An embedding formalism for CFTs in general states on curved backgrounds

Enrico Parisini,* Kostas Skenderis,† and Benjamin Withers‡
Mathematical Sciences and STAG Research Centre,
University of Southampton, Highfield, Southampton SO17 1BJ, UK

We present a generalisation of the embedding space formalism to conformal field theories (CFTs) on non-trivial states and curved backgrounds, based on the ambient metric of Fefferman and Graham. The ambient metric is a Lorentzian Ricci-flat metric in $d+2$ dimensions and replaces the Minkowski metric of the embedding space. It is canonically associated with a d-dimensional conformal manifold, which is the physical spacetime where the CFT_d lives. We propose a construction of CFT_d correlators in non-trivial states and on curved backgrounds using appropriate geometric invariants of the ambient space as building blocks. As a test of the formalism we apply it to thermal 2-point functions and find exact agreement with a holographic computation and expectations based on thermal operator product expansions (OPEs).

It is important to understand quantum field theory in curved backgrounds and nontrivial states. This is the case both for purely theoretical reasons and also because it has many applications in a wide range of physical scenarios, from condensed matter systems at finite temperature, to out of equilibrium physics (*e.g.* the quark-gluon plasma), to semiclassical black hole physics and cosmological observables.

A special class of quantum field theories are CFTs. They appear as fixed points under Renormalisation Group flow, and at second-order phase transitions, meaning that they are ubiquitous in nature. They are also of considerable theoretical interest since they enter the anti-de Sitter/conformal field theory (AdS/CFT) correspondence.

The kinematical constraints of CFTs in (conformally) flat spacetimes in vacuum have been solved long ago [1, 2]. In particular both 2-point and 3-point functions of primary operators are fixed by conformal symmetry up to constants, while higher-point functions are fixed up to functions of cross-ratios. The analogue of these results for CFTs in curved spacetime and nontrivial states is not available and the purpose of this *Letter* is to fill in this gap. In particular we will propose a framework for solving the kinematical constraints and apply it to scalar 2-point functions.

The embedding space. The embedding space formalism takes advantage of the fact that the conformal group $SO(1,d+1)$ in d dimensions coincides with the Lorentz group in $(d+2)$ dimensions [3–6]. Imposing conformal invariance on CFT observables on any d-dimensional conformally-flat background $\mathcal{M}$ simply reduces to demanding Lorentz invariance on the embedding space $\mathbb{R}^{1,d+1}$. This construction is realized by mapping $\mathcal{M}$ to a projective section of the lightcone $X^M X_M = 0$ in Minkowski space.[1] For instance, taking the CFT background to be flat space $g_{(0)ij} = \delta_{ij}$, each point $x^i \in \mathcal{M} = \mathbb{R}^d$ can be mapped to a null ray in $\mathbb{R}^{1,d+1}$ according to

$$X^M(t, x^i) = t\left(\frac{1+x^2}{2}, x^i, \frac{1-x^2}{2}\right), \tag{1}$$

with $t \in \mathbb{R}$ and where $t = 1$ corresponds to the isometric embedding of $\mathbb{R}^d$. A linear $SO(1,d+1)$ transformation maps null rays to null rays and this is equivalent to standard conformal transformations on $\mathbb{R}^d$. Considering scalar correlators, the only building blocks that can be constructed on $\mathbb{R}^{1,d+1}$ out of the positions of the insertions are the scalars $X_{ij} = -2\, X_i \cdot X_j$ of dimension -2. As a consequence, using a scaling argument 2-point functions of scalar operators O of dimension Δ are fixed modulo a constant to

$$\langle O(X_1) O(X_2) \rangle = \frac{C_\Delta}{(X_{12})^\Delta}, \tag{2}$$

where C_Δ is a theory specific constant (which may be set to one by appropriately normalising O). After projecting back to $\mathbb{R}^d$ by setting $t = 1$, the invariant reduces to $X_{12}|_{t=1} = |x_{12}|^2$ where $x_{12}^i = x_1^i - x_2^i$, recovering the known expression. With similar reasoning one can efficiently obtain the form of tensorial correlators and higher-point functions.

We now generalise the embedding space formalism to apply to the case of CFTs in non-conformally flat backgrounds $g_{(0)}$ and non-trivial states by using the geometrical construction known as the *ambient space* [7, 8]. The ambient space allows one to impose the constraints of Weyl invariance *in lieu* of full conformal symmetries by finding Weyl invariants on a d-dimensional manifold as diffeomorphism invariants in $d+2$ dimensions. We use this technology to solve the kinematical constraints obeyed by correlators in such CFTs.[2]

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[1] Throughout this letter, small Latin indices $i, j \ldots$ are d-dimensional, small Greek indices $\mu, \nu \ldots$ denote the $d+1$ directions on hyperbolic spaces while capital Latin letters $M, N, A, B \ldots$ denote the $d+2$ embedding or ambient directions. We will sometimes write X^M as the following triplet $X^M = (X^0, X^i, X^{d+1})$.

[2] Other uses of the ambient space in physics include (but are not

The ambient space. To introduce the ambient space construction we first rewrite the flat metric on $\mathbb{R}^{1,d+1}$ with the new coordinates (t, ρ, x^i),

$$Y^M(t,\rho,x^i) = t\left(\frac{1-2\rho+x^2}{2}, x^i, \frac{1+2\rho-x^2}{2}\right). \quad (3)$$

The result is the Minkowski$_{d+2}$ metric in *ambient form*

$$\eta_{MN}dY^M dY^N = 2\rho dt^2 + 2t dt d\rho + t^2\delta_{ij}dx^i dx^j. \quad (4)$$

The surface $\rho = 0$ where $X^M(t,x^i) = Y^M(t,0,x^i)$ describes the lightcone and in this limit we recover (1) together with its degenerate structure. The region $\rho < 0$ corresponds to the future and past of the lightcone, while $\rho > 0$ covers the points with a space-like separation from the origin. The coordinate $t = Y^+ = Y^0 + Y^{d+1}$ defines various sections of the lightcone. (4) admits a homothety $T = t\partial_t$, a null vector which geometrises scaling transformations of the d-dimensional theory.

The ambient space generalises (4) so that it applies to a general background $g_{(0)}$ and general states. There are two key ingredients. First the generalised spacetime should possess a homothety $T = t\partial_t$ and a nullcone structure at $\rho = 0$. Second it should be Ricci flat. The most general metric satisfying these conditions up to diffeomorphisms is of the form [7, 8]

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 g_{ij}(x,\rho)dx^i dx^j, \quad (5)$$

for which the Ricci tensor $\widetilde{R}_{MN} = 0$. In cases where the Riemann tensor vanishes this reduces to the embedding space (4). Given a boundary metric $g_{ij}(x,0) = g_{(0)ij}(x)$ the Ricci-flat condition can be solved in the neighbourhood of $\rho = 0$,

$$g(x,\rho) = g_{(0)} + \ldots + \rho^{d/2}\left(g_{(d)} + h_{(d)}\log\rho\right) + \ldots, \quad (6)$$

where all the terms up to the order displayed are locally determined by $g_{(0)}$ except $g_{(d)}$. $h_{(d)ij}$ is only present for even d. $g_{(d)}$ carries information about the state. As will become clear momentarily, in situations where AdS/CFT applies, the holographic dictionary gives $g_{(d)ij} \sim \langle T_{ij}\rangle$ [16].

To gain some intuition about these geometries, one can perform the coordinate transformation $\rho = -r^2/2$, $t = s/r$, with $s, r > 0$, covering only the region interior to the future nullcone. The ambient metric becomes

$$\tilde{g} = -ds^2 + s^2\left[\frac{dr^2 + g_{ij}(x,r)dx^i dx^j}{r^2}\right]. \quad (7)$$

The Ricci-flat condition implies that the term in brackets is an asymptotically locally (Euclidean) AdS (ALAdS) spacetime [7, 8]. The coordinate s geometrises the AdS radius and thus scaling dimensions with respect to the ambient homothety T coincide with engineering dimensions. Minkowski space can be foliated by hyperbolic slices and (7) is the generalisation to Ricci-flat spacetimes retaining the homothety T.

Weyl transformations are induced by a specific class of ambient diffeomorphisms [8] and the ambient connection $\widetilde{\nabla}_M$ induces a Weyl connection on the nullcone at $\rho = 0$. This is the precise sense in which Weyl transformations are realized on the ambient space. Here we restrict to non-spinning operators; for spinning operators one may use tractor calculus [17, 18].

Proposal. CFT correlators obey specific transformation rules under Weyl transformations and they should have the right singularity structure at short-distances, reproducing the known flat space behaviour. Let us consider the case of scalar 2-point functions, generalising the embedding space result (2). Let X denote coordinates on the null cone in ambient space at $t = 1$. Then $\langle O(X_1)O(X_2)\rangle$ is a scalar bilocal which depends on the positions of the insertions X_1, X_2 and has the right weight with respect to Weyl transformations. There are now two main differences relative to the embedding formalism. Firstly, on a curved background $\tilde{g}$ it does not make sense to take an inner product between two position vectors. Secondly, $\tilde{g}$ in general has non-zero curvature.

Regarding the first point, on a flat ambient space the homothetic vector is given by $T = X^M\partial_M$ and thus in this case $X_{12} = -2X_1\cdot X_2$ is also equal to $X_{12} = -2T_1\cdot T_2$. Moving away from flat space, we view T as the generalisation of a position vector. To construct an inner product in this case we first parallel transport T_1 from X_1 to X_2 along an ambient space geodesic, giving $\hat{T}_1$ in the tangent space at X_2. Then we generalise X_{12} by defining the following dimension -2 scalar invariant $\widetilde{X}_{12} = -2\,\hat{T}_1\cdot T_2$; a short computation shows [19]

$$\widetilde{X}_{12} = \ell(X_1,X_2)^2, \quad (8)$$

where $\ell(X_1,X_2)$ is the geodesic length between the two points. Clearly, $\widetilde{X}_{12} = X_{12}$ in the case of flat space. Note also that although the insertions X_1, X_2 lie on the nullcone, the geodesics connecting them generically pass through the ALAdS slices and are thus affected by the CFT state.

Moving to the second point, non-vanishing ambient curvature entails that more scalar bilocals than $\widetilde{X}_{12}$ may be constructed. The additional ingredients are the ambient Riemann tensor $\widetilde{R}_{MNPQ}$, ambient covariant derivatives $\widetilde{\nabla}$, and $\hat{T}_1, T_2$. These ingredients are subject to the constraints $\widetilde{\nabla}_M T_N = \tilde{g}_{MN}$ and $T^M\widetilde{R}_{MNPQ} = 0$. We denote $\mathcal{I}^{(k)}$ the general linear combination (with constant coefficients) of all curvature invariants of weight zero (w.r.t. ambient homothety) with k ambient Riemann ten-

limited to) higher spin theories and holographic anomalies [9–15].

sors.[3] Note that depending on the example when we evaluate these invariants on the ambient background some of the terms in $\mathcal{I}^{(k)}$ may become linearly dependent.

With all scalar bilocals in hand, we can write an expression for a general 2-point function of scalars O of dimension Δ which has the right transformation properties and the correct singularity behaviour as follows,

$$\langle O(X_1)O(X_2)\rangle = \frac{C_\Delta}{(\widetilde{X}_{12})^\Delta} \lim_{\substack{\rho\to 0\\ t\to 1}} \left[1+\sum_{k=1}^{\infty}\mathcal{I}^{(k)}\right], \qquad (9)$$

where we are meant to keep all terms which are singular in the limit $x_{12}\to 0$. This has the correct singular behaviour in the flat space limit. We also note that the 2-point function (9) is analytic in $g_{(0)}$, as it should be.

In the large Δ limit we expect from usual saddle-point arguments that the 2-point function is well approximated by the geodesic length in ALAdS$_{d+1}$ connecting the two insertion points. One can show [19] that geodesic lengths in ALAdS$_{d+1}$ are related to those of ambient space geodesics through

$$(\widetilde{X}_{12})^{-\Delta} = r^{-2\Delta}e^{-\Delta L_{AdS}}\big|_{r=0}\,, \qquad (10)$$

where L_{AdS} is the geodesic distance between the boundary insertions on the ALAdS$_{d+1}$ slice of unit radius. Thus only the leading term in (9) will remain in the large Δ limit. The terms in the sum provide the finite-Δ corrections.

As a final remark, observe that in the case of more than one ambient geodesic connecting the two insertions, new independent invariants may be associated to different geodesics, and one should sum over geodesics. Note however that under mild conditions there is a unique geodesic that connects any two points on the boundary [20, 21].

Thermal CFTs. We now consider the example of holographic CFTs at finite temperature living on $S^1_\beta\times\mathbb{R}^{d-1}$. We parametrize such background with coordinates $x^i=(\tau,x^a)$ with $a=2\dots d$, $\tau\sim\tau+\beta$ and denote $|x|=\sqrt{\tau^2+x^2}$. The inverse temperature β introduces a new scale that breaks conformal invariance and thus this appears as one of the simplest settings where we can test our proposal. We work perturbatively in $1/\beta$ and since β is the only scale this corresponds equivalently to a short-distance or low-temperature expansion.

Using holography we can write down the ambient metric as (7) where the $d+1$ dimensional metric in square brackets is given by the Euclidean AdS planar black brane,

$$\frac{1}{z^2}\left[\frac{dz^2}{1-\frac{z^d}{z_H^d}}+\left(1-\frac{z^d}{z_H^d}\right)d\tau^2+\delta_{ab}dx^a dx^b\right], \qquad (11)$$

where $\beta=4\pi z_H/d$. Given the ambient metric we can construct the ambient building blocks discussed above in this specific example. The use of the holographic metric here does not necessarily mean that the 2-point function constructed through (9) using such invariants will only apply to holographic thermal CFTs; the solution to the kinematical constraints at strong coupling may provide a basis for the general solution. We will see shortly that in the present context this is indeed the case.

The ambient geodesic distance between the two insertions can be easily computed perturbatively in $1/\beta$, yielding for the first two orders when $d=4$,

$$\begin{aligned}\widetilde{X}_{12} = |x|^2\Bigg[1 &+ \frac{\pi^4|x|^2\left(x^2-3\tau^2\right)}{120\beta^4} + \\ &- \frac{\pi^8|x|^4\left(91\tau^4-98\tau^2x^2+19x^4\right)}{201600\beta^8} + O(\beta)^{-12}\Bigg],\end{aligned} \qquad (12)$$

which matches the expressions for the AdS geodesic distance given in [22, 23] through (10).

The remaining invariants can be constructed in terms of the Riemann tensor evaluated at X_2. We build scalar bilocals involving k curvatures as follows,

$$(\hat{T}_1)^\ell\otimes(\widetilde{\nabla})^{r_1}\widetilde{R}\otimes\cdots\otimes(\widetilde{\nabla})^{r_k}\widetilde{R}, \qquad (13)$$

where $\widetilde{R}$ is the ambient Riemann curvature tensor at X_2. Their scaling dimension is $q=2k+r-\ell$, where $r=\sum_i^k r_i$ and ℓ the number of $\hat{T}_1$ vectors. One can show that q is always even from geometric identities. To make a term of dimension zero which contributes to $\mathcal{I}^{(k)}$ we multiply by $(\widetilde{X}_{12})^{q/2}$.

In this example and to first order in $g_{(d)}$,

$$\begin{aligned}\widetilde{R}_{\rho jk\rho} &= \frac{d}{4}\left(\frac{d}{2}-1\right)g_{(d)jk}\,\rho^{\frac{d}{2}-2}t^2, \\ \widetilde{R}_{ijkl} &= \frac{d}{4}\left[\delta_{il}g_{(d)jk}+\delta_{jk}g_{(d)il}-(l\leftrightarrow k)\right]\rho^{\frac{d}{2}-1}t^2,\end{aligned} \qquad (14)$$

where $g_{(d)jk}\sim\langle T_{jk}\rangle$, the expectation value of the holographic energy momentum tensor. (14) is a special case of a more general formula that can be derived using the Fefferman-Graham expansion depending on the data $\{g_{(0)},g_{(d)}\}$. Note that $\widetilde{R}_{MNPQ}=O(\beta)^{-d}$ and so in general $\mathcal{I}^{(k)}=O(\beta)^{-kd}$. However, one can show that due to Ricci flatness and Bianchi identities the contribution of order β^{-d} in $\mathcal{I}^{(1)}$ vanishes. In fact, as we show in [19], $\mathcal{I}^{(1)}=0$. Therefore in our proposal (9) the term at $O(\beta)^{-d}$ is fully accounted for by the leading term coming

[3] Where sequences of covariant derivatives appear in the construction of $\mathcal{I}^{(k)}$, e.g. $\nabla_a\nabla_b\dots$ acting on the same object we symmetrise their indices. This makes the counting of Riemann tensors unambiguous.

from the geodesic length, $(\widetilde{X}_{12})^{-\Delta}$, and in particular the coefficient of the term at $O(\beta)^{-d}$ is fully fixed. This is in line with expectations from the thermal OPE described below and results in the literature [22, 24].

At next order β^{-2d}, there are contributions from $(\widetilde{X}_{12})^{-\Delta}$ and $\mathcal{I}^{(2)}$. To this order a complete basis is provided by terms with two Riemann tensors $\{e_0, e_1, e_2\}$, so that $\mathcal{I}^{(2)} = c_0e_0 + c_1e_1 + c_2e_2$, where in $d = 4$,

$$\begin{aligned} e_0 &= \mathcal{R}^{(0)}_{AC}\,\mathcal{R}^{(0)AC} = \frac{3}{4}\frac{|x|^8}{z_H^8} + \dots, \\ e_1 &= \mathcal{R}^{(1)}_{AC}\,\mathcal{R}^{(0)AC} = -\frac{|x|^6}{z_H^8}(3\tau^2+7x^2) + \dots, \\ e_2 &= \mathcal{R}^{(1)}_{AC}\,\mathcal{R}^{(1)AC} = 4\frac{|x|^4}{z_H^8}(3\tau^4+16\tau^2x^2+17x^4) + \dots, \end{aligned} \tag{15}$$

where the ellipses denote $O(\beta)^{-12}$ corrections and

$$\mathcal{R}^{(r)}_{AC} \equiv \hat{T}_1^{M_1}\dots\hat{T}_1^{M_r}\,\hat{T}_1^U\hat{T}_1^V\,\widetilde{\nabla}_{M_1}\dots\widetilde{\nabla}_{M_r}\widetilde{R}_{AUCV}. \tag{16}$$

The polynomials on the right hand side of (15) form a basis of polynomials of order 8 constructed from x^i under contraction with the available boundary tensors, namely δ_{ij} and the thermal $\langle T_{ij}\rangle_\beta$ appearing in (14). In fact, they are linear combinations of the Gegenbauer polynomials appearing below in the discussion of thermal OPEs. Thus, in this case, as advertised, the expression of the 2-point function holds for any thermal 2-point function, even though the ambient metric was constructed using an AdS metric related to holographic thermal CFTs.

With all ambient invariants constructed to the required order in $d = 4$ we have for the proposal (9),

$$\begin{aligned} \langle O(\tau,x)O(0)\rangle_\beta = \frac{C_\Delta}{|x|^{2\Delta}}\Bigg[1 - \frac{\pi^4\Delta\left(x^2-3\tau^2\right)|x|^2}{120\beta^4} + \frac{\pi^8|x|^4}{\beta^8}\Bigg(\left(c_0 + \frac{\Delta(63\Delta+170)}{30240}\right)\frac{3}{4}|x|^4 \\ - \left(c_1 + \frac{\Delta(14\Delta+39)}{25200}\right)|x|^2(3\tau^2+7x^2) + 4\left(c_2 + \frac{\Delta(7\Delta+20)}{201600}\right)(3\tau^4+16\tau^2x^2+17x^4)\Bigg) + O(\beta)^{-12}\Bigg], \end{aligned} \tag{17}$$

which is determined up to three numbers c_0, c_1, c_2.

Thermal OPEs. A CFT in any state in a short-distance / high-energy limit should have an OPE expansion. In nontrivial states 1-point functions can be nonzero and therefore 2-point functions could be expressed as an expansion in terms of these 1-point functions. In the case of the thermal OPE this has been discussed in [25–27], leading to

$$\langle O(\tau,x)O(0)\rangle_\beta \sim \frac{1}{|x|^{2\Delta}}\sum_{\phi\in O\times O} a_\phi\, C_J^{(\nu)}(\eta)\left(\frac{|x|}{\beta}\right)^{\Delta_\phi}, \tag{18}$$

where the sum is over the operators in the theory in the $O\times O$ channel with a non-vanishing VEV in the thermal state. Here Δ_ϕ and J are the dimension and the spin of the exchanged operator, while $C_J^{(\nu)}$ are the Gegenbauer polynomials with order $\nu = d/2-1$ depending on $\eta = \tau/|x|$. The constants a_ϕ are related to the dynamics and are not determined by symmetries.

It was suggested in [22, 27, 28] that the low-lying operators contributing to the sum in (18) in the state dual to a black brane belong to two classes: a) multi-stress tensors $:T^n(J):$, constructed as the traceless symmetric contractions of T_{ij}, with dimension nd; b) double-twist operators $:O\Box^n\partial_{i_1}...\partial_{i_\ell}O:$ with dimension $2\Delta+2n+\ell$ and even ℓ. We then expect the thermal correlator (18) to take the form

$$\begin{aligned} \langle O(\tau,x)O(0)\rangle_\beta = \frac{1}{|x|^{2\Delta}}\sum_{n=0}^{\infty}\sum_{\substack{J=0\\ J\,\text{even}}}^{2n} a_{n,J}^{(T)}\,C_J^{(\nu)}(\eta)\left(\frac{|x|}{\beta}\right)^{nd} \\ + \frac{1}{\beta^{2\Delta}}\sum_{n=0}^{\infty}\sum_{\substack{\ell=0\\ \ell\,\text{even}}}^{\infty} a_{n,\ell}^{(OO)}\,C_\ell^{(\nu)}(\eta)\left(\frac{|x|}{\beta}\right)^{2n+\ell}. \end{aligned} \tag{19}$$

The double-twist contributions in (19) are non-singular as $|x|\to 0$. To the required order in β the singular pieces of (19) match (17) for any Δ with the following identification,

$$a_{0,0}^{(T)} = C_\Delta, \qquad a_{1,0}^{(T)} = 0, \qquad a_{1,2}^{(T)} = \frac{\Delta}{120}C_\Delta, \tag{20}$$

$$a_{2,0}^{(T)} = \left(\frac{3c_0}{4} - 6c_1 + 52c_2 + \frac{\Delta(7\Delta+18)}{201600}\right)C_\Delta, \tag{21}$$

$$a_{2,2}^{(T)} = \left(c_1 - 15c_2 + \frac{\Delta(7\Delta+12)}{201600}\right)C_\Delta, \tag{22}$$

$$a_{2,4}^{(T)} = \left(c_2 + \frac{\Delta(7\Delta+20)}{201600}\right)C_\Delta. \tag{23}$$

The connection between the OPE thermal blocks in (19) and ambient invariants can be understood using factorization $\langle :T^n:\rangle \sim \langle T\rangle^n$ and the appearance of $\langle T_{ij}\rangle$ in (14). From this point of view the connection will continue to hold to all orders $O(\beta)^{-nd}$ through an appropriate set of curvature invariants up to $\mathcal{I}^{(n)}$. Note that $a_{1,0}^{(T)}, a_{1,2}^{(T)}$

get contributions only from the geodesic distance in line with our earlier observation that $\mathcal{I}^{(1)} = 0$.

Thermal holographic computation. As an explicit check of the proposal (9) in the case of thermal CFTs (17) we now compute the 2-point function using a holographic bulk computation. We solve

$$\Box\Phi = \Delta(\Delta - d)\Phi, \tag{24}$$

on the $d+1$ dimensional background (11) subject to Dirichlet boundary conditions and regularity in the interior. We normalise such that $C_\Delta = 1$.

In the case of odd $(d+2\Delta)$ one can solve (24) analytically to arbitrarily high order in $1/\beta$ in Fourier space, and for concreteness we present results for $d = 4, \Delta = 3/2$. We find perfect agreement with (17) which determines the following coefficients of the ambient proposal, and thus the thermal OPE coefficients through (20)–(23),

$$c_0 = -\frac{53}{1575}, \quad c_1 = -\frac{11}{1120}, \quad c_2 = -\frac{11}{16800}. \tag{25}$$

In this case the thermal OPE (19) contains a finite contribution at $O(\beta)^{-3}$ from the double-twist spectrum, and one may wonder if it is compatible with the holographic computation. To see this contribution we work non-perturbatively in $1/\beta$. The reason is that while $\beta \sim z_H$ appears explicitly in the metric, it is also present implicitly as the periodicity of τ which complicates the perturbative approach in Fourier space. We work directly in position space and find the regular interior solution to (24) at finite β, where the Dirichlet data is a delta function at $\tau = x = 0$. This leads to a $3d$ partial differential equation which we solve numerically. Details are provided in [19]. The resulting 2-point function is presented in figure 1. We plot the quantity $\tau^{2\Delta}\langle O(\tau, x=0)O(0)\rangle_\beta - 1$ where the -1 subtracts the leading singularity of the 2-point function. In this case the leading behaviour of this subtracted quantity as $\tau \to 0$ will be $a^{(OO)}_{0,0}\tau^3/\beta^3$ followed by the subleading behaviour $3a^{(T)}_{1,2}\tau^4/\beta^4$. The red line clearly shows the contribution of the double-twist $:OO:$ with $a^{(OO)}_{0,0} \simeq 1.1$. The numerics are also consistent with the subleading term with the determined value $a^{(T)}_{1,2}$ in (20) within the accuracy of the numerics.

Returning to the case of general d, for non-integer Δ we have also computed the order β^{-d} contribution exactly,

$$\langle OO\rangle_\beta = \frac{1}{|x|^{2\Delta}}\left[1 + \lambda_1\left(x^2 - (d-1)\tau^2\right)\frac{|x|^{d-2}}{\beta^d}\right] + O(\beta)^{-2d}, \tag{26}$$

where $\lambda_1 = \left(\frac{4\pi}{d}\right)^d \frac{\sqrt{\pi}(-1)^{d+1}\Delta\,\Gamma\left(-\frac{d}{2}-\frac{1}{2}\right)\sin(\pi(d-\Delta))}{2^{d+2}\Gamma\left(1-\frac{d}{2}\right)\tan\left(\frac{\pi d}{2}\right)\sin(\pi\Delta)}$ which determines the OPE coefficients $a^{(T)}_{0,0} = 1$, $a^{(T)}_{1,0} = 0$, $a^{(T)}_{1,2} = -2\lambda_1/(d-2)$.

Conclusions. We have presented a prescription to solve the kinematical constraints of scalar 2-point functions for CFTs in general backgrounds and states. The construction is based on a generalisation of the embedding space formalism and utilises geometric invariants of the ambient space. We tested the construction in the case of holographic CFTs in a thermal state finding exact agreement, and along the way confirmed expectations from thermal OPEs. It would be interesting to apply the formalism to other backgrounds and states. Natural generalisations of this proposal include higher-point functions and higher-spin operators leveraging tractor calculus. We hope to return to these topics in future work.

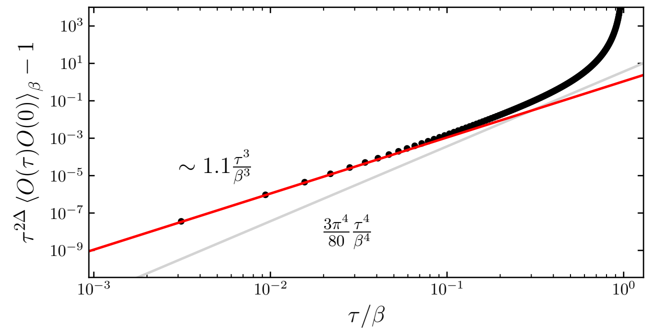

Figure 1. Holographic 2-point function at $x = 0$ for $d = 4, \Delta = 3/2$ on a log-log plot (black dots). The leading behaviour is τ^3/β^3 consistent with nontrivial double-twist contributions (red line). The subleading behaviour is also indicated (grey line).

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* e.parisini@soton.ac.uk
† k.skenderis@soton.ac.uk
‡ b.s.withers@soton.ac.uk

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