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Homotopy Classes of H-maps between Lie Groups

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Abstract

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In this thesis we study the homotopy classes of maps between two compact, simply-connected, simple Lie groups G and L, with a view to classifying when all of these maps are homotopy equivalent to H-maps.

We do this by studying the the homotopy classes of maps $G \to L$, and the homotopy classes of H-maps $G \to L$, as well as the homotopy classes of maps $A_G \to L$ for a related space A_G. By finding homotopy decompositions for these sets of classes, we may compare the decompositions, describe $H[G, L]$, and give sufficient conditions for all maps $G \to L$ to be homotopy equivalent to H-maps. We extend this in some cases to give group isomorphisms between the groups of classes of maps.

We draw together work by Theriault and Grbić on power maps and self maps of low rank Lie groups, as well as work of Kishimoto and Kaji on homotopy nilpotency and older results of James, Cohen and Neisendorfer, and Mimura, Nishida and Toda.

Contents

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. None of this work has been published before submission

Signed:... Date:..................

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Chapter 1

Introduction

1.1 Background and motivation

Lie groups, as introduced by Lie in the late 19^{th} century, are sets G endowed with both the structure of a group and the structure of a differentiable manifold, such that the group multiplication is compatible with the structure of the manifold. A Lie group G is compact or connected if the underlying manifold is compact or connected respectively. Let K be a maximal compact subgroup of G. Then G is connected if and only if K is connected, and simply-connected if and only if K is simply-connected; hence it is sufficient from a homotopy-theoretic point of view to consider compact Lie groups. A Lie group is simple if it is non-homotopy commutative and has no proper closed normal subgroups of dimension greater than zero. A compact, connected Lie group is locally isomorphic to a product of tori and simple, non-homotopy commutative Lie groups; as such the classification problem of such groups reduces to that of simple groups.

An interesting field of research is classifying spaces of Lie groups, in particular producing maps $BG \to BL$ for two Lie groups G and L. Looping a map of classifying spaces of Lie groups would give a loop map $G \to L$. Rather than attempting to classify loop maps, in this thesis we consider H-maps $G \to L$, which also carry multiplicative structure.

Key notions when studying multiplicative structure in Lie groups are those of homotopy commutativity and homotopy nilpotency, which may be thought of as a measure of how non-homotopy commutative the Lie group is. Recent work in this area includes that of Hamanaka and Kono [9, 10], Hasui, Kishimoto, Miyauchi and Ohsita [11], Kishimoto [17], Kishimoto, Kono and Tsutaya [18], and Kishimoto, Ohsita and Takeda [19].

Cohen and Neisendorfer [3] have done much work into constructing certain p-local H-spaces. A theorem of theirs states that for X a simply-connected CW-complex with

$\ell < p-1$ cells in odd dimensions, localised at p, there exists an H-space Y such that

$$H_*(Y,\mathbb{Z}) \cong \Lambda(\tilde{H}_*(X;\mathbb{Z})), \tag{1.1}$$

and a map $\iota : X \to Y$ such that ι_* is the inclusion of the generating set in homology. Theriault [36] proved that for $\ell < p-2$, Y is homotopy associative and homotopy commutative. Grbić and Theriault [8] use these spaces in studying the self-maps of Lie groups, expanding on the work of Mimura and Oshima [26] who determined $[SU(3), SU(3)]$ and $[Sp(2), Sp(2)]$. They prove the following theorem, identifying cases when every self map of a Lie group G is homotopic to an H-map.

Theorem A. *Let p be an odd prime and let G be a homotopy commutative Lie group localised at p, so as to be homotopy commutative. There is a group isomorphism $[G,G] \cong H[G,G]$ in the following cases:*

- *$G = SU(n)$ and $n \leq 7$, $2n < p$, and $n^2 - 1 < 2p$;*
- *$G = Sp(n)$ and $n \leq 13$, $4n < p$ and $2n^2 + n < 2p$;*
- *$G = Spin(2n+1)$ and $n \leq 13$, $4n < p$, and $2n^2 + n < 2p$;*
- *$G = Spin(2n)$ and $n \leq 6$, $4(n-1) < p$, and $2n^2 - n < 2p$;*
- *$G = G_2$ and $p = 5$.*

Our aim is to generalise this theorem, to find cases where every map $G \to L$ is homotopic to an H-map where G is as in Theorem A and L is another Lie group localised at the prime p.

Further results on H-maps of Lie groups were proved by Theriault [38], who used results of Kaji and Kishimoto [16] on homotopy nilpotency and Samelson products together with methods by Cohen and Neisendorfer [3] to prove the following.

Theorem B. *Let L be a p-regular simply-connected, compact, simple Lie group. Then localised at p, the p^{th} power map on L is an H-map for primes $p \geq 5$.*

This was done using the decomposition of p-regular Lie groups as products of odd-dimensional spheres. Russhard [31] extended this result to the case where $SU(n)$ is a product of spheres and sphere bundles over spheres.

We will show that techniques exist to classify H-maps $G \to L$ when both G and L are homotopy commutative, and in certain cases when not. These results will produce a group of H-maps.

1.2 Summary of results

Our first result is the following, relating the p-local spaces constructed by Cohen and Neisendorfer [3] to the concept of universality:

Lemma C (to appear as Lemma 4.5). *Let Z be a homotopy associative, homotopy commutative H-space, and $i : X \to Y$. If $\ell < p-2$, then any map $f : Y \to Z$ has an H-map extension $\bar{f} : X \to Z$, such that $\bar{f} \simeq f \circ i$, that is unique up to homotopy. That is, Y is universal for X.*

Results of Mimura, Nishida and Toda [24, 25] describing homotopy decompositions of Lie groups were built on by Theriault [37] to give, for a torsion-free Lie group G, a co-H-space $A(G)$ such that $H_*(G) \simeq \Lambda(H_*(A(G)))$ and a map $g : A(G) \to G$ inducing the inclusion of the generating set in homology. Such spaces were used by Grbić and Theriault [8] to produce results on self-maps of Lie groups. We instead consider maps $G \to L$ for two different Lie groups.

Proposition D (to appear as Proposition 4.11). *Let G and L be compact, simply-connected, simple Lie groups, localised at a prime p such that both G and L are homotopy commutative. Then there exists a co-H-space $A \simeq \bigvee_{i=1}^{p-1} A_i(G)$ such that $[A, L] \cong H[G, L]$.*

Combining this with a decomposition for $[G, L]$ when G and L are both p-regular and homotopy commutative, we reach the following corollary.

Corollary E (to appear as Corollary 4.13). *Let G and L be compact, connected, simply-connected, simple Lie groups, and let p be a prime such that G and L are both homotopy commutative and p-regular. Let the mod-p decompositions of G and L be $G \simeq S^{d_1} \times \ldots \times S^{d_r}$ and $L \simeq S^{b_1} \times \ldots \times S^{b_s}$ respectively. Then, localised at p,*

$$[G, L] \cong H[G, L] \oplus \bigoplus_{\substack{(k_1,\ldots,k_i)\in V_r \\ j=1,\ldots,s}} [S^{d_{k_1}+\ldots+d_{k_i}}, S^{b_j}] \tag{1.2}$$

$$= H[G, L] \oplus \bigoplus_{\substack{(k_1,\ldots,k_i)\in V_r \\ j=1,\ldots,s}} \pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j}) \tag{1.3}$$

where $V_r = \{(k_1, \ldots, k_i) | 1 \leq k_1 < \ldots < k_i \leq r, 2 \leq i \leq r\}$. Hence, $[G, L] \cong H[G, L]$ if and only if each homotopy group of the form $\pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j})$ for $i \geq 2$ is trivial. That is, if there are no non-trivial homotopy groups that are in $[G, L]$ but not in $H[G, L]$. □

By calculating the homotopy groups in the expression above, we are then able to discern whether $[G, L] \cong H[G, L]$, and also to give a decomposition of these spaces.

We then move on to a more general case, removing the requirement that all spaces are homotopy commutative. For (X, i, Y) a retractile triple, define the space F and the map h by the homotopy fibration $F \xrightarrow{h} \Omega\Sigma X \xrightarrow{\bar{i}} Y$, where $\bar{i}$ is the H-map extension of i given by the James construction.

Lemma F (to appear as Lemma 5.4)**.** *Let (X, i, Y) be a retractile triple, and Z be a homotopy associative, homotopy commutative H-space. For a map $f : X \to Z$, let $\bar{f} : \Omega\Sigma X \to Z$ be the H-map extension of f given by the James construction. Suppose that the composition $F \xrightarrow{h} \Omega\Sigma X \xrightarrow{\bar{f}} Z$ is null homotopic. Then there is an isomorphism of groups $[X, Z] \cong H[Y, Z]$.*

For a compact, simply-connected, simple Lie group G, localised at p such that G is p-regular of type $(m_1, \ldots, m_k)$, we define the related space A to be the wedge sum $\bigvee_1^k S^{2m_i - 1}$. We then use the above lemma, along with some results of Theriault [38], to recover our conditions for $[A, L] \cong H[G, L]$ using different methods.

We are also now able to consider the outlier cases in which homotopy commutativity does not imply p-regularity, in order to provide a more full spectrum of results.

Proposition G (to appear as Proposition 6.5)**.** *Localised at the prime 3, there is a group isomorphism*

$$[Sp(2), Sp(2)] \cong H[Sp(2), Sp(2)] \oplus \mathbb{Z}_3. \tag{1.4}$$

Proposition H (to appear as Proposition 6.6)**.** *Let L be a compact, simply-connected, simple Lie group. Suppose that L is homotopy commutative at the prime 5. Then we have a group isomorphism*

$$[G_2, L] \cong H[G_2, L] \oplus \pi_{14}(L). \tag{1.5}$$

Further, if L is an exceptional Lie group, we have a group isomorphism

$$[G_2, L] \cong H[G_2, L]. \tag{1.6}$$

Kaji and Kishimoto [16] have proved results on the homotopy nilpotency of p-regular Lie groups, and described the Samelson products in G. We use these results, combined with further results of Theriault [38], to consider $[A, L]$ when G is p-regular but not homotopy commutative.

Let X be a space with a basepoint x_0. Then the *(reduced) suspension* of X is the quotient space

$$\Sigma X = (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I). \tag{1.7}$$

This can be thought of as taking the cylinder $X \times I$ and collapsing each end ($X \times \{0\}$ and $X \times \{1\}$ respectively) as well as the line $\{x_0\} \times I$. There is a natural transformation $X \to \Omega\Sigma X$ mapping the space to its loop suspension, which we denote by E.

Let $\mu_i : S^{2m_i-1} \hookrightarrow A \xrightarrow{E} \Omega\Sigma A$, where E is the suspension map. Define x_i to be the composite $x_i : S^{2m_i-1} \xrightarrow{\mu_i} \Omega\Sigma A \xrightarrow{\Omega j} G$, where j is the adjoint of the inclusion $A \hookrightarrow G$.

The least dimensional p-torsion homotopy group of S^3 is $\pi_{2p}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$; denote its generator by $\alpha_1 : S^{2p} \to S^3$, and by abuse of notation let $\alpha_1 : S^{m+2p-3} \to S^m$ be the

$(m-3)$-fold suspension of α_1 for $m \geq 3$. If $\langle x_i, x_j \rangle$ is a non-trivial Samelson product, we define the corresponding map $b_{i,j}$ to be the composition

$$b_{i,j} : S^{2m_i+2m_j-2} \xrightarrow{\alpha_1} S^{2m_i+2m_j-2p+1} \hookrightarrow A \xrightarrow{E} \Omega\Sigma A. \tag{1.8}$$

We then define $\bar{\alpha}$ to be the composite

$$\bar{\alpha} : S^{4p-3} \xrightarrow{\alpha_1} S^{2p} \xrightarrow{\alpha_1} S^3 \hookrightarrow A \xrightarrow{E} \Omega\Sigma A. \tag{1.9}$$

We use these definitions to prove the following theorem, which allows us to extend a map $f : A \to L$ to an H-map $\hat{f} : G \to L$. This enables us to prove later results relating $[A, L]$ and $[G, L]$ despite the lack of homotopy commutativity of L.

Theorem I (to appear as Theorem 7.4)**.** *Let G and L be compact, simply-connected, simple Lie groups. Suppose that G and L are localised at an odd prime p such that G is p-regular but not homotopy commutative. Let $f : A \to L$ be a map, and let $\bar{f} : \Omega\Sigma A \to L$ be the induced H-map extension of Theorem 3.8. Suppose that the following compositions are null homotopic:*

$$\bar{f} \circ \langle \mu_i, \mu_j \rangle \qquad \bar{f} \circ b_{i,j} \qquad \bar{f} \circ \bar{\alpha}. \tag{1.10}$$

Then there exists a unique H-map $\hat{f} : G \to L$ extending f, such that $\hat{f} \circ i \simeq \bar{f}$.

Corollary J (to appear as Corollary 7.5)**.** *Suppose that for any $f : A \to L$, the H-map extension $\bar{f} : \Omega\Sigma A \to L$ from Theorem 3.8 has the property that the compositions in (7.16) are null homotopic. Then there is a one-to-one correspondence $\Phi : [A, L] \to H[G, L]$, sending $f : A \to L$ to its H-map extension $\hat{f} : G \to L$.* □

These are then used on the classical groups to find decompositions for $H[G, L] \cong [A, L]$ as products of homotopy groups of L, under the following conditions:

- $G = SU(m)$, $L = SU(n)$ for $2m \leq n$, $m \leq p < n$;
- $G = Sp(m)$, $L = Sp(n)$ for $4m \leq n$, $2m \leq p < 2n$;
- $G = Spin(m)$, $L = Spin(n)$ for $4m - 3 \leq n$, $2m - 1 \leq p < 2n - 1$ with m and n odd.

Our final results consider when there is in fact a group isomorphism $H[G, L] \cong [A, L]$. Initially, we prove that there exists a homotopy commutative diagram

$$\begin{array}{ccccccc}
A \vee A & \xrightarrow{f \vee g} & L \vee L & \longrightarrow & L \times L & \xrightarrow{m} & L \\
\big\downarrow E & & & & & & \\
\Omega\Sigma(A \vee A) & & & \xrightarrow{\gamma} & & & L \\
\big\downarrow \Omega i & & & & & & \\
\Omega\Sigma A \times \Omega\Sigma A & & & \xrightarrow{\bar{\gamma}} & & & L \\
\big\downarrow r \times r & & & & & & \\
G \times G & & & \xrightarrow{\hat{\gamma}} & & & L
\end{array} \tag{1.11}$$

for H-maps γ, $\bar{\gamma}$ and $\hat{\gamma}$.

This is done through a series of lemmas, resulting in the following.

Corollary K (to appear as Corollary 8.11)**.** *Suppose that (A, i, G) is a retractile triple and there are maps $f, g : A \to L$ where L is a homotopy associative H-space. Given the following hypotheses:*

- *the composite $\Omega(\Sigma A \wedge A) \xrightarrow{\Omega[i_1, i_2]} \Omega(\Sigma A \vee \Sigma A) \xrightarrow{\gamma} L$, where i_1, i_2 are the inclusions into the first and second summands respectively, is null homotopic;*
- *there is a homotopy fibration*
$$\Omega R \xrightarrow{\Omega\rho} \Omega\Sigma A \xrightarrow{r} G \tag{1.12}$$
with the property that $\bar{f} \circ \Omega\rho$ and $\bar{g} \circ \Omega\rho$ are null homotopic,

the map Φ given by Corollary J has the property that $\Phi(f + g) \simeq \Phi(f) + \Phi(g)$.

Finally, we prove that this corollary applies to the classical groups under the same conditions as before, yielding group isomorphisms $[A, L] \cong H[G, L]$.

1.3 Overview

Chapter 2 introduces localisation, p-regularity, homotopy commutativity and homotopy nilpotency. The definitions and results in this section will be vital throughout the thesis, as p-regularity and homotopy commutativity (or lack thereof), are necessary conditions for many following results. Chapter 3 concentrates on the James construction and its properties, along with the Hilton-Milnor theorem and Whitehead and Samelson products.

In Chapter 4 we see some results on decomposing the sets of homotopy classes of maps $G \to L$, and the p-local H-spaces constructed by Cohen and Neisendorfer [3]. Here we prove our results when both G and L are both homotopy commutative, and some example calculations are also given.

Chapter 5 sets up some theory preparing for the non-commutative case, which is then used in Chapter 6 to recover a more general version of our earlier results, using different methods. We also consider the case where homotopy commutativity does not imply p-regularity.

In Chapter 7 we use results on homotopy nilpotency to prove our results on the non-homotopy commutative case, giving conditions for $H[G,L] \cong [A,L]$ in the classical groups. This is then extended to a group homomorphism in Chapter 8.

Chapter 2

Localisation and Homotopy Commutativity

Two of the most important properties of a space in this thesis are those of p-regularity and homotopy commutativity. These conditions have a great effect on the decompositions of the spaces $[G, L]$ and $H[G, L]$.

2.1 Localisation and p-regularity

Localisation of CW-complexes was defined by Mimura, Nishida and Toda in [24], where more detailed definitions of localisation at a set of primes P, and many further results can be found. Here we simplify to localisation at a single prime p.

Definition 2.1. Let X, Y be simply-connected CW-complexes. For a prime p, X is called *p-equivalent* to Y if and only if there exists a map $f : X \to Y$ such that the induced map

$$f_* : H_*(Y; \mathbb{Z}/p\mathbb{Z}) \to H_*(Y; \mathbb{Z}/p\mathbb{Z}) \tag{2.1}$$

is an isomorphism. The map f is then called a *p-equivalence*. Equivalently, f is a p-equivalence if the induced map

$$f^* : H^*(Y; \mathbb{Z}/p\mathbb{Z}) \to H^*(X; \mathbb{Z}/p\mathbb{Z}). \tag{2.2}$$

is an isomorphism.

Definition 2.2. Let $\{X_i, f_i\}$ be a sequence, where each X_i is a subcomplex of X_{i+1}, $X_0 = X$, and $f_i : X_{i-1} \to X_i$. We call $\{X_i, f_i\}$ a *p-sequence* if:

1. each f_i is a p-equivalence, and

2. for any n, i, and any prime q such that $(q, p) = 1$, there exists some $N > i$ such that $(f_N \circ \ldots \circ f_i)_* : H_n(X_{i-1}; \mathbb{Z}/q\mathbb{Z}) \to H_n(X_N; \mathbb{Z}/q\mathbb{Z})$ is trivial.

Definition 2.3. Let X be a simply-connected CW-complex and let $\{X_i, f_i\}$ be a p-sequence. Then the *localisation* of X at p, denoted $X_{(p)}$, is defined to be

$$X_{(p)} = \bigcup_i X_i. \tag{2.3}$$

Let X and Y be simply-connected CW complexes, and $f : X \to Y$ be a given map. Then Mimura, Nishida and Toda [24] showed that as we may assume f is cellular, $f^{(n)} : X^{(n)} \to Y^{(n)}$, it induces a map $\ell_p(f^{(n)}) : X^{(n)}_{(p)} \to Y^{(n)}_{(p)}$ which is unique up to homotopy. Thus, they obtain a map $\ell_{(p)}(f) : X_{(p)} \to Y_{(p)}$. The following theorem of Mimura, Nishida and Toda [24] provides some useful properties of localisation.

Theorem 2.4 (Properties of Localisation)**.** *Let X, Y be simply-connected CW complexes, and let $f : X \to Y$ be a map.*

1. *The localisation $X_{(p)}$ is uniquely determined up to homotopy type.*
2. *There is a natural inclusion $X \hookrightarrow X_{(p)}$.*
3. *The map f induces a map $f_{(p)} : X_{(p)} \to Y_{(p)}$, which is unique up to homotopy.*
4. *If f is a p-equivalence, then $f_{(p)}$ is a homotopy equivalence.* □

We now look at some results of Mimura and Toda on the p-regularity of Lie groups. When a group is p-regular, it has a homotopy decomposition that is simpler to work with, so will be a very useful condition.

Definition 2.5. In [27], Mimura and Toda construct spaces $B_n(p)$, each of which is an S^{2n+1}-bundle over $S^{2n+1+2(p-1)}$. A Lie group G is said to be *quasi-p-regular* if it is p-equivalent to a product of spheres and spaces of type $B_n(p)$.

In the case that G is p-equivalent to a product of spheres, we say that G is *p-regular*.

The following table shows the results of Mimura and Toda [27] on the conditions on p for p-regularity and quasi-p-regularity of compact, connected, simply-connected Lie groups.

	quasi-p-regularity	p-regularity
$SU(n)$	$p > \frac{n}{2}$	$p > n-1$
$Sp(n)$	$p > n$	$p > 2n-1$
$Spin(n)$	$p > \frac{n-1}{2}$	$p \geq n-1$
G_2	$p \geq 5$	$p \geq 7$
F_4	$p \geq 5$	$p \geq 13$
E_6	$p \geq 5$	$p \geq 13$
E_7	$p \geq 11$	$p \geq 19$
E_8	$p \geq 11$	$p \geq 31$

Theorem 2.6 (Hopf's Theorem [13])**.** *Let G be a compact, connected, simple Lie group. Then*

$$H^*(G, \mathbb{Q}) \cong \Lambda(x_{d_1}, \ldots, x_{d_r}), \tag{2.4}$$

where deg $x_{d_i} = d_i = 2n_i - 1$, *r is the rank of G, and dim* $G = \Sigma_i d_i$. □

Definition 2.7. The *type* of G is the r-tuple $(d_1, \ldots, d_r)$.

Suppose G is p-regular. Then there is a p-equivalence $f : G \to S$, where S is a product of spheres. Then S is exactly $S^{d_1} \times \ldots \times S^{d_r}$, as proved by Serre [34] for classical groups and Kumpel [20] for exceptional groups. By Theorem 2.4, localising at p gives us a homotopy equivalence $f_{(p)} : G_{(p)} \xrightarrow{\simeq} S_{(p)}$. Thus, localised at p, we have

$$G \simeq S^{d_1} \times \ldots \times S^{d_r}. \tag{2.5}$$

Recall that $d_i = 2n_i - 1$, so the spheres in the above decomposition are all odd-dimensional.

Thus, by localising at a prime p such that we have p-regularity, the compact, connected, simply-connected Lie groups decompose in the following way [24]:

$SU(n)$	$S^3 \times S^5 \times \ldots \times S^{2n-1}$
$Sp(n)$	$S^3 \times S^7 \times \ldots \times S^{4n-1}$
$Spin(2n+1)$	$S^3 \times S^7 \times \ldots \times S^{4n-1}$
$Spin(2n)$	$S^3 \times S^7 \times \ldots \times S^{4n-5} \times S^{2n-1}$
G_2	$S^3 \times S^{11}$
F_4	$S^3 \times S^{11} \times S^{15} \times S^{23}$
E_6	$S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}$
E_7	$S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}$
E_8	$S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}$.

The dimensions of the spheres in these decompositions correspond to the generators in homology; $H_*(G) \cong \Lambda(x_{d_1}, \ldots, x_{d_r})$.

2.2 Homotopy commutativity and nilpotency

Definition 2.8. A topological group H is said to be *homotopy commutative* if the commutator map $H \times H \to H$, defined by $(x, y) \mapsto xyx^{-1}y^{-1}$, is null homotopic.

Although in general a compact, connected, simply-connected Lie group G is not homotopy commutative [1], for certain primes p, the localisation $G_{(p)}$ is homotopy commutative. In the group of homotopy classes $[G \times G, G]$, the commutator map represents a class of finite order, as proved by Mislin [28]. Then, by identifying primes that do not divide this finite order, McGibbon [21] proved the following theorem.

Theorem 2.9. *A Lie group G is (p-locally) homotopy commutative in the following cases:*

$$\begin{aligned} &SU(n) \text{ if } p > 2n; && G_2 \text{ if } p \geq 13;\\ &Sp(n) \text{ if } p > 4n; && F_4, E_6 \text{ if } p \geq 29;\\ &Spin(2n+1) \text{ if } p > 4n; && E_7 \text{ if } p \geq 37;\\ &Spin(2n) \text{ if } p > 4(n-1); && E_8 \text{ if } p \geq 61; \end{aligned}$$

and the additional instances G_2 at $p = 5$ and $Sp(2)$ at $p = 3$. □

Recalling the results on p-regularity from Section 2.1, note that homotopy commutativity is in almost all cases a stronger condition than p-regularity. The only exceptions to this are $Sp(2)$ at $p = 3$ and G_2 at $p = 5$, which are homotopy commutative but not p-regular. These outlying cases are dealt with in Propositions 6.5 and 6.6.

There are various ways to generalise homotopy commutativity. For example, the notion of higher commutativity gives a gradation between homotopy commutativity and strict commutativity; this was formulated by Sugawara [35], and further work has been done by Williams [41] and Saumell [32], who used this idea to generalise McGibbon's Theorem 2.9.

A generalisation of homotopy commutativity that we will later use is that of homotopy nilpotency, which considers the non-homotopy commutativity of a loop space.

Definition 2.10. Let A be a connected loop space, and let $\gamma_k : A^{k+1} \to A$ denote the iterated commutator map

$$\gamma_k = \gamma \circ (1 \times \gamma) \circ \cdots \circ (1 \times 1 \times \cdots \times 1 \times \gamma). \tag{2.6}$$

Then A is *homotopy nilpotent* if there exists some positive integer N such that γ_N is null homotopic.

The *homotopy nilpotency class* of a homotopy nilpotent loop space A is denoted $\mathrm{nil}(A)$, and is defined to be the least integer n such that γ_n is null homotopic. In this way we can measure to what extent a group fails to be homotopy commutative. Notice that $\mathrm{nil}(A) = 1$ when A is homotopy commutative.

Chapter 3

Universality and the James Construction

3.1 Universality

Definition 3.1. Let T be a space, and let U be a homotopy associative, homotopy commutative H-space. Let $i : T \to U$ be a map. We say that U is *universal* for T if, for any map $f : U \to V$ where V is a homotopy associative, homotopy commutative H-space, there is a unique H-map extension (up to homotopy) $\bar{f} : U \to V$ such that $f \simeq \bar{f} \circ i$. This is illustrated in the below homotopy commutative diagram.

$$\begin{array}{ccc} T & \xrightarrow{f} & V \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \bar{f}} & \\ U & & \end{array} \tag{3.1}$$

If T has a universal space U, then U is unique.

At current, there is no general way to find a universal space for a given space T. However, there are some known examples constructed using *ad hoc* methods. For example, for $p \geq 5$, S^{2n+1} is universal for itself, and is also homotopy associative and homotopy commutative. Gray [7] showed that $[S^{2n+1}, S^{2m+1}] \cong H[S^{2n+1}, S^{2m+1}]$; in fact, his methods can be used to prove $[S^{2n+1}, Z] \cong H[S^{2n+1}, Z]$ for a homotopy associative, homotopy commutative H-space Z.

3.2 The James Construction

The James construction allows us to extend a map $X \to Y$ to an H-map $\Omega\Sigma X \to Y$, which is a very useful tool when looking at the class of H-maps.

Definition 3.2. Let X be a pointed topological space, and let $J_k(X) = X^k/\sim$, where the relation $\sim$ is defined by

$$(x_1, \ldots, x_{j-1}, *, x_j + 1, \ldots, x_k) \sim (x_1, \ldots, x_j - 1, x_j + 1, *, \ldots, x_k). \tag{3.2}$$

Then the *James construction*, denoted $J(X)$, is defined by $J(X) = \lim_{\to k} J_k(X)$.

Note that $X = J_1(X)$, and we have a canonical inclusion $X \hookrightarrow J(X)$.

Remark 3.3. The James construction $J(X)$ is a topological monoid, and since any map $X \to M$ from X to a topological monoid M has a unique extension to a morphism of topological monoids $J(X) \to M$, we see that the canonical inclusion $X \hookrightarrow J(X)$ is universal with respect to maps $X \to M$.

Observe that by definition, $J_k(X)/J_{k-1}(X) = X^{\wedge k}$, where $X^{\wedge k}$ is the k-fold smash product of X with itself. Then the kth *fat wedge* of X is defined to be

$$FW_k(X) = \{(x_1, x_2, \ldots, x_k) \in X^{\wedge k} | x_j = * \text{ for some } j\}. \tag{3.3}$$

We then have a commutative diagram

$$\begin{array}{ccccc} FW_k(X) & \hookrightarrow & X^k & \twoheadrightarrow & X^{\wedge k} \\ \downarrow & & \downarrow & & \| \\ J_{k-1}(X) & \hookrightarrow & J_k(X) & \twoheadrightarrow & X^{\wedge k}. \end{array} \tag{3.4}$$

The following theorem [29] allows us to decompose the suspension of a product.

Proposition 3.4. *For connected CW-complexes X and Y, there is a weak homotopy equivalence*

$$\Sigma(X \times Y) \sim \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y). \tag{3.5}$$

□

Using the above theorem and induction on k, the top right epimorphism in (3.4) splits after suspension, and thus the bottom epimorphism also splits. This gives us the following lemma.

Lemma 3.5. *If X has the homotopy type of a connected CW-complex, then there is a homotopy equivalence $\Sigma J(X) \simeq \bigvee_{k=1}^{\infty} \Sigma(X^{\wedge k})$.* □

Let I denote the unit interval and consider the loop space ΩX of loops $\omega(t)$, $t \in I$. Multiplication in this space is homotopy associative, but not strictly associative; we may define a new set as follows, in which multiplication is strictly associative.

Definition 3.6. The *Moore loop space* $\Omega' X$ is defined by

$$\Omega' X = \{(s, \omega) \in \mathbb{R}^+ \times X^{\mathbb{R}^+} | \omega(0) = * \text{ and } \omega(t) = * \text{ for } t \geq s\}. \tag{3.6}$$

The multiplication in $\Omega' X$ is strictly homotopy associative, where the product of paths of lengths s and s' respectively has length $s + s'$. There is an obvious inclusion $\Omega X \hookrightarrow \Omega' X$, and a map $\Omega' X \to \Omega X$ which reparametrizes a path to have domain $[0, 1]$. These maps are both H-maps, and inverse homotopy equivalences, so ΩX and $\Omega' X$ are homotopy equivalent.

By the universal property of the James construction, as mentioned in Remark 3.3, the composition $X \to \Omega\Sigma X \xrightarrow{\simeq} \Omega'\Sigma X$ can be extended to a map $J(X) \to \Omega'\Sigma X$.

This gives us the following useful result:

Lemma 3.7. *If X has the homotopy type of a connected CW-complex, then the composite $J(X) \to \Omega'\Sigma X \to \Omega\Sigma X$ is a homotopy equivalence, and so $J(X) \simeq \Omega\Sigma X$.* □

Let f be a map $X^{\wedge m} \to Y$, and define the map $\bar{f} : J(X) \to J(Y)$ by

$$\bar{f}_k|_{J_k(X)}(x_1, \ldots, x_k) = \prod f(x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_m}) \in J_{\binom{k}{m}}(Y), \tag{3.7}$$

where the product is taken over subsets $\{x_{i_1}, \ldots, x_{i_m}\}$ of $\{x_1, \ldots, x_k\}$ ordered right lexographically. The collection of such maps on each $J_k(X)$ induce a map $\bar{f} : J(X) \to J(Y)$. This map $\bar{f}$ is known as the *James combinatorial extension* of $J_m(X) \to X^{\wedge m} \xrightarrow{f} Y$, and is unique up to homotopy. This is expressed in the following theorem:

Theorem 3.8 (James [14])**.** *Let X be path-connected and let Y be a path-connected, homotopy associative H-space. Then a map $f : X \to Y$ extends to an H-map $\bar{f} : \Omega\Sigma X \to Y$, where $\bar{f}$ is the unique H-map such that $\bar{f} \circ E \simeq f$.*

In particular, this gives us a one-to-one correspondence between homotopy classes of maps $X \to Y$ and homotopy classes of H-maps $\Omega\Sigma X \to Y$, and the homotopy class of any H-map $\Omega\Sigma X \to Y$ is determined by its restriction to X. □

Remark 3.9. In the above, we have Y a homotopy associative H-space, and an H-map extension

$$\bar{f}(x_1, \ldots, x_k) = f(x_1) \ldots f(x_k), \tag{3.8}$$

where the order of the multiplication does not matter by homotopy associativity.

If instead we take Z an H-space, not necessarily homotopy associative, then we can still define an extension

$$\bar{f}(x_1, \ldots, x_k) = ((f(x_1)f(x_2))f(x_3))f(x_4) \ldots \tag{3.9}$$

in which we choose an order to multiply. However, this extension may not be unique.

In the special case where f is the identity map, the James combinatorial extension is $H_m : J(X) \to J(X^{\wedge m})$, referred to as the m^{th} *James-Hopf invariant map*. Consider the composites of the form

$$J_k(X) \hookrightarrow J(X) \xrightarrow{H_j} J(X^{\wedge j}) \hookrightarrow J\left(\bigvee_{m=1}^{\infty} X^{\wedge m}\right). \tag{3.10}$$

Taking the product of these maps over $j = 1, 2, \ldots, k$ (in order) gives us maps

$$J_k(X) \to J\left(\bigvee_{m=1}^{\infty} X^{\wedge m}\right). \tag{3.11}$$

These can then be combined to form a map $J(X) \to J(\bigvee_{m=1}^{\infty} X^{\wedge m})$; compose this with the homotopy equivalence $J(\bigvee_{m=1}^{\infty} X^{\wedge m}) \simeq \Omega\Sigma(\bigvee_{m=1}^{\infty} X^{\wedge m})$, and adjoint to yield an explicit homotopy equivalence for Lemma 3.5.

3.2.1 The Hilton-Milnor Theorem

The following proposition [33], somewhat similar to Proposition 3.4, allows us to decompose the loops on $X \vee \Sigma Y$.

Proposition 3.10. *Let X and Y be connected CW-complexes. Then*

$$\Omega(X \vee \Sigma Y) \simeq \Omega X \times \Omega\Sigma(Y \vee (\Omega X \wedge Y)). \tag{3.12}$$

□

Recall that by Lemma 3.5, for a connected CW-complex X we have $\Sigma J(X) \simeq \bigvee_{k=1}^{\infty} \Sigma(X^{\wedge k})$. Combining this with Lemma 3.7, which gives us $J(X) \simeq \Omega\Sigma X$, and the above proposition, we get the Hilton-Milnor theorem.

As proved by Hilton in [12] and later generalised by Milnor [22], the Hilton-Milnor theorem allows us to decompose the loop space of a wedge sum in terms of loop spaces of smash products of the factors. Hilton's version used spheres for spaces X and Y, rather than the more general version stated below.

Theorem 3.11 (The Hilton-Milnor theorem)**.** *Let X and Y be connected CW-complexes. Then*

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma\left(\bigvee_{j=0}^{\infty} (X^{\wedge j} \wedge Y)\right). \tag{3.13}$$

□

If we reduce the conditions to have X and Y connected but not necessarily CW-complexes, then we still have a weak equivalence (3.13). This theorem can be applied repeatedly to continue to decompose the final term, although calculations soon become unwieldy. The Hilton-Milnor theorem has been further generalised by Gray [6] and Porter [30].

3.2.2 Whitehead and Samelson Products

The following definitions give us products on homotopy classes of maps. Let X, Y, Z have the homotopy types of connected CW-complexes, and let Z be an H-group. Let $f : X \to Z$ and $g : Y \to Z$, and consider the commutator map $X \times Y \to Z$ given by

$$(x, y) \mapsto f(x)g(y)f(x)^{-1}g(y)^{-1}. \tag{3.14}$$

The restriction of the commutator map to the wedge $X \vee Y$ is null homotopic, and so we have an induced map $\langle f, g\rangle : X \wedge Y \to Z$.

The H-group Z retracts off $\Omega\Sigma Z$, a loop space, so the composition $Z \xrightarrow{E} \Omega\Sigma Z \xrightarrow{i} Z$ is the identity map on Z. Since $\Omega\Sigma Z$ is a loop space, it has homotopy inverses, so for any map into Z we may compose with E, take inverses in $\Omega\Sigma Z$, and then compose with i to return to Z. This gives us homotopy inverses, so the sequence

$$[X \vee Y, Z] \leftarrow [X \times Y, Z] \leftarrow [X \wedge Y, Z] \tag{3.15}$$

splits, and therefore the homotopy class of $\langle f, g\rangle$ is uniquely determined by those of f and g.

Definition 3.12. We call $\langle f, g\rangle$ the *Samelson product* of f and g.

Remark 3.13. If $f : S^m \to Z$ and $g : S^n \to Z$, then the Samelson product $\langle f, g\rangle : S^{n+m} \to Z$ gives the homotopy groups $\pi_*(Z)$ the structure of a graded Lie algebra [29].

The following definition, due to Whitehead [40], gives a product $\Sigma(X \wedge Y) \to Z$ defined using adjunctions on the Samelson product.

Definition 3.14. Let $Z = \Omega C$, and let $f' : \Sigma X \to C$, $g' : \Sigma Y \to C$. Let $f : X \to Z$ and $g : Y \to Z$ be the adjoints of f' and g' respectively. Then take the Samelson product $\langle f, g\rangle$ of f and g; the *Whitehead product* is defined to be the adjoint of $\langle f, g\rangle$, and is denoted $[f', g'] : \Sigma(X \wedge Y) \to Z$.

Define $\omega_k : \Sigma X^{\wedge k} \to \Sigma X$ to be the k-fold Whitehead product of the identity map 1 with itself:

$$\omega_k = [1, [1, \ldots [1, 1] \ldots]. \tag{3.16}$$

For example, $\omega_3 = [1, [1, 1]]$.

3.2.3 The expanded Hilton-Milnor theorem

We start this section with some background on Lie algebras.

Definition 3.15. Let $\mathfrak{g}$ be a vector space over a field F, together with an operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that:

- is bilinear, so $[ax+by,z] = a[x,z] + b[y,z]$ and $[z,ax+by] = a[z,x] + b[z,y]$ for all $a,b \in F$ and all $x,y,z \in \mathfrak{g}$,
- has the property that $[x,x] = 0$ for all $x \in \mathfrak{g}$, and
- satisfies the Jacobi identity $[x,[y,z]] + [y,[z,x]] + [z,x,y]] = 0$ for all $x,y,z \in \mathfrak{g}$.

Then $[\cdot,\cdot]$ is a *Lie bracket*, and $\mathfrak{g}$ together with $[\cdot,\cdot]$ is a *Lie algebra*.

Now let S be a set and $\mathfrak{g}$ a Lie algebra, and let $i : S \to \mathfrak{g}$ be a morphism of sets. Suppose that for any Lie algebra $\mathfrak{h}$ with a morphism of sets $j : S \to \mathfrak{h}$, there is a unique Lie algebra morphism $k : \mathfrak{g} \to B$ such that $j = k \circ i$. Then i is the *universal morphism*, and $\mathfrak{g}$ is the *free Lie algebra* on S. For any set S, we may generate a free Lie algebra $\mathfrak{g}(S)$.

To describe the result of iterating the Hilton-Milnor theorem countably many times, we must introduce some notation, following that of [29].

Let X and Y be connected, and let $L = L_1$ be a finite ordered list of elements, with first element x_1. Let ι_X and ι_Y be the compositions $\iota_X : X \hookrightarrow X \vee Y \xrightarrow{E} \Omega\Sigma(X \vee Y)$ and $\iota_Y : Y \hookrightarrow X \vee Y \to \Omega\Sigma(X \vee Y)$ respectively. Writing $\mathrm{ad}(\alpha)(\beta) = \langle \alpha, \beta \rangle$, we can form iterated Samelson products

$$\mathrm{ad}(\iota_X)^i(\iota_Y) : X^{\wedge i} \wedge Y \to \Omega\Sigma(X \vee Y). \tag{3.17}$$

Definition 3.16. A *Hall basis* is an ordered basis for the ungraded free Lie algebra generated by the set L, and is defined as follows:

1. Let $B_1 = \{x_1\}$, and $L_2 = \{\mathrm{ad}(x_1)^i(x) | i \geq 0, x \in L_1, x \neq x_1\}$.
2. Order L_2 by from shortest to longest by bracket length (for example, $\langle x_1, \langle x_1, x \rangle\rangle$ is of length 3), and let x_2 be the first element of L_2.
3. Let $B_2 = B_1 \cup \{x_2\}$, and $L_3 = \{\mathrm{ad}(x_2)^i(x) | i \geq 0, x \in L_2, x \neq x_2\}$.
4. Continue in this manner, repeating steps 2 and 3 countably often. Define $B := \bigcup_{n=1}^{\infty} B_n$.

Then B is a Hall basis generated by L.

Now let B be a Hall basis generated by the set $\{\iota_X, \iota_Y\}$. Let $\omega = \omega(\iota_X, \iota_Y)$ be an element of B (and thus, an iterated Samelson product), and write $\omega(X, Y)$ for its domain. For example, if $\omega(\iota_X, \iota_Y) = \langle \iota_X, \langle \iota_X, \iota_Y \rangle \rangle$, then the domain is $X \wedge (X \wedge Y)$.

Theorem 3.17 (The expanded Hilton-Milnor Theorem)**.** *Suppose X and Y are connected spaces. Then there is a weak equivalence*

$$\prod_{\omega \in B} \Omega\Sigma(\omega(X, Y)) \to \Omega\Sigma(X \vee Y) \tag{3.18}$$

which is the multiplicative extension of $\omega(\iota_X, \iota_Y)$ on the ω factor, defined by multiplying maps in order according to B. □

Chapter 4

The Homotopy Commutative Case

4.1 Decomposing Homotopy Classes of Maps between Lie Groups

Our main goal is to identify when maps between Lie groups are homotopy equivalent to H-maps. We will begin with some more general results on H-spaces, and then see how these can be applied to the compact, simply-connected, simple Lie groups that we are considering. Initially we decompose $[X, Y]$ for X and Y H-spaces, and then use this decomposition to give an expression for $[G, L]$ in terms of homotopy groups of L.

We begin with the following theorem of Grbić and Theriault [8].

Lemma 4.1. *Let X and Y be H-spaces, and suppose that Y is homotopy associative and homotopy commutative. Then the group structure of $[X, Y]$ is preserved when restricting to $H[X, Y]$, and the inclusion $H[X, Y] \hookrightarrow [X, Y]$ is a group homomorphism.*

To illustrate the method, we reproduce the proof as given in [8].

Proof. We prove this lemma by showing that $H[X, Y]$ is a subgroup of $[X, Y]$; the results will follow. The identity element of $[X, Y]$ is the constant map to a point, and since this is an H-map, it is in $[X, Y]$. Since Y is homotopy associative and homotopy commutative, both the multiplication $\mu : Y \times Y \to Y$ and the inverse $-1 : Y \to Y$ are H-maps.

Let $f, g : X \to Y$ be H-maps, and consider the composite

$$f + g : X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y. \tag{4.1}$$

Each of Δ, $f \times g$ and μ is an H-map, and therefore $f + g$ is also an H-map and $H[X, Y]$ is closed under addition. Since -1 and f are both H-maps, the composite $-f : X \xrightarrow{f} Y \xrightarrow{-1} Y$

is also an H-map, and hence $H[X,Y]$ is closed under inverses. Thus $H[X,Y]$ is a subgroup of $[X,Y]$ so the group structure is preserved, and the inclusion $H[X,Y] \hookrightarrow [X,Y]$ is a group homomorphism. $\square$

The following theorem of Grbić and Theriault [8] allows us to use a decomposition of a suspension ΣX to then decompose homotopy classes $[X,Y]$.

Theorem 4.2. *Let X be a space, such that $\Sigma X \simeq \bigvee_{i=1}^{t} \Sigma X_i$. Let Y be a homotopy associative H-space. Then, as sets,*

$$[X,Y] \cong \prod_{i=1}^{t} [X_i, Y]. \tag{4.2}$$

If additionally Y is homotopy commutative, then the above isomorphism is of groups.

Proof. We split the proof into 2 cases: the simpler case where Y is a loop space, and the more general case where Y is a homotopy associative H-space. We will then consider the additional condition that Y is homotopy commutative.

Case 1 ($Y = \Omega Z$). Consider the following sequence of isomorphisms:

$$\begin{aligned} [X, \Omega Z] &\cong [\Sigma X, Z] \\ &\cong \left[\bigvee_{i=1}^{t} \Sigma X_i, Z \right] \\ &\cong \prod_{i=1}^{t} [\Sigma X_i, Z] \\ &\cong \prod_{i=1}^{t} [X_i, \Omega Z]. \end{aligned}$$

The first isomorphism is by adjunction, and is an isomorphism of groups. We have the second isomorphism from the homotopy decomposition given above; note that this is just an isomorphism of sets. The third is a standard property of mapping spaces, and adjunction again gives us the final isomorphism. These last two isomorphisms are also as groups.

Case 2 (Y a homotopy associative H-space). Let $f : X \to Y$ represent a homotopy class in $[X,Y]$. Then, using the James construction (see Theorem 3.8), there is a unique H-map $\bar{f} : \Omega\Sigma X \to Y$ such that $\bar{f} \circ E \simeq f$. Recall that E denotes the suspension map. Now let $g : \bigvee_{i=1}^{t} X_i \to Y$ represent a homotopy class in $[\bigvee_{i=1}^{t} X_i, Y]$, and proceed similarly: there exists a unique H-map $\bar{g} : \Omega\Sigma(\bigvee_{i=1}^{t} X_i) \to Y$ extending g, such that $\bar{g} \circ E \simeq g$.

We have a homotopy equivalence $e : \Sigma X \simeq \Sigma(\bigvee_{i=1}^t X_i)$, so now define a map

$$\begin{aligned}\phi : [X, Y] &\to \left[\bigvee_{i=1}^{t} X_i, Y\right] \\ f &\mapsto \bar{f} \circ \Omega(e^{-1}) \circ E\end{aligned}$$

so that $\phi(f) : \bigvee_{i=1}^t X_i \xrightarrow{E} \Omega\Sigma(\bigvee_{i=1}^t X_i) \xrightarrow{\Omega(e^{-1})} \Omega\Sigma X \xrightarrow{\bar{f}} Y$. We similarly define a map

$$\begin{aligned}\psi : \left[\bigvee_{i=1}^{t} X_i, Y\right] &\to [X, Y] \\ g &\mapsto \bar{g} \circ \Omega e \circ E\end{aligned}$$

so that $\psi(g) : X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega e} \Omega\Sigma(\bigvee_{i=1}^t X_i) \xrightarrow{\bar{g}} Y$.

We summarise the above situation in the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\quad E \quad} & \Omega\Sigma X & \xrightarrow{\quad \bar{f} \quad} & Y \\ & & \big\downarrow{\scriptstyle \Omega e} & & \| \\ \bigvee_{i=1}^t X_i & \xrightarrow{E} & \Omega\Sigma(\bigvee_{i=1}^t X_i) & \xrightarrow{\bar{g}} & Y \end{array} \tag{4.3}$$

Note that the top row is homotopic to f, the bottom row is homotopic to g, and $\phi(f)$ and $\psi(g)$ 'cross over' in the middle of the diagram.

We will prove that $[X, Y] \cong [\bigvee_{i=1}^t X_i, Y]$ by showing that ϕ is a bijection. In fact, we will show that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity maps of $[\bigvee_{i=1}^t X_i, Y]$ and $[X, Y]$ respectively.

Let $f : X \to Y$, and let $g = \phi(f) = \bar{f} \circ \Omega(e^{-1}) \circ E$. Since $\bar{f}$ and $\Omega(e^{-1})$ are both H-maps, their composite is also an H-map, and thus $\bar{f} \circ \Omega(e^{-1})$ is an H-map whose composite with E is homotopic to g. But $\bar{g}$ was the unique H-map such that $\bar{g} \circ E \simeq g$, and therefore we have $\bar{g} \simeq \bar{f} \circ \Omega(e^{-1})$.

Then we have $(\psi \circ \phi)(f) = \psi(\phi(f)) = \psi(g) = \bar{g} \circ \Omega e \circ E$, and combining this with the above gives us that

$$\begin{aligned}(\psi \circ \phi)(f) &\simeq \bar{g} \circ \Omega e \circ E \\ &\simeq (\bar{f} \circ \Omega(e^{-1})) \circ \Omega e \circ E \\ &\simeq \bar{f} \circ E \\ &\simeq f.\end{aligned}$$

A similar argument for $\phi \circ \psi$ shows that it is also the identity map, and thus ϕ is a bijection.

We have proved that $[X, Y] \cong \prod_{i=1}^{t}[X_i, Y]$ as sets; it remains to prove that when Y is homotopy commutative, the isomorphism is of groups. Suppose that Y is homotopy commutative, then we need to show that ϕ is a group isomorphism. Let f_1 and f_2 represent homotopy classes in $[X, Y]$. By Theorem 3.8, we have a unique H-map $\overline{(f_1 + f_2)}$ such that $\overline{(f_1 + f_2)} \circ E \simeq f_1 + f_2$. By definition, we have

$$\phi(f_1 + f_2) = \overline{(f_1 + f_2)} \circ \Omega(e^{-1}) \circ E. \tag{4.4}$$

We may also apply Theorem 3.8 to f_1 and f_2 separately to get H-maps $\bar{f}_1$ and $\bar{f}_2$ such that $\bar{f}_1 \circ E \simeq f_1$ and $\bar{f}_2 \circ E \simeq f_2$. By Lemma 4.1, homotopy commutativity of Y implies that $\bar{f}_1 + \bar{f}_2$ is also an H-map. Note also that

$$(\bar{f}_1 + \bar{f}_2) \circ E \simeq (\bar{f}_1 \circ E) + (\bar{f}_2 \circ E) \simeq f_1 + f_2. \tag{4.5}$$

By uniqueness of $\overline{(f_1 + f_2)}$, we have $\overline{(f_1 + f_2)} \simeq \bar{f}_1 + \bar{f}_2$, and so

$$\begin{aligned}
\phi(f_1 + f_2) &\simeq \overline{(f_1 + f_2)} \circ \Omega(e^{-1}) \circ E \\
&\simeq (\bar{f}_1 + \bar{f}_2) \circ \Omega(e^{-1}) \circ E \\
&\simeq (\bar{f}_1 \circ \Omega(e^{-1}) \circ E) + (\bar{f}_2 \circ \Omega(e^{-1}) \circ E) \\
&\simeq \phi(f_1) + \phi(f_2).
\end{aligned}$$

Then ϕ is a group homomorphism, and since it is also a bijection ϕ is a group isomorphism. □

The compact, simply-connected, simple Lie groups satisfy the conditions of the above theorem. In fact, they are even loop spaces. There are however many spaces which are not loop spaces, but still satisfy the conditions of Theorem 4.2. For example, the families of p-local finite torsion-free H-spaces constructed by Cohen and Neisendorfer in [3] and Cooke, Harper and Zabrodsky in [4] were given conditions for homotopy associativity and commutativity by Theriault in [36].

Let G and L be compact, connected, simply-connected Lie groups and take a prime p such that both G and L are homotopy commutative. Then as we saw in Section 2.1, G and L are each at least quasi-p-regular. Suppose G is p-regular and write its decomposition

$$G \simeq S^{d_1} \times \ldots \times S^{d_r}. \tag{4.6}$$

Remark 4.3. The suspension of a product of spaces $X = X_1 \times \ldots \times X_r$ is given by

$$\Sigma(X_1 \times \ldots \times X_r) \simeq \bigvee_{T_r} \Sigma(X_{\ell_1} \wedge \ldots \wedge X_{\ell_i}), \tag{4.7}$$

where $T_r = \{(\ell_1, \ldots, \ell_i) | 1 \leq \ell_1 < \ldots < \ell_i < r, 1 \leq i \leq r\}$.

Substituting $X = G_{(p)}$, where G is p-regular, shows that G satisfies the conditions of Theorem 4.2.

Using Theorem 4.2 and Remark 4.3, we see that, localised at p,

$$\begin{aligned}\Sigma G &\simeq \Sigma(S^{d_1} \times \ldots \times S^{d_r}) \\ &\simeq \bigvee_{T_r} \Sigma(S^{d_{\ell_1}} \wedge \ldots \wedge S^{d_{\ell_i}}) \\ &\simeq \bigvee_{T_r} \Sigma S^{d_{\ell_1}+\ldots+d_{\ell_r}}\end{aligned}$$

and therefore

$$[G, L] \cong \bigoplus_{T_r} [S^{d_{\ell_1}+\ldots+d_{\ell_i}}, L]. \tag{4.8}$$

So the space $[G, L]$ is isomorphic to the direct sum of homotopy groups of L, each corresponding to an element of T_r.

Example 4.1. *Consider $G = G_2$. For $p \geq 13$, G_2 is both p-regular and homotopy commutative, and we have $G_2 \simeq S^3 \times S^{11}$. The rank of G_2 is 2, so we need $T_2 = \{(1), (2), (1, 2)\}$. Thus, for L a homotopy commutative Lie group, we have*

$$\begin{aligned}[G_2, L] &\cong \bigoplus_{T_2} [S^{d_{\ell_1}+\ldots+d_{\ell_i}}, L] \\ &\cong [S^3, L] \oplus [S^{11}, L] \oplus [S^{3+11}, L] \\ &\cong \pi_3(L) \oplus \pi_{11}(L) \oplus \pi_{14}(L).\end{aligned}$$

4.2 H-maps between Lie groups

We now move on to look at the subset $H[X, Y]$, the homotopy classes of H-maps $X \to Y$. Assume X and Y have the homotopy types of CW-complexes, unless otherwise stated. Then by [15], Y has both left and right homotopy inverses. If Y is homotopy associative, then these inverses coincide, and therefore $[X, Y]$ is a group. If additionally Y is homotopy commutative, then the group is abelian.

4.2.1 Construction of p-local H-spaces

In [34], Serre decomposes ΩS^{2n}, localised at odd primes, into $S^{2n-1} \times \Omega S^{4n-1}$. Cohen and Neisendorfer [3] then extend this to other examples. This was also proved using different methods by Cooke, Harper and Zabrodsky [4].

Theorem 4.4. *Let p be an odd prime, and let X be a simply-connected CW-complex, with $\ell < p - 1$ cells all in odd dimensions. Localise at p. Then there exists an H-space Y such that:*

1. *With mod-p coefficients,* $H_*(Y) \cong \Lambda(\tilde{H}_*(X))$.
2. *There exists a map* $\iota : X \to Y$ *such that the induced map* ι_* *is the inclusion of the generating set in homology.*

A further result of Theriault [36] says that if $\ell < p-2$ *then* Y *is homotopy associative and homotopy commutative.*

We give an overview of the proof of Theorem 4.4 using Cohen and Neisendorfer's approach.

Note that X is a localisation at an odd prime of odd-dimensional cells, in dimensions $n_1, \dots, n_\ell$. Thus, $H_*(X)$ has basis $1, u_1, \dots, u_\ell$, where each u_i has degree n_i and $V := \tilde{H}_*(X)$ has basis $u_1, \dots, u_\ell$. Since $H_*(X)$ is a trivial coalgebra, $H_*(\Omega\Sigma X)$ is the primitively generated tensor algebra $T(V) = T(u_1, \dots, u_\ell)$.

Cohen and Neisendorfer [3] construct a space R and a map ρ such that we have the following homotopy fibration sequence:

$$\Omega R \to \Omega\Sigma X \xrightarrow{\rho} Y \xrightarrow{*} R \to \Sigma X, \tag{4.9}$$

where Y is such that $H_*(Y)$ is the primitively generated exterior algebra $\Lambda(V) = \Lambda(u_1, \dots, u_\ell)$. We also have a decomposition $\Omega\Sigma X \simeq Y \times \Omega R$. Note that ρ is an H-map if $\ell < p-2$ [36].

The space R can be written $\bigvee_{i=2}^{\ell+1} R_i$, and we have maps $\theta_i : R_i \to \Sigma X$. Define $\theta = \bigvee_{i=2}^{\ell+1} \theta_i$. Then $R_i \xrightarrow{\theta_i} \Sigma X$ factors as $R_i \hookrightarrow \Sigma X^{\wedge i} \xrightarrow{\omega_\ell} \Sigma X$, and R_i retracts off $\Sigma X^{\wedge i}$.

Define ι to be the composition $X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\rho} Y$. In [3], we see that the suspension $\Sigma\iota : \Sigma X \to \Sigma Y$ admits a retraction $\Sigma Y \to \Sigma X$, and thus ρ has a section $s : Y \to \Omega\Sigma X$. Then the H-space multiplication on Y is given by the composition

$$Y \times Y \xrightarrow{s \times s} \Omega\Sigma X \times \Omega\Sigma X \xrightarrow{m} \Omega\Sigma X \xrightarrow{\rho} Y, \tag{4.10}$$

where m is the standard multiplication on $\Omega\Sigma X$.

Lemma 4.5. *Using the notation above, if* $\ell < p-2$, *then* Y *is universal for* X *(See Definition 3.1).*

Proof. By Theorem 4.4, we have that Y is homotopy associative and homotopy commutative. Let f be a map $X \to Z$, where Z is a homotopy associative, homotopy commutative H-space. Recall that $\Omega\Sigma X \simeq Y \times \Omega R$.

Using the James construction, we obtain an H-map extension $\bar{f} : \Omega\Sigma X \to Z$, giving a homotopy commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \downarrow E & \nearrow \bar{f} & \\ \Omega\Sigma X & & \end{array} \tag{4.11}$$

Now consider the composite $\bar{f} \circ \Omega\theta$; recall that θ factors as

$$R = \bigvee_{i=2}^{\ell+1} R_i \hookrightarrow \bigvee_{i=2}^{\ell+1} \Sigma X^{\wedge i} \xrightarrow{\bigvee_i \omega_i} \Sigma X, \tag{4.12}$$

where ω_i is the i-fold Whitehead product.

Now consider $\Omega\Sigma X^{\wedge i} \xrightarrow{\Omega\omega_i} \Omega\Sigma X \xrightarrow{\bar{f}} Z$. We claim that $\bar{f} \circ \Omega\omega_i \simeq *$. By the universal property of the James construction, it suffices to show that the composition

$$X^{\wedge i} \xrightarrow{E} \Omega\Sigma X^{\wedge i} \xrightarrow{\Omega\omega_i} \Omega\Sigma X \xrightarrow{\bar{f}} Z \tag{4.13}$$

is null homotopic. Note that $\Omega\omega_i \circ E$ is the adjoint of ω_i, and is thus a Samelson product. Since H-maps are natural for Samelson products, $\bar{f} \circ \Omega\omega_i \circ E$ is also a Samelson product, which map trivially into homotopy commutative H-spaces and thus the composition (4.13) is null homotopic. This proves our claim.

Consider the composition

$$\Omega\left(\bigvee_{i=2}^{\ell+1} \Sigma X^{\wedge i}\right) \xrightarrow{\Omega(\bigvee_i \omega_i)} \Omega\Sigma X \xrightarrow{\bar{f}} Z. \tag{4.14}$$

Both $\bar{f}$ and $\Omega(\bigvee_i \omega_i)$ are H-maps, and so the homotopy class of the above composition is determined by composing with E,

$$\bigvee_{i=2}^{\ell+1} X^{\wedge i} \xrightarrow{E} \Omega\left(\bigvee_{i=2}^{\ell+1} \Sigma X^{\wedge i}\right) \xrightarrow{\Omega(\bigvee_i \omega_i)} \Omega\Sigma X \xrightarrow{\bar{f}} Z. \tag{4.15}$$

Using the above argument on each wedge summand, we see that the composition is null homotopic when restricted to each factor (see (4.13)), and therefore the whole composition $\bar{f} \circ \Omega(\bigvee_i \omega_i)$ is null homotopic. Then since θ is defined by factoring it through $\bigvee_{i=2}^{\ell+1} \Sigma X^{\wedge i}$, the fact that $\bar{f} \circ \Omega(\bigvee_{i=2}^{\ell+1} \omega_i)$ is null homotopic implies that the composition

$$\Omega R \xrightarrow{\Omega\theta} \Omega\Sigma X \xrightarrow{\bar{f}} Z \tag{4.16}$$

is null homotopic as well.

Now we have an H-map $Y \times \Omega R \simeq \Omega\Sigma X \xrightarrow{\bar{f}} Z$, whose restriction to ΩR is null homotopic. Let $e : Y \times \Omega R \xrightarrow{\simeq} \Omega\Sigma X$ be the above homotopy equivalence, and consider the following diagram, where the top row is homotopic to e.

$$\begin{array}{ccccccc} Y \times \Omega R & \xrightarrow{s\times\Omega\theta} & & & \Omega\Sigma X \times \Omega\Sigma X & \xrightarrow{m_{\Omega\Sigma X}} & \Omega\Sigma X \\ \downarrow{\scriptstyle m_1} & & & & \downarrow{\scriptstyle \bar{f}\times\bar{f}} & & \downarrow{\scriptstyle \bar{f}} \\ Y & \xrightarrow{\bar{f}} & Z & \xhookrightarrow{i_1} & Z \times Z & \xrightarrow{m_Z} & Z \end{array} \tag{4.17}$$

Consider the left square in the above diagram. In the upper direction we have

$$Y \times \Omega R \xrightarrow{s\times\Omega\theta} \Omega\Sigma X \times \Omega\Sigma X \xrightarrow{\bar{f}} Z \times Z, \tag{4.18}$$

and in the lower direction we have

$$Y \times \Omega R \xrightarrow{m_1} Y \xrightarrow{\bar{f}} Z \xrightarrow{i_1} Z \times Z. \tag{4.19}$$

The lower direction is homotopic to $\bar{f}$, and since $Y \times \Omega R \simeq \Omega\Sigma X$ and $\Omega\theta \circ \bar{f}$ is null homotopic, the upper direction is also homotopic to $\bar{f}$ and so the left square homotopy commutes. The right square also homotopy commutes as $\bar{f}$ is an H-map, so the whole diagram is homotopy commutative.

Define $\hat{f} \simeq \bar{f} \circ s$, and note that $\hat{f}$ is homotopic to the bottom row of the above diagram.

We would like for $\hat{f}$ to extend $\bar{f}$, so that $\bar{f} \simeq \hat{f} \circ \rho$. Consider the following homotopy commutative diagram, where the top row is homotopic to e and the bottom row is homotopic to the identity map on Y.

$$\begin{array}{ccccc} Y \times \Omega R & \xrightarrow{s\times\Omega\theta} & \Omega\Sigma X \times \Omega\Sigma X & \xrightarrow{m_{\Omega\Sigma X}} & \Omega\Sigma X \\ \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle \rho\times\rho} & & \downarrow{\scriptstyle \rho} \\ Y & \xhookrightarrow{i_1} & Y \times Y & \xrightarrow{m_Y} & Y. \end{array} \tag{4.20}$$

The right square homotopy commutes as $\ell < p - 2$ which means that ρ is an H-map. The outer square around the diagram also homotopy commutes; the lower direction is the projection onto Y, and since s is a section for ρ and $\rho \circ \Omega\theta \simeq *$, the upper direction is also projection onto Y. Thus the whole diagram is homotopy commutative.

Then $\rho \circ e \simeq m_1$. Combining this with diagram (4.17) as follows, we see that $\bar{f} \simeq \hat{f} \circ \rho$.

$$\begin{array}{ccccc} \Omega\Sigma X & \xrightarrow{e^{-1}} & Y \times \Omega R & \xrightarrow{e} & \Omega\Sigma X \\ & \searrow^{\rho} & \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle \bar{f}} \\ & & Y & \xrightarrow{\hat{f}} & Z \end{array} \tag{4.21}$$

To show that $\hat{f}$ is an H-map, consider the following diagram, where m_Y, m_Z denote the multiplications on Y and Z respectively, and μ is the loop multiplication on $\Omega\Sigma X$. Note that we have the middle square because $\bar{f}$ is an H-map.

$$\begin{array}{ccccccc} Y\times Y & \xrightarrow{s\times s} & \Omega\Sigma X\times\Omega\Sigma X & \xrightarrow{\mu} & \Omega\Sigma X & \xrightarrow{\rho} & Y \\ & \searrow^{\hat{f}\times\hat{f}} & \downarrow{\scriptstyle \bar{f}\times\bar{f}} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \hat{f}} \\ & & Z\times Z & \xrightarrow{m_Z} & Z & = & Z \end{array} \tag{4.22}$$

Recall that $\bar{f} \simeq \hat{f}\circ\rho$, and that s is a section for ρ. Then by definition of $\hat{f}$, we have $\bar{f}\circ s \simeq \hat{f}\circ\rho\circ s \simeq \hat{f}$, and so the triangle homotopy commutes. The middle square of the diagram homotopy commutes because $\bar{f}$ is an H-map, and the right square homotopy commutes since $\bar{f} \simeq \hat{f}\circ\rho$. Thus the whole diagram is homotopy commutative.

Observe that the top row of the diagram is the definition of m_Y, and so the homotopy commutativity of the diagram shows that $\hat{f}$ is an H-map.

It remains to check uniqueness. Suppose that $g, h : Y \to Z$ are such that $f \simeq g\circ\iota = g\circ\rho\circ E$ and $f \simeq h\circ\iota = h\circ\rho\circ E$. Now consider

$$\begin{array}{ccccc} \Omega\Sigma X & \xrightarrow{\rho} & Y & \underset{h}{\overset{g}{\rightrightarrows}} & Z. \\ {\scriptstyle E}\uparrow & \nearrow_{f} & & & \\ X & & & & \end{array} \tag{4.23}$$

Since both $g\circ\rho$ and $h\circ\rho$ are H-maps, and we have $g\circ\rho\circ E \simeq f \simeq h\circ\rho\circ E$, the universal property of the James construction implies that $g\circ\rho \simeq h\circ\rho$. Now composing with $s : Y \to \Omega\Sigma X$ gives us that $g \simeq h$. □

We now move on to some results more specifically about Lie groups.

Mimura, Nishida and Toda [25] proved the following theorem on decomposing Lie groups:

Theorem 4.6. *Let G be a simply-connected, simple Lie group, without p-torsion and of type $(d_1, \ldots, d_r)$. Then G has an irreducibly mod-p decomposition into r spaces, excepting the case $G = Spin(2n)$ which has an irreducibly mod-p decomposition into $r+1$ spaces.*

We have already seen decompositions of Lie groups into products of spheres in Section 2.1.

Definition 4.7. The factors of the mod-p decomposition of G are known as *mod-p Stiefel complexes $B^k_m(p)$*, having the following properties:

1. $H^*(B^k_m(p); \mathbb{Z}_p) \cong \Lambda(x_{2m+1}, x_{2m+1+q}, \ldots, x_{2m+1+(k-1)q})$ with $q = 2(p-1)$, and

2. there exists a map $f : B^k_m(p) \to SU(m+1+(k-1)(p-1))/SU(m)$ inducing an epimorphism in $\mathbb{Z}_p$-cohomology.

The following theorem of Theriault [37] offers more detail about decomposition of Lie groups.

Theorem 4.8. *Let G be a compact, connected, simply-connected, simple Lie group which is torsion free. There is a co-H-space $A(G)$ such that $H_*(G) \simeq \Lambda(H_*(A(G)))$, and a map $g : A(G) \to G$ inducing the inclusion of the generating set in homology. In addition:*

1. *We have the homotopy decompositions*

$$A(G) \simeq \bigvee_{i=1}^{p-1} A_i(G), \qquad G \simeq \prod_{i=1}^{p-1} \bar{B}_i(G). \tag{4.24}$$

2. *For each i, $H_*(A_i(G))$ consists of elements of $H_*(A(G))$ of degrees $2i + 2j(p-1) + 1$ for $j \geq 0$, and so each $H_*(\bar{B}_i(G)) \cong \Lambda(\tilde{H}_*(A_i(G)))$.*

3. *The composition*

$$A_i(G) \to A(G) \to G \to \bar{B}_i(G) \tag{4.25}$$

induces the inclusion of the generating set in homology. □

Remark 4.9. By [23], $SU(n)$ and $Sp(n)$ are torsion-free, and $Spin(n)$ has no torsion for $n \leq 6$ and only 2-torsion for $n \geq 7$. The exceptional groups have torsion only in the following cases: G_2 at $p = 2$, F_4, E_6, E_7 at $p = 2, 3$ and E_8 at $p = 2, 3, 5$.

Lemma 4.10. *If each $A_i(G)$ has $\ell < p-2$ cells all in odd dimensions, then for each $\bar{B}_i(G)$ there exists a homotopy equivalent H-space $B_i(G)$ that is homotopy associative, homotopy commutative, and universal for $A_i(G)$. Further, $\prod_i B_i(G)$ is universal for $\bigvee_i A_i(G)$.*

The following proof follows those found in [36, 37].

Proof. For simplicity, we write $A_i := A_i(G)$, $\bar{B}_i := \bar{B}_i(G)$ and so on. By Theorem 4.8, we have $A \simeq \bigvee_{i=1}^{p-1} A_i \to G \simeq \prod_{i=1}^{p-1} \bar{B}_i$. Fix an A_i. It has $\ell < p-2$ cells in odd dimensions, so we may apply Theorem 4.4 and construct the following homotopy fibration sequence:

$$\Omega R_i \to \Omega\Sigma A_i \xrightarrow{\rho} B_i \xrightarrow{*} R_i \xrightarrow{\theta_i} \Sigma A_i. \tag{4.26}$$

We also get a decomposition $\Omega\Sigma A \simeq B_i \times \Omega R_i$, and thus there is a map $B_i \to \Omega\Sigma A_i$. Note that B_i is homotopy associative and homotopy commutative.

By Theorem 4.4, we have a map $\alpha_i : A_i \xrightarrow{E} \Omega\Sigma A_i \to B_i$ that induces the inclusion of the generating set in homology. Also, by Theorem 4.8, we have a map $\bar{\alpha}_i : A_i \to \bar{B}_i$, which also induces the inclusion of the generating set in homology.

Recall from Remark 3.9 that although $\bar{B}_i$ may not be homotopy associative we may still use the James construction to extend $\bar{\alpha}_i$ to a map $\Omega\Sigma A_i \to \bar{B}_i$, as in the following diagram.

$$\begin{array}{ccc} A_i & \xrightarrow{\bar{\alpha}_i} & \bar{B}_i \\ \Big\downarrow{\scriptstyle E} & \nearrow{\scriptstyle \hat{\alpha}_i} & \\ \Omega\Sigma A_i & & \end{array} \tag{4.27}$$

Note however that $\hat{\alpha}_i$ may not be an H-map. We now have that the composition $B_i \to \Omega\Sigma A_i \to \bar{B}_i$ induces an isomorphism on the generating set in homology. Then we must also have an induced isomorphism on the generating set in cohomology. Since in cohomology we have a multiplication, $B_i \to \Omega\Sigma A_i \to \bar{B}_i$ induces an isomorphism on the whole cohomology algebra, and then by Whitehead's theorem there must be a homotopy equivalence $B_i \simeq \bar{B}_i$. Universality of B_i follows directly from Lemma 4.5. It remains to show that $\prod_i B_i$ is universal for $\bigvee_i A_i$.

Consider $B_1 \times B_2$ and $A_1 \vee A_2$, where B_1 is universal for A_1 and B_2 is universal for A_2. Note that $B_1 \times B_2$ inherits a homotopy associative, homotopy commutative H-structure from B_1 and B_2. We have inclusions $\iota_1 : A_1 \to B_1$ and $\iota_2 : A_2 \to B_2$, so now define ι to be the composite

$$\iota : A_1 \vee A_2 \xrightarrow{\iota_1 \vee \iota_2} B_1 \vee B_2 \hookrightarrow B_1 \times B_2, \tag{4.28}$$

where the second map is the inclusion of the wedge into the product.

Let $f : A_1 \vee A_2 \to Z$ be a map from $A_1 \vee A_2$ to a homotopy associative, homotopy commutative H-space Z. Then f is determined by its restrictions $f_1 : A_1 \to Z$ and $f_2 : A_2 \to Z$. By universality of B_1 and B_2, these may be extended to unique H-maps $\bar{f}_1 : B_1 \to Z$ and $\bar{f}_2 : B_2 \to Z$, such that $\bar{f}_1 \circ \iota_1 \simeq f_1$ and $\bar{f}_2 \circ \iota_2 \simeq f_2$. Let m denote the multiplication on Z, and define $\bar{f}$ to be the composite

$$\bar{f} : B_1 \times B_2 \xrightarrow{\bar{f}_1 \times \bar{f}_2} Z \times Z \xrightarrow{m} Z. \tag{4.29}$$

Since Z is homotopy associative and homotopy commutative, m is an H-map, and so $\bar{f}$ is a composition of H-maps and hence also an H-map.

Consider the following homotopy commutative diagram, in which ∇ is the fold map.

$$\begin{array}{ccccccc} A_1 \vee A_2 & \xrightarrow{\iota_1 \vee \iota_2} & B_1 \vee B_2 & \xrightarrow{\bar{f}_1 \vee \bar{f}_2} & Z \vee Z & & \\ & & \big\downarrow & & \big\downarrow & \searrow{\scriptstyle \nabla} & \\ & & B_1 \times B_2 & \xrightarrow{\bar{f}_1 \times \bar{f}_2} & Z \times Z & \xrightarrow{m} & Z \end{array} \tag{4.30}$$

The lower direction is the definition of $\bar{f} \circ \iota : A_1 \vee A_2 \to Z$, and the top row is

$$(\bar{f}_1 \circ \iota_1) \vee (\bar{f}_2 \circ \iota_2) \simeq f_1 \vee f_2. \tag{4.31}$$

Thus the upper direction, $A_1 \vee A_2 \to B_1 \vee B_2 \to Z \vee Z \to Z$ is homotopic to f, and therefore $\bar{f} \circ \iota \simeq f$. It remains to show uniqueness.

Suppose that $g : B_1 \times B_2 \to Z$ is another H-map such that $g \circ \iota \simeq f$, and let g_1, g_2 be the composites

$$g_1 : B_1 \hookrightarrow B_1 \times B_2 \xrightarrow{g} Z, \tag{4.32}$$

$$g_2 : B_2 \hookrightarrow B_1 \times B_2 \xrightarrow{g} Z, \tag{4.33}$$

where the left maps are the inclusions of the first and second factors, respectively. These inclusions are H-maps, and therefore g_1 and g_2 are also H-maps. By homotopy associativity and homotopy commutativity of Z, g is homotopic to the sum of its restrictions, so we have $g \simeq B_1 \times B_2 \xrightarrow{g_1 \times g_2} Z \times Z \xrightarrow{m} Z$. Then since $f \simeq g \circ \iota$, f is homotopic to the composition

$$A_1 \vee A_2 \xrightarrow{\iota_1 \vee \iota_2} B_1 \vee B_2 \hookrightarrow B_1 \times B_2 \xrightarrow{g_1 \times g_2} Z \times Z \xrightarrow{m} Z. \tag{4.34}$$

Restricting to A_1 gives $g_1 \circ \iota_1 \simeq f_1$, and then by universality of B_1 for A_1, we have $g_1 \simeq \bar{f}_1$. Similarly, we get $g_2 \simeq \bar{f}_2$, and therefore $g \simeq m \circ (g_1 \times g_2) \simeq m \circ (\bar{f}_1 \times \bar{f}_2) \simeq f$. By induction, $\prod_i B_i$ is universal for $\bigvee_i A_i$. □

This was used by Grbić and Theriault [8] to produce many results on self maps of Lie groups; in particular, identifying when $[G, G] \cong H[G, G]$. We will instead consider the case $[G, L] \cong H[G, L]$, for two Lie groups G and L.

4.2.2 When is $[G, L] \cong H[G, L]$?

The following proposition utilises the space $A(G)$ to help us identify properties of $H[G, L]$.

Proposition 4.11. *Let G and L be compact, connected, simply-connected, simple Lie groups, localised at a prime p such that both G and L are homotopy commutative. Suppose also that G is torsion-free. Then there exists a co-H-space $A \simeq \bigvee_{i=1}^{p-1} A_i(G)$ such that $[A, L] \cong H[G, L]$.*

Proof. By Theorem 4.8, we have a co-H-space A such that $H_*(G) \cong \Lambda(\tilde{H}_*(A))$, with the homotopy decomposition $A \simeq \bigvee_{i=1}^{p-1} A_i(G)$. Let $\bar{B}_i = \prod_{i=1}^{p-1} \bar{B}_i(G)$; then we have a homotopy decomposition $G \simeq \bar{B}_i$.

Recall from Chapter 2 that the dimensions of the spheres in the decomposition of G correspond to the generators in homology; thus,

$$\Lambda(\tilde{H}_*(A)) \cong H_*(G) \cong \Lambda(x_{d_1}, \ldots, x_{d_r}) \tag{4.35}$$

where $(d_1, \ldots, d_r)$ is the type of G. Then $A \simeq S^{d_1} \vee \ldots \vee S^{d_r}$. The prime p is such that G is homotopy commutative, so for each G we have $r < p-1$:

- for $SU(n), r = n-1 < 2n \leq p-1$,
- for $Sp(n)$ and $Spin(2n+1), r = n < 4n \leq p-1$,
- for $Spin(2n), r = n < 4(n-1) \leq p-1$,
- for $G_2, r = 2 < 12 \leq p-1$,
- for $F_4, r = 4 < 28 \leq p-1$,
- for $E_6, r = 6 < 28 \leq p-1$,
- for $E_7, r = 7 < 36 \leq p-1$,
- for $E_8, r = 8 < 60 \leq p-1$.

Then $A \simeq \bigvee_{i=1}^{p-1} A_i(G)$ where $A_i(G) = S^{d_i}$ for $1 \leq i \leq r$, and $A_i(G)$ trivial otherwise. Each $A_i(G)$ is either an odd-dimensional sphere or trivial, so the conditions of Lemma 4.10 are satisfied.

Now by Lemmas 4.5 and 4.10, we have a homotopy associative, homotopy commutative H-space B that is universal for A. Let g be a map $A \to B$, and let h be a map $A \to L$. Since L is homotopy associative and homotopy commutative, by universality there is a unique H-map extension $\bar{h} : B \to L$, such that $\bar{h} \simeq h \circ g$. Thus,

$$[A, L] \cong H\left[\prod_i B_i(G), L\right]. \tag{4.36}$$

Then since G is homotopy commutative, the equivalence $G \simeq \prod_i B_i(G)$ is an equivalence of H-spaces, and hence $[A, L] \cong H[G, L]$ as required. □

Suppose that G is p-regular, which homotopy commutativity implies in all cases excepting $L = Sp(2)$ at $p = 3$ and $L = G_2$ at $p = 5$ (we return to these in Propositions 6.5 and 6.6).

Note that $H[G, L] \cong [A, L] \cong [S^{d_1}, L] \oplus \ldots \oplus [S^{d_r}, L]$, and recall the set T_r, defined in Remark 4.3 to be

$$T_r = \{(\ell_1, \ldots, \ell_i) | 1 \leq \ell_1 < \ldots < \ell_i < r, 1 \leq i \leq r\}. \tag{4.37}$$

The elements of the group $H[G, L]$ correspond to the elements of T_r of length 1, so now to determine whether $H[G, L] \cong [G, L]$, we need to find out if the homotopy groups corresponding to the remaining elements are trivial.

Example 4.2. *We return to the case of Example 4.1, where we saw that for $G = G_2$, $p \geq 13$ (so that G_2 is p-regular, homotopy commutative and torsion-free) and L homotopy commutative, we have*

$$\begin{aligned}[G_2, L] &\cong \bigoplus_{T_2}[S^{d_{\ell_1}+\ldots+d_{\ell_i}}, L] \\ &\cong [S^3, L] \oplus [S^{11}, L] \oplus [S^{3+11}, L] \\ &\cong \pi_3(L) \oplus \pi_{11}(L) \oplus \pi_{14}(L).\end{aligned}$$

Here $\Lambda(x_3, x_{11}) \cong H_*(G_2) \cong \Lambda(\tilde{H}_*(A))$, so we have $H[G, L] \cong [A, L] \cong [S^3, L] \oplus [S^{11}, L]$. Hence

$$[G_2, L] \cong H[G_2, L] \oplus [S^{14}, L] \tag{4.38}$$

and so $[G_2, L] \cong H[G_2, L]$ whenever $\pi_{14}(L)$ is trivial. We return to this in Section 4.2.3.

Corollary 4.12. *Let G and L be compact, connected, simply-connected, simple Lie groups, and let p be a prime such that G and L are both homotopy commutative and p-regular. Suppose that G is torsion-free. Let the mod-p decompositions of G and L be $G \simeq S^{d_1} \times \ldots \times S^{d_r}$ and $L \simeq S^{b_1} \times \ldots \times S^{b_s}$ respectively. By Remark 4.3 and the subsequent discussion, localised at p we have*

$$[G, L] \cong \bigoplus_{T_r}[S^{d_{\ell_1}+\ldots+d_{\ell_i}}, L]. \tag{4.39}$$

and thus now that L is p-regular we have

$$[G, L] \cong \bigoplus_{\substack{(\ell_1,\ldots,\ell_i)\in T_r \\ j=1,\ldots,s}} [S^{d_{\ell_1}+\ldots+d_{\ell_i}}, S^{b_j}] \tag{4.40}$$

where $T_r = \{(\ell_1, \ldots, \ell_i) | 1 \leq \ell_1 < \ldots < \ell_i < r, 1 \leq i \leq r\}$. □

Thus when looking at decompositions of $[G, L]$, we are really only looking at homotopy groups of spheres. As observed earlier, the elements of T_r of length 1 correspond to the group $[A, L] \cong H[G, L]$, so let V_r be the set of elements of T_r of length at least 2. Corollary 4.12 can then be used to find the cases where $H[G, L] \cong [G, L]$.

Corollary 4.13. *Let G and L be compact, connected, simply-connected, simple Lie groups, and let p be a prime such that G and L are both homotopy commutative and p-regular. Suppose that G is torsion-free. Let the mod-p decompositions of G and L be $G \simeq S^{d_1} \times \ldots \times S^{d_r}$ and $L \simeq S^{b_1} \times \ldots \times S^{b_s}$ respectively. Then, localised at p,*

$$[G, L] \cong H[G, L] \oplus \bigoplus_{\substack{(k_1,\ldots,k_i)\in V_r \\ j=1,\ldots,s}} [S^{d_{k_1}+\ldots+d_{k_i}}, S^{b_j}] \tag{4.41}$$

$$= H[G, L] \oplus \bigoplus_{\substack{(k_1,\ldots,k_i)\in V_r \\ j=1,\ldots,s}} \pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j}) \tag{4.42}$$

where $V_r = \{(k_1, \ldots, k_i) | 1 \leq k_1 < \ldots < k_i \leq r, 2 \leq i \leq r\}$. *Hence,* $[G, L] \cong H[G, L]$ *if and only if each homotopy group of the form* $\pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j})$ *for* $i \geq 2$ *is trivial. That is, if there are no non-trivial homotopy groups that are in* $[G, L]$ *but not in* $H[G, L]$. □

There are 2 cases in which a group is homotopy commutative when localised at p, but is not p-regular: they are $Sp(2)$ at the prime 3, and G_2 at the prime 5. These cases will be dealt with later, in Propositions 6.5 and 6.6 respectively.

4.2.3 Example calculations

In this section, we will see how the previous results can be used to discern whether $[G, L] \cong H[G, L]$, and to calculate what these groups are.

To calculate the homotopy groups of the odd-dimensional spheres, we use the following theorem of Toda [39].

Theorem 4.14 (Toda's Theorem)**.** *The homotopy groups of spheres, localised at a prime p, are given by the following formulae. The stable groups:*

$$\pi_{2m-1+k}(S^{2m-1}; p) \approx \begin{cases} \mathbb{Z}_p & \text{for } k = 2q(p-1)-1, q = 1, 2, \ldots, p-1 \text{ and } m \geq 2, \\ \mathbb{Z}_p & \text{for } k = 2q(p-1)-2, q = 2, 3, \ldots, p-1 \text{ and } q \geq m \geq 2, \\ 0 & \text{otherwise, for } k < 2p(p-1)-2. \end{cases} \tag{4.43}$$

The non-trivial unstable groups:

$$\pi_{2m-1+2p(p-1)-2}(S^{2m-1}; p) \approx \begin{cases} \mathbb{Z}_{p^2} & \text{for } p \geq m \geq 3, \\ \mathbb{Z}_p & \text{for } m = 2 \text{ and for } m > p. \end{cases} \tag{4.44}$$

$$\pi_{2m-1+2p(p-1)-1}(S^{2m-1}; p) \approx \begin{cases} \mathbb{Z}_{p^2} & \text{for } m \geq 3, \\ \mathbb{Z}_p & \text{for } m = 2. \end{cases} \tag{4.45}$$

We begin with the classical Lie groups. Recall from Section 2.1 that when a compact, connected, simply-connected classical Lie group G is p-regular for some odd prime p, its localisation at p splits in the following way:

$$SU(n) \simeq S^3 \times S^5 \times \ldots \times S^{2n-1} \tag{4.46}$$
$$Sp(n) \simeq S^3 \times S^7 \times \ldots \times S^{4n-1} \tag{4.47}$$
$$Spin(2n+1) \simeq S^3 \times S^7 \times \ldots \times S^{4n-1} \tag{4.48}$$
$$Spin(2n) \simeq S^3 \times S^7 \times \ldots \times S^{4n-5} \times S^{2n-1}. \tag{4.49}$$

Example 4.3. *Consider* $SU(3)$ *and* $SU(4)$. *For* $SU(n)$ *to be homotopy commutative, we require* $p \geq 2n$. *Let* $p \geq 11$, *so that both* $SU(3)$ *and* $SU(4)$ *are homotopy commutative, and* *localised at p we have*

$$SU(3) \simeq S^3 \times S^5 \tag{4.50}$$
$$SU(4) \simeq S^3 \times S^5 \times S^7. \tag{4.51}$$

Note that for $p \geq 11$, $SU(3)$ and $SU(4)$ are also p-regular as we have $p > n-1$.

Now consider $[SU(3), SU(4)]$. Then we have $d_1 = 3$, $d_2 = 5$ and $b_1 = 3$, $b_2 = 5$, $b_3 = 7$ and by Corollary 4.13,

$$[SU(3), SU(4)] \cong H[SU(3), SU(4)] \oplus \bigoplus_{\substack{(k_1,\dots,k_i)\in V_2\\ j=1,2,3}} \pi_{d_{k_1}+\dots+d_{k_i}}(S^{b_j}) \tag{4.52}$$

where $V_2 = \{(k_1, \dots, k_i) | 1 \leq k_1 < \dots < k_i \leq 2, 2 \leq i \leq 2\} = \{(3,5)\}$.

We need to find the homotopy groups of the form $\pi_{d_{k_1}+\dots+d_{k_i}}(S^{b_j})$, with $(k_1, \dots, k_i) \in V_2$, and determine whether any are non-trivial. Thus, we need $\pi_{3+5}(S^{b_j}) = \pi_8(S^{b_j})$, where $b_j = 3, 5, 7$.

By Toda's Theorem 4.14, we see that the dimensions of all non-trivial homotopy groups are at least $2(p-1)$, and since here $p \geq 11$ all non-trivial homotopy groups are of dimension at least $2(p-1) = 2(11-1) = 20$. Hence $\pi_8(S^{b_j})$ is trivial for all b_j, and, for $p \geq 11$ we have $[SU(3), SU(4)] \cong H[SU(3), SU(4)]$. That is, each map $SU(3) \to SU(4)$ has an H-map extension that is unique up to homotopy.

We will now calculate the group $[SU(3), SU(4)] \cong H[SU(3), SU(4)]$. By Corollary 4.12,

$$[SU(3), SU(4)] \cong \bigoplus_{\substack{(k_1,\dots,k_i)\in T_2\\ j=1,\dots,3}} [S^{d_{k_1}+\dots+d_{k_i}}, S^{b_j}] \tag{4.53}$$
$$\cong \bigoplus_{\substack{(k_1,\dots,k_i)\in T_2\\ j=1,\dots,3}} \pi_{d_{k_1}+\dots+d_{k_i}}(S^{b_j}). \tag{4.54}$$

where $T_2 = \{(k_1, \dots, k_i) | 1 \leq k_1 < \dots < k_i < 2, 1 \leq i \leq 2\} = \{(3), (5), (3,5)\}$. Then we want to consider the homotopy groups $\pi_3(S^{b_j})$, $\pi_5(S^{b_j})$ and $\pi_{3+5}(S^{b_j})$ for $b_j = 3, 5, 7$.

We have already seen that the homotopy group corresponding to $(3,5)$, that is, $\pi_{3+5}(S^{b_j}) = \pi_8(S^{b_j})$, is trivial for all b_j. Since $\pi_n(S^m)$ is trivial for $m > n$, we are left with the following:

$$[SU(3), SU(4)] \cong H[SU(3), SU(4)] \tag{4.55}$$
$$\cong \pi_3(S^3) \times \pi_5(S^3) \times \pi_5(S^5) \tag{4.56}$$
$$\cong \mathbb{Z} \times \mathbb{Z}, \tag{4.57}$$

since $\pi_5(S^3) = 0$ when localised at an odd prime. A similar argument shows that we also have $[SU(4), SU(3)] \cong H[SU(4), SU(3)]$ for $p \geq 11$.

We now look at an example with $Sp(n)$. This illustrates that although they are fairly strong conditions, homotopy commutativity and p-regularity alone are not sufficient conditions for $[G, L] \cong H[G, L]$. This is also a case where there is a single outlying value for p, so that $[G, L] \cong H[G, L]$ for $p = q$ and $p \geq r$, with at least 1 prime between q and r.

Example 4.4. *Consider $[Sp(6), Sp(m)]$ where $2 \leq m \leq 6$. For $Sp(n)$ to be homotopy commutative, we need $p > 4n$, and for $Sp(n)$ to be p-regular we need $p > 2n - 1$. Let $p \geq 29$, so that $Sp(6)$ and $Sp(m)$ are both homotopy commutative and p-regular. Localised at p, we have the following splittings:*

$$Sp(2) \simeq S^3 \times S^7 \tag{4.58}$$
$$Sp(3) \simeq S^3 \times S^7 \times S^{11} \tag{4.59}$$
$$Sp(4) \simeq S^3 \times S^7 \times S^{11} \times S^{15} \tag{4.60}$$
$$Sp(5) \simeq S^3 \times S^7 \times S^{11} \times S^{15} \times S^{19} \tag{4.61}$$
$$Sp(6) \simeq S^3 \times S^7 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \qquad = S^{d_1} \times \ldots \times S^{d_6}. \tag{4.62}$$

By Corollary 4.13, we have

$$[Sp(6), Sp(m)] \cong H[Sp(6), Sp(m)] \oplus \bigoplus_{\substack{(k_1,\ldots,k_i)\in V_6 \\ j=1,\ldots,s}} [S^{d_{k_1}+\ldots+d_{k_i}}, S^{b_j}] \tag{4.63}$$
$$= H[Sp(6), Sp(m)] \oplus \bigoplus_{\substack{(k_1,\ldots,k_i)\in V_6 \\ j=1,\ldots,s}} \pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j}) \tag{4.64}$$

where $V_6 = \{(k_1, \ldots, k_i) | 1 \leq k_1 < \ldots < k_i \leq 6, 2 \leq i \leq 6\}$ and the S^{b_j} are spheres in the decomposition of $Sp(m)$.

Then for $[Sp(6), Sp(m)] \cong H[Sp(6), Sp(m)]$, we need all homotopy groups of the form

$$\pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j}), \tag{4.65}$$

with $(k_1, \ldots, k_i) \in V_6$, to be trivial.

At $p = 29$, Toda's Theorem 4.14 shows that there are no non-trivial homotopy groups of the form $\pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j})$.

Now let $p = 31$. Again by Toda's Theorem 4.14, the only non-trivial homotopy group of the form $\pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j})$ is

$$\pi_{3+7+11+19+15+23}(S^{19}) = \pi_{78}(S^{19}) \simeq \mathbb{Z}_{31}. \tag{4.66}$$

Since S^{19} appears in the decomposition of $Sp(5)$ and $Sp(6)$, $p = 31$ is insufficient to give $[Sp(6), Sp(m)] \cong H[Sp(6), Sp(m)]$ for $m = 5, 6$. However, S^{19} does not appear in the decomposition of $Sp(m)$ for $2 \le m \le 4$, and thus for $2 \le m \le 4$ and $p = 31$ we in fact have $[Sp(6), Sp(m)] \cong H[Sp(6), Sp(m)]$.

Now let $p = 37$. The only non-trivial homotopy group of the form $\pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j})$ is

$$\pi_{3+7+11+19+15+23}(S^7) = \pi_{78}(S^7) \simeq \mathbb{Z}_{37}. \tag{4.67}$$

For $2 \le m \le 6$, $Sp(m)$ contains S^7 in its decomposition, so $p = 37$ is insufficient to give $[Sp(6), Sp(m)] \cong H[Sp(6), Sp(m)]$.

Finally, let $p \ge 41$. Then any non-trivial homotopy group must be of dimension at least $2(41-1) = 80$. The highest dimension of any homotopy group in the expansion $[Sp(6), L]$ for any L is

$$3 + 7 + 11 + 15 + 19 + 23 = 78 < 80, \tag{4.68}$$

so there will be no non-trivial homotopy groups of the form $\pi_{d_{k_1}+\ldots+d_{k_i}}(S^{b_j})$ and for all $p \ge 41$ we have $[Sp(6), Sp(m)] \cong H[Sp(6), Sp(m)]$.

Thus $[Sp(6), Sp(m)] \cong H[Sp(6), Sp(m)]$ for $p \ge 41$, $2 \le m \le 6$, and the additional case $p = 31$, $2 \le m \le 4$.

We now look at some examples involving exceptional groups. Since the exceptional groups do not have the same regularity in their mod-p decompositions as the classical groups, calculations are more cumbersome and must be done on a case-by-case basis. Note that below p is such that G is torsion-free (see Remark 4.9).

Recall that when an exceptional group G is p-regular, its localisation at p splits in the following way:

$$G_2 \simeq S^3 \times S^{11} \tag{4.69}$$
$$F_4 \simeq S^3 \times S^{11} \times S^{15} \times S^{23} \tag{4.70}$$
$$E_6 \simeq S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23} \tag{4.71}$$
$$E_7 \simeq S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35} \tag{4.72}$$
$$E_8 \simeq S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}. \tag{4.73}$$

The following lemma of Grbić and Theriault [8] considers the case of self maps $G_2 \to G_2$, at $p = 5$. In this case, G_2 is homotopy commutative but not p-regular.

Lemma 4.15. *Let $p = 5$. Then there is a group isomorphism $[G_2, G_2] \cong H[G_2, G_2]$.* □

The following lemma generalises the result of Grbić and Theriault; it will be generalised further in Chapter 6.

Lemma 4.16. *Suppose $p = 5$ or $p \geq 13$. Then $[G_2, G_2] \cong H[G_2, G_2]$.*

Proof. By Lemma 4.15, at $p = 5$ we have $[G_2, G_2] \cong H[G_2, G_2]$. Suppose now that $p \geq 13$.

Recall from examples 4.1 and 4.2 that $[G_2, G_2] \cong H[G_2, G_2] \oplus [S^{14}, G_2]$. Since $p \geq 13$, G_2 is p-regular and so $G_2 \simeq S^3 \times S^{11}$. Then

$$[S^{14}, G_2] = \pi_{14}(G_2) \cong \pi_{14}(S^3) \oplus \pi_{14}(S^{11}). \tag{4.74}$$

By Toda's Theorem 4.14, the dimensions of non-trivial unstable homotopy groups are at least $2p(p-1) \geq 312$, so they do not include $\pi_{14}(S^3)$ or $\pi_{14}(S^{11})$. Suppose, towards a contradiction, that $\pi_{14}(S^3)$ is a non-trivial stable homotopy group. Then

$$14 = 2m - 1 + k = 3 + k, \tag{4.75}$$

so $k = 11$. Since 11 is odd, this then implies that $11 = k = 2q(p-1) - 1$, and so $q(p-1) = 6$. But $p \geq 13$ so this is impossible, and hence $\pi_{14}(S^3)$ is trivial. Similarly, if $\pi_{14}(S^{11})$ were a non-trivial stable homotopy group we would have $k = 3$, $q(p-1) = 2$, which is impossible and so $\pi_{14}(S^{11})$ is also trivial. Thus

$$[G_2, G_2] \cong H[G_2, G_2] \oplus \pi_{14}(S^3) \oplus \pi_{14}(S^{11}) \cong H[G_2, G_2]. \tag{4.76}$$

□

Chapter 5

Non-homotopy-commutative spaces

Having now dealt with the case where both G and L are homotopy commutative, we now work toward removing this condition. In this section, we will build up the background methodology around non-homotopy commutative spaces.

5.1 Retractile triples

Definition 5.1. Let $i : X \to Y$ be a map of path-connected spaces where Y is a homotopy associative H-space. The triple (X, i, Y) is *retractile* if:

1. $H_*(Y) \cong \Lambda(\tilde{H}_*(X))$;
2. i_* is the inclusion of the generating set;
3. the H-map $\bar{i} : \Omega\Sigma X \to Y$ extending i given by Theorem 3.8 has a right homotopy inverse $s : Y \to \Omega\Sigma X$.

We now set up some notation, which will be used for the rest of this section. Suppose (X, i, Y) is a retractile triple. Define the space F and the map h by the homotopy fibration $F \xrightarrow{h} \Omega\Sigma X \xrightarrow{\bar{i}} Y$.

Denote the loop multiplication on $\Omega\Sigma X$ by m. Since $\bar{i}$ has a right homotopy inverse, the composition

$$e : Y \times F \xrightarrow{s\times h} \Omega\Sigma X \times \Omega\Sigma X \xrightarrow{m} \Omega\Sigma X \tag{5.1}$$

is a homotopy equivalence.

Now let Z be a homotopy associative H-space and $f : X \to Z$ be a map. Again by Theorem 3.8 we have a map $\bar{f} : \Omega\Sigma X \to Z$, which is the unique H-map extending f such that $\bar{f} \circ E \simeq f$. Define $\hat{f}$ by the composition $\hat{f} : Y \xrightarrow{s} \Omega\Sigma X \xrightarrow{\bar{f}} Z$. Then we have the following proposition of Theriault [38].

Proposition 5.2. *Let (X, i, Y) be a retractile triple. Using the above notation, if the composition $F \xrightarrow{h} \Omega\Sigma X \xrightarrow{\bar{f}} Z$ is null homotopic, then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega\Sigma X & \xrightarrow{\bar{f}} & Z \\ \downarrow{\scriptstyle \bar{i}} & \nearrow{\scriptstyle \hat{f}} & \\ Y & & \end{array} \tag{5.2}$$

and $\hat{f}$ is an H-map. Furthermore, $\hat{f}$ is the unique H-map, up to homotopy, such that $\hat{f} \circ i \simeq f$. □

Corollary 5.3. *Let (X, i, Y) be a retractile triple, Z a homotopy associative H-space, and suppose that Proposition 5.2 holds for all maps $f : X \to Z$. Then there is a one-to-one correspondence between $[X, Z]$ and $H[Y, Z]$.*

We now consider when this bijection is in fact a group isomorphism.

5.2 An extension to group isomorphisms

Recall Lemma 4.1, which said that if Y is a path-connected H-space and Z is a path-connected, homotopy associative, homotopy commutative H-space, then $H[Y, Z]$ is a subgroup of $[Y, Z]$.

Lemma 5.4. *Suppose that (X, i, Y) is a retractile triple. If Z is a homotopy associative, homotopy commutative H-space and Proposition 5.2 holds for all $f : X \to Z$, then the bijection $[X, Z] \cong H[Y, Z]$ in Corollary 5.3 is an isomorphism of groups.*

Proof. Let the map $\Psi : [X, Z] \to H[Y, Z]$ be given by $\Psi(f) = \hat{f}$. We will show that it is a group isomorphism.

Let f and g be maps $X \to Z$, and note that by Lemma 4.1, $H[Y, Z]$ is a subgroup of $[Y, Z]$. The sum $f + g$ is defined by the composition

$$f + g : X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Z \times Z \xrightarrow{m} Z, \tag{5.3}$$

where m is the multiplication on Z.

Consider the following diagram:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\Delta} & X \times X & \xrightarrow{f \times g} & Z \times Z & \xrightarrow{m} & Z \\
\downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E \times E} & \nearrow{\scriptstyle \bar{f} \times \bar{g}} & & & \\
\Omega\Sigma X & \xrightarrow{\Delta} & \Omega\Sigma X \times \Omega\Sigma X & & {\scriptstyle \hat{f} \times \hat{g}} \nearrow & & \\
\downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle \bar{i} \times \bar{i}} & \nearrow & & & \\
Y & \xrightarrow{\Delta} & Y \times Y. & & & &
\end{array}
\tag{5.4}
$$

In the upper triangle the maps $\bar{f}$ and $\bar{g}$ are H-map extensions of f and g respectively given by Theorem 3.8, and in the bottom triangle the maps $\hat{f}$ and $\hat{g}$ are H-maps extending f and g respectively by the hypothesis that Proposition 5.2 holds for all maps $X \to Z$. The two squares at left homotopy commute by the naturality of the diagonal map, so the whole diagram is homotopy commutative. Note that the top row is $f + g$ and the composite $m \circ (\hat{f} \times \hat{g}) \circ \Delta$ is $\hat{f} + \hat{g}$. Thus (5.4) implies that $f + g \simeq (\hat{f} + \hat{g}) \circ \bar{i} \circ E$.

On the other hand, the hypothesis that Proposition 5.2 holds for all maps $X \to Z$ implies that we may extend $f + g$ to an H-map $\widehat{f+g} : Y \to Z$ such that $\widehat{f+g} \circ \bar{i} \circ E \simeq f + g$. Combining the two homotopies, we see that

$$\widehat{f+g} \circ \bar{i} \circ E \simeq f + g \simeq (\hat{f} + \hat{g}) \circ \bar{i} \circ E. \tag{5.5}$$

Note that $i = \bar{i} \circ E$, so $\widehat{f+g} \circ i \simeq (\hat{f} + \hat{g}) \circ i$.

Since $\hat{f}$ and $\hat{g}$ are H-maps, and by Lemma 4.1 $H[Y, Z]$ is a subgroup of $[Y, Z]$, the map $\hat{f} + \hat{g}$ is also an H-map. Thus by (5.5), $\widehat{f+g}$ and $\hat{f} + \hat{g}$ are two H-maps extending $f + g$, so by the uniqueness property in Proposition 5.2, $\widehat{f+g} \simeq \hat{f} + \hat{g}$.

We now have

$$\Psi(f + g) = \widehat{f+g} \simeq \hat{f} + \hat{g} = \Psi(f) + \Psi(g), \tag{5.6}$$

and so $\Psi : [X, Z] \to H[Y, Z]$ is a group homomorphism. As Ψ is also a bijection, it is therefore a group isomorphism. □

5.3 Investigating A_G

From now on, all spaces and maps will be localised at an odd prime p, and homology will be taken with mod-p coefficients.

Let G be a compact, simply-connected, simple Lie group, localised at p such that G is p-regular. Suppose that the type of G is $(m_1, \ldots, m_k)$; we define the space A_G to be the wedge sum

$$A_G = \bigvee_{i=1}^{k} S^{2m_i-1}. \tag{5.7}$$

In what follows, we will often simplify notation to $A := A_G$. Let $i : A \hookrightarrow G$ be the inclusion of the wedge into the product, and let $j : \Sigma A \to BG$ be its adjoint. There is an induced homotopy fibration sequence

$$\cdots \to \Omega Q \xrightarrow{\Omega\gamma} \Omega\Sigma A \xrightarrow{\Omega j} G \xrightarrow{\partial} Q \xrightarrow{\gamma} \Sigma A \xrightarrow{j} BG \tag{5.8}$$

which defines the space Q and the maps γ and ∂.

Lemma 5.5. *The triple (A, i, G) is retractile, where $i \simeq \Omega j \circ E$.*

Proof. Observe first that $H_*(G) \cong \Lambda(\tilde{H}_*(A))$. Since j is the adjoint of i, the composition $A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\Omega j} G$ is homotopic to i. Then in homology, $i_* : H_*(A) \to H_*(\Omega\Sigma A) \to H_*(G)$ induces the inclusion of the generating set.

It remains to check that $\bar{i}$ has a right homotopy inverse, where $\bar{i} : \Omega\Sigma A \to G$ is the H-map extension of Theorem 3.8. Here, since j is the adjoint of i, $\bar{i}$ is the unique H-map extension of i and since Ωj is also an H-map extension of i we have $\bar{i} \simeq \Omega j$.

Note that $\Omega\Sigma A \simeq \prod_{i=1}^{k} S^{2m_i-1} \times \prod(\text{other factors})$, by repeated application of the Hilton-Milnor theorem 3.11. We would like for there to be a factorisation as in the following diagram:

$$\begin{array}{ccc} & & A \\ & \swarrow & \big\downarrow E \\ G \simeq \prod_{i=1}^{k} S^{2m_i-1} & \xrightarrow{\quad s \quad} & \Omega\Sigma A \end{array} \tag{5.9}$$

This may not be the case, however. Instead we move to homology.

The suspension map $E : A \to \Omega\Sigma A$ induces the map $E_* : H_*(A) \to H_*(\Omega\Sigma A)$, which is the identity on the generating set. The m^{th} James-Hopf invariant map (see Section 3.2) is $H_m : J(A) \to J(A^{\wedge m})$, or equivalently $H_m : \Omega\Sigma A \to \Omega\Sigma A^{\wedge m}$. Consider the composite

$$A \xrightarrow{E} \Omega\Sigma A \xrightarrow{H_m} \Omega\Sigma(A^{\wedge m}). \tag{5.10}$$

Every H_m is zero on the generating set, so this composite is null homotopic. Then there exists a map $\lambda : H_*(A) \to H_*(\prod S^{2m_i+1})$, such that $s_* \circ \lambda \simeq E_*$. This λ exists algebraically, but may not be induced by a map of spaces.

We now have the following homotopy commutative diagram.

$$\begin{array}{ccccc} & & H_*(A) & & \\ & \swarrow^{\lambda} & \downarrow E_* & \searrow^{i_*} & \\ H_*(\prod S^{2m_i-1}) & \xrightarrow{s_*} & H_*(\Omega\Sigma A) & \xrightarrow{(\Omega j)_*} & H_*(G) \end{array} \tag{5.11}$$

Thus $(\Omega j \circ s)_* \circ \lambda \simeq i_*$, and hence is the inclusion of the generating set. Then $(\Omega j \circ s)_*$ is the identity on the generating set. Dualising to cohomology, we see that $(\Omega j \circ s)^*$ is an algebra map which is an isomorphism on the generating set, so $(\Omega j \circ s)^*$ is an isomorphism. Therefore $(\Omega j \circ s)_*$ is an isomorphism and $\Omega j \circ s$ is a homotopy equivalence by Whitehead's Theorem. □

The Bott-Samelson Theorem [2] can be used to determine the homology of the loop suspension $\Omega\Sigma A$.

Theorem 5.6 (Bott-Samelson Theorem). *If X is connected and the reduced homology of $\bar{H}_*(X, R)$ is free over a coefficient ring R, then there is an isomorphism of algebras*

$$T(\bar{H}_*(X;R)) \to H_*(\Omega\Sigma X;R) \tag{5.12}$$

where $T(V)$ denotes the tensor algebra generated by a module V. Further, if $L(V)$ is the free graded Lie algebra generated by V, then $T(V)$ is isomorphic to $UL(V)$, the universal enveloping algebra.

Let $L = \langle \tilde{H}_*(A) \rangle$ be the free Lie algebra generated by $\tilde{H}_*(A)$. By the Bott-Samelson Theorem, $H_*(\Omega\Sigma A) \cong UL$, where UL is the universal enveloping algebra of L. Now let $[L, L]$ be the free Lie algebra generated by the brackets in L. As in [38], there is an isomorphism of $\mathbb{Z}/p\mathbb{Z}$-vector spaces $H_*(\Omega Q) \cong U[L, L]$, where $U[L, L]$ is the universal enveloping algebra of $[L, L]$. Elements of $\tilde{H}_*(A)$ are all in odd degree; let $\{u_1, \ldots, u_k\}$ be a basis for $\tilde{H}_*(A)$ where $|u_i| = 2n_i - 1$. Cohen and Neisendorfer [3] gave a basis for $[L, L]$ in this case, given by the elements

$$[u_i, u_j], [u_{t_1}, [u_i, u_j]], [u_{t_2}, [u_{t_1}, [u_i, u_j]]], \ldots \tag{5.13}$$

where $1 \leq j \leq i \leq k$ and $1 \leq t_\ell < t_{\ell-1} < \cdots < t_2 < t_1 < i$. Note that the basis elements have bracket lengths from 2 through $k+1$.

Each element u_i, regarded as an element of $H_*(\Omega\Sigma A)$, is in the image of the Hurewicz homomorphism via the composite $\mu_i : S^{2m_i-1} \hookrightarrow A \xrightarrow{E} \Omega\Sigma A$. Therefore, the Lie bracket $[u_i, u_j]$ is in the image of the Hurewicz homomorphism via the Samelson product

$$\langle \mu_i, \mu_j \rangle : S^{2m_i+2m_j-2} \cong S^{2m_i-1} \wedge S^{2m_j-1} \to \Omega\Sigma A. \tag{5.14}$$

Let $\nu_i : S^{2m_i} \to \Sigma A$ be the adjoint of μ_i, and note that the adjoint of the Samelson product $\langle \mu_i, \mu_j \rangle$ is the Whitehead product $[\nu_i, \nu_j] : S^{2m_i+2m_j-1} \to \Sigma A$. Analogous statements exist for higher bracket lengths.

For $\ell \geq 2$, let $\mathcal{I}_\ell$ be an indexing set enumerating the basis elements of $[L, L]$ of bracket length ℓ. Note that by (5.13), $\mathcal{I}_\ell = \emptyset$ for $\ell > k+1$. Then each $\alpha \in \mathcal{I}_\ell$ represents a bracket $[u_{t_{\ell-2}}, \ldots [u_{t_1}, [u_i, u_j]] \ldots]$, which is the Hurewicz image of the iterated Samelson product $\langle \mu_{t_{\ell-2}}, \ldots \langle \mu_{t_1} \langle \mu_i, \mu_j \rangle \rangle \ldots \rangle$. The adjoint of this iterated Samelson product is the iterated Whitehead product $w_{\ell,\alpha} = [\nu_{t_{\ell-2}}, \ldots [\nu_{\ell_1}[\nu_i, \nu_j]] \ldots]$, which is a map $w_{\ell,\alpha} : S^{\delta_\alpha+1} \to \Sigma A$ for $\delta_\alpha = (2m_{t_{\ell-2}} - 1) + \cdots + (2m_{t_1} - 1) + (2m_i - 1) + (2m_j - 1)$. Let $R_\ell = \bigvee_{\alpha \in \mathcal{I}_\ell} S^{\delta_\alpha}$, and let $\lambda_\ell : \Sigma R_\ell \to \Sigma A$ be the wedge sum of the $w_{\ell,\alpha}$.

Let $R = \bigvee_{\ell=2}^{k+1} R_\ell$, and let $\lambda : \Sigma R \to \Sigma A$ be the wedge sum of the λ_ℓ. Note that we now have a new homotopy fibration sequence

$$\cdots \to \Omega\Sigma R \xrightarrow{\Omega\lambda} \Omega\Sigma A \xrightarrow{r} F \xrightarrow{\delta} \Sigma R \xrightarrow{\lambda} \Sigma A \tag{5.15}$$

defining the space F and the maps r, δ.

Theorem 5.7 (Theriault [38]). *Let G be a simply-connected, simple, compact p-regular Lie group.*

1. *If G is homotopy commutative, then there is a homotopy commutative diagram*

$$\begin{array}{ccc} & & \Sigma R \\ & \swarrow^{e} & \downarrow \lambda \\ Q & \xrightarrow{\gamma} & \Sigma A \end{array} \tag{5.16}$$

   where $e : \Sigma R \to Q$ is a homotopy equivalence.

2. *If G is not homotopy commutative, then there is a homotopy commutative diagram*

$$\begin{array}{ccc} & & \Sigma R \\ & \swarrow^{e} & \downarrow \lambda' \\ Q & \xrightarrow{\gamma} & \Sigma A \end{array} \tag{5.17}$$

   where $e : \Sigma R \to Q$ is a homotopy equivalence and λ' is a sum of:

   (a) *iterated Whitehead products, and*

   (b) *maps $\omega + a$, where ω is an iterated Whitehead product and a is a suspension of the stable map $\alpha_1 : S^{2p} \to S^3$.* □

Chapter 6

Revisiting homotopy commutativity

In this section we will briefly return to the case where G and L are homotopy commutative, and recover the results of Chapter 4. The case where they are not will be dealt with in Chapters 7 and 8.

Theorem 6.1. *Let G and L be compact, simply-connected, simple Lie groups, localised at an odd prime p such that both G and L are homotopy commutative. If Proposition 5.2 holds for all maps $f : A \to L$, then there is a group isomorphism $H[G, L] \cong [A, L]$.*

Proof. By Lemma 5.5, there is a retractile triple (A, i, G) where $i = \Omega j \circ E$. Let f be a map $A \to L$. Then by Theorem 3.8, there is a homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & L \\ \downarrow{\scriptstyle E} & \nearrow{\scriptstyle \bar{f}} & \\ \Omega\Sigma A & & \end{array} \tag{6.1}$$

where $\bar{f}$ is an H-map. By construction, λ is a wedge sum of Whitehead products, and so the composition $R \xrightarrow{E} \Omega\Sigma R \xrightarrow{\Omega\lambda} \Omega\Sigma A$ is a wedge sum of Samelson products. Then, since $\bar{f}$ is an H-map, the composition $\bar{f} \circ \Omega\lambda \circ E$ is also a wedge sum of Samelson products. Since we have assumed L is homotopy commutative, $\bar{f} \circ \Omega\lambda \circ E$ is null homotopic and by Theorem 3.8, $\bar{f} \circ \Omega\lambda$ is also null homotopic. We have satisfied the conditions of Proposition 5.2, which implies that there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\Sigma A & \xrightarrow{\bar{f}} & L \\ \downarrow{\scriptstyle \Omega j} & \nearrow{\scriptstyle \hat{f}} & \\ G & & \end{array} \tag{6.2}$$

where $\hat{f}$ is an H-map. Recall that $i \simeq \Omega j \circ E$, so that we may combine diagrams (6.1) and (6.2) to give $f \simeq \hat{f} \circ i$. Since this is true for all maps $f : A \to L$, by Corollary 5.3

there is a bijection $\Psi : [A, L] \to H[G, L]$, given by $\Psi(f) = \hat{f}$. Lemma 5.4 then implies that Ψ is a group isomorphism. □

In most cases, homotopy commutativity implies p-regularity. The exceptions to this are $Sp(2)$ at the prime 3, and G_2 at the prime 5. For convenience, we treat these cases separately in Propositions 6.5 and 6.6.

6.1 The p-regular case

For now, we consider the case where G is p-regular. Recall Theorem 4.2; if X is a space such that $\Sigma X \simeq \bigvee_{i=1}^{t} \Sigma X_i$, and Y is a homotopy associative H-space, then as sets $[X, Y] \cong \prod_{j=1}^{t} [X_j, Y]$. If additionally Y is homotopy commutative, the above isomorphism is of groups.

Recall also that if G is p-regular, then $G \simeq \prod_{i=1}^{k} S^{2m_i-1}$, where $(m_1, \ldots, m_k)$ is the type of G, and $A_G \simeq \bigvee_{i=1}^{k} S^{2m_i-1}$. Note that the supension of a product of spaces $X = X_1 \times \ldots \times X_k$ is given by

$$\Sigma(X_1 \times \ldots \times X_k) \simeq \bigvee_{T_k} \Sigma(X_{r_1} \wedge \ldots \wedge X_{r_j}),$$

where $T_k = \{(r_1, \ldots, r_j) | 1 \leq r_1 < \ldots < r_j \leq k, 1 \leq j \leq k\}$.

Let $X = G$, and each $X_i = S^{2m_{r_i}-1}$ for $1 \leq i \leq k$. Thus $\Sigma G \simeq \bigvee_{T_k} \Sigma(X_{r_1} \wedge \ldots \wedge X_{r_j})$, and we may again use Theorem 4.2 to yield the following.

Lemma 6.2. *If G is p-regular, then we have the following bijections, where $X_{r_i} = S^{2m_{r_i}-1}$ for $1 \leq i \leq k$:*

$$[G, L] \cong \prod_{T_k} [X_{r_1} \wedge \ldots \wedge X_{r_j}, L]$$
$$[A_G, L] \cong \prod_{i=1}^{k} [S^{2m_i-1}, L].$$

If L is homotopy commutative, then both isomorphisms are of groups. □

Recall that the set T_k was defined to be $T_k = \{(r_1, \ldots, r_j) | 1 \leq r_1 < \ldots < r_j \leq k, 1 \leq j \leq k\}$. Let T be the set $\{(1), \ldots, (k)\}$ and $T'_k = T_k \setminus T$.

Proposition 6.3. *Let G and L be localised at an odd prime p such that G and L are both p-regular and homotopy commutative. Then we have group isomorphisms $[A_G, L] \cong H[G, L]$, and*

$$[G, L] \cong \prod_{T_k'} [X_{r_1} \wedge \ldots \wedge X_{r_j}, L] \oplus H[G, L] \tag{6.3}$$

where $X_{r_i} = S^{2m_{r_i}-1}$.

Proof. By Theorem 6.1, we have a group isomorphism $[A_G, L] \cong H[G, L]$, and by Lemma 6.2 we have the group isomorphisms $[G, L] \cong \prod_{T_k} [X_{r_1} \wedge \ldots \wedge X_{r_j}, L]$ and $[A_G, L] \cong \prod_{i=1}^{k} [S^{2m_i-1}, L]$.

Note that T is in one-to-one correspondence with terms of the decomposition of $[A_G, L]$, and T_k' consists of elements of T_k with $j \geq 2$. Hence, we may 'separate' the decomposition of $[G, L]$ to give the group isomorphism

$$[G, L] \cong \prod_{T_k'} [X_{r_1} \wedge \ldots \wedge X_{r_j}, L] \oplus H[G, L].$$

□

Consider a single term of the above decomposition, $[X_{r_1} \wedge \ldots \wedge X_{r_j}, L]$. Since the X_i are all spheres, this can be written

$$[S^{2m_{r_1}-1} \wedge \ldots \wedge S^{2m_{r_j}-1}, L] \cong [S^{2(m_{r_1}+\ldots+m_{r_j})-j}, L],$$

which is just a homotopy group of L.

Corollary 6.4. *If additionally L is p-regular of type $(n_1, \ldots, n_\ell)$, then*

$$[X_{r_1} \wedge \ldots \wedge X_{r_j}, L] \cong \prod_{i=1}^{\ell} [S^{2(m_{r_1}+\ldots+m_{r_j})-j}, S^{2n_i-1}], \tag{6.4}$$

which is a product of homotopy groups of spheres. Then if each of these homotopy groups is trivial for $j \geq 2$, we have a group isomorphism $[G, L] \cong H[G, L]$. □

6.2 When homotopy commutativity does not imply p-regularity

We now return to the outlier cases, where G is homotopy commutative but not p-regular.

Proposition 6.5. *At the prime 3, there is a group isomorphism*

$$[Sp(2), Sp(2)] \cong H[Sp(2), Sp(2)] \oplus \mathbb{Z}_3. \tag{6.5}$$

Proof. Assume throughout that homology is in mod-3 coefficients. Localised at 3, we have $H_*(Sp(2)) \cong \Lambda(x_3, x_7)$ with $\mathscr{P}^1_*(x_7) = x_3$. Let A be the 7-skeleton of $Sp(2)$.

By [27], there is a homotopy equivalence $\Sigma Sp(2) \simeq \Sigma A \vee \Sigma S^{10}$, and so by Theorem 4.2 we have $[Sp(2), L] \cong [A, L] \oplus [S^{10}, L]$. Applying Corollary 5.3 gives us the following bijection

$$[Sp(2), L] \cong H[Sp(2), L] \oplus \pi_{10}(L). \tag{6.6}$$

Since $\pi_{10}(Sp(2)) \cong \mathbb{Z}_{120}$, and $\mathbb{Z}_{120}$ localised at 3 is $\mathbb{Z}_3$, by Lemma 5.4 and homotopy commutativity of $Sp(2)$ (6.5) is in fact a group isomorphism. □

Proposition 6.6. *Let L be a compact, simply-connected, simple Lie group. Suppose that L is homotopy commutative at the prime 5. Then we have a group isomorphism*

$$[G_2, L] \cong H[G_2, L] \oplus \pi_{14}(L). \tag{6.7}$$

Further, if L is an exceptional Lie group, we have a group isomorphism

$$[G_2, L] \cong H[G_2, L]. \tag{6.8}$$

Proof. We proceed similarly to the proof of Proposition 6.5. Assume throughout that homology is in mod-5 coefficients. At 5, we have $H_*(G_2) \cong \Lambda(x_3, x_{11})$ with $\mathscr{P}^1_*(x_{11}) = x_3$. Let A be the 11-skeleton of G_2.

Note that $\Sigma G_2 \simeq \Sigma A \vee \Sigma S^{14}$, so Theorem 4.2 tells us that $[G_2, L] \cong [A, L] \oplus [S^{14}, L]$. By Corollary 5.3, we have a bijection $[G_2, L] \cong H[G_2, L] \oplus \pi_{14}(L)$, and for $L = G_2, F_4, E_6, E_7, E_8$ we have $\pi_{14}(L) = 0$, so we indeed have a bijection $[G_2, L] \cong H[G_2, L]$.

By Theorem 6.1, since L is homotopy commutative these bijections are actually group isomorphisms. □

Note that Proposition 6.6 further generalises Lemma 4.16, which considered only self maps $[G_2, G_2]$.

Example 6.1. *At the prime 5, the following groups are homotopy commutative: G_2, $SU(2)$, $Spin(3)$ and $Spin(4)$. This gives us the following group isomorphisms:*

$$\begin{aligned}
[G_2, G_2] &\cong H[G_2, G_2] \\
[G_2, SU(2)] &\cong H[G_2, SU(2)] \oplus \pi_{14}(SU(2)) && \cong H[G_2, SU(2)] \oplus (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}_{84} \\
[G_2, Spin(3)] &\cong H[G_2, Spin(3)] \oplus \pi_{14}(Spin(3)) && \cong H[G_2, Spin(2)] \oplus (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}_{84} \\
[G_2, Spin(4)] &\cong H[G_2, Spin(4)] \oplus \pi_{14}(Spin(4)) && \cong H[G_2, Spin(4)] \oplus ((\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}_{84})^2.
\end{aligned}$$

Then since $(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_{84}$ is trivial when localised at 5, we have

$$\begin{aligned}
[G_2, G_2] &\cong H[G_2, G_2] \\
[G_2, SU(2)] &\cong H[G_2, SU(2)] \\
[G_2, Spin(3)] &\cong H[G_2, Spin(3)] \\
[G_2, Spin(4)] &\cong H[G_2, Spin(4)].
\end{aligned}$$

Chapter 7

Extension to the non-homotopy commutative case

Let L be a compact, simply-connected, simple Lie group. We consider maps $f : A_G \to L$, in the hope of finding something out about $[G, L]$. Note that we can use Theorem 3.8 to extend f to an H-map $\bar{f} : \Omega\Sigma A \to L$. Now, however, we will need to consider the Samelson products on G. Throughout, homology will be in $\mathbb{Z}/p\mathbb{Z}$ coefficients.

7.1 Homotopy nilpotency

Recall from Section 5.3 the map $\mu_i : S^{2m_i-1} \hookrightarrow A \xrightarrow{E} \Omega\Sigma A$. Define x_i to be the composite

$$x_i : S^{2m_i-1} \xrightarrow{\mu_i} \Omega\Sigma A \xrightarrow{\Omega j} G, \tag{7.1}$$

and note that since $\Omega j \simeq \bar{\imath}$ by Lemma 5.5 the composition x_i is, up to homotopy, the inclusion of the ith factor of the product. As Ωj is an H-map into a homotopy associative H-space, composing a Samelson product on $\Omega\Sigma A$ with Ωj gives a Samelson product on G. Thus $\Omega j \circ \langle \mu_i, \mu_j \rangle \simeq \langle x_i, x_j \rangle$.

The least dimensional p-torsion homotopy group of S^3 is $\pi_{2p}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$; denote its generator by $\alpha_1 : S^{2p} \to S^3$, and by abuse of notation let $\alpha_1 : S^{m+2p-3} \to S^m$ be the $(m-3)$-fold suspension of α_1 for $m \geq 3$. The following theorem of Kaji and Kishimoto [16] gives us information about the Samelson products of G.

Theorem 7.1. *Suppose that G is p-regular but not homotopy commutative. Then the following hold:*

1. *at least one Samelson product $\langle x_i, x_j \rangle$ is non-trivial;*

2. *if $\langle x_i, x_j \rangle$ is non-trivial, then it is homotopic to a non-zero multiple of the composite*

$$a_{i,j} : S^{2m_i+2m_j-2} \xrightarrow{\alpha_1} S^{2m_i+2m_j-2p+1} \to G, \tag{7.2}$$

and we write $\langle x_i, x_j \rangle \simeq z_{i,j} \cdot a_{i,j}$ for some $z_{i,j} \in \mathbb{Z}/p\mathbb{Z}$;

3. *the Samelson product $\langle x_t, \langle x_i, x_j \rangle \rangle$ is only non-trivial when $2m_t + 2m_i + 2m_j = 4p$;*

4. *if $\langle x_t, \langle x_i, x_j \rangle \rangle$ is non-trivial, then it is homotopic to a non-zero multiple of the composite*

$$a : S^{4p-3} \xrightarrow{\alpha_1} S^{2p} \xrightarrow{\alpha_1} S^3, \tag{7.3}$$

and we write $\langle x_t, \langle x_i, x_j \rangle \rangle \simeq z_{t,i,j} \cdot a$ for some $z_{t,i,j} \in \mathbb{Z}/p\mathbb{Z}$. □

By Theorem 7.1 there exists at least one non-trivial $\langle x_i, x_j \rangle$. Recalling the map λ in Section 5.3, each Samelson product $\langle \mu_i, \mu_j \rangle$ factors through $\Omega\lambda$ by definition, so the composition $\Omega\Sigma R \xrightarrow{\Omega\lambda} \Omega\Sigma A \xrightarrow{\bar{f}} L$ may be non-trivial, depending on L. Unlike in Theorem 6.1, we do not necessarily have an H-map extension $\hat{f} : G \to L$. Instead, we will use the above theorem together with some results on homotopy nilpotency (as introduced in Section 2.2) to modify λ, and obtain an extension. The following theorem of Kaji and Kishimoto [16] gives us values for the homotopy nilpotency of G.

Theorem 7.2. *Let G be of type $(m_1, \ldots, m_k)$. Then we have the following values for the homotopy nilpotency of G:*

1. *if $\frac{3}{2}m_k < p < 2m_k$ then $nil(G) = 2$;*

2. *if $m_k \leq p \leq \frac{3}{2}m_k$ then $nil(G) = 3$, unless (G, p) is one of $(F_4, 17)$, $(E_6, 17)$, $(E_8, 41)$ and $(E_8, 43)$;*

3. *if $m_k \leq p \leq \frac{3}{2}m_k$ and (G, p) is one of $(F_4, 17)$, $(E_6, 17)$, $(E_8, 41)$ and $(E_8, 43)$, then $nil(G) = 2$.* □

7.2 Nontrivial Samelson products

Recall that R was defined to be $R = \bigvee_{\ell=2}^{k+1} R_\ell$, where each R_ℓ is a wedge of spheres, and $\lambda = \bigvee_{\ell=1}^{k+1} \lambda_\ell$ where each λ_ℓ is a wedge sum of Whitehead products of length ℓ. By Theorem 7.2, $\mathrm{nil}(G) \leq 3$, so if $\ell \geq 4$ then the composite $R_\ell \xrightarrow{E} \Omega\Sigma R_\ell \xrightarrow{\Omega\lambda_\ell} \Omega\Sigma A \xrightarrow{\Omega j} G$ is null homotopic. Then for each $\ell \geq 4$, we have a lift

$$\begin{array}{ccc} & & R_\ell \\ & \swarrow^{\phi_\ell} & \downarrow{\scriptstyle \Omega\lambda_\ell \circ E} \\ \Omega Q & \xrightarrow{\Omega\gamma} & \Omega\Sigma A \end{array} \tag{7.4}$$

for some map ϕ_ℓ. It remains to consider $\ell = 2$ and $\ell = 3$.

For $\ell = 2$, consider the composition

$$R_2 \xrightarrow{E} \Omega\Sigma R_2 \xrightarrow{\Omega\lambda_2} \Omega\Sigma A \xrightarrow{\Omega j} G. \tag{7.5}$$

It is a wedge sum of Samelson products $\langle x_i, x_j \rangle$ for $i \geq j$, some of which may be null homotopic. By Theorem 7.1, at least one is not. Hence, define $R_{2,1}$ to be the wedge of spheres in R_2 whose corresponding Samelson products are null homotopic, and $R_{2,2}$ to be the wedge of spheres in R_2 with nontrivial Samelson products. Then we have $R_2 = R_{2,1} \vee R_{2,2}$, and the restriction of (7.5) to $R_{2,1}$ is null homotopic. We want to modify the restriction of $\Omega\lambda_2 \circ E$ to $R_{2,2}$ such that it will also be null homotopic when composed with Ωj.

Let $\langle x_i, x_j \rangle$ be a nontrivial Samelson product, and define the corresponding map $b_{i,j}$ by the composition

$$b_{i,j} : S^{2m_i+2m_j-2} \xrightarrow{\alpha_1} S^{2m_i+2m_j-2p+1} \hookrightarrow A \xrightarrow{E} \Omega\Sigma A. \tag{7.6}$$

Notice that $a_{i,j} = \Omega j \circ b_{i,j}$, where $a_{i,j}$ is from Theorem 7.1.

By Theorem 7.1, $\langle x_i, x_j \rangle \simeq z_{i,j} \cdot a_{i,j}$ for some $z_{i,j} \in \mathbb{Z}/p\mathbb{Z}$. Define a 'difference map' $d_{i,j} = \langle \mu_i, \mu_j \rangle - z_{i,j} \cdot b_{i,j}$, and observe that there are homotopies

$$\begin{aligned} \Omega j \circ d_{i,j} &= \Omega j \circ (\langle \mu_i, \mu_j \rangle - z_{i,j} \cdot b_{i,j}) && (7.7)\\ &\simeq \Omega j \circ \langle \mu_i, \mu_j \rangle - z_{i,j} \cdot \Omega j \circ b_{i,j} && (7.8)\\ &\simeq \langle x_i, x_j \rangle - z_{i,j} \cdot a_{i,j} && (7.9)\\ &\simeq *. && (7.10) \end{aligned}$$

Thus the composite $\Omega j \circ d_{i,j}$ is null homotopic, and $d_{i,j}$ lifts through $\Omega\gamma$.

For each summand of $R_{2,2}$, there is a difference map $d_{i,j}$. Let $D_2 : R_{2,2} \to \Omega\Sigma A$ be the wedge sum of all the $d_{i,j}$, and observe that $\Omega j \circ D_2$ is null homotopic. Now consider the map $\theta_2 : R_{2,1} \vee R_{2,2} \to \Omega\Sigma A$, defined to be the wedge sum of $\Omega\lambda_2 \circ E$ on $R_{2,1}$ and D_2 on $R_{2,2}$. We have a lift

$$\begin{array}{ccc} & & R_2 \\ & \swarrow^{\phi_2} & \downarrow{\scriptstyle \theta_2} \\ \Omega Q & \xrightarrow{\Omega\gamma} & \Omega\Sigma A \end{array} \tag{7.11}$$

for some map ϕ_2.

We proceed similarly for $\ell = 3$; let $R_{3,1}$ (respectively $R_{3,2}$) be the wedge of spheres in R_3 whose corresponding Samelson products compose trivially (nontrivially) with Ωj. Then $R_3 = R_{3,1} \vee R_{3,2}$. Observe that if $\mathrm{nil}(G) = 2$, then $R_{3,2} = *$. If $\mathrm{nil}(G) = 3$, then we use Theorem 7.1 to describe the nontrivial Samelson products.

Let $\langle x_t, \langle x_i, x_j \rangle\rangle$ be a nontrivial Samelson product. Then we have $2m_t + 2m_i + 2m_j = 4p$, and $\langle x_t, \langle x_i, x_j \rangle\rangle$ is homotopic to a nonzero multiple of $S^{4p-3} \xrightarrow{\alpha_1} S^{2p} \xrightarrow{\alpha_1} S^3$, so we may write $\langle x_t, \langle x_i, x_j \rangle\rangle \simeq z_{t,i,j} \cdot a$ for some $z_{t,i,j} \in \mathbb{Z}/p\mathbb{Z}$. Define $\bar{\alpha}$ to be the composite

$$\bar{\alpha} : S^{4p-3} \xrightarrow{\alpha_1} S^{2p} \xrightarrow{\alpha_1} S^3 \hookrightarrow A \xrightarrow{E} \Omega\Sigma A. \tag{7.12}$$

Then by definition of the maps x_i and the fact that Ωj is an H-map, $\langle x_t, \langle x_i, x_j \rangle\rangle$ is homotopic to $\Omega j \circ \langle \mu_t, \langle \mu_i, \mu_j \rangle\rangle$. We define a 'difference map'

$$d_{t,i,j} = \langle \mu_t, \langle \mu_i, \mu_j \rangle\rangle - z_{t,i,j} \cdot \bar{\alpha}. \tag{7.13}$$

Observe that there are homotopies

$$\begin{aligned}
\Omega j \circ d_{t,i,j} &= \Omega j \circ (\langle \mu_t, \langle \mu_i, \mu_j \rangle\rangle - z_{t,i,j} \cdot \bar{\alpha}) \\
&\simeq \Omega j \circ \langle \mu_t \langle \mu_i, \mu_j \rangle\rangle - z_{t,i,j} \cdot \Omega j \circ \bar{\alpha} \\
&\simeq \langle x_t, \langle x_i, x_j \rangle\rangle - z_{t,i,j} \cdot a \\
&\simeq *.
\end{aligned}$$

Collect the $d_{t,i,j}$ to obtain a map $D_3 : R_{3,2} \to \Omega\Sigma A$ such that $\Omega j \circ D_3 \simeq *$, and now define $\theta_3 : R_{3,1} \vee R_{3,2} \to \Omega\Sigma A$ to be the wedge sum of $\Omega\lambda_3 \circ E$ on $R_{3,1}$ and D_3 on $R_{3,2}$. Then, for some map ϕ_3, we obtain a lift

$$\begin{array}{ccc}
 & & R_3 \\
 & \overset{\phi_3}{\swarrow} & \downarrow \theta_3 \\
\Omega Q & \xrightarrow{\Omega\gamma} & \Omega\Sigma A.
\end{array} \tag{7.14}$$

Let θ be the wedge sum of maps θ_2, θ_3 and $\Omega\lambda_\ell \circ E$ for $4 \leq \ell \leq k+1$, and let ϕ be the wedge sum of maps ϕ_ℓ for $2 \leq \ell \leq k+1$. Taking adjoints, denote $\vartheta = ev \circ \Sigma\theta$ and $\varphi = ev \circ \Sigma\phi$, where ev is the evaluation map. The following result of Theriault [38] pulls the above construction together.

Theorem 7.3. *Let G and L be compact, simply-connected, simple Lie groups. The map $\varphi : \Sigma R \to Q$ is a homotopy equivalence and the following two diagrams homotopy commute:*

$$\begin{array}{ccc}
 & & R \\
 & \overset{\phi}{\swarrow} & \downarrow \theta \\
\Omega Q & \xrightarrow{\Omega\gamma} & \Omega\Sigma A
\end{array}
\qquad
\begin{array}{ccc}
 & & \Sigma R \\
 & \overset{\varphi}{\swarrow} & \downarrow \vartheta \\
Q & \xrightarrow{\gamma} & \Sigma A
\end{array} \tag{7.15}$$

Consequently, there is a homotopy fibration $\Sigma R \xrightarrow{\vartheta} \Sigma A \xrightarrow{j} BG$.

7.2.1 Conditions for $[A, L] \cong H[G, L]$

Recall that i was defined to be the composite $A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\Omega j} G$. The following theorem will give us a useful framework for proving later results on the classical Lie groups.

Theorem 7.4. *Let G and L be compact, simply-connected, simple Lie groups. Suppose that G and L are localised at an odd prime p such that G is p-regular but not homotopy commutative. Let $f : A \to L$ be a map, and let $\bar{f} : \Omega\Sigma A \to L$ be the induced H-map extension of Theorem 3.8. Suppose that the following compositions are null homotopic:*

$$\bar{f} \circ \langle \mu_i, \mu_j \rangle \qquad \bar{f} \circ b_{i,j} \qquad \bar{f} \circ \bar{\alpha}. \tag{7.16}$$

Then there exists a unique H-map $\hat{f} : G \to L$ extending f, such that $\hat{f} \circ i \simeq \bar{f}$.

Proof. By definition, θ is a wedge sum of maps $\Omega\lambda_\ell \circ E$ on $R_{2,1}$, $R_{3,1}$ and R_ℓ for $4 \leq \ell \leq k+1$, and maps D_2, D_3 on $R_{2,2}$ and $R_{3,2}$. Note that since $\bar{f} \circ \langle \mu_i, \mu_j \rangle$ is null homotopic and $\bar{f}$ is an H-map, the composition $\bar{f} \circ \langle \mu_t, \langle \mu_i, \mu_j \rangle \rangle$ is also null homotopic. Each $\Omega\lambda_\ell \circ E$ is a wedge sum of iterated Samelson products, obtained by bracketing with $\langle \mu_i, \mu_j \rangle$. By assumption $\bar{f} \circ \langle \mu_i, \mu_j \rangle$ is null homotopic, so as $\bar{f}$ is an H-map, $\bar{f} \circ \Omega\lambda_\ell \circ E$ is null homotopic for all ℓ. By definition, each map in D_2 is a difference $\langle \mu_i, \mu_j \rangle - b_{i,j}$ and each map in D_3 is a difference $\langle \mu_t, \langle \mu_i, \mu_j \rangle \rangle - \bar{\alpha}$. Since $\bar{f}$ is an H-map, it distributes on the left, so as each of $\bar{f} \circ \langle \mu_i, \mu_j \rangle \rangle$, $\hat{f} \circ b_{i,j}$ and $\hat{f} \circ \bar{\alpha}$ is null homotopic, the composites $\hat{f} \circ D_2$ and $\hat{f} \circ D_3$ are null homotopic. Combining this, the composition $R \xrightarrow{\theta} \Omega\Sigma A \xrightarrow{\bar{f}} L$ is null homotopic.

Consider the composite $\Omega\Sigma R \xrightarrow{\Omega\vartheta} \Omega\Sigma A \xrightarrow{\bar{f}} L$. Since $\Omega\vartheta$ and $\bar{f}$ are H-maps, so is the composite. Theorem 3.8 therefore implies that the homotopy class of $\bar{f} \circ \Omega\vartheta$ is determined by the homotopy class of $\bar{f} \circ \Omega\vartheta \circ E$. But by definition of ϑ we have $\Omega\vartheta \circ E = \theta$, and in the first paragraph it was shown that $\bar{f} \circ \theta$ is null homotopic. Therefore $\bar{f} \circ \Omega\vartheta$ is null homotopic.

By Theorem 7.3, there is a homotopy fibration $\Sigma R \xrightarrow{\vartheta} \Sigma A \xrightarrow{j} BG$. Looping, we obtain a homotopy fibration

$$\Omega\Sigma R \xrightarrow{\Omega\vartheta} \Omega\Sigma A \xrightarrow{\Omega j} G. \tag{7.17}$$

By Lemma 5.5, (A, i, G) is retractile. Define $\hat{f}$ by the composition $\hat{f} : G \xrightarrow{s} \Omega\Sigma A \xrightarrow{\bar{f}} L$, and observe that by Proposition 5.2 there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\Sigma A & \xrightarrow{\bar{f}} & L \\ \big\downarrow{\scriptstyle \Omega j} & \nearrow{\scriptstyle \hat{f}} & \\ G & & \end{array} \tag{7.18}$$

where $\hat{f}$ is an H-map and it is the unique H-map, up to homotopy, such that $\hat{f} \circ i \simeq f$. □

Corollary 7.5. *Suppose that for any $f : A \to L$, the H-map extension $\bar{f} : \Omega\Sigma A \to L$ from Theorem 3.8 has the property that the compositions in (7.16) are null homotopic. Then there is a one-to-one correspondence $[A, L] \cong H[G, L]$, sending $f : A \to L$ to its H-map extension $\hat{f} : G \to L$.* □

7.2.2 Results on the classical Lie groups

Theorem 7.6. *Let $2m \leq n$, and consider $H[SU(m), SU(n)]$. Localised at a prime p such that $m \leq p < n$, there is a bijection*

$$H[SU(m), SU(n)] \cong \left[\bigvee_{i=1}^{m-1} S^{2m_i-1}, SU(n)\right] \tag{7.19}$$

$$\cong \pi_3(SU(n)) \times \pi_5(SU(n)) \times \ldots \times \pi_{2m-1}(SU(n)). \tag{7.20}$$

Proof. Note that the condition $m \leq p < n$ means that $SU(m)$ is p-regular, and $SU(n)$ is not. Let $A = \Sigma\mathbb{C}P^{m-1}$, and note that $A \simeq S^3 \vee \ldots \vee S^{2m-1}$. Let $f : A \to SU(n)$. We will prove that each of the compositions in (7.16) is null homotopic.

Step 1: The composition $\bar{f} \circ b_{i,j}$ is null homotopic.

The map $b_{i,j}$ is defined by

$$b_{i,j} : S^{2m_i+2m_j-2} \xrightarrow{\alpha_1} S^{2m_i+2m_j-2p+1} \hookrightarrow A \xrightarrow{E} \Omega\Sigma A. \tag{7.21}$$

Thus $\bar{f} \circ b_{i,j}$ represents a map in $\pi_{2m_i+2m_j-2}(SU(n))$. Since $m_i \leq m$ and $2m \leq n$, we have $2m_i + 2m_j - 2 \leq 2n - 2$, and by Bott periodicity all homotopy groups of the form $\pi_{2k}(SU(n))$ are trivial for $2 \leq 2k < 2n$. Hence $\hat{f} \circ b_{i,j}$ is null homotopic.

Step 2: The composition $\bar{f} \circ \bar{\alpha}$ is null homotopic.

By definition, the map $\bar{\alpha}$ is the composite

$$\bar{\alpha} : S^{4p-3} \xrightarrow{\alpha_1} S^{2p} \xrightarrow{\alpha_1} S^3 \hookrightarrow A \xrightarrow{E} \Omega\Sigma A. \tag{7.22}$$

This map factors through S^{2p}, and by Bott periodicity $\pi_{2k}(SU(n))$ is trivial for $2 \leq 2k < 2n$. Thus, since $p < n$, $\pi_{2p}(SU(n))$ is trivial and $\bar{f} \circ \bar{\alpha}$ is null homotopic.

Step 3: The composition $\bar{f} \circ \langle \mu_i, \mu_j \rangle$ is null homotopic.

We would like the composition

$$S^{2m_i-1} \wedge S^{2m_j-1} \xrightarrow{\langle \mu_i, \mu_j \rangle} \Omega\Sigma A \xrightarrow{\bar{f}} SU(n) \tag{7.23}$$

to be null homotopic. Let $\gamma : SU(n) \to SU(\infty)$, and consider $\gamma_* : [X, SU(n)] \to [X, SU(\infty)]$, where $\dim(X) \leq 2m_i + 2m_j - 2 \leq 2m - 2$.

The fibre of γ is $\Omega(SU(\infty)/SU(n)) \simeq S^{2n+1} \cup \cdots$, implying that γ_* is an isomorphism if $\dim(X) \leq 2n + 1$. But we have assumed that $\dim(X) \leq 2m - 2 < 2n + 1$, and so γ_* is indeed an isomorphism.

Thus, the homotopy class of $\bar{f} \circ \langle \mu_i, \mu_j \rangle$ is determined by the homotopy class of $\gamma \circ \bar{f} \circ \langle \mu_i, \mu_j \rangle$. Then since γ and $\bar{f}$ are H-maps, the composition $\gamma \circ \bar{f} \circ \langle \mu_i, \mu_j \rangle$ is a Samelson product $S^{2m_i - 1} \wedge S^{2m_j - 1} \to SU(\infty)$. Since $SU(\infty)$ is an infinite loop space, it is homotopy commutative. Thus, the composition $\gamma \circ \bar{f} \circ \langle \mu_i, \mu_j \rangle$ is null homotopic.

The conditions of Theorem 7.4 have been met, so the map $f : A \to SU(n)$ can be extended to a unique H-map $\hat{f} : SU(m) \to SU(n)$. As this is true for any map $f : A \to SU(n)$, by Corollary 7.5 we have a bijection $H[SU(m), SU(n)] \cong [A, SU(n)]$. Since A is homotopy equivalent to a wedge of spheres $S^3 \vee S^5 \ldots \vee S^{2m-1}$, we indeed have

$$H[SU(m), SU(n)] \cong \pi_3(SU(n)) \times \pi_5(SU(n)) \times \ldots \times \pi_{2m-1}(SU(n)). \tag{7.24}$$

□

Theorem 7.7. *Let $4m \leq n$, and consider $H[Sp(m), Sp(n)]$. Localised at a prime p such that $2m \leq p < 2n$, there is a bijection*

$$H[Sp(m), Sp(n)] \cong \left[\bigvee_{i=1}^{2m} S^{2m_i - 1}, Sp(n) \right] \tag{7.25}$$

$$\cong \pi_3(Sp(n)) \times \pi_7(Sp(n)) \ldots \times \pi_{4m-1}(Sp(n)). \tag{7.26}$$

Proof. Note that the condition $2m \leq p < 2n$ means that $Sp(m)$ is p-regular, and $Sp(n)$ is not. Let $A_{Sp(m)} = S^3 \vee S^7 \vee \ldots \vee S^{4m-1}$, and let $f : A_{Sp(m)} \to Sp(n)$. We will prove that each of the compositions in (7.16) is null homotopic.

Step 1: The composition $\bar{f} \circ b_{i,j}$ is null homotopic.

The map $b_{i,j}$ is defined by

$$b_{i,j} : S^{2m_i + 2m_j - 2} \xrightarrow{\alpha_1} S^{2m_i + 2m_j - 2p + 1} \hookrightarrow A_{Sp(m)} \xrightarrow{E} \Omega\Sigma A_{Sp(m)}. \tag{7.27}$$

Thus $\hat{f} \circ b_{i,j}$ represents a map in $\pi_{2m_i + 2m_j - 2}(Sp(n))$. Localised at an odd prime, $Sp(n)$ retracts off $SU(2n)$. There is a $\mathbb{Z}$ summand in $\pi_{4j-1}(Sp(n))$ for $1 \leq j \leq n$, and $\pi_k(Sp(n))$ is trivial for $k < 4n - 1$ otherwise. Since $2m_i + 2m_j - 2$ is not of the form $4j - 1$, we see that $\hat{f} \circ b_{i,j}$ is null homotopic.

Step 2: The composition $\bar{f} \circ \bar{\alpha}$ is null homotopic.

The map $\bar{\alpha}$ factors through S^{2p}, so we may again use the retraction of $Sp(n)$ off $SU(2n)$ to see that $\pi_{2p}(Sp(n))$ is trivial, and so $\bar{f} \circ \bar{\alpha}$ is null homotopic.

Step 3: The composition $\bar{f} \circ \langle \mu_i, \mu_j \rangle$ is null homotopic.

We would like the composition

$$S^{2m_i-1} \wedge S^{2m_j-1} \xrightarrow{\langle \mu_i, \mu_j \rangle} \Omega\Sigma A \xrightarrow{\bar{f}} Sp(n) \tag{7.28}$$

to be null homotopic. Let $\gamma : Sp(n) \to Sp(\infty)$, and consider $\gamma_* : [X, Sp(n)] \to [X, Sp(\infty)]$, where $\dim(X) \leq 2m_i + 2m_j - 2 \leq 4m - 2$.

The fibre of γ is $\Omega(Sp(\infty)/Sp(n)) \simeq S^{4n+3} \cup \cdots$, implying that γ_* is an isomorphism if $\dim(X) \leq 4n + 3$. But we have assumed that $\dim(X) \leq 4m - 2 < 4n + 3$, and so γ_* is indeed an isomorphism.

Thus, the homotopy class of $\bar{f} \circ \langle \mu_i, \mu_j \rangle$ is determined by the homotopy class of $\gamma \circ \bar{f} \circ \langle \mu_i, \mu_j \rangle$. Then since γ and $\bar{f}$ are H-maps, the composition $\gamma \circ \bar{f} \circ \langle \mu_i, \mu_j \rangle$ is a Samelson product $S^{2m_i-1} \wedge S^{2m_j-1} \to Sp(\infty)$. When $Sp(n)$ is mapped into $Sp(\infty)$, it behaves as though it is homotopy commutative (note the similarity to how Sp(m) behaves as though it is homotopy commutative when mapped into $Sp(n)$). Thus, the composition $\gamma \circ \bar{f} \circ \langle \mu_i, \mu_j \rangle$ is null homotopic.

We have now met the conditions of Theorem 7.4, and thus the map $f : A \to Sp(n)$ can be extended to a unique H-map $\hat{f} : Sp(m) \to Sp(n)$. As this is true for any map $f : A \to Sp(n)$, by Corollary 7.5 we have a bijection $H[Sp(m), Sp(n)] \cong [A, Sp(n)]$. Since $Sp(m)$ is p-regular of type $(2, 4, \ldots, 2m)$, A is a wedge of spheres $S^3 \vee S^7 \vee \ldots \vee S^{4m-1}$, and so we indeed have

$$H[Sp(m), Sp(n)] \cong \pi_3(Sp(n) \times \pi_7(Sp(n)) \times \ldots \times \pi_{4m-1}(Sp(n)). \tag{7.29}$$

□

Corollary 7.8. *Consider $H[Spin(m), Spin(n)]$, such that m and n are both odd and $4m - 3 \leq n$. Localised at a prime p such that $2m - 1 \leq p < 2n - 1$, we have*

$$H[Spin(m), Spin(n)] \cong [A(Spin(m)), Spin(n)] \tag{7.30}$$
$$\cong \pi_3(Spin(n)) \times \pi_7(Spin(n)) \ldots \times \pi_{4m-1}(Spin(n)). \tag{7.31}$$

Proof. For odd m, Friedlander [5] gave a homotopy equivalence

$$Spin(m) \simeq Sp\left(\frac{m-1}{2}\right). \tag{7.32}$$

This means that in the case where m is odd, $A_{Spin(m)} = A_{Sp(\frac{m-1}{2})}$. Throughout, we will take $m_i = 2i$, so that $A_{Sp(m)} = \bigvee_{i=1}^{2m} S^{2m_i-1}$.

Consider $H[Sp(\frac{m-1}{2}), Sp(\frac{n-1}{2})]$. Let $M = \frac{m-1}{2}$ and $N = \frac{n-1}{2}$. Suppose that $4m - 3 \leq n$, and observe that this implies that

$$4(m-1)+1 \leq n \implies 4(m-1) \leq n-1 \tag{7.33}$$

$$\implies 4\left(\frac{m-1}{2}\right) \leq \frac{n-1}{2} \tag{7.34}$$

$$\implies 4M \leq N. \tag{7.35}$$

Then we have satisfied the conditions of Theorem 7.7, and there is hence a bijection

$$H[Sp(M), Sp(N)] \cong \left[\bigvee_{i=1}^{2M} S^{2m_i-1}, Sp(N)\right] \tag{7.36}$$

or, equivalently, there is a bijection

$$H[Spin(m), Spin(n)] \cong \left[\bigvee_{i=1}^{m-1} S^{2m_i-1}, Spin(n)\right] \tag{7.37}$$

$$\cong \pi_3(Spin(n)) \times \pi_7(Spin(n)) \ldots \times \pi_{4m-1}(Spin(n)). \tag{7.38}$$

□

Since in the above theorems G is p-regular, $[G, L]$ also decomposes as a product of homotopy groups of L and so we may compare these decompositions to discern whether $[G, L] \cong H[G, L]$.

Chapter 8

A group isomorphism

Recall the map $\Phi : [A, L] \to [G, L]$ that sends a map $f : A \to L$ to its H-map extension $\hat{f} : G \to L$. As we have seen in Section 7.2.2, Φ is a bijection when

- $G = SU(m)$, $L = SU(n)$ for $2m \leq n$, $m \leq p < n$;
- $G = Sp(m)$, $L = Sp(n)$ for $4m \leq n$, $2m \leq p < 2n$;
- $G = Spin(m)$, $L = Spin(n)$ for $4m - 3 \leq n$, $2m - 1 \leq p < 2n - 1$ with m and n odd.

These are the conditions given in Theorems 7.6 - 7.8. We now consider when, given these conditions, the map Φ is also a group homomorphism (and hence an isomorphism). In the setting of Chapter 5, this was easily done using Lemma 5.4; however this relies heavily on the homotopy commutativity of L, which we no longer assume.

However, we still might expect Φ to be a group homomorphism here, due to the properties exploited in the previous section. The strategy has been to take L large enough compared to G that it 'seems homotopy commutative' from the viewpoint of G. That is, we take L large enough that non-trivial Samelson products in G are killed off by mapping into L. Our hope is that this will also mean that Φ respects the group operation, although a different method will be needed than that of Lemma 5.4.

8.1 The general case

Let X and Y be path-connected spaces, and let Z be a path-connected, homotopy associative H-space. Then by the Hilton-Milnor Theorem 3.11, we have a homotopy equivalence

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \prod_{\alpha \in \mathcal{I}} (\Omega \Sigma X^{\wedge \alpha_1} \wedge Y^{\alpha_2}), \tag{8.1}$$

where $\mathcal{I}$ runs over a module basis for the free Lie algebra $L\langle u, v\rangle$ on two generators, and $\alpha \in \mathcal{I}$ corresponds to an iterated bracket in which α_1, α_2 respectively record the number of instances of u and v that appear in α.

We may realise this homotopy equivalence in the following way. Let i_1, i_2 be the inclusions of the left and right summands respectively,

$$i_1 : \Sigma X \to \Sigma X \vee \Sigma Y \tag{8.2}$$

$$i_2 : \Sigma Y \to \Sigma X \vee \Sigma Y. \tag{8.3}$$

For $\alpha \in \mathcal{I}$, let

$$w_\alpha : \Sigma X^{\wedge \alpha_1} \wedge Y^{\wedge \alpha_2} \to \Sigma X \vee \Sigma Y \tag{8.4}$$

be the Whitehead product formed by replacing each instance of u and v in α with i_1 and i_2 respectively. Then the map

$$e : \prod_{\alpha \in \mathcal{I}} \Omega(\Sigma X^{\wedge \alpha_1} \wedge Y^{\wedge \alpha_2}) \to \Omega(\Sigma X \vee \Sigma Y) \tag{8.5}$$

formed by taking the product of the maps Ωw_α is a homotopy equivalence.

Observe that the factors corresponding to the Lie algebra basis elements u and v themselves are $\Omega\Sigma X$ and $\Omega\Sigma Y$, and the restriction of e to these factors is Ωi_1 and Ωi_2 respectively. Let $\mathcal{I}' = \mathcal{I} - \{u, v\}$ and let e' be the restriction of e to the factors indexed by $\mathcal{I}'$.

Lemma 8.1. *Let W be a homotopy associative H-space and suppose that*

$$\theta : \Omega(\Sigma X \vee \Sigma Y) \to W \tag{8.6}$$

is an H-map. If $\theta \circ \Omega[i_1, i_2]$ is null homotopic, then $\theta \circ e'$ is null homotopic.

Proof. Recall that e' is a product of maps Ωw_α, where each α is in $\mathcal{I}'$ and so corresponds to a generic bracket involving both u and v. In particular, it expands on $[u, v]$, which corresponds to $[i_1, i_2]$.

Consider the composition $\theta \circ \Omega w_\alpha$. By hypothesis θ is an H-map and W is homotopy associative, so by the James Construction (Theorem 3.8), in order to show that $\theta \circ \Omega w_\alpha$ is null homotopic it suffices to show that $\theta \circ \Omega w_\alpha \circ E$ is null homotopic. That is, we wish to show that the composition

$$X^{\wedge \alpha_1} \wedge Y^{\wedge \alpha_2} \xrightarrow{E} \Omega(\Sigma X^{\wedge \alpha_1} \wedge Y^{\wedge \alpha_2}) \xrightarrow{\Omega w_\alpha} \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\theta} W \tag{8.7}$$

is null homotopic.

Note that $\Omega w_\alpha \circ E$ is the Samelson product s_α that is adjoint to w_α. Define maps μ and ν by the following compositions, so that μ and ν are adjoint to i_1 and i_2 respectively.

$$\mu : X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega i_1} \Omega(\Sigma X \vee \Sigma Y), \tag{8.8}$$

$$\nu : Y \xrightarrow{E} \Omega\Sigma Y \xrightarrow{\Omega i_2} \Omega(\Sigma X \vee \Sigma Y). \tag{8.9}$$

Then we may form the Samelson product $s_\alpha(\mu, \nu)$. As θ is an H-map, we have

$$\theta \circ s_\alpha(\mu, \nu) \simeq s_\alpha(\theta \circ \mu, \theta \circ \nu). \tag{8.10}$$

By hypothesis, $\theta \circ \Omega[i_1, i_2]$ is null homotopic. Then we also have $\theta \circ \Omega[i_1, i_2] \circ E \simeq *$, and this is homotopic to $\theta \circ \langle \mu, \nu \rangle$. Then, as $\theta \circ \langle \mu, \nu \rangle$ is homotopic to $\langle \theta \circ \mu, \theta \circ \nu \rangle$, we see that the latter Samelson product is also null homotopic.

For all $\alpha \in \mathcal{I}'$, we have that $\theta \circ \Omega w_\alpha$ is null homotopic. As θ is an H-map, $\theta \circ e'$ is homotopic to the product of the maps $\theta \circ \Omega w_\alpha$ for $\alpha \in \mathcal{I}'$. Hence $\theta \circ e'$ is null homotopic. □

The Hilton-Milnor Theorem 3.11 implies the following.

Lemma 8.2. *There is a homotopy fibration*

$$\prod_{\alpha \in \mathcal{I}'} \Omega(\Sigma X^{\wedge \alpha_1} \wedge Y^{\wedge \alpha_2}) \xrightarrow{e'} \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\Omega i} \Omega\Sigma X \times \Omega\Sigma Y \tag{8.11}$$

where i is the inclusion of the wedge into the product.

Proof. Define the space F by the homotopy fibration

$$F \to \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\Omega i} \Omega\Sigma X \times \Omega\Sigma Y. \tag{8.12}$$

Consider the following diagram:

$$\begin{array}{ccccc} & & \prod_{\alpha \in \mathcal{I}'} \Omega(\Sigma X^{\wedge \alpha_1} \wedge Y^{\wedge \alpha_2}) & & \\ & \swarrow^{\lambda} & \downarrow e' & & \\ F & \longrightarrow & \Omega(\Sigma X \vee \Sigma Y) & \xrightarrow{\Omega i} & \Omega X \times \Omega Y. \end{array} \tag{8.13}$$

We would like to find a lift λ of e', $\lambda : \prod_{\alpha \in \mathcal{I}'} \Omega(\Sigma X^{\wedge \alpha_1} \wedge Y^{\wedge \alpha_2}) \to F$.

Consider the special case $\Omega(\Sigma X \wedge \Sigma Y) \xrightarrow{\Omega[i_1, i_2]} \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\Omega i} \Omega\Sigma X \times \Omega\Sigma Y$. Before looping, this is the cofibration

$$\Sigma X \wedge Y \xrightarrow{[i_1, i_2]} \Sigma X \vee \Sigma Y \xrightarrow{i} \Sigma X \times \Sigma Y, \tag{8.14}$$

and is hence null homotopic. This implies that $i \circ [i_1, i_2] \simeq *$, so $\Omega i \circ \Omega[i_1, i_2] \simeq *$ and then by Lemma 8.1 $\Omega i \circ e'$ is null homotopic.

Now recall that e' is a restriction of the homotopy equivalence e, leaving out the two factors $\Omega\Sigma X$ and $\Omega\Sigma Y$. These correspond to the loops on the inclusions $i_1 : \Sigma X \hookrightarrow \Sigma X \vee \Sigma Y$ and $i_2 : \Sigma Y \hookrightarrow \Sigma X \vee \Sigma Y$ respectively. Once looped, we can then multiply to get a homotopy equivalence defined by the composite

$$\Omega\Sigma X \times \Omega\Sigma Y \xrightarrow{\Omega i_1 \cdot \Omega i_2} \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\Omega i} \Omega\Sigma X \times \Omega\Sigma Y. \tag{8.15}$$

Thus, since e is a homotopy equivalence, F has the homotopy type of $\prod_{\alpha \in \mathcal{I}'} (\Omega\Sigma X^{\wedge\alpha_1} \wedge Y^{\wedge\alpha_2})$. We now have the following homotopy commutative diagram, where the bottom map is a left homotopy inverse for e' (which we have since e is a homotopy equivalence):

$$\begin{array}{ccccc} & & \prod_{\alpha\in\mathcal{I}'} \Omega(\Sigma X^{\wedge\alpha_1} \wedge Y^{\wedge\alpha_2}) & & \\ & \swarrow^{\lambda} & \downarrow{\scriptstyle e'} & \searrow^{=} & \\ F & \longrightarrow & \Omega(\Sigma X \vee \Sigma Y) & \longrightarrow & \prod_{\alpha\in\mathcal{I}'} \Omega(\Sigma X^{\wedge\alpha_1} \wedge Y^{\wedge\alpha_2}) \end{array} \tag{8.16}$$

Hence $\prod_{\alpha\in\mathcal{I}'} \Omega(\Sigma X^{\wedge\alpha_1} \wedge Y^{\wedge\alpha_2})$ retracts off F; but since they have the same homotopy type, λ must be a homotopy equivalence. Therefore we indeed have a homotopy fibration as stated in (8.11). □

For $f \in [X, Z]$ and $g \in [Y, Z]$, where Z is assumed to be homotopy associative, let h denote the composite

$$h : X \vee Y \xrightarrow{f \vee g} Z \vee Z \to Z \times Z. \tag{8.17}$$

By Theorem 3.8, we may extend h to an H-map $\bar{h} : \Omega\Sigma(X \vee Y) \to Z \times Z$.

Lemma 8.1 immediately implies the following.

Corollary 8.3. *Suppose that the composite $\bar{h} \circ \Omega[i_1, i_2]$ is null homotopic. Then the composition*

$$\prod_{\alpha\in\mathcal{I}'} \Omega(\Sigma X^{\wedge\alpha_1} \wedge Y^{\wedge\alpha_2}) \xrightarrow{e'} \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\bar{h}} Z \times Z \tag{8.18}$$

is null homotopic. □

We specialise now to the case where $X = Y = A$ and $Z = L$.

Given maps $f, g : A \to L$, we aim to construct a homotopy commutative diagram

$$\begin{array}{ccccccc}
A \vee A & \xrightarrow{f \vee g} & L \vee L & \longrightarrow & L \times L & \xrightarrow{m} & L \\
\downarrow{\scriptstyle E} & & & & & \nearrow_{\gamma} & \\
\Omega\Sigma(A \vee A) & & & & & & \\
\downarrow{\scriptstyle \Omega i} & & & & & \nearrow_{\bar{\gamma}} & \\
\Omega\Sigma A \times \Omega\Sigma A & & & & & & \\
\downarrow{\scriptstyle r \times r} & & & & & \nearrow_{\hat{\gamma}} & \\
G \times G & & & & & &
\end{array} \tag{8.19}$$

for some H-maps γ, $\bar{\gamma}$ and $\hat{\gamma}$. We do this through the following series of lemmas.

Corollary 8.4. *There is a homotopy commutative diagram*

$$\begin{array}{ccccccc}
A \vee A & \xrightarrow{f \vee g} & L \vee L & \longrightarrow & L \times L & \xrightarrow{m} & L \\
\downarrow{\scriptstyle E} & & & & & \nearrow_{\gamma} & \\
\Omega\Sigma(A \vee A) & & & & & &
\end{array} \tag{8.20}$$

where γ is an H-map.

Proof. By the James construction (Theorem 3.8), the composition

$$h : A \vee A \xrightarrow{f \vee g} L \vee L \to L \times L \xrightarrow{m} L \tag{8.21}$$

extends to an H-map $\gamma : \Omega\Sigma(A \vee A) \to L$, which is the unique H-map such that $\gamma \circ E \simeq h$. □

Lemma 8.5. *Suppose that the composite $\Omega(\Sigma A \wedge A) \xrightarrow{\Omega[i_1, i_2]} \Omega(\Sigma A \vee \Sigma A) \xrightarrow{\gamma} L$ is null homotopic. Then there is a homotopy commutative diagram*

$$\begin{array}{ccc}
\Omega\Sigma(A \vee A) & \xrightarrow{\gamma} & L \\
\downarrow{\scriptstyle \Omega i} & \nearrow_{\bar{\gamma}} & \\
\Omega\Sigma A \times \Omega\Sigma A & &
\end{array} \tag{8.22}$$

where $\bar{\gamma}$ is an H-map.

Proof. By Corollary 8.4, γ is an H-map. Then we may use Lemma 8.1 to find that the composite

$$\Omega(\Sigma A^{\wedge \alpha_1} \wedge A^{\wedge \alpha_2}) \xrightarrow{e'} \Omega(\Sigma A \vee \Sigma A) \xrightarrow{\gamma} L \tag{8.23}$$

is null homotopic.

By Lemma 8.2, we have a fibration

$$\prod_{\alpha\in\mathcal{I}'}\Omega(\Sigma A^{\wedge\alpha_1}\wedge A^{\wedge\alpha_2})\xrightarrow{e'}\Omega\Sigma(A\vee A)\xrightarrow{\Omega i}\Omega\Sigma A\times\Omega\Sigma A. \tag{8.24}$$

Then, since $\gamma\circ e'$ is null homotopic, we may employ Proposition 5.2 to yield an H-map $\bar{\gamma}:\Omega\Sigma A\times\Omega\Sigma A\to L$ extending γ, as in (8.22). □

Define $\bar{f}:\Omega\Sigma A\to L$ to be the H-map extension of f, as given by the James construction. Then $\bar{f}$ is the unique H-map such that $\bar{f}\circ E\simeq f$. Define $\bar{g}:\Omega\Sigma A\to L$ similarly.

Lemma 8.6. *The H-map $\bar{\gamma}$ in Lemma 8.5 is homotopic to $m\circ(\bar{f}\times\bar{g})$.*

Proof. As $\bar{\gamma}$ is an H-map it is determined by its restriction to each factor $\Omega\Sigma A$. Let $\bar{\gamma}_1:\Omega\Sigma A\to L$ and $\bar{\gamma}_2:\Omega\Sigma A\to L$ be the restrictions to the left and right factors respectively. Since γ is an H-map, so are $\bar{\gamma}_1$ and $\bar{\gamma}_2$. Then, since $\bar{\gamma}_1$ is an H-map, the James construction (Theorem 3.8) implies that the homotopy class of $\bar{\gamma}_1$ is determined by the composite $A\xrightarrow{E}\Omega\Sigma A\xrightarrow{\bar{\gamma}_1}L$.

By Corollary 8.4 and Lemma 8.5, this is homotopic to the composite

$$A\hookrightarrow A\vee A\xrightarrow{f\vee g}L\vee L\hookrightarrow L\times L\xrightarrow{m}L, \tag{8.25}$$

which when considering only the first factor is homotopic to f. Then $E\circ\bar{\gamma}_1\simeq f$, which by the uniqueness property of the James construction implies that $\bar{\gamma}_1\simeq\bar{f}$. Similarly, $\bar{\gamma}_2\simeq\bar{g}$, and we see that $\bar{\gamma}\simeq m\circ(\bar{f}\times\bar{g})$. □

Lemma 8.7. *Suppose that (A,i,G) is a retractile triple, and there is a homotopy fibration*

$$\Omega R\xrightarrow{\Omega\rho}\Omega\Sigma A\xrightarrow{r}G \tag{8.26}$$

with the property that $\bar{f}\circ\Omega\rho$ and $\bar{g}\circ\Omega\rho$ are null homotopic. Then there is a homotopy commutative diagram

$$\begin{array}{ccc}\Omega\Sigma A\times\Omega\Sigma A & \xrightarrow{\bar{\gamma}} & L\\ \downarrow{\scriptstyle r\times r} & \nearrow{\scriptstyle\hat{\gamma}} & \\ G\times G & & \end{array} \tag{8.27}$$

where $\hat{\gamma}$ is an H-map.

Proof. There is a homotopy fibration

$$\Omega R\times\Omega R\xrightarrow{\Omega\rho\times\Omega\rho}\Omega\Sigma A\times\Omega\Sigma A\xrightarrow{r\times r}G\times G. \tag{8.28}$$

By hypothesis, $\bar{f}\circ\Omega\rho$ and $\bar{g}\circ\Omega\rho$ are null homotopic. By Lemma 8.6, $\bar{\gamma}\simeq m\circ(\bar{f}\times\bar{g})$, and by Lemma 8.5 $\bar{\gamma}$ is an H-map, implying that $\bar{\gamma}\circ(\Omega\rho\times\Omega\rho)$ is null homotopic.

Since we have a homotopy fibration (8.28) and the composition

$$\Omega R \times \Omega R \xrightarrow{\Omega\rho\times\Omega\rho} \Omega\Sigma A \times \Omega\Sigma A \xrightarrow{\bar{\gamma}} L \tag{8.29}$$

is null homotopic, Proposition 5.2 tells us that the diagram (8.27) homotopy commutes, and that $\hat{\gamma}$ is an H-map. $\square$

By Theorem 5.7, the map r has a section $s : G \to \Omega\Sigma A$. Define $\hat{f}$ and $\hat{g}$ to be the composites

$$\hat{f} : G \xrightarrow{s} \Omega\Sigma A \xrightarrow{\bar{f}} L \tag{8.30}$$

$$\hat{g} : G \xrightarrow{s} \Omega\Sigma A \xrightarrow{\bar{g}} L. \tag{8.31}$$

Lemma 8.8. *Given the hypotheses of Lemmas 8.5 and 8.7, the H-map $\hat{\gamma}$ is homotopic to $m \circ (\hat{f} \times \hat{g})$.*

Proof. As $s \circ r$ is homotopic to the identity map on G and $\gamma \simeq \hat{\gamma} \circ (r \times r)$, the definitions of $\hat{f}$ and $\hat{g}$ yield

$$\begin{aligned} \hat{\gamma} &\simeq \hat{\gamma} \circ (\Omega j \times \Omega j) \circ (s \times s) && (8.32)\\ &\simeq \gamma \circ (s \times s) && (8.33)\\ &\simeq m \circ (\bar{f} \times \bar{g}) \circ (s \times s) && (8.34)\\ &\simeq m \circ (\hat{f} \times \hat{g}). && (8.35) \end{aligned}$$

In particular, as $\hat{\gamma}$ is an H-map so is $m \circ (\hat{f} \times \hat{g})$. $\square$

Lemma 8.9. *There is a homotopy commutative diagram*

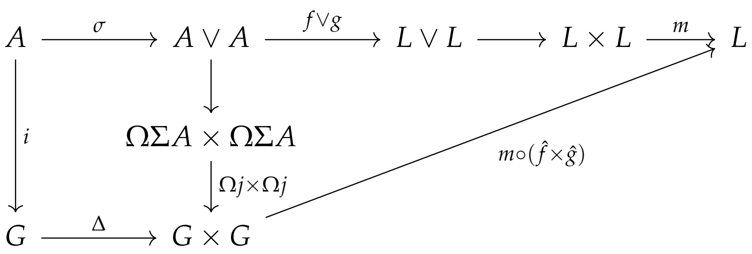
$$\begin{array}{ccc} A & \xrightarrow{\sigma} & A \vee A \\ \Big\downarrow E & & \Big\downarrow E \\ & & \Omega(\Sigma A \vee \Sigma A) \\ & & \Big\downarrow \Omega i \\ \Omega\Sigma A & \xrightarrow{\Delta} & \Omega\Sigma A \times \Omega\Sigma A. \end{array} \tag{8.36}$$

Proof. Observe that when $A \vee A \xrightarrow{E} \Omega\Sigma(A \vee A) \xrightarrow{\Omega i} \Omega\Sigma A \times \Omega\Sigma A$ is restricted to the left copy of A, one obtains $A \xrightarrow{E} \Omega\Sigma A \xrightarrow{i_1} \Omega\Sigma A \times \Omega\Sigma A$ where i_1 is the inclusion of the first factor, and similarly the restriction to the right copy of A is $A \xrightarrow{E} \Omega\Sigma A \xrightarrow{i_2} \Omega\Sigma A \times \Omega\Sigma A$ where i_2 is the inclusion of the second factor. Thus $\Omega i \circ E$ is homotopic to

$$A \vee A \to A \times A \xrightarrow{E\times E} \Omega\Sigma A \times \Omega\Sigma A. \tag{8.37}$$

Next consider the composite $A \xrightarrow{\sigma} A \vee A \to A \times A$. Since σ is a comultiplication this is homotopic to the diagonal map. Thus the upper direction around Diagram (8.36) is homotopic to $A \xrightarrow{\Delta} A \times A \xrightarrow{E \times E} \Omega\Sigma A \times \Omega\Sigma A$. The naturality of the diagonal implies that this is homotopic to the composite $A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\Delta} \Omega\Sigma A \times \Omega\Sigma A$, which is the lower direction around the Diagram (8.36). Hence the diagram homotopy commutes. □

Proposition 8.10. *Suppose that there are maps $f, g : A \to L$ where L is a homotopy associative H-space. Given hypotheses as in Lemmas 8.5 and 8.7, there is a homotopy commutative diagram*



(8.38)

Proof. The homotopy commutativity of the left rectangle follows from Lemma 8.9. The top triangle homotopy commutes by Lemmas 8.5 and 8.6, and the bottom triangle homotopy commutes by Lemmas 8.7 and 8.8. Thus the diagram as a whole is homotopy commutative. □

Recall the map $\Phi : [A, L] \to H[G, L]$ sending f to $\hat{f}$.

Corollary 8.11. *Suppose that there are maps $f, g : A \to L$ where L is a homotopy associative H-space. Given hypotheses as in Lemmas 8.5 and 8.7, the map Φ has the property that*

$$\Phi(f + g) \simeq \Phi(f) + \Phi(g). \tag{8.39}$$

Proof. Consider the homotopy commutative diagram 8.38. Observe that the top row is the definition of $f + g$, and the composite $m \circ (\hat{f} \times \hat{g}) \circ \Delta$ is the definition of $\Phi(f) + \Phi(g)$. Thus the diagram shows that $(\Phi(f) + \Phi(g)) \circ i \simeq f + g$. Since both $m \circ (\hat{f} \times \hat{g})$ and Δ are H-maps, so is $\Phi(f) + \Phi(g)$. Then we have $\Phi(f + g) \simeq \Phi(f) + \Phi(g)$. □

Theorem 8.12. *Suppose that L is a homotopy associative H-space. Given hypotheses as in Lemmas 8.5 and 8.7, then $H[G, L]$ is a subgroup of $[G, L]$, and Φ is a group isomorphism.*

Proof. Let $\phi, \rho : G \to L$ be H-maps. Since we have a 1-1 correspondence between $[A, L]$ and $H[G, L]$, we have $\phi \simeq \hat{f}$ and $\rho \simeq \hat{g}$ for some choices of $f, g : A \to L$. As we have seen, $\Phi(f) + \Phi(g)$ is an H-map, so $\phi + \rho : G \xrightarrow{\Delta} G \times G \xrightarrow{\phi \times \rho} L \times L \xrightarrow{m} L$ is an H-map and $H[G, L]$ is closed under addition.

The inverse of f is $-f$, which corresponds to an H-map $\Phi(-f) : G \to L$. Corollary 8.11 implies that

$$\begin{aligned} \Phi(f) + \Phi(-f) &\simeq \Phi(f + (-f)) && (8.40) \\ &\simeq \Phi(*) && (8.41) \\ &\simeq *, && (8.42) \end{aligned}$$

so $\Phi(-f) \simeq -\Phi(f)$ and $H[G, L]$ is closed under taking inverses.

The identity map is an element of $H[G, L]$, so $H[G, L]$ is a group and Corollary 8.11 shows that Φ is a group homomorphism. Since Φ is also a bijection, it is hence an isomorphism. □

8.2 Application to the classical Lie groups

Note that in the following proposition, conditions on m, n and p have been chosen so that G is p-regular and L is not.

Proposition 8.13. *The hypotheses of Lemmas 8.5 and 8.7 hold in the following cases:*

- $G = SU(m)$, $L = SU(n)$ *for* $2m \leq n$, $m \leq p < n$;
- $G = Sp(m)$, $L = Sp(n)$ *for* $4m \leq n$, $2m \leq p < 2n$;
- $G = Spin(m)$, $L = Spin(n)$ *for* $4m - 3 \leq n$, $2m - 1 \leq p < 2n - 1$ *with* m *and* n *odd.*

Proof. Consider the composite

$$\Omega(\Sigma A \wedge A) \xrightarrow{\Omega[i_1, i_2]} \Omega(\Sigma A \vee \Sigma A) \xrightarrow{\gamma} L. \tag{8.43}$$

This is a composite of H-maps from a loop suspension to a homotopy associative H-space, and thus the James construction implies it is null homotopic if and only if the composite

$$A \wedge A \xrightarrow{E} \Omega(\Sigma A \wedge A) \xrightarrow{\Omega[i_1, i_2]} \Omega(\Sigma A \vee \Sigma A) \xrightarrow{\gamma} L \tag{8.44}$$

is null homotopic. Note that the composition $\Omega[i_1, i_2] \circ E$ is homotopic to the Samelson product $\langle \Omega i_1 \circ E, \Omega i_2 \circ E \rangle$.

By definition of γ, we have the homotopy commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{E} & \Omega\Sigma A & \xrightarrow{\Omega i_1} & \Omega(\Sigma A \vee \Sigma A) \\ & \searrow^{f} & & & \downarrow \gamma \\ & & & & L. \end{array} \tag{8.45}$$

Similarly, we also have $\gamma \circ \Omega i_2 \circ E \simeq g$.

Since γ is an H-map, it preserves Samelson products, and thus the composite (8.43) is homotopic to the Samelson product $\langle f, g \rangle$.

We will treat the cases for each group separately here. Let $G = SU(m)$, $L = SU(n)$ for $2m \leq n$, $m \leq p < n$. Then $A = \Sigma\mathbb{C}P^{m-1} = S^3 \vee S^5 \vee \ldots \vee S^{2m-1}$, and so $A \wedge A$ has dimension $2(2m-1) = 4m-2$. Note that as $SU(\infty)$ is an infinite loop space, it is homotopy commutative, and so since $SU(n) \to SU(m)$ is $(2n-1)$-connected we have that $L = SU(n)$ is homotopy commutative through dimension $2n-1$. But $2m \leq n$ by hypothesis, and so $4m-2 \leq 2n-1$. As Samelson products map trivially into homotopy commutative H-groups, $\langle f, g \rangle$ is null homotopic, implying that (8.44) is null homotopic, thus the conditions of Lemma 8.5 are met.

Now let $G = Sp(m)$, $L = Sp(n)$ for $4m \leq n$, $2m \leq p < 2n$. In this case

$$A \simeq S^3 \vee S^7 \vee \ldots \vee S^{4m-1}, \tag{8.46}$$

and $A \wedge A$ has dimension $2(4m-1) = 8m-2$. Then since $Sp(\infty)$ is homotopy commutative and $Sp(n) \to Sp(m)$ is $(4n+1)$-connected, we have that $L = Sp(n)$ is homotopy commutative through dimension $4n+1$. But $4m \leq n$ by hypothesis, and so $8m-1 \leq 4n+1$. Then as before, $\langle f, g \rangle$ is null homotopic, so (8.44) is null homotopic and the conditions of Lemma 8.5 are met.

Recall from the proof of Corollary 7.8 that since m and n are both odd, there is a homotopy equivalence

$$Spin(m) \simeq Sp\left(\frac{m-1}{2}\right), \tag{8.47}$$

and thus, since the conditions of Lemma 8.5 are met for $G = Sp(m)$, $L = Sp(n)$ for $4m \leq n$, $2m \leq p < 2n$, they are also met for the equivalent conditions $G = Spin(m)$, $L = Spin(n)$ for $4m-3 \leq n$, $2m-1 \leq p < 2n-1$ with m and n odd.

We now turn to the conditions of Lemma 8.7. For each case in the statement of this theorem, conditions are such that G is p-regular. Theorem 7.3 gives us a homotopy fibration $\Sigma R \xrightarrow{\vartheta} \Sigma A \xrightarrow{j} BG$. Looping this, we get a homotopy fibration $\Omega\Sigma R \xrightarrow{\Omega\vartheta} \Omega\Sigma A \xrightarrow{r} G$.

Consider now the composition $\Omega\Sigma R \xrightarrow{\Omega\vartheta} \Omega\Sigma A \xrightarrow{\bar{f}} L$. Recall from the proof of Theorem 7.4 that if G and L are localised at an odd prime p such that G is p-regular but not homotopy commutative, and the following compositions are null homotopic:

$$\bar{f} \circ \langle \mu_i, \mu_j \rangle \qquad \bar{f} \circ b_{i,j} \qquad \bar{f} \circ \bar{\alpha}, \tag{8.48}$$

then $\bar{f} \circ \Omega\vartheta$ is null homotopic.

It was then proved in Theorems 7.6 - 7.8 that the compositions (8.48) are indeed null homotopic under the hypotheses of this theorem. Hence the hypotheses of Lemma 8.7 hold. □

Applying Theorem 8.12 to the above cases, combined with Theorems 7.6 - 7.8 gives us the following results for group isomorphisms $H[G, L] \cong [A, L]$ for classical Lie groups under certain conditions. Since G is p-regular, $[G, L]$ may be decomposed as a product of homotopy groups of L, and this may be used to discern whether $[G, L] \cong H[G, L]$.

Theorem 8.14. *Let $2m \leq n$, and consider $H[SU(m), SU(n)]$. Localised at a prime p such that $m \leq p < n$, there is a group isomorphism*

$$H[SU(m), SU(n)] \cong \left[\bigvee_{i=1}^{m-1} S^{2m_i-1}, SU(n)\right] \tag{8.49}$$

$$\cong \pi_3(SU(n)) \times \pi_5(SU(n)) \ldots \times \pi_{2m-1}(SU(n)). \tag{8.50}$$

□

Theorem 8.15. *Let $4m \leq n$, and consider $H[Sp(m), Sp(n)]$. Localised at a prime p such that $2m \leq p < 2n$, there is a group isomorphism*

$$H[Sp(m), Sp(n)] \cong \left[\bigvee_{i=1}^{2m} S^{2m_i-1}, Sp(n)\right] \tag{8.51}$$

$$\cong \pi_3(Sp(n)) \times \pi_7(Sp(n)) \ldots \times \pi_{4m-1}(Sp(n)). \tag{8.52}$$

□

Corollary 8.16. *Consider $H[Spin(m), Spin(n)]$, such that m and n are both odd and $4m - 3 \leq n$. Localised at a prime p such that $2m - 1 \leq p < 2n - 1$, there is a group isomorphism*

$$H[Spin(m), Spin(n)] \cong [A(Spin(m)), Spin(n)]. \tag{8.53}$$

□

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