Jerry Ericksen: Liquid Crystal Pioneer

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Abstract

In the 1960s Jerry Ericksen made major contributions to the construction of the continuum theory of nematic liquid crystals. This paper gives a brief summary of his work and the consequent giant impact on the field.

Key Words: Liquid Crystals; Ericksen; Nematic; Continuum Theory

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**Pioneering ideas, formulations, and techniques in modern
continuum mechanics – in memory of Jerald Laverne Ericksen**

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1 Introduction

No tribute to Jerry Ericksen's research work in the mathematical sciences can be complete without serious contemplation of his seminal contribution to the mathematical formulation of the continuum theory of liquid crystals. In this paper we shall firstly summarize some of his important work. We shall also make a start at providing some a more extensive context within which historians of science in the future may be able to make a more complete contextual analysis of its significance. In the interstices we shall give a brief introduction to liquid crystals which should enable the naive reader to follow our key ideas.

Jerry has left us an extensive archive. We hope that this archive will reveal more both about Jerry's life and about his work. Examining it should do much to complete, and perhaps to some extent supersede, the sketches drawn in this article. But it will take time and is a project for another day.

This volume is mainly directed at experts, and the mathematical level of other contributions is concomitantly elevated. This paper, by contrast, will be relatively mathematically unsophisticated. Our goal will be that even those outside the field may at least appreciate some of the flavor of Jerry's work and why workers within the field regarded him with such respect, We beg the forbearance of experts who might otherwise be inclined to neglect the ordinary language which we employ in the place of mathematical formalism.

The reader will find some mathematics here, of course, because Jerry was, as all readers are aware, a consummate mathematician. But he was not only a mathematician. Behind the equations lay a powerful physical intuition. This provided an intellectual structure which enabled him to seek a fundamentals upon which practical calculations could subsequently be based. Perhaps as importantly, in doing so, he provided a route map, which would also be used by subsequent workers to construct analogous theories for similar materials at a later stage.

The 1960s were an "Age of Heroes" for liquid crystals. During this period – including perhaps a few years before and a few years after – a branch of science, founded as a curiosity in the late 1880s, was reaching its scientific maturity. In the early 1900s there appeared several classic monographs [1–4] on liquid crystals, which raised many questions but provided few answers. These monographs – written in German and French, and thus (sadly!) relatively inaccessible to the modern reader – are of great historical interest, but provide little useful to the beginning graduate student in the 2020s. By the 1970s, however, the first textbooks on liquid crystals [5, 6] were appearing on the market, and even the first textbook generation remains useful half a century later. And a major influence on the content of these textbooks, high up on the "Liquid Crystal Honor List", is Jerry Ericksen.

The organization of this paper is as follows. In §2 we present some historical focused background to liquid crystals, which should explain to the naïve reader enough to follow the subsequent sections. In §3 we examine some of Ericksen's key contributions to liquid crystal science. After some slightly hagiographic front matter, the scientific core of this presentation is presented in 3.1. In §4 we attempt a (rather limited!) contextualization of what has gone before.

2 Liquid Crystals

Readers familiar with liquid crystals should omit this section, which may well merely repeat for them long-familiar concepts in a language which they find condescending, unhelpful and oversimplified. We give here a brief outline, with a historical focus, enabling " nematically naïve" readers to make sense of what follows. Fuller accounts of different aspects of the early

history of liquid crystals can be found elsewhere [7–19]. The most important key idea is that "liquid crystals" are not crystals at all. The misnomer is the result of a historical mistake, which subsequent scientific rectification failed to eliminate. However, as with other complex fluids, a coherent description requires the use of ideas from both theories of fluids and of solids.

Liquid crystals are thermodynamic phases which interpose themselves between the crystalline and liquid phases in some materials. Early workers misread the appearance of some colors which appeared in diffraction patterns as indicating crystallinity, when in fact only local anisotropy could rigorously be inferred. The inference of liquidity came from flow observations, and in general (although not universally), indeed liquid crystals are fluid in the sense that they do not sustain a shear stress without flowing. The principal liquid crystal phases are the *nematic* phase, in which there is a local preferred direction, and *smectic* phases in which there is also some layering. A chiral version of the nematic phase, in which the preferred direction is uniform within a layer, and lies within the layer plane, but twists in a helix as one proceeds perpendicular to the layer, is known as the *cholesteric* phase.

Most discussion in this paper concerns nematics. The unit vector in the local preferred direction is known as the *director* $\hat{\mathbf{n}} \equiv -\hat{\mathbf{n}}$, although we note that the use of this term dates only from the modern period. Nevertheless we shall use the term anachronistically in our historical discussion. The first attempts at creating a continuum theory were made in the 1920s and 1930s by the Swedish theoretical physicist Carl Wilhelm Oseen (1879-1944). Oseen's papers not only dealt with continuum theory (he called it "the distortion theory"), but also the link between the continuum theory and the microscopic intermolecular interaction. In addition, as at that stage essentially all experiments were optical, he studied the optical properties of the nematic and cholesteric phases.

To a contemporary student, Oseen's articles are quite hard going. This is partly because he does not use the tensor or vector notations in use nowadays, partly because of his insistence on labeling his quantities with Gothic letters, and partly, it must be admitted, because the calculations are quite difficult anyway. Summaries of Oseen's conclusions can be found in a book [20], a long review article [21], and in a contribution to 1933 conference held in London [22], his only paper in English. Oseen constructed an elastic theory of nematic free energy more or less as we use it nowadays.

This theory was recapitulated by F.C. Frank [23] in 1958. This classic paper revitalized studies of liquid crystals after a period of quiescence. We give here (a slightly modified version of) Frank's formula * for the liquid crystal free energy density associated with changes in the director $\hat{n}$ is

$$
\begin{aligned}
F_{\text{elastic}} = {} & \frac{1}{2} K_{11} \left(\nabla \cdot \hat{\mathbf{n}} \right)^2 + \frac{1}{2} K_{22} \left(\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}) \right)^2 \\
& + \frac{1}{2} K_{33} \left(\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}) \right)^2 - K_{24} \nabla \cdot \left[\hat{\mathbf{n}} (\nabla \cdot \hat{\mathbf{n}}) + \hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}) \right].
\end{aligned}
\tag{1}
$$

The contributions to this free energy are known respectively as splay, twist, bend and saddle-splay, and the constants K_{ij} are known as (Frank-Oseen) elastic constants.

The static distortion theory could be tested by considering alignment in a thin infinite sample of thickness d, in the presence of a magnetic field which adds a term of the form $-\chi_A (\mathbf{H}.\hat{\mathbf{n}})^2$ per unit volume to the free energy density (1). This would favor a director parallel or antiparallel to **H**. Suppose there is a competition between Dirichlet boundary conditions, favoring, say, an in-plane director (known nowadays as *strong anchoring*), and the magnetic field, favoring a director perpendicular to the sample.

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*The notation here corresponds to that of Selinger [24], rather than that of Frank [23].

In 1929, the Russian physicist Vsevolod Konstatinovich Frederiks (1885–1944) and his graduate student Valentina Zolina showed experimentally that there is a threshold field H_c below which the boundary condition wins completely, and the director remains in the plane of the layer [25]. For fields $H > H_c$, the director in the sample begins to realign perpendicular to the layer, and only for fields $H \gg H_c$ is the director within the sample more or less parallel to the field, with thin boundary layers of thickness $\sim \dfrac{H_c}{H}d$ close to the wall.

The German (later Brazilian) physicist Hans Zocher (1893–1969) was able to explain this experiment using Oseen's framework. More importantly, he was unable to explain them using the competing "Swarm Theory". This was championed by the Dutch theoretical physicist Leonard Ornstein (1880–1941), and had previously been regarded favorably by the then small liquid crystal community. Zocher found that

$$H_c = \frac{\pi}{d}\sqrt{\frac{K}{\chi_A}}, \tag{2}$$

where K is an elastic constant; which elastic constant depends on the exact geometry under consideration. The Frederiks experiment (as it came to be known) can thus be regarded as the *experimentum crucis* for the distortion theory, and H_c is known as the *Frederiks threshold*.

In his 1931 Ph.D. thesis one of Oseen's graduate students, Adolf Anzelius (1894-1979), attempted to generalize this static theory to dynamics [26], but eventually it turned out that his stress tensor lacked some key elements. Even Oseen [22] himself was publicly somewhat critical. The task of constructing a dynamic continuum theory was made considerably harder by the lack of reproducible data. However, in careful experiments the Polish physicist Marian Mięsowicz (1907-1992) was able to identify three separate nematic viscosities η_1, η_2, η_3 depending on the relative configuration of the directors (constrained by strong magnetic fields) and the shear flows. Finally, another relevant experiment by the Russian physicist Victor Tsvetkov [27] (1907–1999) in 1939 measured the rotational viscosity, associated with the director attempting to follow a rotating magnetic field, while the sample as a whole remained at rest.

There then followed a quiescent period in liquid crystal research, for reasons which have been discussed in the literature, and probably have to do with the war and subsequent research concentration on other matters. The baton was picked up again by Frank in 1958, as discussed above.

3   Ericksen and liquid crystals

A number of those contributing at the highest level to the theory of liquid crystals, in particular Charles Frank (1911-1998) and Pierre-Gilles de Gennes (1932-2007), only lent their talents to the liquid crystal community. Frank's principal academic contributions came in his microscopic studies of crystal dislocations [11, 28], while de Gennes spread his genius across the whole of what came to be known as Soft Matter physics [29, 30]. In both cases the magnitude of their non-liquid-crystal *oeuvre* dwarfed their liquid crystal work, at least in terms of the number of pertinent pages. As other contributors to this volume will readily attest, Jerry Ericksen was likewise a member of this polymathic company. Notwithstanding his iconic status as one of the founders of modern liquid crystal continuum theory, a mere 33 papers out of the total of 145 included in his 2005 CV [31] were devoted to studies of liquid crystals or anisotropic fluids.

Ericksen's route toward the study of liquid crystals would be regarded as unexpected by many outside the mathematical sciences. He reports in his autobiographical sketch [31] that around 1959...

> ...I decided to formulate a simple, properly invariant theory of a fluid with a single preferred direction. I wondered whether there were any real fluids that might be roughly described by it. Chemists are likely to know about such things, so I asked Bernard Coleman, who has such a background. He suggested and told me a little about liquid crystals, materials I had never heard of. Soon, I made my first contact with a person working on these, James Fergason, a physicist then working for Westinghouse in Baltimore...

> ...We learned of a symposium on these at a huge American Chemistry Society meeting, so we participated and made contact with the other participants. At the time, most of those interested in liquid crystals were chemists. However, it was not long before a number of physicists became involved, stimulated by the creation of the Orsay Group, headed by Pierre De Gennes...

So, primarily Ericksen was concerned with providing a consistent foundation for a continuum theory of fluids with some internal orientational variables. Only secondarily was he led by a colleague of a colleague, to the liquid crystal community, who would be able to provide experimental examples of his initially mythical fluids. It was only after that, that he was completely convinced that it would be fruitful to concentrate on liquid crystal studies.

The liquid crystal community has created a permanent memorial to Ericksen by endowing some key concepts with his name. We talk about the *Ericksen stress*, the *Ericksen number*, *Ericksen inequalities*, and, of course, the everyday workhorse liquid crystal continuum theory is the *Ericksen-Leslie theory*.

3.1 Overview

Initial work on anisotropic fluids

Ericksen's first few papers on anisotropic fluids [32–34] were concerned with constructing a general continuum theory. Although his contacts had shown that liquid crystals were experimental candidates, he was cautious in insisting that this was not a theory of liquid crystals *per se*. The goal was to use general principles of force and couple balance, entropy production, together with symmetry principles. One such, entitled "Transversely Isotropic Fluids" [33] starts:

> As is noted by *Noll* (1), most theories of anisotropic fluids do not satisfy invariance conditions which any continuum theory of classical mechanics should. The purpose of this paper is to present and discuss some features of what may be the simplest properly invariant theory of anisotropic fluids, a type of theory which is virtually unexplored. ...

Who the miscreant uninvariant competitors are, is left deliberately vague, although the extract above is accompanied by mention of papers by Noll [35], Oldroyd [36] and Green and Rivlin [37]. As Walter Noll (1925-2017), Albert Green (1912-1999) and Ronald Rivlin (1915-2005) were close colleagues and presumably would not have been offended by mere professional disagreements, they seem unlikely candidates.

Much effort was devoted to what approximations were allowable and/or fruitful. Should the direction vector $\mathbf{n}$ be regarded as constant, or would flow fields affect not only its direction but its magnitude? Could one ignore the "molecular inertia" associated with rotation of $\mathbf{n}$ and merely balance couples? What were the correct forms of the constitutive relation linking the stress tensor t_{ij} to the rate of strain tensor d_{ij}, and likewise the local rotational force density g_i to flow and directional characteristics?

Clearly the relation had to involve the vector n itself, as well as its covariant time derivative[*] $N_i = \dfrac{Dn_i}{Dt} - w_{ij}n_j$, where $w_{ij} = \frac{1}{2}\left(\partial_j v_i - \partial_i v_j\right)$ is the vorticity tensor. One early attempt at this

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[*]We have substituted the modern notation N for Ericksen's contemporary $\hat{n}$.

relation [34] required as many as 12 viscosity coefficients:

$$t_{ij} = \left(\alpha_0 + \alpha_1 d_{kk} + \alpha_2 d_{km} n_k n_m + \alpha_3 N_k n_k\right)\delta_{ij}$$
$$+ \left(\alpha_4 + \alpha_5 d_{kk} + \alpha_6 d_{km} n_k n_m + \alpha_7 N_k n_k\right) n_i n_j \tag{3}$$
$$+ \alpha_8 d_{ij} + \alpha_9 d_{ik} n_k n_j + \alpha_{10} d_{jk} n_j n_k + \alpha_{11} n_i N_j + \alpha_{12} n_j N_i.$$

However, when the fluid is incompressible, the director supposed to always have unit modulus, and the molecular inertia ignored, the number of viscosities reduces to only 4. The constitutive equation for g_i starts with 4 rotational viscosities γ_m, and also includes 4 of the α's, but when all the approximations are applied, this reduces to:

$$g_i = \mu\left(d_{ij} n_j - d_{km} n_k n_m n_i\right), \tag{4}$$

with a single rotational viscosity μ.

At this stage, calculations only included flow. Neither magnetic fields, known from early work to rotate the liquid crystal director, nor the effect of the elastic forces in the liquid crystal, were included. The idea, worked through to some extent in a number of papers [33, 34, 38, 39], was that the flow fields by themselves would provide some orienting forces for the director, although the calculations would have to be carried out self-consistently.

Ericksen Stress
However, by the end of 1961 Ericksen was ready to include the elastic forces in his theory [40, 41]. Again Ericksen's own words [41] are superior to those of the present writers:

Hydrostatic Theory of Liquid Crystals

> ...This permits us to include as a special case the hydrostatic theory presented by FRANK [23]; which is a revised static version of the hydrodynamical theory of OSEEN [22]. ...
> ...In studies of the static behavior of liquid crystals, it is common to ignore or to treat in an obscure way the stress tensor and other representations of force. Our treatment emphasizes these. From the viewpoint of the development of general continuum mechanics, the theory seems interesting. It is one of the few mechanical theories which involve an asymmetric stress tensor. ...
> [Reference numbers for this paper, rather than in ref.[41].]

Two points in this introduction stand out. The first is the asymmetric stress tensor, which is, as Ericksen points out here, rare but not forbidden in continuum theory. This is not the first time that Ericksen has addressed this issue; already in ref.[34], he wrote the equation

$$n_i g_j - n_j g_i + t_{ij} - t_{ji} = 0 \ . \tag{5}$$

The symmetry of the stress tensor in fluids without internal orientational variables (i.e. usual fluids) is the dynamical consequence of the inability of an infinitesimally small volume to sustain a couple. Asymmetry indicates the existence of such a couple. Eq.(5) indicates how in an anisotropic fluid, the couple due to the asymmetric stress tensor can be balanced by an explicit torque.

The second point concerns the manner in which the theoretical physicists Oseen and Frank had neglected the stress tensor. Both the earlier treatment of Oseen [20–22] and its later renaissance by Frank [23] deal with free energies. The equilibrium is determined by minimizing this free energy, which in turn is performed by constructing Euler-Lagrange equations subject to constraints

such as $|\hat{\mathbf{n}}| = 1$. Neither really thought about the dynamic consequences of this minimization. Implicitly they suppose that the process of orientational relaxation to equilibrium can be decoupled from fluid motion. But given that the dynamical stress tensor contains orientational contributions, these may in fact be coupled, and the equilibrium equations must apply not only to orientational equilibrium but also to hydrostatic equilibrium.

In other words, there must be an elastic stress tensor with the properties that (i) its associated equilibrium equation is equivalent to the Euler-Lagrange equation, and (ii) it must also contribute to the total stress tensor in the dynamical equations away from equilibrium. Using a virtual work argument, Ericksen derives the following elastic contribution to the stress tensor:

$$\alpha_{ik} = -\frac{\delta F_{\text{elastic}}}{\delta n_{r,k}} n_{r,i} \; , \tag{6}$$

where F_{elastic} is the Oseen-Frank elastic free energy density, and commas indicate a partial derivative index. This was a key contribution to the final dynamical theory and is labeled by de Gennes [5], and subsequently more generally in the literature, as the *Ericksen stress*.

Ericksen Inequalities

Having discussed the implications of the elastic theory for the construction of the stress tensor, Ericksen then (1962) immediately turned his attention [42] to the the Euler-Lagrange equations that followed from the Frank-Oseen elastic free energy, eq.(1). There are four coefficients, $K_{11}, K_{22}, K_{33}, K_{24}$, the last of which is associated with a total derivative. Apparently a second derivative of the director with respect to space seems to enter the associated term. But in fact it does not; after suitable expansion, the second derivatives cancel out. Notwithstanding our formulation of Frank's theory in eq.(1), in his 1958 paper [23] the second derivatives have been removed and only first derivatives appear explicitly. We add in passing that another apparently analogous term $K_{13}\nabla \cdot [\hat{\mathbf{n}}(\nabla \cdot \hat{\mathbf{n}})]$ has often been discussed in the literature, but in this case the second derivative is irreducible, and the term is in some sense inadmissible at the level of a second gradient expansion [24].

The key point about the K_{24} term is that because it is a total derivative, it does not enter the Euler-Lagrange equations, nor by consequence, the associated elastic stress tensor. The term only enters the solution of the director problem through boundary conditions, which, as Ericksen observed, caused considerable problems when thinking about how to measure it. In fact, Oseen [43] had made a similar observation as long ago as 1924. Oseen noted that in the elastic theory of solids the 21 apparent constants reduced to 15, and

> ...the explanation of the contradiction has been given by Lord Kelvin. ...

However, unhelpfully, Oseen fails to give us a reference to the relevant paper by Kelvin!

Though motivated by the physical problem, Ericksen is here concerned by the mathematics. The K_{24} term is what he calls a "nilpotent energy", but what nowadays we would label as "null Lagrangian". What kind of terms can enter the energy functional and not appear in the resulting equations? This general problem has been extensively discussed subsequently in the mathematical literature (see e.g. [44, 45]).

He returns to the null Lagrangian problem in 1966 [46]. In modern language, the null term can be re-expressed in terms of a quadratic expansion in irreducible director curvatures. If the undistorted nematic is in equilibrium, the coefficients in the quadratic re-expansion must all be positive. These coefficients are combinations of the elastic constants K_{ij}. The consequence is:

$$K_{11} > K_{24}; \quad K_{22} > K_{24}; \quad K_{24} > 0, \tag{7}$$

results which have come to be known as the *Ericksen inequalities*.

Ericksen-Leslie Theory

The was a short hiatus between 1963 and 1966 during which Ericksen worked less on liquid crystals. His interest in liquid crystal problems was rekindled by the arrival of Frank Leslie (1935-2000) as a sabbatical visitor at Johns Hopkins in August 1966. Leslie, originally from Dundee in Scotland, and eventually to finish his career as Professor of Applied Mathematics at Strathclyde University in Glasgow, was at the time a lecturer in the mathematics department in Newcastle, England. His immediate boss was Professor Albert Green, who, as we have seen, was a specialist in the dynamics of complex fluids and a colleague of Ericksen's.

It was Green who had recommended Leslie to Ericksen. Under Green's tutelage, Leslie had rederived a version of the dynamical part of Ericksen's liquid crystal theory [47], but had noted that

> ... The theory described in this paper allows a fluid to have a preferred direction at each point ... materials called liquid crystals possess such preferred directions. However since the present theory does not reduce to Frank's hydrostatic theory [23], it would appear inadequate to describe these materials. ...

The story of the creation of the Ericksen-Leslie theory has been told elsewhere [17, 48], but bears repetition here. It was Ericksen's suggestion that Leslie generalize his 1966 paper [47] (itself partly based on Ericksen's previous work) so as to complete the hydrodynamic theory of nematic liquid crystals. After some discussion, it appeared that two key inputs would have to be (a) the explicit inclusion of the Frank-Oseen elastic theory, and (b) the only attempt so far to create a dynamic theory of liquid crystal, that of Anzelius [26].

Luckily, by that time Ericksen had been able to get hold of a copy of Anzelius's Ph.D thesis from Uppsala University. The good news was that the thesis was not in Swedish. The bad news was that it was in German, which had been the scientific *lingua franca* of the time. Leslie's German was rudimentary, but the library did own a rather broad German dictionary. Leslie repaired to the library. After a month, with thesis in one hand and dictionary in the other, he had made some degree of progress. But it was all painfully slow.

We do not here give an exposition of the full theory, which can be found by now in many textbooks (see e.g. [49]), but confine ourselves to a few remarks. Leslie's new theory modified and synthesized both Anzelius's and Ericksen's previous work. The dynamical part of the stress tensor t_{ij}^D now included 6 viscosities :

$$t_{ij}^D = \alpha_1 n_i n_j n_k n_p d_{kp} + \alpha_2 n_i N_j + \alpha_3 n_j N_i + \alpha_4 d_{ij} + \alpha_5 n_i n_k d_{kj} + \alpha_6 n_j n_k d_{ki}. \tag{8}$$

The α_i have come to be known as *Leslie viscosities*. The local viscous body torque was given by $\mathbf{G} = \hat{\mathbf{n}} \times \mathbf{g}$, where

$$g_i = \gamma_1 N_i + \gamma_2 n_j d_{ji}. \tag{9}$$

In addition the coupling between the fluid motion and its rotation dictated some relations between the rotational and fluid viscosities:

$$\gamma_1 = \alpha_3 - \alpha_2; \qquad\qquad \gamma_2 = \alpha_6 - \alpha_5. \tag{10}$$

We can compare the new theory to the previous attempts. The theory is able to make contact with the old shear measurements of both Mięsowicz [50, 51] and the rotational measurements of Tsvetkov [27] mentioned in §2 As compared to Anzelius, the formula for $\underline{t}^D$ includes a dependence on the covariant derivative $\mathbf{N}$. Anzelius had not included any director time derivatives. He did

include a viscous torque, but in a inconsistent fashion, as the stress tensor remained symmetric. Thus in contrast to Anzelius, but in agreement with Ericksen's insights, $\underline{t}^D$ was now properly asymmetric. Finally. as compared to Leslie's previous paper [47], elastic forces had been included. The result was a theory that now did reduce to the Frank theory in the hydrostatic limit, and also (as it was designed to be) to the Navier–Stokes equations in the absence of anisotropy, in which case the only remaining viscosity is α_4

Leslie's paper [52] was only submitted after his return to the UK, and was not published until 1968. It was modestly entitled "Some constitutive equations for liquid crystals". Acknowledging his strong intellectual debt to Ericksen, Leslie did suggest that his name be included as an author. Ericksen would have none of it. It was his rather generous policy not to include his name on papers written by his graduate students and postdoctoral workers. He remarks in his autobiography [31] that he had had some bad experiences in collaborating on publications, and he confirmed his policy in correspondence with one of the present authors.

So the paper appeared under Leslie's by-line alone. But Ericksen was universally acknowledged by colleagues to have been its grandfather, despite, so to speak, not being named on its birth certificate. According to current ethical publication convention, he should have been included as an author of the paper. It was only Ericksen who was surprised that the theory came to be called the *Ericksen-Leslie theory*. Given the background, the historical judgment is surely right.

A final touch to the theory was added by Parodi [53], using the symmetry of Onsager coefficients innon-equilibrium thermodynamics, in the language of which the Ericksen-Leslie theory can be rephrased. This gives the relation $\alpha_6 - \alpha_5 = \alpha_2 + \alpha_3$. In that context, it is amusing to note Ericksen's remark in an earlier paper [34]:

> We feel that the basic equations are applicable to fluids for which (n may not be variable). We have less faith in the Clausius-Duhem inequality and Onsager relations. ...

We are led to wonder whether Ericksen's "less faith" of the Onsager reciprocal relations may have played a role in missing this relation, which is well verified by experiment.

Ericksen number
There are three physical sources of director alignment. We consider, as in §2, a sample of thickness d, in field H, and with flows of characteristic magnitude $v$subject to no slip boundary conditions.

Two of the sources of director alignment are static – "strong anchoring" (i.e the Dirichlet boundary condition fixing the director at the boundary) and an imposed magnetic field. The competition between them has been considered in §2. The critical field $H_c = \dfrac{\pi}{d}\sqrt{\dfrac{K}{\chi_A}}$ from eq.(2) might be expressed in more modern non-dimensional language by defining the non-dimensional field as a *Frederiks number*

$$\mathscr{F}\imath = \left(\frac{\chi_A}{K}\right)^{1/2} dH. \tag{11}$$

Thus the elastic energy wins for low $\mathscr{F}\imath$ and the director is aligned parallel to the boundary conditions. The magnetic energy wins for high $\mathscr{F}\imath$, and the director is then aligned along the field. The transition occurs at critical $\mathscr{F}\imath_c \sim \pi$ with boundary layers of thickness $\delta \sim \dfrac{\mathscr{F}\imath_c}{\mathscr{F}\imath}d$ within which the nematic reorients to be consistent with the surface strong anchoring.

With the establishment of the Ericksen-Leslie theory, it was natural for both both Ericksen [54–56] and Leslie [52] to extend the Frederiks effect to consider competition between flow, magnetic fields, and surface effects. A key fluid mechanics result in the absence of a field was the so-called

flow alignment in a shear flow [52]. The director orients in a direction making an angle θ_0 with the direction of flow in the plane of the shear, where the angle depends on the ratio of two of the Leslie viscosities, specifically

$$\tan^2 \theta = \frac{\alpha_3}{\alpha_2}. \tag{12}$$

The simplest case considers the competition between the surface alignment and shear flow or Poiseuille flow. Ericksen [56] remarks that it would be expected that for thin samples, the surface effects would dominate, while for thick samples, the bulk flow alignment solutions would prevail. Following this paradigm, and by analogy with the Reynolds number $\mathscr{R}e = \dfrac{\rho dv}{\eta}$ (ρ is density, v a typical velocity, η a characteristic dynamic viscosity) which governs the transition from laminar to turbulent flow, de Gennes [5] was led to define the *Ericksen number*, a usage which others have been content to follow (see e.g [49]).

$$\mathscr{E}r = \frac{\eta dv}{K}. \tag{13}$$

$\mathscr{E}r$ can be regarded as the ratio of the effects of viscous to elastic forces, and is the non-dimensional quantity governing the transition between a sample well-aligned with the boundary orientation and one subject to flow alignment. For small $\mathscr{E}r$, the surface alignment effects extend through the whole sample. By contrast, for large $\mathscr{E}r$, the surface alignment effects only affect thin boundary layers.

Verification of Ericksen-Leslie theory
The true test of a dynamical theory, no matter how aesthetically attractive, is not its physical plausibility, nor yet its mathematical consistency, though both are necessarily important. But the sufficiency test is whether the theory passes tests of experiment.

An early strong hint, which was taken as verification given its *a priori* lack of plausibility, came from observations of Poiseuille flow in nematics by Fisher and Frederickson [57]. Atkin [58] was able to compare predictions of the Ericksen-Leslie theory of nematics, with analogous predictions for the behavior of an isotropic viscoelastic fluid by Coleman and Noll [59]. The experimentalists measure the *apparent viscosity*, which for Poiseuille flow is

$$\eta_{\mathrm{app}} = \pi \frac{PR^4}{8Q}, \tag{14}$$

where Q is the fluid volume flow per unit time, R is the tube radius, and P is the pressure gradient.

Then if the nematic were behaving like a viscoelastic fluid, one would expect [59] $\eta_{\mathrm{app}} = f(QR^{-3})$. By contrast the Ericksen-Leslie theory predicts [58] $\eta_{\mathrm{app}} = g(QR^{-1})$, where f, g are unknown functions which permit flow curves for different flow rates and tube sizes to collapse onto a single curve. The different *scaling* predictions of the two types of theory provide quite a stringent test of nematic hydrodynamics, which the Ericksen-Leslie theory was able to pass successfully.

3.2 Ericksen's influence on the development of liquid crystal science

Most contemporary scientists, even mathematicians, often publish jointly. A good first indicator of manner in which the ideas of individual researchers have penetrated their chosen field can be gathered by examining – or rather, by asking your computer to examine, for otherwise it will take too long – the network of their publication links. A link here indicates the existence of a

joint publication, with nodes in the network representing authors of articles. Most workers in the mathematical sciences, for example, will be aware of their *Erdös Number* – the shortest number of connected nodes in the publication network required to link them with the famous Hungarian mathematician Paul Erdös (1913-1996). Only a diminishing tribe of mathematical aristocrats can boast an Erdös number of 1.

But Jerry Ericksen possesses a very restricted local network. From his 145 publications up to 2005, I can find only 7 co-authors – some extremely distinguished, to be sure – but all from before his liquid crystal period. By then a bad experience had convinced him to eschew joint publication. From the formal publication record there were no – (*zero*!) – liquid crystal collaborators. The computational sociologist would draw a blank. But as we know all too well, sometimes it behoves us to think before we calculate and not afterward. The links were present, but must be examined with greater subtlety than merely by counting.

Nevertheless some counting does yield results. One measure of his influence on the development of liquid crystal science and technology comes from a search on Google Scholar using the terms "Ericksen–Leslie" or "Leslie–Ericksen". This yields some 7000 entries. Two early paper on anisotropic fluids [32, 40] have been cited over 1000 times, while Leslie's 1968 paper [52], for which we assert Ericksen's moral co-authorship, has been cited on almost 2000 occasions. A number of other articles have citation rates well into the hundreds. Ericksen himself would surely have vigorously repudiated citation scores as a measure of scientific excellence. But these scores do serve as a sociological indication of scientific influence, and at least to some extent, this may be regarded as an imperfect surrogate for quality.

Ericksen continued his work in liquid crystals for another thirty years, dividing his time between "pure" mathematical and "applied" engineering problems. Only part of his enormous influence on the field follows directly from the Ericksen-Leslie theory. His close comradeship with his erstwhile apprentice Frank Leslie continued until Leslie's premature death in June 2000. Fig.1, a picture of Leslie and Ericksen taken sometime in the late 1970s when Ericksen was visiting Glasgow, attests to their continuing collaboration.

It is important to recall that at the University of Minnesota, where he arrived in 1982, Jerry was a joint faculty member of the Department of Aerospace Engineering and Mechanics, and the School of Mathematics. And through his active and prominent role at the newly founded "Institute for Mathematics and its Applications" of the National Science Foundation, both Jerry the Mathematician and Jerry the Engineer played an active role in organizing, encouraging, exchanging views and mentoring younger workers and postdoctoral fellows

Given that the theory's paradigmatic status, many hundreds of papers have been written – indeed, are being written – by mathematicians discussing the conditions under which solutions of the Ericksen-Leslie equations exist, are unique etc. (e.g. [60]). But these studies were started by colleagues of Ericksen's and in some sense under his tutelage.

Thus in 1986 Hardt et al [61] was studied the nature of the solutions to the Frank-Oseen static liquid crystal configurations. Then in 1992 Bethuel et al [62] studied these equations in a cylindrical tube, examining a bifurcation to broken symmetry solutions which avoid a singular defect line, known as " escape in the third dimension". This work came out of experiments by Patricia Cladis (1938-2017) and Maurice Kléman (1934-2021) in Paris [63, 64]; the physicists Cladis and Kléman were able to locate the correct solution, but more by accident and physical intuition than by design. Studies of the mathematical structure of the solutions of the Ericksen-Leslie model itself started with a 1995 study by Lin and Liu in Communications on Pure and Applied Mathematics [65].

Figure 1: Frank Leslie and Jerry Ericksen. The photograph dates from the late 1970s when Ericksen was visiting Glasgow. Courtesy Ellen Leslie.

Ericksen the engineer kept in close contact with experimentalists. Patricia Cladis at Bell Labs was close; her studies [66, 67] of flow in nematic systems were directly inspired by Ericksen. Another close colleague also at Bell Labs, was the optical physicist Dwight Berreman who constructed numerical models to solve the Ericksen-Leslie equations in device cells. In fact, there has been a long subsequent tradition addressing the question of the construction of efficient numerical methods for Ericksen-Leslie problems (e.g. [68]). Finally no mention of Ericksen's colleagues would miss his interaction with the Liquid Crystal Institute at Kent State University, founded in 1965 by the chemist Glenn H. Brown (1915-1995), and still one of the leading centers of liquid crystal work.

4   Afterword

We stop our detailed discussion of Ericksen's contribution at this point. The decade of work which led up to the creation of the Ericksen-Leslie theory seems in historical hindsight to have been designed with the launch of the liquid crystal display (or simply "LCD") industry in mind (see e.g. [12, 69, 70] for discussions of this progress). In the 1990s and 2000s, flat screens, many of which used liquid crystal technology, rapidly replaced the bulky cathode-ray-tube-based TVs of the 1950s. The development of smart phones in recent years likewise depends almost totally on LCD's.

The continuum theory of liquid crystals has been an essential tool not only for device modeling, but also in device design. For once it was known that the theory accurately modeled device characteristics, designers were able to specify desirable properties (e.g. low voltage switching, rapid switching times etc.). These in turn depended on specific elastic constants or viscosities, influencing theoretical physicists and chemists to seek links between molecular and macroscopic properties (e.g. [71, 72]).

Thus the Ericksen-Leslie theory is by now standard. All liquid crystal textbooks (e.g. [5, 6,

49, 73], but there are many others) include chapters on the basic equations, together with a consideration of some simple cases to show which parameters are important where. The theory has also been extended, using essentially the Ericksen philosophy, to a whole host of smectic and chiral phases (e.g. [49, 74]). The theory has also been rederived using a Lagrangian and a Rayleigh dissipation function [75], and also using a Poisson bracket approach [76].

Notwithstanding its continuing success, there are outstanding problems, and the theory has not been without controversy. An alternative generalized continuum theory (the so-called "Harvard theory") was published in 1972 by Martin, Parodi and Pershan [77]. According to the authors of this paper, their formulation would apply to fluids, crystals and liquid crystals. Only following some rather heavy calculations was de Gennes [5] able to show that for liquid crystals, this approach was equivalent to that of Ericksen and Leslie.

A more serious challenge came from the distinguished Turkish Princeton engineering scientist A. Cemal Eringen (1921-2009). Eringen had his own general formulation of complex fluid dynamics, which he called "micropolar fluid theory" [78], and which has been immensely influential. The early application to liquid crystals was published in 1971 by Lee and Eringen [79, 80], with some later specific comparisons with Ericksen-Leslie by Lee and Eringen [81–83] and by Rymarz [84].

Here is not the place to discuss this comparison in detail. It appears (after pages and pages of calculations!) that if the Eringen theory is slightly adapted, and some limits are taken, then the two theories do appear to give identical results. But in addition to purely academic questions, there do seem to be some personal issues involved. It seems that Eringen and Ericksen were not in later life the best of friends. Eringen certainly felt insulted that his approach had been ignored by the community. In his 1993 comparison with theory with Ericksen-Leslie [83], he remarks rather bitterly that Leslie's review [85], with *268* (!) references, did not see fit to mention his work [82], but did draw attention to critical remarks about papers of Lee and Eringen [79, 80].

What is the case, however, is that the Ericksen-Leslie approach seems not yet to be able to treat compressible or highly stressed liquid crystals, and even in the incompressible case fails as soon as defects are present. In many practical cases, defect lines are ubiquitous, and indeed it is their presence which causes the dramatic colored characteristic patterns (so-called "textures") visible through crossed polarizers under the microscope. Generalizations of Ericksen-Leslie, which either could be regarded as more complete Eringen theories or time-dependent Landau theories, and which employ the so-called Q-tensor approach [86–88], have been extensively used. The existence of "active liquid crystals" in living organisms, playing a role in embryonic development [89], implies that Jerry Ericksen's influence on liquid crystal science will continue for some time yet.

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