e^{\frac{-\beta E_n}{2}} |n\rangle_L |n\rangle_R \tag{2.8.2}$$

where $|n\rangle_L$ and $|n\rangle_R$ are the energy eigenstates in the CFT_L and CFT_R. The holographic dual to the TFD is the two-sdied eternal black hole in AdS. So taking the partial trace is identical to considering only one side of the black hole exterior. As an example, a metric written in null coordinates is [23],

$$ds^2 = -a(uv)dudv + r^2(uv)dx^i dx^i \tag{2.8.3}$$

where the horizons locate at $u = 0$ and $v = 0$.

The holographic description of the OTOC is given by bulk scattering with insertion of boundary operators W and V. A useful convention to use is to switch the sign of the time $t \to -t$ and the bulk *in* and *out* states are,

$$|\Psi_{in}\rangle = W(-t)V(0)|TFD\rangle, \quad |\Psi_{out}\rangle = V(0)W(-t)|TFD\rangle \tag{2.8.4}$$

To write the bulk states in momentum space we first Fourier transform the boundary-to-bulk propagators along the horizons to get the wavefunctions,

$$\psi_V(p^u, x) = \int dv e^{\frac{ia_0 p^u v}{2}} \langle\phi_V(u, v, x)V(0)\rangle|_{u=0} \tag{2.8.5}$$

$$\psi_W(p^v, x) = \int du e^{\frac{ia_0 p^v u}{2}} \langle\phi_W(u, v, x)W(-t)\rangle|_{v=0} \tag{2.8.6}$$

where ϕ_V and ϕ_W are the bulk fields dual to V and W.

The *in* state written in the null momentum space is,

$$|\Psi_{in}\rangle = \int \psi_V(p_4^u, x_4)\psi_W(p_3^v, x_3) |p_3^v, x_3; p_4^u, x_4\rangle_{in}, \tag{2.8.7}$$

similarly the *out* state is,

$$|\Psi_{out}\rangle = \int \psi_V(p_2^u, x_2)\psi_W(p_1^v, x_1) |p_1^v, x_1; p_2^u, x_2\rangle_{out}, \tag{2.8.8}$$

where the *in* and *out* label the states that are defined on different Cauchy slices and $|p^u, x,\rangle$ and $|p^v, x,\rangle$ are defined on $u = 0$ horizon and $v = 0$ horizon. The overlap of the bulk *in* and *out* states defines the bulk OTOC,

$$\int \psi_V^*(p_2^u, x_2)\psi_W^*(p_1^v, x_2)\psi_V(p_4^u, x_4)\psi_W(p_3^v, x_3)_{out}\left\langle p_1^v, x_1; p_2^u, x_2 | p_3^v, x_3; p_4^u, x_4\right\rangle_{in} \tag{2.8.9}$$

The S-matrix element ${}_{out}\left\langle p_1^v, x_1; p_2^u, x_2 | p_3^v, x_3; p_4^u, x_4\right\rangle_{in}$ in the high energy regime where the centre of mass energy s is large and fixed t, the forward scattering dominates with $p_3^v \approx p_1^v$ and $p_4^u \approx p_2^u$. Then we can use the eikonal approximation we discussed in section 2.6. The caveat in here is the S-matrix formalism is formally defined for asymptotically flat spacetime with *in* and *out* states defined at the past and future infinity. However, for large l_{AdS} and t we can approximate the amplitude to be similar to the usual eikonal scattering. So the *out* state is related to the *in* state by a phase factor,

$$\left|p_2^u, x_2; p_1^v, x_1\right\rangle_{out} \approx e^{i\delta}\left|p_2^u, x_2; p_1^v, x_1\right\rangle_{in} \tag{2.8.10}$$

This eikonal phase δ in our gravity setting can be approximated by the classical action,

$$\delta \sim I_{cl}^{(2)} \tag{2.8.11}$$

where $I_{cl}^{(2)}$ is the quadratic perturbation of the classical action with metric perturbation sourced by the highly boosted particles inserted from the boundary along the horizons. This metric perturbation can be obtained by calculating the back reaction of the particle. For t large enough, the particles are highly boosted along the horizons, the energy momentum tensors are,

$$T_{uu} = \frac{a_0}{2r_0^{d-1}} p_1^v \delta(u)\delta^{d-1}(x - x_1) \tag{2.8.12}$$

$$T_{vv} = \frac{a_0}{2r_0^{d-1}} p_2^u \delta(v)\delta^{d-1}(x - x_2). \tag{2.8.13}$$

These can be inserted in the linearized Einstein equation to obtain the back reaction perturbations h_{uu} and h_{vv}. The action up to second order in perturbation is,

$$\begin{aligned} I_{cl} &= I_0 + \delta I + \delta I_{p_1^v} + \delta I_{p_2^u} \\ I_{cl} &= I_0 + \int d^{d+1}x \delta g \cdot \frac{\delta I}{\delta g} + \delta g \cdot \frac{\delta I_{p_1^v}}{\delta g} + \delta g \cdot \frac{\delta I_{p_2^u}}{\delta g} \\ I_{cl}^{(2)} &= \frac{1}{2}\int d^{d+1}x\sqrt{-g} h \cdot \mathcal{D}^2 h + h_{uu}T^{uu} + h_{vv}T^{vv} \end{aligned} \tag{2.8.14}$$

where $\mathcal{D}^2$ is a differential operator in terms of g and I_0 factor is absorbed by the path integral normalisation. Since the metric perturbations satisfied the linearized Einstein equation, the first term is equal but opposite to the last two terms. So the eikonal phase is

$$\delta = \frac{1}{2}\int d^{d+1}x\sqrt{-g} h_{uu}T^{uu}. \tag{2.8.15}$$

The shock wave perturbation induced by particle 2 is,

$$h_{uu} = \frac{8\pi G_N a_0}{r_0^{d-3}} p_2^v \delta(u) f(x - x_1) \tag{2.8.16}$$

where f satisfy the following equation derived from linearized Einstein equation,

$$(-\partial_x^2 + \mu^2) f(x) = \delta^{d-1}(x) \tag{2.8.17}$$

where $f(x)$ is then Green's function for the transverse part of linearised Einstein equation. We will calculation the exact form of this Green's function in chapter 5. At large argument $f(x)$ has the asymptotic form,

$$f(|x|) = \frac{\mu^{\frac{d-4}{2}}}{2(2\pi|x|)^{\frac{d-2}{2}}} e^{-\mu|x|} \tag{2.8.18}$$

and $\mu^2 = \frac{2\pi(d-1)r_0}{\beta}$. Using (2.8.15) and combining all the terms, we obtain the eikonal phase,

$$\delta = \frac{4\pi G_N}{r_0^{d-3}} s f(b) \tag{2.8.19}$$

where $s = a_0 p_1^v p_2^u$ and $b = x_1 - x_2$ is the impact parameter of the 2-2 scattering. In the Kruskal null coordinate u and v, we can see the bulk particle sourced by $W(-t)$ are highly boosted along one of the horizon by a factor of $e^{\frac{2\pi t}{\beta}}$. So we can see both the metric perturbation and eikonal phase is proportional to the boost factor

$$h_{uu} \sim e^{\frac{2\pi t}{\beta}} \tag{2.8.20}$$

in the other frame along the other horizon particle sourced by $V(0)$ is boosted. The boost factor originates from the change of variable from t, r to u, v. It is related to the inverse temperature by the g_{tt} component near the horizon. Hence the eikonal phase also have the exponential behaviour

$$\delta \sim G_N e^{\frac{2\pi t}{\beta}} e^{-\mu b} \tag{2.8.21}$$

This is the first indication of the chaotic exponential growth behaviour, suggesting the Lyapunov exponent λ_L is

$$\lambda_L = \frac{2\pi}{\beta}. \tag{2.8.22}$$

The eikonal phase becomes of order one, $\delta \sim 1$, when at $t = t_*$ so the scrambling time equals to

$$t_* = \frac{\beta}{2\pi} \log \frac{1}{G_N} \tag{2.8.23}$$

From (2.8.21)the butterfly velocity v_B is

$$v_B = \frac{\lambda_L}{\mu} \tag{2.8.24}$$

The exact behaviour can be obtained by the OTOC, see examples in [23].

The interpretation of the bulk OTOC is similar to the one used for the quantum system. We will look at the case for all the one sided boundary operators insertions. First we construct the *in* state by inserting $V(0)$ on the CFT as a boundary condition for the bulk field. Then we insert $W(-t)$ and for large enough t the W particle generates a shock wave along the past horizon and it shift the trajectory of the V after crossing the shock wave. Hence the position of V on the boundary changes.

For the *out* state we insert the $W(-t)$ first and produced a shock wave. Then we insert $V(0)$, now the boundary position of V is unchanged, only its past before crossing the shock wave is shifted. The boundary positions of V is different between the *in* and *out* state. Therefore the overlap is small for large t.

In the two sided perspective is we can define the *in* state on a past Cauchy slice with W and V insertions on opposite boundary. Then we have two particles on the same Cauchy slice with momenta along different horizons. Similarly we can define the *out* state on a future Cauchy slice with W and V insertions on opposite boundary. Again we have two particles on the same Cauchy slice with momenta along different horizon. So the *out* state can be thought of as the final state of a scattering in the past and the *in* state is pre-scattering state. The OTOC is then the high energy scattering S-matrix element of these boosted particles along the horizons.

Both the explicit examples of OTOC calculation and heuristic argument point towards the eikonal phase being the key to diagnose the chaotic behaviour of a system. The eikonal phase is essentially the Fourier transform of tree level amplitude in transverse momentum bases to impact parameter bases. In the Regge limit, it has a power law behaviour,

$$\delta(s, b) \sim s^{j-1}, \tag{2.8.25}$$

where j is the spin of the intermediate particle. In the gravitational scattering processes, the spin-2 graviton will dominate at high s regime.

2.8.1 Stringy correction

There has been a conjecture for the bound on chaos which set the upper bound of the Lyapunov exponent,

$$\lambda_L \leq \frac{2\pi}{\beta}. \tag{2.8.26}$$

This proposed bound is saturated by Einstein gravity [67].

One of the way to introduce stringy correction is simulate the OTOC using the four point function of closed string tachyon vertex operators

$$\mathcal{A} = \int d^2z \, \langle V_4(0,0)V_2(z,\bar{z})V_3(1,1)V_1(\infty,\infty)\rangle \tag{2.8.27}$$

where the worldsheet positions of three out of four vertex operators are fixed by conformal symmetry and the remaining position is integrated over the worldsheet. The scale we need to be aware of are the string length scale l_s and the string coupling g_s. From the coefficient of the effective action one can deduce the gravitational constant is related to the string scales by

$$G_N \sim g_s^2 l_s^{D-2}. \tag{2.8.28}$$

The standard procedure of calculating the string amplitude is by operator product expansion (OPE) of the vertex operators. The indexing of the vertex operators hinted that we are taking the OPE as $z \to 0$ so intermediate string imitate the t-channel intermediate particle in the 2-2 forward scattering. In flat target space, the result is the two tachyons merge to form a excited closed string which leads to the flat space Regge behaviour

$$\mathcal{A} \sim s^{2+\frac{\alpha' t}{2r_0^2}} \tag{2.8.29}$$

where $t < 0$ and $\alpha' = l_s^2$ being the standard Regge slope in string theory. In curved spacetime, particularly in AdS, we see the transverse part of linearised Einstein equation in (2.8.17) has a term μ^2. By change of variable, one can absorb the μ^2 term into the momentum. Using this analogy, we can see the shift in the Regge behaviour as

$$\mathcal{A} \sim s^{2+\frac{\alpha'(t-\mu^2)}{2r_0^2}}. \tag{2.8.30}$$

This shift in momentum changes the Regge intercept from 2 to

$$\alpha(t=0) = 2 - \frac{\alpha'\mu^2}{2r_0^2} \tag{2.8.31}$$

so the spin of the Reggeon decreases to be below spin-2. This also indicates the Lyapunov exponent becomes

$$\frac{2\pi}{\beta}(1 - \frac{\alpha'\mu^2}{2r_0^2}) \tag{2.8.32}$$

falling below the conjectured chaos bound. We have provided an heuristic justification of the stringy correction to chaos. The precise calculation can be found in [23].

CHAPTER 3

Renormalized Entanglement Entropy and Curvature Invariants

3.1 Introduction and summary

Viewed from the perspective of quantum field theory, entanglement entropy is an unusual quantity. Entanglement entropy is usually expressed as a regulated quantity, with the regulator being a short distance cutoff but the regulated power law divergences depend on the details of the regulation scheme. Accordingly the main focus is on the so-called universal terms, the coefficients of logarithmic divergences, as these are related to the coefficients of the Weyl anomaly of the stress energy tensor.

For condensed matter and quantum information applications, quantum field theory is used as an intermediate tool to describe a system with an inherent lattice cutoff. In such contexts the short distance regulator has a physical interpretation as the lattice spacing. If quantum field theory is used to describe a continuum system, there is no inherent physical cutoff: in quantum field theory we work with renormalized quantities, rather than regulated quantities. Renormalized entanglement entropy has been developed in [54, 70, 71, 72, 73, 74, 75].

The focus in this chapter will be on the holographic definition of renormalized entanglement entropy in terms of the renormalized area of entangling surfaces, as shown in (3.2.1) and (3.3.2). Renormalized entanglement entropy can however be defined in generality using the replica approach, which is in practice almost always used for explicit computations of entanglement entropy in quantum field theory, see for example [76, 77, 78]. The bare entanglement entropy is expressed as

$$S = -\mathrm{Lim}_{n\to 1}\left(\partial_n\left[\mathrm{Tr}(\rho^n)\right]\right) \tag{3.1.1}$$

where ρ is the density matrix of the (reduced) state. This expression can be written in terms of partition functions as

$$S = -\mathrm{Lim}_{n\to 1}\left(\partial_n\left[Z(n) - nZ(1)\right]\right) \tag{3.1.2}$$

where $Z(1)$ denotes the partition function and $Z(n)$ denotes the partition function on the replica space (n copies of the original space joined together cyclically). The renormalized entanglement entropy can then be defined as

$$S_{\mathrm{ren}} = -\mathrm{Lim}_{n\to 1}\left(\partial_n\left[Z_{\mathrm{ren}}(n) - nZ_{\mathrm{ren}}(1)\right]\right). \tag{3.1.3}$$

Here the partition function $Z_{\mathrm{ren}}(1)$ is renormalized using any method of renormalization. The partition function on the replica space inherits the same UV divergence structure and thus the renormalized $Z_{\mathrm{ren}}(n)$ can be defined without ambiguities from the original renormalization scheme.

In [79] Page characterised information recovery from black holes in terms of the time dependence of the entanglement entropy of the Hawking radiation. A number of recent works, such as [21, 80, 22], have discussed how the Page curve for Hawking radiation can be recovered from semiclassical geometry. It is interesting to note that these discussions inherently rely on a finite (renormalized) notion of entanglement entropy, as defined above.

The UV divergences in the bare entanglement entropy are associated physically with local entanglement at the boundary of the entangling region. The renormalized entanglement entropy is instead associated with non-local entanglement between the entangling region and its complement. The behaviour of renormalized entanglement entropy in various phases of holographically realised quantum field theories was explored in [70].

Renormalized entanglement entropy is computed holographically in terms of the renormalized area of minimal surfaces. The latter topics has been explored right from the very early days of the AdS/CFT correspondence [81, 82], as it is also relevant to the holographic computation of Wilson loops. Within the mathematics community, there has been considerable study of renormalized areas of surfaces, see for example [83, 84, 85, 86, 87, 88]. Connections between renormalized areas, entanglement and the Willmore functional have been explored within both the mathematics and the physics communities [89, 90].

The main goal of this chapter is to demonstrate how the renormalized entanglement entropy can be expressed in terms of the Euler characteristic and other conformal invariants in odd-dimensional UV conformal field theories dual to gravity in even dimensions. The restriction to even dimensions is for the usual reason: conformal field theories in even dimensions have conformal anomalies, and accordingly the renormalized entanglement entropy is not a conformal invariant. For AdS_4/CFT_3, the required geometric analysis is already contained in [83]; here we interpret these mathematics results physically, particularly in terms of the F quantity. We then generalize the approach of [83] to AdS_6/CFT_5 dualities.

We show that the renormalized entropy $S(\Sigma)$ for a static entangling surface Σ in an asymptotically AdS_{2n} spacetime has the following structure:

$$S(\Sigma) \sim (-1)^{n+1}\mathcal{F}_n\,\chi(\Sigma) - \sum_r \mathcal{W}_r(\Sigma) - \sum_p \mathcal{H}_p(\Sigma) - \sum_q \mathcal{I}_q(\tilde{B}). \tag{3.1.4}$$

In this and all subsequent expressions S refers to the renormalized entanglement entropy i.e. for notational brevity we drop the subscript. The Euler invariant of the entangling surface is denoted $\chi(\Sigma)$ and $\mathcal{F}_n$ is a numerical coefficient. In everything that follows we implicitly work with spacetimes with constant negative Ricci curvature, i.e. no matter or gauge fields, but the generalization of our results to include bulk stress energy tensors would be straightforward.

The contributions $\mathcal{W}_r$ are expressed in terms of the pullback of the Weyl curvature to the surface. Each such contribution is individually finite and conformally invariant; finiteness generically requires that appropriate boundary terms are included. For $n = 2$ there is one single such contribution, linear in the Weyl tensor while for $n = 3$, there are two terms, linear and quadratic in the Weyl tensor. For general n terms up to and including order $(n-1)$ arise.

The contributions $\mathcal{H}_p$ are expressed in terms of scalar invariants built from the extrinsic curvature. Again, each such contribution is individually finite and conformally invariant, with boundary terms generically being required. For AdS_{2n} there are contributions up to and including order $2(n-1)$ in the extrinsic curvature; all such contributions involve an even number of extrinsic curvatures. For $d > 5$ there are $\mathcal{I}_q$ renormalized integrals containing products of Weyl and extrinsic curvature.

The general structure of the renormalized entropy/area and the decomposition using Gauss-Codazzi relation will apply in all even dimensions. The explicit terms that arise would need to be calculated for dimensions greater than or equal to eight, and the associated positivity properties proven.

While the gravity calculation can be carried out in all even dimensions, we should note however that the quantum field theory interpretation of the results in AdS_{2n} with $n \geq 4$ is unclear and there are no conformal field theories in dimension $n \geq 7$.

We note that relations between a renormalized entanglement entropy, the Euler invariant and curvature invariants has been considered in earlier works [71, 72, 73, 74, 75]. However, the underlying approach of these works is somewhat different: the renormalized entanglement entropy is not defined by using the boundary terms induced by the variational problem at the conformal boundary [91] as in [54, 70], following the standard approach to holographic renormalization [43, 92], but instead by adding Chern forms as boundary terms. However, the results coincide for $AAdS_4$; the Chern form and counterterm for the codimension two minimal surface renormalized area are identical as illustrated in [71, 83]. When the bulk entangling surface is a codimension two asymptotically hyperbolic slice of $AAdS_6$, $\Sigma = A\mathbb{H}^4$, by discarding all quantity extrinsic to Σ, the renormalized area formula (4.2.11) reproduces Anderson's four dimensional renormalized volume formula [93]

$$\chi(\Sigma) = \frac{3}{4\pi^2}\mathcal{A}(\Sigma) + \frac{1}{32\pi^2}\int_{\Sigma} |W^{(4)}|^2 \tag{3.1.5}$$

where $\mathcal{A}(\Sigma)$ is the renormalized area and $W^{(4)}$ is the Weyl tensor intrinsic to the four dimensional hyperbolic space Σ. In this case our results should coincide with [72, 73, 74, 75].

More generally, as pointed out by [94], the Kounterterm approach differs from the holographic renormalization procedure when the boundary Weyl tensor of the asymptotically locally AdS spacetime is non-vanishing. Using the Gauss-Codazzi relations, the boundary Weyl tensor is related to projections of the bulk Weyl tensor. In (4.2.11), the projections of the bulk Weyl contributes to the renormalized area. Hence, we anticipate that our results could differ from the Kounterterm approach and it would be interesting to compare the results in higher dimensions.

The expression (3.1.4) has several immediate physical applications. Firstly, for entangling surfaces in AdS_{2n} all $\mathcal{W}_r$ contributions are zero, due to the vanishing of the Weyl tensor. Umbilic minimal surfaces have zero extrinsic curvature, and thus the renormalized entanglement entropy reduces to the Euler invariant term. Entangling surfaces associated with spherical entangling regions (discussed extensively in [51]) are indeed umbilic and thus their renormalized entanglement entropies are proportional to their Euler invariants (which are one for all n).

In [51] it was shown that the finite contributions to the entanglement entropy of spherical regions compute the F quantities [95] in odd dimensional conformal field theories. Renormalized entanglement entropy enables these finite contributions to be extracted elegantly, in a manifestly scheme independent manner [54, 96]. By expressing the renormalized entanglement entropy in the form (3.1.4), it is manifest that the coefficients of proportionality $\mathcal{F}_n$ of the Euler invariants directly compute the F quantities.

The second immediate application of (3.1.4) is to variations of the entanglement entropy under changes in the background geometry (state of quantum field theory) and changes in the shape of the entangling region. The expression (3.1.4) can be used to give an elegant proof of the first law of entanglement entropy, generalizing the work of [57] as one no longer needs to restrict to normalizable metric perturbations.

The first variation of the entanglement entropy around spherical entangling regions in AdS takes a particularly simple and elegant form. Since such variations do not change the topology of the entangling surface, the Euler invariant contribution does not change. All contributions from the extrinsic curvature are quadratic or higher order; since the extrinsic curvature vanishes to leading order, this means the contributions $\mathcal{H}_p$ do not contribute to first variations (but do contribute to the second variations). By analogous reasoning, the only contribution from the Weyl terms $\mathcal{W}_r$ comes from the term that is linear in the Weyl tensor. Thus we arrive at

$$\delta S \propto \frac{-1}{4G_{2n}}\delta\mathcal{W} \tag{3.1.6}$$

where G_{2n} is the Newton constant and

$$\delta\mathcal{W} = \int_{\Sigma} d^{2(n-1)}x\sqrt{g}\;\delta\widetilde{W}_{1212} - \int_{\partial\Sigma} d^{2n-3}x\sqrt{h}\delta W_{1212} + \cdots \tag{3.1.7}$$

where $\delta\widetilde{W}_{1212}$ is the pullback of the normal components of the bulk linearized Weyl curvature in an orthonormal frame and δW_{1212} is the pullback of the normal components of the boundary linearized Weyl curvature in an orthonormal frame. The boundary terms are such that $\delta\mathcal{W}$ is a finite conformal invariant for a generic non-normalizable metric perturbation. Note that the boundary term vanishes for AdS_4. The ellipses denote additional boundary terms expressed in terms of higher powers of the boundary Weyl curvature that are required for $n > 3$.

In a future work [2] we will show in detail how $\delta\mathcal{W}$ can be related to the renormalized stress tensor defined in [43] and hence to the variation in the energy; this gives a generalized proof of the first law [57] in a simple and elegant way.

The plan of this chapter is as follows. In section 3.2 we consider static entangling surfaces in asymptotically locally AdS_4 spacetimes; the relevant mathematical results were derived in [83]. In section 3.3 we analyse static entangling surfaces in asymptotically locally AdS_6 spacetimes; the main result of this section is the explicit form of the renormalized area in terms of finite conformal invariants (4.2.11). Details of the asymptotic analysis are contained within the appendix. In section 3.4 we express the renormalized entanglement entropy for spherical entangling regions in terms of the Euler invariant and show that linearized variations can be expressed in terms of the conformal invariant that is linear in the Weyl tensor. We conclude in section 3.5.

3.2 Asymptotically AdS_4

Consider a codimension two static minimal surface Σ with boundary $\partial\Sigma$ in an asymptotically locally AdS_4 spacetime. The renormalized entanglement entropy $S(\Sigma)$ is expressed in terms of the renormalized area $\mathcal{A}(\Sigma)$ as

$$S(\Sigma) = \frac{\mathcal{A}(\Sigma)}{4G_4} \tag{3.2.1}$$

where G_4 is the four-dimensional Newton constant. The renormalized area is [54]

$$\mathcal{A}(\Sigma) = \int_\Sigma d^2x\sqrt{g} - \int_{\partial\Sigma} dx\sqrt{h}. \tag{3.2.2}$$

Here g is the metric on the minimal surface and h is the metric at the boundary of the minimal surface.

It was shown in [83] that the renormalized area can be expressed in terms of the Euler characteristic of the surface and an integral of local invariants. The analysis of [83] was for two dimensional minimal surfaces in $(d+1)$-dimensional asymptotically locally hyperbolic Einstein spaces i.e. Euclidean signature. This analysis demonstrated that

$$\mathcal{A}(\Sigma) = -2\pi\chi(\Sigma) - \frac{1}{2}\int_\Sigma d^2x\sqrt{g}|K^s|^2 + \int_\Sigma d^2x\sqrt{g}\widetilde{W}_{3434} \tag{3.2.3}$$

where $\widetilde{W}_{3434}$ is the Weyl curvature of the bulk metric evaluated on any orthonormal basis for the tangent space of the entangling surface and the bulk curvature is normalised to satisfy $R_{\mu\nu} = -dG_{\mu\nu}$. Here K^s_{ij} are the components of the second fundamental form; the index s runs over the directions orthogonal to the surface i.e. $s = 1, 2$ in the case of a four-dimensional bulk geometry. Note that the minimal condition implies that K^s is trace free. Each term in (3.2.3) is individually finite: the integrands in the last two terms fall off sufficiently quickly near the conformal boundary that the integrals do not have divergent contributions [83].

In the case of a static Ryu-Takayanagi entangling surface, the extrinsic curvature in the time direction is zero and by tracelessness of the Weyl curvature the renormalized area reduces to

$$\mathcal{A}(\Sigma) = -2\pi\chi(\Sigma) - \frac{1}{2}\int_\Sigma d^2x\sqrt{g}|K|^2 - \int_\Sigma d^2x\sqrt{g}\widetilde{W}_{1212} \tag{3.2.4}$$

where K_{ij} is the extrinsic curvature of the surface along a spatial section and $\widetilde{W}_{1212}$ is the

Weyl curvature evaluated on an orthonormal basis for the normal space of Σ. Writing the Weyl tensor in this way is to match with our higher dimensional result shown in the later section.

3.2.1 Disk entangling region

Let us now consider the renormalized entanglement entropy in particular contexts. In pure AdS_4 the Weyl tensor vanishes and therefore

$$S(\Sigma) = -\frac{\pi}{2G_4}\chi(\Sigma) - \frac{1}{8G_4}\int_\Sigma d^2x\sqrt{g}|K|^2 \tag{3.2.5}$$

Consider a single entangling region in the boundary, which is topologically a disk. The corresponding Ryu-Takayanagi surface has the same topology and accordingly its Euler characteristic $\chi(\Sigma) = 1$. The renormalized entanglement entropy for such surfaces therefore satisfies

$$S(\Sigma) \leq -\frac{\pi}{2G_4} \tag{3.2.6}$$

with equality in the case of $K_{ij} = 0$. Minimal surfaces satisfy $K = 0$; surfaces that in addition satisfy $K_{ij} = 0$, i.e. the traceless part of the extrinsic curvature vanishes, are called umbilic. Umbilic surfaces are locally spherical; the normal curvatures in all directions are equal.

In the specific case of a disk entangling region, the entangling surface indeed has zero extrinsic curvature and is umbilic. This can be seen by changing from Poincaré coordinates:

$$ds^2 = \frac{1}{\rho^2}\left(-dt^2 + d\rho^2 + dr^2 + r^2 d\phi^2\right) \tag{3.2.7}$$

to new coordinates adapted to the entangling surface:

$$\rho = R\sin\theta \qquad r = R\cos\theta \tag{3.2.8}$$

so that

$$ds^2 = \frac{1}{R^2\sin^2\theta}\left(-dt^2 + dR^2 + R^2(d\theta^2 + \cos^2\theta d\phi^2)\right). \tag{3.2.9}$$

The induced metric on an entangling surface of constant t and R can thus be written as

$$ds^2 = \frac{1}{\sin^2\theta}\left(d\theta^2 + \cos^2\theta d\phi^2\right), \tag{3.2.10}$$

which is independent of both t and R, demonstrating that the extrinsic curvatures are zero.

For a disk entangling region $\mathcal{D}$, the renormalized entropy is thus directly proportional to the Euler characteristic of the entangling surface. As discussed in [54, 96], the renormalized entropy is also related to the F quantity of the corresponding 3d CFT and hence

$$F = -S(\mathcal{D}) = \frac{\pi}{2G_4}\chi(\mathcal{D}) \tag{3.2.11}$$

and the representation of the entanglement entropy in terms of a topological invariant emphasises that this quantity does not depend on any choice of renormalization scheme.

Now let us consider linearized perturbations around the disk entangling surface in AdS$_4$. Linear and quadratic perturbations around generic minimal surfaces in asymptotically hyperbolic manifolds were discussed in detail in [83]. The analysis of [83] however simplifies considerably for perturbations around the disk entangling surface as both the Weyl and extrinsic curvatures vanish at leading order. Accordingly the only term in the linearized variation is

$$\delta S = \frac{-1}{4G_4}\int d^2x\sqrt{g}\;\delta\widetilde{W}_{1212} \tag{3.2.12}$$

In a subsequent work [2] we will show how $\delta\tilde{W}_{1212}$ can be related to the renormalized stress tensor constructed in [43] and hence to the variation in the energy; this gives a generalized proof of the first law [57].

3.2.2 Strip entangling region

Consider now a strip entangling region $\mathcal{S}$ in pure AdS$_4$. Using the following Poincaré coordinates

$$ds^2 = \frac{1}{\rho^2}\left(-dt^2 + d\rho^2 + dx^2 + dy^2\right), \tag{3.2.13}$$

the entangling surface for a strip entangling region along the y direction can be expressed as

$$\frac{d\rho}{dx} = \mp\frac{\sqrt{\rho_c^4 - \rho^4}}{\rho^2} \tag{3.2.14}$$

where ρ_c is the turning point of the surface and $-$ for $0 \leq x \leq \frac{L_x}{2}$ and $+$ for $-\frac{L_x}{2} \leq x \leq 0$. The width of the strip L_x along the x direction is related to ρ_c as

$$L_x = 2\int_0^{\rho_c}\frac{\rho^2}{\sqrt{\rho_c^4 - \rho^4}}d\rho = 2\sqrt{\pi}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\rho_c. \tag{3.2.15}$$

Here implicitly we assume that $L_x \ll L_y$, where L_y is the length of the strip, so that contributions from the corners and short sides are negligible. The renormalized area $\mathcal{A}(\mathcal{S})$ is then given by

$$\mathcal{A}(\mathcal{S}) = -\frac{2L_y}{\rho_c}\sqrt{\pi}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} = -\frac{L_y L_x}{\rho_c^2}. \tag{3.2.16}$$

Since for large L_y the Euler characteristic is negligible and in the limit of the infinite strip $\chi(\mathcal{S}) = 0$, and the Weyl curvature vanishes for pure AdS, the renormalized area (4.2.10) is given in terms of the integral of the extrinsic curvature over the surface.

Using (3.2.14) we can pullback the AdS_4 metric onto $\mathcal{S}$ to give:

$$ds^2 = \frac{1}{\rho(x)^2}\left(\frac{\rho_c^4}{\rho(x)^4}dx^2 + dy^2\right), \tag{3.2.17}$$

where implicitly ρ is expressed in terms of x. The push forward of the unit spatial normal vector is

$$n_2 = \frac{\rho^3}{\rho_c^2}\left(\frac{\partial}{\partial\rho} \pm \frac{\sqrt{\rho_c^4 - \rho^4}}{\rho^4}\frac{\partial}{\partial x}\right). \tag{3.2.18}$$

The interpretation of the two signs is as follows. Let the strip extend from $x = -\frac{1}{2}L_x$ to $x = \frac{1}{2}L_x$. For $x > 0$, the normal to the entangling surface points in the direction of increasing x (positive sign) while for $x < 0$ the normal points in the direction of decreasing x (negative sign). Accordingly the induced metric can be written as

$$\begin{aligned} G^{\mathcal{S}}_{\mu\nu}dx^\mu dx^\nu &= \left(G_{\mu\nu} - n_{2\mu}n_{2\nu}\right)dx^\mu dx^\nu \\ &= \frac{1}{\rho^2}\left(\frac{\rho_c^4 - \rho^4}{\rho_c^4}d\rho^2 - 2\frac{\rho^2\sqrt{\rho_c^4-\rho^4}}{\rho_c^4}d\rho dx + \frac{\rho^4}{\rho_c^4}dx^2 + dy^2\right). \end{aligned} \tag{3.2.19}$$

The temporal extrinsic curvature vanishes and the spatial extrinsic curvature is given by

$$K_{\mu\nu}dx^\mu dx^\nu = \frac{\rho_c^4 - \rho^4}{\rho_c^4}d\rho^2 \mp 2\frac{\rho^2\sqrt{\rho_c^4-\rho^4}}{\rho_c^6}d\rho dx + \frac{\rho^4}{\rho_c^6}dx^2 - \frac{1}{\rho_c^2}dy^2, \tag{3.2.20}$$

The trace of the extrinsic curvature can be easily read off and satisfies the required minimality condition, $K = 0$. From (4.2.10), the only non vanishing term of the renormalized area is

$$\mathcal{A}(\mathcal{S}) = -\frac{1}{2}\int_{\mathcal{S}} d^2x\sqrt{g}K^{\mu\nu}K_{\mu\nu} = -\frac{1}{2}\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}}dy\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}}dx\frac{\rho_c^2}{\rho^4}\left(\frac{2\rho^4}{\rho_c^4}\right) = -\frac{L_yL_x}{\rho_c^2}. \tag{3.2.21}$$

Note that $K^{\mu\nu}K_{\mu\nu}$ takes the same value for either sign in (3.2.20). This matches with the explicit result for the renormalized area of the minimal surface extends from the strip entangling region in (3.2.16).

3.3 Asymptotically AdS$_6$

Consider a codimension two static minimal surface Σ with boundary $\partial\Sigma$ in an asymptotically locally AdS$_6$ spacetime. The renormalized entanglement entropy $S(\Sigma)$ is expressed in terms of the renormalized area $\mathcal{A}(\Sigma)$ as

$$S(\Sigma) = \frac{\mathcal{A}(\Sigma)}{4G_6} \tag{3.3.1}$$

where G_6 is the six-dimensional Newton constant. The renormalized area is [54]

$$\begin{aligned}\mathcal{A}(\Sigma) &= \int_\Sigma d^4x\sqrt{g} - \frac{1}{3}\int_{\partial\Sigma} d^3x\sqrt{h} \\ &\quad -\frac{1}{9}\int_{\partial\Sigma} d^3x\sqrt{h}\left(\hat{R}_{aa} - \frac{1}{2}k^2 - \frac{5}{8}\hat{R}\right).\end{aligned} \tag{3.3.2}$$

Here g is the metric on the minimal surface and h is the metric at the boundary of the minimal surface. $\hat{R}_{aa}$ is the curvature of the metric on the boundary of the asymptotically locally AdS$_6$ spacetime, projected to the subspace orthogonal to $\partial\Sigma$. $\hat{R}$ is the Ricci scalar of the boundary curvature and k^2 is the square of the extrinsic curvature of $\partial\Sigma$ embedded into ∂M, the boundary of the asymptotically locally AdS$_6$ spacetime M. The counterterms are sufficient for bulk dimension less than or equal to six; additional divergences arise in higher dimensions [54].

Using the Chern-Gauss-Bonnet theorem, the Euler invariant for a four-dimensional manifold with boundary consists of a bulk contribution

$$\chi(\Sigma) = \frac{1}{32\pi^2}\int_\Sigma d^4x\sqrt{g}\left(\mathcal{R}^{ijkl}\mathcal{R}_{ijkl} - 4\mathcal{R}_{ij}\mathcal{R}^{ij} + \mathcal{R}^2\right) \tag{3.3.3}$$

(where $\mathcal{R}$ refers to the intrinsic curvature of the manifold) with boundary contributions that may be expressed as in [97]:

$$\begin{aligned}&+\frac{1}{4\pi^2}\int_{\partial\Sigma} d^3x\sqrt{h}\Big(\mathcal{R}_{ijkl}\mathcal{K}^{ik}n^jn^k - \mathcal{R}^{ij}\mathcal{K}_{ij} - \mathcal{K}\mathcal{R}_{ij}n^in^j + \frac{1}{2}\mathcal{K}\mathcal{R} \\ &\qquad + \frac{1}{3}\mathcal{K}^3 - \mathcal{K}\mathrm{Tr}(\mathcal{K}^2) + \frac{2}{3}\mathrm{Tr}(\mathcal{K}^3)\Big).\end{aligned} \tag{3.3.4}$$

The above formulae used different sign convention to [97] and are further explained in the appendix. Note that this form for the boundary contributions was derived in the context of analysing conformal anomalies on manifolds with boundary.

By construction both functionals (3.3.2) and (3.3.3) are finite. However, there are clear conceptual differences between the boundary terms. In the case of the renormalised area, the boundary terms are counterterms, expressed covariantly in terms of Dirichlet data at the conformal boundary. This implies that the boundary terms have to be expressed only in terms of the intrinsic curvature of the conformal boundary, and the extrinsic curvature of the boundary of the entangling surface, embedded into the conformal boundary.

By contrast, the Euler invariant is expressed entirely in terms of quantities that are intrinsic to the entangling surface itself, with no reference to the embedding of the surface into the six-dimensional bulk manifold. Here the boundary terms are not expressed in terms of Dirichlet data at the boundary of the surface, but involve the extrinsic curvature of the

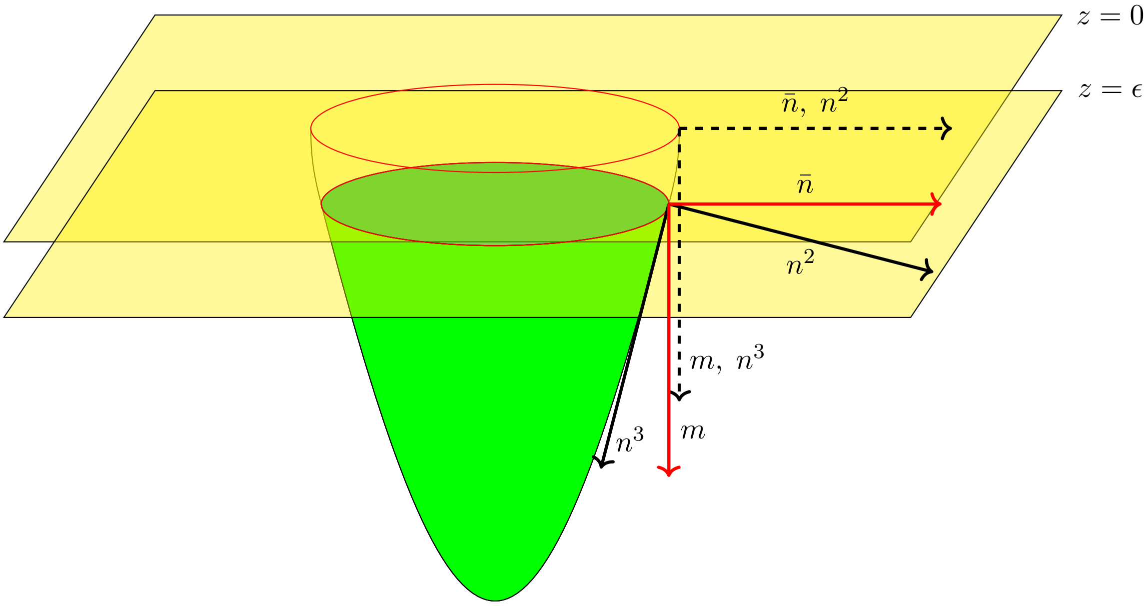


Figure 3.3.1: In this diagram the temporal direction, n^1, is suppressed. We can identify two distinct sets of normal directions: n^2 is the normal of the Σ and n^3 is the normal of $\partial\Sigma$ within Σ, while $\bar{n}$ is the normal of $\partial\Sigma$ on ∂M and m is the normal of ∂M in M. On the regulated boundary $\partial M|_{z=\epsilon}$, n^2 and n^3 are not equal to $\bar{n}$ and m respectively. However the normal space is manifestly spanned by both $\{n^s, s = 1, 2, 3\}$ and $\{n^1, \bar{n}, m\}$.

boundary.

The goal of this section is to relate the renormalised area to the Euler invariant, through the use of Gauss-Codazzi relations and asymptotic analysis. Related analysis was carried out in the mathematics literature in [86, 89] but these works did not use explicit counterterms to define the renormalized area.

3.3.1 Geometric preliminaries

The extrinsic curvatures of the entangling surface Σ are defined by

$$K^s_{\mu\nu} = g^\rho_\mu g^\sigma_\nu \nabla_\rho n^s_\sigma \tag{3.3.5}$$

where the normals to Σ are n^s with $s = 1, 2$; we will denote by n^1 the normal in the time direction. Similarly, the extrinsic curvature of the boundary entangling surface $\partial\Sigma$ embedded into Σ is defined by

$$\mathcal{K}_{ij} = -h^k_i h^l_j \nabla_k n^3_l \tag{3.3.6}$$

where n^3 is the associated inward pointing normal, as shown in Figure 3.3.1.

We can define a second set of normals to $\partial\Sigma$, $(n^1, \bar{n}, m)$, where $\bar{n}$ is the normal of $\partial\Sigma$ lying within ∂M and m is the normal of ∂M in M. The extrinsic curvatures corresponding to this second set of normals are defined as

$$k_{ij} = h^k_i h^l_j \nabla_k \bar{n}_l \qquad\qquad k^\perp_{ij} = h^k_i h^l_j \nabla_k m_l \tag{3.3.7}$$

The two sets of normal vectors (n_1, n_2, n_3) and $(n_1, \bar{n}, m)$ can be related by coordinate transformations.

$$\begin{aligned} n^2 &= A\bar{n} + A^\perp m \\ n^3 &= -A^\perp \bar{n} + Am \end{aligned} \tag{3.3.8}$$

where $A^2 + {A^\perp}^2 = 1$ and $A, A^\perp \in \mathbb{R}$. The induced metric on $\partial\Sigma$ can also be obtained from either set of normal

$$h_{\mu\nu} = G_{\mu\nu} + n^1_\mu n^1_\nu - n^2_\mu n^2_\nu - n^3_\mu n^3_\nu = G_{\mu\nu} + n^1_\mu n^1_\nu - \bar{n}_\mu \bar{n}_\nu - m_\mu m_\nu. \tag{3.3.9}$$

Note that it is often convenient to work in an orthonormal basis, $G_{\mu\nu} e^\mu_M e^\nu_N = \eta_{MN}$, such that the induced metric on Σ is $e^A e^A = n^3 n^3 + e^a e^a = g_{\mu\nu} dx^\mu dx^\nu$ and on $\partial\Sigma$ is $e^a e^a = h_{ij} dx^i dx^j$.

As in [83], we will work in Fefferman-Graham coordinate systems for asymptotically locally

AdS metrics:

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{dz^2}{z^2} + \gamma_{\alpha\beta}dx^\alpha dx^\beta \tag{3.3.10}$$

and the metric γ admits the expansion

$$\gamma_{\alpha\beta} = z^{-2}\left(\bar{\gamma}^{(0)}_{\alpha\beta} + z^2\bar{\gamma}^{(2)}_{\alpha\beta} + \cdots\right) \tag{3.3.11}$$

Note that this implicitly assumes that the entangling surface is contained within the Fefferman-Graham coordinate patch.

For static manifolds Σ and $\partial\Sigma$ in AdS we can then express the normals as

$$n^1 = \frac{dt}{z}, \quad m = \frac{dz}{z}, \quad \bar{n} = \frac{\bar{\alpha}d\bar{f}}{z}. \tag{3.3.12}$$

(For non-static surfaces one would need to parameterise the timelike normal as $n^1 = \frac{\alpha_\tau d\bar{\tau}}{z}$.) The corresponding normal vector fields are

$$n_1 = e_1 = z\frac{\partial}{\partial t}, \quad e_m = z\frac{\partial}{\partial z}, \quad e_{\bar{n}} = \frac{z}{\bar{\alpha}}\frac{\partial}{\partial \bar{f}} \tag{3.3.13}$$

where $\bar{\alpha}$ is a function of $(z, \bar{f})$ only. Using these relations one can decompose bulk curvatures into quantities that are intrinsic and extrinsic to the surface. For example, the Lie bracket for the normal vector fields has structure constant F^P_{MN} such that $[e_M, e_N] = F^P_{MN}e_P$. Only the following components are non-vanishing

$$F^1_{m1} = -F^1_{1m} = 1, \quad F^{\bar{n}}_{m\bar{n}} = -F^{\bar{n}}_{\bar{n}m} = 1 - \bar{\beta} \tag{3.3.14}$$

where $\bar{\beta} = \frac{z\partial_z\bar{\alpha}}{\bar{\alpha}}$. From these expressions we can then work out the connections and curvature tensors in terms of quantities defined on Σ.

3.3.2 Gauss-Codazzi relations

In this section we collect together identities relating the bulk curvature with the intrinsic and and extrinsic curvatures of the entangling surface. First let us note the following relation for the bulk curvature: since the manifold is Einstein with negative cosmological constant, we can express the Riemann curvature in terms of the Weyl curvature as

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + G_{\mu\rho}G_{\nu\sigma} - G_{\mu\sigma}G_{\nu\rho} \tag{3.3.15}$$

where $G_{\mu\nu}$ is the metric on $\mathcal{M}$. In particular, the Weyl curvature vanishes for anti-de Sitter spacetime itself.

In this section we will implement Gauss-Codazzi relations for the codimension two surface, taking into account both (unit) normals to the entangling surface by n^s_μ with $s = 1, 2$. In the context of Ryu-Takayanagi surfaces the extrinsic curvatures in time directions are trivial, but the analysis carried out in this section is more general and does not pick out a distinguished coordinate system for the normal directions.

The Gauss-Codazzi relations then state that:

$$g^\kappa_\mu g^\lambda_\nu g^\tau_\rho g^\eta_\sigma \; R_{\kappa\lambda\tau\eta} = \mathcal{R}_{\mu\nu\rho\sigma} + \sum_{s=1}^{2} (-1)^s (K^s_{\mu\sigma} K^s_{\nu\rho} - K^s_{\mu\rho} K^s_{\nu\sigma}) \tag{3.3.16}$$

where the extrinsic curvatures are defined above in (3.3.5). (Note that it is often convenient to choose adapted coordinates for the hypersurface.)

The pullback of the bulk curvature can be expressed as

$$g^\kappa_\mu g^\lambda_\nu g^\tau_\rho g^\eta_\sigma \; R_{\kappa\lambda\tau\eta} = g^\kappa_\mu g^\lambda_\nu g^\tau_\rho g^\eta_\sigma \; W_{\kappa\lambda\tau\eta} + g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}, \tag{3.3.17}$$

using $g^{\mu\nu} n^s_\mu = 0$. In what follows it is convenient to use a compressed notation to denote the pulled back Weyl curvature as

$$\widetilde{W}_{\mu\nu\rho\sigma} \equiv g^\kappa_\mu g^\lambda_\nu g^\tau_\rho g^\eta_\sigma \; W_{\kappa\lambda\tau\eta}. \tag{3.3.18}$$

Contraction of the Gauss-Codazzi relations gives

$$g^\kappa_\mu g^\lambda_\nu R_{\kappa\lambda} + g^\kappa_\mu g^\lambda_\nu R_{\kappa\tau\lambda\eta} \sum_{s=1}^{2} (-1)^{s-1} n^\tau_s n^\eta_s = \mathcal{R}_{\mu\nu} + \sum_{s=1}^{2} (-1)^s K^s_{\mu\rho} K^{s\ \rho}_{\ \nu}, \tag{3.3.19}$$

where here and in the rest of this sectoin we show the normal index s as a subscript to improve the clarity of equations. Contracting further gives

$$R + 2 g^\kappa_\mu g^\lambda_\nu R_{\kappa\lambda} \sum_{s=1}^{2} (-1)^{s-1} n^\mu_s n^\nu_s - 2 R_{\mu\nu\rho\sigma} n^\mu_1 n^\nu_2 n^\rho_1 n^\sigma_2 = \mathcal{R} + \sum_{s=1}^{2} (-1)^s K^s_{\mu\rho} K^{s\ \mu\rho} \tag{3.3.20}$$

where we use the fact that the surface is minimal so $K^s = 0$. In our case, the background manifold is Einstein, for which the Ricci curvature can conveniently be normalised as

$$R_{\mu\nu} = -d G_{\mu\nu} \tag{3.3.21}$$

for asymptotically locally AdS$_{(d+1)}$ spacetimes. Using the fact that $g_{\mu\nu} n^\nu_s = 0$, we can thus write

$$\mathcal{R}_{\mu\nu} + \sum_{s=1}^{2} (-1)^s K^s_{\mu\rho} K^{s\ \rho}_{\ \nu} = -(d-2) g_{\mu\nu} - g^\lambda_\mu g^\tau_\nu W_{\lambda\rho\tau\sigma} \sum_{s=1}^{2} (-1)^s n^\rho_s n^\sigma_s, \tag{3.3.22}$$

and

$$\mathcal{R} + \sum_{s=1}^{2} (-1)^s K^s_{\mu\nu} K^{s\mu\nu} = -(d-2)(d-1) - 2W_{\mu\nu\rho\sigma} n_1^\mu n_2^\nu n_1^\rho n_2^\sigma, \tag{3.3.23}$$

where we use the fact that the dimension of the entangling surface is $(d-1)$.

For notational convenience we will define the combinations

$$H_{\mu\nu\rho\sigma} = \sum_{s=1}^{2} (-1)^s (K^s_{\mu\sigma} K^s_{\nu\rho} - K^s_{\mu\rho} K^s_{\nu\sigma}), \tag{3.3.24}$$

as well as

$$\widetilde{W}_{\mu n \nu n} = g^\lambda_\mu g^\tau_\nu W_{\lambda\rho\tau\sigma} \sum_{s=1}^{2} (-1)^s n_s^\mu n_s^\nu \tag{3.3.25}$$

and

$$\widetilde{W}_{1212} = W_{\mu\nu\rho\sigma} n_1^\mu n_2^\nu n_1^\rho n_2^\sigma. \tag{3.3.26}$$

The Gauss-Codazzi relations can then be used to rewrite the bulk term in the Euler invariant as follows. The Riemann curvature terms give

$$\begin{aligned} \mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} &= 2(d-1)(d-2) + 4H + H_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} \\ &\quad -4\widetilde{W} + \widetilde{W}_{\mu\nu\rho\sigma} \widetilde{W}^{\mu\nu\rho\sigma} - 2\widetilde{W}_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma}. \end{aligned} \tag{3.3.27}$$

The Ricci curvature terms give

$$\begin{aligned} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} &= (d-2)^2 (d-1) + 2(d-2) H + H_{\mu\nu} H^{\mu\nu} \\ &\quad +2(d-2)\widetilde{W}_{nn} + \widetilde{W}_{\mu n \nu n} \widetilde{W}^{\mu n \nu n} + 2\widetilde{W}_{\mu n \nu n} H^{\mu\nu} \end{aligned} \tag{3.3.28}$$

while the Ricci scalar terms give

$$\begin{aligned} \mathcal{R}^2 &= (d-2)^2 (d-1)^2 + 2(d-2)(d-1) H + H^2 \\ &\quad +4(d-2)(d-1)\widetilde{W}_{1212} + 4\widetilde{W}_{1212}^2 + 4H\widetilde{W}_{1212}. \end{aligned} \tag{3.3.29}$$

Here H and $H_{\mu\nu}$ can be expressed as

$$H_{\mu\nu} = \sum_{s=1}^{2} (-1)^s K^s_{\mu\sigma} K^{s}{}_{\nu}{}^{\sigma} \qquad H = \sum_{s=1}^{2} (-1)^s K^s_{\mu\nu} K^{s\mu\nu}. \tag{3.3.30}$$

Combining these terms for the case of $d = 5$ (AdS_6), we obtain an expression for the Euler invariant of the form:

$$\chi(\Sigma) = \frac{1}{32\pi^2} \int_\Sigma d^4x \sqrt{g} \left(24 + \Delta\chi(K^s, \widetilde{W}) \right) + \frac{1}{4\pi^2} \int_{\partial\Sigma} d^3x \sqrt{h} \partial E_4 \tag{3.3.31}$$

where the functional appearing in the volume term takes the form

$$\begin{aligned}\Delta\chi =& 4H + 8\widetilde{W}_{1212} + H^2 - 4H_{\mu\nu}H^{\mu\nu} + H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma} \\ &+ 4\widetilde{W}_{1212}^2 - 4\widetilde{W}_{\mu n\nu n}\widetilde{W}^{\mu n\nu n} + \widetilde{W}_{\mu\nu\rho\sigma}\widetilde{W}^{\mu\nu\rho\sigma} \\ &+ 4H\widetilde{W}_{1212} - 8H^{\mu\nu}\widetilde{W}_{\mu n\nu n} - 2H^{\mu\nu\rho\sigma}\widetilde{W}_{\mu\nu\rho\sigma}\end{aligned} \tag{3.3.32}$$

The notation chosen for the volume term reflects the fact that $\Delta\chi$ vanishes for spherical entangling surfaces ($K^s_{\mu\nu} = 0$) in pure AdS ($W_{\mu\nu\rho\sigma}$), as we discuss in the next section.

To simplify this expression we have used the following expressions for the projected and contracted Weyl tensor

$$\begin{aligned}&\widetilde{W} = \widetilde{W}^{AB}{}_{AB} = W^{AB}{}_{AB} && \widetilde{W}_{22} = W^{A}{}_{2A2} \\ &\widetilde{W}_{1212} = W_{1212} && \widetilde{W}_{11} = W^{A}{}_{1A1}.\end{aligned} \tag{3.3.33}$$

Since Weyl tensor is traceless, we can write the curvatures in (3.3.33) in terms of each other as

$$\begin{aligned}W^{AB}{}_{AB} &= W^{MN}{}_{MN} - 2\left(W^{A1}{}_{A1} + W^{A2}{}_{A2} + W^{12}{}_{12}\right) \\ \widetilde{W} &= 2\widetilde{W}_{11} - 2\widetilde{W}_{22} + 2\widetilde{W}_{1212}\end{aligned} \tag{3.3.34}$$

and thus

$$\widetilde{W} = -\widetilde{W}_{nn} = -2\widetilde{W}_{1212} \tag{3.3.35}$$

Therefore the contributions linear in the Weyl tensor can be written in terms of the projection of the Weyl tensor onto $N\Sigma$, $\widetilde{W}_{1212}$.

We now need to express the boundary contributions to the Euler density integral in terms of extrinsic curvatures and the Weyl tensor. We first define the extrinsic curvature, $\mathscr{K}$, of ∂M embedded into M,

$$\mathscr{K}_{\mu\nu} = (\delta^\rho_\mu - m^\rho_\mu)(\delta^\sigma_\nu - m^\sigma_\nu)\nabla_\rho m_\sigma. \tag{3.3.36}$$

Using the definition of the extrinsic curvature of $\partial\Sigma$ pointing out of the boundary $k^\perp_{\mu\nu} := h^\rho_\mu h^\sigma_\nu \nabla_\rho m_\sigma$, one can show that

$$\mathscr{K}_{\mu\nu} = k^\perp_{\mu\nu} - (1-\bar{\beta})\bar{n}_\mu\bar{n}_\nu + n^1_\mu n^1_\nu. \tag{3.3.37}$$

The trace of $\mathscr{K}$ is

$$\mathscr{K} = k^\perp - 2 + \bar{\beta} \tag{3.3.38}$$

and the trace of the product is

$$\mathscr{K}_{\mu\nu}\mathscr{K}^{\mu\nu} = k^{\perp}_{\mu\nu}k^{\perp\,\mu\nu} + 2 - 2\bar{\beta} + \bar{\beta}^2 \tag{3.3.39}$$

The intrinsic curvature of the boundary ∂M, $\hat{R}_{\mu\nu\rho\sigma}$, is related to the projection and contraction of Weyl tensor and $\mathscr{K}$ by the Gauss-Codazzi equation, giving the following relations

$$\begin{aligned}
\hat{R}_{1\bar{n}1\bar{n}} &= 1 + W_{1\bar{n}1\bar{n}} - \mathscr{K}_{1\bar{n}}\mathscr{K}_{\bar{n}1} + \mathscr{K}_{11}\mathscr{K}_{\bar{n}\bar{n}} \\
\hat{R}_{11} &= d - 1 - W_{1m1m} - \mathscr{K}_1{}^{\rho}\mathscr{K}_{\rho 1} + \mathscr{K}_{11}\mathscr{K} \\
\hat{R}_{\bar{n}\bar{n}} &= -d + 1 - W_{\bar{n}m\bar{n}m} - \mathscr{K}_{\bar{n}}{}^{\rho}\mathscr{K}_{\rho\bar{n}} + \mathscr{K}_{\bar{n}\bar{n}}\mathscr{K} \\
\hat{R} &= -d(d-1) - \mathscr{K}_{\mu\nu}\mathscr{K}^{\mu\nu} + \mathscr{K}^2 \\
\hat{R}_{ij} &= -(d-1)h_{ij} - W_{imjm} - \mathscr{K}_i{}^{\rho}\mathscr{K}_{\rho j} + \mathscr{K}_{ij}\mathscr{K} \\
\hat{R}_{i\bar{n}j\bar{n}} &= -h_{ij} + W_{i\bar{n}j\bar{n}} - \mathscr{K}_{i\bar{n}}\mathscr{K}_{\bar{n}j} + \mathscr{K}_{ij}\mathscr{K}_{\bar{n}\bar{n}} \\
\hat{R}_{i1j1} &= h_{ij} + W_{i1j1} - \mathscr{K}_{i1}\mathscr{K}_{1j} + \mathscr{K}_{ij}\mathscr{K}_{11}.
\end{aligned} \tag{3.3.40}$$

Substituting $\mathscr{K}$ terms using (3.3.37), (3.3.38) and (3.3.39) we obtain

$$\begin{aligned}
\hat{R}_{1\bar{n}1\bar{n}} &= W_{1\bar{n}1\bar{n}} + \bar{\beta} \\
\hat{R}_{11} &= d - 2 + k^{\perp} - W_{1m1m} + \bar{\beta} \\
\hat{R}_{\bar{n}\bar{n}} &= -d + 2 - k^{\perp} - W_{\bar{n}m\bar{n}m} + (k^{\perp} - 1)\bar{\beta} \\
\hat{R} &= -d(d-1) + 2 - 4k^{\perp} + {k^{\perp}}^2 - k^{\perp}_{ij}k^{\perp\,ij} + 2(k^{\perp} - 1)\bar{\beta} \\
\hat{R}_{ij} &= -(d-1)h_{ij} - 2k^{\perp}_{ij} - k^{\perp}_i{}^k k^{\perp}_{kj} + k^{\perp}_{ij}k^{\perp} - W_{imjm} + k^{\perp}_{ij}\bar{\beta} \\
\hat{R}_{i\bar{n}j\bar{n}} &= -h_{ij} - k^{\perp}_{ij} + W_{i\bar{n}j\bar{n}} + k^{\perp}_{ij}\bar{\beta} \\
\hat{R}_{i1j1} &= h_{ij} + k^{\perp}_{ij} + W_{i1j1}.
\end{aligned} \tag{3.3.41}$$

The intrinsic curvature $\mathcal{R}$ terms on the surface given in (3.3.4) are related to the curvatures of the boundary of the entangling surface $\partial\Sigma$, $\overline{R}$, by additional Gauss-Codazzi relations:

$$\mathcal{K}\left(\frac{1}{2}\mathcal{R} - \mathcal{R}_{\mu\nu}n^{3\,\mu}n^{3\,\nu}\right) = \frac{1}{2}\mathcal{K}\left(\overline{R} - \mathcal{K}^2 + \mathcal{K}_{ij}\mathcal{K}^{ij}\right) \tag{3.3.42}$$

and

$$-\mathcal{K}^{ij}\mathcal{R}_{\mu i\nu j}(g^{\mu\nu} - n^{3\,\mu}n^{3\,\nu}) = -\mathcal{K}^{ij}\left(\overline{R}_{ij} + \mathcal{K}_{ik}\mathcal{K}^k_j - \mathcal{K}_{ij}\mathcal{K}\right). \tag{3.3.43}$$

We can again use Gauss-Codazzi relations to transform relate quantities on $\partial\Sigma$ to quantities in ∂M:

$$\begin{aligned}\overline{R} &= \hat{R} + 2\hat{R}_{11} - 2\hat{R}_{\bar{n}\bar{n}} - 2\hat{R}_{1\bar{n}1\bar{n}} + k^2 - k_{ij}k^{ij} \\ \overline{R}_{ij} &= \hat{R}_{ij} - \hat{R}_{i\bar{n}j\bar{n}} + \hat{R}_{i1j1} + k_{ij}k - k_i^{\ k}k_{kj}.\end{aligned} \tag{3.3.44}$$

Expressing Riemann tensors in terms of Weyl tensors gives

$$\begin{aligned}\overline{R} &= -(d-1)(d-4) + k^2 - k_{ij}k^{ij} + k^{\perp}_{ij}k^{\perp\, ij} - 2\left(W_{1m1m} + W_{1\bar{n}1\bar{n}} - W_{\bar{n}m\bar{n}m}\right) \\ \overline{R}_{ij} &= -(d-1)h_{ij} + k_{ij}k - k_i^k k_{kj} + k^{\perp}_{ij}k - k_i^{\perp\, k}k^{\perp}_{kj} + W_{i1j1} - W_{i\bar{n}j\bar{n}} - W_{imjm}.\end{aligned} \tag{3.3.45}$$

Specialising to $d = 5$ these expressions reduce to:

$$\begin{aligned}\overline{R} &= -4 + k^2 - k_{ij}k^{ij} + k^{\perp}_{ij}k^{\perp\, ij} - 2\left(W_{1m1m} + W_{1\bar{n}1\bar{n}} - W_{\bar{n}m\bar{n}m}\right) \\ \overline{R}_{ij} &= -4h_{ij} + k_{ij}k - k_i^k k_{kj} + k^{\perp}_{ij}k - k_i^{\perp\, k}k^{\perp}_{kj} + W_{i1j1} - W_{i\bar{n}j\bar{n}} - W_{imjm}.\end{aligned} \tag{3.3.46}$$

The decomposition of $\mathcal{K}$ into $k, k^{\perp}$ is straightforward:

$$\mathcal{K}_{ij} = +A^{\perp}k_{ij} - Ak^{\perp}_{ij}. \tag{3.3.47}$$

Our final expression for the boundary contributions to the Euler density can be written in terms of projections of the Weyl tensor and extrinsic curvatures of $\partial\Sigma$ tangent and normal to ∂M,

$$\begin{aligned}\partial E_4 = &-(A^{\perp}k - Ak^{\perp})\left(W_{1\bar{n}1\bar{n}} + W_{1m1m} - W_{\bar{n}m\bar{n}m}\right) \\ &+ (A^{\perp}k^{ij} - Ak^{\perp\, ij})\left(-W_{i1j1} + W_{imjm} + W_{i\bar{n}j\bar{n}}\right) \\ &+ Ak^{\perp} - \left(\frac{A}{2} - \frac{A^3}{6}\right)k^{\perp 3} - \left(A - \frac{A^3}{3}\right)k^{\perp}_{ij}k^{\perp\, jk}k^{\perp\ i}_{k} - \left(-\frac{3A}{2} + \frac{A^3}{2}\right)k^{\perp}k^{\perp}_{ij}k^{\perp\, ij} \\ &- A^{\perp}k - \left(-\frac{A^{\perp}}{2} - \frac{A^2A^{\perp}}{2}\right)kk^{\perp 2} - \left(\frac{A^{\perp}}{2} - \frac{A^2A^{\perp}}{2}\right)kk^{\perp}_{ij}k^{\perp\, ij} \\ &- \left(-A^{\perp} + A^2A^{\perp}\right)k_{ij}k^{\perp\, jk}k^{\perp\ i}_{k} - \left(A^{\perp} - A^2A^{\perp}\right)k^{\perp}k_{ij}k^{\perp\, ij} \\ &- \left(\frac{A}{2} - \frac{AA^{\perp 2}}{2}\right)k^2k^{\perp} - \left(-\frac{A}{2} + \frac{AA^{\perp 2}}{2}\right)k_{ij}k^{ij}k^{\perp} \\ &- \left(A - AA^{\perp 2}\right)k_{ij}k^{jk}k^{\perp\ i}_{k} - \left(-A + AA^{\perp 2}\right)kk_{ij}k^{\perp\, ij} \\ &- \left(-\frac{A^{\perp}}{2} + \frac{A^{\perp 3}}{6}\right)k^3 - \left(-A^{\perp} + \frac{A^{\perp 3}}{3}\right)k_{ij}k^{jk}k^{i}_{k} - \left(\frac{3A^{\perp}}{2} - \frac{A^{\perp 3}}{2}\right)kk_{ij}k^{ij}.\end{aligned} \tag{3.3.48}$$

We will use this expression in what follows, comparing the boundary terms in the Euler characteristic with those in the renormalized area.

3.3.3 Asymptotic analysis

The ultimate goal is to express the Euler characteristic as a linear combination of the renormalized area $\mathcal{A}(\Sigma)$ and other finite contributions i.e.

$$\chi(\Sigma) = \frac{3}{4\pi^2}\mathcal{A}(\Sigma) + \cdots \tag{3.3.49}$$

where the ellipses denote contributions that are finite term by term. In this section we will show that the finite contributions are such that

$$\begin{aligned} \mathcal{A}(\Sigma) \;&=\; \frac{4\pi^2}{3}\chi(\Sigma) - \frac{1}{6}\mathcal{H}(\Sigma) - \frac{1}{3}\mathcal{W}(\Sigma) \\ &-\frac{1}{24}\int_\Sigma d^4x\sqrt{g}\left(H^2 - 4H_{\mu\nu}H^{\mu\nu} + H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma}\right. \\ &\left.+4\widetilde{W}_{1212}^2 - 4\widetilde{W}_{\mu n\nu n}\widetilde{W}^{\mu n\nu n} + \widetilde{W}_{\mu\nu\rho\sigma}\widetilde{W}^{\mu\nu\rho\sigma}\right), \end{aligned} \tag{3.3.50}$$

where the finite terms $\mathcal{H}(\Sigma)$ and $\mathcal{W}(\Sigma)$ are defined in (3.3.68) and (3.3.69), respectively, This formula is the direct generalisation of the corresponding expression for two-dimensional surfaces given in (3.2.3).

To determine the terms arising in this expression, we need to consider the asymptotic analysis of the bulk and boundary terms in the Euler characteristic and the renormalized area. To compare terms between the Euler characteristic and the renormalized area, it is convenient to convert quantities written with respect to quantities intrinsic to Σ into quantities expressed with respect to ∂M and $\partial\Sigma$. Intuitively, it is apparent that the extrinsic curvature $K^{(2)}$ of Σ and $\mathcal{K}$ of $\partial\Sigma$ can be expressed in terms of the two extrinsic curvatures $k, k^\perp$. Indeed, by decomposing the metric and normal of Σ into boundary components we can write $K^{(2)}$ as a combination of $k, k^\perp$ plus additional terms.

The integration in the M is regulated by restricting integration up to the regulated boundary $\partial M_\epsilon := M|_{z=\epsilon}$. The regulated divergences in Euler characteristic integral

$$\chi(\Sigma_\epsilon) = \frac{1}{32\pi^2}\int_{\Sigma_\epsilon} d^4x\sqrt{g}\,(24 + \Delta\chi) + \frac{1}{4\pi^2}\int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\,\partial E_4 \tag{3.3.51}$$

come from the bulk terms up to order z^4 and boundary terms up to order z^3. Clearly by construction all such divergences cancel, as the Euler characteristic is finite, but to compare with the renormalized area we need to identify which terms are finite and which include regulated divergences. In what follows we will show that each term in

$$\begin{aligned} \int_{\Sigma_\epsilon} d^4x\sqrt{g}\left(H^2 - 4H_{\mu\nu}H^{\mu\nu} + H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma} + 4\widetilde{W}_{1212}^2\right. \\ \left.-4\widetilde{W}_{\mu n\nu n}\widetilde{W}^{\mu n\nu n} + \widetilde{W}_{\mu\nu\rho\sigma}\widetilde{W}^{\mu\nu\rho\sigma}\right) \end{aligned} \tag{3.3.52}$$

is individually finite, while the other bulk contributions

$$\frac{1}{32\pi^2}\int_{\Sigma_\epsilon} d^4x\sqrt{g}\left(24+4H+8\widetilde{W}_{1212}\right) \tag{3.3.53}$$

each have regulated divergences. As we will be comparing regulated divergences of the Euler characteristic with those in the renormalised area, and the latter assumes static embedding, we will set $K^{(1)}_{\mu\nu}=0$ for the rest of this section.

We need to calculate the asymptotic expansions of the geometric quantities appearing in the Euler characteristic. We begin with the normal vectors defined in (3.3.8). If we expand $A^\perp$ in z and apply the boundary condition of $A^\perp(z=0)=0$ we obtain the leading term in the z power series to be $A^\perp = zA^\perp_{(0)}+\cdots$. Similarly, the leading term in the A asymptotic series is $A=1+\cdots$. From the relation $A^2+{A^\perp}^2=1$ we can thus conclude that the asymptotic expansion for A and $A^\perp$ is

$$\begin{aligned} A &= 1-\frac{1}{2}z^2{A^\perp_{(0)}}^2+O(z^4) \\ A^\perp &= zA^\perp_{(0)}+z^3A^\perp_{(2)}+O(z^5) \end{aligned} \tag{3.3.54}$$

Hence $A, A^\perp$ have even and odd power series of z respectively.

The asymptotic analysis for extrinsic curvatures is worked out in the appendix. The trace of the extrinsic curvature behaves as

$$K^{(2)} \sim O(z) \tag{3.3.55}$$

while the trace of the product of extrinsic curvature

$$K^{(2)}_{\mu\nu}K^{(2)\mu\nu} \sim O(z^2). \tag{3.3.56}$$

Accordingly H is of order z^2 but terms quadratic in H are of order z^4 so do not contribute to the regulated divergences. We can write explicit expansions

$$(K^{(2)})^2 = z^2\left(\bar{k}^2_{(0)}+9{A^\perp_{(0)}}^2-6A^\perp_{(0)}\bar{k}_{(0)}\right)+O(z^4) \tag{3.3.57}$$

and

$$K^{(2)}_{ij}K^{(2)ij} = z^2\left(\bar{k}_{(0)ij}\bar{k}^{ij}_{(0)}+3{A^\perp_{(0)}}^2-2A^\perp_{(0)}\bar{k}_{(0)}\right)+O(z^6) \tag{3.3.58}$$

where

$$k=\bar{k}_{(0)}z+\cdots, \tag{3.3.59}$$

and

$$k_{ij}k^{ij} = z^2\bar{k}_{(0)ij}\bar{k}^{ij}_{(0)}+\cdots \tag{3.3.60}$$

According the regulated divergences from the term linear in H gives

$$\int_{\Sigma_\epsilon} d^4x\sqrt{g}H \to \int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\ \left(\epsilon^2\left(\bar{k}_{(0)ij}\bar{k}^{ij}_{(0)} - \bar{k}^2_{(0)} + 4A^{\perp}_{(0)}\bar{k}_{(0)} - 6{A^{\perp}_{(0)}}^2\right) + \cdots\right) \quad (3.3.61)$$

where the ellipses denote terms that do not contribute in the limit $\epsilon \to 0$.

Let us now consider the asymptotic behaviour of the projections and contractions of the Weyl tensors. In our gauge choice $\widetilde{W}, \widetilde{W}_{11}, \widetilde{W}_{22}$ and $\widetilde{W}_{1212}$ are of order $O(z^2)$ and hence only terms linear in the Weyl tensor contribute to the regulated divergences. If the Weyl tensor admits an expansion

$$\widetilde{W}_{1212} = z^2\overline{W}_{1212} + \cdots, \quad (3.3.62)$$

leading to regulated divergences

$$\int_{\Sigma_\epsilon} d^4x\sqrt{g}\widetilde{W}_{1212} \to \int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\ \epsilon^2\overline{W}_{1212} \quad (3.3.63)$$

The regulated divergences (3.3.53) are obtained from combining (3.3.61) and (3.3.63)

$$\begin{aligned}&\frac{2}{3\pi^2}\int_{\Sigma_\epsilon} d^4x\sqrt{g} \quad (3.3.64)\\ &+\frac{1}{8\pi^2}\int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\epsilon^2\left(\left[\bar{k}_{(0)ij}\bar{k}^{ij}_{(0)} - \bar{k}^2_{(0)} + 4A^{\perp}_{(0)}\bar{k}_{(0)} - 6{A^{\perp}_{(0)}}^2\right] + 2\overline{W}_{1212}\right).\end{aligned}$$

Here we do not explicitly analyse the regulated divergences of the first (area) term, as this was already done in [54], as we will use below. The expression above can be simplified using the minimal condition for the surface: as explained in the appendix, $K^{(2)} = 0$ implies that

$$A^{\perp}_{(0)} = \frac{1}{3}\bar{k}_{(0)} \quad (3.3.65)$$

and therefore $A^{\perp}_{(0)}$ can be eliminated.

Let us now consider the regulated divergences of the boundary terms in the Euler characteristic. As mentioned in the beginning of the section, only terms of order $O(z^3)$ in ∂E_4 contribute to divergences. These terms are analysed in the appendix; the regulated divergences take the form

$$\int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\partial E_4 \to -\int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\left(1 + \epsilon^2(\overline{W}_{1212} - \frac{1}{3}\bar{k}^2_{(0)} + \frac{1}{2}\bar{k}_{(0)ij}\bar{k}^{ij}_{(0)})\right). \quad (3.3.66)$$

By construction the regulated divergences of the boundary terms in the Euler characteristic cancel those from the bulk terms.

We can now express the regulated contributions in (3.3.64) and (3.3.66) in terms of the renormalized area

$$\mathcal{A}(\Sigma_\epsilon) = \int_{\Sigma_\epsilon} d^4x\sqrt{g} + \frac{1}{3}\int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\Big(-1 + \frac{1}{6}k^2\Big), \tag{3.3.67}$$

and two other integrals that are finite in the limit of $\epsilon \to 0$:

$$\mathcal{H}(\Sigma_\epsilon) := \int_{\Sigma_\epsilon} d^4x\sqrt{g}H - \int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}\left(k_{ij}k^{ij} - \frac{1}{3}k^2\right); \tag{3.3.68}$$

and

$$\mathcal{W}(\Sigma_\epsilon) := \int_{\Sigma_\epsilon} d^4x\sqrt{g}\widetilde{W}_{1212} - \int_{\partial\Sigma_\epsilon} d^3x\sqrt{h}W_{1212}. \tag{3.3.69}$$

The regulated terms in the Euler characteristic then combine to give

$$\frac{3}{4\pi^2}\mathcal{A}(\Sigma_\epsilon) + \frac{1}{8\pi^2}\mathcal{H}(\Sigma_\epsilon) + \frac{1}{4\pi^2}\mathcal{W}(\Sigma_\epsilon), \tag{3.3.70}$$

and thus, reinstating the bulk contributions to the Euler characteristic that are individually finite, we obtain the final expression for the renormalized area (4.2.11).

Note that the extra counterterms for the renormalized area integral (3.3.2) vanishes in the limit $z \to 0$. It can be seen from (3.3.41). As $W_{1m1m}, W_{\bar{n}m\bar{n}m} \sim O(z^4)$ the individual Ricci terms are

$$\begin{aligned} \hat{R} &= -8\bar{\beta} + O(z^4) \\ \hat{R}_{11} &= \bar{\beta} + O(z^4) \\ \hat{R}_{\bar{n}\bar{n}} &= -4\bar{\beta} + O(z^4) \end{aligned} \tag{3.3.71}$$

Since the definition of the projected Ricci curvature $\hat{R}_{aa}$ is

$$\begin{aligned} \hat{R}_{aa} &:= -\hat{R}_{11} + \hat{R}_{\bar{n}\bar{n}} \\ \hat{R}_{aa} &:= -5\bar{\beta} + O(z^4), \end{aligned} \tag{3.3.72}$$

the Ricci counterterms $\hat{R}_{aa} - \frac{5}{8}\hat{R}$ is

$$-\hat{R}_{11} + \hat{R}_{\bar{n}\bar{n}} - \frac{5}{8}\hat{R} = 0 + O(z^4). \tag{3.3.73}$$

The order of this term is great than z^3 therefore it vanishes in the boundary integral as $z \to 0$.

3.4 Spherical entangling surface in AdS$_6$ and linear perturbations

Consider AdS$_6$ written in Poincaré coordinates as:

$$ds^2 = \frac{1}{\rho^2}\left(-dt^2 + d\rho^2 + dr^2 + r^2 d\Omega^2\right) \tag{3.4.1}$$

We can introduce new coordinates adapted to the entangling surface $\mathcal{S}$ associated with a spherical entangling region

$$\rho = R\sin\theta \qquad r = R\cos\theta \tag{3.4.2}$$

so that

$$ds^2 = \frac{1}{R^2\sin^2\theta}\left(-dt^2 + dR^2 + R^2(d\theta^2 + \cos^2\theta d\Omega^2)\right). \tag{3.4.3}$$

The induced metric on the entangling surface $\mathcal{S}$ of constant t and R can thus be written as

$$ds^2 = \frac{1}{\sin^2\theta}\left(d\theta^2 + \cos^2\theta d\Omega^2\right). \tag{3.4.4}$$

This parameterisation makes manifest that the extrinsic curvatures of $\mathcal{S}$ within $\mathcal{M}$ are zero: the induced metric is independent of the coordinates t and R. One can then change coordinates as $u = -\log(\tan(\theta/2))$ to write the induced metric as

$$ds^2 = du^2 + \sinh^2 u d\Omega^2, \tag{3.4.5}$$

i.e. making manifest that the metric on the entangling surface is global AdS with unit radius.

The bulk contribution to the Euler invariant is thus

$$\frac{\Omega_3}{32\pi^2}\left(16 + e^{3\bar{u}} - 9e^{\bar{u}} + \cdots\right) \tag{3.4.6}$$

where we have regulated the boundary at $u = \bar{u} \gg 1$, and dropped terms that are zero when $\bar{u} \to \infty$. Here $\Omega_3 = 2\pi^2$ is the volume of a three sphere of unit radius.

Calculation of the boundary contributions to the Euler invariant (3.3.4) is more complicated. We need the following expressions:

$$\begin{aligned}
\mathcal{R}_{ijkl}\mathcal{K}^{jl}n^i n^k &= -3\frac{\cosh(u)}{\sinh(u)}; \qquad \mathcal{K} = 3\frac{\cosh(u)}{\sinh(u)}; \\
\mathcal{R}_{ij}\mathcal{K}^{ij} &= -9\frac{\cosh(u)}{\sinh(u)}; \qquad \mathcal{R}_{ij}n^i n^j = -3; \\
\mathrm{Tr}(\mathcal{K}^2) &= 3\frac{\cosh^2(u)}{\sinh^2(u)}; \qquad \mathrm{Tr}(\mathcal{K}^3) = 3\frac{\cosh^3(u)}{\sinh^3(u)}.
\end{aligned} \tag{3.4.7}$$

Combining these we obtain the following contribution from (3.3.4)

$$-\frac{\Omega_3}{4\pi^2}\Big(\cosh^3(\bar{u}) - 3\cosh(\bar{u})\Big) = -\frac{\Omega_3}{32\pi^2}\left(e^{3\bar{u}} - 9e^{\bar{u}} + \cdots\right) \tag{3.4.8}$$

where in the second expression we have dropped all terms that go to zero as $\bar{u} \to \infty$.

Combining bulk and boundary terms we obtain

$$\chi(\mathcal{S}) = 1, \tag{3.4.9}$$

which is indeed the Euler invariant for a half ball.

Let us now turn to the computation of the renormalized entanglement entropy. The regulated bulk contribution is proportional to the regulated volume of the entangling surface

$$\frac{\Omega_3}{4G_6}\left(\frac{2}{3} + \frac{1}{24}e^{3\bar{u}} - \frac{3}{8}e^{\bar{u}} + \cdots\right) \tag{3.4.10}$$

where the ellipses denote terms that vanish as $\bar{u} \to \infty$. The first counterterm gives

$$-\frac{\Omega_3}{4G_6}\left(\frac{1}{24}e^{3\bar{u}} - \frac{1}{8}e^{\bar{u}} + \cdots\right) \tag{3.4.11}$$

while the second counterterm gives

$$\frac{\Omega_3}{4G_6}\left(\frac{1}{4}e^{\bar{u}} + \cdots\right). \tag{3.4.12}$$

The counterterms, as expected, remove divergent contributions while not adding further finite contributions and thus

$$S(\mathcal{S}) = \frac{\pi^2}{3G_6} \equiv \frac{\pi^2}{3G_6}\chi(\Sigma), \tag{3.4.13}$$

i.e. the renormalized entanglement entropy is proportional to the Euler invariant, as shown in (3.1.4), with the coefficient of proportionality being the F quantity in the dual CFT_5.

Next let us consider the variation of the entanglement entropy under linear perturbations around the spherical entangling surface in AdS. Since the Weyl curvature and the extrinsic curvatures are zero at leading order, only terms linear in the curvatures can contribute to the first variation of the entropy. From (4.2.11) we obtain

$$\delta S = -\frac{1}{12G_6}\delta\mathcal{W} \tag{3.4.14}$$

where $\mathcal{W}$ is linear in the Weyl curvature and is defined in (3.3.69).

As we will show in a future work [2], this expression allows for an elegant derivation of the first law of entanglement entropy, generalizing the discussions in [57]. Since we work with renormalized quantities, we do not need to assume specific fall off conditions for metric perturbations; the perturbations can be non-normalizable as well as normalizable. It is straightforward to relate $\delta\mathcal{W}$ to the renormalized stress tensor defined in [43] and hence to the variation in the energy.

3.5 Conclusions and outlook

In this chapter we have shown that the renormalized area of static minimal surfaces in asymptotically locally AdS_{2n} spacetimes can be expressed in terms of the Euler invariant and renormalized curvature invariants. It is perhaps unsurprising that renormalized integrals of extrinsic and intrinsic curvature invariants arise. Indeed, renormalized curvature integrals on asymptotically locally hyperbolic manifolds have been considered in the mathematics literature; see for example [98].

There is however a key difference between our definitions of renormalized curvature invariants and those in the mathematics literature. Here we follow the standard holographic renormalization approach, identifying explicit boundary counterterms. By contrast, the mathematics literature identifies the "renormalized" integrals as the finite terms in a regulated expansion around the conformal boundary. While the latter gives equivalent results when counterterms do not make finite contributions, there will generically be finite contributions from counterterms (for example, if we add matter or gauge fields in the bulk).

Our results can be used to infer certain bounds on the renormalized entanglement entropy for given topology. Earlier discussions on bounds on renormalized entanglement entropy in asymptotically locally AdS_4 spacetimes using inverse mean curvature flow techniques can be found in [99]. For entangling surfaces of disk topology in AdS_4, the bound is given in (3.2.5): the entanglement entropy is negative and the absolute value of the renormalized entanglement entropy is minimised for the disk, which has zero extrinsic curvature.

Note that the expression (3.2.5) is closely analogous to the Willmore energy $\mathcal{E}_w$, which measures how much a closed two surface Σ embedded into R^3 deviates from the round two sphere:

$$\mathcal{E}_w = \int_\Sigma d^2x |K|^2 - 2\pi\chi(\Sigma) \tag{3.5.1}$$

The Willmore energy is positive semi-definite and zero for a round two sphere.

For entangling surfaces that are topologically disks in asymptotically locally AdS_4 manifolds (cf pure AdS_4), there is no such bound: from (4.2.10), the Weyl curvature term is not negative definite. Indeed, if one considers linearized perturbations around AdS_4, one can show that this term is positive for all metric perturbations that give rise to positive energy [2].

Now let us turn to the renormalized entanglement entropy for asymptotically locally AdS_6 spacetimes. From (4.2.11), this reduces in AdS_6 to

$$\begin{aligned} S(\Sigma) &= \frac{\pi^2}{3G_6}\chi(\Sigma) - \frac{1}{24G_6}\mathcal{H}(\Sigma) \\ &\quad -\frac{1}{96G_6}\int_\Sigma d^4x\sqrt{g}\left(H^2 - 4H_{\mu\nu}H^{\mu\nu} + H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma}\right). \end{aligned} \tag{3.5.2}$$

Even restricting to entangling surfaces of fixed topology, this expression does not seem to have bounds, in accordance with the discussions of higher dimensional Willmore functionals in [86, 89].

For example, consider perturbations around a spherical entangling surface in AdS_6; the change in the renormalized entanglement entropy is

$$\delta S = -\frac{1}{24G_6}\delta\mathcal{H} = -\frac{1}{24G_6}\int_\Sigma d^4x\sqrt{g}\delta H + \frac{1}{24G_6}\int_{\partial\Sigma} d^3x\sqrt{h}\left(\delta k_{ij}\delta k^{ij} - \frac{1}{3}(\delta k)^2\right), \tag{3.5.3}$$

i.e. it is quadratic in the extrinsic curvature. The bulk curvature integrand is non-positive but the renormalized curvature invariant does not manifestly have any negativity bounds. Hence, in 3d holographic CFTs, disk regions minimise the magnitude of the renormalized entanglement entropy in the conformal vacuum while in 5d holographic CFTs spherical regions do not necessarily do so. It would be interesting to understand the implications of this directly from field theory.

Throughout this chapter we have been considering static RT entangling surfaces [7] although our analysis of the Chern-Gauss-Bonnet integrals is applicable to generic asymptotically locally AdS manifolds. It would be interesting to extend our analysis of renormalized entanglement entropy to HRT surfaces [8].

While we have focussed on connected entangling regions, our expressions for renormalized entanglement entropy are equally applicable to disconnected regions. If we consider n widely separated disk/spherical entangling regions in pure AdS then from (3.2.5) and (3.5.2) the entanglement entropy is proportional to n/G. It would be interesting to explore how the renormalized entanglement entropy changes as these regions approach each other and intersect. In AdS_4 the extremum once all regions intersect would be a single disk region with entropy proportional to $1/G$ and the renormalized entanglement entropy may satisfy monotonicity properties under deformations of disconnected regions into a single connected region.

3.A Notation and terminology

In this appendix we collect together notation and terminology.

We denote the curvature of the asymptotically locally AdS manifold $\mathcal{M}$ (metric $G_{\mu\nu}$) with boundary $\partial\mathcal{M}$ (metric $\gamma_{\alpha\beta}$) as $R_{\mu\nu\rho\sigma}$. The intrinsic curvature of the boundary of $\partial\mathcal{M}$ is denoted by $\hat{R}_{\alpha\beta\gamma\delta}$. The entangling surface Σ with metric g has boundary $\partial\Sigma$ with metric h. The intrinsic curvature of the surface Σ is denoted $\mathcal{R}_{ijkl}$. The intrinsic curvature of the surface $\partial\Sigma$ is denoted $\bar{R}_{ijkl}$.

We also need to distinguish between four distinct extrinsic curvatures: the extrinsic curvatures of Σ embedded into M (K^s), of $\partial\Sigma$ embedded into Σ ($\mathcal{K}$), of $\partial\Sigma$ embedded into ∂M (k^s), of $\partial\Sigma$ embedded into M ($k^{\perp}$) orthogonal to ∂M and of ∂M embedded into M ($\mathscr{K}$), where $s = 1, 2$ denote the normals to the entangling surface with the boundary condition $K^s = k^s$ on $\partial\Sigma$. Note that we write $k^{(2)} = k$ and in the static case $K^{(1)} = k^{(1)} = 0$.

3.B Asymptotic analysis

The boundary metric γ has a Fefferman-Graham expansion therefore the metrics g and h on Σ and $\partial\Sigma$ have their own Fefferman-Graham expansion,

$$g_{ij} = \frac{1}{z^2}\bar{g}_{ij} = \frac{1}{z^2}\left(\bar{g}^{(0)}_{ij} + z^2\bar{g}^{(2)}_{ij} + \cdots\right) \tag{3.B.1}$$

and

$$h_{ij} = \frac{1}{z^2}\bar{h}_{ij} = \frac{1}{z^2}\left(\bar{h}^{(0)}_{ij} + z^2\bar{h}^{(2)}_{ij} + \cdots\right). \tag{3.B.2}$$

Hence $\bar{\alpha}$ has an even power series of z,

$$\bar{\alpha} = \bar{\alpha}^{(0)} + z^2\bar{\alpha}^{(2)} + O(z^4). \tag{3.B.3}$$

Then

$$\bar{\beta} = \frac{2z^2\bar{\alpha}^{(2)}}{\bar{\alpha}^{(0)}} + O(z^4). \tag{3.B.4}$$

Similarly, for e^a_i has an even power series of z,

$$e^a_{\ i} = \frac{1}{z}\bar{e}^a_{\ i} = \frac{1}{z}\left(\bar{e}^{(0)a}_{\quad\ i} + z^2\bar{e}^{(2)a}_{\quad\ i} + \cdots\right) \tag{3.B.5}$$

The extrinsic curvature $k^{\perp}$ of $\partial\Sigma$ pointing out of ∂M,

$$\begin{aligned}
k^{\perp}_{ab} &= (\nabla m)_{ab} \\
&= -\Gamma^{m}_{ab} \\
&= e_{(b}{}^{k} m \cdot \partial e_{a)k} \\
&= z\partial_z \left(\frac{1}{z}\right) e_{(b}{}^{k}\bar{e}_{a)k} + e_{(b}{}^{k}\partial_z \bar{e}_{a)k} \\
&= -\delta_{ab} + 2\bar{e}^{(0)}_{(b}{}^{k}\bar{e}^{(2)}_{a)k} + O(z^4) \\
k^{\perp}_{ab} &= -\delta_{ab} + O(z^4)
\end{aligned} \tag{3.B.6}$$

where we used the fact $\bar{e}_{(b}^{k}\bar{e}_{a)k} = \delta_{ab}$ which implies $\bar{e}^{(0)}{}_{b}{}^{k}\bar{e}^{(2)}{}_{ak} + \bar{e}^{(2)}{}_{b}{}^{k}\bar{e}^{(0)}{}_{ak} = 0$. Transforming back to coordinate basis,

$$k^{\perp}_{ij} = -h_{ij} + O(z^2) \tag{3.B.7}$$

The other extrinsic curvature k of $\partial\Sigma$ lying within ∂M,

$$k_{ab} = e_{(b}{}^{k}\bar{n} \cdot \partial e_{a)k} \sim O(z). \tag{3.B.8}$$

In coordinate basis,

$$k_{ij} \sim O(z^{-1}) \tag{3.B.9}$$

3.B.1 Asymptotic analysis for bulk Euler density

Starting from the definition of the extrinsic curvature

$$K^{(2)}_{\mu\nu} = (h^{\rho}_{\mu} + n^3_{\mu}n^{3\,\rho})(h^{\sigma}_{\nu} + n^3_{\nu}n^{3\,\sigma})\nabla_{\rho}(A\bar{n}_{\sigma} + A^{\perp}m_{\sigma}). \tag{3.B.10}$$

Expanding the bracket and grouping the terms tangent and normal to $\partial\Sigma$

$$\begin{aligned}
K^{(2)}_{\mu\nu} = Ak_{\mu\nu} + A^{\perp}k^{\perp}_{\mu\nu} + h^{\rho}_{\mu}n^3_{\nu}\Big(&-A^{\perp}\partial_{\rho}A + A\partial_{\rho}A^{\perp} + m^{\sigma}\nabla_{\rho}\bar{n}_{\sigma} \\
&- AA^{\perp}(\bar{n}^{\sigma}\nabla_{\sigma}\bar{n}_{\rho} - m^{\sigma}\nabla_{\sigma}m_{\rho}) - {A^{\perp}}^2[\bar{n},m]_{\rho}\Big) + n^3_{\mu}n^3_{\nu}n^3_{\sigma}[n^3,n^2]^{\sigma}.
\end{aligned} \tag{3.B.11}$$

As mentioned before, $\partial_i A = 0$ and $[\bar{n}, m] \in N\partial\Sigma$ so the terms with one index tangent and one index normal to $\partial\Sigma$ vanish. Then expand the Lie bracket of n^2, n^3 and use the relation $\partial_{\rho}A^{\perp} = -\frac{A}{A^{\perp}}\partial_{\rho}A$, the expression simplifies to

$$K^{(2)}_{\mu\nu} = Ak^{(2)}_{\mu\nu} + A^{\perp}k^{\perp}_{\mu\nu} + n^3_{\mu}n^3_{\nu}\Big[\partial_{\rho}A\Big(\bar{n}^{\rho} + m^{\rho}(-\frac{A}{A^{\perp}} + A^2 - AA^{\perp})\Big) - \Big(1-\bar{\beta}\Big)A^{\perp}\Big]. \tag{3.B.12}$$

Finally defining the coefficient in the $n^3_\mu n^3_\nu$ component in $K^{(2)}_{\mu\nu}$

$$L := \partial_\rho A\Big(\bar{n}^\rho + m^\rho(-\frac{A}{A^\perp} + A^2 - AA^\perp)\Big) - \Big(1-\bar{\beta}\Big)A^\perp, \tag{3.B.13}$$

the extrinsic curvature has tangent components,

$$K^{(2)}_{ij} = Ak_{ij} + A^\perp k^\perp_{ij} \tag{3.B.14}$$

and normal components,

$$K^{(2)}_{\bar{n}m} = -AA^\perp L, \quad K^{(2)}_{\bar{n}\bar{n}} = -{A^\perp}^2 L, \quad K^{(2)}_{mm} = -A^2 L. \tag{3.B.15}$$

Using (3.3.54), expanding L in z,

$$\begin{aligned} L = &-\frac{z^3 A^\perp_{(0)}\partial_{\bar{f}}A^\perp_{(0)}}{\bar{\alpha}_0} - z^2 {A^\perp_{(0)}}^2\left(-\frac{1}{zA^\perp_{(0)}} + z\left(1 + \frac{A^\perp_{(2)}}{{A^\perp_{(0)}}^2} - \frac{{A^\perp_{(0)}}^2}{2}\right)\right) \\ &- zA^\perp_{(0)} - z^3 A^\perp_{(2)} + \frac{2z^3\bar{\alpha}_2 A^\perp_{(0)}}{\bar{\alpha}_0} + O(z^5) \\ L = &-z^3\left(\frac{A^\perp_{(0)}\partial_{\bar{f}}A^\perp_{(0)}}{\bar{\alpha}_0} + {A^\perp_{(0)}}^2 + 2A^\perp_{(2)} - \frac{{A^\perp_{(0)}}^4}{2} - \frac{2\bar{\alpha}_2 A^\perp_{(0)}}{\bar{\alpha}_0}\right) + O(z^5). \end{aligned} \tag{3.B.16}$$

Therefore trace of the extrinsic curvature,

$$K^{(2)} = Ak + A^\perp k^\perp + L \sim O(z) \tag{3.B.17}$$

and trace of the product of extrinsic curvature,

$$\begin{aligned} K^{(2)}_{\mu\nu}K^{(2)\mu\nu} &= A^2 k_{\mu\nu}k^{\mu\nu} + {A^\perp}^2 k^\perp_{\mu\nu}k^{\perp\mu\nu} + 2AA^\perp k_{\mu\nu}k^{\perp\mu\nu} + L^2 \\ K^{(2)}_{\mu\nu}K^{(2)\mu\nu} &= k_{\mu\nu}k^{\mu\nu} + 3{A^\perp}^2 - 2A^\perp k + O(z^4) \sim O(z^2). \end{aligned} \tag{3.B.18}$$

From $(3.B.17)$ and $(3.B.18)$ we observed that up to $O(z^4)$ only H contains divergent integrals. Further expanding in z,

$$\begin{aligned} {K^{(2)}}^2 &= k^2 + 9z^2{A^\perp_{(0)}}^2 - 6zA^\perp_{(0)}k + O(z^4) \\ {K^{(2)}}^2 &= z^2\left(\bar{k}_0^2 + 9{A^\perp_{(0)}}^2 - 6A^\perp_{(0)}\bar{k}_0\right) + O(z^4) \end{aligned} \tag{3.B.19}$$

and

$$\begin{aligned} K^{(2)}_{ij}K^{(2)ij} &= k_{ij}k^{ij} + 3z^2{A^\perp_{(0)}}^2 - 2zA^\perp_{(0)}k + O(z^6) \\ K^{(2)}_{ij}K^{(2)ij} &= z^2\left(\bar{k}_{0ij}\bar{k}_0^{ij} + 3{A^\perp_{(0)}}^2 - 2A^\perp_{(0)}\bar{k}_0\right) + O(z^6). \end{aligned} \tag{3.B.20}$$

The leading order term in the Taylor expansion of the extrinsic curvature $\bar{k}_{(0)}$ can be written in terms of the leading order term in the Fefferman-Graham expansion of boundary induced metric $\bar{h}_{ij}^{(0)}$

$$\bar{k}_{(0)} = \left[z^{-1}k\right]_{z=0} = \frac{1}{\bar{\alpha}^{(0)2}}\bar{h}^{(0)ij}\partial_{\bar{f}}\bar{h}_{ij}^{(0)} \tag{3.B.21}$$

and

$$\bar{k}_{(0)ij}\bar{k}_{(0)}^{ij} = \left[z^{-2}k_{ij}k^{ij}\right]_{z=0} = \frac{1}{\bar{\alpha}^{(0)2}}\bar{h}^{(0)ik}\partial_{\bar{f}}\bar{h}_{ij}^{(0)}\bar{h}^{(0)jl}\partial_{\bar{f}}\bar{h}_{kl}^{(0)}. \tag{3.B.22}$$

3.B.2 Asymptotic analysis for boundary Euler density

Now consider the regulated divergences in the boundary terms in the Euler characteristic. From power counting the dominant order of each term in (3.3.48), only the following terms contain divergent integral

$$\begin{aligned}
\partial E_4 =& Ak^{\perp}\left(W_{1\bar{n}1\bar{n}} + W_{1m1m} - W_{\bar{n}m\bar{n}m}\right) \\
&- Ak^{\perp\, ij}\left(-W_{i1j1} + W_{imjm} + W_{i\bar{n}j\bar{n}}\right) \\
&+ Ak^{\perp} - \left(\frac{A}{2} - \frac{A^3}{6}\right)k^{\perp\,3} - \left(A - \frac{A^3}{3}\right)k_{ij}^{\perp}k^{\perp\,jk}k_{k}^{\perp\,i} \\
&- \left(-\frac{3A}{2} + \frac{A^3}{2}\right)k^{\perp}k_{ij}^{\perp}k^{\perp\,ij} \\
&- A^{\perp}k - \left(-\frac{A^{\perp}}{2} + \frac{A^2A^{\perp}}{2}\right)kk^{\perp\,2} - \left(\frac{A^{\perp}}{2} - \frac{A^2A^{\perp}}{2}\right)kk_{ij}^{\perp}k^{\perp\,ij} \\
&- \left(-A^{\perp} + A^2A^{\perp}\right)k_{ij}k^{\perp\,jk}k_{k}^{\perp\,i} - \left(A^{\perp} - A^2A^{\perp}\right)k^{\perp}k_{ij}k^{\perp\,ij} \\
&- \frac{A}{2}k^2k^{\perp} + \frac{A}{2}k_{ij}k^{ij}k^{\perp} + Ak_{ij}k^{jk}k_{k}^{\perp\,i} + Akk_{ij}k^{\perp\,ij}.
\end{aligned} \tag{3.B.23}$$

Simplifying by substituting the leading order term of $k_{ij}^{\perp} = -h_{ij}$,

$$\begin{aligned}
\partial E_4^{div} =& -3\left(W_{1\bar{n}1\bar{n}} + W_{1m1m} - W_{\bar{n}m\bar{n}m}\right) \\
&+ h^{ij}\left(-W_{i1j1} + W_{imjm} + W_{i\bar{n}j\bar{n}}\right) \\
&- 3A + \frac{27A}{2} - \frac{9A^3}{2} + 3A - A^3 + \frac{27A}{2} + \frac{9A^3}{2} \\
&- A^{\perp}k + \frac{9A^{\perp}}{2}k - \frac{9A^2A^{\perp}}{2}k - \frac{3A^{\perp}}{2}k + \frac{3A^2A^{\perp}}{2}k \\
&+ A^{\perp}k - A^2A^{\perp}k - 3A^{\perp}k + 3A^2A^{\perp}k \\
&+ \frac{3}{2}k^2 - \frac{3}{2}k_{ij}k^{ij} + k_{ij}k^{ij} - Ak^2.
\end{aligned} \tag{3.B.24}$$

Expanding the induced metric and using tracelessness of the Weyl tensor,

$$\partial E_4^{div} = -\left(W_{1\bar{n}1\bar{n}} + W_{1m1m} - W_{\bar{n}m\bar{n}m} + A^3 + A^{\perp}k - \frac{1}{2}k^2 + \frac{1}{2}k_{ij}k^{ij}\right). \tag{3.B.25}$$

To look at the detail asymptotic behaviour of the Weyl tensor we need the expression of Riemann tensor in Fefferman-Graham gauge. Particularly for $W_{1\bar{n}1\bar{n}}, W_{1m1m}, W_{\bar{n}m\bar{n}m}$

$$R_{tztz} = -\frac{1}{z^4}\bar{\gamma}_{tt} + \frac{1}{4z^2}\left(-2\bar{\gamma}''_{tt} + \bar{\gamma}'_{t\mu}\bar{\gamma}^{\mu\nu}\bar{\gamma}'_{\nu t}\right) + \frac{1}{2z^3}\bar{\gamma}'_{tt} \tag{3.B.26}$$

$$R_{\bar{f}z\bar{f}z} = -\frac{1}{z^4}\bar{\gamma}_{\bar{f}\bar{f}} + \frac{1}{4z^2}\left(-2\bar{\gamma}''_{\bar{f}\bar{f}} + \bar{\gamma}'_{\bar{f}\mu}\bar{\gamma}^{\mu\nu}\bar{\gamma}'_{\nu\bar{f}}\right) + \frac{1}{2z^3}\bar{\gamma}'_{\bar{f}\bar{f}} \tag{3.B.27}$$

$$\begin{aligned} R_{t\bar{f}t\bar{f}} = {} & \frac{1}{z^4}\left(\bar{\gamma}^2_{t\bar{f}} - \bar{\gamma}_{tt}\bar{\gamma}_{\bar{f}\bar{f}}\right) + \hat{R}_{t\bar{f}t\bar{f}}[\gamma] + \frac{1}{4z^2}\left(\bar{\gamma}'^2_{t\bar{f}} - \bar{\gamma}'_{tt}\bar{\gamma}'_{\bar{f}\bar{f}}\right) \\ & + \frac{1}{2z^3}\left(\bar{\gamma}'_{tt}\bar{\gamma}_{\bar{f}\bar{f}} + \bar{\gamma}_{tt}\bar{\gamma}'_{\bar{f}\bar{f}} - 2\bar{\gamma}_{t\bar{f}}\bar{\gamma}'_{t\bar{f}}\right) \end{aligned} \tag{3.B.28}$$

$$R_{t\bar{f}tz} = \frac{1}{2z^2}\left(D_t\bar{\gamma}'_{t\bar{f}} - D_{\bar{f}}\bar{\gamma}'_{tt}\right) \tag{3.B.29}$$

where $' = \partial_z$ and D is the covariant derivative on ∂M. Note in Fefferman-Graham gauge, the derivatives of the metric scale as $\bar{\gamma}'_{\mu\nu} \sim O(z)$ and $\bar{\gamma}''_{\mu\nu} \sim O(1)$. In our gauge

$$\begin{aligned} W_{1212} &= z^4 {A^{\perp}}^2 R_{tztz} + \frac{z^4 A^2}{\bar{\alpha}^2} R_{t\bar{f}t\bar{f}} + \frac{2AA^{\perp}}{\bar{\alpha}} R_{t\bar{f}tz} \\ W_{1212} &= z^2\left(\frac{z^2}{\bar{\alpha}_0^2}\hat{R}_{t\bar{f}t\bar{f}} - \frac{2\bar{\alpha}_2}{\bar{\alpha}_0}\right) + O(z^4) \end{aligned} \tag{3.B.30}$$

and

$$\begin{aligned} W_{1\bar{n}1\bar{n}} &= \frac{z^4}{\bar{\alpha}^2} R_{t\bar{f}t\bar{f}} - 1 \\ W_{1\bar{n}1\bar{n}} &= z^2\left(\frac{z^2}{\bar{\alpha}_0^2}\hat{R}_{t\bar{f}t\bar{f}} - \frac{2\bar{\alpha}_2}{\bar{\alpha}_0}\right) + O(z^4) \\ W_{1\bar{n}1\bar{n}} &\sim W_{1212} \end{aligned} \tag{3.B.31}$$

where the equivalence is up to order $O(z^4)$. Similarly,

$$\begin{aligned} W_{\bar{n}m\bar{n}m} &= \frac{z^4}{\bar{\alpha}^2} R_{\bar{f}z\bar{f}z} + 1 \\ W_{\bar{n}m\bar{n}m} &= 1 + \frac{z^4}{\bar{\alpha}_0^2}\left(-\frac{\bar{\alpha}_0^2}{z^4} - \frac{2\bar{\alpha}_0\bar{\alpha}_2}{2z^2} + \frac{2\bar{\alpha}_0\bar{\alpha}_2}{z^3}\right) + O(z^4) \\ W_{\bar{n}m\bar{n}m} &\sim 0 \end{aligned} \tag{3.B.32}$$

and

$$
\begin{aligned}
W_{1m1m} &= z^4 R_{tztz} - 1 \\
W_{1m1m} &\sim 0.
\end{aligned}
\tag{3.B.33}
$$

Although for our gauge $\bar{\gamma}'_{tt} = 0$, in non-static boundary one can replace the normalization constant of the spacelike unit normal, $\bar{\alpha}$, to the normalization constant of timelike orthonormal basis, α_τ, and W_{1m1m} should also vanish up to $O(z^4)$.

The minimal condition for Σ is equivalent to having a vanishing trace for the extrinsic curvature $K^{(2)} = 0$; the vanishing of Lie derivative of the volume form of Σ with respect to the normal n^2, $(3.B.17)$ implies

$$
\begin{aligned}
A^{\perp} &= \frac{-Ak - L}{k^{\perp}} \\
A^{\perp}_{(0)} &= \frac{\bar{k}_{(0)}}{3}.
\end{aligned}
\tag{3.B.34}
$$

3.C Chern Gauss Bonnet Formula

The construction of the Chern Gauss Bonnet Formula follows from [100, 97, 101]. The connection 1-form ω_{ab} is defined by

$$
\omega^a_{\ b} = \Gamma^a_{bc} e^c, \quad de^a = -\omega^a_{\ b} \wedge e^b, \quad de_b = \omega^a_{\ b} e_a \tag{3.C.1}
$$

and the curvature 2-form Ω_{ab} is defined by

$$
\Omega^a_{\ b} = \frac{1}{2}\mathcal{R}^a_{\ bcd} e^c \wedge e^d, \quad \Omega^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b}. \tag{3.C.2}
$$

Consider a vector field X of a d dimensional manifold M with zeros of the vector field $I \subset M$. For a $d-1$ sphere bundle $\pi : SM \to M$, one can identify a map, by the normalized vector field, from the $M \setminus I$ to SM such that $\hat{X} \in \Gamma\left(M \setminus I, SM\right)$. The d form $\Omega_p \in \wedge^d T^*_p M$,

$$
\Omega_p = \frac{1}{2^d \pi^{\frac{d}{2}} \left(\frac{d}{2}\right)!} \epsilon_{a_1 \cdots a_d} \Omega^{a_1 a_2} \wedge \cdots \wedge \Omega^{a_{d-1} a_d}, \tag{3.C.3}
$$

is exact when pullback to SM

$$
\pi^* \Omega = -d\Pi. \tag{3.C.4}
$$

The exact form on SM is

$$\Pi = \frac{1}{\pi^{\frac{d}{2}}} \sum_{k=0}^{\frac{d}{2}-1} \frac{1}{1 \cdot 3 \cdots (d-2k-1) k! 2^{\frac{d}{2}+k}} \Phi_k \tag{3.C.5}$$

The Φ_k are $d-1$ forms on SM

$$\Phi_k = \epsilon_{a_1 \cdots a_d} u^{a_1} \theta^{a_2} \wedge \cdots \wedge \theta^{a_{d-2k}} \wedge \pi^* \Omega^{a_{d-2k+1} a_{d-2k+2}} \wedge \cdots \wedge \pi^* \Omega^{a_{d-1} a_d} \tag{3.C.6}$$

where u is an unit tangent vector of M pullback to SM and the 1-form θ is defined by

$$\theta^a = du^a + u^b \pi^* \omega^a_b. \tag{3.C.7}$$

Then using Stoke's theorem and the fact $\hat{X}^* \pi^* = id_M$

$$\begin{aligned} \int_M \Omega &= \int_{\hat{X}(M)} \pi^* \Omega = - \int_{\hat{X}(M)} d\Pi \\ &= - \int_{\hat{X}(M \setminus \cup_{x \in I} B_x)} d\Pi - \int_{\hat{X}(\cup_{x \in I} B_x)} d\Pi \\ &= - \int_{\hat{X}(\partial M)} \Pi + \int_{\hat{X}\left(\cup_{x \in I} \partial \bar{B}_x\right)} \Pi - \int_{\cup_{x \in I} B_x} \Omega \\ &\overset{\lim_{r \to 0}}{=} - \int_{\hat{X}(\partial M)} \Pi + \int_{\hat{X}\left(\cup_{x \in I} \partial \bar{B}_x\right)} \Pi \\ &= - \int_{\partial M} \hat{X}^* \Pi + \int_{\cup_{x \in I} \partial \bar{B}_x} \hat{X}^* \Pi \\ &= - \int_{\partial M} \hat{X}^* \Pi + \sum_{x \in I} deg_x(\hat{X}) \int_{\partial \bar{B}_x} \Pi \\ &= - \int_{\partial M} \hat{X}^* \Pi + \sum_{x \in I} index_x(\hat{X}), \end{aligned} \tag{3.C.8}$$

rearranging the above equation and apply the Poincare-Hopf theorem we get the Chern-Gauss-Bonnet formula

$$\int_M \Omega + \int_{\partial M} \hat{X}^* \Pi = \chi(M). \tag{3.C.9}$$

3.C.1 d=4 Riemannian manifold with boundary

Our entangling surface is a 4 dimensional Riemannian manifold.

$$\Omega = \frac{1}{32\pi^2}\epsilon_{abcd}\Omega^{ab}\wedge\Omega^{cd} \tag{3.C.10}$$

$$\Omega = \frac{1}{128\pi^2}\epsilon_{abcd}\mathcal{R}^{ab}{}_{ef}\mathcal{R}^{cd}{}_{gh}e^e e^f e^g e^h$$

$$\Omega = \frac{1}{128\pi^2}\epsilon_{abcd}\epsilon^{efgh}\mathcal{R}^{ab}{}_{ef}\mathcal{R}^{cd}{}_{gh}e^1 e^2 e^3 e^4$$

$$\Omega = \frac{1}{32\pi^2}\left(\mathcal{R}^{ab}{}_{cd}\mathcal{R}^{cd}{}_{ab} - 4\mathcal{R}^a_b\mathcal{R}^b{}_a + \mathcal{R}^2\right)dV \tag{3.C.11}$$

We can choose the vector field $\hat{X}$ to be the inward pointing unit normal of the boundary $n = e_4$ then

$$n^*\Pi = \frac{1}{\pi^2}\left(\frac{1}{12}n^*\Phi_0 + \frac{1}{8}n^*\Phi_1\right). \tag{3.C.12}$$

The 3-form Φ_0 is explicitly written in terms of extrinsic curvature of the boundary ∂M as,

$$n^*\Phi_0 = \epsilon_{4ijk}\omega^i{}_4\wedge\omega^j{}_4\wedge\omega^k_4 \tag{3.C.13}$$

$$n^*\Phi_0 = \epsilon_{ijk}\epsilon^{lpq}\mathcal{K}^i_l\mathcal{K}^j_p\mathcal{K}^k_q e^1 e^2 e^3$$

$$n^*\Phi_0 = \left(\mathcal{K}^3 - 3\mathcal{K}\mathrm{Tr}(\mathcal{K}^2) + 2\mathrm{Tr}(\mathcal{K}^3)\right)dS \tag{3.C.14}$$

to get to the second line we used $\omega^i{}_4$ on the boundary is related to the extrinsic curvature

$$\omega^i{}_4 = -\mathcal{K}^i_j e^j. \tag{3.C.15}$$

Similarly the 3-form Φ_1 is written in terms of the intrinsic curvature of M and the extrinsic curvature of ∂M,

$$n^*\Phi_1 = \epsilon_{4ijk}\omega^i{}_4\wedge\Omega^{jk} \tag{3.C.16}$$

$$n^*\Phi_1 = \frac{1}{2}\epsilon_{ijk}\epsilon^{lpq}\mathcal{K}^i_l\mathcal{R}^{jk}{}_{pq}e^1 e^2 e^3$$

$$n^*\Phi_1 = \mathcal{K}\mathcal{R} - 2\mathcal{K}\mathcal{R}_{ab}n^a n^b - 2\mathcal{K}^{ab}\mathcal{R}_{ab} + 2\mathcal{R}_{abcd}\mathcal{K}^{ac}n^b n^d \tag{3.C.17}$$

Combining the $(3.C.14)$ and $(3.C.17)$ and using the coordinate on Σ, $x^i,\ i = 1,2,3,4$, we recover (3.3.3)

$$\int_\Sigma \Omega = \frac{1}{32\pi^2}\int_\Sigma d^4x\left(\mathcal{R}^{ijkl}\mathcal{R}_{ijkl} - 4\mathcal{R}_{ij}\mathcal{R}^{ij} + \mathcal{R}^2\right) \tag{3.C.18}$$

and (3.3.4)

$$\int_{\partial\Sigma} n^*\Pi = \frac{1}{4\pi^2}\int_{\partial\Sigma} d^3x\sqrt{h}\Big(\mathcal{R}_{ijkl}\mathcal{K}^{ik}n^jn^k - \mathcal{R}^{ij}\mathcal{K}_{ij} - \mathcal{K}\mathcal{R}_{ij}n^in^j + \frac{1}{2}\mathcal{K}\mathcal{R} + \frac{1}{3}\mathcal{K}^3 - \mathcal{K}\mathrm{Tr}(\mathcal{K}^2) + \frac{2}{3}\mathrm{Tr}(\mathcal{K}^3)\Big). \tag{3.C.19}$$

Note that (3.3.4) is independent of the orientation of the normal n because the extrinsic curvature is only defined by the outward pointing normal.

CHAPTER 4

Renormalized First Law of Entanglement Entropy

4.1 Introduction

The first law of entanglement entropy states that the variation of the entanglement entropy S_B is equal to the variation of the modular energy $\langle H_B \rangle$,

$$\delta S_B = \delta \langle H_B \rangle . \tag{4.1.1}$$

The holographic first law of entanglement entropy first demonstrated in [57] is applicable to field theories that admit a holographic description. In the analysis of [57] the righthandside of (4.1.1) is by construction finite, as it derives from the standard holographic renormalization expressions for energy [43]. The authors of [57] work with a regulated entanglement entropy and restrict variations such that the left hand side of (4.1.1) has no ultraviolet divergences. The goal of this paper is to demonstrate the holographic first law in general situations, without imposing restrictions on variations, making use of the consistent renormalization procedure for the entanglement entropy developed in [54]. Holographic renormalization of entanglement entropy has been discussed in a number of other works, including [70, 73, 102, 74, 75].

Entanglement entropy in quantum field theory is divergent due to the correlations across the boundary of the entangling region. Holographically, the entanglement entropy is captured by the area of the Ryu-Takayangi surface [7], which is also divergent due to the infinite volume of the entangling surface in the bulk spacetime. In both situations the entropy can be systematically renormalized, inheriting its renormalization scheme from that for the partition function of the theory. The renormalized holographic entanglement entropy in [54] can be derived from the holographically renormalized action [43] using the replica trick. We will show it is necessary to use the renormalized entanglement entropy on the left hand side of (4.1.1) to obtain the correct finite contributions when one considers general linear perturbations.

The covariant charge formalism can be used to give an elegant discussion of the holographic first law [57]. In the covariant formalism both sides of (4.1.1) can be expressed as integrals of charge densities over entangling surfaces. We will review this approach in the following section. The charge associated with the change in modular energy used in [57] was renormalized, following the earlier works of [92, 91]. However, the charge associated with the change in entropy was not renormalized; its variation was finite in [57] due to constraints on the asymptotic falloff of metric perturbations. In this paper we construct a renormalized charge corresponding to the change in entropy such that the integral version of the holographic first law applies to generic metric perturbations.

At a technical level, one can understand the construction of this charge as follows. Onshell, the density of the conserved charge is defined in terms of the current density as

$$\boldsymbol{J} = d\boldsymbol{Q} \tag{4.1.2}$$

where $\boldsymbol{J}$ and $\boldsymbol{Q}$ are differential forms. The charge density clearly has an intrinsic ambiguity: additional exact terms in $\boldsymbol{Q}$ will not change the current. In our context, the exact term ambiguity in the density of the conserved charge contributes to the entanglement entropy (and modular energy) at the boundary of the entangling surface. The holographic counterterms fix the ambiguity in the density of the conserved charges, with the matching of renormalization schemes for energy and entropy ensuring that the first law holds. Relative to the expressions given for the entropy in [57], our expressions have additional boundary terms. Our general expressions are applicable to variations of the entanglement entropy associated with generic variations of the bulk metric i.e. perturbations of the non-normalizable terms in the metric.

Boundary terms in the construction of charges using the covariant phase space formalism have been discussed recently in [103]. The boundary counterterms associated with holographic charges were constructed using Hamiltonian renormalisation methods in [91]. There are key conceptual differences in the entropy variation that require us to generalize relative to both of these works. The vector used to construct the entropy variation is no longer Killing. In [91] the goal was to compute conserved charges for black holes and accordingly any variations considered would preserve the non-normalizable modes of the background. In our case the non-normalizable modes are not held fixed: the metric perturbations can be such that the non-normalizable modes vary, corresponding to deforming not just the state of the dual field theory, but the theory itself. This different physical setup leads to differences in the counterterms arising in the analysis of the covariant phase space construction, which are explained in detail in Appendix C.

The structure of the paper is the following. In section 4.2.1 we review the holographic renormalized entanglement entropy, introducing the notion of renormalized area integral for codimension two minimal surfaces in $AlAdS$ that allows us to express the renormalized entanglement entropy functional in terms of certain conformal invariants. In section 4.2.2 we summarise the covariant formalism or Hamiltonian formalism for holographic renormalization and conserved charges and explain in 4.2.3 the first law of holographic entanglement entropy, explaining the constraints imposed on variations in previous works. In section 4.3 we explore the infinitesimal version of the first law i.e. the radius of the entangling region is infinitesimal, for general variations, explaining the differences between odd and even dimensions. In odd dimensions the variation of the renormalized entropy can be expressed elegantly in terms of the pullback of the Weyl tensor variation.

We demonstrate the integral version of the renormalized first law in section 4.4. We first need to introduce in 4.4.1 the proper definition of the conserved charges and their integrals: we demonstrate how the equivalence relations between conserved charges need to be generalized to include appropriate counterterms once one allows for generic variations. In section 4.4.2, we use these conserved charges and their equivalence relations to derive the renormalized first law of entanglement entropy. This general proof is illustrated using two examples in $d = 3$, 4 and 5. We end the paper with discussion of implications and applications of our results.

4.2 Review of renormalized entanglement entropy, holographic charges and first law

In this section we briefly review the definition of renormalized entanglement entropy and holographic charges, and describe the first law of entanglement entropy.

4.2.1 Renormalized entanglement entropy

One of the main goals of this work is to generalise first laws for holographic entanglement entropy, relaxing assumptions on boundary conditions for bulk metric perturbations. One of the tools that will be used in our analysis is renormalized entanglement entropy; this is relevant as general boundary conditions for bulk metric perturbations are associated with UV divergences in the regulated entanglement entropy. Working with quantities that are consistently renormalized allows us to work systematically with such setups.

Renormalized entanglement entropy was discussed extensively in [54], with explicit formulae for holographic renormalized entanglement entropy being derived. A convenient way to construct expressions for renormalized entanglement entropy is via the replica trick. Using the replica trick, entanglement entropy can be derived as the limit of Rényi entropy

$$S = -\alpha\partial_\alpha[\log \mathcal{Z}(\alpha) - \alpha \log \mathcal{Z}(1)]_{\alpha=1} \tag{4.2.1}$$

where $\mathcal{Z}(\alpha)$ is the partition function on the α fold cover manifold. In much of the condensed matter literature this approach is applied to UV regulated quantities, with the UV regulator being interpreted in terms of the lattice scale of the discrete condensed matter system of interest. However, from a quantum field theory perspective, it is much more natural to work directly with renormalized quantities, ie.e. $\mathcal{Z}$ is the renormalized partition function.

In holography the partition function is computed to leading order from the onshell bulk action i.e.

$$\mathcal{Z}_{grav} = e^{-I_{grav}}, \tag{4.2.2}$$

where I_{grav} denotes the onshell gravitational action. Applying the holographic dictionary and the replica trick to the *renormalized* gravity action one obtains a formal definition of the renormalized holographic entanglement entropy.

$$S = \alpha\partial_\alpha[I_{ren}(\alpha) - \alpha I_{ren}(1)]_{\alpha=1} \tag{4.2.3}$$

This approach thus directly relates the renormalization scheme for the partition function (gravitational action) to the scheme for the entanglement entropy.

To obtain a finite value for the gravitational action, one needs to use holographic renormalization. The renormalized action can then be obtained by the procedure of regularization and the introduction of appropriate covariant boundary counterterms

$$I_{ren} = I_{reg} - I_{ct} \tag{4.2.4}$$

For pure gravity with negative cosmological constant, the renormalized action in $(d+1)$ dimensions takes the form [43]

$$\begin{aligned} I_{ren} = {} & \frac{1}{16\pi G_{d+1}} \int_{\mathcal{M}_{z\geq\epsilon}} d^{d+1}x\sqrt{g}(R + d(d-1)) \\ & - \frac{1}{16\pi G_{d+1}} \int_{\tilde{\mathcal{M}}_\epsilon} d^d x\sqrt{\tilde{g}}[K + 2(1-d)\mathcal{R} + \frac{1}{2-d}\mathcal{R} \\ & \qquad - \frac{1}{(d-4)(d-2)^2}(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} - \frac{d}{4(d-1)}\mathcal{R}^2) - \log\epsilon a_{(d)} + \cdots] \end{aligned} \tag{4.2.5}$$

In these expressions the bulk manifold is regulated using a radial coordinate $z \geq \epsilon$; R

denotes the curvature of the bulk manifold while K and $\mathcal{R}$ refer to the extrinsic and intrinsic curvature of the boundary manifold respectively. Here the given counterterms suffice for $d \leq 5$; expressions for the additional counterterms required for $d > 5$ can be found in [43]. Logarithmic counterterms associated with conformal anomalies arise for d even, and explicit expressions for these can also be found in [43].

One can then derive the renormalized entanglement entropy from the renormalized action, making use of the following expressions for the integrals of curvature invariants, expressed as series in powers of $(1-\alpha)$ [104, 105]:

$$
\begin{aligned}
\int_{\mathcal{M}_\alpha} d^{d+1}x\sqrt{g}\mathcal{R}_\alpha &= \alpha \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\mathcal{R} + 4\pi(1-\alpha)\int_{\tilde{B}} d^{d-1}x\sqrt{\gamma} \\
\int_{\mathcal{M}_\alpha} d^{d+1}x\sqrt{g}\mathcal{R}_\alpha^2 &= \alpha \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\mathcal{R}^2 + 8\pi(1-\alpha)\int_{\tilde{B}} d^{d-1}x\sqrt{\gamma}\mathcal{R} \\
\int_{\mathcal{M}_\alpha} d^{d+1}x\sqrt{g}\mathcal{R}_{\alpha\mu\nu}\mathcal{R}_\alpha^{\mu\nu} &= \alpha \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \\
&\quad + 4\pi(1-\alpha)\int_{\tilde{B}} d^{d-1}x\sqrt{\gamma}(\mathcal{R}_{\mu\nu}n^\mu \cdot n^\nu - \frac{1}{2}(TrK)^2)
\end{aligned}
\tag{4.2.6}
$$

Using these replica curvature integrals the explicit expression for the holographic renormalized entanglement entropy becomes [54]

$$
\begin{aligned}
S_{ren} = &\frac{1}{4G_{d+1}}\int_{\tilde{B}} d^{d-1}x\sqrt{\gamma} - \frac{1}{4(d-2)G_{d+1}}\int_{\partial\tilde{B}} d^{d-2}x\sqrt{\tilde{\gamma}} \\
&- \frac{1}{4(d-2)(d-4)G_{d+1}}\int_{\partial\tilde{B}} d^{d-2}x\sqrt{\tilde{\gamma}}\Big(\mathcal{R}_{\mu\nu}n^\mu \cdot n^\nu - \frac{1}{2}(TrK)^2 - \frac{d}{2(d-1)}\mathcal{R}\Big)
\end{aligned}
\tag{4.2.7}
$$

where $\mathcal{R}$ is the Ricci scalar of the metric $g_{\mu\nu}$, $\mathcal{R}_{aa} = \sum_a (-1)^a \mathcal{R}_{\mu\nu} n_a^\mu n_a^\nu$ is the projection of the Ricci tensor on the subspace orthogonal to $\partial\tilde{B}$ with temporal and spatial normals n_a^μ, $a = 1, 2$, $\tilde{\gamma}$ is the determinant of the induced metric on $\partial\tilde{B}$ and $k^2 = \sum_a K_a K_a$ with K_a trace of the extrinsic curvature corresponding to the two normals n_a^μ. Here $\tilde{B}$ denotes the entangling surface with boundary $\partial\tilde{B}$. The counterterms given here are sufficient for $d < 6$, but can straightforwardly be computed for $d \geq 6$. For d even there are logarithmic counterterms related to conformal anomalies, see [54] for details.

When the CFT dimension d is odd, the renormalized entanglement entropy can be written in terms of the Euler characteristic and other renormalized curvature invariants of the bulk entangling surface, see chapter 3, in particular (3.1.4),

$$
S_{ren}(\tilde{B}) \sim (-1)^{n+1}\mathcal{F}_n\, \chi(\tilde{B}) - \sum_r \mathcal{W}_r(\tilde{B}) - \sum_p \mathcal{H}_p(\tilde{B}) - \sum_q \mathcal{I}_q(\tilde{B}). \tag{4.2.8}
$$

where $\mathcal{W}_r$ are renormalized integral of projections of the Weyl curvature, $\mathcal{H}_p$ are renormalized integral of even powers of the extrinsic curvature and for $d > 5$ there are $\mathcal{I}_q$

renormalized integrals containing products of Weyl and extrinsic curvature. More explicitly the renormalized entanglement entropy proportional to the renormalized area of the bulk entangling surface $\tilde{B}$

$$S_{ren}(\tilde{B}) = \frac{\mathcal{A}(\tilde{B})}{4G_{d+1}} \tag{4.2.9}$$

and renormalized area integral in $d = 3$ is

$$\mathcal{A}(\tilde{B}) = -2\pi\chi(\tilde{B}) - \frac{1}{2}\int_{\tilde{B}} d^2x\sqrt{g}|K|^2 - \int_{\tilde{B}} d^2x\sqrt{g}W_{1212}, \tag{4.2.10}$$

and in $d = 5$ is

$$\begin{aligned}\mathcal{A}(\tilde{B}) \quad = \quad & \frac{4\pi^2}{3}\chi(\tilde{B}) - \frac{1}{6}\mathcal{H}(\tilde{B}) - \frac{1}{3}\mathcal{W}(\tilde{B}) \\ & -\frac{1}{24}\int_{\tilde{B}} d^4x\sqrt{g}\Big(H^2 - 4H_{\mu\nu}H^{\mu\nu} + H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma} \\ & +4W_{1212}^2 - 4W_{\mu n\nu n}\widetilde{W}^{\mu n\nu n} + W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}\Big).\end{aligned} \tag{4.2.11}$$

In what follows we will find these geometric expressions for renormalized entanglement entropy useful. Note in particular that these will simplify considerably in the context of first variations around AdS backgrounds.

4.2.2 Hamiltonian formalism and charges in AdS

In this section we review the description of Wald Hamiltonians [106, 107, 108, 109, 110] and charges in anti-de Sitter spacetimes. Our review follows closely the work of [92, 91], and more details may be found in these references. The Wald approach assumes that the gravitational theory is described by a diffeomorphism covariant Lagrangian d-form $\mathbf{L}(\psi)$, where $\mathbf{L}(\psi)$ will depend both on the metric and other fields, denoted collectively as ψ. In the anti-de Sitter context we work with a renormalized Lagrangian

$$\boldsymbol{L}^{ren} = \boldsymbol{L} - d\boldsymbol{B} \tag{4.2.12}$$

where $\boldsymbol{L}$ is the bulk Lagrangian form and $\boldsymbol{B}$ may be viewed as the combination of the Gibbon-Hawking term and boundary counterterms. The onshell regular Lagrangian is exact i.e.

$$\boldsymbol{L}^{onshell} = -\frac{d(\boldsymbol{\varepsilon}_a n^a \lambda)}{16\pi G_{d+1}}. \tag{4.2.13}$$

where n^a is the outward normal in the asymptotic radial direction,

$$n = -\frac{dz}{z}, \tag{4.2.14}$$

$\boldsymbol{\varepsilon}$ is the volume form and $\boldsymbol{\varepsilon}_{a_1 \dots a_n}$ is a $(d-n+1)$ form

$$\boldsymbol{\varepsilon}_{a_1 \dots a_n} = \frac{1}{(d-n+1)!} \varepsilon_{a_1 \dots a_n b_1 \dots b_{d-n+1}} dx^{b_1} \wedge \dots \wedge dx^{b_{d-n+1}}, \tag{4.2.15}$$

with the orientation

$$\varepsilon_{ztx^1 \dots} = +\sqrt{-g}. \tag{4.2.16}$$

In the Hamiltonian formalism, fields can be expanded asymptotically near the conformal boundary in series of dilatation eigenfunctions with ascending weight, see appendix 4.B.1 for more detailed explanation. The general structure of the boundary term is then

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}^{GH} - \boldsymbol{B}^{ct} \\ &= -\frac{\boldsymbol{\varepsilon}_a n^a}{8\pi G_{d+1}} \left(K - (K_{ct} - \lambda_{ct})\right) \\ &= \frac{-\boldsymbol{\varepsilon}_a n^a}{8\pi G_{d+1}} \left(K_{(d)} + \lambda_{ct}\right). \end{aligned} \tag{4.2.17}$$

where typically counterterms contribute up to terms at the d^{th} order i.e.

$$O_{ct} \sim \sum_{i<d} O_(i). \tag{4.2.18}$$

Variations can then be expressed as

$$\delta \boldsymbol{L}^{ren} = \delta \boldsymbol{L} - d\delta \boldsymbol{B} \tag{4.2.19}$$

$$= \boldsymbol{E}^{\psi} \delta\psi + d\boldsymbol{\Theta}[\delta\psi] - d\delta \boldsymbol{B} \tag{4.2.20}$$

where $\boldsymbol{E}^{\psi}$ denotes the equations of motion and $\boldsymbol{\Theta}$ is the symplectic potential. This expression can be rewritten as

$$\delta \boldsymbol{L}^{ren} = \boldsymbol{E}^{\psi} \delta\psi + d\boldsymbol{\Theta}^{ren}[\delta\psi] \tag{4.2.21}$$

where the renormalized symplectic potential form can be expressed as

$$\boldsymbol{\Theta}^{ren}[\delta\psi] = \boldsymbol{\varepsilon}_a n^a \pi^{\mu\nu}_{(d)} \delta\gamma_{\mu\nu} \tag{4.2.22}$$

The canonical momentum $\pi_{\mu\nu}$ can be expressed in terms of the extrinsic curvature as

$$\pi^{\mu\nu} = -\frac{1}{16\pi G_{d+1}} \sqrt{\gamma} \left(K^{\mu\nu} - K\gamma^{\mu\nu}\right). \tag{4.2.23}$$

If we expand both sides of the equality in the dilatation eigenfunction expansion, we can match the dilatation weights and obtain,

$$\pi^{\mu\nu}_{(n)} = -\frac{1}{16\pi G_{d+1}} \left(K^{\mu\nu}_{(n)} - K_{(n)} \gamma^{\mu\nu} \right). \tag{4.2.24}$$

In (4.2.22) $\pi^{\mu\nu}_{(d)}$ is the d^{th}-term in the dilatation eigenfunction series of the conjugate momentum with respect to the metric and this is in turn related to the renormalized CFT stress tensor as

$$2\pi^{\mu\nu}_{(d)} = -\frac{1}{16\pi G_{d+1}} T^{\mu\nu}_{ren}, \tag{4.2.25}$$

i.e. the first variation of the renormalized action is

$$\delta I^{onshell}_{ren} = \frac{-1}{32\pi G_{d+1}} \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} T^{\mu\nu}_{ren} \delta\gamma_{\mu\nu}. \tag{4.2.26}$$

Now let us consider the asymptotic behaviour of metric perturbations. Expressing the AdS$_{d+1}$ metric as

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} \eta_{\mu\nu} dx^{\mu\nu} \tag{4.2.27}$$

where η is the Minkowski metric, only the normalizable mode is allowed to vary under a Dirichlet condition and

$$\delta\gamma_{\mu\nu} = z^{d-2} \delta\gamma_{(d)\ \mu\nu} + O(z^{d-1}), \qquad z \to 0. \tag{4.2.28}$$

For d odd, using the tracelessness of the stress tensor and absence of trace anomaly, the Dirichlet boundary condition can be automatically generalized to a conformal Dirichlet boundary condition which fixes the conformal class only. In the present of a conformal anomaly, for the onshell action to be stationary under perturbations a representative of the conformal class has to be fixed.

Now let us turn to Noether charges. If the field variation is induced by a vector field ξ, we can define the Noether current form as

$$\boldsymbol{J}[\xi] = \boldsymbol{\Theta}[\delta_\xi \psi] - \iota_\xi \boldsymbol{L} \tag{4.2.29}$$

where ι_ξ contracts ξ with the first index of $\boldsymbol{L}$. The exterior derivative of the Noether current is proportional to the equation of motion

$$d\boldsymbol{J}[\xi] = -\boldsymbol{E}^\psi \delta_\xi \psi, \tag{4.2.30}$$

and thus vanishes onshell. Hence we can define the Noether charge form $\boldsymbol{Q}[\xi]$ as the exact term in the Noether current

$$\boldsymbol{J}[\xi] = d\boldsymbol{Q}[\xi] - \boldsymbol{N}[\xi] \tag{4.2.31}$$

where

$$d\boldsymbol{N}[\xi] = \boldsymbol{E}^{\psi}\delta_{\xi}\psi. \tag{4.2.32}$$

There is another conserved charge in $AlAdS$ induced by ξ called the holographic charge. Using (4.2.25) and the fact that the renormalized CFT stress tensor is conserved, we can construct the total relativistic momentum of the boundary system. The holographic charge form $\boldsymbol{\mathcal{Q}}[\xi]$ is defined by

$$\boldsymbol{\mathcal{Q}}[\xi] = -\boldsymbol{\varepsilon}_{ab} n^a 2\pi^{bc}_{(d)}\xi_c. \tag{4.2.33}$$

and this form is integrated over a timeslice at the boundary to obtain the holographic charge. This can be interpreted as the renormalized relativistic momentum along the ξ direction.

In Lemma 4.1 in [91] it was proved that for any asymptotically locally anti-de Sitter space $\mathcal{M}$ the two definitions of charges corresponding to asymptotic conformal Killing vector, ξ, on a spatial slice on the conformal boundary, $\partial\mathcal{M}\cap C$, are equivalent i.e.

$$-\int_{\partial\mathcal{M}\cap C}\boldsymbol{\mathcal{Q}}[\xi] = \int_{\partial\mathcal{M}\cap C}\mathbf{Q}^{full}[\xi]. \tag{4.2.34}$$

where

$$\mathbf{Q}^{full}[\xi] = \mathbf{Q}[\xi] - \iota_{\xi}\mathbf{B}. \tag{4.2.35}$$

Note that this equivalence is defined up to exact terms since $\partial\mathcal{M}\cap C$ is a cycle and the asymptotic conformal Killing vector ξ has the following fall off condition:

$$\xi^z = O(z^d), \qquad \xi^{\mu} = \zeta^{\mu}(1 + O(z^{d+2})) \tag{4.2.36}$$

where ζ is a boundary conformal Killing vector. We will later need to generalise this equivalence to less restrictive fall off conditions on the vector field.

4.2.3 Holographic first law of entanglement entropy

In this section we briefly review the first law of entanglement entropy. Given a reduced density matrix ρ_B the modular Hamiltonian H_B is given by

$$\rho_B = e^{-H_B} \tag{4.2.37}$$

Under a small variation of the entanglement entropy, we obtain the relation

$$\delta S_B = \delta \langle H_B \rangle \tag{4.2.38}$$

and the equivalence between $\delta \langle H_B \rangle$ and the change in energy δE gives the first law of entanglement entropy.

Following [51], we now review relevant properties of the modular Hamiltonian and modular flow for CFTs on Minkowski space. There is a symmetry group associated with the modular Hamiltonian: the modular group, a group of one-parameter transformations of the form

$$U_B(s) = e^{-isH_B} \tag{4.2.39}$$

where ∂_s is called the modular flow. For QFTs on Minkowski space, the modular flow generates a boost. In null coordinates $X^{\pm}$ this is given by

$$X^{\pm}(s) = X^{\pm} e^{\pm 2\pi s}. \tag{4.2.40}$$

For an accelerated observer in Rindler coordinates, the state is thermal in τ where the longitudinal part of the metric is given by

$$dX^{+}dX^{-} = -\frac{\rho^2}{R^2} d\tau^2 + d\rho^2, \tag{4.2.41}$$

where R relates to the imaginary time periodicity i.e. $\beta = 2\pi R$. The thermal density matrix of the state is

$$\rho_{\mathcal{R}} = \frac{e^{\beta H_\tau}}{Tr(e^{\beta H_\tau})}. \tag{4.2.42}$$

The modular flow generator is $2\pi R \partial_\tau$ and the modular Hamiltonian is given by $2\pi R H_\tau + \log Tr(e^{\beta H_\tau})$.

For a spatial ball B of radius R centred at $x^i = 0$, $t = 0$ on d-dimensional Minkowski space, we can conformally map the causal development of the spatial ball $D(B)$ to the Rindler wedge. This conformal map $X^\mu \to x^\mu$ can also map the modular flow (4.2.40) to

$$x^{\pm}(s) = R\frac{(R + x^{\pm}) - e^{\mp 2\pi s}(R - x^{\pm})}{(R + x^{\pm}) + e^{\mp 2\pi s}(R - x^{\pm})} \tag{4.2.43}$$

and hence the modular flow generator ∂_s as ζ_B

$$\partial_s = \frac{\pi}{R}\left((R^2 - t^2 - \vec{x}^2)\partial_t - 2tx^i\partial_i\right) \tag{4.2.44}$$

$$\zeta_B = \frac{i\pi}{R}(R^2 P_t + K_t) \tag{4.2.45}$$

where P_t and K_t are the time translation and special conformal transformation generators, respectively.

Since the modular Hamiltonian is the translation operator in s, on B we get

$$H_B = \int_B d^{d-1}x T^{ts}. \tag{4.2.46}$$

We can write (4.2.46) in covariant form as

$$H_B = \int_B d\sigma^\mu T_{\mu\nu}\zeta_B^\nu \tag{4.2.47}$$

and the modular energy as

$$E_B = \int_B d\sigma^\mu \langle T_{\mu\nu}\rangle \zeta_B^\nu. \tag{4.2.48}$$

The entanglement entropy of region B can be calculated holographically by the area of the corresponding bulk entangling surface $\tilde{B}$.

A CFT in the vacuum state on the causal wedge $D(B)$ can also be mapped conformally to a CFT in a thermal state on the hyperbolic cylinder. This can be easily seen from writing the Rindler metric as:

$$ds^2 = \frac{\rho^2}{R^2}\left(-d\tau^2 + \frac{R^2}{\rho^2}(d\rho^2 + dX^i dX^i)\right) = \frac{\rho^2}{R^2} ds^2_{\mathbb{R}\times\mathbb{H}^{d-1}}. \tag{4.2.49}$$

As discussed in [57], the first law of entanglement entropy of the CFT thermal state on hyperbolic cylinder can be related to the first law of black hole dynamics via holography. Essentially, the CFT on hyperbolic cylinder $\mathbb{R}\times\mathbb{H}^{d-1}$ is dual to the Rindler AdS_{d+1} black hole exterior and the bulk entangling surface $\tilde{B}$ can be viewed as the black hole horizon. The perturbation of entanglement entropy δS_B is equal to the perturbation of black hole entropy calculated from the Wald functional δS_{Wald} where

$$S_{Wald} = -2\pi \int_{\mathcal{H}} d\sigma \frac{\delta\mathcal{L}}{\delta R^{ab}_{\ \ cd}} n^{ab} n_{cd}. \tag{4.2.50}$$

Changing back to the interpretation in terms of a Minkowski boundary, we label Σ as the bulk region enclosed by B and $\tilde{B}$ and the bulk causal wedge of Σ as $D(\Sigma)$. We extend the boundary modular flow to the bulk as the Killing vector

$$\xi_B = -\frac{2\pi}{R}(t-t_0)[z\partial_z + (x^i - x_0^i)\partial_i] + \frac{\pi}{R}[R^2 - z^2 - (t-t_0)^2 - (\vec{x}-\vec{x}_0)^2]\partial_t. \tag{4.2.51}$$

Note that this Killing vector does not satisfy (4.2.36) but instead has the weaker fall-off behaviour

$$\xi_B^z = O(z), \qquad \xi_B^\mu = \zeta_B^\mu(1 + O(z^2)). \tag{4.2.52}$$

One can check ξ_B vanishes on $\tilde{B}$.

It was shown in [57] that for metric perturbations limited to normalizable modes, $\delta g_{\mu\nu} = z^{d-2}h^{(d)}_{\mu\nu}$, one get the infinitesimal first law of entanglement entropy as

$$\begin{aligned}\delta \langle T_{tt} \rangle &= \frac{d^2-1}{2\pi\Omega_{d-2}} \lim_{R\to 0}\left(\frac{1}{R^d}\delta S_B\right)\\ \delta \langle T_{tt}\rangle &= \frac{d}{16\pi G_{d+1}} h^{(d)}_{tt}\\ \delta \langle T_{\mu\nu}\rangle &= \frac{d}{16\pi G_{d+1}} h^{(d)}_{\mu\nu}\end{aligned} \tag{4.2.53}$$

where the tracelessness condition of $h^{(d)}_{\mu\nu}$ is used to go from the second line to the final covariant expression. The final expression matches the holographic dictionary between stress tensor and normalizable metric coefficient found in [43].

The covariant first law of entanglement entropy utilises the charges associated with with energy and entropy corresponding to the bulk Killing vector ξ_B introduced in section 4.2.2. The entanglement entropy is

$$S_B^{grav} = \int_{\tilde{B}} \mathbf{Q}^{full}[\xi_B] \tag{4.2.54}$$

and the modular energy is

$$E_B^{grav} = -\int_B \boldsymbol{\mathcal{Q}}[\xi_B]. \tag{4.2.55}$$

Limiting to the variation involving only the normalizable mode with no boundary variation on $\partial B = \partial\tilde{B}$ and using (4.2.34),

$$-\int_B \delta\boldsymbol{\mathcal{Q}}[\xi_B] = \int_B \delta\mathbf{Q}^{full}[\xi_B]. \tag{4.2.56}$$

The off-shell difference is expressed in terms of the Einstein equations

$$\begin{aligned}\delta E_B^{grav} - \delta S_B^{grav} &= \int_B \delta\boldsymbol{Q}^{full}[\xi_B] - \int_{\tilde{B}} \delta\boldsymbol{Q}^{full}[\xi_B] \\ &= \int_\Sigma d\delta\mathbf{Q}^{full}[\xi_B] = \int_\Sigma \delta\mathbf{J}^{full}[\xi_B] = \int_\Sigma -2\boldsymbol{\varepsilon}^a\delta E_{ab}\xi_B^a,\end{aligned} \tag{4.2.57}$$

hence recovering the first law of entanglement entropy onshell. We also obtain the version of (4.2.34)

$$-\int_B \delta\boldsymbol{Q}[\xi_B] = \int_{\tilde{B}} \delta\boldsymbol{Q}^{full}[\xi_B]. \tag{4.2.58}$$

Note there are many caveats regarding boundary terms and fall-off condition when we allow the variation of the non-normalizable modes. We shall address them in the following sections.

4.3 Infinitesimal renormalized first law

In this section we will discuss the renormalized version of the first law of entanglement entropy in the infinitesimal limit, for $\mathcal{M} = AdS_{d+1}$ with spherical boundary entangling surfaces $\partial\tilde{B} = S^{d-2}$ in $d \leq 6$. We begin by collecting together expressions for the renormalized entanglement entropy of such spherical regions. We derive the infinitesimal renormalized first law of entanglement entropy in $AlAdS_{d+1}$ for odd d and explain its connection with the variation of the renormalized integral of a curvature invariant. Since the renormalized entanglement entropy in even dimensions is scheme dependent, we postpone the proof of the generalized first law in even d to section 4.4.3.2 to avoid repetitions.

4.3.1 Spherical entangling region in AdS

The metric of AdS_{d+1} on the Poincare patch may be parameterized as

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} g_{\mu\nu} dx^\mu dx^\nu, \tag{4.3.1}$$

where $g_{\mu\nu} = \eta_{\mu\nu}$ is flat Minkowski metric with signature $(-,+,\cdots,+)$. In the case of spherical entangling regions, the $(d-1)$-dimensional bulk extending entangling surface $\tilde{B}$ with boundary $\partial\tilde{B}$ as the entangling surface of the boundary CFT can be described by

$$r^2 + z^2 = R^2 \tag{4.3.2}$$

where r is the radial coordinate on the boundary and R is the radius of the spherical entangling region. The induced metric on the entangling surface in AdS_{d+1} is then

$$ds^2 = \frac{R^2}{z^2 r^2} dz^2 + \frac{r^2}{z^2} d\Omega_{d-2}^2. \tag{4.3.3}$$

where $d\Omega_{d-2}^2$ is the standard unit sphere metric. Another convenient choice of coordinates (w, u) are defined by

$$r = w\cos u, \qquad\qquad z = w \sin u \tag{4.3.4}$$

so the AdS_{d+1} metric can be written as

$$ds^2 = \frac{1}{w^2 \sin^2 u}(dw^2 - dt^2) + \frac{1}{\sin^2 u} dz^2 + \frac{\cos^2 u}{\sin^2 u} d\Omega_{d-2}^2 \tag{4.3.5}$$

and the induced metric on $\tilde{B}$ becomes

$$ds^2 = \frac{1}{\sin^2 u} dz^2 + \frac{\cos^2 u}{\sin^2 u} d\Omega_{d-2}^2. \tag{4.3.6}$$

The regularised bulk contribution to the entanglement entropy for such an entangling surface is then

$$\begin{aligned} S_B^{reg} &= \frac{1}{4G_{d+1}} \int_{\tilde{B}_\epsilon} d^{d-1}x \sqrt{\gamma} \\ &= \frac{\Omega_{d-2}}{4G_{d+1}} \int_\epsilon^R dz \, \frac{R r^{d-3}}{z^{d-1}} \qquad (4.3.7) \\ &= \frac{\Omega_{d-2}}{4G_{d+1}} \int_\xi^{\frac{\pi}{2}} du \, \frac{\cos^{d-2} u}{\sin^{d-1} u} \qquad (4.3.8) \end{aligned}$$

where Ω_{d-2} is the area of $(d-2)$-dimensional unit sphere and $R \sin\xi \equiv \epsilon$.

The divergent contributions are of the form ϵ^{-n} except in even d where there are extra logarithmic terms. Focussing first on odd d, from (4.2.7) we know the counterterms for $d < 6$ are

$$\begin{aligned} S_B^{ct} &= \frac{1}{4(d-2)G_{d+1}} \int_{\partial \tilde{B}_\epsilon} d^{d-2}x \sqrt{\tilde{\gamma}} \left[1 + \frac{1}{(d-2)(d-4)} (\mathcal{R}_{aa} - \frac{1}{2}k^2 - \frac{d}{2(d-1)}\mathcal{R}) \right] \qquad (4.3.9) \\ &= \frac{\Omega_{d-2}}{4(d-2)G_{d+1}} \frac{r^{d-2}}{\epsilon^{d-2}} \left[1 - \frac{(d-2)\epsilon^2}{2(d-4)r^2} \right]. \end{aligned}$$

Note that the intrinsic curvature terms do not contribute here since the boundary metric is flat, but they will contribute to the variation of the entanglement entropy under metric perturbations later. For odd d, using the definition of the entangling surface one obtains counterterm contributions

$$S_B^{ct} = \frac{\Omega_{d-2}}{4G_{d+1}} \left[\frac{R^{d-2}}{(d-2)\epsilon^{d-2}} - \frac{R^{d-4}}{(d-4)\epsilon^{d-4}} + \cdots \right]. \tag{4.3.10}$$

Combing (4.3.10) with (4.3.7) we get

$$\begin{aligned} d = 3: &\qquad S_{ren} = -\frac{\pi}{2G_4} \qquad (4.3.11) \\ d = 5: &\qquad S_{ren} = \frac{\pi^2}{3G_6}. \end{aligned}$$

For $d = 4$, the regularized entanglement entropy from (4.3.7) is

$$S_B^{reg} = \frac{\Omega_2}{4G_5}\left[-\frac{\ln R}{2} + \frac{\ln(R^2)}{2} + \frac{R(R^2-\epsilon^2)^{\frac{1}{2}}}{2\epsilon^2} + \frac{\ln \epsilon}{2} - \frac{\ln\left(R^2 + R(R^2-\epsilon^2)^{\frac{1}{2}}\right)}{2}\right] \quad (4.3.12)$$

and the corresponding the full set of counterterms, including the logarithmic counterterm, gives

$$\begin{aligned} S_B^{ct} &= \frac{1}{8G_5}\int_{\partial\tilde{B}} d^2x\sqrt{\tilde{\gamma}} - \frac{\ln\epsilon}{16G_5}\int_{\partial\tilde{B}} d^2x\sqrt{\tilde{\gamma}}(\mathcal{R}_{aa} - \frac{1}{2}k^2 - \frac{2}{3}\mathcal{R}) \\ &= \frac{\Omega_2}{8G_5}\left[\frac{r^2}{\epsilon^2} + \ln\epsilon\right]. \end{aligned} \quad (4.3.13)$$

Combining these we obtain the renormalized entanglement entropy in $d = 4$

$$S_B^{ren} = \frac{\Omega_2}{4G_5}\left[-\frac{\ln 2R}{2} + \frac{1}{4R}\right]. \quad (4.3.14)$$

Since the action in this case has logarithmic counterterms there is an intrinsic scheme dependence in the renormalised entanglement entropy, which is completely determined by the scheme chosen for the renormalization of the action.

4.3.2 First law and variation of modular energy

We now consider the variation of the entanglement entropy under a linear perturbation of the bulk metric. We will express the perturbed metric in radial gauge so that

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \quad (4.3.15)$$

A general perturbation $h_{\mu\nu}$ can be expanded near the conformal boundary as

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + z^2 h^{(2)}_{\mu\nu} + \cdots + z^d h^{(d)}_{\mu\nu} + z^d \log z \tilde{h}^{(d)}_{\mu\nu} + \cdots \quad (4.3.16)$$

where the logarithmic terms arise in even d and all coefficients in the expansion can be expressed in terms of the pair of data $(h^{(0)}_{\mu\nu}, h^{(d)}_{\mu\nu})$ using the Einstein equations.

The goal of this section is to show that the change in the renormalized entropy δS_B^{ren} under such metric perturbations is equal to the change in modular energy i.e.

$$\delta E_B = \delta S_B^{ren}. \quad (4.3.17)$$

In the previous literature [57], the first law was derived by restricting the variation of metric to only normalizable modes i.e. imposing $h^{(0)}_{\mu\nu} = 0$ with $h^{(d)}_{\mu\nu} \neq 0$. Accordingly, the change in the entanglement entropy δS_B is finite even without including the counterterms. Here we will derive the first law for general perturbations for which $h^{(0)}_{\mu\nu}$ is not necessarily zero; from a QFT perspective a general bulk metric perturbation corresponds to changing the background for the dual QFT as well as the state in the theory.

We will first demonstrate the renormalized first law in the infinitesimal limit where the radius of the boundary entangling region B tends to zero $R \to 0$. The modular energy may be approximated by

$$\delta E_B = \int_B d\sigma^\mu \delta T^{ren}_{\mu\nu} \xi^\nu_B \qquad (4.3.18)$$
$$\delta E_B \xrightarrow{R\to 0} \frac{2\pi R^d \Omega_{d-2}}{d^2-1} \delta T^{ren}_{tt}.$$

From holographic renormalization [43], the variation of the renormalized energy momentum tensor for odd d is

$$\delta T^{ren}_{\mu\nu} = \frac{d}{16\pi G_{d+1}} h^{(d)}_{\mu\nu}. \qquad (4.3.19)$$

In even dimensions the relation between the renormalized stress tensor and the coefficients of the asymptotic expansion is more complicated, capturing the conformal anomalies. For example, in $d = 4$

$$\delta T^{ren}_{\mu\nu} = \frac{1}{16\pi G_5} \left(4h^{(4)}_{\mu\nu} + 6\tilde{h}^{(4)}_{\mu\nu} \right), \qquad (4.3.20)$$

i.e. there is an additional contribution associated with the coefficient of the logarithmic term $\tilde{h}^{(4)}$. At linearized order we can express $\tilde{h}^{(4)}$ in terms of the curvature $R^{(0)}$ of the perturbation of the QFT metric $h^{(0)}$ as

$$\tilde{h}^{(4)}_{\mu\nu} = -\frac{1}{48}\partial_\mu\partial_\nu R^{(0)} + \frac{1}{16}\partial^\rho\partial_\rho R^{(0)}_{\mu\nu} - \frac{1}{96}(\partial^\rho\partial_\rho R^{(0)})\eta_{\mu\nu}. \qquad (4.3.21)$$

The infinitesimal first law of entanglement entropy for general variation is thus equivalent to showing that the variation of renormalized entanglement entropy can be expressed in terms of the renormalized stress tensor as

$$\delta S^{ren}_B = \frac{2\pi R^d \Omega_{d-2}}{d^2-1} \delta T^{ren}_{tt} \qquad (4.3.22)$$

4.3.3 Infinitesimal first law for odd d

We shall focus on odd d. The linearized variation of regularized entanglement entropy can be expressed in Cartesian spatial coordinates as

$$\delta S^{reg}_B = \frac{R}{8G_{d+1}} \int_{\tilde{B}_\epsilon} d^{d-1}x \frac{1}{z^d}(h_{ii} - \hat{x}^i\hat{x}^j h_{ij}), \qquad (4.3.23)$$

where i runs over the spatial indices of the d-dimensional Minkowski space.

To obtain the infinitesimal version of the first law, we consider the limit $R \to 0$. To evaluate the integrals explicitly it is more convenient to use the $(w,\, u)$ coordinates in (4.3.4), in terms of which the variation of regularized entanglement entropy is

$$\delta S_B^{reg} = \frac{1}{8G_{d+1}} \int_{\xi}^{\pi/2} du \int_{S^{d-2}} d\Omega_{d-2} \frac{\cos^{d-2} u}{\sin^{d-1} u} (\delta^{ij} - \cos^2 u \hat{x}^i \hat{x}^j) h_{ij}. \tag{4.3.24}$$

For the variation of the counterterms we need the linearized variation of the spatial extrinsic curvature, which can be expressed as

$$\delta K_2 = -\frac{(d-2)z}{2r} \hat{x}^i \hat{x}^j h_{ij} + \frac{z}{2} \partial_r (h_{ii} - \hat{x}^i \hat{x}^j h_{ij}) \tag{4.3.25}$$

and the variation of a specific combination of Ricci tensors,

$$-\delta \mathcal{R}_{tt} + \delta \mathcal{R}_{rr} - \frac{d}{2(d-1)} \delta \mathcal{R} = (d-2) \left(h_{ii}^{(2)} - \hat{x}^i \hat{x}^j h_{ij}^{(2)} \right). \tag{4.3.26}$$

The latter equality holds at linearized level, see equation (4.3.31) below.

Substituting the above expressions into the variation of (4.3.9) we get the following expression for the counterterms in general $d \leq 6$ to first order:

$$\begin{aligned} \delta S_B^{ct} = \frac{1}{4(d-2)G_{d+1}} \int_{S^{d-2}} d\Omega_{d-2} \Bigg[& \frac{1}{2} \frac{r^{d-2}}{\epsilon^{d-2}} \left(h_{ii} - \hat{x}^i \hat{x}^j h_{ij} \right) - \frac{(d-2)}{4(d-4)} \frac{r^{d-4}}{\epsilon^{d-4}} \left(h_{ii} - 3\hat{x}^i \hat{x}^j h_{ij} \right) \\ & + \frac{1}{(d-4)} \frac{r^{d-2}}{\epsilon^{d-4}} \left(h_{ii}^{(2)} - \hat{x}^i \hat{x}^j h_{ij}^{(2)} \right) - \frac{1}{2(d-4)} \frac{r^{d-3}}{\epsilon^{d-4}} \left(\hat{x}^k \partial_k h_{ii} - \hat{x}^i \hat{x}^j \hat{x}^k \partial_k h_{ij} \right) \Bigg]. \end{aligned} \tag{4.3.27}$$

In the (w, u) coordinate system, the area integral for the bulk entangling surface $\tilde{B}_\epsilon$ is expressed in terms of an integral over the asymptotic angular coordinate u and the spatial angular coordinates. We can thus evaluate the integral up to the upper limit $u = \frac{\pi}{2}$ even when expanding around $R = 0$.

The Taylor expansion around $x^i = 0$ at each order of the Fefferman-Graham expansion can be written as:

$$\begin{aligned} h_{\mu\nu}^{(n)}(x^i) = h_{\mu\nu}^{(n)}(0) + R\hat{x}^i \partial_i h_{\mu\nu}^{(n)}(0) + \frac{R^2 \hat{x}^i \hat{x}^j}{2!} \partial_i \partial_j h_{\mu\nu}^{(n)}(0) + \frac{R^3 \hat{x}^i \hat{x}^j \hat{x}^k}{3!} \partial_i \partial_j \partial_k h_{\mu\nu}^{(n)}(0) \\ + \frac{R^4 \hat{x}^i \hat{x}^j \hat{x}^k \hat{x}^l}{4!} \partial_i \partial_j \partial_k \partial_l h_{\mu\nu}^{(n)}(0) + \ldots \end{aligned} \tag{4.3.28}$$

The Fefferman-Graham expansion also becomes an expansion in R,

$$h_{\mu\nu} = h_{\mu\nu}^{(0)} + R^2 \sin^2 u h_{\mu\nu}^{(2)} + \cdots \tag{4.3.29}$$

We can now expand (4.3.24) using (4.3.29) and (4.3.28) up to R^d. The two angular integrals du, $d\Omega_{d-2}$ can be evaluated independently for each term in the expansion. In the appendix we give a general formula (4.A.5) for integrating over products of unit vectors over S^n. Together with (4.A.8), generic terms in the expansion after the spatial angular $d\Omega_{d-2}$ integral take the form

$$(\partial^2)^m h_{ii}^{(n)}, \qquad (\partial^2)^m \partial_i \partial_j h_{ij}^{(n)}. \tag{4.3.30}$$

All the non-normalizable modes are related to the first term in the Fefferman-Graham expansion $h^{(0)}$ through the Einstein equation [43]. For $h^{(2)}$ we have

$$h_{\mu\nu}^{(2)} = -\frac{1}{d-2}\Big(\delta\mathcal{R}_{\mu\nu} - \frac{1}{2(d-1)}\delta\mathcal{R}\eta_{\mu\nu}\Big) \tag{4.3.31}$$

The linear variation of the Ricci tensor is,

$$\delta\mathcal{R}_{\mu\nu} = -\frac{1}{2}\partial_\mu\partial_\nu h^{(0)\sigma}{}_{\sigma} + \partial_\sigma\partial_{(\mu} h^{(0)\sigma}{}_{\nu)} - \frac{1}{2}\partial^\sigma\partial_\sigma h_{\mu\nu}^{(0)} \tag{4.3.32}$$

We can use the above information to express $h^{(2)}$ in terms of derivatives of $h^{(0)}$ and $h^{(4)}$ in terms of derivatives of $h^{(2)}$ as

$$h_{ii}^{(2)} = \frac{1}{2(d-2)}\left(\partial_i\partial_i h_{jj}^{(0)} - \partial_i\partial_j h_{ij}^{(0)}\right). \tag{4.3.33}$$

In $d = 5$ we will also require the following relation between $h^{(4)}$ and $h^{(2)}$,

$$h_{ii}^{(4)} = \frac{1}{4}\partial_j\partial_j h_{ii}^{(2)} - \frac{1}{4}\partial_i\partial_j h_{ij}^{(2)}. \tag{4.3.34}$$

We can follow appendix 4.A.2 and 4.A.3 to obtain

$$\delta S_B^{ren} = \frac{dR^d\Omega_{d-2}}{8(d-1)(d+1)G_{d+1}} h_{ii}^{(d)} \tag{4.3.35}$$

Here by working with the renormalized quantities we recover the first law of entanglement entropy for general linearized variations of the metric, including both non-normalizable and normalizable modes.

4.3.4 Curvature invariants formula

The first variation of the entanglement entropy around spherical entangling regions in AdS$_{d+1}$ with d odd can be expressed in a particularly simple and elegant geometric form, using the expression for the renormalized entanglement entropy in terms of curvature and topological invariants (4.2.8). Since such variations do not change the topology of the entangling surface, the topological Euler invariant contribution does not change. All contributions from the extrinsic curvature are quadratic or higher order; since the extrinsic curvature vanishes to leading order, and this means the contributions $\mathcal{H}_p$ do not contribute to first variations (but do contribute to second variations). By analogous reasoning, the only contribution from the Weyl terms $\mathcal{W}_r$ comes from the term that is linear in the Weyl tensor. Thus we arrive at

$$\delta S^{ren} \propto -\frac{1}{4G_{2n}}\delta\mathcal{W} \tag{4.3.36}$$

where G_{2n} is the Newton constant (with $2n = d+1$) and

$$\delta\mathcal{W} = \int_\Sigma d^{2(n-1)}x\sqrt{g}\,\delta W_{1212} - \int_{\partial\Sigma} d^{2n-3}x\sqrt{h}\,\delta \mathcal{W}_{1212} + \cdots \tag{4.3.37}$$

where δW_{1212} is the pullback of the normal components of the bulk linearized Weyl curvature in an orthonormal frame and $\delta \mathcal{W}_{1212}$ is the pullback of the normal components of the boundary linearized Weyl curvature in an orthonormal frame. The boundary terms are such that $\delta\mathcal{W}$ is a finite conformal invariant for a generic non-normalizable metric perturbation. Note that the boundary term vanishes for AdS_4. The ellipses denote additional boundary terms expressed in terms of higher powers of the boundary Weyl curvature that are required for $n > 3$.

The variation of renormalized entanglement entropy for $d = 3, 5$ is

$$d = 3: \qquad \delta S^{ren}(\tilde{B}) = -\tfrac{1}{4G_4}\delta\mathcal{W}(\tilde{B}) \tag{4.3.38}$$

$$d = 5: \qquad \delta S^{ren}(\tilde{B}) = -\tfrac{1}{12G_6}\delta\mathcal{W}(\tilde{B}) \tag{4.3.39}$$

In Poincaré coordinates the linear variation of the Weyl tensor δW_{abcd} is,

$$\delta W_{\mu\nu\rho\sigma} = \frac{1}{z^2}\mathcal{R}[\eta+h]_{\mu\nu\rho\sigma} + \frac{1}{2z^3}(h'_{\mu\rho}\eta_{\nu\sigma} + h'_{\nu\sigma}\eta_{\mu\rho} - h'_{\mu\sigma}\eta_{\nu\rho} - h'_{\nu\rho}\eta_{\mu\sigma}) \tag{4.3.40}$$

$$\delta W_{\mu\nu\rho z} = \frac{1}{2z^2}[\partial_\mu h'_{\nu\rho} - \partial_\nu h'_{\mu\rho}] \tag{4.3.41}$$

$$\delta W_{\mu z\nu z} = -\frac{1}{2z^2}h''_{\mu\nu} + \frac{1}{2z^3}h'_{\mu\nu} \tag{4.3.42}$$

where $\mathcal{R}[\eta+h]_{\mu\nu\rho\sigma}$ is the Riemann tensor for boundary metric $\eta_{\mu\nu} + h_{\mu\nu}$. In Poincaré coordinates, the two unit normals are

$$n_1 = z\frac{\partial}{\partial z} \tag{4.3.43}$$

$$n_2 = \frac{z}{\sqrt{r^2+z^2}}\left(z\frac{\partial}{\partial z} + r\hat{x}^i\frac{\partial}{\partial x^i}\right) \tag{4.3.44}$$

Then the projection of Weyl tensor onto $N\tilde{B}$, δW_{1212}, is

$$\delta W_{1212} = \frac{z^4}{r^2+z^2}\left(z^2\delta W_{tztz} + r^2\hat{x}^i\hat{x}^j\delta W_{titj} + 2zr\hat{x}^i\delta W_{tzti}\right) \tag{4.3.45}$$

The bulk Weyl integral becomes

$$\int_{\tilde{B}} d^{d-1}x\sqrt{\gamma}\, W_{1212} = \int_{\xi}^{\frac{\pi}{2}} du \int_{S^{d-2}} d\Omega_{d-2} \frac{R^2 \cos^{d-2} u}{\sin^{d-5} u} \left(z^2 W_{tztz} + r^2 \hat{x}^i \hat{x}^j W_{titj} + 2zr\hat{x}^i W_{tzti}\right) \tag{4.3.46}$$

and the boundary Weyl integral is

$$\int_{\partial\tilde{B}} d^{d-2}x\sqrt{\tilde{\gamma}}\, W_{1212} = \int_{S^{d-2}} d\Omega_{d-2} \frac{R^2 \cos^{d-2} u}{\sin^{d-6} u} \left(z^2 W_{tztz} + r^2 \hat{x}^i \hat{x}^j W_{titj} + 2zr\hat{x}^i W_{tzti}\right) \tag{4.3.47}$$

Substituting (4.3.40) – (4.3.42) into the above integrals

$$\begin{aligned}\int_{\tilde{B}} d^{d-1}x\sqrt{\gamma}\, \delta W_{1212} = \int_{\xi}^{\frac{\pi}{2}} du \int_{S^{d-2}} d\Omega_{d-2} R^2 \bigg(& \frac{\cos^{d-2} u}{\sin^{d-5} u}\Big[-\frac{1}{2}h''_{tt} + \frac{1}{2R\sin u}h'_{tt}\Big] \\ & + \frac{\cos^{d} u}{\sin^{d-3} u}\hat{x}^i\hat{x}^j\Big[\mathcal{R}_{titj} + \frac{1}{2R\sin u}(h'_{tt}\eta_{ij} + h'_{ij}\eta_{tt})\Big] \\ & + \frac{\cos^{d-1} u}{\sin^{d-4} u}\hat{x}^i[\partial_t h'_{ti} - \partial_i h'_{tt}]\bigg)\end{aligned} \tag{4.3.48}$$

and

$$\begin{aligned}\int_{\tilde{B}} d^{d-2}x\sqrt{\tilde{\gamma}}\, \delta W_{1212} = \int_{S^{d-2}} d\Omega_{d-2} R^2 \bigg(& \frac{\cos^{d-2} u}{\sin^{d-6} u}\Big[-\frac{1}{2}h''_{tt} + \frac{1}{2R\sin u}h'_{tt}\Big] \\ & + \frac{\cos^{d} u}{\sin^{d-4} u}\hat{x}^i\hat{x}^j\Big[\mathcal{R}_{titj} + \frac{1}{2R\sin u}(h'_{tt}\eta_{ij} + h'_{ij}\eta_{tt})\Big] \\ & + \frac{\cos^{d-1} u}{\sin^{d-5} u}\hat{x}^i[\partial_t h'_{ti} - \partial_i h'_{tt}]\bigg)\end{aligned} \tag{4.3.49}$$

where $' = \frac{\partial}{\partial z}$ Up to order R^d, the relevant components of the integrand are obtained by Taylor expanding about the origin and eliminating the odd components as the it is integrated over S^{d-2}. In appendix 4.A.4, we expand $\mathcal{R}_{titj}$ into linear perturbation $h_{\mu\nu}$ then further relate the higher order non-normalizable modes $h_{\mu\nu}^{(n<d)}$ to the lower order non-normalizable modes via the Einstein equation. Finally, we can see all the lower order non-normalizable modes perturbation are cancelled and the renormalized Weyl integral is

$$d = 3: \qquad \delta\mathcal{W} = -\frac{3R^3\Omega_1}{16G_4}h_{tt}^{(3)} \tag{4.3.50}$$

$$d = 5: \qquad \delta\mathcal{W} = -\frac{5R^5\Omega_3}{16G_6}h_{tt}^{(5)}. \tag{4.3.51}$$

Then substituting (4.3.50), (4.3.51) into (4.3.38), (4.3.39) to get the renormalized entanglement entropy. We recovered the infinitesimal first law of entanglement entropy in (4.3.22) for variation that includes perturbation of non-normalizable modes,

$$d = 3: \qquad \delta S_B^{ren} = \frac{3R^3\Omega_1}{48G_4} h_{tt}^{(3)} \tag{4.3.52}$$

$$d = 5: \qquad \delta S_B^{ren} = \frac{5R^5\Omega_3}{192G_6} h_{tt}^{(5)}. \tag{4.3.53}$$

Again the fact that the non-normalizable modes do not contribute indicates the finite contribution of the entanglement entropy in odd d is universal and scheme independent.

4.3.5 Cancellation of divergences in $d = 4$

We now turn from odd dimensional boundaries to even dimensions and show how the cancellation of divergences of the renormalized entanglement entropy works in $d = 4$. A general perturbation of the boundary metric $h_{\mu\nu}$ can be expanded around the boundary $z = 0$

$$h_{\mu\nu} = h_{\mu\nu}^{(0)}(r,\theta,\phi) + z^2 h_{\mu\nu}^{(2)}(r,\theta,\phi) + \cdots \tag{4.3.54}$$

Since on $\tilde{B}$ the coordinate r is a function of z. The coefficient in the expansion of the metric perturbation can be further expanded around $r = R$. For $h_{\mu\nu}^{(0)}(r,\theta,\phi)$ the expansion is

$$\begin{aligned} h_{\mu\nu}^{(0)}(r,\theta,\phi) &= h_{\mu\nu}^{(0)}(R,\theta,\phi) + (r-R)\partial_r h_{\mu\nu}^{(0)}(R,\theta,\phi) + \cdots & (4.3.55) \\ &= h_{\mu\nu}^{(0)}(R,\theta,\phi) - \frac{z^2}{2R}\partial_r h_{\mu\nu}^{(0)}(R,\theta,\phi) + \cdots & (4.3.56) \end{aligned}$$

So the variation of the regularized entanglement entropy in polar coordinates for $d = 4$ is,

$$\begin{aligned} \delta S_B^{reg} =& \frac{1}{8G_5}\int_\epsilon^R dz \int_{S^2} d\Omega_2 \Big[\frac{1}{z^3}\Big(h_{\theta\theta}^{(0)} + \frac{1}{\sin^2\theta}h_{\Phi\Phi}^{(0)}\Big) + \frac{1}{z}\Big(-\frac{1}{2R^2}h_{\theta\theta}^{(0)} - \frac{1}{2R^2\sin^2\theta}h_{\phi\phi}^{(0)} \\ &+ h_{rr}^{(0)} + \frac{1}{R^2}h_{\theta\theta}^{(0)} + \frac{1}{R^2\sin^2\theta}h_{\phi\phi}^{(0)} - \frac{1}{2R}\partial_r h_{\theta\theta}^{(0)} - \frac{1}{2R\sin^2\theta}\partial_r h_{\phi\phi}^{(0)} + h_{\theta\theta}^{(2)} + \frac{1}{\sin^2\theta}h_{\phi\phi}^{(2)}\Big)\Big] \end{aligned} \tag{4.3.57}$$

Evaluating the z integral and the divergent terms are,

$$\begin{aligned} (\delta S_B^{reg})^{div} =& \frac{1}{8G_5}\int_{S^2} d\Omega_2 \Big[\frac{1}{2\epsilon^2}\Big(h_{\theta\theta}^{(0)} + \frac{1}{\sin^2\theta}h_{\Phi\Phi}^{(0)}\Big) - \ln\epsilon\Big(-\frac{1}{2R^2}h_{\theta\theta}^{(0)} - \frac{1}{2R^2\sin^2\theta}h_{\phi\phi}^{(0)} \\ &+ h_{rr}^{(0)} + \frac{1}{R^2}h_{\theta\theta}^{(0)} + \frac{1}{R^2\sin^2\theta}h_{\phi\phi}^{(0)} - \frac{1}{2R}\partial_r h_{\theta\theta}^{(0)} - \frac{1}{2R\sin^2\theta}\partial_r h_{\phi\phi}^{(0)} + h_{\theta\theta}^{(2)} + \frac{1}{\sin^2\theta}h_{\phi\phi}^{(2)}\Big)\Big] \end{aligned} \tag{4.3.58}$$

We can see that (4.3.58) is identical to (4.A.83) so the divergences of the regularized entanglement entropy will be removed by the counterterms in the renormalization procedure.

$$(\delta S_B^{reg})^{div} = (\delta S_B^{ct})^{div}. \tag{4.3.59}$$

More explicitly, in Cartesian coordinate, the set of counterterms from (4.2.7) is

$$\delta S_B^{ct} = \frac{1}{16G_5}\int_{S^2} d\Omega_2 \Big[(h_{ii} - \hat{x}^i\hat{x}^j h_{ij})(\frac{r^2}{\epsilon^2} + \ln\epsilon) + \ln\epsilon\Big(- \delta\mathcal{R}_{tt} + \delta\mathcal{R}_{rr} - \frac{2}{3}\delta\mathcal{R}\Big) - \ln\epsilon\Big(- 2h_{ii} + \frac{1}{r}\partial_r(r^2 h_{ii} - x^i x^j h_{ij})\Big)\Big]. \tag{4.3.60}$$

Following section 4.A.5 we get

$$\delta S_B^{ct} = \frac{1}{16G_5}\int_{S^2} d\Omega_2 \frac{r^2}{\epsilon^2}(h_{ii} - \hat{x}^i\hat{x}^j h_{ij}) + \ln\epsilon\Big[(h_{ii}^{(0)} - \hat{x}^i\hat{x}^j h_{ij}^{(0)}) - 2R^2(h_{ii}^{(2)} - \hat{x}^i\hat{x}^j h_{ij}^{(2)}) - \big(- r\hat{x}^k\partial_k h_{ii}^{(0)} + 2\hat{x}^i\hat{x}^j h_{ij}^{(0)} + r\hat{x}^i\hat{x}^j\hat{x}^k\partial_k h_{ij}^{(0)}\big)\Big]. \tag{4.3.61}$$

Note that there are finite contributions from the first term in (4.3.61).

4.4 Integral renormalized first law

Under general variations of the boundary metric where both the non-normalizable and normalizable modes are not fixed, we need to modify the relation between the conserved charges (4.2.34) and the associated first law. Since the spatial slice Σ where the charges are defined has a boundary, we cannot neglect the total derivative terms. In fact the boundary terms capture all the divergent behaviour of the Noether charge and act as counterterms.

As mentioned in the section 4.2.2, the asymptotic conformal Killing vector used to define the Noether charges in [91] has to follow the fall off condition (4.2.36) which our modular flow generator ξ_B in (4.2.51) does not satisfy. We shall see later that all these extra terms are essential to match the universal divergences of the entanglement entropy.

Charges defined on asymptotic boundary $\tilde{B}$ and entangling surface B have asymptotic behaviours analogous to the entanglement entropy. In $(d+1)$ even spacetime dimensions, the finite charges are universal. For $(d+1)$ odd spacetime dimensions, the finite parts are scheme dependent, and change covariantly under changes of scheme. Hence, the first law of entanglement entropy in odd bulk dimensions requires appropriate finite counterterms.

4.4.1 Charges in the entangling region

The Noether charge form $\boldsymbol{Q}[\xi]$ is the exact term in the conserved current form induced by the vector ξ. For pure Einstein gravity (with or without cosmological constant), it can be expressed as

$$\begin{aligned}\boldsymbol{Q}[\xi] &= -\frac{1}{16\pi G_N}\star d\xi \\ &= -\frac{1}{16\pi G_N}\boldsymbol{\varepsilon}_{ab}\nabla^a\xi^b,\end{aligned} \tag{4.4.1}$$

up to exact terms. Since the extra exact terms will introduce boundary terms in the integral over B and $\tilde{B}$ respectively, we will treat (4.4.1) as the definition of $\boldsymbol{Q}[\xi]$ to avoid confusion. In asymptotically locally AdS, the full expression for the Noether charge form is then written as

$$\boldsymbol{Q}^{full}[\xi] = \boldsymbol{Q}[\xi] - \iota_\xi \boldsymbol{B} \tag{4.4.2}$$

with $\boldsymbol{B}$ defined as

$$\boldsymbol{B} = -\frac{1}{8\pi G_N}\boldsymbol{\varepsilon}_a n^a \left(K_{(d)} + \lambda_{ct}\right) \tag{4.4.3}$$

where n is the radial unit normal pointing outwards from the asymptotic boundary $\partial\mathcal{M}$.

The holographic charge form $\boldsymbol{\mathcal{Q}}[\xi]$ is defined in terms of the d^{th} term in the dilatation eigenfunction expansion of the canonical momentum, $\pi^{bc}_{(d)}$, through the following expression

$$\boldsymbol{\mathcal{Q}}[\xi] = -\boldsymbol{\varepsilon}_{ab} n^a 2\pi^{bc}_{(d)}\xi_c. \tag{4.4.4}$$

In our setup, the full Noether current form $\boldsymbol{J}^{full}[\xi_B]$ is induced by the bulk modular flow of a bulk Killing vector ξ_B. The full Noether charge on the spatial slice Σ_ϵ can be thought of as the charge captured by the surface from the current:

$$\begin{aligned} Q^{full}[\xi_B] &= \int_{\Sigma_\epsilon} \boldsymbol{J}^{full}[\xi_B] \\ &= \int_{\Sigma_\epsilon} \boldsymbol{\Theta}[\delta_{\xi_B}\phi] - \iota_{\xi_B}\boldsymbol{L}^{onshell} \\ &= -\int_{\Sigma_\epsilon} \iota_{\xi_B}\boldsymbol{L}^{onshell} \end{aligned} \tag{4.4.5}$$

As shown in (4.2.31), the onshell Noether current form is exact.

In (4.2.54, 4.2.55), we defined the bulk entanglement entropy by an integral of the Noether charge form over $\tilde{B}_\epsilon$ and the modular energy through an integral of the holographic charge form over B_ϵ. In order to relate the two we need to generalize (4.2.34) to

$$\int_{B_\epsilon} \boldsymbol{Q}^{full}[\xi] - \boldsymbol{\Delta}[\xi] = -\int_{B_\epsilon} \boldsymbol{\mathcal{Q}}[\xi]. \tag{4.4.6}$$

Here $\boldsymbol{\Delta}$ captures the counterterms associated with renormalizing the divergences of $\boldsymbol{Q}^{full}$; this term is needed as the quantity on the righthandside, $\boldsymbol{\mathcal{Q}}$, is renormalized. We could redefine the Noether charge on the lefthandside to include these counterterms, but in what follows we keep track of the contributions separately to emphasise how the counterterm contributions arise.

The counterterms need to be included here because of our more general falloff conditions for the perturbations. This contribution vanishes in [91] because of the stricter fall-off condition of ξ which makes the radial derivative of ξ vanishes and the counterterms integrate to zero as the integral is over a surface with no boundary. In [57] this term vanishes due to the falloff conditions imposed on the metric perturbations.

The conserved charge forms $\boldsymbol{Q}^{full}$ and $\boldsymbol{\mathcal{Q}}$ can be interpreted as Hamiltonian potentials, as explained in detail in appendix 4.C. $\boldsymbol{\Delta}$ in this context is the difference of the counterterm contributions of the two Hamiltonian potentials. In the covariant phase space formalism, given an action with boundary terms, one can obtain the presymplectic current through variation of the Lagrangian and boundary terms. The presymplectic current maps the vector field in the configuration space to the Hamiltonian potential.

We are interested in renormalized quantities and there are two ways to see how the counterterms arise in the Hamiltonian potential. The first approach is to use the renormalized action that includes the counterterms, and then obtain the presymplectic current and Hamiltonian potential. We denote this as the full Hamiltonian potential because it is equal to the full Noether charge form when $\pi^{\mu\nu}_{(d)} = 0$

$$\boldsymbol{H}^{full}[\xi] = \boldsymbol{Q}[\xi] + \boldsymbol{b}^{GH}[\xi] - \boldsymbol{b}^{ct}[\xi] \tag{4.4.7}$$
$$= \boldsymbol{H}^{GH}[\xi] - \boldsymbol{b}^{ct}[\xi] \tag{4.4.8}$$
$$= \boldsymbol{Q}^{full}[\xi], \tag{4.4.9}$$

where $\boldsymbol{b}^{GH}$ and $\boldsymbol{b}^{ct}$ represent the Gibbon-Hawking boundary term and counterterm contribution. Here $\boldsymbol{H}^{GH}$ is Hamiltonian potential obtained from the action that only includes the Gibbon-Hawking boundary term.

The second way to see how the counterterms arise is to renormalize the Gibbon-Hawking Hamiltonian potential $\boldsymbol{H}^{GH}$ directly by subtracting the lower order terms in the dilatation eigenvalue expansion. The renormalized Gibbon-Hawking Hamiltonian potential is given in terms of the holographic charge form when $\pi^{\mu\nu}_{(d)} = 0$

$$\boldsymbol{H}^{GH}_{(d)}[\xi] = \boldsymbol{H}^{GH}[\xi] - \boldsymbol{H}^{GH}_{ct}[\xi] \tag{4.4.10}$$
$$= -\boldsymbol{\mathcal{Q}}[\xi]. \tag{4.4.11}$$

Hence we can interpret $\boldsymbol{\Delta}$ as the difference of the two aforementioned Hamiltonian potentials

$$\boldsymbol{\Delta}[\xi] = \boldsymbol{H}^{GH}_{ct}[\xi] - \boldsymbol{b}^{ct}[\xi] \tag{4.4.12}$$

As we will see this term in the perturbed case this is exact and represents the counterterms for the entanglement entropy.

On B_ϵ, for bulk Killing vector, ξ_B the full Noether charge form is

$$\boldsymbol{Q}^{full}[\xi_B]|_{B\epsilon} = -\frac{1}{8\pi G_N}\boldsymbol{\varepsilon}_{zt}n^z\left(-z\partial_z\xi^t_B + K^t_\mu\xi^\mu_B - K_{(d)}\xi^t_B - \lambda_{ct}\xi^t_B\right) \tag{4.4.13}$$

where the first term on the right hand side was neglected in [91] due to the falloff condition (4.2.36). The holographic charge form is

$$\begin{aligned}\boldsymbol{\mathcal{Q}}[\xi_B]|_{B\epsilon} &= -\boldsymbol{\varepsilon}_{zt}n^z 2\pi^{tc}_{(d)}\xi_{B\,c}\\ &= \frac{1}{8\pi G_N}\boldsymbol{\varepsilon}_{zt}n^z\left(K^{ta}_{(d)} - K_{(d)}\gamma^{ta}\right)\xi_{B\,a}\end{aligned} \tag{4.4.14}$$

where we used (4.2.24) to express the holographic charge in terms of the d^{th} term in the dilatation eigenfunction expansion of the extrinsic curvature. The difference in the charges $\boldsymbol{\Delta}[\xi_B]$ is

$$\boldsymbol{\Delta}[\xi_B]|_{B\epsilon} = -\frac{1}{8\pi G_N}\boldsymbol{\varepsilon}_{zt}n^z\left(-z\partial_z\xi^t_B + K_{ct\,\mu}^{\ \ t}\xi^\mu_B - \lambda_{ct}\xi^t_B\right). \tag{4.4.15}$$

It is important to remember that this expression is only valid when ξ_B is Killing. We shall see later the perturbed difference of the charges $\delta\Delta[\xi_B]$ admits an extra term as ξ_B is no longer Killing. In appendix 4.B.1, we follow [92] and derive the explicit dilatation eigenfunction expansion for K^μ_ν and λ. In the unperturbed setting, the boundary metric is d-dimensional Minkowski metric $\eta_{\mu\nu}$. Only the zeroth term in the dilatation eigenfunction expansion is non-vanishing,

$$K_{(0)\,\nu}^{\ \ \mu} = \delta^\mu_\nu, \qquad\qquad \lambda_{(0)} = 1. \tag{4.4.16}$$

From (4.4.14) we know the holographic charge is zero

$$\boldsymbol{\mathcal{Q}}[\xi_B]|_{B\epsilon} = 0. \tag{4.4.17}$$

Then $\boldsymbol{\Delta}[\xi_B]$ simply equals the Noether charge

$$\begin{aligned}\boldsymbol{\Delta}[\xi_B]|_{B\epsilon} = \boldsymbol{Q}[\xi_B]|_{B\epsilon} &= -\frac{1}{8\pi G_N}\boldsymbol{\varepsilon}_{zt}n^z\left(-z\partial_z\xi^t_B\right)\\ &= \frac{z^3}{4G_N}\boldsymbol{\varepsilon}_{zt}\end{aligned} \tag{4.4.18}$$

We can now turn our attention to the Noether charge form on the entangling surface $\tilde{B}\epsilon$. As explained in section 4.2.3, the integral of the Noether charge form over $\tilde{B}_\epsilon$ can be interpreted as both the entropy of the Rindler black hole and the entanglement entropy of boundary region B_ϵ. Since ξ_B vanishes on $\tilde{B}_\epsilon$,

$$\begin{aligned}\boldsymbol{Q}^{full}[\xi_B]|_{\tilde{B}_\epsilon} &= \boldsymbol{Q}[\xi_B]|_{\tilde{B}_\epsilon} \\ &= \frac{1}{8\pi G_N}\boldsymbol{\varepsilon}_{wt}\partial^w\xi_B^t \end{aligned} \tag{4.4.19}$$

where we used the w coordinate in (4.3.4) and the Killing condition. Integrating over $\tilde{B}_\epsilon$,

$$\begin{aligned}\int_{B_\epsilon}\boldsymbol{Q}^{full}[\xi_B] &= \frac{1}{8\pi G_N}\int_{B_\epsilon} du\, d\Omega_{d-2}\frac{\cos^{d-2}u}{\sin^{d-1}u}\frac{2\pi}{w}\left(z^2+r^2\right) \\ &= \frac{R}{4G_N}\int_{B_\epsilon} du\, d\Omega_{d-2}\frac{\cos^{d-2}u}{\sin^{d-1}u} \\ &= S_B^{reg} \end{aligned} \tag{4.4.20}$$

where we use (4.3.8) to identify the second line with the regulated entanglement entropy.

4.4.2 Variation of charges

The variations of $\boldsymbol{Q}^{full}$ and $\boldsymbol{\mathcal{Q}}$ differ from the previous literature [57] when we allow variations of the non-normalizable modes. For general perturbations of $\gamma_{\mu\nu}$, the linear variation of the Noether charge form is

$$\begin{aligned}\delta\boldsymbol{Q}[\xi_B] &= \frac{-1}{16\pi G_N}\delta\left[\boldsymbol{\varepsilon}_{ab}\nabla^a\xi_B^b\right] \\ &= \frac{-1}{16\pi G_N}\boldsymbol{\varepsilon}_{ab}\left[\frac{\delta\gamma}{2}\nabla^a\xi_B^b - \delta g^{ac}\nabla_c\xi_B^b + g^{ac}\delta\Gamma^b_{dc}\xi_B^d\right] \\ &= \frac{-z^2}{8RG_N}\Bigg[\boldsymbol{\varepsilon}_{ti}\left(x^i h_{kk} - x^j h_{ij} - (R^2-z^2-\vec{x}^2)\partial_t h_{it}\right) \\ &\qquad + \boldsymbol{\varepsilon}_{tz}\left(z h_{kk} + (R^2-z^2-\vec{x}^2)(-\frac{2}{z}h_{tt}+\partial_z h_{tt})\right)\Bigg] \end{aligned} \tag{4.4.21}$$

Using the coordinates (4.3.4) on $\tilde{B}_\epsilon$ we get the integral

$$\begin{aligned}\int_{\tilde{B}_\epsilon}\delta\boldsymbol{Q}[\xi_B] &= \frac{1}{8RG_N}\int_{\tilde{B}_\epsilon} d^{d-1}x\frac{1}{z^d}(R^2h_{kk} - x^ix^jh_{ij}) \\ &= \frac{1}{8RG_N}\int_{\tilde{B}_\epsilon} d^{d-1}x(R^2-\vec{x}^2)^{-\frac{d}{2}}(R^2h_{kk} - x^ix^jh_{ij}) \end{aligned} \tag{4.4.22}$$

which is equal to the linear variation of the holographic entanglement entropy

$$\int_{\tilde{B}_\epsilon}\delta\boldsymbol{Q}[\xi_B] = \delta S_B^{reg} \tag{4.4.23}$$

For the variation of the full Noether charge form, we need to evaluate the boundary term $\delta\boldsymbol{B}$. This term is related to the presymplectic form $\boldsymbol{\Theta}[\delta\phi]$ by

$$\begin{aligned}\boldsymbol{\Theta}[\delta\phi] &= \frac{\boldsymbol{d^d x}}{8\pi G_N}\delta\left(-\sqrt{\gamma}\lambda\right) \\ &= \boldsymbol{d^d x}\left[\frac{1}{8\pi G_N}\delta\left(-\sqrt{\gamma}K\right) + \pi^{\mu\nu}\delta\gamma_{\mu\nu}\right] \\ &= \boldsymbol{d^d x}\left[\frac{1}{8\pi G_N}\delta\left(-\sqrt{\gamma}\left(K_{(d)} + \lambda_{ct}\right)\right) + \pi^{\mu\nu}_{(d)}\delta\gamma_{\mu\nu}\right] \\ &= \delta\boldsymbol{B} + \boldsymbol{\varepsilon}_{\partial\mathcal{M}_\epsilon}\pi^{\mu\nu}_{(d)}\delta\gamma_{\mu\nu}.\end{aligned} \tag{4.4.24}$$

where the d-form $\boldsymbol{d^d x}$ is

$$\boldsymbol{d^d x} = \frac{1}{d!}dx^0 \wedge \cdots \wedge dx^{d-1}. \tag{4.4.25}$$

The variation of the full Noether charge form is then

$$\begin{aligned}\delta\boldsymbol{Q}^{full}[\xi_B] &= \delta\boldsymbol{Q}[\xi_B] - \iota_{\xi_B}\delta\boldsymbol{B} \\ &= \delta\boldsymbol{Q}[\xi_B] - \iota_{\xi_B}\boldsymbol{\Theta}[\delta\phi] - \iota_{\xi_B}\boldsymbol{\varepsilon}_{\partial\mathcal{M}_\epsilon}\pi^{\mu\nu}_{(d)}\delta\gamma_{\mu\nu}.\end{aligned} \tag{4.4.26}$$

The linear variation of the holographic charge is

$$\delta\boldsymbol{\mathcal{Q}}[\xi]|_{B_\epsilon} = -\boldsymbol{\varepsilon}_{zt}n^z 2\delta\pi_{(d)\,t}^{\;t}\xi_B^t. \tag{4.4.27}$$

This is related to the renormalized boundary energy momentum tensor via

$$2\delta\pi^{\mu\nu}_{(d)} = -\delta T^{\mu\nu}_{ren}. \tag{4.4.28}$$

Substituting this expression into (4.4.27) the integral of the variation of holographic charge form $\delta\boldsymbol{\mathcal{Q}}[\xi_B]$ on the boundary ball region B_ϵ is equal to the variation of modular energy

$$-\int_{B_\epsilon}\delta\boldsymbol{\mathcal{Q}}[\xi_B] = \delta E_B. \tag{4.4.29}$$

To express the variation of modular energy in terms of dilatation eigenfunction expansion of extrinsic curvature we vary (4.2.23) to obtain

$$\delta\pi_{(d)\,\nu}^{\;\mu} = \frac{-1}{16\pi G_N}\left(\delta K_{(d)\,\nu}^{\;\mu} - \delta K_{(d)}\delta^\mu_\nu\right) \tag{4.4.30}$$

$$\delta\pi_{(d)\,t}^{\;t} = \frac{1}{16\pi G_N}\delta K_{(d)\,i}^{\;i}. \tag{4.4.31}$$

Using the tracelessness of $\delta K_{(d)\,\nu}^{\;\mu}$ at the linear level we can write $\delta\boldsymbol{\mathcal{Q}}[\xi]$ on B_ϵ as

$$\delta\boldsymbol{\mathcal{Q}}[\xi_B]|_{B_\epsilon} = \frac{1}{8\pi G_N}\boldsymbol{\varepsilon}_{zt}n^z\delta K_{(d)\,t}^{\;t}\xi_B^t. \tag{4.4.32}$$

(This expression holds for all d, with conformal anomalies present if we write out $K_{(d)}{}^{t}{}_{t}$ in terms of $g^{(n)}_{\mu\nu}$ and $\tilde{g}^{(d)}_{\mu\nu}$, see for example (4.4.65).) The variation of the full Noether charge form $\delta \boldsymbol{Q}^{full}$ on B_ϵ is

$$\begin{aligned}\delta \boldsymbol{Q}^{full}[\xi_B]|_{B\epsilon} = \frac{z}{8\pi G_N}\boldsymbol{\varepsilon}_{zt}\Big[&\left(\frac{\delta\gamma}{2}(K^t_\mu \xi^\mu_B - \frac{1}{z}\partial^z \xi^t_B) - \frac{1}{2z}\partial_t \xi^z_B \delta\gamma^{tt} + \xi^t_B \delta K^t_t\right) \\ &- \xi^t_B \left(\frac{\delta\gamma}{2} + \delta K_{(d)} + \delta\lambda_{ct}\right)\Big].\end{aligned} \tag{4.4.33}$$

By using (4.4.30), we can obtain the relation between $\delta \boldsymbol{Q}^{full}$ and $\delta \boldsymbol{\mathcal{Q}}$.

Similarly to (4.4.6), this revised version of (4.2.58) receives a contribution $\delta\boldsymbol{\Delta}[\xi_B]$. We get

$$\int_{B_\epsilon} \delta \boldsymbol{Q}^{full}[\xi_B] = \int_{B_\epsilon} -\delta \boldsymbol{\mathcal{Q}}[\xi_B] + \delta\boldsymbol{\Delta}[\xi_B]. \tag{4.4.34}$$

The latter term takes the form

$$\delta\boldsymbol{\Delta}[\xi_B] = \frac{z}{8\pi G_N}\boldsymbol{\varepsilon}_{zt}\left[-\frac{1}{2z}\partial^z \xi^t_B \delta\gamma - \frac{1}{2z}\partial_t \xi^z_B \delta\gamma^{tt} + \xi^t_B \left(\delta K^t_t - \delta\lambda\right)_{ct}\right]. \tag{4.4.35}$$

Note that we can understand why this term arises for two reasons. Firstly, ξ_B is no longer Killing with respect to the perturbed metric and secondly ξ_B has a weaker falloff condition (4.2.52) instead of (4.2.36). Here we use an abbreviated notation:

$$\delta\gamma = \gamma^{\mu\nu}\delta\gamma_{\mu\nu}, \qquad \delta K^\mu_\nu = \delta\left(\gamma^{\mu\sigma}K_{\sigma\nu}\right). \tag{4.4.36}$$

In terms of Hamiltonian potentials, the $\delta\boldsymbol{\Delta}$ term is

$$\delta\boldsymbol{\Delta}[\xi_B] = \delta \boldsymbol{H}^{GH}_{ct}[\xi_B] - \delta \boldsymbol{b}^{ct}[\xi_B]. \tag{4.4.37}$$

We further describe the origin of each term in the appendix 4.C and expressed $\delta\boldsymbol{\Delta}$ in (4.C.33) using the formalism of [103].

Substituting the unperturbed flat boundary metric and the bulk Killing vector we obtain

$$\begin{aligned}\delta\boldsymbol{\Delta}[\xi_B] &= \frac{d^{d-1}x\sqrt{-\gamma}}{8\pi G_N}\left[\frac{\pi z^2}{R}(-h_{tt}+h_{ii}) + \frac{\pi z^2}{R}h_{tt} + \frac{\pi(R^2-z^2-\vec{x}^2)}{R}\left(\delta K^t_t - \delta\lambda\right)_{ct}\right] \\ \delta\boldsymbol{\Delta}[\xi_B] &= \frac{d^{d-1}x\, z^{-d}}{8RG_N}\left[z^2 h_{ii} + (R^2-z^2-\vec{x}^2)\left(\delta K^t_t - \delta\lambda\right)_{ct}\right].\end{aligned} \tag{4.4.38}$$

The variation of the onshell boundary Lagrangian, $\delta\lambda$, is related to the variation of the extrinsic curvature, δK, via the canonical momentum in (4.2.23)

$$\begin{aligned} -\frac{1}{8\pi G_N}\delta\left[\sqrt{\gamma}\lambda\right] &= -\frac{1}{8\pi G_N}\delta\left[\sqrt{\gamma}K\right] + \pi^{\mu\nu}\delta\gamma_{\mu\nu} \\ -\frac{\sqrt{\gamma}}{8\pi G_N}\left(\frac{\lambda}{2}\delta\gamma + \delta\lambda\right) &= -\frac{\sqrt{\gamma}}{8\pi G_N}\left(\frac{K}{2}\delta\gamma + \delta K + \frac{K^{\mu\nu} - K\gamma^{\mu\nu}}{2}\delta\gamma_{\mu\nu}\right) \\ \frac{\lambda}{2}\delta\gamma + \delta\lambda &= \delta K + \frac{K^{\mu\nu}}{2}\delta\gamma_{\mu\nu}. \end{aligned} \tag{4.4.39}$$

For flat boundary metrics we have

$$\delta\lambda = \delta K. \tag{4.4.40}$$

We then get the following simplified expression for all dimension

$$\begin{aligned} \delta\boldsymbol{\Delta}[\xi_B] &= \frac{z^{-d}}{8RG_N}\left[z^2 h_{ii} + (R^2 - z^2 - \vec{x}^2)\left(\delta K^t_t - \delta K\right)_{ct}\right] \\ &= \frac{z^{-d}}{8RG_N}\left[z^2 h_{ii} - (R^2 - z^2 - \vec{x}^2)\delta K_{ct}{}^i_i\right]. \end{aligned} \tag{4.4.41}$$

The extrinsic curvature counterterm means that all the terms appear earlier in the dilatation eigenfunction expansion, i.e.

$$\begin{aligned} \delta K_{\mu\nu} &= \delta K_{(0)\,\mu\nu} + \delta K_{(2)\,\mu\nu} + \cdots + \log z^2 \delta\tilde{K}_{(d)\,\mu\nu} + \delta K_{(d)\,\mu\nu} + \cdots \\ \delta K_{\mu\nu} &= \delta K_{ct\,\mu\nu} + \delta K_{(d)\,\mu\nu} + \cdots \end{aligned} \tag{4.4.42}$$

From (4.4.34) we can deduce that the divergence of the full Noether charge integral is equal to the divergence of the correction term integral,

$$\left(\int_{B_\epsilon} \delta\boldsymbol{Q}^{full}[\xi_B]\right)^{div} = \int_{B_\epsilon} \delta\boldsymbol{\Delta}^{div}[\xi_B]. \tag{4.4.43}$$

In order to see how this divergence is equivalent to the divergence in the entanglement entropy, we need to use the Stoke's theorem of the full Noether charge form on Σ_ϵ,

$$\int_{B_\epsilon} \delta\boldsymbol{Q}^{full}[\xi_B] - \int_{\tilde{B}_\epsilon} \delta\boldsymbol{Q}^{full}[\xi_B] = \int_{\Sigma_\epsilon} d\delta\boldsymbol{Q}^{full}[\xi_B]. \tag{4.4.44}$$

The exterior derivative of the variation of Noether charge form can be deduced from

(4.2.29) and (4.2.31),

$$\begin{aligned}
d\delta \boldsymbol{Q}[\xi_B] &= \delta \boldsymbol{J} - \delta \boldsymbol{N} &(4.4.45)\\
&= \delta\boldsymbol{\Theta}[\delta_{\xi_B}\psi] - \iota_{\xi_B}\delta\boldsymbol{L} + \delta\boldsymbol{N}\\
&= \delta\boldsymbol{\Theta}[\delta_{\xi_B}\psi] - \iota_{\xi_B} d\delta\boldsymbol{\Theta}[\delta\psi] - \iota_{\xi_B}\boldsymbol{E}^{\psi}\delta\psi + \delta\boldsymbol{N}\\
&= \delta\boldsymbol{\Theta}[\delta_{\xi_B}\psi] - \mathcal{L}_{\xi_B}\boldsymbol{\Theta}[\delta_{\xi_B}\phi] + d\iota_{\xi_B}\delta\boldsymbol{\Theta}[\delta\phi] - \iota_{\xi_B}\boldsymbol{E}^{\psi}\delta\phi + \delta\boldsymbol{N}\\
&= \boldsymbol{\omega}(\delta\psi, \delta_{\xi_B}\psi) + d\iota_{\xi_B}\delta\boldsymbol{\Theta}[\delta\psi] - \iota_{\xi_B}\boldsymbol{E}^{\phi}\delta\psi + \delta\boldsymbol{N}\\
&= d\iota_{\xi_B}\delta\boldsymbol{\Theta}[\delta\psi] - \iota_{\xi_B}\boldsymbol{E}^{\phi}\delta\psi + \delta\boldsymbol{N} &(4.4.46)
\end{aligned}$$

where $\boldsymbol{\omega}(\delta_1\psi, \delta_2\psi)$ is the symplectic form

$$\boldsymbol{\omega}(\delta_1\psi, \delta_2\psi) = \delta_2\boldsymbol{\Theta}[\delta_1\psi] - \delta_1\boldsymbol{\Theta}[\delta_2\psi] \tag{4.4.47}$$

and it vanishes when ξ_B is Killing. Note the last two terms are off-shell terms. We first write out (4.4.34) as

$$\int_{B_\epsilon} \delta\boldsymbol{Q}[\xi_B] - \iota_{\xi_B}\boldsymbol{\Theta}[\delta\phi] = \int_{B_\epsilon} -\delta\boldsymbol{\mathcal{Q}}[\xi_B] + \delta\boldsymbol{\Delta}[\xi_B]. \tag{4.4.48}$$

Now substitute (4.4.46) and (4.4.48) into (4.4.44),

$$\begin{aligned}
\int_{B_\epsilon} \delta\boldsymbol{Q}^{full}[\xi_B] - \int_{\tilde{B}_\epsilon} \delta\boldsymbol{Q}^{full}[\xi_B] &= \int_{\Sigma_\epsilon} d\iota_{\xi_B}\delta\boldsymbol{\Theta}[\delta\phi] + \delta\boldsymbol{N} - \iota_{\xi_B}\boldsymbol{E}^{\phi}\delta\phi - d\iota_{\xi_B}\delta\boldsymbol{B}\\
\int_{B_\epsilon} \delta\boldsymbol{Q}[\xi_B] - \iota_{\xi_B}\boldsymbol{\Theta}[\delta\phi] - \int_{\tilde{B}_\epsilon} \delta\boldsymbol{Q}[\xi_B] &= \int_{\Sigma_\epsilon} -\iota_{\xi_B}\boldsymbol{E}^{\phi}\delta\phi + \delta\boldsymbol{N}\\
\int_{B_\epsilon} -\delta\boldsymbol{\mathcal{Q}}[\xi_B] + \delta\boldsymbol{\Delta}[\xi_B] &= \int_{\tilde{B}_\epsilon} \delta\boldsymbol{Q}[\xi_B] + \int_{\Sigma_\epsilon} -\iota_{\xi_B}\boldsymbol{E}^{\phi}\delta\phi + \delta\boldsymbol{N}. &(4.4.49)
\end{aligned}$$

Onshell we get

$$\int_{B_\epsilon} -\delta\boldsymbol{\mathcal{Q}}[\xi_B] = \int_{\tilde{B}_\epsilon} \delta\boldsymbol{Q}[\xi_B] - \int_{B_\epsilon} \delta\boldsymbol{\Delta}[\xi_B]. \tag{4.4.50}$$

Since the left hand side is manifestly finite we have

$$\begin{aligned}
\left(\int_{\tilde{B}_\epsilon} \delta\boldsymbol{Q}[\xi_B]\right)^{div} &= \int_{B_\epsilon} \delta\boldsymbol{\Delta}^{div}[\xi_B] &(4.4.51)\\
\delta S_B^{div} &= \int_{B_\epsilon} \delta\boldsymbol{\Delta}^{div}[\xi_B] &(4.4.52)
\end{aligned}$$

Therefore the integral of $\delta\boldsymbol{\Delta}$ on the boundary ball region can be thought of as the counterterm of the entanglement entropy. In the next section we will show that the finite part of $\delta\boldsymbol{\Delta}$ matches with the counterterm of the entanglement entropy as well. Hence we get the integral first law of entanglement entropy:

$$\delta E_B = \delta S_B^{ren}. \tag{4.4.53}$$

Finite counterterms contribute only when the CFT dimension is even. This is an expected result as the finite part of the entanglement entropy is scheme dependent in even d. Similarly, the left hand side is related to the renormalized energy momentum tensor which is also scheme dependent for even d. For odd d, the finite part of the renormalized entanglement entropy is universal We will see explicit examples in the following section.

The implication of $\delta\boldsymbol{\Delta}$ acting as the density of the entanglement entropy counterterms is that $\delta\boldsymbol{\Delta}$ is exact,

$$\delta\boldsymbol{\Delta}[\xi_B] = d\delta\boldsymbol{S}_B^{ct}, \tag{4.4.54}$$

where $\delta\boldsymbol{S}_B^{ct}$ is the $(d-2)$-form that integrates to the entanglement entropy counterterm. This means the full Hamiltonian potential $\delta\boldsymbol{H}^{full}$ and the renormalized Gibbon-Hawking Hamiltonian potential $\delta\boldsymbol{H}_{(d)}^{GH}$ is equal up to an exact term. In the usual context of conserved quantities, this is attributed to the exact term ambiguity. Since the potential is integrated over a boundary manifold, which itself does not have boundary, the exact term ambiguity will not contribute to the conserved charge. For us, both the entanglement entropy and modular energy is defined as the integral of a manifold that does have boundary, so the exact term difference is no longer an ambiguity. Note the counterterm of the entanglement entropy is obtained systemically from the renormalized action through the replica trick, this indicates this exact term difference can be calculated from the renormalized action directly. In the Hamiltonian holographic renormalization framework, we show how to obtain $\delta\boldsymbol{\Delta}$ from the counterterms contribution of the Hamiltonian potentials in appendix 4.C.

4.4.3 Examples: generalized first law in $AlAdS_{d+1}$

We have shown that the variation of the modular energy δE_B is equal to the integral of the holographic charge form over the boundary ball region B_ϵ in (4.4.29) and the variation of the entanglement entropy δS_B is equal to the integral of the Noether charge form over the bulk entangling surface $\tilde{B}_\epsilon$ in (4.4.23). To complete the generalized first law of entanglement entropy (4.4.53) for generic variations of the boundary metric $\delta\gamma_{\mu\nu}$ in $AlAdS_{d+1}$, we only need to check that the integral of the term $\delta\boldsymbol{\Delta}[\xi_B]$ over B_ϵ is the counterterm of the entanglement entropy,

$$\int_{B_\epsilon} \delta\boldsymbol{\Delta}[\xi_B] = \delta S_B^{ct}. \tag{4.4.55}$$

In the following subsections, we will demonstrate this equality up to dimension $d = 5$, thus implying the renormalized first law (4.4.53), with scheme dependence of renormalized entropy and energy systematically matched.

4.4.3.1 $d = 3$

The terms in the dilatation eigenfunction expansion of the extrinsic curvature variation are related to the Fefferman-Graham expansion of the boundary metric variation. For $d = 3$, we only need to include terms up to $O(z^4)$ as higher order terms will not contribute to calculations in the limit $\epsilon \to 0$:

$$\delta K_{(0)}{}^{\mu}{}_{\nu} = 0 \tag{4.4.56}$$

$$\delta K_{(2)}{}^{\mu}{}_{\nu} = -z^2 \eta^{\mu\sigma} h_{(2)\,\sigma\nu} + O(z^4) \tag{4.4.57}$$

$$\delta K_{(3)}{}^{\mu}{}_{\nu} = -\frac{3}{2} z^3 \eta^{\mu\sigma} h_{(3)\,\sigma\nu} + O(z^4). \tag{4.4.58}$$

In this case, the counterterm $\delta K_{ct}{}^{\mu}{}_{\nu}$ is just the second term in the dilatation eigenfunction expansion $\delta K_{(2)}{}^{\mu}{}_{\nu}$. Hence the counterterm from (4.4.38) gives

$$\delta \mathbf{\Delta}[\xi_B] = \frac{\boldsymbol{d^2x} z^{-3}}{8RG_N} \left[z^2 h_{ii} - (R^2 - z^2 - \vec{x}^2) \left(-z^2 h_{(2)\,ii} \right) \right]. \tag{4.4.59}$$

Keeping the terms up to $O(z)$ we have,

$$\delta \mathbf{\Delta}[\xi_B] = \frac{\boldsymbol{d^2x}}{8RG_N} \left[\frac{1}{z} \left(h_{(0)\,ii} + (R^2 - \vec{x}^2) h_{(2)\,ii} \right) \right]. \tag{4.4.60}$$

We see that for this example in odd dimensions, $\delta \mathbf{\Delta}[\xi_B]$ has no term of order z^0 and there is as expected no finite counterterm contribution to the entanglement entropy.

To see the identification of the integral of $\delta \mathbf{\Delta}[\xi_B]$ over B_ϵ with the ordinary entanglement entropy counterterm in (4.3.27), we need to use (4.B.29) with the result

$$\int_{B_\epsilon} \delta \mathbf{\Delta}[\xi_B] = \frac{1}{8RG_N} \int_{S^1} d\Omega_1 \frac{r}{\epsilon} \left(h_{(0)\,ii} - \hat{x}^i \hat{x}^j h_{(0)\,ij} \right). \tag{4.4.61}$$

This matches with the variation in the counterterm (4.3.27) exactly. Hence we have satisfied (4.4.55) confirming that the general variation of the modular energy is the variation of the renormalized entanglement entropy.

4.4.3.2 $d = 4$

For $d = 4$, in addition to including the logarithmic term in dilatation eigenfunction expansion, we also have to include terms up to $O(z^6)$ to evaluate both the divergent and finite contributions:

$$\delta K_{(0)\,\nu}^{\ \ \mu} = 0 \tag{4.4.62}$$

$$\delta K_{(2)\,\nu}^{\ \ \mu} = -z^2\eta^{\mu\sigma}h_{(2)\,\sigma\nu} - z^2\eta^{\mu\sigma}\delta D_{(2)\,\sigma\nu} + O(z^6) \tag{4.4.63}$$

$$\delta\tilde{K}_{(4)\,\nu}^{\ \ \mu} = -2z^4\eta^{\mu\sigma}\tilde{h}_{(4)\,\sigma\nu} + O(z^6) \tag{4.4.64}$$

$$\delta K_{(4)\,\nu}^{\ \ \mu} = -2z^4\eta^{\mu\sigma}h_{(4)\,\sigma\nu} - z^4\eta^{\mu\sigma}\tilde{h}_{(4)\,\sigma\nu} + z^4\eta^{\mu\sigma}\delta D_{(2)\,\sigma\nu} + O(z^6) \tag{4.4.65}$$

where we use the notation

$$\delta D_{(n)\,\mu\nu} = \delta\left[\int g_{(n)\sigma\rho}\frac{g_{(n)\mu\nu}}{g_{(0)\sigma\rho}}\right]. \tag{4.4.66}$$

It turns out at linear level that the second order term $\delta D_{(2)\,\mu\nu}$ is related to the coefficient of the logarithmic term in the Fefferman-Graham expansion as

$$\delta D_{(2)\,\mu\nu} = -2\tilde{h}_{(4)\,\mu\nu}, \tag{4.4.67}$$

and hence it is also traceless. Then the relevant terms in the dilatation eigenfunction expansion for the extrinsic curvature are

$$\delta K_{(0)\,\nu}^{\ \ \mu} = 0 \tag{4.4.68}$$

$$\delta K_{(2)\,\nu}^{\ \ \mu} = -z^2\eta^{\mu\sigma}h_{(2)\,\sigma\nu} + 2z^4\eta^{\mu\sigma}\tilde{h}_{(4)\,\sigma\nu} + O(z^6) \tag{4.4.69}$$

$$\delta\tilde{K}_{(4)\,\nu}^{\ \ \mu} = -2z^4\eta^{\mu\sigma}\tilde{h}_{(4)\,\sigma\nu} + O(z^6) \tag{4.4.70}$$

$$\delta K_{(4)\,\nu}^{\ \ \mu} = -2z^4\eta^{\mu\sigma}h_{(4)\,\sigma\nu} - 3z^4\eta^{\mu\sigma}\tilde{h}_{(4)\,\sigma\nu} + O(z^6). \tag{4.4.71}$$

The counterterm from (4.4.38) is then

$$\delta\boldsymbol{\Delta}[\xi_B] = \frac{\boldsymbol{d^3}xz^{-4}}{8RG_N}\left[z^2h_{ii} - (R^2 - z^2 - \vec{x}^2)\left(-z^2h_{(2)\,ii} + 2z^4\tilde{h}_{(4)\,ii} - 2z^4\log z^2\tilde{h}_{(4)\,ii}\right)\right]. \tag{4.4.72}$$

Neglecting the $O(z)$ terms as they vanish in the limit $\epsilon \to 0$ we have

$$\delta\boldsymbol{\Delta}[\xi_B] = \frac{\boldsymbol{d^3x}}{8RG_N}\left[\frac{1}{z^2}\left(h_{(0)\,ii} - (R^2 - \vec{x}^2)h_{(2)\,ii}\right) + 2(R^2 - \vec{x}^2)(1 - \log z^2)\tilde{h}_{(4)\,ii}\right]. \tag{4.4.73}$$

Finally we need to transform this integral on boundary ball region B_ϵ into a surface integral on the sphere ∂B_ϵ via the manipulation of $h_{(n)\,\mu\nu}$ in appendix 4.B.2. First we use (4.B.29) to turn the coefficient of ϵ^{-2} divergences into a surface integral

$$\begin{aligned}\int_{B_\epsilon} \delta\boldsymbol{\Delta}[\xi_B] = \frac{1}{8RG_N}\Bigg[&\frac{(R^2-\epsilon^2)^{\frac{3}{2}}}{2\epsilon^2}\int_{S^2} d\Omega_2\,\left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) \\ &+ \int_{B_\epsilon} d^3x\, h_{(2)\,ii} \\ &- 2(1-\log\epsilon^2)\int_{B_\epsilon} d^3x\,(R^2-\vec{x}^2)\tilde{h}_{(4)\,ii}\Bigg].\end{aligned} \tag{4.4.74}$$

For the coefficient of the logarithmic divergence, we use (4.B.33) to turn the $\tilde{h}_{(4)\,ii}$ integral into integrals of $h_{(2)\,ii}$ then use (4.B.35) to turn the remaining volume integral into a surface integral of $h_{(0)\,ii}$. The final result is

$$\begin{aligned}\int_{B_\epsilon} \delta\boldsymbol{\Delta}[\xi_B] = \frac{1}{8RG_N}\int_{S^2} d\Omega_2\Bigg[&\frac{r^3}{2\epsilon^2}\left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) \\ &+ \log\epsilon^2\Bigg(\frac{r^3}{2}(\hat{x}^i\hat{x}^j h_{(2)\,ij} - h_{(2)\,ii}) \\ &+ \frac{r}{4}(h_{(0)\,ii} - 3\hat{x}^i\hat{x}^j h_{(0)\,ij} + x^j\partial_j h_{(0)\,ii} - \hat{x}^i\hat{x}^j x^k\partial_k h_{(0)\,ij})\Bigg) \\ &- \frac{r^3}{2}(\hat{x}^i\hat{x}^j h_{(2)\,ij} - h_{(2)\,ii})\Bigg]\end{aligned} \tag{4.4.75}$$

Note that there are, as expected, finite contributions. Comparing with (4.3.61) we can see this term is exactly the counterterm for the entanglement entropy. Therefore in $AlAdS_5$ we have satisfied (4.4.55). The renormalized stress tensor $T^{ren}_{\mu\nu}$ in (4.3.20) has a scheme dependent term proportional to $\tilde{h}_{(d)\,\mu\nu}$ that originates from the variation of anomaly term in the counterterm action. Therefore the finite counterterm in the entanglement entropy is necessary to match the contribution associated with the holographic conformal anomaly.

4.4.3.3 $d = 5$

The $d = 5$ case is very similar to the above example but without the logarithmic terms. The dilatation eigenfunction expansion for the variation of the extrinsic curvature is

$$\delta K_{(0)\,\nu}^{\ \mu} = 0 \tag{4.4.76}$$

$$\delta K_{(2)\,\nu}^{\ \mu} = -z^2\eta^{\mu\sigma}h_{(2)\,\sigma\nu} - z^2\eta^{\mu\sigma}\delta D_{(2)\,\sigma\nu} + O(z^6) \tag{4.4.77}$$

$$\delta K_{(4)\,\nu}^{\ \mu} = -2z^4\eta^{\mu\sigma}h_{(4)\,\sigma\nu} + z^4\eta^{\mu\sigma}\delta D_{(2)\,\sigma\nu} + O(z^6) \tag{4.4.78}$$

where at linear level we have

$$\delta D_{(2)\,ii} = \frac{2}{3}h_{(4)\,ii}. \tag{4.4.79}$$

Then the relevant dilatation eigenfunction expansion terms, up to $O(z^6)$, are

$$\delta K_{(0)\,i}^{\;i} = 0 \tag{4.4.80}$$

$$\delta K_{(2)\,i}^{\;i} = -z^2 h_{(2)\,ii} - \frac{2}{3} z^2 h_{(4)\,ii} + O(z^6) \tag{4.4.81}$$

$$\delta K_{(4)\,i}^{\;i} = -\frac{4}{3} z^4 h_{(4)\,ii} + O(z^6). \tag{4.4.82}$$

The counterterm from (4.4.38) gives

$$\delta\mathbf{\Delta}[\xi_B] = \frac{\boldsymbol{d^4x} z^{-5}}{8RG_6}\left[z^2 h_{ii} - (R^2 - z^2 - \vec{x}^2)\left(-z^2 h_{(2)\,ii} - 2z^4 h_{(4)\,ii}\right)\right]. \tag{4.4.83}$$

Neglecting the $O(z)$ terms as they vanish in the limit $\epsilon \to 0$ we have,

$$\delta\mathbf{\Delta}[\xi_B] = \frac{\boldsymbol{d^4x}}{8RG_6}\left[\frac{1}{z^3}\left(h_{(0)\,ii} - (R^2 - \vec{x}^2) h_{(2)\,ii}\right) + \frac{2}{z}(R^2 - \vec{x}^2) h_{(4)\,ii}\right]. \tag{4.4.84}$$

Now evaluate the integral of the correction following in appendix $4.B.2$. We use $(4.B.29)$ and $(4.B.38)$ to get

$$\begin{aligned}\int_{B_\epsilon} \delta\mathbf{\Delta}[\xi_B] =& \frac{1}{8RG_6}\Bigg[\frac{(R^2-\epsilon^2)^2}{3\epsilon^3}\int_{S^3} d\Omega_3 \left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) + \frac{1}{\epsilon}\int_{B_\epsilon} d^4x\, h_{(2)\,ii} \\ &+ \frac{(R^2-\epsilon^2)^2}{\epsilon}\int_{S^3} d\Omega_3 \left(h_{(2)\,ii} - \hat{x}^i\hat{x}^j h_{(2)\,ij}\right) - \frac{3}{\epsilon}\int_{B_\epsilon} d^4x\, h_{(2)\,ii}\Bigg].\end{aligned} \tag{4.4.85}$$

The remaining volume integral of $h_{(2)\,ii}$ can be converted to surface integral via $(4.B.40)$,

$$\begin{aligned}\int_{B_\epsilon} \delta\mathbf{\Delta}[\xi_B] =& \frac{1}{8RG_6}\int_{S^3} d\Omega_3 \Bigg[\frac{R^4}{3\epsilon^3}\left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) - \frac{2R^2}{3\epsilon}\left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) \\ &+ \frac{R^4}{\epsilon}\left(h_{(2)\,ii} - \hat{x}^i\hat{x}^j h_{(2)\,ij}\right) - \frac{R^2}{3\epsilon}\left(h_{(0)\,ii} - 4\hat{x}^i\hat{x}^j h_{(0)\,ij} + x^j\partial_j h_{(0)\,ii} - \hat{x}^i\hat{x}^j x^k \partial_k h_{(0)\,ij}\right)\Bigg].\end{aligned} \tag{4.4.86}$$

After rearranging we arrive at the final expression of the correction term

$$\begin{aligned}\int_{B_\epsilon} \delta\mathbf{\Delta}[\xi_B] =& \frac{1}{8RG_6}\int_{S^3} d\Omega_3 \Bigg[\frac{R^4}{3\epsilon^3}\left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) - \frac{R^2}{\epsilon}\left(h_{(0)\,ii} - 2\hat{x}^i\hat{x}^j h_{(0)\,ij}\right) \\ &+ \frac{R^4}{\epsilon}\left(h_{(2)\,ii} - \hat{x}^i\hat{x}^j h_{(2)\,ij}\right) - \frac{R^2}{3\epsilon}\left(x^j\partial_j h_{(0)\,ii} - \hat{x}^i\hat{x}^j x^k \partial_k h_{(0)\,ij}\right)\Bigg].\end{aligned} \tag{4.4.87}$$

This is in fact identical to the counterterm in (4.3.27) when taking the limit $\epsilon \to 0$ and satisfying (4.4.55). Note that in (4.3.27) one has to expand $r = \sqrt{R^2 - \epsilon^2}$ to arrive at (4.4.87). Since d is odd, there is no finite counterterm and the renormalized first law is scheme independent.

4.5 Conclusions and outlook

In this paper we have proven the renormalized first law of holographic entanglement entropy, in both infinitesimal and covariant versions, for generic variations of the metric. The original proofs of the first law of holographic entanglement entropy assumed that only normalisable modes of the metric were varied, corresponding to changing the state in the dual conformal field theory. Our proof extends to non-normalisable variations of the metric, corresponding to changing the background metric for the dual conformal theory.

When the boundary dimension d is odd, both the renormalized stressed tensor and renormalized area of the entangling surface are scheme independent and the holographic conformal anomaly is absent. When the boundary dimension d is even, there are finite contributions from counterterms and one needs to ensure that the same renormalization scheme is used for the stress tensor and entanglement entropy; this follows immediately from the approach taken in [54] because the counterterms for the entanglement entropy are derived from the counterterms for the action given in [43] using the replica trick. In our setup the background about which we are perturbing is conformally flat and thus there are no explicit contributions from the conformal anomaly at linear order.

A motivation to include the non-normalisable variations of the metric is that we need to allow variation of the boundary metric within its conformal class to preserved bulk diffeomorphism. This is because the PBH tranformation, which is a bulk diffeomorphism [91], induces a Weyl transformation on the conformal boundary. However in the presence of the anomaly, we must pick a representative of the conformal class for the variational problem to be well defined. Hence the above bulk diffeomorphism is required to be broken. As we discuss below, the first law admits an extra term under the broken diffeomorphism.

The first law can also be derived using the covariant phase space approach, building on [57], as well discussions of the covariant phase space formalism in the presence of boundaries [103] and boundary counterterm contributions to conserved charges [91]. The generalization to non-normalizable variations of the bulk metric, corresponding to deforming the background metric for the dual CFT, induces specific counterterms in the covariant phase space construction. We explain in detail how these relate to the boundary terms in [103]. Note that in the context of the laws of black holes one would fix the non-normalizable modes and therefore the our analysis differs from the renormalized black hole charge analysis of [91]. The first law of entanglement entropy takes a similar form as in the first law of black holes thermodynamics in [91]. In the presence of anomaly and for generic representatives, the first law of entanglement entropy will admit an extra term corresponding to the anomaly. If we consider the variation of the representative, this will induce inhomogeneous transformation of the d^{th} term of dilatation eigenfunction expansion of the canonical momentum and induced an explicit conformal anomaly term for $\delta \boldsymbol{Q}$. We will have this type of first law of entanglement entropy,

$$\delta E_B = \delta_\sigma E_B + \delta S_B^{ren}, \tag{4.5.1}$$

where δ_σ is the variation induced by the varying the Weyl factor. This extra term is analogous to the first law of black hole thermodynamics in $AlAdS$ where

$$\delta M = \delta_\sigma M + T\delta S \tag{4.5.2}$$

where M, S and T are the mass, entropy and temperature of the black hole. Since we have the conformal class, $[g_{(0)}]$, with representative as $g_{(0)} = \eta$, the holographic conformal anomaly is zero up to quadratic perturbation. This is the reason, there is no explicit anomaly term in the renormalized first law.

While the focus of this paper has been on proving the holographic first law of entanglement entropy for non-normalisable bulk metric variations, our methodology could be extended to many analyses within holographic information theory. One could clearly explore perturbations of the surface itself, following [111, 112, 113, 114]. The extension to higher derivative gravity theories would be straightforward in principle although one may need to resolve analogous technical ambiguities to those encountered in [115, 116]. Analyses of local reconstruction in the bulk from boundary entanglement such as [117, 118] assume normalizable fall offs of metric perturbations (corresponding to CFT states), but our approach facilitates the discussion of marginal and indeed even irrelevant deformations. To include the latter, one would simply add in the bulk field corresponding to the irrelevant operator, and compute renormalized quantities perturbatively in the irrelevant deformation. Other analyses where our methodology would be useful to extend the class of theories/states under consideration include discussions of subregion complexity and the first law of complexity [119, 120] as well as analyses of the relation of holographic entanglement entropy to inverse mean curvature flow [121].

Finally, let us consider the expression for the variation of the entanglement entropy in terms of the Weyl tensor (4.3.36). This relation could have been anticipated from the known relationship between the Einstein sector of conformal (Weyl) gravity and Einstein gravity [122, 123]. Up to a topological term the renormalized action for Einstein gravity is proportional to the Weyl squared term [93, 123, 124]. Accordingly, the Wald entropy functional for the AdS Rindler black hole on the black hole horizon $\tilde{H}_\epsilon$ gives

$$S_{Wald} \propto \int_{\tilde{H}_\epsilon} W^{abcd} n_{ab} n_{cd} \tag{4.5.3}$$

where n_{ab} is the binormal for the codimension two surface $\tilde{H}_\epsilon$. Using the standard Casini, Huerta and Myers approach [51] we can then map this entropy to the entanglement entropy for a spherical region in a flat background. The computations in this paper relate to the first variation of this entropy under bulk metric variations and using the CHM map we immediately obtain the first term of the Weyl integral in (4.3.37)

$$\delta S^{\text{ren}} \propto \int_{\tilde{B}_\epsilon} \delta W_{1212}. \tag{4.5.4}$$

This relation holds in all even bulk spacetime dimensions, even though the expressions for the renormalized entanglement entropy become increasingly complex expressions of the Euler characteristic and curvature invariants of the entangling surface in higher dimensions [1]. The variation manifestly simplifies to just this one term for linear variations of a spherical surface around a background with zero Weyl curvature. Working to higher order in the variations, and in more general setups, one should make use of the full form of the renormalized area in terms of Euler characteristic and curvature invariants in [1] to understand the underlying geometric structure.

4.A Infinitesimal first law

4.A.1 Useful identities

In this appendix, we provide some useful identities that are used in section 4.3. First we give angular integrals of the unit vectors,

$$\int_{S^d} d\Omega_d \hat{x}^{odd} = 0 \tag{4.A.1}$$

$$\int_{S^d} d\Omega_d \hat{x}^i \hat{x}^j = \frac{\Omega_d}{d+1}\delta^{ij} \tag{4.A.2}$$

$$\int_{S^d} d\Omega_d \hat{x}^i \hat{x}^j \hat{x}^k \hat{x}^l = \frac{\Omega_d}{(d+3)(d+1)}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{4.A.3}$$

$$\int_{S^d} d\Omega_d\, \hat{x}^i \hat{x}^j \hat{x}^k \hat{x}^l \hat{x}^p \hat{x}^q = \frac{15\Omega_d}{(d+5)(d+3)(d+1)}\delta^{(ij}\delta^{kl}\delta^{pq)} \tag{4.A.4}$$

$$\int_{S^d} d\Omega_d\, \hat{x}^{i_1} \hat{x}^{i_2} \cdots \hat{x}^{i_{2n-1}} \hat{x}^{i_{2n}} = \Omega_d \prod_{r=1}^{n} \frac{2r-1}{(d+2r-1)}\delta^{(i_1 i_2} \cdots \delta^{i_{2n-1} i_{2n})}. \tag{4.A.5}$$

Since the angular integral of unit vectors is expressed as symmetrized Kronecker deltas, it is also useful to have the expression of the symmetrized Kronecker deltas contracted with derivatives of the metric perturbation:

$$\delta^{(ij}\delta^{kl)}\partial_k\partial_l h_{ij} = \frac{1}{3}\partial_k\partial_k h_{ii} + \frac{2}{3}\partial_i\partial_j h_{ij} \tag{4.A.6}$$

$$\delta^{(ij}\delta^{kl}\delta^{pq)}\partial_k\partial_l\partial_p\partial_q h_{ij} = \frac{1}{5}\partial_k\partial_k\partial_l\partial_l h_{ii} + \frac{4}{5}\partial_k\partial_k\partial_i\partial_j h_{ij} \tag{4.A.7}$$

$$\delta^{(i_1 i_2} \cdots \delta^{i_{2n-1} i_{2n})}\partial_{i_3}\partial_{i_4}\cdots\partial_{i_{2n-1}}\partial_{i_{2n}} h_{i_1 i_2} = \frac{(\partial^2)^{n-2}}{2n-1}\left[\partial^2 h_{i_1 i_1} + (2n-2)\partial_{i_1}\partial_{i_2} h_{i_1 i_2}\right]. \tag{4.A.8}$$

4.A.2 Explicit variation in $d = 3$

In this section we will show the procedure used to calculate the variation of the variation of regularized entanglement entropy and variation of the counterterms in $d = 3$. Here we continue the calculation from (4.3.24). First we consider the leading order in the Taylor expansion which has no derivatives and perform the angular integrals (4.A.2) to get

$$\delta S_B^{reg}(\partial^0) = \frac{\Omega_1}{8G_4}\int_\xi^{\pi/2} du \frac{\cos u}{\sin^2 u}(1 - \frac{\cos^2 u}{2})h_{ii}(z, 0, 0), \tag{4.A.9}$$

where

$$h_{ii}(z) = h_{ii}^{(0)} + R^2 \sin^2 u h_{ii}^{(2)} + R^3 \sin^3 u h_{ii}^{(3)}. \tag{4.A.10}$$

After performing the u integrals we get

$$\begin{aligned}\delta S_B^{reg}(\partial^0) = \frac{\Omega_1}{8G_4}\Big[&\frac{1}{2}\Big(\frac{1}{\sin\xi} - \sin\xi\Big)h_{ii}^{(0)} + \Big(\frac{2}{3} - \frac{\sin\xi}{2} - \frac{\sin^3\xi}{6}\Big)R^2 h_{ii}^{(2)} \\ &+ \Big(\frac{3}{8} - \frac{\sin^2\xi}{4} - \frac{\sin^4\xi}{8}\Big)R^3 h_{ii}^{(3)}\Big]\end{aligned} \tag{4.A.11}$$

We also need to evaluate the higher derivative terms in the Taylor expansion. For our purposes we need only the Taylor expansion of $h_{ij}^{(0)}(x, y)$. The contribution of the one derivative term of the variation is

$$\delta S_B^{reg}(\partial^1) = \frac{R}{8G_4}\int_\xi^{\pi/2} du \int_0^{2\pi} d\phi \frac{\cos^2 u}{\sin^2 u}(\delta^{ij} - \cos^2 u \hat{x}^i \hat{x}^j)\hat{x}^k \partial_k h_{ij}^{(0)}. \tag{4.A.12}$$

Using the angular integrals (4.A.1) we can deduce $\delta S_B^{reg}(\partial^1)$ vanishes.

The contribution of the leading two derivative terms of the variation is

$$\delta S_B^{reg}(\partial^2) = \frac{R^2}{8G_4}\int_\xi^{\pi/2} du \int_0^{2\pi} d\phi \frac{\cos^3 u}{\sin^2 u}(\delta^{ij} - \cos^2 u \hat{x}^i \hat{x}^j)\frac{\hat{x}^k \hat{x}^l}{2!}\partial_k \partial_l h_{ij}^{(0)}. \tag{4.A.13}$$

We evaluate the angular integrals by substituting the results from (4.A.2) and (4.A.3),

$$\delta S_B^{reg}(\partial^2) = \frac{R^2 \Omega_1}{8G_4}\int_\xi^{\pi/2} du \frac{\cos^3 u}{\sin^2 u}\frac{1}{4}\partial_j \partial_j h_{ii}^{(0)} - \frac{\cos^5 u}{\sin^2 u}\frac{1}{16}\Big(\partial_j \partial_j h_{ii}^{(0)} + 2\partial_i \partial_j h_{ij}^{(0)}\Big). \tag{4.A.14}$$

Evaluating the u integral and rearranging the derivatives of metric variation we get

$$\begin{aligned}\delta S_B^{reg}(\partial^2) &= \frac{R^2\Omega_1}{8G_4}\Big[\frac{1}{4}\Big(-2+\frac{1}{\sin\xi}+\sin\xi\Big)\partial_j\partial_j h_{ii}^{(0)} \\ &\quad -\frac{1}{16}\Big(-\frac{8}{3}+\frac{1}{\sin\xi}+2\sin\xi+\frac{\sin^3\xi}{3}\Big)\Big(\partial_j\partial_j h_{ii}^{(0)}+2\partial_i\partial_j h_{ij}^{(0)}\Big)\Big] \\ \delta S_B^{ref}(\partial^2) &= \frac{R^2\Omega_1}{32G_4}\Big[\Big(-\frac{4}{3}+\frac{3}{4\sin\xi}+\frac{\sin\xi}{2}-\frac{\sin^3\xi}{12}\Big)\partial_j\partial_j h_{ii}^{(0)} \\ &\quad \Big(\frac{4}{3}-\frac{1}{2\sin\xi}-\sin\xi-\frac{\sin^3\xi}{6}\Big)\partial_i\partial_j h_{ij}^{(0)}\Big] \end{aligned} \tag{4.A.15}$$

We would like to take the limit of $\xi\to 0$ so we need to check the divergences are cancelled out by counterterms in (4.3.27). We first evaluate the leading order terms in the Taylor series with no derivatives:

$$\delta S_B^{ct}(\partial^0) = \frac{1}{8G_4}\int_0^{2\pi} dz\phi\frac{\cos\xi}{\sin\xi}(h_{ii}-\hat{x}^i\hat{x}^j h_{ij}). \tag{4.A.16}$$

Evaluating the angular integral and expanding around $\xi = 0$ we get

$$\delta S_B^{ct}(\partial^0) = -\frac{\Omega_1}{16G_4}\Big[\frac{1}{\sin\xi}-\frac{\sin\xi}{2}\Big]\times\Big[h_{ii}^{(0)}+R^2\sin^2\xi h_{ii}^{(2)}+R^3\sin^3\xi h_{ii}^{(3)}\Big] \tag{4.A.17}$$

We now evaluate contributions from the subleading one derivative terms in the Taylor expansion of $h_{ij}^{(0)}(z)$

$$\delta S_B^{ct}(\partial^1) = -\frac{R}{8G_4}\int_0^{2\pi}\frac{\cos^2\xi}{\sin\xi}\Big[\hat{x}^k\partial_k h_{ii}^{(0)}-\hat{x}^i\hat{x}^j\hat{x}^k\partial_k h_{ij}^{(0)}\Big]. \tag{4.A.18}$$

Using the angular integrals $(4.A.1)$ we can deduce $\delta S_B^{ct}(\partial^1)$ vanishes. The next leading two derivative contribution is

$$\delta S_B^{ct}(\partial^2) = -\frac{R^2}{8G_4}\int_0^{2\pi}\frac{\cos^3\xi}{\sin\xi}\Big[\hat{x}^k\hat{x}^l\partial_k\partial_l h_{ii}^{(0)}-\hat{x}^i\hat{x}^j\hat{x}^k\hat{x}^l\partial_k\partial_l h_{ij}^{(0)}\Big] \tag{4.A.19}$$

After evaluating the angular integrals we obtain

$$\delta S_B^{ct}(\partial^2) = -\frac{R^2\Omega_1}{128G_4}\frac{\cos^3\xi}{\sin\xi}\Big[3\partial_j\partial_j h_{ii}^{(0)}-2\partial_i\partial_j h_{ij}^{(0)}\Big]. \tag{4.A.20}$$

Combining the variations of the regularized entanglement entropy and the variation of the counterterms we get the following. For ∂^0 terms we have

$$\begin{aligned}\delta S_B^{reg}(\partial^0)+\delta S_B^{ct}(\partial^0) &= \frac{\Omega_1}{8G_4}\Big[\frac{1}{2\sin\xi}h_{ii}^{(0)}+\frac{2}{3}R^2 h_{ii}^{(2)}+\frac{3}{8}R^3 h_{ii}^{(3)}\Big] \\ &\quad -\frac{\Omega_1}{16G_4}\frac{1}{\sin\xi}h_{ii}^{(0)}, \end{aligned} \tag{4.A.21}$$

and for ∂^2 terms have

$$\delta S_B^{reg}(\partial^2)+\delta S_B^{ct}\partial^2)=\frac{R^2\Omega_1}{32G_4}\Big[\Big(-\frac{4}{3}+\frac{3}{4\sin\xi}\Big)\partial_j\partial_j h_{ii}^{(0)}+\Big(\frac{4}{3}-\frac{1}{2\sin\xi}\Big)\partial_i\partial_j h_{ij}^{(0)}\Big]$$
$$-\frac{R^2\Omega_1}{128G_4}\frac{1}{\sin\xi}\Big[3\partial_j\partial_j h_{ii}^{(0)}-2\partial_i\partial_j h_{ij}^{(0)}\Big]. \tag{4.A.22}$$

Gathering all terms together we obtain the variation of renormalized entanglement entropy,

$$\delta S_B^{ren}=\frac{\Omega_1}{8G_4}\Big[-\frac{R^2}{3}\partial_j\partial_j h_{ii}^{(0)}+\frac{R^2}{3}\partial_i\partial_j h_{ij}^{(0)}+\frac{2}{3}R^2h_{ii}^{(2)}+\frac{3}{8}R^3h_{ii}^{(3)}\Big]. \tag{4.A.23}$$

Then from (4.3.33) we can express $h^{(2)}$ in terms of $h^{(0)}$ and the variation of the renormalized entanglement entropy becomes

$$\delta S_B^{ren}=\frac{\Omega_1}{8G_4}\Big[-\frac{R^2}{3}\partial_j\partial_j h_{ii}^{(0)}+\frac{R^2}{3}\partial_i\partial_j h_{ij}^{(0)}+\frac{2}{3}R^2\Big(\frac{1}{2}\partial_j\partial_j h_{ii}^{(0)}-\frac{1}{2}\partial_i\partial_j h_{ij}^{(0)}\Big)+\frac{3}{8}R^3h_{ii}^{(3)}\Big]$$
$$=\frac{3R^3\Omega_1}{64G_4}h_{ii}^{(3)}, \tag{4.A.24}$$

which is the result stated in (4.3.35) for $d=3$.

4.A.3 Explicit variation in $d=5$

Following the same approaches as in the section above, we continue the calculation from (4.3.24) for $d=5$. The variation of the regularized entanglement entropy to leading order of the near boundary approximation, the zero derivative terms in the Taylor expansion give

$$\delta S_B^{reg}(\partial^0)=\frac{\Omega_3}{8G_6}\int_\xi^{\pi/2}du\Big[\frac{\cos^3 u}{\sin^4 u}\Big(1-\frac{\cos^2 u}{4}\Big)h_{ii}\Big]. \tag{4.A.25}$$

Using the Fefferman Graham expansion and evaluating the u integral we get

$$\delta S_B^{reg}(\partial^0)=\frac{\Omega_3}{8G_6}\Big[\Big(\frac{1}{4\sin^3\xi}-\frac{1}{2\sin\xi}\Big)h_{ii}^{(0)}+\Big(\frac{3}{4\sin\xi}-\frac{4}{3}\Big)R^2h_{ii}^{(2)}+\frac{8}{15}R^4h_{ii}^{(4)}+\frac{5}{24}R^5h_{ii}^{(5)}\Big]. \tag{4.A.26}$$

The two derivative terms give

$$\delta S_B^{reg}(\partial^2)=\frac{R^2\Omega_3}{8G_6}\int_\xi^{\pi/2}du\frac{\cos^5 u}{2!\sin^4 u}\Big[\frac{1}{4}\partial_k\partial_k h_{ii}-\frac{\cos^2 u}{24}(\partial_k\partial_k h_{ii}+2\partial_i\partial_j h_{ij})\Big] \tag{4.A.27}$$

Using the Fefferman Graham expansion and evaluating the u integral we get

$$\delta S_B^{reg}(\partial^2)=\frac{\Omega_3}{8G_6}\Big[\Big(\frac{5}{144\sin^3\xi}-\frac{3}{16\sin\xi}+\frac{2}{9}\Big)R^2\partial_k\partial_k h_{ii}^{(0)}+\Big(-\frac{1}{72\sin^3\xi}+\frac{1}{8\sin\xi}-\frac{2}{9}\Big)R^2\partial_i\partial_j h_{ij}^{(0)}$$
$$-\Big(\frac{5}{48\sin\xi}-\frac{4}{15}\Big)R^4\partial_k\partial_k h_{ii}^{(2)}+\Big(-\frac{1}{24\sin\xi}+\frac{2}{15}\Big)R^4\partial_i\partial_j h_{ij}^{(2)}\Big]. \tag{4.A.28}$$

The four derivative terms are

$$\delta S_B^{reg}(\partial^4) = \frac{R^4\Omega_3}{8G_6}\int_\xi^{\pi/2} du \frac{\cos^7 u}{4!\sin^4 u}\left[\frac{1}{8}\partial_k\partial_k\partial_l\partial_l h_{ii} - \frac{\cos^2 u}{64}\left(\partial_k\partial_k\partial_l\partial_l h_{ii} + 4\partial_k\partial_k\partial_i\partial_j h_{ij}\right)\right]. \tag{4.A.29}$$

Using the Fefferman Graham expansion and evaluating the u integral we get

$$\begin{aligned}\delta S_B^{reg}(\partial^4) =& \frac{\Omega_3}{8G_6}\Bigg[\left(\frac{7}{4608\sin^3\xi} - \frac{5}{384\sin\xi} + \frac{1}{45}\right)R^4\partial_k\partial_k\partial_l\partial_l h_{ii}^{(0)} \\ &+ \left(-\frac{1}{1152\sin^3\xi} + \frac{1}{96\sin\xi} - \frac{1}{45}\right)R^4\partial_k\partial_k\partial_i\partial_j h_{ij}^{(0)}\Bigg].\end{aligned} \tag{4.A.30}$$

We have thus obtained all the divergent and finite terms for the variation of regularized entanglement entropy up to R^5. Note that for the δS_B^{reg} only even derivatives survive the angular integrals. This is no longer the case for the counterterms δS_B^{ct} as some terms in (4.3.27) contain an odd number of directional vectors $\hat{x}$.

For the variation of the counterterms, (4.3.27), in the near boundary approximation, the leading order zero derivative terms are

$$\delta S_B^{ct}(\partial^0) = \frac{\Omega_3}{8G_6}\left[\left(-\frac{1}{4\sin^3\xi} + \frac{1}{2\sin\xi}\right)h_{ii}^{(0)} - \frac{3}{4\sin\xi}R^2 h_{ii}^{(2)}\right]. \tag{4.A.31}$$

The one derivative terms comes from the variation of the extrinsic curvature, corresponding to the last terms in (4.3.27). Note that one derivative means up to and including the first derivative terms in the Taylor expansion:

$$\delta S_B^{ct}(\partial^1) = \frac{1}{24G_6}\int_{S^3} d\Omega_3 \frac{r^3}{z}\left[\hat{x}^k\hat{x}^l\partial_k\partial_l h_{ii} - \hat{x}^i\hat{x}^j\hat{x}^k\hat{x}^l\partial_k\partial_l h_{ij}\right]. \tag{4.A.32}$$

After the integration only the following terms remain

$$\delta S_B^{ct}(\partial^1) = \frac{\Omega_3}{8G_6}\left[\frac{5}{72\sin\xi}R^2\partial_k\partial_k h_{ii}^{(0)} - \frac{1}{36\sin\xi}R^2\partial_i\partial_j h_{ij}^{(0)}\right]. \tag{4.A.33}$$

Similar procedures are used for higher derivative terms. The order two derivative terms are

$$\begin{aligned}\delta S_B^{ct}(\partial^2) =& \frac{\Omega_3}{8G_6}\Bigg[\left(-\frac{5}{144\sin^3\xi} + \frac{17}{144\sin\xi}\right)R^2\partial_k\partial_k h_{ii}^{(0)} \\ &+ \left(+\frac{1}{72\sin^3\xi} - \frac{7}{72\sin\xi}\right)R^2\partial_i\partial_j h_{ij}^{(0)} - \frac{5}{48\sin\xi}R^4\partial_k\partial_k h_{ii}^{(2)} + \frac{1}{24\sin\xi}R^4\partial_i\partial_j h_{ij}^{(2)}\Bigg].\end{aligned} \tag{4.A.34}$$

The order three derivative terms are

$$\delta S_B^{ct}(\partial^3) = \frac{1}{24G_6}\int_{S^3} d\Omega_3 \frac{r^5}{3!z}\left[\hat{x}^k\hat{x}^l\hat{x}^p\hat{x}^q\partial_k\partial_l\partial_p\partial_q h_{ii} - \hat{x}^i\hat{x}^j\hat{x}^k\hat{x}^l\hat{x}^p\hat{x}^q\partial_k\partial_l\partial_p\partial_q h_{ij}\right] \tag{4.A.35}$$

Since only integrals with even directional vectors are non vanishing, there is no term of the form $\partial^3 h^{(0)}$. The remaining relevant terms are

$$\delta S_B^{ct}(\partial^3) = \frac{\Omega_3}{8G_6}\Big[\frac{7}{1152\sin\xi}R^4\partial_k\partial_k\partial_l\partial_l h_{ii}^{(0)} - \frac{1}{288\sin\xi}R^4\partial_k\partial_k\partial_i\partial_j h_{ij}^{(0)}\Big]. \tag{4.A.36}$$

and the four derivative terms

$$\begin{aligned}\delta S_B^{ct}(\partial^4) = -\frac{\Omega_3}{8G_6}\Big[&\Big(-\frac{7}{4608\sin^3\xi}+\frac{17}{4608\sin\xi}\Big)R^4\partial_k\partial_k\partial_l\partial_l h_{ii}^{(0)}\\ &+\Big(\frac{1}{1152\sin^3\xi}-\frac{1}{144\sin\xi}\Big)R^4\partial_k\partial_k\partial_i\partial_j h_{ij}^{(0)}\Big].\end{aligned} \tag{4.A.37}$$

We have thence obtained all the relevant counterterms.

For notational simplicity we express $h_{ii} = h$, $\partial_k\partial_k = \partial^2$ and $\partial_i\partial_j h_{ij} = h(\partial^2)$. To compute the renormalized entanglement entropy we arrange all the relevant terms at order R^n:

Order R^0

$$\delta S_{ren}(R^0) = \frac{\Omega_3}{8G_6}\Big[\Big(\frac{1}{4\sin^3\xi}-\frac{1}{2\sin\xi}\Big)h^{(0)} + \Big(-\frac{1}{4\sin^3\xi}+\frac{1}{2\sin\xi}\Big)h^{(0)}\Big] \tag{4.A.38}$$

Order R^2

$$\begin{aligned}\delta S_{ren}(R^2) = \frac{R^2\Omega_3}{8G_6}\Big[&\Big(\frac{3}{4\sin\xi}-\frac{4}{3}\Big)h^{(2)} + \Big(\frac{5}{144\sin^3\xi}-\frac{3}{16\sin\xi}+\frac{2}{9}\Big)\partial^2 h^{(0)}\\ &+\Big(-\frac{1}{72\sin^3\xi}+\frac{1}{8\sin\xi}-\frac{2}{9}\Big)h^{(0)}(\partial^2) - \frac{3}{4\sin\xi}h^{(2)} + \frac{5}{72\sin\xi}\partial^2 h^{(0)} - \frac{1}{36\sin\xi}h^{(0)}(\partial^2)\\ &+\Big(-\frac{5}{144\sin^3\xi}+\frac{17}{144\sin\xi}\Big)\partial^2 h^{(0)} + \Big(+\frac{1}{72\sin^3\xi}-\frac{7}{72\sin\xi}\Big)h^{(0)}(\partial^2)\Big]\end{aligned} \tag{4.A.39}$$

Order R^4

$$\begin{aligned}\delta S_{ren}(R^4) = \frac{R^4\Omega_3}{8G_6}\Big[&\frac{8}{15}h^{(4)} + \Big(\frac{5}{48\sin\xi}-\frac{4}{15}\Big)\partial^2 h^{(2)} + \Big(-\frac{1}{24\sin\xi}+\frac{2}{15}\Big)h^{(2)}(\partial^2)\\ &+\Big(\frac{7}{4608\sin^3\xi}-\frac{5}{384\sin\xi}+\frac{1}{45}\Big)\partial^2\partial^2 h^{(0)} + \Big(-\frac{1}{1152\sin^3\xi}+\frac{1}{96\sin\xi}-\frac{1}{45}\Big)\partial^2 h^{(0)}(\partial^2)\\ &-\frac{5}{48\sin\xi}\partial^2 h^{(2)} + \frac{1}{24\sin\xi}h^{(2)}(\partial^2) + \frac{7}{1152\sin\xi}\partial^2\partial^2 h^{(0)} - \frac{1}{288\sin\xi}\partial^2 h^{(0)}(\partial^2)\\ &+\Big(-\frac{7}{4608\sin^3\xi}+\frac{1}{44\sin\xi}\Big)\partial^2\partial^2 h^{(0)} + \Big(\frac{1}{1152\sin^3\xi}-\frac{1}{144\sin\xi}\Big)\partial^2 h^{(0)}(\partial^2)\Big]\end{aligned} \tag{4.A.40}$$

Order R^5

$$\delta S_{ren}(\partial^0) = \frac{\Omega_3}{8G_6}\frac{5}{24}R^5 h^{(5)}_{ii} \tag{4.A.41}$$

Using (4.3.33) and(4.3.34) to express all the higher order term in the Fefferman-Graham expansion in terms of lower order ones, we find that below order R^5 the variation of renormalized entanglement entropy is zero. More explicitly for each orders we have

Order R^2

$$\begin{aligned}
\delta S_{ren}(R^2) = \frac{R^2\Omega_3}{8G_6}\Bigg[&\left(\frac{1}{8\sin\xi}-\frac{2}{9}\right)(\partial^2 h^{(0)} - h^{(0)}(\partial^2)) + \left(\frac{5}{144\sin^3\xi}-\frac{3}{16\sin\xi}+\frac{2}{9}\right)\partial^2 h^{(0)} \\
&+\left(-\frac{1}{72\sin^3\xi}+\frac{1}{8\sin\xi}-\frac{2}{9}\right)h^{(0)}(\partial^2) - \frac{1}{8\sin\xi}(\partial^2 h^{(0)} - h^{(0)}(\partial^2)) + \frac{5}{72\sin\xi}\partial^2 h^{(0)} \\
&-\frac{1}{36\sin\xi}h^{(0)}(\partial^2) + \left(-\frac{5}{144\sin^3\xi}+\frac{17}{144\sin\xi}\right)\partial^2 h^{(0)} + \left(+\frac{1}{72\sin^3\xi}-\frac{7}{72\sin\xi}\right)h^{(0)}(\partial^2)\Bigg] \\
\delta S_{ren}(R^2) = 0&
\end{aligned} \tag{4.A.42}$$

Order R^4

$$\begin{aligned}
\delta S_{ren}(R^4) = \frac{R^4\Omega_3}{8G_6}\Bigg[&\frac{2}{15}(\partial^2 h^{(2)} - h^{(2)}(\partial^2)) - \left(\frac{5}{48\sin\xi}-\frac{4}{15}\right)\partial^2 h^{(2)} + \left(-\frac{1}{24\sin\xi}+\frac{2}{15}\right)h^{(2)}(\partial^2) \\
&+\left(\frac{7}{4608\sin^3\xi}-\frac{5}{384\sin\xi}+\frac{1}{45}\right)\partial^2\partial^2 h^{(0)} + \left(-\frac{1}{1152\sin^3\xi}+\frac{1}{96\sin\xi}-\frac{1}{45}\right)\partial^2 h^{(0)}(\partial^2) \\
&-\frac{5}{48\sin\xi}\partial^2 h^{(2)} + \frac{1}{24\sin\xi}h^{(2)}(\partial^2) + \frac{7}{1152\sin\xi}\partial^2\partial^2 h^{(0)} - \frac{1}{288\sin\xi}\partial^2 h^{(0)}(\partial^2) \\
&+\left(-\frac{7}{4608\sin^3\xi}+\frac{1}{144\sin\xi}\right)\partial^2\partial^2 h^{(0)} + \left(\frac{1}{1152\sin^3\xi}-\frac{1}{144\sin\xi}\right)\partial^2 h^{(0)}(\partial^2)\Bigg],
\end{aligned} \tag{4.A.43}$$

gathering all the terms, it simplifies to

$$\begin{aligned}
\delta S_{ren}(R^4) =& \frac{R^4\Omega_3}{8G_6}\Bigg[-\frac{1}{45}\partial^2(\partial^2 h^{(0)} - h^{(0)}(\partial^2)) \\
& \frac{1}{45}\partial^2\partial^2 h^{(0)} - \frac{1}{45}\partial^2 h^{(0)}(\partial^2)\Bigg] \\
\delta S_{ren}(R^4) =& 0
\end{aligned} \tag{4.A.44}$$

Order R^5

This is the only order ≤ 5 that is non-vanishing,

$$\delta S_{ren}(\partial^0) = \frac{\Omega_3}{8G_6}\frac{5}{24}R^5 h^{(5)}_{ii}, \tag{4.A.45}$$

which matches with (4.3.35) for $d = 5$.

4.A.4 Renormalized Weyl integrals

This appendix provides the calculation details for section 4.3.4. In (4.3.48) and (4.3.49), the Weyl integrals are given in terms of the Riemann tensor of the boundary of AdS, $\mathcal{R}_{\mu\nu\rho\sigma}$, and we need to expand $\mathcal{R}_{\mu\nu\rho\sigma}$ into linear perturbation $h_{\mu\nu}$. For $\mathcal{R}_{titj}$, we have the following expression

$$\mathcal{R}_{titj} = \frac{1}{2}\left(\partial_t\partial_j h_{ti} + \partial_t\partial_i h_{tj} - \partial_t\partial_t h_{ij} - \partial_i\partial_j h_{tt}\right) \tag{4.A.46}$$

$$\begin{aligned}\mathcal{R}_{titj} =& \left(1 + \frac{r^2\hat{x}^k\hat{x}^l}{2}\partial_k\partial_l\right)\frac{1}{2}\left(\partial_t\partial_j h^{(0)}_{ti} + \partial_t\partial_i h^{(0)}_{tj} - \partial_t\partial_t h^{(0)}_{ij} - \partial_i\partial_j h^{(0)}_{tt}\right)\\ &+ \frac{z^2}{2}\left(\partial_t\partial_j h^{(2)}_{ti} + \partial_t\partial_i h^{(2)}_{tj} - \partial_t\partial_t h^{(2)}_{ij} - \partial_i\partial_j h^{(2)}_{tt}\right).\end{aligned} \tag{4.A.47}$$

The $d = 3$ integral

For $d = 3$, we do not need the subleading term in the Taylor expansion of the metric perturbation as in (4.3.28). Also in $d = 3$ the boundary integral (4.3.49) is vanishing in the limit of $\xi \to 0$. After we substitute (4.A.47) into (4.3.48) the renormalized Weyl integral $\mathcal{W}$ is mixed with different orders in the Fefferman-Graham expansion. Explicitly we have

$$\begin{aligned}\mathcal{W} =& \int_\xi^{\frac{\pi}{2}} du \int_{S^1} d\Omega_1 \Big[-\frac{3R^3\cos u \sin^3 u}{2}h^{(3)}_{tt} + \frac{3R^3\cos^3 u \sin u \hat{x}^i\hat{x}^j}{2}\left(h^{(3)}_{tt}\eta_{ij} - h^{(3)}_{ij}\right) \\ &+ \frac{R^2\cos^3 u\hat{x}^i\hat{x}^j}{2}\left(\partial_t\partial_j h^{(0)}_{ti} + \partial_t\partial_i h^{(0)}_{tj} - \partial_t\partial_t h^{(0)}_{ij} - \partial_i\partial_j h^{(0)}_{tt} + 2h^{(2)}_{tt}\eta_{ij} - 2h^{(2)}_{ij}\right)\Big].\end{aligned} \tag{4.A.48}$$

After integrating over the circle we obtain

$$\begin{aligned}\mathcal{W} = \Omega_1\Big[& -\frac{3R^3}{8}h^{(3)}_{tt} + \frac{3R^3}{16}\left(2h^{(3)}_{tt} - h^{(3)}_{ii}\right) \\ &+ \frac{R^2}{6}\left(2\partial_t\partial_i h^{(0)}_{ti} - \partial_t\partial_t h^{(0)}_{ii} - \partial_i\partial_i h^{(0)}_{tt} + 4h^{(2)}_{tt} - h^{(2)}_{ii}\right)\Big].\end{aligned} \tag{4.A.49}$$

By solving the Einstein equation order by order in the Fefferman-Graham expansion, we can deduced $h^{(n)}$ for $n < d$ from $h^{(0)}$. This gives

$$h^{(2)}_{ii} = \frac{1}{2}\Big(\partial_k\partial_k h^{(0)}_{ii} - \partial_i\partial_j h^{(0)}_{ij}\Big) \tag{4.A.50}$$

$$h^{(2)}_{tt} = \frac{1}{4}\Big(\partial_k\partial_k h^{(0)}_{ii} - \partial_i\partial_j h^{(0)}_{ij} + \partial_t\partial_t h^{(0)}_{ii} + \partial_k\partial_k h^{(0)}_{tt} - 2\partial_t\partial_i h^{(0)}_{ti}\Big). \tag{4.A.51}$$

Using the above two expression sfor $h^{(2)}_{\mu\nu}$, we can easily simplify the renormalized Weyl integral as

$$\mathcal{W} = -\frac{3R^3\Omega_1}{16}h^{(3)}_{tt} \tag{4.A.52}$$

which is the result stated in section 4.3.4.

The $d = 5$ integral

For $d = 5$, we need the subleading term in the Taylor expansion of the metric perturbation as in (4.3.28). The relevant metric perturbation derivatives are

$$h''_{tt} = \left(1 + \frac{r^2\hat{x}^k\hat{x}^l}{2}\partial_k\partial_l\right) 2h^{(2)}_{tt} + 12z^2 h^{(4)}_{tt} + 20z^3 h^{(5)}_{tt} \tag{4.A.53}$$

$$h'_{\mu\nu} = \left(1 + \frac{r^2\hat{x}^k\hat{x}^l}{2}\partial_k\partial_l\right) 2zh^{(2)}_{\mu\nu} + 4z^3 h^{(4)}_{\mu\nu} + 5z^4 h^{(5)}_{\mu\nu} \tag{4.A.54}$$

$$\partial_\mu h'_{\nu\rho} = 2zr\hat{x}^k\partial_k\partial_\mu h^{(2)}_{\nu\rho}, \tag{4.A.55}$$

where $'$ represent the radial derivative ∂_z. The renormalized Weyl integral $\mathcal{W}$ becomes

$$\begin{aligned}\mathcal{W} = \int_{S^3} d\Omega_3\Big[&-\frac{8R^4}{15}h^{(4)}_{tt} - \frac{15R^5}{24}h^{(5)}_{tt} \\ &+ \left(-\frac{4R^2}{3}\hat{x}^i\hat{x}^j - \frac{4R^2}{5}\hat{x}^i\hat{x}^j\hat{x}^k\hat{x}^l\partial_k\partial_l\right) \\ &\quad\times\left(\partial_t\partial_j h^{(0)}_{ti} + \partial_t\partial_i h^{(0)}_{tj} - \partial_t\partial_t h^{(0)}_{ij} - \partial_i\partial_j h^{(0)}_{tt} + 2h^{(2)}_{tt}\eta_{ij} + 2h^{(2)}_{ij}\eta_{tt}\right) \\ &+ \frac{4R^4}{15}\hat{x}^i\hat{x}^j\left(\partial_t\partial_j h^{(2)}_{ti} + \partial_t\partial_i h^{(2)}_{tj} - \partial_t\partial_t h^{(2)}_{ij} - \partial_i\partial_j h^{(2)}_{tt} + 4h^{(4)}_{tt}\eta_{ij} + 4h^{(4)}_{ij}\eta_{tt}\right) \\ &+ \frac{5R^5}{12}\hat{x}^i\hat{x}^j\left(h^{(5)}_{tt}\eta_{ij} + h^{(5)}_{ij}\eta_{tt}\right) \\ &+ \frac{16R^4}{15}\hat{x}^i\hat{x}^k\partial_k\left(\partial_t h^{(2)}_{ti} - \partial_i h^{(2)}_{tt}\right)\Big].\end{aligned} \tag{4.A.56}$$

After integrating this over the S^3 using (4.A.5) we get

$$\begin{aligned}\mathcal{W} = \Omega_3\Big[\frac{R^2}{3}&\Big(-2\partial_t\partial_i h^{(0)}_{ti} + \partial_t\partial_t h^{(0)}_{ii} + \partial_i\partial_i h^{(0)}_{tt} - 8h^{(2)}_{tt} + 2h^{(2)}_{ii}\Big) \\ +\frac{R^4}{30}&\Big(-6\partial_t\partial_k\partial_k\partial_i h^{(0)}_{ti} + \partial_t\partial_t\partial_k\partial_k h^{(0)}_{ii} + 2\partial_t\partial_t\partial_i\partial_j h^{(0)}_{ij} + 3\partial_k\partial_k\partial_l\partial_l h^{(0)}_{tt} \\ &\quad - 22\partial_k\partial_k h^{(2)}_{tt} - 2\partial_t\partial_t h^{(2)}_{ii} + 2\partial_k\partial_k h^{(2)}_{ii} + 4\partial_i\partial_j h^{(2)}_{ij} + 12\partial_t\partial_i h^{(2)}_{ti} \\ &\quad + 16h^{(4)}_{tt} - 8h^{(4)}_{ii}\Big) \\ +\frac{R^5}{48}&\Big(-10h^{(5)}_{tt} - 5h^{(5)}_{ii}\Big)\Big].\end{aligned} \tag{4.A.57}$$

Following the lower dimensional case, we need to related the terms of different orders in Fefferman-Grahm expansion to see the cancellation between divergent pieces. By solving the Einstein equations order by order in the Fefferman-Graham expansion, we can deduced $h^{(n)}$ for $n < d$ from $h^{(0)}$. Hence,

$$h^{(2)}_{ii} = \frac{1}{6}\Big(\partial_k\partial_k h^{(0)}_{ii} - \partial_i\partial_j h^{(0)}_{ij}\Big) \tag{4.A.58}$$

$$h^{(2)}_{tt} = \frac{1}{24}\Big(\partial_k\partial_k h^{(0)}_{ii} - \partial_i\partial_j h^{(0)}_{ij} + 3\partial_t\partial_t h^{(0)}_{ii} + 3\partial_k\partial_k h^{(0)}_{tt} - 6\partial_t\partial_i h^{(0)}_{ti}\Big) \tag{4.A.59}$$

$$h^{(2)}_{ti} = \frac{1}{6}\Big(\partial_t\partial_i h^{(0)}_{jj} - \partial_t\partial_j h^{(0)}_{ij} - \partial_i\partial_j h^{(0)}_{tj} + \partial_k\partial_k h^{(0)}_{ti}\Big) \tag{4.A.60}$$

$$h^{(4)}_{ii} = \frac{1}{4}\Big(\partial_k\partial_k h^{(2)}_{ii} - \partial_i\partial_j h^{(2)}_{ij}\Big) \tag{4.A.61}$$

$$h^{(4)}_{tt} = \frac{1}{4}\Big(\partial_k\partial_k h^{(2)}_{tt} - \partial_t\partial_t h^{(2)}_{ii}\Big) \tag{4.A.62}$$

$$\begin{aligned}\partial_i\partial_j h^{(2)}_{ij} = \frac{1}{24}\Big(&3\partial_k\partial_k\partial_l\partial_l h^{(0)}_{ii} - 3\partial_k\partial_k\partial_i\partial_j h^{(0)}_{ij} + \partial_t\partial_t\partial_k\partial_k h^{(0)}_{ii} \\ &- 3\partial_k\partial_k\partial_l\partial_l h^{(0)}_{tt} + 63\partial_t\partial_k\partial_k\partial_i h^{(0)}_{ti} - 4\partial_t\partial_t\partial_i\partial_j h^{(0)}_{ij}\Big)\end{aligned} \tag{4.A.63}$$

$$\partial_t\partial_i h^{(2)}_{it} = \frac{1}{6}\Big(\partial_t\partial_t\partial_k\partial_k h^{(0)}_{ii} - \partial_t\partial_t\partial_i\partial_j h^{(0)}_{ij}\Big) \tag{4.A.64}$$

Substituting the above expressions for $h^{(n)}_{\mu\nu}$, we can easily simplify the renormalized Weyl integral as

$$\mathcal{W} = -\frac{5R^5\Omega_3}{16}h^{(5)}_{tt} \tag{4.A.65}$$

which is the result stated in section 4.3.4.

4.A.5 Variations in AdS_5

Here we will fill in the computational details of section 4.3.5 to show that the divergences of the variation of regularized entanglement entropy and variation of the counterterms match. In (4.3.57), the variation of regularized entanglement entropy was given in terms of both $h^{(0)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$. In order to compare with the counterterm we will first express $h^{(2)}_{\mu\nu}$

as function of $h^{(0)}_{\mu\nu}$.

Since the perturbed metric of AdS_5 satisfies the Einstein equation, the metric perturbation can be expanded and solved order by order in an asymptotic series. Using the results in [43],

$$g^{(2)}_{\mu\nu} = -\frac{1}{2}\left(\mathcal{R}_{\mu\nu}[g^{(0)}] - \frac{1}{6}\mathcal{R}[g^{(0)}]g^{(0)}_{\mu\nu}\right). \tag{4.A.66}$$

In $d = 4$, we only need to consider terms of order up to z^2, hence we have

$$h^{(2)}_{\mu\nu} = -\frac{1}{2}\left(\mathcal{R}_{\mu\nu}[\eta + h^{(0)}] - \frac{1}{6}\mathcal{R}[\eta + h^{(0)}](\eta_{\mu\nu} + h^{(0)}_{\mu\nu})\right). \tag{4.A.67}$$

Since the Ricci tensor of $\eta_{\mu\nu}$ vanishes, to first order of h the Ricci tensor of $g^{(0)}$ is just the first order variation. For our interests the relevant terms then become

$$\frac{\sin\theta}{z}h^{(2)}_{\theta\theta} = -\frac{\sin\theta}{2z}\delta\mathcal{R}_{\theta\theta} + \frac{R^2\sin\theta}{12z}\delta\mathcal{R} \tag{4.A.68}$$

$$\frac{1}{\sin\theta z}h^{(2)}_{\phi\phi} = -\frac{1}{2\sin\theta z}\delta\mathcal{R}_{\phi\phi} + \frac{R^2\sin\theta}{12z}\delta\mathcal{R} \tag{4.A.69}$$

Using this expression, we can write the divergent term of the regularized entanglement entropy in (4.3.58) in terms of $h^{(0)}_{\mu\nu}$.

Now we need to evaluate the variation of the counter terms and check all the divergences are cancelled. The induced metric $\widetilde{\gamma}$ of the regularised entangling surface $\partial\tilde{B}_\epsilon = \tilde{B}|_{z=\epsilon}$ is

$$ds^2 = \frac{R^2 - \epsilon^2}{\epsilon^2}(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.A.70}$$

Then the variation of the volume form is

$$\begin{aligned}\delta\sqrt{\widetilde{\gamma}} &= \frac{1}{2}\sqrt{\widetilde{\gamma}}\widetilde{\gamma}^{ij}\delta\widetilde{\gamma}_{ij} \\ &= \frac{\sin\theta}{2}\left(\frac{1}{\epsilon^2}h^{(0)}_{\theta\theta} + \frac{1}{\epsilon^2\sin^2\theta}h^{(0)}_{\phi\phi}\right)\end{aligned} \tag{4.A.71}$$

To calculate the variation of the counterterms we need to embed $(\partial\tilde{B}, \widetilde{\gamma})$ into $(AdS_5|_{z=\epsilon}, \widetilde{G})$ and find its unit normals which are

$$n_1 = \frac{dt}{\epsilon}, \qquad n_2 = \frac{dr}{\epsilon}. \tag{4.A.72}$$

The extrinsic curvature $K_{\mu\nu}$ is defined by $\frac{1}{2}\mathcal{L}_n\widetilde{\gamma}_{\mu\nu}$. The trace of the extrinsic curvature is then

$$K = \widetilde{G}^{\mu\nu}K_{\mu\nu} = \frac{1}{2}\widetilde{\gamma}^{ij}\mathcal{L}_n\widetilde{\gamma}_{ij} = \mathcal{L}_n \ln\sqrt{\widetilde{\gamma}} \tag{4.A.73}$$

In time independent situations, K_1 vanishes. The extrinsic curvature corresponding to the radial normal is

$$K_2 = \epsilon \partial_r \ln \frac{r^2 \sin\theta}{\epsilon^2} = \frac{2\epsilon}{r} \tag{4.A.74}$$

Although we are only taking linear order of metric variation which leaves the direction of the normals unchanged, the coefficients of unit normals n_a vary. Specifically for n_2

$$\delta n_2 = \delta(n^r \partial_r) = \delta\left(\sqrt{\frac{1}{\widetilde{G}_{rr}}}\partial_r\right) = -\frac{\delta \widetilde{G}_{rr}}{2\widetilde{G}_{rr}^{\frac{3}{2}}}\partial_r \tag{4.A.75}$$

The variation of K_2 can be related to the variation of the metric g as

$$\begin{aligned}\delta K_2 &= \delta n_2\big(\ln\sqrt{\widetilde{\gamma}}\big) + n_2\delta\big(\ln\sqrt{\widetilde{\gamma}}\big) \\ &= -\frac{\delta\widetilde{G}_{rr}}{2\widetilde{G}_{rr}^{\frac{3}{2}}}\partial_r \ln\sqrt{\widetilde{\gamma}} + \epsilon\partial_r\left(\frac{1}{2}\widetilde{\gamma}^{ij}\delta\widetilde{\gamma}_{ij}\right) \\ &= -\frac{\epsilon}{r}h^{(0)}_{rr} - \frac{\epsilon}{r^3}h^{(0)}_{\theta\theta} - \frac{\epsilon}{r^3\sin^2\theta}h^{(0)}_{\phi\phi} + \frac{\epsilon}{2r^2}\partial_r h^{(0)}_{\theta\theta} + \frac{\epsilon}{2r^2\sin^2\theta}\partial_r h^{(0)}_{\phi\phi}\end{aligned} \tag{4.A.76}$$

Keeping only the divergence, the structure of the variation of the third term in (4.3.9) is

$$\delta\big(\sqrt{\widetilde{\gamma}}k^2\big) = \delta\big(\sqrt{\widetilde{\gamma}}\big)K_2^2 + \sqrt{\widetilde{\gamma}}\delta\big(K_2^2\big) \tag{4.A.77}$$

Separating the terms in $(4.A.77)$,

$$\delta\big(\sqrt{\widetilde{\gamma}}\big)K_2^2 = \sin\theta\Big(\frac{2}{R^2}h^{(0)}_{\theta\theta} + \frac{2}{R^2\sin^2\theta}h^{(0)}_{\phi\phi}\Big) \tag{4.A.78}$$

$$\begin{aligned}2\sqrt{\widetilde{\gamma}}K_2\delta K_2 = \ &\sin\theta\Big(-4h^{(0)}_{rr} - \frac{4}{R^2}h^{(0)}_{\theta\theta} - \frac{4}{R^2\sin^2\theta}h^{(0)}_{\phi\phi} \\ &+ \frac{2}{R}\partial_r h^{(0)}_{\theta\theta} + \frac{2}{R\sin^2\theta}\partial_r h^{(0)}_{\phi\phi}\Big)\end{aligned} \tag{4.A.79}$$

The remaining terms are the variation of Ricci scalar and projected Ricci tensor. Note that $\mathcal{R}_{aa}$ in [104, 105] was given in a Euclidean setting. After Wick rotating the normal direction back to Lorentzian signature, we obtain

$$\begin{aligned}\mathcal{R}_{aa} &= \mathcal{R}_{\mu\nu}(in_1^\mu)(in_1^\nu) + \mathcal{R}_{\mu\nu}n_2^\mu n_2^\nu \\ \mathcal{R}_{aa} &= z^2(-\mathcal{R}_{tt} + \mathcal{R}_{rr}).\end{aligned} \tag{4.A.80}$$

Again we use the fact that our unperturbed spacetime is flat so the variation of these terms is

$$\begin{aligned}\delta(\mathcal{R}_{aa} - \frac{2}{3}\mathcal{R}) &= \delta\mathcal{R}_{\mu\nu}n_a^\mu n_a^\nu - \frac{2}{3}\delta\mathcal{R} \\ &= -\frac{z^2}{3}\delta\mathcal{R}_{tt} + \frac{z^2}{3}\delta\mathcal{R}_{rr} - \frac{2z^2}{3}\left(\frac{1}{r^2}\delta\mathcal{R}_{\theta\theta} + \frac{1}{r^2\sin^2\theta}\delta\mathcal{R}_{\phi\phi}\right) \\ &= \frac{z^2}{3}\delta\mathcal{R} - \frac{z^2}{r^2}\delta\mathcal{R}_{\theta\theta} - \frac{z^2}{r^2\sin^2\theta}\delta\mathcal{R}_{\phi\phi} \end{aligned} \tag{4.A.81}$$

notice there is an abuse of notation where in the first line $\delta\mathcal{R} = \widetilde{G}^{\mu\nu}\delta\mathcal{R}_{\mu\nu}$ and in the last line $\delta\mathcal{R} = g^{(0)\mu\nu}\delta\mathcal{R}_{\mu\nu}$. Using $(4.A.67)$ we can write $(4.A.81)$ in terms of $h^{(2)}$,

$$\delta\mathcal{R}_{\mu\nu}n_a^\mu n_a^\nu - \frac{2}{3}\delta\mathcal{R} = \frac{2z^2}{r^2}h^{(2)}_{\theta\theta} + \frac{2z^2}{r^2\sin^2\theta}h^{(2)}_{\phi\phi} \tag{4.A.82}$$

The divergent contributions to the counterterms are

$$\begin{aligned}(\delta S_B^{ct})^{div} =& \frac{1}{8G_5}\int_{S^2} d\Omega_2\Big[\frac{1}{2\epsilon}\Big(h^{(0)}_{\theta\theta} + \frac{1}{\sin^2\theta}h^{(0)}_{\phi\phi}\Big) + \frac{\ln\epsilon}{2}\Big(\frac{1}{R^2}h^{(0)}_{\theta\theta} + \frac{1}{R^2\sin^2\theta}h^{(0)}_{\phi\phi} \\ & - 2h^{(0)}_{rr} - \frac{2}{R^2}h^{(0)}_{\theta\theta} - \frac{2}{R^2\sin^2\theta}h^{(0)}_{\phi\phi} + \frac{1}{R}\partial_r h^{(0)}_{\theta\theta} + \frac{1}{R\sin^2\theta}\partial_r h^{(0)}_{\phi\phi} \\ & - 2h^{(2)}_{\theta\theta} - \frac{2}{\sin^2\theta}h^{(2)}_{\phi\phi}\Big)\Big]\end{aligned} \tag{4.A.83}$$

which matches with (4.3.58).

4.B Asymptotic expansions and integrals

4.B.1 Dilatation eigenfunction expansion

Under dilatation transformation $x^\mu \to \Omega x^\mu$, the boundary metric transforms as

$$\gamma_{\mu\nu} \to \Omega^2\gamma_{\mu\nu}. \tag{4.B.1}$$

In terms of infinitesimal operator

$$\gamma_{\mu\nu} \to (1 + \epsilon\delta_D)\gamma_{\mu\nu} \tag{4.B.2}$$

where $1 + \epsilon = \Omega$. The dilatation operator for the boundary metric γ is then

$$\delta_D = 2\int d^dx\gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}} \tag{4.B.3}$$

which replaces $\gamma_{\mu\nu}$ with $2\gamma_{\mu\nu}$ as the dilatation weight of the metric is 2. The dilatation operator in general contains all fields that transform non-trivially under dilatation. For our purposes we will actually only consider pure gravitational systems so the dilatation operator only contains the metric $\gamma_{\mu\nu}$. In the radial gauge the extrinsic curvature depends only on $\gamma_{\mu\nu}$ and it curvature can be expanded in Fefferman-Graham coefficients as

$$\begin{aligned} K_{\mu\nu}[\gamma] &= -\frac{z}{2}\partial_z\gamma_{\mu\nu} \\ &= z^{-2}g^{(0)}_{\mu\nu} - z^2 g^{(4)}_{\mu\nu} + \cdots + \frac{2-d}{2}\tilde{g}^{(d)}_{\mu\nu}\log z^2 - \tilde{g}^{(d)}_{\mu\nu} + \frac{2-d}{2}g^{(d)}_{\mu\nu} + \cdots \end{aligned} \tag{4.B.4}$$

and in dilatation eigenfunction expansion

$$K_{\mu\nu}[\gamma] = K_{(0)\ \mu\nu}[\gamma] + K_{(2)\ \mu\nu}[\gamma] + \cdots + \tilde{K}_{(d)\ \mu\nu}[\gamma]\log z^2 + K_{(d)\ \mu\nu}[\gamma] + \cdots \tag{4.B.5}$$

where the logarithmic terms are only present for even d. The dilatation eigenfunctions transform according to their order: we have homogenous transformations for $K_{(n<d)\ \mu\nu}$ and $\tilde{K}_{(d)\ \mu\nu}$,

$$\delta_D K_{(n)\ \mu\nu} = (2-n)K_{(n)\ \mu\nu} \tag{4.B.6}$$

and inhomogenous transformations for $K_{(d)\ \mu\nu}$,

$$\delta_D K_{(d)\ \mu\nu} = (2-d)K_{(d)\ \mu\nu} - 2\tilde{K}_{(d)\ \mu\nu}. \tag{4.B.7}$$

The origin of the inhomogenous transformation will become obvious when we relate the two expansions. To do that we need to express the radial derivative in terms of functional derivative of $\gamma_{\mu\nu}$

$$-z\partial_z = -z\partial_z|_{\gamma_{\mu\nu}=const} + \int d^dx 2K_{\mu\nu}[\gamma]\frac{\delta}{\delta\gamma_{\mu\nu}}. \tag{4.B.8}$$

Let us drop the first term as we are considering field that does not depend on z explicitly. We know from $(4.B.6)$ that the zeroth term in the dilatation eigenfunction expansion $K_{(0)\ \mu\nu}[\gamma]$ is proportional to $\gamma_{\mu\nu}$ then comparing with the leading term in $(4.B.4)$ we can deduce

$$K_{(0)\ \mu\nu}[\gamma] = \gamma_{\mu\nu}. \tag{4.B.9}$$

We see that expanding the extrinsic curvature in $(4.B.8)$ the radial derivative is related to the dilatation operator by

$$-z\partial_z = \delta_D + \delta_{(2)} + \cdots \tag{4.B.10}$$

where

$$\delta_{(n)} = \int d^dx 2K_{(n)\ \mu\nu}[\gamma]\frac{\delta}{\delta\gamma_{\mu\nu}}. \tag{4.B.11}$$

Taylor expanding the $K_{(n)\,\mu\nu}[\gamma]$ about $z^{-2}g^{(0)}_{\mu\nu}$

$$K_{(n)\,\mu\nu}[\gamma] = K_{(n)\,\mu\nu}[z^{-2}g^{(0)}] + \int g^{(2)}_{\rho\sigma}\frac{\delta K_{(n)\,\mu\nu}}{\delta\gamma_{\mu\nu}}|_{\gamma=z^{-2}g^{(0)}} + \cdots \tag{4.B.12}$$

Since $K_{(n)\mu\nu}[z^{-2}g^{(0)}]$ are also dilatation eigenfunctions, we can rescale the metric to get rid of the implicit z dependence. Using the integrated transformation of $(4.B.6)$ for $K_{(n<d)\,\mu\nu}$ and $\tilde{K}_{(d)\,\mu\nu}$,

$$K_{(n)\,\mu\nu}[z^{-2}g^{(0)}] = z^{n-2}K_{(n)\,\mu\nu}[g^{(0)}]. \tag{4.B.13}$$

Now that we know at the leading order we can write the dilatation operator in terms of the radial derivative $\delta_D \sim -z\partial_z$ for implicit z dependence terms then,

$$-z\partial_z\left(\tilde{K}_{(d)\,\mu\nu}[\gamma]\log z^2 + K_{(d)\,\mu\nu}[\gamma]\right) \sim \delta_D\left(\tilde{K}_{(d)\,\mu\nu}[\gamma]\log z^2 + K_{(d)\,\mu\nu}[\gamma]\right). \tag{4.B.14}$$

Note the bracket term depends on z through γ only because of the diffeomorphism invariance of the bulk action. Expanding the bracket we get

$$-z\partial_z\tilde{K}_{(d)\,\mu\nu}\log z^2 - 2\tilde{K}_{(d)\,\mu\nu} - z\partial_z K_{(d)\,\mu\nu} \sim \delta_D\tilde{K}_{(d)\,\mu\nu}\log z^2 + \delta_D K_{(d)\,\mu\nu} \tag{4.B.15}$$

and for all n at leading order of z we have

$$-z\partial_z K_{(n)\,\mu\nu}[z^{-2}g^{(0)}] \sim (2-n)K_{(n)\,\mu\nu}[z^{-2}g^{(0)}]. \tag{4.B.16}$$

Hence matching the leading order terms in $(4.B.15)$ we get back the inhomogenous transformation in $(4.B.7)$. After all the steps above we arrive at the z expansion of the dilatation eigenfunctions,

$$K_{(0)\,\mu\nu}[\gamma] = z^{-2}g^{(0)}_{\mu\nu} + g^{(2)}_{\mu\nu} + \cdots \tag{4.B.17}$$

$$K_{(2)\,\mu\nu}[\gamma] = K_{(2)\,\mu\nu}[g^{(0)}] + z^2\int g^{(2)}_{\rho\sigma}\frac{\delta K_{(2)\,\mu\nu}}{\delta g^{(0)}_{\mu\nu}} + \cdots \tag{4.B.18}$$

and so on. The final steps to relate the Fefferman-Graham coefficients to the dilatation eigenfunctions is to express $K_{(n)\,\mu\nu}[g^{(0)}]$ in terms of $g^{(m)}_{\mu\nu}$. In general $K_{(n)\,\mu\nu}[g^{(0)}]$ are obtained by comparing with the z^{n-2} in $(4.B.4)$, i.e. for $d > 4$

$$\begin{aligned} z^0: \quad K_{\mu\nu}[\gamma] &= g^{(2)}_{\mu\nu} + K_{(2)\,\mu\nu}[g^{(0)}] \\ &= 0 \end{aligned} \tag{4.B.19}$$

$$\begin{aligned} z^2: \quad K_{\mu\nu}[\gamma] &= z^2 g^{(4)}_{\mu\nu} + \int g^{(2)}_{\rho\sigma}\frac{\delta K_{(2)\,\mu\nu}}{\delta g^{(0)}_{\mu\nu}} + z^2 K_{(4)\,\mu\nu}[g^{(0)}] \\ &= -z^2 g^{(4)}_{\mu\nu} \end{aligned} \tag{4.B.20}$$

so we get

$$K_{(2)\,\mu\nu}[g^{(0)}] = -g^{(2)}_{\mu\nu} \tag{4.B.21}$$

$$K_{(4)\,\mu\nu}[g^{(0)}] = -2g^{(4)}_{\mu\nu} + \int g^{(2)}_{\rho\sigma}\frac{\delta g^{(2)}_{\mu\nu}}{\delta g^{(0)}_{\mu\nu}}. \tag{4.B.22}$$

For larger n, there will be functional derivative terms coming from the Taylor expansion in $(4.B.12)$ at the z^n order, for example

$$z^{n-2}\int g^{(n-2)}\cdot\frac{\delta K_{(2)}}{\delta g^{(0)}},\cdots, z^{n-2}\left(\int\cdots\int\right)^m (g^{(p_1)}\cdots g^{(p_m)})\cdot(\frac{\delta}{\delta g^{(0)}}\cdots\frac{\delta}{\delta g^{(0)}})K_{(q)},\cdots \tag{4.B.23}$$

where $q + p_1 + \cdots + p_m = n$. Of course when onshell all $g^{n<d}_{\mu\nu}$ and $\tilde{g}^{(d)}_{\mu\nu}$ are functions of $g^{(0)}_{\mu\nu}$. Order by order, we can write all the dilatation eigenfunctions in terms of the terms Fefferman-Graham expansion.

4.B.2 Volume integrals of $h_{(n)}$

This appendix will address some technical steps omitted in section 4.4.3. In those examples, the integral term in (4.4.55) is given by a volume integral over B_ϵ. We know the counterterm is given by surface integral over the regulated boundary of the entangling surface $\partial\tilde{B}_\epsilon$. Since $\partial\tilde{B}_\epsilon = \partial B_\epsilon$, we need to express the integral term as a surface integral over ∂B_ϵ. In the following we will show the relation between volume and surface integrals of the terms in the Fefferman-Graham expansion.

The leading term in the Fefferman-Graham expansion, $h_{(0)\,\mu\nu}$, is part of the boundary data hence should be treated as independent variable. Nonetheless, we can express them as combination of total derivatives and moment density of the derivatives of $h_{(0)\,\mu\nu}$. For the spatial trace $h_{(0)\,ii}$ we have

$$\begin{aligned}(d-2)h_{(0)\,ii} = \partial_i\left(x^i h_{(0)\,jj} - x^j h_{(0)\,ij} - \frac{\vec{x}^2}{2}\left(\partial_i h_{(0)\,jj} - \partial_i h_{(0)\,ij}\right)\right)\\ + \frac{\vec{x}^2}{2}\left(\partial_i\partial_i h_{(0)\,jj} - \partial_i\partial_j h_{(0)\,ij}\right).\end{aligned} \tag{4.B.24}$$

From the Einstein equation the last bracket above is related to $h_{(2)\,ii}$ by (4.3.33) and we get

$$h_{(0)\,ii} = \frac{1}{d-2}\partial_i\left(x^i h_{(0)\,jj} - x^j h_{(0)\,ij} - \frac{\vec{x}^2}{2}\left(\partial_i h_{(0)\,jj} - \partial_i h_{(0)\,ij}\right)\right) + \vec{x}^2 h_{(2)\,ii}. \tag{4.B.25}$$

Integrating over B_ϵ, we obtain a surface integral and a second moment of $h_{(2)\,ii}$ over B_ϵ,

$$\int_{B_\epsilon} d^{d-1}x\, h_{(0)\,ii} = \frac{1}{d-2}\int_{\partial B_\epsilon} d^{d-2}x\hat{x}^i \left(x^i h_{(0)\,jj} - x^j h_{(0)\,ij} - \frac{\vec{x}^2}{2}\left(\partial_i h_{(0)\,jj} - \partial_i h_{(0)\,ij}\right)\right) \tag{4.B.26}$$
$$+ \int_{B_\epsilon} d^{d-1}x\, \vec{x}^2 h_{(2)\,ii}.$$

Since ∂B_ϵ is a sphere of radius $\vec{x}^2 = R^2 - \epsilon^2$, we can reverse the surface integral for the last terms in the first line to get back a volume integral of $h_{(2)\,ii}$ over B_ϵ.

$$\begin{aligned}\int_{B_\epsilon} d^{d-1}x\, h_{(0)\,ii} = &\frac{(R^2-\epsilon^2)^{\frac{d-1}{2}}}{d-2}\int_{\partial B_\epsilon} d\Omega_{d-2}\left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) \\ &- \int_{B_\epsilon} d^{d-1}x\,(R^2-\epsilon^2-\vec{x}^2) h_{(2)\,ii}.\end{aligned} \tag{4.B.27}$$

Gathering the terms that appear in the integral correction terms we get

$$\int_{B_\epsilon} d^{d-1}x \left(h_{(0)\,ii} + (R^2-\vec{x}^2)h_{(2)\,ii}\right) = \frac{(R^2-\epsilon^2)^{\frac{d-1}{2}}}{d-2}\int_{S^{d-2}} d\Omega_{d-2}\left(h_{(0)\,ii} - \hat{x}^i\hat{x}^j h_{(0)\,ij}\right) \tag{4.B.28}$$
$$+ \int_{B_\epsilon} d^{d-1}x\, \epsilon^2 h_{(2)\,ii}.$$

For $d > 3$, the integral correction term contains higher order terms in the Fefferman-Graham expansion. In general, the n^{th} order terms are second derivative of $(n-2)^{th}$. The following expressions for evaluating volume integral of a generic second derivative of a tensor will be useful later on. First the second moment of such a derivative is

$$\int_{B_\epsilon} d^{d-1}x\, \vec{x}^2 \partial_i\partial_j A_{ij} = \int_{B_\epsilon} d^{d-1}x\, \partial_i\left(\vec{x}^2\partial_j A_{ij} - 2\vec{x}^j A_{ij}\right) + 2A_{ii} \tag{4.B.29}$$

then the shifted second moment is

$$\begin{aligned}\int_{B_\epsilon} d^{d-1}x\,(R^2-\vec{x}^2)\partial_i\partial_j A_{ij} &= \int_{B_\epsilon} d^{d-1}x\,\partial_i\left(R^2\partial_j A_{ij} - \vec{x}^2\partial_j A_{ij} + 2\vec{x}^j A_{ij}\right) - 2A_{ii} \\ &= \epsilon^2\int_{\partial B_\epsilon} d^{d-2}x\,\hat{x}^i\partial_j A_{ij} + \int_{\partial B_\epsilon} d^{d-2}x\, 2\hat{x}^i x^j A_{ij} - \int_{B_\epsilon} d^{d-1}x\, 2A_{ii} \\ &= 2r^{d-1}\int_{S^{d-2}} d\Omega_{d-2}\,\hat{x}^i\hat{x}^j A_{ij} - 2\int_{B_\epsilon} d^{d-1}x\, A_{ii} \\ &+ \epsilon^2 r^{d-2}\int_{S^{d-2}} d\Omega_{d-2}\,\hat{x}^i\partial_j A_{ij}.\end{aligned} \tag{4.B.30}$$

The $d = 4$ examples in section 4.4.3.2, we have integral of the form of $(4.B.31)$ where

$$\tilde{h}_{(4)\,ii} = \partial_i\partial_j A_{ij} \tag{4.B.31}$$

and

$$A_{ij} = \frac{1}{8}\left(h_{(2)\,ij} - \delta_{ij} h_{(2)\,kk}\right). \tag{4.B.32}$$

Neglecting the $O(\epsilon^2)$ term since they are irrelevant in (4.4.75) we get,

$$\begin{aligned}\int_{B_\epsilon} d^3x\,(R^2 - \vec{x}^2)\tilde{h}_{(4)\,ii} = \frac{(R^2-\epsilon^2)^{\frac{3}{2}}}{4}\int_{S^2} d\Omega_2\,(\hat{x}^i\hat{x}^j h_{(2)\,ij} - h_{(2)\,ii}) \\ + \frac{1}{2}\int_{B_\epsilon} d^3x\,h_{(2)\,ii}.\end{aligned} \tag{4.B.33}$$

As seen in (4.3.33), $h_{(2)\,ii}$ is the second derivative of $h_{(0)\,ij}$, the last volume integral can be easily turned into surface integral,

$$\begin{aligned}\int_{B_\epsilon} d^3x\,h_{(2)\,ii} &= \frac{(R^2-\epsilon^2)^{\frac{1}{2}}}{4}\int_{S^2} d\Omega_2 \hat{x}^i(\partial_i h_{(0)\,jj} - \partial_j h_{(0)\,ij}) \\ &= \frac{r}{4}\int_{S^2} d\Omega_2 \hat{x}^i\Big(\partial_i h_{(0)\,jj} - \partial_r h_{(0)\,ir} \\ &- \frac{1}{r^2}\partial_\theta h_{(0)\,\theta i} - \frac{1}{r^2\sin^2\theta}\partial_\phi h_{(0)\,\phi i} - \frac{\cos\theta}{r^2\sin\theta} h_{(0)\,\theta i} - \frac{2}{r} h_{(0)\,ri}\Big) \\ &= \frac{r}{4}\int_{S^2} d\phi d\theta \sin\theta\Big(\hat{x}^i\partial_i h_{(0)\,jj} - \hat{x}^i\hat{x}^j\hat{x}^k\partial_k h_{(0)\,ij} \\ &+ \frac{1}{r^3} h_{(0)\,\theta\theta} + \frac{1}{r^3\sin^2\theta} h_{(0)\,\phi\phi} - \frac{2\hat{x}^i\hat{x}^j}{r} h_{(0)\,ij}\Big)\end{aligned} \tag{4.B.34}$$

where we went from the first line to the second line by evaluating $\partial_j h_{(0)\,ij}$ in polar coordinates. From the second line to the third line we integrate by parts and we transform r coordinate to Cartesian. Finally we can transform the angular coordinate into Cartesian coordinates,

$$\begin{aligned}\int_{B_\epsilon} d^3x\,h_{(2)\,ii} = \frac{r}{4}\int_{S^2} d\Omega_2\Big[h_{(0)\,ii} - 3\hat{x}^i\hat{x}^j h_{(0)\,ij} \\ + x^j\partial_j h_{(0)\,ii} - \hat{x}^i\hat{x}^j x^k\partial_k h_{(0)\,ij}\Big].\end{aligned} \tag{4.B.35}$$

For the $d = 4$ example in section 4.4.3.3, we have integrals of the form of $(4.B.31)$ where

$$h_{(4)\,ii} = \partial_i\partial_j B_{ij} \tag{4.B.36}$$

and

$$B_{ij} = \frac{1}{4}\left(\delta_{ij} h_{(2)\,kk} - h_{(2)\,ij}\right). \tag{4.B.37}$$

Neglecting the $O(\epsilon^2)$ term since they are irrelevant in (4.4.84) we get

$$\int_{B_\epsilon} d^4x\,(R^2-\vec{x}^2)h_{(4)\,ii} = \frac{(R^2-\epsilon^2)^{\frac{5}{2}}}{2}\int_{S^3} d\Omega_3\,(h_{(2)\,ii}-\hat{x}^i\hat{x}^j h_{(2)\,ij}) - \frac{3}{2}\int_{B_\epsilon} d^4x\,h_{(2)\,ii} \tag{4.B.38}$$

Following the steps in $(4.B.34)$ we can evaluate the volume integral of $h_{(2)\,ii}$,

$$\begin{aligned}\int_{B_\epsilon} d^4x\,h_{(2)\,ii} &= \frac{(R^2-\epsilon^2)^{\frac{3}{2}}}{6}\int_{S^3} d\Omega_3\hat{x}^i(\partial_i h_{(0)\,jj}-\partial_j h_{(0)\,ij}) \\ &= \frac{r^3}{6}\int_{S^3} d\Omega_3\hat{x}^i\Big(\partial_i h_{(0)\,jj}-\hat{x}^j\hat{x}^k\partial_k h_{(0)\,ij} \\ &- \frac{1}{r^2}\partial_{\theta_1}h_{(0)\,\theta_1 i}-\frac{1}{r^2\sin^2\theta_1}\partial_{\theta_2}h_{(0)\,\theta_2 i}-\frac{1}{r^2\sin^2\theta_1\sin^2\theta_2}\partial_\phi h_{(0)\,\phi i} \\ &- \frac{2\cos\theta_1}{r^2\sin\theta_1}h_{(0)\,\theta_1 i}-\frac{\cos\theta_2}{r^2\sin^2\theta_1\sin\theta_2}h_{(0)\,\theta_2 i}-\frac{3}{r}h_{(0)\,ri}\Big) \\ &= \frac{r^3}{6}\int_{S^3} d\phi d\theta_1 d\theta_2\sin^2\theta_1\sin\theta_2\Big(\hat{x}^i\partial_i h_{(0)\,jj}-\hat{x}^i\hat{x}^j\hat{x}^k\partial_k h_{(0)\,ij} \\ &+ \frac{1}{r^3}h_{(0)\,\theta_1\theta_1}+\frac{1}{r^3\sin^2\theta_1}h_{(0)\,\theta_2\theta_2}+\frac{1}{r^3\sin^2\theta_1\sin^2\theta_2}h_{(0)\,\phi\phi}-\frac{3\hat{x}^i\hat{x}^j}{r}h_{(0)\,ij}\Big)\end{aligned} \tag{4.B.39}$$

Finally transforming into Cartesian coordinate we get

$$\int_{B_\epsilon} d^4x\,h_{(2)\,ii} = \frac{r^3}{6}\int_{S^3} d\Omega_3\Big[h_{(0)\,ii}-4\hat{x}^i\hat{x}^j h_{(0)\,ij} + x^j\partial_j h_{(0)\,ii}-\hat{x}^i\hat{x}^j x^k\partial_k h_{(0)\,ij}\Big]. \tag{4.B.40}$$

4.C Covariant phase space Hamiltonian

In this section we follow the formalism in [103] but here we consider the renormalized action, as well as different conditions on the vector. The variational problem of a Lagrangian theory with bulk and boundary terms requires the variation of both the bulk and boundary terms to be zero onshell. Therefore the sum of the presymplectic potential and the variation of the boundary terms should be exact on the boundary of the manifold

$$\boldsymbol{\Theta}[\delta\phi]-\delta\boldsymbol{B} = d\boldsymbol{C}[\delta\phi]. \tag{4.C.1}$$

The presymplectic current can be expressed as

$$\boldsymbol{\omega}[\delta_1\phi,\delta_2\phi] = \delta_1\left(\boldsymbol{\Theta}[\delta_2\phi]-d\boldsymbol{C}[\delta_2\phi]\right) \tag{4.C.2}$$

where δ is the exterior derivative on the configuration space. In Einstein gravity with cosmological constant and Gibbons-Hawking boundary term, without imposing any boundary condition, we get

$$\boldsymbol{\Theta}[\delta g] - \delta \boldsymbol{B}^{GH} = d\boldsymbol{C}^{GH}[\delta g] + \boldsymbol{\pi} \cdot \delta g. \tag{4.C.3}$$

The exact contribution $\boldsymbol{C}^{GH}[\delta g]$ captures the variation of the metric in the normal direction and the canonical momentum term captures the usual variation of the induced metric. Hence one can eliminate this term by imposing a radial gauge condition. However, as we will see later the variation of $\boldsymbol{C}^{GH}$ will have a non zero contribution. On B_ϵ we get

$$\boldsymbol{C}^{GH}[\delta g] = -\frac{\boldsymbol{\varepsilon}_{\mu\nu}}{16\pi G_N}\gamma^{\nu\sigma} n^\mu n^\rho \delta g_{\sigma\rho} \tag{4.C.4}$$

$$= -\frac{\boldsymbol{\varepsilon}_{zt}}{16\pi G_N}\delta g^{tz}. \tag{4.C.5}$$

The variation of the Hamiltonian along the vector field ξ can be constructed from the presymplectic form $\tilde{\Omega}$

$$\delta H[\xi] = -\iota_{X_\xi}\tilde{\Omega} \tag{4.C.6}$$

where X_ξ is the configuration space vector that takes the one form in configuration space to the Lie derivative in configuration space

$$X_\xi(\delta\phi) = \mathcal{L}_{X_\xi}\phi. \tag{4.C.7}$$

The Lie derivative in configuration space only varies the dynamical fields along ξ direction and the Lie derivative in spacetime varies both the dynamical fields and background fields along the ξ direction. Any tensor is called covariant under the diffeomorphism induced by ξ if the two Lie derivatives coincide

$$\mathcal{L}_{X_\xi}T = \mathcal{L}_\xi T. \tag{4.C.8}$$

In general, the normal is constructed from a background function,

$$n \propto df, \tag{4.C.9}$$

such that the level sets of the function define a foliation. Anything that distinguishes the normal direction from other directions is not covariant unless we impose an extra condition on ξ,

$$\mathcal{L}_\xi f = \xi(f) = 0 \tag{4.C.10}$$

which implies the normal direction of ξ vanishes. We label the difference between the two Lie derivatives of $\boldsymbol{C}$ along generic ξ by

$$\boldsymbol{D}[\xi] = \mathcal{L}_\xi\boldsymbol{C} - \mathcal{L}_{X_\xi}\boldsymbol{C}. \tag{4.C.11}$$

Since the presymplectic form is given by the integral of the presymplectic form, ω, on the Cauchy surface $\mathcal{C}$, we can express the variation of the Hamiltonian as

$$\delta H[\xi] = \int_{\mathcal{C}} -\iota_{X_\xi}\boldsymbol{\omega}. \tag{4.C.12}$$

Through some algebra in a generic theory onshell we get

$$-X_\xi \cdot \boldsymbol{\omega} = d\left(\delta\boldsymbol{Q}[\xi] - \iota_\xi\delta\boldsymbol{B} - \delta\iota_{X_\xi}\boldsymbol{C} - \iota_\xi\boldsymbol{\pi}\cdot\delta\phi - \boldsymbol{D}[\xi] + d\iota_\xi\boldsymbol{C}\right). \tag{4.C.13}$$

Let us define the Hamiltonian potential as the density over $\partial\mathcal{C}$ so

$$\delta\boldsymbol{H}[\xi] = \delta\boldsymbol{Q}[\xi] - \iota_\xi\delta\boldsymbol{B} - \delta\iota_{X_\xi}\boldsymbol{C} - \iota_\xi\boldsymbol{\pi}\cdot\delta\phi - \boldsymbol{D}[\xi] + d\iota_\xi\boldsymbol{C}. \tag{4.C.14}$$

We can see that the Hamiltonian potential has an exact term ambiguity because the Hamiltonian is defined to be the integral of the Hamiltonian form over a manifold with no boundary. We will now show that the full Noether charge form is a well defined Hamiltonian potential of the renormalized action. Since we have found that the holographic charge form is equal to the full Noether charge form up to an exact term, the Hamiltonian defined through holographic charge form is the full Noether charge. In the context of the first law of entanglement entropy, neither the entanglement entropy nor the modular energy is a Hamiltonian or a conserved charge, and hence the exact term difference matters. Here we will derive an expression for $\delta\boldsymbol{\Delta}[\xi_B]$ in terms of the quantities defined above.

In [103], the case of Einstein gravity with cosmological constant and Gibbons-Hawking boundary term was considered. The boundary condition imposed was

$$\boldsymbol{\pi}\cdot\delta g = 0 \tag{4.C.15}$$

and restricting normal direction of ξ to be identically zero. Under these conditions, the variation of the Hamiltonian potential is

$$\delta\boldsymbol{H}^{BY}[\xi] = \delta\left(\boldsymbol{\varepsilon}_{\mu\nu}n^\mu T^\nu_\sigma\xi^\sigma\right) \tag{4.C.16}$$

$$= \delta\left(\boldsymbol{\varepsilon}_{\mu\nu}n^\mu 2\pi^\nu_\sigma\xi^\sigma\right) \tag{4.C.17}$$

$$= -\delta\boldsymbol{\mathcal{Q}}[\xi] \tag{4.C.18}$$

where $T_{\mu\nu}$ is the Brown York stress tensor given by

$$T^{\mu\nu} = -\frac{1}{8\pi G}\left(K^{\mu\nu} - \gamma^{\mu\nu}K\right). \tag{4.C.19}$$

In our case, not only we do not impose the boundary condition $(4.C.15)$, we also need to use the vector field ξ_B which will introduce a term relating to the normal component of ξ_B.

The Hamiltonian potential from Einstein gravity with Gibbons-Hawking boundary term is

$$\begin{aligned}\delta\boldsymbol{H}^{GH}[\xi_B] &= \delta\boldsymbol{Q}[\xi_B] - \iota_{\xi_B}\delta\boldsymbol{B}^{GH} - \delta\iota_{X_{\xi_B}}\boldsymbol{C}^{GH} - \iota_{\xi_B}\boldsymbol{\pi}\cdot\delta g - \boldsymbol{D}^{GH}[\xi_B] + d\iota_{\xi_B}\boldsymbol{C}^{GH} \\ &= \delta\boldsymbol{Q}[\xi_B] - \iota_{\xi_B}\delta\boldsymbol{B}^{GH} - \delta\iota_{X_{\xi_B}}\boldsymbol{C}^{GH} - \iota_{\xi_B}\boldsymbol{\pi}\cdot\delta g \\ &= \delta\boldsymbol{H}^{BY}[\xi_B] - \frac{\boldsymbol{\varepsilon}_{\mu\nu}n^\mu\tau^\nu}{16\pi G_N}\delta\gamma\tau^\alpha\partial_\alpha(\xi_B^\beta n_\beta) - \iota_{\xi_B}\boldsymbol{\pi}\cdot\delta g\end{aligned} \tag{4.C.20}$$

where τ is the future pointing timelike normal vector. To get to the last line we also used the following properties for the Killing vector ξ_B and in radial gauge,

$$\iota_{\xi_B}\boldsymbol{C}^{GH} = \boldsymbol{D}^{GH}[\xi_B] = 0. \tag{4.C.21}$$

When we consider the renormalized action there are additional counterterms in the full Hamiltonian potential

$$\begin{aligned}\delta\boldsymbol{H}^{full}[\xi] = \delta\boldsymbol{Q}[\xi] - \iota_\xi\delta\boldsymbol{B}^{GH} - \delta\iota_{X_\xi}\boldsymbol{C}^{GH} - \iota_\xi\boldsymbol{\pi}\cdot\delta g - \boldsymbol{D}^{GH}[\xi] + d\iota_\xi\boldsymbol{C}^{GH} \\ + \iota_\xi\delta\boldsymbol{B}^{ct} + \delta\iota_{X_\xi}\boldsymbol{C}^{ct} + \iota_\xi\boldsymbol{\pi}^{ct}\cdot\delta g + \boldsymbol{D}^{ct}[\xi] - d\iota_\xi\boldsymbol{C}^{ct}.\end{aligned} \tag{4.C.22}$$

Simplifying the above equation by gathering the boundary terms we get

$$\delta\boldsymbol{H}^{full}[\xi] = \delta\boldsymbol{Q}[\xi] + \delta\boldsymbol{b}^{GH}[\xi] - \delta\boldsymbol{b}^{ct}[\xi]. \tag{4.C.23}$$

Hence the Gibbon-Hawking Hamiltonian potential is related to the full Hamiltonian potential by

$$\delta\boldsymbol{H}^{GH}[\xi] = \delta\boldsymbol{H}^{full}[\xi] + \delta\boldsymbol{b}^{ct}[\xi]. \tag{4.C.24}$$

From $(4.C.4)$ and $(4.C.11)$ we can deduce

$$\boldsymbol{C}^{GH} = \boldsymbol{C}^{ct}, \qquad\qquad \boldsymbol{D}^{GH} = \boldsymbol{D}^{ct} \tag{4.C.25}$$

then we have

$$\delta\boldsymbol{H}^{full}[\xi] = \delta\boldsymbol{Q}^{full}[\xi] - \iota_\xi\boldsymbol{\pi}_{(d)}\cdot\delta g. \tag{4.C.26}$$

The full Hamiltonian potential is equal to the full Noether charge when the last term is zero. For a conformal Killing vector we can apply the tracelessness condition on $\boldsymbol{\pi}^{\mu\nu}_{(d)}$. In our case, the unperturbed $\boldsymbol{\pi}^{\mu\nu}_{(d)}$ is zero by itself, so we can relax all boundary condition on $\delta g_{\mu\nu}$.

By inspecting the dilatation eigenvalue expansion of $(4.C.20)$, the renormalized Brown-York Hamiltonian potential $\delta\boldsymbol{H}^{BY}_{(d)}[\xi_B]$ can be expressed in terms of $\delta\boldsymbol{H}^{GH}[\xi_B]$ and its counterterm,

$$\delta \boldsymbol{H}^{BY}_{(d)}[\xi_B] - \iota_{\xi_B}\boldsymbol{\pi}_{(d)} \cdot \delta g = \delta \boldsymbol{H}^{GH}[\xi_B] - \delta \boldsymbol{H}^{GH}_{ct}[\xi_B]. \tag{4.C.27}$$

In our setting $\boldsymbol{\pi}^{\mu\nu}_{(d)} = 0$ so the renormalized Brown-York Hamiltonian potential is obtained by subtracting the lower order terms in the dilatation eigenvalue expansion of the Gibbons-Hawking Hamiltonian potential. This should be distinguished from the full Hamiltonian that is constructed form the renormalized Lagrangian or action. These two procedures of obtaining the Hamiltonian are equivalent if the difference between Hamiltonian potentials is exact. We will see in the following how the two renormalization procedures differ in the context of entanglement entropy and modular energy.

First we use (4.C.20) and (4.C.22), to relate the two Hamiltonian potentials, $\delta \boldsymbol{H}^{BY}[\xi_B]$ and $\delta \boldsymbol{H}^{full}[\xi_B]$, by

$$\delta \boldsymbol{H}^{BY}[\xi_B] = \delta \boldsymbol{H}^{full}[\xi_B] - \iota_{\xi_B}\delta \boldsymbol{B}^{ct} - \delta \iota_{X_{\xi_B}}\boldsymbol{C}^{ct} + \frac{\boldsymbol{\varepsilon}_{\mu\nu} n^\mu \tau^\nu}{16\pi G_N}\delta\gamma\tau^\alpha\partial_\alpha(\xi^\beta_B n_\beta) \tag{4.C.28}$$

The difference between the two Hamiltonian potentials is not exact, and this implies $\delta \boldsymbol{H}^{BY}[\xi_B]$ is not a proper Hamiltonian potential that integrates to give the Hamiltonian induced by ξ_B. However, we shall see that the renormalized Brown-York Hamiltonian potential or the holographic charge form is an appropriate Hamiltonian potential. Let us first express it in terms of the full Hamiltonian potential and all the counterterms,

$$\delta \boldsymbol{H}^{BY}_{(d)}[\xi_B] = \delta \boldsymbol{H}^{full}[\xi_B] - \iota_{\xi_B}\delta \boldsymbol{B}^{ct} - \delta \iota_{X_{\xi_B}}\boldsymbol{C}^{ct} - \iota_{\xi_B}\boldsymbol{\pi}_{ct} \cdot \delta g - \delta \boldsymbol{H}^{GH}_{ct}[\xi_B] \tag{4.C.29}$$

$$\delta \boldsymbol{H}^{BY}_{(d)}[\xi_B] = \delta \boldsymbol{H}^{full}[\xi_B] - \delta \boldsymbol{\Delta}[\xi_B]. \tag{4.C.30}$$

The difference in the Hamiltonian potentials is non zero in general.

We can express the difference in Hamiltonian potentials as

$$\delta \boldsymbol{\Delta}[\xi_B] = \delta \boldsymbol{H}^{GH}_{ct}[\xi_B] + \iota_{\xi_B}\delta \boldsymbol{B}^{ct} + \delta \iota_{X_{\xi_B}}\boldsymbol{C}^{ct} + \iota_{\xi_B}\boldsymbol{\pi}_{ct} \cdot \delta g \tag{4.C.31}$$

$$= \delta \boldsymbol{H}^{GH}_{ct}[\xi_B] - \delta \boldsymbol{b}^{ct}[\xi_B]. \tag{4.C.32}$$

Hence, the physical interpretation of $\delta\boldsymbol{\Delta}$ is the difference of counterterms in the two renormalization procedure where $\delta\boldsymbol{b}^{ct}$ is the counterterms contribution of the Hamiltonian potential derived from the renormalized action and $\delta\boldsymbol{H}^{GH}_{ct}$ is the counterterm of the Hamiltonian potential derived from the bare action. More explicitly the we have the expression that matches with (4.4.38),

$$\delta \boldsymbol{\Delta}[\xi_B] = -\frac{\boldsymbol{\varepsilon}_{\mu\nu} n^\mu \tau^\nu}{16\pi G_N}\delta\gamma\tau^\alpha\partial_\alpha(\xi^\beta_B n_\beta) + \delta \iota_{X_{\xi_B}}\boldsymbol{C}^{ct} + \iota_{\xi_B}\delta \boldsymbol{B}^{ct} + \delta\left(\boldsymbol{\varepsilon}_{\mu\nu} n^\mu 2\pi^\nu_{ct\,\sigma}\xi^\sigma\right), \tag{4.C.33}$$

with

$$\delta\iota_{X_{\xi_B}}\boldsymbol{C}^{ct} = \frac{z\boldsymbol{\varepsilon}_{zt}}{16\pi G_N}\delta g^t_t \partial_z \xi^t_B. \tag{4.C.34}$$

Let us now dissect $(4.C.33)$ term by term.

The first term captures the non-covariant variation of the normal direction. In [103] this term is absent as they restrict the diffeomorphism generator to preserve covariance of the normal. In [91], this term is absent as a stronger fall-off condition is imposed.

The second term captures the variation of the diffeomorphism of the metric in the normal direction. This term is non-vanishing because ξ_B is no longer Killing in the perturbed metric. Hence we do not see the equivalent of this term in the unperturbed $\boldsymbol{\Delta}[\xi_B]$ from (4.4.15). The last two terms are the standard counterterm contributions from the full Hamiltonian potential and Brown-York Hamiltonian potential. The non trivial result we found is that this difference is exact, the exterior derivative of the density of the entanglement entropy counterterms,

$$\delta\boldsymbol{\Delta}[\xi_B] = d\delta\boldsymbol{S}^{ct}_B. \tag{4.C.35}$$

Then the Hamiltonian defined by the renormalized Brown-York Hamiltonian potential is the same as the full Hamiltonian potential,

$$\begin{aligned}\delta\mathcal{H}[\xi_B] &= \int_{\partial\mathcal{C}} \delta\boldsymbol{H}^{BY}_{(d)}[\xi_B] && (4.C.36)\\ &= \int_{\partial\mathcal{C}} \delta\boldsymbol{H}^{full}[\xi_B] - d\delta\boldsymbol{S}^{ct}_B \\ &= \int_{\partial\mathcal{C}} \delta\boldsymbol{H}^{full}[\xi_B] \\ \delta\mathcal{H}[\xi_B] &= \delta H^{full}[\xi_B]. && (4.C.37)\end{aligned}$$

For entanglement entropy and modular energy this difference matters because the integral is over a manifold with boundary that turns the exact term into the appropriate counterterm for the entanglement entropy. This analysis establishes the first law of renormalized entanglement entropy.

CHAPTER 5

Classical String Correction to Holographic Chaos

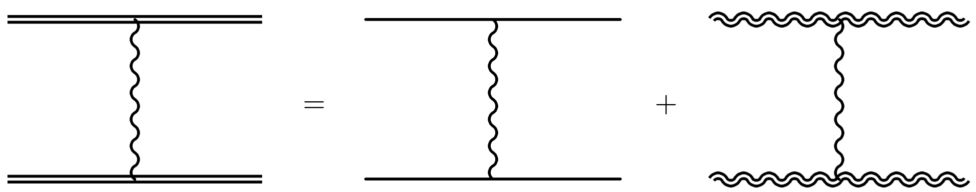
5.1 Introduction

In the original AdS/CFT correspondence, the bulk classical gravity theory acts as the effective field theory of a quantum string theory. One would expect at certain energy scale, the stringy behaviour will start to become significant. Since the quantum chaos on the boundary quantum theory is characterised by the high energy scattering in the bulk spacetime, one would expect there to be stringy correction to the scrambling behaviour in the gravitational theory. There have been investigations on the quantum string correction to the Lyapunov exponent of holographic chaos [23]. In [23], they considered Tachyon-Tachyon scattering in curved spacetime to simulate the high energy scattering in gravity. The exchange of stringy Pomeron in curved spacetime introduced a subleading term in the Lyapunov exponent that reduces the magnitude of the scrambling rate and increases the scrambling time. This supports conjectured chaos bound in [67],

$$\lambda_L \leq \frac{2\pi}{\beta}, \tag{5.1.1}$$

which is saturated by Einstein gravity.

In this chapter, we are investigating the classical strings scattering process to probe the possible correction from stringy scattering to chaos. In particular, we are interested in the transverse oscillation contribution to the Lyapunov exponent. We will follow closely



(5.1.2)

Figure 5.1.1: These are the schematic Feynman diagrams representing the tree level forward scattering with graviton, denoted by the wiggly line, as the intermediate particle. On the left, the doubled straight lines represent the full classical string. In the middle, the solid straight lines represent the zero mode of the classical string which is identical to a particle following the centre of mass motion of the string. On the right, the doubled wiggly lines represent oscillation modes of the classical string.

with the elastic eikonal scattering approach in [23] where the eikonal phase is equal to the classical gravity action. Hence we aim to obtain the classical gravity action with linear perturbation of the metric being sourced by the energy momentum tensors generated from the motion of highly energetic classical strings with large momentum component along the horizons of the two sided black hole.

The aim is to explore the possible corrections originate from the stringy nature in AdS/CFT. We should emphasise the following exploration of classical string scattering is only one of many stringy corrections one can include. In particular we are focusing on the dependence of s in the eikonal phase which is related to Regge behaviour mention in (2.5.1). The semi-classical 2-2 scattering in the shock wave picture is related to the perturbative quantum gravity picture; the shock waves induced by the backreaction of the particles act as the intermediate particles in the Feynman diagrams. Hence the Regge behaviour is governed by the spin-2 gravitons exchange. In our picture, we are looking the backreaction induced by the classical string. Therefore the intermediate particle is still the spin-2 graviton. However, we are taking a deviation from Tachyon or scalar field; we introduce transverse oscillations in the classical strings, see figure 5.1.1. The qualitative result from this calculation can give us insight into the full quantum string scattering of excited states. Nonetheless, our methodology of calculating the backreaction of classical string is also an interesting result on its own.

5.1.1 Overview of approach

In section 2.8, we saw the Lyapunov exponent was directly related to power law behaviour of the eikonal phase. For the stringy correction in [23], which we reviewed in section 2.8.1, they focused on the shift in Regge intercept due to the curvature of the spacetime. However, in this chapter we are focusing on the correction induced by replacing the incoming scalar particles to classical strings. The approach we are taking follows directly from the shock wave calculation of the eikonal phase in section 2.8 but replaces the high energy particle energy momentum tensors with high energy classical string energy momentum tensors. In this section, we are going to outline our approach step by step before presenting the calculating in the later part of the chapter.

We would like to set up two closed strings propagating along the two horizons of the two sided AdS planar black hole separately. So the closed string action is given by the Polyakov action with the two sided AdS planar black hole as the target. Then we solve the worldsheet Polyakov action variational problem with respect to the string coordinates and the worldsheet metric to get the equations of motion and the constraints in the lightcone gauge and flat worldsheet gauge. We can solve for the equations of motion of the string to obtain the expansion of the oscillation modes.

With the string solutions, we can obtain the spacetime energy momentum tensors that are used to obtain the backreaction. We need to vary the string action with respect to the spacetime metric to get the spacetime energy momentum tensor. Then we substitute the string solutions back to the energy momentum tensor and expand it to the subleading order. The subleading energy momentum tensors are the sources for subleading backreactions. Both are responsible for the corrections to the eikonal phase.

Given the spacetime energy momentum tensor, we need the linearised Einstein equations to find the backreaction, i.e. shock wave geometry. The covariant de Donder gauge renders the simplest form of linearised modified Einstein equations which have the Laplacians as the only second derivative operators. Some of the Einstein equations have only the Laplacian of the derivatives of the transverse coordinates, others have the full Laplacian including the null derivatives.

Here we give a technical description on how we solved the linearised Einstein equations. Since the energy momentum tensors are singular in one of the null coordinate, we split the metric perturbation into singular and non-singular parts which correspond to the Green's function of transverse Laplacian and the full Laplacian respectively. By looking at the null coordinates dependence of the energy momentum tensors, we can split the metric perturbation according to the null coordinates dependence.

Because the interaction occurs at the bifurcation point, we would like to set both null coordinates to zero. However, we have to be careful with the singular distributional functions, i.e. delta function and its derivative. After manipulation of the type $u\partial_u\delta(u) = -\delta(u)$, we can then drop the terms with explicit null coordinates as coefficients. The Einstein equations simplified to one independent and 3 pairs of coupled sets of second order linear PDE's with constant coefficients.

We then Fourier transform the Einstein equations which then become a set of algebraic equations with momenta as coefficients. We solve the algebraic equations and write the Fourier transform of the metric perturbations in terms of the momenta and the Fourier transform of the energy momentum tensors.

Finally, after evaluating the integrals of the inverse Fourier transform, we obtain the metric perturbations which are the backreactions. By the argument presented in section 2.8, the eikonal phase is equal to the onshell action. So we evaluate the onshell action with the subleading metric perturbations and subleading energy momentum tensor to obtain the next-to-subleading eikonal phase induced by the classical string oscillating.

5.1.2 Overview of result

We found the corrections to eikonal phase due to the contributions of transverse classical string oscillations to be insignificant or negligible. More specifically, these type of subleading contributions to the eikonal phase do not grow when the centre of mass energy increases,

$$\delta_{sub}(s, \mathbf{b}) \sim O\left(s^{-n}\right), \qquad n \geq 0. \tag{5.1.3}$$

Due to the scaling of the centre of mass energy s, these types of corrections are neglected in high energy scattering. This result matches with our expectation as the intermediate particle remains spin-2.

5.2 High energy classical strings scattering in curved spacetime

5.2.1 Setup

The original metric, $g_{\mu\nu}$, of the AdS_{d+1} planar black hole is

$$ds^2 = -a(uv)dudv + r^2(uv)\delta_{ij}dx^i dx^j. \tag{5.2.1}$$

The metric perturbation $h_{\mu\nu}$ induces the perturbation in the Ricci tensor as

$$R_{\mu\nu}[g+h] = R^{(0)}_{\mu\nu}[g] + R^{(1)}_{\mu\nu}[g,h] + \cdots . \tag{5.2.2}$$

The unperturbed Ricci tensors is

$$R^{(0)}_{\mu\nu} = -\frac{(D-1)}{l^2_{AdS}} g_{\mu\nu}. \tag{5.2.3}$$

Using this relation we can work out explicitly the connection between a, r, a', r', the dimension and the "radius" of the AdS at the horizons $u = 0$ or $v = 0$,

$$\frac{r'}{r} = -\frac{(D-1)a}{4l_{AdS}^2} \tag{5.2.4}$$

$$\frac{a'}{a^2} = \frac{(D-1)(D-4)}{4l_{AdS}^2}. \tag{5.2.5}$$

To evaluate the curvature tensor perturbation, we find the covariant de Donder gauge helpful,

$$\nabla^\mu \bar{h}_{\mu\nu} = 0 \tag{5.2.6}$$

where $\bar{h}$ is

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h. \tag{5.2.7}$$

After fixing the gauge of the metric perturbation, the linear perturbation of the Ricci tensor is

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\nabla^2 h_{\mu\nu} + R^\rho{}_{\mu\nu}{}^\sigma h_{\rho\sigma} + R^\sigma_{(\mu} h_{\nu)\sigma} \tag{5.2.8}$$

which its components are listed in appendix $5.A$.

5.2.2 Classical string dynamics

The motion of the classical string in the two sided black hole background is governed by equations of motion of the string. The analysis of classical string follows from standard bosonic string theory but omits the quantisation of the oscillation modes, i.e. without promoting the coefficients of the oscillation modes to quantum operators. We start off by finding the equations of motion of the classical string from the Polykov string action. Given a general background metric $g_{\mu\nu}$, the Polykov string action is,

$$\mathcal{S}_{string} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma}\gamma^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \tag{5.2.9}$$

where γ_{ab} is the worldsheet metric, σ^a is the worldsheet coordinate and X^μ are the string coordinates in the background manifold. Since the worldsheet metric has three gauge freedoms, we can fix the metric to be flat $\gamma_{ab} = \eta_{ab} = diag(-1, 1)$. The classical string equation of motion is derived from varying the action with respect to the sting X^μ,

$$E_\mu = g_{\mu\nu} \partial^a \partial_a X^\nu + g_{\mu\nu} \Gamma^\nu_{\sigma\rho} \partial^a X^\sigma \partial_a X^\rho = 0. \tag{5.2.10}$$

The action also has to be invariant to the variation of the worldsheet metric γ_{ab} as it is fixed by the gauge condition, hence the constraints are,

$$Z_{ab} = g_{\mu\nu}\left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\eta_{ab}\partial^c X^\mu \partial_c X^\nu\right). \tag{5.2.11}$$

On either horizons, including only the first leading correction and using the lightcone gauge the equation of motion simplifies to

$$E^u = \partial^a\partial_a U = 0, \qquad\qquad E^i = \partial^a\partial_a X^i = 0 \tag{5.2.12}$$

and the constraints become

$$Z_{\tau\sigma} = -\frac{a}{2}\left(\frac{2\pi\alpha'}{l}p^v\right)\partial_\sigma U + r^2\partial_\tau X^i\partial_\sigma X^i \tag{5.2.13}$$

$$-2Z_{\sigma\sigma} = 2Z_{\tau\tau} = -a\left(\frac{2\pi\alpha'}{l}p^u\right)\partial_\tau U + r^2\left(\partial_\tau X^i\partial_\tau X^i + \partial_\sigma X^i\partial_\sigma X^i\right) = 0 \tag{5.2.14}$$

where the worldsheet spatial coordinates are parametrised in the domain $\sigma = [0, l)$. The Fourier mode expansion of the solutions of the equation of motion are,

$$V = \frac{2\pi\alpha'}{l}p^v\tau \tag{5.2.15}$$

$$U = U_0 + \frac{2\pi\alpha'}{l}p^u\tau + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{i\alpha'}{n}\left(\alpha^u_n e^{\frac{i2\pi n}{l}(\tau+\sigma)} + \tilde{\alpha}^u_n e^{\frac{i2\pi n}{l}(\tau-\sigma)}\right) \tag{5.2.16}$$

$$X^i = X^i_0 + \frac{2\pi\alpha'}{l}p^i\tau + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{i\alpha'}{n}\left(\alpha^i_n e^{\frac{i2\pi n}{l}(\tau+\sigma)} + \tilde{\alpha}^i_n e^{\frac{i2\pi n}{l}(\tau-\sigma)}\right), \tag{5.2.17}$$

where we have taken the lightcone gauge in the V coordinate. From the constraints, the spatial oscillation modes $\alpha^i_n,\ \tilde{\alpha}^i_n$ are related to the mass square of the string,

$$-ap^v p^u + r^2 p^i p^i = -m^2 = 4r^2\sum_{n\neq 0}\alpha^i_n\alpha^i_{-n} = 4r^2\sum_{n\neq 0}\tilde{\alpha}^i_n\tilde{\alpha}^i_{-n}, \tag{5.2.18}$$

and the null oscillation modes $\alpha^u_n,\ \tilde{\alpha}^u_n$ become functions of the spatial oscillation modes,

$$\alpha^u_n = \frac{2r^2p^i}{p^v}\alpha^i_n - \frac{2r^2}{ap^v}\sum_{p\neq 0}\alpha^i_n\alpha^i_{n-p} \tag{5.2.19}$$

$$\tilde{\alpha}^u_n = \frac{2r^2p^i}{p^v}\tilde{\alpha}^i_n - \frac{2r^2}{ap^v}\sum_{p\neq 0}\tilde{\alpha}^i_n\tilde{\alpha}^i_{n-p}. \tag{5.2.20}$$

As the strings are boosted along the past horizon, $u = 0$, towards the bifurcation surface, the ingoing momentum is much larger than the other direction. Without loss of generality, we take the transverse spatial centre of mass momentum to zero $p^i = 0$, and $|p^v| \gg |p^u|$.

Then the outgoing momentum p^u is,

$$p^u = \frac{4r^2}{ap^v}\sum_{n\neq 0}\alpha_n^i\alpha_{-n}^i = \frac{4r^2}{ap^v}\sum_{n\neq 0}\widetilde{\alpha}_n^i\widetilde{\alpha}_{-n}^i, \tag{5.2.21}$$

and the outgoing null oscillation modes α_n^u, $\widetilde{\alpha}_n^u$ are

$$\alpha_n^u = -\frac{2r^2}{ap^v}\sum_{p\neq 0}\alpha_n^i\alpha_{n-p}^i \tag{5.2.22}$$

$$\widetilde{\alpha}_n^u = -\frac{2r^2}{ap^v}\sum_{p\neq 0}\widetilde{\alpha}_n^i\widetilde{\alpha}_{n-p}^i. \tag{5.2.23}$$

The exact analysis can be repeated for the string propagating along the $v = 0$ horizon and taking the lightcone gauge in the U coordinate.

We can take the functional derivative of the Polykov string action (5.2.9) with respect to the spacetime metric to obtain the spacetime energy momentum tensor,

$$T_{\mu\nu} = -\frac{1}{2\pi\alpha'\sqrt{-g}}\int d^2\sigma\left[-\partial_\tau X_\mu\partial_\tau X_\nu + \partial_\sigma X_\mu\partial_\sigma X_\nu\right]\delta^{(D)}(x^\rho - X^\rho). \tag{5.2.24}$$

Then we can substitute the string solutions into the argument of the delta functions and Taylor expand about the centre of mass coordinates. Inside the worldsheet integral we can manipulate the delta functions as follow,

$$\int_0^l d\sigma\int_{-\infty}^{\infty} d\tau\;\delta^{(D)}\left(x^\mu - X^\mu(\tau,\sigma)\right)f(\sigma,\tau) \tag{5.2.25}$$

$$= \frac{l}{2\pi\alpha'|p^v|}\int_0^l d\sigma\;\left[\delta(u-U_0) - \frac{vp^u}{p^v}\delta'(u-U_0)\right]\delta(x^i - X_0^i)f\left(\sigma, \frac{lv}{2\pi\alpha' p^v}\right) \tag{5.2.26}$$

where δ' denotes the derivative of the delta function. To the leading order in α' we neglected the explicit α_n^μ and $\widetilde{\alpha}_n^\mu$ terms inside the delta functions but they are implicit in p^u. The integrals in (5.2.24) can be evaluated in the series expansion. Therefore the components of the spacetime energy momentum tensor for the string with lightcone gauge in the V coordinate up to subleading order in $(p^v)^{-1}$ are

$$T_{uu} = \frac{a|p^v|}{2r^{D-2}}\delta(x^i - X_0^i)\delta(u-U_0) - \frac{a|p^u|}{2r^{D-2}}v\delta(x^i - X_0^i)\delta'(u-U_0) \tag{5.2.27}$$

$$T_{uv} = \frac{a|p^u|}{2r^{D-2}}\delta(x^i - X_0^i)\delta(u-U_0) \tag{5.2.28}$$

$$T_{ij} = \frac{8}{ar^{D-6}|p^v|}\sum_{n\neq 0}\alpha_n^{(i}\widetilde{\alpha}_n^{j)}e^{\frac{i2nv}{\alpha' p^v}}\delta(x^i - X_0^i)\delta(u-U_0) \tag{5.2.29}$$

$$T_{vv} = T_{ui} = T_{vj} = 0. \tag{5.2.30}$$

5.3 Einstein equations

In curved background, the Einstein equations even in the linearised form are complicated and highly coupled in between different components of metric perturbation. In addition, the form of the energy momentum tensor is highly non-trivial with terms like the derivative of delta fucntion. The goal of this section is to write the linearised Einstein equation into a set of second order PDE with differential operators being only the Laplacian of the full set of coordinates or the Laplacian of the subset of the transverse coordinates. On our way to obtain the set of equations, we list out the approximations and assumptions needed. At the leading order, governed by the centre of mass momenta of the strings, we recovered the linearised Einstein equation in [23].

The linearised Einstein equation is

$$R^{(1)}_{\mu\nu} + \frac{(D-1)}{l^2_{AdS}} h_{\mu\nu} = 8\pi \Big(T_{\mu\nu} - \frac{T}{D-2} g_{\mu\nu} \Big) \tag{5.3.1}$$

where $T_{\mu\nu}$ is the energy momentum tensor,

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I}{\delta g_{\mu\nu}}. \tag{5.3.2}$$

For convenience we write the above as

$$\mathcal{G}_{\mu\nu} = 8\pi \mathcal{T}_{\mu\nu} \tag{5.3.3}$$

where the modified Einstein tensor is

$$\mathcal{G}_{\mu\nu} := R^{(1)}_{\mu\nu} + \frac{(D-1)}{l^2_{AdS}} h_{\mu\nu} \tag{5.3.4}$$

and the modified energy momentum tensor is

$$\mathcal{T}_{\mu\nu} := T_{\mu\nu} - \frac{T}{D-2} g_{\mu\nu}. \tag{5.3.5}$$

The components of the modified energy momentum tensor up to subleading order are

$$\mathcal{T}_{uu} = \frac{a|p^v|}{2r^{D-2}} \delta(x^i - X^i_0)\delta(u - U_0) - \frac{2}{r^{D-4}|p^v|} \sum_{n\neq 0} \alpha^i_n \alpha^i_{-n} \delta(x^i - X^i_0)\delta'(u - U_0) \tag{5.3.6}$$

$$\mathcal{T}_{uv} = \frac{4}{r^{D-4}|p^v|(D-2)} \sum_{n\neq 0} \left(\frac{D-4}{2} \alpha^i_n \alpha^i_{-n} + \alpha^i_n \tilde{\alpha}^i_n e^{\frac{i2nv}{\alpha' p^v}} \right) \delta(x^i - X^i_0)\delta(u - U_0) \tag{5.3.7}$$

$$\mathcal{T}_{ij} = \frac{8}{ar^{D-6}|p^v|} \sum_{n\neq 0} \left(\alpha^{(i}_n \tilde{\alpha}^{j)}_n e^{\frac{i2nv}{\alpha' p^v}} - \frac{\delta_{ij}}{(D-2)} \left(-\alpha^i_n \alpha^i_{-n} + \alpha^i_n \tilde{\alpha}^i_n e^{\frac{i2nv}{\alpha' p^v}} \right) \right) \times \delta(x^i - X^i_0)\delta(u - U_0) \tag{5.3.8}$$

$$\mathcal{T}_{vv} = \mathcal{T}_{vi} = \mathcal{T}_{uj} = 0. \tag{5.3.9}$$

We would like to expand the energy momentum tensor in the high energy, low curvature and small oscillation classical string limit. The perturbation parameters are p^v, $\frac{1}{l_{AdS}^2}$, α_n^i and $\tilde{\alpha}_n^i$. We introduce the following notation to separate the leading and subleading terms with different dependence on v. $\mathcal{T}_{\mu\nu}^{(0)}$ denotes the leading terms, $t_{\mu\nu}^{(v)}$ and $t_{\mu\nu}^{(\not v)}$ are the v dependent and v independent terms of the subleading $\mathcal{T}_{\mu\nu}^{(1)}$. Namely,

$$\mathcal{T}_{uu}^{(0)} = \frac{a|p^v|}{2r^{D-2}}\delta(x^i - X_0^i)\delta(u - U_0) \tag{5.3.10}$$

$$t_{uu}^{(v)} = -\frac{a_0|p^u|}{2r^{D-2}}v\delta(x^i - X_0^i)\delta'(u - U_0) \tag{5.3.11}$$

$$t_{uv}^{(v)} = \frac{4}{r^{D-4}|p^v|(D-2)}\sum_{n\neq 0}\alpha_n^i\tilde{\alpha}_n^i e^{\frac{i2nv}{\alpha' p^v}}\delta(x^i - X_0^i)\delta(u - U_0) \tag{5.3.12}$$

$$t_{uv}^{(\not v)} = \frac{(D-4)a_0|p^u|}{2(D-2)r_0^{D-2}}\sum_{n\neq 0}\alpha_n^i\alpha_{-n}^i\delta(x^i - X_0^i)\delta(u - U_0) \tag{5.3.13}$$

$$t_{ij}^{(v)} = \frac{8}{a_0 r_0^{D-6}|p^v|}\sum_{n\neq 0}\left(\alpha_n^{(i}\tilde{\alpha}_n^{j)} - \frac{\delta_{ij}}{(D-2)}\alpha_n^i\tilde{\alpha}_n^i\right)e^{\frac{i2nv}{\alpha' p^v}}\delta(x^i - X_0^i)\delta(u - U_0) \tag{5.3.14}$$

$$t_{ij}^{(\not v)} = \frac{2|p^u|}{(D-2)r_0^{D-4}}\sum_{n\neq 0}\delta_{ij}\delta(x^i - X_0^i)\delta(u - U_0). \tag{5.3.15}$$

Setting $u = 0$, but keeping $u\partial_u h_{\mu\nu}$, the components of the modified linearised Einstein tensor $\mathcal{G}_{\mu\nu}$ at first order of $\frac{v}{l_{AdS}}$ are,

$$\begin{aligned}
\mathcal{G}_{uu} =& -\frac{1}{2}\partial^\sigma\partial_\sigma h_{uu} - \frac{(D-1)(D-2)}{4l_{AdS}^2}u\partial_u h_{uu} - \frac{(D-1)(5D-18)}{4l_{AdS}^2}v\partial_v h_{uu} \\
&- \frac{(D-1)a_0}{2r_0^2 l_{AdS}^2}v\partial_k h_{uk} \qquad (5.3.16)\\
\mathcal{G}_{vv} =& -\frac{1}{2}\partial^\sigma\partial_\sigma h_{vv} - \frac{(D-1)(5D-18)}{4l_{AdS}^2}u\partial_u h_{vv} - \frac{(D-1)(D-2)}{4l_{AdS}^2}v\partial_v h_{vv} \qquad (5.3.17)\\
\mathcal{G}_{uv} =& -\frac{1}{2}\partial^\sigma\partial_\sigma h_{uv} - \frac{(D-1)(3D-10)}{4l_{AdS}^2}u\partial_u h_{uv} - \frac{(D-1)(3D-10)}{4l_{AdS}^2}v\partial_v h_{uv} \\
&- \frac{(D-1)a_0}{4r_0^2 l_{AdS}^2}v\partial_k h_{vk} - \frac{(D-1)(D-4)}{l_{AdS}^2}h_{uv} - \frac{(D-1)a_0}{4r_0^2 l_{AdS}^2}h_{kk} \qquad (5.3.18)\\
\mathcal{G}_{ij} =& -\frac{1}{2}\partial^\sigma\partial_\sigma h_{ij} - \frac{(D-1)(D-6)}{4l_{AdS}^2}u\partial_u h_{ij} - \frac{(D-1)(D-6)}{4l_{AdS}^2}v\partial_v h_{ij} \\
&- \frac{(D-1)}{l_{AdS}^2}v\partial_{(i}h_{j)u} - \frac{2(D-1)r_0^2}{a_0 l_{AdS}^2}\delta_{ij}h_{uv} - \frac{(D-1)}{l_{AdS}^2}h_{ij} \qquad (5.3.19)\\
\mathcal{G}_{ui} =& -\frac{1}{2}\partial^\sigma\partial_\sigma h_{uj} - \frac{(D-1)(D-4)}{4l_{AdS}^2}u\partial_u h_{ui} - \frac{(D-1)(D-4)}{4l_{AdS}^2}v\partial_v h_{ui} \\
&- \frac{(D-1)a_0}{4r_0^2 l_{AdS}^2}v\partial_k h_{ki} - \frac{(D-1)}{l_{AdS}^2}v\partial_i h_{uv} - \frac{(D-1)(D-4)}{4l_{AdS}^2}h_{ui} \qquad (5.3.20)\\
\mathcal{G}_{vi} =& -\frac{1}{2}\partial^\sigma\partial_\sigma h_{vj} - \frac{(D-1)(D-4)}{4l_{AdS}^2}u\partial_u h_{vi} - \frac{(D-1)(D-4)}{4l_{AdS}^2}v\partial_v h_{vi} \\
&- \frac{(D-1)}{4l_{AdS}^2}v\partial_i h_{vv} + \frac{(D-1)(D-8)}{4l_{AdS}^2}h_{vi}. \qquad (5.3.21)
\end{aligned}$$

We neglect the homogenous solution as they correspond to the background gravitational wave in vacuum or associated to the incoming asymptotic solution in potential scattering. Therefore up to subleading order h_{vv} and h_{vi} vanish,

$$\mathcal{T}_{vv}, \mathcal{T}_{vi} = 0 \to \mathcal{G}_{vv}, \mathcal{G}_{vi} = 0 \to h_{vv} = 0 \to h_{vi} = 0. \qquad (5.3.22)$$

Since we are interested in the eikonal approximation of two particles moving along the separate horizons and scatter at the bifurcation surface, the interaction or the exchange of graviton is instantaneous. Hence we can set the limit of $v \to 0$ and drop all terms with explicit v dependence, only keeping $v\partial_v h_{\mu\nu}$ terms in the PDE in order to match the components of the energy momentum tensor with monomial v^1 dependence. Then up to subleading order h_{ui} vanishes,

$$\mathcal{T}_{ui} = 0 \to \mathcal{G}_{ui} = 0 \to h_{ui} \sim 0 \qquad (5.3.23)$$

The v dependence of $\mathcal{T}_{ij}, \mathcal{T}_{uv}$ is of exponential form so we can drop all the explicit v terms and the u dependence of $\mathcal{T}_{ij}, \mathcal{T}_{uv}$ is a delta function $\delta(u)$. We can now split the metric perturbation into singular and non-singular parts

$$h_{uv} = h_{uv}^{(S)} + h_{uv}^{(NS)}, \qquad\qquad h_{ij} = h_{ij}^{(S)} + h_{ij}^{(NS)} \tag{5.3.24}$$

where

$$h_{uv}^{(S)}, h_{ij}^{(S)} \propto \delta(u). \tag{5.3.25}$$

The singular parts have the distribution property of delta function

$$u\partial_u h_{uv}^{(S)} = -h_{uv}^{(S)}, \qquad\qquad u\partial_u h_{ij}^{(S)} = -h_{ij}^{(S)}. \tag{5.3.26}$$

The leading and subleading order of $\mathcal{T}_{uu}$ is v independent and linear in v respectively. So we need to keep the $v\partial_v h_{uu}$ term. As the leading and subleading order is singular in u we can use the distribution property of the delta function and its derivative,

$$u\partial_u h_{uu} = -h_{uu}. \tag{5.3.27}$$

It is convenient to split the subleading metric perturbation $h_{uu}^{(1)}$ into a constant and linear term in v

$$h_{uu}^{(1)} = h_{uu}^{(NL)} + v h_{uu}^{(L)}. \tag{5.3.28}$$

One can deduce

$$h_{uu}^{(0)} \propto \delta(u), \qquad h_{uu}^{(NL)} \propto \delta''(u), \qquad h_{uu}^{(L)} \propto \delta'(u). \tag{5.3.29}$$

Then the modified Einstein equation simplifies to

$$-\frac{1}{2}\partial^k\partial_k h_{uu}^{(0)} + \bar{F} h_{uu}^{(0)} = 8\pi \mathcal{T}_{uu}^{(0)} \tag{5.3.30}$$

$$-\frac{1}{2}\partial^k\partial_k h_{uu}^{(NL)} + \tilde{E} h_{uu}^{(NL)} = -\frac{1}{a_0}\partial_u h_{uu}^{(L)} \tag{5.3.31}$$

$$-\frac{1}{2}\partial^i\partial_i h_{uu}^{(L)} + \bar{E} h_{uu}^{(L)} = 8\pi t_{uu}^{(v)} \tag{5.3.32}$$

$$-\frac{1}{2}\partial^\sigma\partial_\sigma h_{ij}^{(NS)} + \tilde{C} h_{ij}^{(NS)} + \bar{D}\delta_{ij} h_{uv}^{(NS)} = 8\pi t_{ij}^{(v)} \tag{5.3.33}$$

$$-\frac{1}{2}\partial^k\partial_k h_{ij}^{(S)} + \bar{C} h_{ij}^{(S)} + \bar{D}\delta_{ij} h_{uv}^{(S)} = 8\pi t_{ij}^{(\not{v})} \tag{5.3.34}$$

$$-\frac{1}{2}\partial^\sigma\partial_\sigma h_{uv}^{(NS)} + \tilde{A} h_{uv}^{(NS)} + \bar{B} h_{kk}^{(NS)} = 8\pi t_{ij}^{(v)} \tag{5.3.35}$$

$$-\frac{1}{2}\partial^k\partial_k h_{uv}^{(S)} + \bar{A} h_{uv}^{(S)} + \bar{B} h_{kk}^{(S)} = 8\pi t_{uv}^{(\not{v})} \tag{5.3.36}$$

where the coefficients are listed in $(5.B.1 - 5.B.9)$. The leading order Einstein equation in (5.3.30) is the full equation for the point particle case, its transverse profile matches with [23]. By Fourier transforming the whole equation then solve for $\tilde{h}_{uu}^{(0)}$ and finally do an inverse Fourier transform, we obtain the solution for $h^{(0)}_{\mu\nu}$,

$$h^{(0)}_{uu} = \frac{4a_0|p^v|\delta(U)}{(2\pi)^{D-3}r_0^{D-4}}\sqrt{\pi}\Omega_{D-4}\Gamma\left(\frac{D-3}{2}\right)\left(\frac{2r_0\sqrt{2\bar{F}}}{x}\right)^{\frac{D-4}{2}} K_{\frac{D-4}{2}}(xr_0\sqrt{2\bar{F}}). \tag{5.3.37}$$

The asymptotic form of $h^{(0)}_{\mu\nu}$ as the argument of the modified Bessel function becomes large is

$$h^{(0)}_{uu} \sim \frac{8\pi a_0|p^v|}{r_0^{D-4}}\delta(U)\frac{\left(r_0\sqrt{2\bar{F}}\right)^{\frac{D-5}{2}}}{2(2\pi x)^{\frac{D-3}{2}}}e^{-xr_0\sqrt{2\bar{F}}}. \tag{5.3.38}$$

This is matches exactly with [23] as

$$2r_0^2\bar{F} = \frac{(D-1)(D-2)r_0^2}{2l_{AdS}^2} \tag{5.3.39}$$

which is the constant μ used in [23]. This is an evidence that validates our current approach.

The subleading metric perturbations follow a similar approach, first Fourier transform all the Einstein equations which become a set of simultaneous equations. Solving the simultaneous to get the expression for $\tilde{h}_{\mu\nu}$ and do the inverse Fourier transform integral to obtain the subleading metric perturbations. The calculation for the subleading terms are more involved, we included the details of the calculation and results in appendix 5.C.

5.4 Onshell action

Previously, we have obtained the backreaction of a high energy classical string propagating along one of the horizon. In the linear gravity approximation, we can separate the backreaction of two classical strings propagating in orthogonal directions. In general, the onshell shell action will include terms that represent a particle self interaction, e.g. a term that is the product of the linear metric perturbation induced by string 1 and energy momentum tensor of string 1,

$$h_{1\mu\nu}T_1^{\mu\nu}. \tag{5.4.1}$$

But recall the Feynman diagrams for eikonal scattering do not include self interaction in the ladder diagrams. These corrections are related to the loop correction to the propagator. Following the standard ladder diagram approach, we will drop the self interaction terms. In the calculation below, we recover the classical action for the particles case in [23]. We will also demonstrate explicitly the subleading classical stringy correction vanishes and obtain the next-to-subleading order correction.

The action of our system with two strings in a curved spacetime is

$$I = I_{EH}[g] + I_1[g] + I_2[g] \tag{5.4.2}$$

where g is the full metric solution that minimises the classical action I.

$$\left.\frac{\delta I}{\delta g^{ab}}\right|_g = 0 \tag{5.4.3}$$

$$\left.\frac{\delta I_{EH}}{\delta g^{ab}}\right|_g + \left.\frac{\delta I_1}{\delta g^{ab}}\right|_g + \left.\frac{\delta I_2}{\delta g^{ab}}\right|_g = 0 \tag{5.4.4}$$

From the perspective of perturbation theory, we introduce a schematic perturbative parameter ϵ to the particles' action,

$$I = I_{EH}[g] + \epsilon I_1[g] + \epsilon I_2[g], \tag{5.4.5}$$

which may be set to 1^-. We can now expand the full solution g into the background g_b and the perturbation h

$$g = g_b + \epsilon h. \tag{5.4.6}$$

From the definition of background solution, we know that g_b minimises I_{EH}

$$\left.\frac{\delta I_{EH}}{\delta g^{ab}}\right|_{g_b} = 0. \tag{5.4.7}$$

The linearised equation of motion can be derived from expanding (5.4.4) using Taylor series in functional derivative to the first order in ϵ,

$$\left.\frac{\delta I_{EH}}{\delta g^{ab}(y)}\right|_{g_b+\epsilon h} + \epsilon \left.\frac{\delta I_1}{\delta g^{ab}(y)}\right|_{g_b+\epsilon h} + \epsilon \left.\frac{\delta I_2}{\delta g^{ab}(y)}\right|_{g_b+\epsilon h} = 0 \tag{5.4.8}$$

$$\left.\frac{\delta I_{EH}}{\delta g^{ab}(y)}\right|_{g_b} - \epsilon \int d^D z \left.\frac{\delta I_{EH}}{\delta g^{cd}(z)\delta g^{ab}(y)}\right|_{g_b} h^{cd}(z) + \epsilon \left.\frac{\delta I_1}{\delta g^{ab}(y)}\right|_{g_b} + \epsilon \left.\frac{\delta I_2}{\delta g^{ab}(y)}\right|_{g_b} = 0$$

$$-\int_z \left.\frac{\delta I_{EH}}{\delta g^{cd}(z)\delta g^{ab}(y)}\right|_{g_b} h^{cd}(z) + \left.\frac{\delta I_1}{\delta g^{ab}(y)}\right|_{g_b} + \left.\frac{\delta I_2}{\delta g^{ab}(y)}\right|_{g_b} = 0. \tag{5.4.9}$$

We define a differential operator $\mathcal{D}^2_{ab}$ by

$$\mathcal{D}^2_{ab}[h] = \frac{1}{\sqrt{-g_b}} \int d^D z \left.\frac{\delta I_{EH}}{\delta g^{cd}(z)\delta g^{ab}(y)}\right|_{g_b} h^{cd}(z) \tag{5.4.10}$$

and the energy momentum tensor of particle 1 by

$$T_{1ab} = \frac{-2}{\sqrt{-g}} \frac{\delta I_1}{\delta g^{ab}(y)} \tag{5.4.11}$$

and identically for particle 2. The linear equation of motion for particles' backreaction is

$$\mathcal{D}^2_{ab}[h] + \frac{T_{1ab}}{2} + \frac{T_{1ab}}{2} = 0. \tag{5.4.12}$$

From the Einstein tensor,

$$\frac{1}{16\pi} G_{ab} = \frac{1}{\sqrt{-g_b}} \frac{\delta I_{EH}}{\delta g^{ab}}, \tag{5.4.13}$$

we can deduce the differential operator is essentially

$$\mathcal{D}^2_{ab}[h] = \frac{-1}{16\pi} \delta G_{ab}[h]. \tag{5.4.14}$$

Then (5.4.12) is the linearised Einstein equation. By linearity we can separate the backreaction by particle 1 from particle 2, hence

$$\delta G_{ab}[h_1] = 8\pi T_{1ab}, \tag{5.4.15}$$

similarly form for particle 2.

As usual, the variation of the action is

$$I[g_b + \epsilon h] = I[g_b] + \delta I[g_b, h]. \tag{5.4.16}$$

The full action can be expanded around the background solution

$$\begin{aligned}
I[g_b + \epsilon h] &= I_{EH}[g_b] - \epsilon \int_y \left.\frac{\delta I_{EH}}{\delta g^{ab}(y)}\right|_{g_b} h^{ab}(y) + \frac{\epsilon^2}{2} \int_z \int_y \left.\frac{\delta I_{EH}}{\delta g^{cd}(z) g^{ab}(y)}\right|_{g_b} h^{ab}(y) h^{cd}(z) + \cdots \qquad (5.4.17)\\
&\quad + \epsilon I_1[g_b] - \epsilon^2 \int_y \left.\frac{\delta I_1}{\delta g^{ab}(y)}\right|_{g_b} h^{ab}(y) + \cdots \\
&\quad + \epsilon I_2[g_b] - \epsilon^2 \int_y \left.\frac{\delta I_2}{\delta g^{ab}(y)}\right|_{g_b} h^{ab}(y) + \cdots \\
I[g_b + \epsilon h] &= I[g_b] + \frac{\epsilon^2}{2} \int d^D y \sqrt{-g_b} \left[h^{ab} \mathcal{D}^2_{ab}[h] + h^{ab} T_{1ab} + h^{ab} T_{2ab} \right] \qquad (5.4.18)\\
I[g_b + \epsilon h] &= I[g_b] + \frac{\epsilon^2}{4} \int d^D y \sqrt{-g_b} h^{ab} \left(T_{1ab} + T_{2ab} \right) \qquad (5.4.19)
\end{aligned}$$

we used (5.4.12) to reach the last line. In the point particle case the two integrands are identical when we set $\epsilon = 1$ the classical action becomes

$$I[g_b + \epsilon h] = I[g_b] + \frac{1}{2} \int d^D y \sqrt{-g_b} h_1^{ab} T_{2ab}. \tag{5.4.20}$$

Hence recovering the action (32) in [23].

5.4.0.1 Stringy correction to classical action

The metric perturbation and energy momentum tensor of string $a = 1, 2$ are expressed as

$$h_{a\mu\nu} = h^{(0)}_{a\mu\nu} + h^{(1)}_{a\mu\nu}, \qquad T_{a\mu\nu} = T^{(0)}_{a\mu\nu} + T^{(1)}_{a\mu\nu}, \tag{5.4.21}$$

where the first term is the leading point particle contribution and the second term is the stringy correction. The leading contribution is the expected

$$\delta I^{(0)} = \frac{16|p_1^v||p_2^u|a_0}{r_0^{D-4}} \left(\frac{r_0\sqrt{2\bar{F}}}{2\pi\Delta x}\right)^{\frac{D}{2}-2} K_{\frac{D}{2}-2}(\Delta x r_0\sqrt{2\bar{F}}) \tag{5.4.22}$$

$$\delta I^{(0)} \sim s \tag{5.4.23}$$

as $a_0 p_1^v p_2^u = s$ where the sign convention is different from (2.5.10).

The supposed subleading contributions vanish,

$$\frac{1}{2}\int d^D y\sqrt{-g_b}\left[h^{(0)}_{1uu}T_2^{(1)uu} + h^{(1)}_{1uu}T_2^{(0)uu}\right] = 0 \tag{5.4.24}$$

because of the mixture of null coordinate and delta function dependence of the metric perturbation and energy momentum tensor that vanish under the integral, i.e.

$$h^{(0)}_{1uu}T_2^{(1)uu} \propto \int v\delta(v) \tag{5.4.25}$$

and

$$h^{(1)}_{1uu}T_2^{(0)uu} \propto \int u\delta(u). \tag{5.4.26}$$

The next-to-subleading classical action can be separated into

$$\delta I^{(2)}_{uu} = \frac{1}{4}\int d^D y\sqrt{-g_b}\left[h^{(1)}_{1uu}T_2^{(1)uu} + h^{(1)}_{2vv}T_1^{(1)vv}\right] \tag{5.4.27}$$

$$\delta I^{(2)}_{uv} = \frac{1}{4}\int d^D y\sqrt{-g_b}\left[h^{(1)}_{1uv}T_2^{(1)uv} + h^{(1)}_{2uv}T_1^{(1)uv}\right] \tag{5.4.28}$$

$$\delta I^{(2)}_{ij} = \frac{1}{4}\int d^D y\sqrt{-g_b}\left[h^{(1)}_{1ij}T_2^{(1)ij} + h^{(1)}_{2ij}T_1^{(1)ij}\right] \tag{5.4.29}$$

Note the index in $\delta I^{(2)}_{\mu\nu}$ are not tensor index, it is only a notation to indicate the which components of metric perturbation and energy momentum tensor contributed. The result of the next-to-subleading classical action can be found in appendix $5.D$. From $(5.D.1 - 5.D.4)$ we can see all the Bessel function argument is independent of the momenta. Hence all the next-to-subleading classical action except the $\delta I^{(NS)}_{ij}$ component scale inversely to s

$$\delta I^{(S)}_{uu}, \delta I^{(S)}_{uv}, \delta I^{(S)}_{ij}, \delta I^{(NS)}_{uv} \sim s^{-1}. \tag{5.4.30}$$

This is because the momenta follow the mass shell condition in (5.2.18) and (5.2.19) as the momenta in the null direction orthogonal to the dominant null momenta scale as

$$|p_1^u p_2^v|, \left|\frac{p_1^u}{p_2^u}\right|, \left|\frac{p_2^v}{p_1^v}\right| \sim \frac{1}{p_1^v p_2^u} \sim s^{-1}. \tag{5.4.31}$$

For $\delta I_{ij}^{(NS)}$, the Bessel fucntions' argument does depend on s but in all limit we see $\delta I_{ij}^{(NS)}$ is at most of order one

$$\delta I_{ij}^{(NS)} \sim O(1). \tag{5.4.32}$$

All the corrections we found will not change the power law behaviour of the eikonal phase with exponent equal to the spin of the intermediate particle minus one, which is $2-1=1$,

$$\delta \sim s^1. \tag{5.4.33}$$

5.5 Conclusions and outlook

Therefore we can conclude that the classical string transverse oscillation modes do not give correction to the Regge intercept of the eikonal phase which means no correction to the Lyapunov exponent. The Regge behaviour is governed by the intermediate particle which in our analysis is still the spin-2 graviton. The next-to-subleading correction in the eikonal phase corresponds to the tree level scattering of the oscillation modes of the classical strings with the exchange of graviton. Through the Polykov string action, it is easy to see the vertex between zero mode of the two strings, represented by the centre of mass position, and a graviton is proportional to s. Hence the zero modes scattering recovers the same eikonal phase as in the point particles scattering. Again by the inspection of the Polykov string action, we see the vertex from the interaction of the oscillation mode of the two strings and the graviton is proportion to the product of two Fourier coefficients, α_n^i, representing the amplitude of the oscillation mode. Therefore for each zero modes-graviton vertex replaced by the oscillation modes-graviton vertex, a factor of s is missing. In the tree level diagrams, there are only two vertices. At the subleading order, we only replace one zero modes-graviton vertex with oscillation modes-graviton vertex. But due to the onshell condition, this vanishes. At the next-to-subleading order, we need to replace both of the zero modes-graviton vertices with oscillation modes-graviton vertices. Hence, in total a factor of s^2 is missing. In this heuristic argument, we can see directly the difference of s between the leading and next-to-subleading (nts) eikonal phase

$$\delta_{nts} \sim s^{-2}\delta_{leading} \tag{5.5.1}$$

$$\delta_{nts} \sim s^{-1} \tag{5.5.2}$$

matching with our explicit calculation. Unless α_n^i, which have the same dimension as momentum, is of order $\sqrt{s}$, the transverse oscillation will not give significant correction to the eikonal phase.

Both the classical string correction we explored above and quantum string correction in [23] are only a subset of possible stringy corrections one can introduce, nonetheless the chaos bound still holds in both cases. There are other types of correction one can introduce. Focusing on the calculation of the OTOC in the bulk picture, one has to look into different way of incorporating stringy correction to the gravitational eikonal scattering.

A natural extension is to investigate the quantum string correction. There are ways of quantising the string in a shock wave background [125] and the correction to the eikonal phase from quantum stringy correction were also found in some background [126]. In the ideal case, one can calculate the scattering of massless closed strings from a stack of D-branes that created the AdS black hole background. In fact there are works on the leading and subleading eikonal phase for high energy strings-brane scattering [127]. The stringy correction comes into the impact parameter, where the impact parameter is shifted by the transverse string position. It is possible to Taylor expand the eikonal phase and express the subleading part in terms of the string spreading.

The previously mentioned outlooks are top-down string theoretic approach. An interesting possible bottom-up approach to probe correction to chaos is by using the Lipatov effective field theory method [128, 129]. By constructing extra gauge invariant terms in the Lagrangian for the Reggeons, one is able to formulate the same Regge behaviour and explore correction to the ladder diagrams.

5.A Curvature Tensor

The Christoffel symbols are

$$\Gamma^u_{uu} = \frac{(D-1)(D-4)a_0 v}{4l_{AdS}^2} \qquad \Gamma^v_{vv} = \frac{(D-1)(D-4)a_0 u}{4l_{AdS}^2} \tag{5.A.1}$$

$$\Gamma^u_{ij} = -\frac{(D-1)r_0^2 u}{2l_{AdS}^2}\delta_{ij} \qquad \Gamma^v_{ij} = -\frac{(D-1)r_0^2 v}{2l_{AdS}^2}\delta_{ij} \tag{5.A.2}$$

$$\Gamma^j_{iu} = -\frac{(D-1)r_0^2 v}{4l_{AdS}^2}\delta_{ij} \qquad \Gamma^j_{iv} = -\frac{(D-1)r_0^2 u}{4l_{AdS}^2}\delta_{ij} \tag{5.A.3}$$

Following (5.2.8) and setting $u=0$ but keeping the $u\delta(u)$ terms,

$$
\begin{aligned}
R^{(1)}_{uu} =& -\frac{D-1}{l^2_{AdS}}h_{uu} - \frac{1}{2}\partial^\sigma\partial_\sigma h_{uu} - \frac{(D-1)(D-2)}{4l^2_{AdS}}u\partial_u h_{uu} \\
&- \frac{(D-1)(5D-18)}{4l^2_{AdS}}v\partial_v h_{vv} - \frac{(D-1)a_0}{2r_0^2 l^2_{AdS}}v\partial_k h_{uk} \\
&- \frac{(D-1)^2(D-2)a_0}{8l^4_{AdS}}v^2 h_{uu} + \frac{(D-1)^2(2D-9)a_0^2}{16r_0^2 l^4_{AdS}}v^2 h_{kk}
\end{aligned}
\tag{5.A.4}
$$

$$
\begin{aligned}
R^{(1)}_{vv} =& -\frac{D-1}{l^2_{AdS}}h_{vv} - \frac{1}{2}\partial^\sigma\partial_\sigma h_{vv} - \frac{(D-1)(5D-18)}{4l^2_{AdS}}u\partial_u h_{vv} \\
&- \frac{(D-1)(D-2)}{4l^2_{AdS}}v\partial_v h_{vv}
\end{aligned}
\tag{5.A.5}
$$

$$
\begin{aligned}
R^{(1)}_{uv} =& -\frac{(D-1)(D-3)}{l^2_{AdS}}h_{uv} - \frac{(D-1)a_0}{4r_0^2 l^2_{AdS}}h_{ii} - \frac{1}{2}\partial^\sigma\partial_\sigma h_{uv} - \frac{(D-1)(3D-10)}{4l^2_{AdS}}u\partial_u h_{uv} \\
&- \frac{(D-1)(3D-10)}{4l^2_{AdS}}v\partial_v h_{vv} - \frac{(D-1)a_0}{4r_0^2 l^2_{AdS}}v\partial_k h_{vk} - \frac{(D-1)^2(D-2)a_0}{16l^4_{AdS}}v^2 h_{vv}
\end{aligned}
\tag{5.A.6}
$$

$$
\begin{aligned}
R^{(1)}_{ij} =& -\frac{2(D-1)r_0^2}{a_0 l^2_{AdS}}\delta_{ij}h_{uv} - \frac{1}{2}\partial^\sigma\partial_\sigma h_{ij} - \frac{(D-1)(D-6)}{4l^2_{AdS}}u\partial_u h_{ij} \\
&- \frac{(D-1)(D-6)}{4l^2_{AdS}}v\partial_v h_{ij} - \frac{(D-1)}{l^2_{AdS}}v\partial_{(i}h_{j)v} - \frac{(D-1)^2 r_0^2}{4l^4_{AdS}}\delta_{ij}v^2 h_{vv}
\end{aligned}
\tag{5.A.7}
$$

$$
\begin{aligned}
R^{(1)}_{ui} =& -\frac{(D-1)(D-8)}{4l^2_{AdS}}h_{ui} - \frac{1}{2}\partial^\sigma\partial_\sigma h_{ui} - \frac{(D-1)(D-4)}{4l^2_{AdS}}u\partial_u h_{ui} \\
&- \frac{(D-1)(D-4)}{4l^2_{AdS}}v\partial_v h_{ui} - \frac{(D-1)a_0}{4r_0^2 l^2_{AdS}}v\partial_k h_{ki} \\
&- \frac{(D-1)}{2l^2_{AdS}}v\partial_i h_{uv} - \frac{(D-1)^2(3D-13)a_0}{16l^4_{AdS}}v^2 h_{vi}
\end{aligned}
\tag{5.A.8}
$$

$$
\begin{aligned}
R^{(1)}_{vi} =& -\frac{(D-1)(D-8)}{4l^2_{AdS}}h_{vi} - \frac{1}{2}\partial^\sigma\partial_\sigma h_{ui} - \frac{(D-1)(D-4)}{4l^2_{AdS}}u\partial_u h_{vi} \\
&- \frac{(D-1)(D-4)}{4l^2_{AdS}}v\partial_v h_{vi} - \frac{(D-1)}{2l^2_{AdS}}v\partial_i h_{vv}
\end{aligned}
\tag{5.A.9}
$$

Since we are interested in the length scale less than length scale of the spacetime to be large, only the terms that depends on the AdS radius up to l^{-2}_{AdS} is consider. So we can neglect the terms above with l^{-4}_{AdS} dependence.

5.B Coefficient in Einstein equations

The coefficients are

$$\bar{A} = -\frac{(D-1)(D-6)}{4l_{AdS}^2} \tag{5.B.1}$$

$$\widetilde{A} = -\frac{(D-1)(D-6)}{l_{AdS}^2} \tag{5.B.2}$$

$$\bar{B} = -\frac{(D-1)}{4l_{AdS}^2}\frac{a_0}{r_0^2} \tag{5.B.3}$$

$$\bar{C} = \frac{(D-1)(D-2)}{4l_{AdS}^2} \tag{5.B.4}$$

$$\widetilde{C} = \frac{(D-1)}{l_{AdS}^2} \tag{5.B.5}$$

$$\bar{D} = -\frac{2(D-1))}{l_{AdS}^2}\frac{r_0^2}{a_0} \tag{5.B.6}$$

$$\bar{E} = \widetilde{A} \tag{5.B.7}$$

$$\tilde{E} = \bar{C} \tag{5.B.8}$$

$$\bar{F} = \frac{(D-1)(D-2)}{4l_{AdS}^2}. \tag{5.B.9}$$

5.C Fourier analysis

We are using the tilde to notate the Fourier transform of the coefficient of delta function or it's derivative in the u direction, so for function

$$Y = y \times (\partial_u)^n \delta(u) \tag{5.C.1}$$

then when we say the "Fourier transform of Y" it is actually the Fourier transform of y

$$\tilde{Y} = \mathcal{F}[y]. \tag{5.C.2}$$

Fourier transform of linear metric perturbation

$$\tilde{h}^{(0)}_{uu} = 16\pi \frac{\tilde{T}^{(0)}_{uu}}{\mathbf{k}^2 + 2\bar{F}} \tag{5.C.3}$$

$$\tilde{h}^{(L)}_{uu} = 16\pi \frac{\tilde{T}^{(0)}_{uu}}{\mathbf{k}^2 + 2\bar{E}} \tag{5.C.4}$$

$$\tilde{h}^{(NL)}_{uu} = -\frac{32\pi}{a_0} \frac{\tilde{T}^{(0)}_{uu}}{(\mathbf{k}^2 + 2\bar{E})(\mathbf{k}^2 + 2\tilde{E})} \tag{5.C.5}$$

$$\tilde{h}^{(S)}_{uv} = 16\pi \left[\frac{(\mathbf{k}^2 + 2\bar{C})\tilde{t}^{(\not{v})}_{uv} - 2\bar{B}\tilde{t}^{(\not{v})}_{ii}}{\mathbf{k}^4 + 2(\bar{A} + \bar{C})\mathbf{k}^2 - 4(D-2)\bar{B}\bar{D}} \right] \tag{5.C.6}$$

$$\tilde{h}^{(NS)}_{uv} = 16\pi \left[\frac{(k^2 + 2\widetilde{C})\tilde{t}^{(v)}_{uv} - 2\bar{B}\tilde{t}^{(v)}_{ii}}{k^4 + 2(\widetilde{A} + \widetilde{C})k^2 - 4(D-2)\bar{B}\bar{D}} \right] \tag{5.C.7}$$

$$\tilde{h}^{(S)}_{ij} = \frac{16\pi}{\mathbf{k}^2 + 2\bar{C}} \left[\tilde{t}^{(\not{v})}_{ij} - 2\delta_{ij} \frac{\bar{D}(\mathbf{k}^2 + 2\bar{C})\tilde{t}^{(\not{v})}_{uv} - 2\bar{B}\bar{D}\tilde{t}^{(\not{v})}_{ii}}{\mathbf{k}^4 + 2(\bar{A} + \bar{C})\mathbf{k}^2 - 4(D-2)\bar{B}\bar{D}} \right] \tag{5.C.8}$$

$$\tilde{h}^{(NS)}_{ij} = \frac{16\pi}{k^2 + 2\widetilde{C}} \left[\tilde{t}^{(v)}_{ij} - 2\delta_{ij} \frac{\bar{D}(k^2 + 2\bar{C})\tilde{t}^{(v)}_{uv} - 2\bar{B}\bar{D}\tilde{t}^{(v)}_{ii}}{k^4 + 2(\widetilde{A} + \widetilde{C})k^2 - 4(D-2)\bar{B}\bar{D}} \right] \tag{5.C.9}$$

where $k^2 = k^\sigma k_\sigma$ and $\mathbf{k}^2 = k^i k_i$. The Fourier transform of the energy momentum tensor components are

$$\tilde{T}^{(0)}_{uu} = \frac{a_0}{2r_0^{D-2}} |p^v| e^{-ik_j x^j} \tag{5.C.10}$$

$$\tilde{t}^{(v)}_{uu} = -\frac{a_0}{2r_0^{D-2}} |p^u| e^{-ik_j x^j} \tag{5.C.11}$$

$$\tilde{t}^{(v)}_{uv} = \frac{8\pi}{2r_0^{D-2}} \frac{\sum_{n\neq 0} \mathcal{A}_n}{|p^v|} e^{-ik_j x^j} \tag{5.C.12}$$

$$\tilde{t}^{(\not{v})}_{uv} = \frac{(D-4)a_0}{2(D-2)r_0^{D-2}} |p^u| e^{-ik_j x^j} \tag{5.C.13}$$

$$\tilde{t}^{(v)}_{ij} = \frac{16\pi}{a_0 r_0^{D-6}} \frac{\sum_{n\neq 0} \bar{\mathcal{A}}^{ij}_n}{|p^v|} e^{-ik_j x^j} \tag{5.C.14}$$

$$\tilde{t}^{(\not{v})}_{ij} = \frac{2}{(D-2)r_0^{D-4}} \delta_{ij} |p^u| e^{-ik_j x^j} \tag{5.C.15}$$

$$\tag{5.C.16}$$

where $\mathcal{A}^{ij}_n$ is the sum of symmetric combination pair of left and right oscillators

$$\mathcal{A}^{ij}_n = \alpha^{(i}_n \tilde{\alpha}^{j)}_n \tag{5.C.17}$$

and $\bar{\mathcal{A}}^{ij}_n$ is the traceless part of $\mathcal{A}^{ij}_n$

$$\bar{\mathcal{A}}^{ij}_n = \mathcal{A}^{ij}_n - \frac{\delta_{ij}}{D-2} \mathcal{A}_n. \tag{5.C.18}$$

5.C.1 Inverse Fourier transform

We see that the Fourier transform of the linear metric perturbations are made up of fraction composed of polynomial of k or $\mathbf{k}$. To assist the calculation of the integral in the inverse Fourier transform, we are decomposing all the terms as fraction of the form

$$\frac{1}{k^2+c}, \qquad \frac{1}{\mathbf{k}^2+c} \tag{5.C.19}$$

The denominators in $(5.C.6 - 5.C.9)$ take similar form and can be factorised as

$$X^2 + 2(\bar{A}+\bar{C})X - 4(D-2)\bar{B}\bar{D} = (X - A_+)(X - A_-) \tag{5.C.20}$$
$$X^2 + 2(\widetilde{A}+\widetilde{C})X - 4(D-2)\bar{B}\bar{D} = (X - A'_+)(X - A'_-) \tag{5.C.21}$$

and

$$A_\pm = -(\bar{A}+\bar{C}) \pm \sqrt{(\bar{A}+\bar{C})^2 + 4(D-2)\bar{B}\bar{D}} \tag{5.C.22}$$
$$A'_\pm = -(\widetilde{A}+\widetilde{C}) \pm \sqrt{(\widetilde{A}+\widetilde{C})^2 + 4(D-2)\bar{B}\bar{D}}. \tag{5.C.23}$$

With this factorisation, we used partial fraction to decompose all the fractions into the form of $(5.C.19)$. Here are the two integrals that are needed for the inverse Fourier transform. First, I the Green's function for the transverse Laplace equation

$$I[Q,x] = \int d^{D-2}k \frac{e^{ik_i x^i}}{k_i k_i - Q}, \tag{5.C.24}$$

second, Δ the Green's function for the full Laplace equation

$$\Delta_n[\lambda, u, x^i] = \frac{2}{i(2\pi)^{D-1}} \int dk_u d^{D-2}k \frac{e^{ik_u u} e^{ik_i x^i}}{k_u - \frac{\alpha' p^v}{2nr_0^2} k_i k_i + \frac{\alpha' p^v}{2n}\lambda}. \tag{5.C.25}$$

The two integrals are related by a integration in u,

$$\int du \Delta_n[\lambda, u, x^i] = \frac{i2}{(2\pi)^{D-2}} \frac{2nr_0^2}{\alpha' p^v} I[r_0^2\lambda, x] \tag{5.C.26}$$
$$\int du e^{\frac{i2mu}{\alpha' q_2^u}} \Delta_n[\lambda, u, x^i] = \frac{i2}{(2\pi)^{D-2}} \frac{2nr_0^2}{\alpha' p^v} I[r_0^2\left(\lambda - \frac{4nm}{\alpha' p^v q^u}\right), x]. \tag{5.C.27}$$

After Fourier transform, solving the algebraic equation and inverse Fourier transform we obtain the metric perturbation

$$h^{(0)}_{uu} = \frac{4a_0|p^v|\delta(u)}{(2\pi)^{D-3}r_0^{D-4}} I[-2r_0^2\bar{F}, x-x_0] \tag{5.C.28}$$

$$h^{(v)}_{uu} = \frac{-4a_0|p^u|\delta'(u)}{(2\pi)^{D-3}r_0^{D-4}(\tilde{E}-\bar{E})}\left(I[-2r_0^2\bar{E}, x-x_0] - I[-2r_0^2\tilde{E}, x-x_0]\right) \tag{5.C.29}$$

$$h^{(S)}_{uv} = \frac{4a_0|p^u|\delta(u)}{(2\pi)^{D-3}r_0^{D-4}}\left(B_+ I[2r_0^2A_+, x-x_0] - B_- I[2r_0^2A_-, x-x_0]\right) \tag{5.C.30}$$

$$\begin{aligned} h^{(S)}_{ij} =& \frac{8|p^u|\delta_{ij}\delta(u)}{(2\pi)^{D-3}r_0^{D-6}}\Big(C_+ I[2r_0^2A_+, x-x_0] + C_- I[2r_0^2A_-, x-x_0] \\ &+ \mathcal{C} - I[-2r_0^2\bar{C}, x-x_0]\Big) \end{aligned} \tag{5.C.31}$$

$$h^{(NS)}_{uv} = \frac{-i16\pi a_0\alpha'\sigma(p^v)}{(D-2)r_0^{D-4}}\sum_{n\neq 0}\mathcal{A}_n e^{\frac{i2nv}{\alpha' p^v}}\left(B'_+\Delta_n[A'_+] + B'_-\Delta_n[A'_-]\right) \tag{5.C.32}$$

$$\begin{aligned} h^{(NS)}_{ij} =& \frac{-i32\pi a_0\alpha'\sigma(p^v)}{(D-2)r_0^{D-6}}\sum_{n\neq 0} e^{\frac{i2nv}{\alpha' p^v}}\Big(\bar{\mathcal{A}}_n\Delta_n[-2\bar{C}] \\ &- \frac{a_0}{(D-2)r_0^2}\delta_{ij}\mathcal{A}_n\left(C'_+\Delta_n[A'_+] + C'_-\Delta_n[A'_-]\right)\Big) \end{aligned} \tag{5.C.33}$$

5.C.2 Fourier integrals

We need to evaluate the integral

$$I[\pm P^2, x^i] = \int_{\mathbb{R}^{D-2}} d^{D-2}k \frac{e^{ik_i x^i}}{k_i k_i \mp P^2}. \tag{5.C.34}$$

Since the integral is diffeomorphic invariant or rotational invariant, we can pick a frame to align with the position vector $\underline{x} = (x, 0, \cdots, 0)$. We can then integrate the k_1 direction by contour integration,

$$\int dk_1 dk_2 \cdots dk_{D-2} \frac{e^{ik_1 x}}{k_1^2 + (k_2^2 + \cdots + k_{D-2}^2 \pm P^2)}. \tag{5.C.35}$$

The corresponding contour integrals are

$$\oint_C dz \frac{e^{izx}}{z^2+\lambda^2} \tag{5.C.36}$$

$$\oint_{C'} dz \frac{e^{izx}}{z^2-\lambda^2} \tag{5.C.37}$$

$$\tag{5.C.38}$$

For upper hemisphere C of radius R, the integral $(5.C.36)$ is

$$\int_{-R}^{R} dr \frac{e^{irx}}{r^2+\lambda^2} + i\int_0^\pi d\theta \frac{Re^{-Rx\sin(\theta)}e^{i(Rx\cos(\theta)+\theta)}}{R^2e^{2i\theta}+\lambda^2}. \tag{5.C.39}$$

In the limit of $R \to \infty$, the angular integral vanishes and the radial integral becomes the k_1 integral. Using the residual theorem on (5.C.36) we can deduce

$$\int_{-\infty}^{\infty} dr \frac{e^{irx}}{r^2+\lambda^2} = \frac{\pi}{\lambda} e^{-\lambda x}. \tag{5.C.40}$$

For (5.C.37) the poles are on the real axis hence we need to take a small semi-circle contour of radius ϵ around poles. For such contour C' of radius R, the integral (5.C.36) is

$$\begin{aligned} P\int_{-R}^{R} dr \frac{e^{irx}}{r^2-\lambda^2} + i\int_{-\pi}^{0} d\theta \left[\frac{\epsilon e^{i\theta} e^{i(\epsilon e^{i\theta}-\lambda)x}}{\epsilon e^{i\theta}(\epsilon e^{i\theta}-2\lambda^2)} + \frac{\epsilon e^{i\theta} e^{i(\epsilon e^{i\theta}+\lambda)x}}{\epsilon e^{i\theta}(\epsilon e^{i\theta}+2\lambda^2)}\right] \\ + i\int_{0}^{\pi} d\theta \frac{Re^{-Rx\sin(\theta)} e^{i(Rx\cos(\theta)+\theta)}}{R^2 e^{2i\theta}+\lambda^2}. \end{aligned} \tag{5.C.41}$$

Similarly, taking the limit $R \to \infty$ and using the residual theorem on (5.C.36) we can deduce

$$P\int_{-R}^{R} dr \frac{e^{irx}}{r^2-\lambda^2} = -\frac{\pi}{\lambda}\sin\lambda x. \tag{5.C.42}$$

Let $\underline{q} = (k_2, \cdots, k_{D-2})$ and convert to polar coordinates,

$$I[-P^2, x^i] = \pi\Omega_{D-4}\int_0^\infty dq \frac{q^{D-4}}{\sqrt{q^2+P^2}} e^{-x\sqrt{q^2+P^2}} \tag{5.C.43}$$

$$I[P^2, x^i] = \pi\Omega_{D-4}\left[\int_P^\infty dq \frac{q^{D-4}}{\sqrt{q^2-P^2}} e^{-x\sqrt{q^2-P^2}} - \int_0^P dq \frac{q^{D-4}}{\sqrt{P^2-q^2}} \sin x\sqrt{P^2-q^2}\right] \tag{5.C.44}$$

After substitute of $q = \sqrt{Q_\pm^2 \pm P^2}$, we can read off the integral from *Table of Integrals, Series and Products [130], section 3.387 equation 6, 7 and section 3.771 equation 6* for $D > 3$

$$I[-P^2, x^i] = \pi^{\frac{1}{2}}\Omega_{D-4}\Gamma\left(\frac{D-3}{2}\right)\left(\frac{2P}{x}\right)^{\frac{D-4}{2}} K_{\frac{D-4}{2}}(xP) \tag{5.C.45}$$

$$I[P^2, x^i] = -\frac{1}{2}\pi^{\frac{3}{2}}\Omega_{D-4}\Gamma\left(\frac{D-3}{2}\right)\left(\frac{2P}{x}\right)^{\frac{D-4}{2}} Y_{\frac{D-4}{2}}(xP) \tag{5.C.46}$$

where $Y_n(z)$ and $K_n(z)$ are the Bessel function and modified Bessel function of the second kind. For $D = 3$, we can skip the q integrals and

$$I[-P^2, x^i] = \frac{\pi}{P} e^{-xP} \tag{5.C.47}$$

$$I[P^2, x^i] = -\frac{\pi}{P}\sin xP, \quad x \neq 0 \tag{5.C.48}$$

5.D Next-to-subleading contribution to onshell action

The subleading onshell action are

$$\delta I_{uu}^{(S)} = \frac{a_0|p_1^u p_2^v|}{2}\left(\frac{\sqrt{2\bar{E}}}{2\pi r_0 x}\right)^{\frac{D}{2}-2} K_{\frac{D}{2}-2}(xr_0\sqrt{2\bar{E}}) + (1 \leftrightarrow 2, u \leftrightarrow v) \tag{5.D.1}$$

$$\delta I_{uv}^{(S)} = \frac{a_0|p_1^u p_2^v|}{2}\left(\frac{1}{2\pi r_0 x}\right)^{\frac{D}{2}-2}\left[-\frac{\pi}{2}B_+ Y_{\frac{D}{2}-2}(xr_0\sqrt{A_+}) + B_- K_{\frac{D}{2}-2}(xr_0\sqrt{-A_-})\right] \tag{5.D.2}$$

$$+ (1 \leftrightarrow 2, u \leftrightarrow v)$$

$$\delta I_{ij}^{(S)} = 4\left|\frac{p_1^u}{p_2^v}\right|\sum_{n\neq 0}\left(\frac{1}{2\pi r_0 x}\right)^{\frac{D}{2}-2}\mathcal{A}_n \times \tag{5.D.3}$$

$$\left[-\frac{\pi}{2}C_+ Y_{\frac{D}{2}-2}(xr_0\sqrt{A_+}) + C_- K_{\frac{D}{2}-2}(xr_0\sqrt{-A_-}) + \mathcal{C} K_{\frac{D}{2}-2}(xr_0\sqrt{2\bar{C}})\right]$$

$$+ (1 \leftrightarrow 2, u \leftrightarrow v)$$

$$\delta I_{uv}^{(NS)} = \frac{4a_0 r_0^2}{(D-2)}\left|\frac{p_1^u}{p_2^v}\right|\left(\frac{1}{2\pi r_0 x}\right)^{\frac{D}{2}-2}\sum_{n\neq 0} n\mathcal{A}_n \times \tag{5.D.4}$$

$$\left[-\frac{\pi}{2}B'_+ Y_{\frac{D}{2}-2}(xr_0\sqrt{A_+}) + B'_- K_{\frac{D}{2}-2}(xr_0\sqrt{-A_-})\right]$$

$$+ (1 \leftrightarrow 2, u \leftrightarrow v)$$

$$\delta I_{ij}^{(NS)} = \frac{32a_0}{|p_1^v p_2^u|}\left(\frac{1}{2\pi r_0 x}\right)^{\frac{D}{2}-2}\sum_{n_1,n_2>0} n_1\left|\frac{4n_1 n_2}{\alpha'^2 p_1^v p_2^u}\right|^{\frac{D}{4}-1} \times \tag{5.D.5}$$

$$\left[-\frac{\pi}{2}\mathcal{M}_{n_1,-n_2} Y_{\frac{D}{2}-2}\left(xr_0\sqrt{\frac{4n_1n_2}{\alpha'^2 p_1^v p_2^u}}\right) + \mathcal{M}_{n_1,n_2} K_{\frac{D}{2}-2}\left(xr_0\sqrt{\frac{4n_1n_2}{\alpha'^2 p_1^v p_2^u}}\right)\right]$$

$$+ (1 \leftrightarrow 2, u \leftrightarrow v)$$

where $B_\pm, B'_\pm, C_\pm, \mathcal{C}$ are constants and $\mathcal{M}_{n_1,n_2}$ is a function $\mathcal{A}_{n_1}^{ij}, \mathcal{A}_{n_2}^{ij}$,

$$\mathcal{M}_{n_1,n_2} = \bar{\mathcal{A}}_{n_1} \cdot \bar{\mathcal{A}}_{-n_2} + \bar{\mathcal{A}}_{-n_1} \cdot \bar{\mathcal{A}}_{n_2} \tag{5.D.6}$$

with

$$B_\pm = \pm\frac{1}{A_+ - A_-}\left(\frac{(D-4)(A_\pm + 2\bar{C})}{D-2} - 4\bar{B}\right) \tag{5.D.7}$$

$$B'_\pm = \pm\frac{A'_\pm + 2\bar{C}}{A'_+ - A'_-} \tag{5.D.8}$$

$$C_\pm = \pm\frac{\bar{D}}{A'_+ - A'_-}\left(-\frac{(D-4)a_0}{(D-2)r_0^2} + \frac{4\bar{B}}{A_+ + 2\bar{C}}\right) \tag{5.D.9}$$

$$C'_\pm = \pm\frac{\bar{D}}{A'_+ - A'_-} \tag{5.D.10}$$

$$\mathcal{C} = \frac{2}{D-2} + \frac{4\bar{B}\bar{D}}{(A_+ + 2\bar{C})(A_- + 2\bar{C})} \tag{5.D.11}$$

5.E Scaling of Bessel functions

For $z \gg \left|\alpha^2 - \frac{1}{4}\right|$,

$$Y_\alpha(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{5.E.1}$$

$$K_\alpha(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z} \tag{5.E.2}$$

For $0 < z \ll \sqrt{\alpha+1}$,

$$Y_\alpha(z) \sim \begin{cases} \frac{2}{\pi}\left(\ln\left(\frac{z}{2}\right) + \gamma\right) & \text{if } \alpha = 0 \\ -\frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{z}\right)^\alpha + \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha \cot(\alpha\pi) & \text{if } \alpha \in \mathbb{Z}^- \\ -\frac{(-1)^\alpha\Gamma(-\alpha)}{\pi}\left(\frac{z}{2}\right)^\alpha & \text{otherwise} \end{cases} \tag{5.E.3}$$

$$K_\alpha(z) \sim \begin{cases} -\ln\left(\frac{z}{2}\right) - \gamma & \text{if } \alpha = 0 \\ \frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{z}\right)^\alpha & \text{if } \alpha\, 0 \end{cases} \tag{5.E.4}$$

where γ is the Euler-Mascheroni constant.

CHAPTER 6

Conclusions

To conclude, we will first present the results obtained from the previous three chapters on renormalised entanglement entropy, first law of entanglement entropy and classical string contribution to chaos. Furthermore, we will discuss about the significant and outlook from the results and methodologies developed in these three chapters.

In chapter 3, we found an expression for the renormlised entanglement entropy in terms of the Euler characteristic and other renormalised curvature invariants of the bulk entangling surface. The expression was derived from the renormalising the area of the minimal surface in asymptotically locally AdS_{2n} case which we also showed it matches with the renormalised entanglement entropy formula derived from the renormalised action. This is due to the universal property of the holographic entanglement entropy in even spacetime dimension. This new formula of renormalised entanglement entropy allows us to access the property of the entangling surface. Most notably, the topology of the bulk entangling surface is directly represented by the Euler characteristic in the renormalised entanglement entropy formula. The term constructed from the extrinsic curvature of the bulk entangling surface takes the form similar to higher dimensional Willmore functionals. Another term in the new expression is the renormalised integral with integrand constructed from the Weyl tensor pull back to the normal space of the bulk entangling surface. This term proved to be useful in the first law of entanglement entropy.

In chapter 4, the renormalised version of the first law of entanglement entropy was derived. Three approaches were taken to show the equivalence between the variation of renormalised entanglement entropy and variation of the modular energy calculated from the renormalised stress tensor. The first two approaches set the radius of the disk like entangling region to be small, resolving to the infinitesimal first law of the entanglement entropy. The variation of the renormalised area density was explicitly shown to match the density of the modular energy given by the renormalised stress tensor. In even spacetime dimensions, we applied the new renormalised entanglement entropy formula developed in chapter 3 to the first law of entanglement entropy which related the variation of renormalised entanglement entropy and variation of the modular energy to the variation of the pull back of the Weyl tensor. The integral version of the first law of entanglement entropy is related to conserved charges in holography. Hence we used the holographic renormalisation procedure in the Hamiltonian formalism, that specialises in obtaining the renormalised Noether charges density, to derive the renormalised integral first law of entanglement entropy. We further explained our findings in the covariant phase space formalism and showed examples in spacetime dimension 4, 5 and 6.

In chapter 5, we considered the modification of the shock wave analysis of holographic chaos from particles scattering to scattering of classical strings with oscillation. Due to the setup, we focused on the high energy gravitational eikonal scattering. To validate our method, we reproduced the exact result for point particle eikonal phase. Then we included particular contributions from the transverse oscillation of the classical string. The leading centre of mass contribution is identical to the standard point particles case. The subleading contributions were shown explicitly to vanish in accordance to the onshell condition. The next-to-subleading contributions were calculated by solving the Einstein equation soured by the energy momentum tensor of the transverse oscillation modes. We saw the correction to the eikonal phase is insignificant hence satisfying the chaos bound. The result from the semi-classical shock wave calculation matches with heuristic argument from perturbative method.

The exploration of renormalised entanglement entropy can be naturally extended into high derivative gravity and can include non-trivial time dependence by switching to the covariant holographic entanglement entropy in the HRT formalism [8]. For higher derivative gravity, there are additional terms in the counterterms action hence more counterterms for renormalised entanglement entropy as found in [54]. The renormalised entanglement entropy formula would then need to be generalised to the renormalised area integral of the HRT surface. The HRT surface, the covariant analogue of the RT surface, is an codimension two extremal surface with minimal area. Also the form of the Weyl integral in renormalised entanglement entanglement formula is related to the entropy integral in conformal gravity. The connections between these as suggested in [122] are interesting directions for future investigation.

Holographic complexity has been following the footsteps of holographic entanglement entropy. In the complexity=volume (CV) proposal, complexity is equal to the volume of a codimension one subspace [13, 12]. Hence the bulk dual of both complexity and entanglement entropy are divergent volume/area of some geometrical object. In the complexity=action (CA) proposal, complexity is equal to the action of some region of the bulk spacetime [14, 15]. Then both complexity and entanglement entropy are evaluations of the action in some regimes in the bulk, and are divergent. The divergent structures of both proposals have been studied in [131]. The first law of complexity was also studied in [132]. After the results and methodologies developed for renormalised entanglement entropy in chapter 3 and 4, we gained insights and found new applications by reformulating the renormalised quantities. Therefore, renormalised complexity will be a great extension of our previous work on renormalised entanglement entropy.

The discovery of maximal chaotic growth in Einstein gravity, posed constraints on the dual CFT [133]. The bulk scattering picture provided clear physical intuition as to how chaos is propagated. Thus, it is interesting to investigate on how different type of bulk interactions change, or not change, the chaotic behaviour. The string scattering problem in curved spacetime by itself deserves attention. For future work, we would like to explore the possible correction to the Lyapunov exponent by considering corrections to the eikonal phase as mentioned in the conclusion of chapter 5.

The entanglement structure of spacetime and chaotic behaviour of gravity inspired a whole new group of interesting theories of gravity as an ensemble [134, 135] or as an emergent property of quantum mechanics [55, 56, 57]. The interests in developing discretised quantum circuit models for holography is growing fast [16, 136, 137]. The tie between the general concepts of holography, quantum information and quantum computing is not only of interest to fundamental theorists but it is becoming a relevant and applicable theory.

This thesis highlighted aspects of quantum information in holography. The technically challenging yet conceptually intriguing realisation of quantum phenomena in gravity is teaching us about the quantum nature of spacetime. With the aforementioned recent developments and future directions, we as a community are getting closer towards a complete picture of quantum gravity.

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