Pure spinor formulation of the superstring and its applications

Nathan Berkovits and Carlos R. Mafra

Abstract The pure spinor formalism for the superstring has the advantage over the more conventional Ramond-Neveu-Schwarz formalism of being manifestly spacetime supersymmetric, which simplifies the computation of multiparticle and multiloop amplitudes and allows the description of Ramond-Ramond backgrounds. In addition to the worldsheet variables of the Green-Schwarz-Siegel action, the pure spinor formalism includes bosonic ghost variables which are constrained spacetime spinors and are needed for covariant quantization using a nilpotent BRST operator.

In this review, several applications of the formalism are described including the explicit computation in D=10 superspace of the general disk amplitude with an arbitrary number of external massless states, genus one amplitudes with up to seven external states, genus two amplitudes with up to five external states, and the low-energy limit of the genus three amplitude with up to four external states. The pure spinor formalism has also been used to covariantly quantize the superstring in an $AdS_5 \times S^5$ background and might be useful for proving the AdS-CFT correspondence in the limit of small AdS radius.

This is an overview written for the “Handbook of Quantum Gravity”, eds. C. Bambi, L. Modesto and I. Shapiro.

Nathan Berkovits
ICTP South American Institute for Fundamental Research, Instituto de Física Teórica, UNESP – Universidade Estadual Paulista, Rua Dr. Bento T. Ferraz 271, 01140-070, São Paulo, SP, Brasil
e-mail: nathan.berkovits@unesp.br

Carlos R. Mafra
STAG Research Centre and Mathematical Sciences, University of Southampton, Highfield, Southampton SO17 1BJ, UK e-mail: c.r.mafra@soton.ac.uk

Keywords

superstring, supersymmetry, amplitudes, pure spinors.

1 Introduction

At the present time, superstring theory is the only formalism available for computing perturbative scattering amplitudes of gravitons without ultraviolet quantum-mechanical divergences. Although comparing these scattering amplitudes with experiments is unlikely in the near future, various properties of these amplitudes such as spacetime supersymmetry and duality symmetry might have testable low-energy implications.

Using the conventional Ramond-Neveu-Schwarz (RNS) formalism of the superstring, the complicated nature of vertex operators for spacetime fermions and the need to sum over spin structures has made it difficult to compute amplitudes involving external fermions or to compute muiltiloop amplitudes. Furthermore, backgrounds involving Ramond-Ramond fields necessary for the AdS-CFT correspondence are difficult to describe in the RNS formalism.

In 2000, a new formalism for the superstring was constructed in which spacetime supersymmetry is manifest and there is no need to sum over spin structures [1]. In addition to the worldsheet variables (x^m, θ^α) of the Green-Schwarz formalism [2] for $m = 0$ to 9 and $\alpha = 1$ to 16, this new formalism includes the fermionic momenta variables d_α of Siegel [3] as well as bosonic ghost variables $(\lambda^\alpha, w_\alpha)$ constrained to satisfy $\lambda\gamma^m\lambda = 0$. This constraint implies that λ^α is a $D = 10$ "pure spinor" as defined by Cartan with 11 independent components, and the conformal anomaly contribution of $+22$ from $(\lambda^\alpha, w_\alpha)$ cancels the conformal anomaly contribution of $+10-32 = -22$ from x^m and $(\theta^\alpha, d_\alpha)$. Generalizing a supersymmetric field theory observation of Howe [4, 5], physical superstring states in this "pure spinor formalism" are defined using the nilpotent BRST operator $Q = \oint dz \lambda^\alpha d_\alpha$ and, unlike in the Green-Schwarz formalism, covariant quantization is straightforward. A similar BRST operator is useful for describing $d = 11$ supergravity [6, 7, 8], and more details on pure spinor applications in supersymmetric field theories can be found in the review of Martin Cederwall [9].

Over the last 20 years, this pure spinor formalism has been used to compute various multiparticle and multiloop amplitudes in superstring theory including several amplitudes which have not yet been computed using the RNS formalism. All amplitudes computed using both the RNS and pure spinor formalisms have been shown to coincide, however, the pure spinor computations are typically much more efficient since there is no sum over spin structures and amplitudes are automatically expressed in $D = 10$ superspace. Nevertheless, a proof of equivalence of the RNS and pure spinor formalisms for the superstring is still lacking.

A promising approach towards proving this equivalence involves a recently constructed formalism for the superstring which includes θ^α and an unconstrained bosonic spacetime spinor worldsheet variable Λ^α, in addition to the usual N=1 worldsheet supersymmetric RNS matter and ghost variables. [10, 11] This new formalism, named the B-RNS-GSS formalism since it combines features of the RNS, Green-Schwarz-Siegel and pure spinor formalisms, is both N=1 worldsheet supersymmetric and D=10 spacetime supersymmetric and acts as a bridge between the RNS and pure spinor formalisms. It can be related to the RNS formalism by treating $(\theta^\alpha, \Lambda^\alpha)$ as non-minimal variables which decouple from the BRST cohomology, and can be related to the pure spinor formalism by "twisting" the N=1 superconformal generators into N=2 superconformal generators so that all worldsheet variables carry integer conformal weight. Work is in progress on computing scattering amplitudes using the B-RNS-GSS formalism and proving that the amplitudes coincide with those computed using the RNS and pure spinor formalisms. Since multiloop amplitude computations using the RNS and pure spinor formalism have different types of subtleties, it is expected that the B-RNS-GSS formalism will be useful for relating these subtleties.

Just as the RNS formalism for the superstring can be described in any curved background which preserves N=1 worldsheet supersymmetry, the pure spinor formalism can be described in any curved background in which the BRST current $\lambda^\alpha d_\alpha$ remains nilpotent and holomorphic [12]. This allows not only the Calabi-Yau backgrounds which can be described using the RNS formalism, but also any curved supergravity background in which the D=10 supergravity equations of motion are satisfied to lowest order in α'. For example, unlike the RNS formalism, the pure spinor formalism can be used to covariantly quantize the superstring in an $AdS_5 \times S^5$ Ramond-Ramond background which is dual to $\mathscr{N} = 4$ $D = 4$ super-Yang-Mills through the AdS-CFT correspondence.

Although this important application will not be discussed in later sections of the review, quantum consistency of the $AdS_5 \times S^5$ background has been proven [13] using the pure spinor formalism. To prove quantum consistency to all orders in α', it was shown using symmetry arguments that any potential BRST anomalies coming from quantum corrections can be cancelled by the addition of local counterterms to the worldsheet action. It was also shown using BRST arguments that the classical non-local conserved currents related to integrability can be extended to quantum non-local conserved currents.

The construction of BRST-invariant vertex operators for half-BPS states in an $AdS_5 \times S^5$ background was recently achieved [14, 15, 16], and work is in progress on using these vertex operators for the computation of scattering amplitudes. The structure of the vertex operators and the pure spinor worldsheet action in an $AdS_5 \times S^5$ background is more complicated than in a flat background, however, the manifest $PSU(2,2|4)$ isometry of the construction should be useful in simplifying the amplitude computations. An important open question is how to generalize the super-Poincaré invariant BRST cohomology methods which are described in this review to BRST cohomology methods with $PSU(2,2|4)$ invariance.

In the limit of small AdS radius, the pure spinor version of the $AdS_5 \times S^5$ worldsheet action has been shown to reduce to a BRST-trivial topological action plus a small $PSU(2,2|4)$-invariant deformation term [17, 18]. In this limit, the dual theory is $\mathcal{N}=4$ $D=4$ super-Yang-Mills at weak coupling, and it has been conjectured that the topological action describes free super-Yang-Mills and the deformation describes the cubic super-Yang-Mills interaction term. The topological action and deformation term are constructed by combining the x^m and λ^α bosonic worldsheet variables of the pure spinor formalism into a twistor-like variable which transforms linearly under the $SO(4,2)\times SO(6)$ bosonic subgroup of $PSU(2,2|4)$. Similar twistor variables have been extremely useful for computing perturbative scattering amplitudes of $\mathcal{N}=4$ $D=4$ super-Yang-Mills [19], and it would not be surprising if the two types of twistor variables are related through the AdS-CFT correspondence.

If this conjecture could be verified, it would provide a proof of the AdS-CFT correspondence in the case of $AdS_5\times S^5$. A proof of the AdS-CFT correspondence in the simpler case of $AdS_3\times S^3$ was established by Eberhardt, Gaberdiel and Gopakumar in [20] using a "hybrid" formalism of the superstring which can be interpreted as a six-dimensional version of the $D=10$ pure spinor formalism. It is very suggestive that twistor-like variables were used in their proof, and that Gaberdiel and Gopakumar were recently able to generalize their twistor-like construction of the spectrum of $AdS_3\times S^3$ at zero radius to the more interesting case of $AdS_5\times S^5$ at zero radius [21].

After a brief review of the pure spinor formalism and the superspace formulation of ten-dimensional super Yang-Mills theory in sections (2.1) and (2.2), section (2.3) will showcase its applications to the computation of scattering amplitudes in a flat background. From the complete genus-zero amplitudes with an arbitrary number of external massless states to the low-energy limit of the massless four-point amplitude at genus three, the pure spinor formalism and related techniques played a crucial role in determining their manifestly supersymmetric forms. Finally, section (2.4) will discuss how these amplitudes have been used to test S-duality conjectures.

2 The pure spinor formalism and scattering amplitudes

2.1 Ten-dimensional super-Yang-Mills theory in superspace

There is a super-Poincaré description of $D=10$ super-Yang-Mills (SYM) in superspace [80, 79] that describes the gluon and gluino states via Lie algebra-valued superfield connections $\mathbb{A}_\alpha(x,\theta)$ and $\mathbb{A}_m(x,\theta)$ satisfying the non-linear constraint $\{\nabla_\alpha,\nabla_\beta\}=\gamma^m_{\alpha\beta}\nabla_m$, where $\nabla_\alpha=D_\alpha-\mathbb{A}_\alpha$ and $\nabla_m=\partial_m-\mathbb{A}_m$ are supercovariant derivatives and

$$D_\alpha=\frac{\partial}{\partial\theta^\alpha}+\frac{1}{2}(\gamma^m\theta)_\alpha\partial_m \tag{1}$$

is the superspace derivative satisfying $\{D_\alpha,D_\beta\}=\gamma^m_{\alpha\beta}\partial_m$. The ten-dimensional superspace coordinates (x,θ) are composed of a $SO(9,1)$ Lorentz vector x^m, where $m=1,\ldots,10$, and a Weyl spinor θ^α, where $\alpha=1,\ldots,16$. In ten dimensions, the Lorentz group has two inequivalent spinor representations, denoted Weyl and anti-

Weyl. They are distinguished by the position of the spinor index, upstairs for Weyl Ψ^α and downstairs for anti-Weyl χ_α which cannot be raised or lowered. The gamma matrices $\gamma^m_{\alpha\beta}$ and $\gamma_m^{\alpha\beta}$ are the 16×16 off-diagonal symmetric Pauli matrices of the 32×32 Dirac matrices Γ^m of the $SO(9,1)$ Clifford algebra $\{\Gamma^m,\Gamma^n\}=2\eta^{mn}\mathbb{I}_{32\times 32}$. They satisfy $\gamma^m_{\alpha\beta}(\gamma^n)^{\beta\rho}+\gamma^n_{\alpha\beta}(\gamma^m)^{\beta\rho}=2\eta^{mn}\delta^\rho_\alpha$.

The non-linear equations of motion following from the above constraint have linearized counterparts written in terms of linearized superfield connections $A_\alpha(x,\theta)$, $A^m(x,\theta)$ and their field-strengths $W^\alpha(x,\theta)$, and $F^{mn}(x,\theta)$,

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta}A_m\,, \qquad D_\alpha A_m = (\gamma_m W)_\alpha + \partial_m A_\alpha$$
$$D_\alpha F_{mn} = \partial_{[m}(\gamma_{n]}W)_\alpha\,, \qquad D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}\,. \tag{2}$$

These linearized superfields will enter the expressions for the massless vertex operators of the pure spinor formalism and will be the main actors in the composition of *pure spinor superspace* expressions to be reviewed below. In this context, it is essential to know how these superfields are expanded in a series of θ variables.

The linearized superfields can be expanded in the so-called Harnad-Shnider gauge $\theta^\alpha A_\alpha(x,\theta)=0$ in terms of the gluon e^m_i and gluino χ^α_i polarizations of a particle state labelled by i [78]. For convenience we strip off the universal plane-wave factor $e^{k\cdot x}$ that carries all the x dependence from the superfields and define their θ-dependent factor as $A^i_\alpha(x,\theta)=A^i_\alpha(\theta)e^{k\cdot x}$ etc. One can show that

$$\begin{aligned}
A^i_\alpha(\theta) &= \frac{1}{2}(\theta\gamma_m)_\alpha e^m_i + \frac{1}{3}(\theta\gamma_m)_\alpha(\theta\gamma^m\chi_i) - \frac{1}{32}(\theta\gamma_m)^\alpha(\theta\gamma^{mnp}\theta)f^i_{np} \\
&\quad + \frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)k_n(\chi^i\gamma_p\theta) + \frac{1}{1152}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\theta\gamma^{pqr}\theta)k^n_i f^{qr}_i + \ldots \\
A^m_i(\theta) &= e^m_i + (\theta\gamma^m\chi_i) - \frac{1}{8}(\theta\gamma^{mpq}\theta)f^{pq}_i + \frac{1}{12}(\theta\gamma^{mnp}\theta)k^n_i(\chi_i\gamma^p\theta) \\
&\quad + \frac{1}{192}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)k^n_i f^{pq}_i - \frac{1}{480}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)k^n_i k^p_i(\chi_i\gamma^q\theta) + \ldots \\
W^\alpha_i(\theta) &= \chi^\alpha_i + \frac{1}{4}(\theta\gamma^{mn})^\alpha f^i_{mn} - \frac{1}{4}(\theta\gamma_{mn})^\alpha k^m_i(\chi_i\gamma^n\theta) - \frac{1}{48}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)k^m_i f^{np}_i \\
&\quad + \frac{1}{96}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)k^m_i k^n_i(\chi_i\gamma^p\theta) - \frac{1}{1920}(\theta\gamma_m{}^r)^\alpha(\theta\gamma_{nr}{}^s\theta)(\theta\gamma_{spq}\theta)k^m_i k^n_i f^{pq}_i + \ldots \\
F^{mn}_i(\theta) &= f^{mn}_i - k^{[m}_i(\chi_i\gamma^{n]}\theta) + \frac{1}{8}(\theta\gamma_{pq}{}^{[m}\theta)k^{n]}_i f^{pq}_i - \frac{1}{12}(\theta\gamma_{pq}{}^{[m}\theta)k^{n]}_i k^p_i(\chi_i\gamma^q\theta) \\
&\quad - \frac{1}{192}(\theta\gamma_{ps}{}^{[m}\theta)k^{n]}_i k^p_i f^{qr}_i(\theta\gamma^s{}_{qr}\theta) + \frac{1}{480}(\theta\gamma^{[m}{}_{ps}\theta)k^{n]}_i k^p_i k^q_i(\chi_i\gamma^r\theta)(\theta\gamma^s{}_{qr}\theta) + \ldots\,,
\end{aligned} \tag{3}$$

where $f^{mn}_i = k^m e^n_i - k^n e^m_i$ is the linearized field strength of the ith gluon and the terms in the ellipsis of order $\theta^{>5}$ will not contribute in pure spinor superspace expressions to be reviewed below.

2.2 Non-minimal pure spinor formalism

It is customary to distinguish two very closely related versions of the pure spinor formalism: minimal [25] and non-minimal [22]. Both are based on the ideas of [1] but the non-minimal incarnation introduces new variables on the worldsheet and admits a simpler "topological" multiloop amplitude prescription.

The left-moving sector of the non-minimal pure spinor formalism is composed of the fields $\partial x^m, p_\alpha, w_\alpha, s^\alpha$ of conformal weight one and of $\theta^\alpha, \lambda^\alpha, \overline{\lambda}_\alpha, r_\alpha$ of conformal weight zero, where $m = 0,1\ldots,9$ and $\alpha = 1,\ldots,16$ are the vector and spinorial indices of $SO(10)$. The world-sheet action is

$$S = \frac{1}{2\pi\alpha'}\int d^2z\left(\partial x^m\overline{\partial}x_m + \alpha' p_\alpha\overline{\partial}\theta^\alpha - \alpha' w_\alpha\overline{\partial}\lambda^\alpha - \alpha'\overline{w}^\alpha\overline{\partial\lambda}_\alpha + \alpha' s^\alpha\overline{\partial}r_\alpha\right), \tag{4}$$

and α' denotes the inverse string tension. The field λ^α is bosonic and satisfies the *pure spinor constraint*

$$\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0\,. \tag{5}$$

The field $\overline{\lambda}_\alpha$ is bosonic while r_α is fermionic and they satisfy the constraints

$$\overline{\lambda}_\alpha\gamma_m^{\alpha\beta}\overline{\lambda}_\beta = 0, \quad \overline{\lambda}_\alpha\gamma_m^{\alpha\beta}r_\beta = 0\,. \tag{6}$$

The OPEs of the matter variables are given by

$$X^m(z,\overline{z})X_n(w,\overline{w}) \sim -\frac{\alpha'}{2}\delta^m_n\ln|z-w|^2, \qquad p_\alpha(z)\theta^\beta(w) \sim \frac{\delta^\beta_\alpha}{z-w}, \tag{7}$$

while the OPEs of the ghost variables do not follow from a free-field calculation due to the constraints above. In certain circumstances, however, the variables $(w_\alpha, \lambda^\alpha)$ can be viewed as a conjugate pair with canonical OPE. The Green–Schwarz constraint $d_\alpha(z)$, the supersymmetric momentum $\Pi^m(z)$ have conformal weight $+1$ and are given by

$$\begin{aligned} d_\alpha(z) &= p_\alpha - \frac{1}{\alpha'}(\gamma^m\theta)_\alpha\partial x_m - \frac{1}{4\alpha'}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta), \\ \Pi^m(z) &= \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta)\,. \end{aligned} \tag{8}$$

These fields satisfy the following OPEs

$$\begin{aligned} d_\alpha(z)d_\beta(w) &\sim -\frac{\gamma^m_{\alpha\beta}\Pi_m}{z-w}, \qquad \Pi^m(z)\Pi^n(w) \sim -\frac{\eta^{mn}}{(z-w)^2}, \\ d_\alpha(z)\Pi^m(w) &\sim \frac{(\gamma^m\partial\theta)_\alpha}{z-w}, \end{aligned} \tag{9}$$

In addition, if $f(x(w),\theta(w))$ does not depend on derivatives $\partial^k x$ nor $\partial^k\theta$

$$d_\alpha(z) f(x(w),\theta(w)) \sim \frac{D_\alpha f}{z-w}, \qquad \Pi^m(z) f(x(w),\theta(w)) \sim -\frac{k^m f}{z-w} \qquad (10)$$

The non-minimal BRST charge

$$Q = \oint (\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha), \qquad (11)$$

can be shown to be nilpotent $Q^2 = 0$ using the OPEs (8) and the pure spinor constraint (5). Physical states are required to be in the cohomology of (11) and it will be shown below that the cohomology is independent on the quartet of non-minimal variables $(\overline{w}^\alpha, \overline{\lambda}_\alpha, s^\alpha, r_\alpha)$.

The constraint (5) implies that the conjugate momentum w_α to the pure spinor λ^α can only appear in gauge-invariant combinations under

$$\delta w_\alpha(z) = \Omega_m(z)(\gamma^m\lambda)_\alpha\,. \qquad (12)$$

The basic gauge-invariant quantities are the current J_λ, the energy momentum tensor T_λ and the Lorentz current N_{mn} given by

$$J_\lambda(z) = w_\alpha\lambda^\alpha\,, \qquad T_\lambda(z) = w_\alpha\partial\lambda^\alpha\,, \qquad N^{mn}(z) = \frac{1}{2}(w\gamma^{mn}\lambda)\,. \qquad (13)$$

Since the conjugate pair $(\lambda^\alpha, w_\alpha)$ is not free due to the pure spinor constraint, the OPEs of these gauge invariants are computed using the $U(5)$ parameterization of λ^α, with the $SO(10)$-covariant result [1],

$$N^{mn}(z)\lambda^\alpha(w) \sim \frac{\frac{1}{2}(\gamma^{mn}\lambda)^\alpha(w)}{z-w}\,, \qquad J_\lambda(z)\lambda^\alpha(w) \sim \frac{\lambda^\alpha(w)}{z-w}\,, \qquad (14)$$
$$N^{mn}(z)J_\lambda(w) \sim \text{regular}\,, \qquad J_\lambda(z)J_\lambda(w) \sim \frac{-4}{(z-w)^2}\,,$$
$$N_{mn}(z)T_\lambda(w) \sim \frac{N_{mn}(w)}{(z-w)}\,, \qquad J_\lambda(z)T_\lambda(w) \sim \frac{-8}{(z-w)^3} + \frac{J_\lambda(w)}{(z-w)^2}\,,$$

$$N^{mn}(z)N_{pq}(w) \sim \frac{N^m{}_p\delta^n_q - N^n{}_p\delta^m_q + N^n{}_q\delta^m_p - N^m{}_q\delta^n_p}{z-w} - \frac{3(\delta^n_p\delta^m_q - \delta^n_q\delta^m_p)}{(z-w)^2}\,,$$
$$T_\lambda(z)T_\lambda(w) \sim \frac{11}{(z-w)^4} + \frac{2T_\lambda(w)}{(z-w)^2} + \frac{\partial T_\lambda(w)}{z-w}\,.$$

Similarly, the constraints (6) imply that the conjugates $\overline{w}^\alpha$ and s^α of conformal weight $+1$ can only appear in gauge-invariant combinations under

$$\delta\overline{w}^\alpha = \overline{\Omega}_m(\gamma^m\overline{\lambda})^\alpha - \phi_m(\gamma^m r)^\alpha\,, \qquad \delta s^\alpha = \phi_m(\gamma^m\overline{\lambda})^\alpha\,, \qquad (15)$$

where $\overline{\Omega}_m$ and ϕ_m are arbitrary parameters. The non-minimal counterparts of the gauge invariants (13) are given by [22],

$$\overline{N}_{mn} = \frac{1}{2}(\overline{w}\gamma_{mn}\overline{\lambda} - s\gamma_{mn}r), \quad \overline{J}_{\overline{\lambda}} = \overline{w}^\alpha\overline{\lambda}_\alpha - s^\alpha r_\alpha, \quad T_{\overline{\lambda}} = \overline{w}^\alpha\partial\overline{\lambda}_\alpha - s^\alpha\partial r_\alpha, \quad (16)$$

with additional gauge invariants

$$S_{mn} = \frac{1}{2}s\gamma_{mn}\overline{\lambda}, \quad S = s^\alpha\overline{\lambda}_\alpha\,. \quad (17)$$

The above gauge invariants are related via the BRST charge

$$\overline{N}_{mn} = QS_{mn}, \quad \overline{J}_{\overline{\lambda}} = QS, \quad T_{\overline{\lambda}} = Q(s^\alpha\partial\overline{\lambda}_\alpha)\,, \quad (18)$$

Therefore, the operator $\overline{q} = \oint \overline{J}_{\overline{\lambda}}$ counting the non-minimal variables is BRST exact, and satisfies $\overline{q}\overline{\lambda}_\alpha = \overline{\lambda}_\alpha$ and $\overline{q}r_\alpha = r_\alpha$. Therefore if a BRST-closed state $Q\Psi = 0$ has non-vanishing non-minimal $\overline{q}$ charge $\overline{q}\Psi = n\Psi$ with $n \neq 0$, it is also BRST-exact; $\Psi = \frac{\overline{q}}{n}\Psi$. And since S and S^{mn} are not closed and $(\overline{N}_{mn}, \overline{J}_{\overline{\lambda}}, T_{\overline{\lambda}})$ are exact, the quartet $(\overline{w}^\alpha, \overline{\lambda}_\alpha, s^\alpha, r_\alpha)$ of non-minimal variables decouples from the cohomology in what is known as the Kugo-Ojima quartet mechanism.

Moreover, the energy-momentum tensor,

$$T(z) = -\frac{1}{\alpha'}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + w_\alpha\partial\lambda^\alpha + \overline{w}^\alpha\partial\overline{\lambda}_\alpha - s^\alpha\partial r_\alpha\,, \quad (19)$$

is related to the BRST charge through the b ghost as $\{Q, b(z)\} = T(z)$, where

$$\begin{aligned} b = {} & s^\alpha\partial\overline{\lambda}_\alpha + \frac{1}{4(\lambda\overline{\lambda})}\left[2\Pi^m(\overline{\lambda}\gamma_m d) - N_{mn}(\overline{\lambda}\gamma^{mn}\partial\theta) - J_\lambda(\overline{\lambda}\partial\theta) - (\overline{\lambda}\partial^2\theta)\right] \\ & + \frac{(\overline{\lambda}\gamma^{mnp}r)}{192(\lambda\overline{\lambda})^2}\left[\frac{\alpha'}{2}(d\gamma_{mnp}d) + 24N_{mn}\Pi_p\right] \\ & - \frac{\alpha'}{2}\frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^m d)N^{np}}{16(\lambda\overline{\lambda})^3} + \frac{\alpha'}{2}\frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\lambda\overline{\lambda})^4}\,. \end{aligned} \quad (20)$$

After extracting the non-minimal $U(1)$ ghost-number current

$$J(z) = w_\alpha\lambda^\alpha - s^\alpha r_\alpha - \frac{2\big((\overline{\lambda}\partial\lambda) + (r\partial\theta)\big)}{(\lambda\overline{\lambda})} + \frac{2(\lambda r)(\overline{\lambda}\partial\theta)}{(\lambda\overline{\lambda})^2} \quad (21)$$

from the double pole of the b ghost with the integrand $\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha$ of the BRST charge, the non-minimal pure spinor formalism was shown in [22] to be a critical $N = 2$ topological string. More precisely, using the terminology $G^+(z) = (\lambda^\alpha d_\alpha + \overline{\varpi}^\alpha r_\alpha)$, $G^-(z) = b(z)$ one can show that $T(z)$, $G^+(z)$, $G^-(z)$ and $J(z)$ satisfy the OPEs [22, 83]

$$T(z)T(w) \sim \frac{2T}{(z-w)^2} + \frac{\partial T}{(z-w)} \quad (22)$$

$$
\begin{aligned}
T(z)G^{+}(w) &\sim \frac{G^{+}}{(z-w)^2} + \frac{\partial G^{+}}{(z-w)}\\
T(z)G^{-}(w) &\sim \frac{2G^{-}}{(z-w)^2} + \frac{\partial G^{-}}{(z-w)}\\
G^{+}(z)G^{-}(w) &\sim \frac{3}{(z-w)^3} + \frac{J}{(z-w)^2} + \frac{T}{(z-w)}\\
T(z)J(w) &\sim \frac{-3}{(z-w)^3} + \frac{J}{(z-w)^2} + \frac{\partial J}{(z-w)}\\
J(z)G^{\pm}(w) &\sim \pm\frac{G^{\pm}}{(z-w)}\\
J(z)J(w) &\sim \frac{3}{(z-w)^2}\\
G^{\pm}(z)G^{\pm}(w) &\sim \text{regular},
\end{aligned}
$$

which identifies them as the generators of a $\hat{c}=3$ $N=2$ twisted topological conformal algebra. As such, not only the BRST charge has to be nilpotent but also the b ghost (see e.g. [85]). A proof that $b^2=0$ with (20) can be found in [83, 84]. Note that the simpler BRST-equivalent $U(1)$ charge $J(z)=w_\alpha\lambda^\alpha-\overline{w}^\alpha\overline{\lambda}_\alpha$ was show in [22] to preserve the essential features of the topological string and therefore can be used instead of (21) to define the ghost-number of pure spinor operators.

2.2.1 Vertex operators and amplitude prescription

Vertex operators for massless open-string states are constructed from the linearized SYM superfields of (2) as

$$
\begin{aligned}
&V=\lambda^\alpha A_\alpha(x,\theta), \\
&U=\partial\theta^\alpha A_\alpha(x,\theta)+\Pi^m A_m(x,\theta)+d_\alpha W^\alpha(x,\theta)+\frac{1}{2}N_{mn}F^{mn}(x,\theta)
\end{aligned}
\tag{23}
$$

and are independent on the non-minimal variables using the quartet mechanism discussed above. V is called the unintegrated vertex and has conformal weight zero and U is called the integrated vertex and has conformal weight $+1$. They are related via the BRST charge (11) by $QU=\partial V$, so the integrated vertex is BRST closed up to a total derivative on the worldsheet. The unintegrated vertex is BRST closed as a consequence of (10), the equation of motion (2), as well as the pure spinor constraint (5)

$$
QV=\lambda^\alpha\lambda^\beta D_\alpha A_\beta=\frac{1}{2}\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta A_m=0\,. \tag{24}
$$

Their closed-string versions are obtained by a double-copy of the open-string vertex operators with the plane-wave factor stripped off, that is $|V|^2=V(\theta)\tilde{V}(\tilde{\theta})e^{k\cdot x}$ where $V(\theta)=\lambda^\alpha A_\alpha(\theta)$ with $A_\alpha(\theta)$ as in (3), and similarly for $|U|^2$.

The prescription to calculate n-point closed-string amplitudes at genus g is

$$\mathscr{A}_{g=0} = \kappa^n e^{-2\lambda} \int_\Sigma \prod_{j=2}^{n-2} d^2 z_j |\langle \mathscr{N}_0 V_1(0) U_j(z_j) V_{n-1}(1) V_n(\infty)\rangle|^2 \tag{25}$$

$$\mathscr{A}_{g=1} = \frac{1}{2}\kappa^n \int_{\Sigma,\mathscr{M}_1} d^2\tau_1 \prod_{j=2}^{n} d^2 z_j |\langle \mathscr{N}(b,\mu_1) V_1(0) U_j(z_j)\rangle|^2 \tag{26}$$

$$\mathscr{A}_{g>1} = \kappa^n e^{2\lambda}(1-\frac{1}{2}\delta_{g,2}) \int_{\Sigma,\mathscr{M}_g} d^{3g-3}\tau \prod_{j=1}^{n} d^2 z_j |\langle U_j(z_j) \prod_{I=1}^{3g-3} (b,\mu_I)\mathscr{N}\rangle|^2 \tag{27}$$

where $U(z)$ is the integrated vertex operator (23), τ_I for $I = 1,\ldots,3g-3$ are the complex Teichmüller parameters with μ_I their associated Beltrami differentials, the b ghost is given by (20) and

$$(b,\mu_I) = \frac{1}{2\pi}\int d^2 z b_{zz} \mu_{I\bar{z}}^{z}, \tag{28}$$

$\mathscr{N}$ is the regularization factor (36) responsible for convergence as $(\lambda\overline{\lambda}) \to \infty$, κ is the normalization of the vertex operators ($\kappa^2 = e^{2\lambda}\pi/\alpha'^2$ by unitarity) and $e^{2(g-1)\lambda}$ is the string coupling constant as in [43]. The factor of $1/2$ in the genus-two amplitude is required because all genus-two curves have a $\mathbb{Z}_2$ symmetry [86]. In addition, $|.|^2$ signifies the holomorphic square of the integrand with the plane waves of the vertex operators dealt with as described above, and it is important to emphasize that all calculations are done in the left- and right-moving sectors separately using the chiral splitting formalism explained below.

Integration of non-zero modes The OPEs in a genus g Riemann surface are used to integrate out the non-zero modes of the fields of conformal weight $+1$. To do this, we first separate off the zero modes as (using $d_\alpha(z)$ to illustrate the procedure)

$$d_\alpha(z) = \hat{d}_\alpha(z) + \sum_{I=1}^{g} d_\alpha^I \,\omega_I(z)\,, \qquad \oint_{A_I} \hat{d}_\alpha = 0 \tag{29}$$

where $\omega_I(z)$ are g holomorphic one-forms satisfying $\oint_{A_I}\omega_J(z)dz = \delta_{IJ}$ and A_I represents the A cycles of the Riemann surface. Then the non-zero modes (indicated by hats) are integrated out via their OPEs. For example,

$$\begin{aligned} \hat{p}_\alpha(z)\,\theta^\beta(y) &\sim \partial_z \ln E(z,y)\,\delta_\alpha^\beta \\ \hat{d}_\alpha(z)\,K\big(x(y),\theta(y)\big) &\sim \partial_z \ln E(z,y)\, D_\alpha K\big(x(y),\theta(y)\big) \\ \hat{\Pi}_m(z)\,K\big(x(y),\theta(y)\big) &\sim -\partial_z \ln E(z,y)\,\partial_m K\big(x(y),\theta(y)\big) \end{aligned} \tag{30}$$

where $E(z,y)$ is the prime form and $K(x,\theta)$ is an arbitrary superfield depending on x and θ, but not on the worldsheet derivatives of these fields. In the limit where $z \to y$, the prime form behaves as $E(z,y) \sim z-y$ and the propagator $\partial_z \ln E(z,y)$ displays its distinctive singular structure $\sim 1/(z-y)$ seen in (8). The OPE of the $x^m(z,\overline{z})$ fields

$$X^m(z,\overline{z})X_n(w,\overline{w}) \sim -\frac{\alpha'}{2}\delta^m_n G(z,w), \tag{31}$$

with $G(z,w)$ the genus-g Green function, couples the left- and right-movers and motivates the chiral splitting techniques developed by D'Hoker and Phong.

Zero-mode integrations The zero-mode integrations that remain after integrating out the non-zero modes via OPEs are performed using

$$\langle \ldots \rangle = \int [d\theta][dr][d\lambda][d\overline{\lambda}] \prod_{I=1}^{g} [dd^I][ds^I][d\overline{w}^I][dw^I]\ldots \tag{32}$$

where [45]

$$\begin{aligned}
&[d\lambda]\, T_{\alpha_1\ldots\alpha_5} = c_\lambda\,(\varepsilon\cdot d^{11}\lambda)_{\alpha_1\ldots\alpha_5}\,, && [dw] = c_w\,(T\cdot\varepsilon\cdot d^{11}w)\\
&[d\overline{\lambda}]\,\overline{T}^{\alpha_1\ldots\alpha_5} = c_{\overline{\lambda}}\,(\varepsilon\cdot d^{11}\overline{\lambda})^{\alpha_1\ldots\alpha_5}\,, && [dr] = c_r\,(\overline{T}\cdot\varepsilon\cdot\partial_r^{11})\\
&[d\overline{w}]\, T_{\alpha_1\ldots\alpha_5} = c_{\overline{w}}\,(\varepsilon\cdot d^{11}\overline{w})_{\alpha_1\ldots\alpha_5} && [ds^I] = c_s\,(T\cdot\varepsilon\cdot\partial_{s^I}^{11})\\
&[d\theta] = c_\theta\, d^{16}\theta && [dd^I] = c_d\, d^{16}d^I.
\end{aligned} \tag{33}$$

with the shorthand $(\varepsilon\cdot d^{11}\lambda)_{\alpha_1\ldots\alpha_5} := \frac{1}{11!}\varepsilon_{\alpha_1\ldots\alpha_{16}} d\lambda^{\alpha_6}\ldots d\lambda^{\alpha_{16}}$, and its contraction $(\overline{T}\cdot\varepsilon\cdot d^{11}\lambda) = \frac{1}{11!5!}\overline{T}^{\alpha_1\ldots\alpha_5}(\varepsilon\cdot d^{11}\lambda)_{\alpha_1\ldots\alpha_5}$ with similar expressions for the others. The expressions of $T_{\alpha_1\ldots\alpha_5}$ and $\overline{T}^{\alpha_1\ldots\alpha_5}$ are given by

$$\begin{aligned}
T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} &= (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\\
\overline{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} &= (\overline{\lambda}\gamma^m)^{\alpha_1}(\overline{\lambda}\gamma^n)^{\alpha_2}(\overline{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}
\end{aligned} \tag{34}$$

and they can be shown to be totally antisymmetric due to the pure spinor constraints and satisfy $T\cdot\overline{T} = 5!\,2^6(\lambda\overline{\lambda})^3$. Finally, the normalizations are given by

$$\begin{aligned}
c_\lambda &= \Big(\frac{\alpha'}{2}\Big)^{-2}\Big(\frac{A_g}{4\pi^2}\Big)^{11/2} & c_w &= \Big(\frac{\alpha'}{2}\Big)^{2}\frac{1}{(2\pi)^{11}Z_g^{11/g}}\\
c_{\overline{\lambda}} &= 2^6\Big(\frac{\alpha'}{2}\Big)^{2}\Big(\frac{A_g}{4\pi^2}\Big)^{11/2} & c_r &= R\Big(\frac{\alpha'}{2}\Big)^{-2}\Big(\frac{2\pi}{A_g}\Big)^{11/2}\\
c_{\overline{w}} &= \Big(\frac{\alpha'}{2}\Big)^{-2}\frac{(\lambda\overline{\lambda})^3}{(2\pi)^{11}}Z_g^{-11/g} & c_s &= \Big(\frac{\alpha'}{2}\Big)^{2}\frac{(2\pi)^{11/2}}{2^6R(\lambda\overline{\lambda})^3}Z_g^{11/g}\\
c_\theta &= \Big(\frac{\alpha'}{2}\Big)^{4}\Big(\frac{2\pi}{A_g}\Big)^{16/2} & c_d &= \Big(\frac{\alpha'}{2}\Big)^{-4}(2\pi)^{16/2}Z_g^{16/g}\,.
\end{aligned} \tag{35}$$

where $A_g = \int d^2z\sqrt{g}$ is the area of the genus-g Riemann surface. Moreover, $Z_g = 1/\sqrt{\det(2\,\mathrm{Im}\,\Omega)}$ where Ω_{IJ} is the period matrix, and R is a free parameter that is used to choose the normalization of the three-point amplitude at genus zero (after which the normalization of all genus-g n-point amplitudes is fixed). As shown in [45], the closed-string amplitudes are independent on the area A_g because the number of bosonic and fermionic variables of conformal weight 0 is the same, and independent on the choice of normalization of the holomorphic one-forms because the number of bosonic and fermionic variables of conformal weight $+1$ is the same. The factor

$$\mathcal{N} = \sum_{I=1}^{3} e^{-(\lambda\overline{\lambda})-(w^I\overline{w}^I)-(r\theta)+(s^I d^I)}. \tag{36}$$

regulates the zero-mode integrations over the non-compact spaces of the bosonic variables $\lambda^\alpha, \overline{\lambda}_\alpha$ and $w^\alpha, \overline{w}_\alpha$ as $(\lambda\overline{\lambda}) \to \infty$ and $(w\overline{w}) \to \infty$ in a manner explained in [22]. The formula for the integration over the pure spinor variables was found in [47] using techniques from algebraic geometry

$$\int [d\lambda][d\overline{\lambda}](\lambda\overline{\lambda})^n e^{-(\lambda\overline{\lambda})} = \left(\frac{A_g}{2\pi}\right)^{11} \frac{\Gamma(8+n)}{7!60}, \tag{37}$$

where $\Gamma(x)$ is the gamma function. The b ghost (20) has factors of $1/(\lambda\overline{\lambda})$ which are not regularized by the regulator (36) as $(\lambda\overline{\lambda}) \to 0$. It was shown in [22] that as long as the integrands diverge slower than $1/(\lambda^{8+3g}\overline{\lambda}^{11})$ the amplitudes are still well-defined due to a compensating factor of $\lambda^{8+3g}\overline{\lambda}^{11}$ arising from $\langle \mathcal{N} \ldots \rangle$ in (32). As explained in [22], this issue is closely related to the existence of the operator

$$\xi = \frac{(\lambda\theta)}{(\lambda\overline{\lambda}) + (r\theta)} = \frac{(\lambda\theta)}{(\lambda\overline{\lambda})} \sum_{n=0}^{11} \left(\frac{(r\theta)}{(\lambda\overline{\lambda})}\right)^n \tag{38}$$

where the Taylor expansion ends at $n = 11$ because there are only 11 degrees of freedom in r_α due to the constraint (6). This operator trivializes the cohomology as $Q\xi = 1$ but $\langle \mathcal{N}\xi(\lambda^3\theta^5)\rangle$ diverges faster than $1/(\lambda^{8+3g}\overline{\lambda}^{11})$, therefore if the integrands were allowed to diverge too fast they would also be BRST-exact. Forbidding such pathological behavior restricts the amplitude prescription to contain at most three b ghosts, or in other words, up to genus two. By regularizing the b ghost to remove the singularity as $(\lambda\overline{\lambda}) \to 0$, an alternative prescription that allows amplitudes at arbitrary genus to be well defined was proposed in [26].

As emphasized in [45], after the integration over $[dd^I][ds^I][dw^I][d\overline{w}^I]$ has been performed, the remaining integrations over $\lambda^\alpha, \overline{\lambda}_\beta, \theta^\delta$ and r_α are the same ones appearing in the prescription of the tree-level amplitudes, and therefore give rise to (non-minimal) pure spinor superspace expressions.

Chiral splitting To address the mixing of left- and right-movers via OPE contractions – an issue that prevents writing the closed-string correlator as an holomorphic square – the chiral splitting procedure [38, 39, 75] introduces loop momenta ℓ_I^m

$$\ell_I^m = \oint_{A_I} dz \Pi^m(z) \tag{39}$$

in order to rewrite conformal correlators of the x^m-field in terms of an integral over ℓ_I. The integrand then becomes a product of left- and right-movers of schematic form $\mathscr{F}_n(z_i,k_i,\ell^I)\overline{\tilde{\mathscr{F}}_n(z_i,-\bar{k}_i,-\ell^I)}$, denoted *chiral blocks*. Chiral blocks have a universal monodromy behavior as the points are moved around one another or circled around the homology cycles of the surface, and these properties can be exploited[1] to propose pure spinor superstring integrands [64, 41]. More precisely, decomposing the chiral blocks into chiral kinematic correlators $\mathscr{K}(z_i,\ell^I)$ and a chiral Koba-Nielsen factor $\mathscr{I}_n$ (to be displayed below) as $\mathscr{F}=\langle\mathscr{K}_n\rangle\mathscr{I}_n$, the expression for the chiral correlator must be invariant under the combined *homology shifts* of vertex positions z_i and loop momenta ℓ^I around the A_I or B_I cycles:

$$\begin{aligned}\mathscr{K}_n(z_i,k_i,\ell^I)&=\mathscr{K}_n(z_i+\delta_{ij}A_J,k_i,\ell^I)\\ \mathscr{K}_n(z_i,k_i,\ell^I)&=\mathscr{K}_n(z_i+\delta_{ij}B_J,k_i,\ell^I-2\pi\delta^I_J k_j)\,.\end{aligned}\tag{40}$$

When viewed as a constraint on the chiral correlator, these invariances can be used as a guide to obtain superstring correlators [62, 63, 64, 41, 42].

2.2.2 Pure spinor superspace

After all the non-zero modes of the worldsheet fields have been integrated out using OPEs, the correlator contains only the zero modes of conformal-weight zero variables. In the minimal pure spinor formalism of [25] that means the zero modes of λ^α and θ^α, while in the non-minimal formalism they can also include $\overline{\lambda}_\alpha$ and r_α variables. In the latter case, one can show that r_α can be converted to supersymmetric derivatives D_α while the pure spinors $\overline{\lambda}_\alpha$ can always be arranged to contract λ^α to produce scalar factors of $(\lambda\overline{\lambda})$ which change the normalization factor. Therefore the zero mode integration (with a constant number of $(\lambda\overline{\lambda})$ factors) can be done with the prescription [1]

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle=2880\,.\tag{41}$$

This motivates the notion of *pure spinor superspace* [27], defined as expressions containing three pure spinors and an arbitrary number of SYM superfields composed of polarizations, momenta and θ^α variables. The prescription (41) justifies the previous claim that terms of order $\theta^{>5}$ in (3) could be safely ignored. As a simple example of pure spinor superspace one can consider extracting the supersymmetric expression of the massless open-string three-point amplitude at genus zero,

$$\begin{aligned}\langle V_1V_2V_3\rangle=\Big[&\frac{1}{64}k^2_m e^1_r e^2_n e^3_s\langle(\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma_p\theta)(\theta\gamma^{pmn}\theta)\rangle\\ &+\frac{1}{18}e^m_1\langle(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^n\chi_2)(\theta\gamma^p\chi_3)\rangle+\text{cyclic}(1,2,3)\Big]\end{aligned}$$

[1] Of course, the monodromy of the chiral blocks play a central role in calculations with the RNS formalism, see e.g.[40], but in this review we will focus on the pure spinor formalism.

$$= \frac{1}{2} e_1^m f_2^{mn} e_3^n + e_m^1 (\chi_2 \gamma^n \chi_3) + \text{cyclic}(1,2,3)\,. \tag{42}$$

where we plugged in the θ expansions of (3) and kept only the terms with θ^5. Moreover, we used

$$\begin{aligned} \langle (\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma_p\theta)(\theta\gamma^{pmn}\theta)\rangle &= 64\delta^{rs}_{mn}, \\ \langle (\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^n\chi_2)(\theta\gamma^p\chi_3)\rangle &= 18(\chi_2\gamma^m\chi_3), \end{aligned} \tag{43}$$

which can be derived from group-theory considerations (see appendix of [28]), momentum conservation $k_1^m + k_2^m + k_3^m = 0$ and the transversality condition $(k_i \cdot e_i) = 0$.

2.2.3 Multiparticle superfields

While four-point scattering amplitudes at one and two can be written down using the (single-particle) SYM superfields, the OPE contractions present at higher points lead to linear combinations of SYM superfields whose patterns are captured by so-called *multiparticle superfields*, describing multiple strings at the same time. Not only they encode the numerators associated to OPE singularities, but they are also designed in a way that removes BRST-exact pieces and total derivatives. The end result displays covariant BRST transformations and generalized Jacobi identities [31] – the latter property is particularly useful for describing Bern-Carrasco-Johansson color/kinematics duality [29].

The two-particle superfields generalizing the standard superfields of (2) are given by [66]

$$\begin{aligned} A^{12}_\alpha &= \tfrac{1}{2}\big[A^2_\alpha(k_2\cdot A_1) + A^m_2(\gamma_m W_1)_\alpha - (1\leftrightarrow 2)\big]\,, \\ A^m_{12} &= \tfrac{1}{2}\big[A^m_2(k_2\cdot A_1) + A^1_p F^{pm}_2 + (W_1\gamma^m W_2) - (1\leftrightarrow 2)\big]\,, \\ W^\alpha_{12} &= \tfrac{1}{4}(\gamma_{mn}W_2)^\alpha F^{mn}_1 + W^\alpha_2(k_2\cdot A_1) - (1\leftrightarrow 2)\,, \\ F^{mn}_{12} &= F^{mn}_2(k_2\cdot A_1) + \tfrac{1}{2}F_2^{[m}{}_p F_1^{n]p} + k_1^{[m}(W_1\gamma^{n]}W_2) - (1\leftrightarrow 2)\,, \end{aligned} \tag{44}$$

and satisfy

$$\begin{aligned} D_\alpha A^{12}_\beta + D_\beta A^{12}_\alpha &= \gamma^m_{\alpha\beta}A^{12}_m + (k_1\cdot k_2)(A^1_\alpha A^2_\beta + A^1_\beta A^2_\alpha)\,, \\ D_\alpha A^m_{12} &= \gamma^m_{\alpha\beta}W^\beta_{12} + k^m_{12}A^{12}_\alpha + (k_1\cdot k_2)(A^1_\alpha A^m_2 - A^2_\alpha A^m_1)\,, \\ D_\alpha W^\beta_{12} &= \tfrac{1}{4}(\gamma_{mn})_\alpha{}^\beta F^{mn}_{12} + (k_1\cdot k_2)(A^1_\alpha W^\beta_2 - A^2_\alpha W^\beta_1)\,, \\ D_\alpha F^{mn}_{12} &= k^{[m}_{12}(\gamma^{n]}W_{12})_\alpha + (k_1\cdot k_2)\big[A^1_\alpha F^{mn}_2 + A^{[n}_1(\gamma^{m]}W_2)_\alpha - (1\leftrightarrow 2)\big]\,. \end{aligned} \tag{45}$$

These equations of motion have the same form as in the single-particle case (2) with additional corrections proportional to $(k^1\cdot k^2)$. The construction of (local) multiparticle superfields of arbitrary multiplicity leads to superfields labelled by words

$P = p_1p_2p_3\ldots$ or by arbitrary nested commutators $P = [\ldots[[p_1,p_2],p_3],\ldots]$ (e.g. A^m_{1234} or $F^{mn}_{[1,[2,3]]}$) and can be found in [66, 32].

2.3 Superstring amplitudes with pure spinors

The pure spinor prescription to compute genus-g amplitudes relies on the basic fact that the OPE analysis of primary operators determines a meromorphic function of the vertex positions due to its poles and residues. In the absence of monodromy such as at genus zero, this completely determines the correlator, but this is no longer true at higher genus. On a surface of higher genus, the existence of holomorphic one-forms implies that the knowledge of the positions and residues of the poles from the OPE analysis no longer suffices to completely determine the correlator; the regular terms contain non-trivial information. In principle, the zero modes provide the additional information to find the complete correlator [74]. However, sometimes this is impractical to follow systematically and the calculation benefits from the practical requirements of *homology* and *BRST* invariance[2] constraints to be discussed below.

Homology invariance The introduction of loop momentum integrals with the chiral splitting formalism had to pass the consistency check that the integrated amplitudes were single-valued as a function $\hat{f}(z_i)$ of the vertex positions z_i *after* the loop momentum was integrated out [39]. However, a stronger constraint was proposed[3] in [67, 63]: that the chiral *integrands*, viewed as a function $f(\ell_I, z_i)$ of both the loop momenta ℓ_I and vertex positions z_i should be strictly single-valued under the monodromies of the loop momentum and the vertex positions as they move around A_I and B_I cycles:

$$f(z_i', \ell_I') = f(z_i, \ell_I), \qquad \begin{cases} A_I\text{-cycle}: & (z_i', \ell_I') = (z_i + \delta_{ij}A_J, \ell_I) \\ B_I\text{-cycle}: & (z_i', \ell_I') = (z_i + \delta_{ij}B_J, \ell_I - 2\pi i \delta_I^J k_j) \end{cases} \tag{46}$$

That is, the chiral integrands should be single-valued *before* the loop momentum is integrated out. This requirement interlocks the different sectors of the integrands with different powers of loop momenta with predictive consequences: it can be used to constrain and obtain the superstring integrands themselves.

This requirement of *homology invariance* was used in [62, 63, 64] to determine the integrands of the five, six and seven-point massless amplitudes at genus one, and in [41] to obtain the massless five-point integrand at genus two.

---

[2] It is worth mentioning that several amplitudes computed in this manner used the “minimal” pure spinor formalism and its simpler pure spinor superspace expressions depending only on the zero modes of λ^α and θ^α (the expressions in the non-minimal formalism also depend on $\overline{\lambda}_\alpha, r_\alpha$ in intermediate stages).

[3] It was initially dubbed “monodromy invariance” and it led to the development of *generalized elliptic integrands* (GEI) in the context of genus-one string amplitudes [63].

BRST invariance Superstring scattering amplitudes must be spacetime supersymmetric and gauge invariant. As explained in detail in [1], the cohomology prescription (41) to integrate out the pure spinor zero modes leads to gauge-invariant and supersymmetric expressions if the pure spinor superspace expression is BRST invariant. Recall that when the BRST charge (11) acts on superfield expressions containing only x^m, θ^α (and possibly λ^α), the OPE (30) implies

$$QK(x,\theta) = \lambda^\alpha D_\alpha K(x,\theta) \tag{47}$$

where D_α is the superspace derivative (1) and λ^α is the pure spinor. As we will see, this equation plays an important role in the study of the BRST cohomology properties of string scattering amplitudes. More precisely, if the outcome of the OPEs among the vertices is written in pure spinor superspace as $\langle\lambda^\alpha\lambda^\beta\lambda^g f_{\alpha\beta\gamma}(e_i,k_i,\xi_i,\theta)\rangle$ where e_i, ξ_i and k_i represent a collection of bosonic and fermionic polarizations and their momenta, then the amplitude prescription will give rise to gauge invariant and supersymmetric expressions if

$$\begin{aligned} Q\big(\lambda^\alpha\lambda^\beta\lambda^g f_{\alpha\beta\gamma}(e_i,k_i,\xi_i,\theta)\big) &= 0\,, \\ \lambda^\alpha\lambda^\beta\lambda^g f_{\alpha\beta\gamma}(e_i,k_i,\xi_i,\theta) &\neq Q\Omega\,. \end{aligned} \tag{48}$$

This implies that the superspace expressions of arbitrary scattering amplitudes must be in the *cohomology of the BRST charge*. This requirement together with the OPE structure of the genus-zero pure spinor prescription is enough to completely determine the tree-level scattering amplitudes of ten-dimensional SYM theory [59, 60].

2.3.1 Genus zero

SYM tree amplitudes The knowledge that the genus-zero superstring amplitudes reduce to ten-dimensional SYM tree amplitudes [81] has a powerful consequence: the tree amplitudes in field theory have the same superfield structure as their string theory counterparts. This led to the suggestion that the BRST cohomology structure of pure spinor superspace expressions inspired by the pure spinor prescription could be used to completely fix the form of the SYM tree amplitudes [59]. Using multiparticle superfields, the first non-vanishing tree amplitudes were found to be

$$\begin{aligned} A(1,2,3) &= \langle V_1V_2V_3\rangle\,, \\ A(1,2,3,4) &= \frac{\langle V_{12}V_3V_4\rangle}{s_{12}} + \frac{\langle V_1V_{23}V_4\rangle}{s_{23}}\,, \\ A(1,2,3,4,5) &= \frac{\langle V_{123}V_4V_5\rangle}{s_{12}s_{45}} + \frac{\langle V_{321}V_4V_5\rangle}{s_{23}s_{45}} + \frac{\langle V_{12}V_{34}V_5\rangle}{s_{12}s_{34}} + \frac{\langle V_1V_{234}V_5\rangle}{s_{23}s_{51}} + \frac{\langle V_1V_{432}V_5\rangle}{s_{34}s_{51}}\,. \end{aligned} \tag{49}$$

The regular structure of the BRST variation of certain non-local multiparticle superfield building blocks M_P, the Berends-Giele currents, and various other hints led to the general n-point expression for SYM tree amplitudes in [60]:

$$A(P,n) = \sum_{XY=P} \langle M_X M_Y M_n \rangle \tag{50}$$

where $XY = P$ represents the sum over all deconcatenations of the word P into the words X and Y (including the empty word provided we define $M_\emptyset := 0$). In this language, the amplitudes (49) become

$$\begin{aligned} A(1,2,3) &= \langle M_1 M_2 M_3 \rangle\,, \\ A(1,2,3,4) &= \langle M_{12} M_3 M_4 \rangle + \langle M_1 M_{23} M_4 \rangle\,, \\ A(1,2,3,4,5) &= \langle M_{123} M_4 M_5 \rangle + \langle M_{12} M_{34} M_5 \rangle + \langle M_1 M_{234} M_5 \rangle\,. \end{aligned} \tag{51}$$

The explicit expressions for the Berends-Giele currents in terms of multiparticle superfields, the first of which are given by

$$M_1 = V_1\,, \qquad M_{12} = \frac{V_{12}}{s_{12}}\,, \qquad M_{123} = \frac{V_{123}}{s_{12}s_{123}} + \frac{V_{321}}{s_{23}s_{123}}\,, \tag{52}$$

can be constructed in a multitude of ways (see [82]). Their BRST variation admit a simple all-order form

$$QM_P = \sum_{XY=P} M_X M_Y\,, \tag{53}$$

from which it easily follows that the superfield expression in (50) is BRST closed. It is also not BRST exact, and therefore it is in the cohomology of the BRST charge. To see this, note that M_P contains a divergent propagator $1/s_P$ in the phase space of $|P|+1 = n$ massless particles, so one cannot write the superfields in (50) as $Q(M_P M_n)$. In other words, $M_P M_n$ is not an allowable BRST ancestor, which explains why $\langle \sum_{XY=P} M_X M_Y M_n \rangle \neq 0$.

The n-point superstring disk correlator The general n-point disk correlator of massless string states was computed in [61] using multiparticle superfield techniques to capture the OPE singularities of vertex operators. The result can be written as a sum over $(n-3)!$ SYM field-theory tree amplitudes (50) as follows

$$\mathscr{A}_n(P) = (2\alpha')^{n-3} \int d\mu_P^n \left[\prod_{k=2}^{n-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} A(1,2,\ldots,n) + \mathrm{perm}(2,3,\ldots,n-2) \right], \tag{54}$$

where $\int d\mu_P^n$ is a shorthand for the integration over the vertex positions with integration domain $D(P)$ and weighted by the genus-zero Koba-Nielsen factor $\int_{D(P)} \prod_{j=2}^{n-2} dz_j \prod_{1\leq i<j}^{n-1} |z_{ij}|^{-2\alpha' s_{ij}}$. This result motivated the development of a method [35] to obtain the α' expansion of the integrals in (54). In addition, the conclusion that there is a $(n-3)!$ basis of tree amplitudes in the work of Bern, Carrasco and Johansson [29] becomes manifest as the left-hand side must reduce, in the limit as $\alpha' \to 0$, to a color-ordered SYM tree amplitude $A(P)$ with arbitrary ordering P, which in turn is expanded in terms of $(n-3)!$ tree amplitudes on the right-hand side. For an in-depth discussion of these matters, see [82].

2.3.2 Genus one

We are now going to showcase some of the results obtained with the pure spinor formalism at genus one. For the open string, the amplitudes have the general form

$$\mathscr{A}_n = \sum_{\text{top}} C_{\text{top}} \int_{D_{\text{top}}} d\tau\, dz_2\, dz_3 \ldots dz_n \int d^D\ell\, |\mathscr{I}_n(\ell)|\, \langle \mathscr{K}_n(\ell) \rangle\,, \tag{55}$$

with $\langle \ldots \rangle$ denoting the zero-mode integration prescription (32), which will be presented in the examples below as pure spinor superspace expressions in terms of zero modes of λ^α and θ^α. The integration domains D_{top} for the modular parameter τ and vertex positions z_j must be chosen according to the topologies of a cylinder or a Möbius strip with associated color factors C_{top}. The integration over loop momenta ℓ must be performed as a consequence of the chiral-splitting method, which, in turn, allows to derive massless closed-string one-loop amplitudes from an integrand of double-copy form

$$\mathscr{M}_n = \int_{\mathscr{F}} d^2\tau\, d^2z_2\, d^2z_3 \ldots d^2z_n \int d^D\ell\, |\mathscr{I}_n(\ell)|^2\, \langle \mathscr{K}_n(\ell) \rangle\, \langle \tilde{\mathscr{K}}_n(-\ell) \rangle\,, \tag{56}$$

with $\mathscr{F}$ denoting the fundamental domain for inequivalent tori with respect to the modular group. Both expressions (55) and (56) involve the universal one-loop Koba–Nielsen factor

$$\mathscr{I}_n(\ell) \equiv \exp\Big(\sum_{i<j}^{n} s_{ij} \log \theta_1(z_{ij}, \tau) + \sum_{j=1}^{n} z_j (\ell \cdot k_j) + \frac{\tau}{4\pi i} \ell^2 \Big)\,, \tag{57}$$

with light-like external momenta k_j and $s_{ij} \equiv k_i \cdot k_j$ as well as $z_{ij} \equiv z_i - z_j$.

The Eisenstein-Kronecker series As pointed out above, knowing the singularity structure of the superstring correlators is not enough to reconstruct the full meromorphic integrand as a function of the vertex positions, as crucial information from the non-singular parts is needed. In [34], a generating series of worldsheet functions was proposed that contained an infinite tower of functions $g^{(n)}(z)$ for $n \geq 0$ on a complex elliptic curve describing a genus-one surface with modulus τ. These functions turn out to have the correct properties to capture both the singular part of superstring correlators with $g^{(1)}$ as well as the non-singular pieces with $g^{(n)}, n \geq 2$. More precisely, these functions are constructed via the Laurent series of the Eisenstein-Kronecker series $F(z, \alpha, \tau)$ [36]

$$F(z, \alpha, \tau) \equiv \frac{\theta_1'(0, \tau)\theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\theta_1(\alpha, \tau)} = \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z, \tau) \tag{58}$$

where $\theta_1(z, \tau)$ is the odd Jacobi theta function ($q = e^{2\pi i \tau}$)

$$\theta_1(z, \tau) \equiv 2iq^{1/8} \sin(\pi z) \prod_{j=1}^{\infty} (1 - q^j) \prod_{j=1}^{\infty} (1 - e^{2\pi i z} q^j) \prod_{j=1}^{\infty} (1 - e^{-2\pi i z} q^j)\,, \tag{59}$$

satisfying $\theta_1(z+1,\tau) = -\theta_1(z,\tau)$ and $\theta_1(z+\tau,\tau) = -e^{-\pi i\tau}e^{-2\pi iz}\theta_1(z,\tau)$ as z is moved around the A or B cycle. In addition, $\theta_1'(z,\tau) = \partial_z\theta_1(z,\tau)$. The functions $g^{(n)}$ for the first few cases are $g^{(0)}(z,\tau) = 1$,

$$g^{(1)}(z,\tau) = \partial \log\theta_1(z,\tau)\,, \quad g^{(2)}(z,\tau) = \frac{1}{2}\Big[(\partial\log\theta_1(z,\tau))^2 - \wp(z,\tau)\Big]\,, \tag{60}$$

where $\wp(z,\tau) = -\partial^2\log\theta_1(z,\tau) - \mathrm{G}_2(\tau)$ is the Weierstrass function and $\mathrm{G}_{2k}(\tau)$ are holomorphic Eisenstein series.

The function $g^{(1)}(z,\tau)$ is singular as $z\to 0$ while all $g^{(n)}(z,\tau)$ with $n\geq 2$ are non-singular in this limit. In addition, all $g^{(n)}(z,\tau)$ are single-valued around the A-cycle as $z\to z+1$ but have non-trivial monodromy around the B-cycle as $z\to z+\tau$

$$g^{(n)}(z+\tau,\tau) = \sum_{k=0}^{n}\frac{(-2\pi i)^k}{k!}g^{(n-k)}(z,\tau)\,. \tag{61}$$

For instance, $g^{(1)}(z+\tau,\tau) = -2\pi i$ and $g^{(2)}(z+\tau,\tau) = -2\pi i g^{(1)}(z,\tau) + \frac{1}{2}(2\pi i)^2$. The singularity structure of these functions as well as their monodromies in a genus-one surface provided valuable information to constrain and obtain [62, 63, 64] the genus-one n-point superstring correlators for $n\leq 7$ using the homology invariance principle discussed above. The shorthand $g^{(n)}_{ij} := g^{(n)}(z_i - z_j,\tau)$ will be used below and it will be convenient to define a linearized B-cycle monodromy operator D

$$D = -\frac{1}{2\pi i}\sum_{j=1}^{n}\Omega_j\delta_j \tag{62}$$

where Ω_j are formal variables that capture the B-cycle monodromies around z_j generated by the formal operator δ_j with action $\delta_j\ell = -2\pi i k_j$ and $\delta_j g^{(n)}_{jm} = -2\pi i g^{(n-1)}_{jm}$ for $n\geq 1$ as well as $\delta_j g^{(0)}_{jm} = 0$ and $\delta_j g^{(n)}_{im} = 0$ for all $i,m\neq j$. As discussed in [63], there is a remarkable duality relating the operator D with the BRST charge Q.

BRST building blocks The other ingredient used to obtain the genus-one superstring correlators was the BRST invariance property of the integrands. This was addressed by the construction of BRST building blocks with covariant BRST transformations, using multiparticle superfields techniques in combination with pure spinor zero-mode analysis and group theory to constrain the appearance of superfields. This led to the definition of multiple BRST building blocks with different BRST transformation properties allowing for the construction of BRST invariants in the pure spinor cohomology.

For instance, the zero-mode sector with four d_α zero modes from the b ghost suggests the scalar building blocks

$$T_{A,B,C} = \frac{1}{3}(\lambda\gamma_m W_A)(\lambda\gamma_n W_B)F^{mn}_C + \mathrm{cyclic}(A,B,C)\,. \tag{63}$$

in terms of multiparticle superfields labelled by words A,B,C. Their BRST variation following (47) is given by ($k_\emptyset \equiv 0$)

$$QT_{A,B,C} = \sum_{\substack{A=XjY \\ Y=R\sqcup S}} (k_X \cdot k_j)\big[V_{XR}T_{jS,B,C} - V_{jR}T_{XS,B,C}\big] + (A \leftrightarrow B,C)\,, \tag{64}$$

where $\sqcup$ denotes the shuffle product defined iteratively by [73]

$$\emptyset \sqcup P = P \sqcup \emptyset := P\,, \qquad iP \sqcup jQ := i(P \sqcup jQ) + j(Q \sqcup iP)\,, \tag{65}$$

for letters i and j, words P and Q with $\emptyset$ representing the empty word. For example, $1 \sqcup 23 = 123 + 213 + 231$.

For an illustration of (64), the BRST variations of all $T_{A,B,C}$ up to multiplicity five are given by

$$\begin{aligned} QT_{1,2,3} &= 0\,, \\ QT_{12,3,4} &= (k_1 \cdot k_2)\big[V_1 T_{2,3,4} - V_2 T_{1,3,4}\big]\,, \\ QT_{123,4,5} &= (k_1 \cdot k_2)\big[V_1 T_{23,4,5} + V_{13} T_{2,4,5} - V_2 T_{13,4,5} - V_{23} T_{1,4,5}\big] \\ &\quad + (k_{12} \cdot k_3)\big[V_{12} T_{3,4,5} - V_3 T_{12,4,5}\big]\,, \\ QT_{12,34,5} &= (k_1 \cdot k_2)\big[V_1 T_{2,34,5} - V_2 T_{1,34,5}\big] + (12 \leftrightarrow 34)\,. \end{aligned} \tag{66}$$

Other zero-mode contributions from the b ghost give rise to tensorial building blocks with an arbitrary number of vector indices. For simplicity, the vector building block has the form

$$T^m_{A,B,C,D} \equiv \big[A^m_A T_{B,C,D} + (A \leftrightarrow B,C,D)\big] + W^m_{A,B,C,D} \tag{67}$$

with

$$W^m_{A,B,C,D} = \frac{1}{12}(\lambda\gamma_n W_A)(\lambda\gamma_p W_B)(W_C \gamma^{mnp} W_D) + (A,B|A,B,C,D) \tag{68}$$

with the notation $(A_1,\ldots,A_p\,|A_1,\ldots,A_n)$ instructing to sum over all possible ways to choose p elements $A_1,A_2,\ldots,A_p$ out of the set $\{A_1,\ldots,A_n\}$, for a total of $\binom{n}{p}$ terms.

The BRST transformation of (67) is given by

$$QT^m_{A,B,C,D} = k^m_A V_A T_{B,C,D} + \sum_{\substack{A=XjY \\ Y=R\sqcup S}} (k_X \cdot k_j)\big[V_{XR} T^m_{jS,B,C,D} - V_{jR} T^m_{XS,B,C,D}\big] + (A \leftrightarrow B,C,D)\,, \tag{69}$$

for example,

$$\begin{aligned} QT^m_{1,2,3,4} &= k^m_1 V_1 T_{2,3,4} + (1 \leftrightarrow 2,3,4)\,, \\ QT^m_{12,3,4,5} &= \big[k^m_{12} V_{12} T_{3,4,5} + (12 \leftrightarrow 3,4,5)\big] + (k_1 \cdot k_2)\big(V_1 T^m_{2,3,4,5} - V_2 T^m_{1,3,4,5}\big)\,. \end{aligned} \tag{70}$$

Other building blocks were defined in [62] to capture the gauge anomaly of the field-theory SYM integrands that disappear in the $SO(32)$ superstring [49, 50].

Four points At genus one, the simplest scattering amplitude with four massless states computed in 1982 by Green and Schwarz [58] was reproduced in a 2004 calculation using the minimal pure spinor formalism [25]. A salient feature of this calculation is the absence of OPE singularities among the vertices; the amplitude is completely determined by the pure spinor zero modes. The result of the correlator in the conventions of (55) is given by

$$\mathcal{K}_4(\ell) = V_1 T_{2,3,4}\,, \tag{71}$$

and its zero-mode evaluation can be written in terms of the tree-level SYM amplitude $A^{\rm SYM}$ as follows

$$\langle V_1 T_{2,3,4}\rangle = s_{12}s_{23}A^{\rm SYM}(1,2,3,4)\,. \tag{72}$$

For Neveu-Schwarz external states, the zero-mode evaluation of (72) yields the famous t_8 tensor, $\langle V_1 T_{2,3,4}\rangle = \frac{1}{2}t_8(f_1,f_2,f_3,f_4)$ where

$$t_8(f_1,f_2,f_3,f_4) = {\rm tr}(f_1f_2f_3f_4) - \frac{1}{4}{\rm tr}(f_1f_2){\rm tr}(f_3f_4) + {\rm cyclic}(2,3,4)\,, \tag{73}$$

and tr represents a trace over Lorentz indices, for example ${\rm tr}(f_1,f_2) = f_1^{mn}f_2^{nm}$.

Five points At five points, an analysis of the structure of the superstring correlator arising from the pure spinor prescription (26) reveals that the it is composed of two sectors: one containing a loop momentum contracting a vectorial combination of superfields and no OPE singularities, and another with no loop momentum and with singularities as vertex positions collide multiplying a collection of superfields with no free vector indices. Combining this information with the BRST transformation properties of scalar (64) and vectorial building blocks (69) as well as the monodromy properties of the functions $g^{(n)}(z,\tau)$ and ℓ^m, yields the proposal for the five-point correlator

$$\begin{aligned}\mathcal{K}_5(\ell,z_i) &= V_1 T^m_{2,3,4,5}\ell^m \\ &\quad + V_{12}T_{3,4,5}g^{(1)}_{12} + (2\leftrightarrow 3,4,5) \\ &\quad + V_1 T_{23,4,5}g^{(1)}_{23} + (2,3|2,3,4,5)\,.\end{aligned} \tag{74}$$

This is BRST invariant up to total worldsheet derivatives since

$$\begin{aligned}Q\mathcal{K}_5(\ell,z_i)\mathcal{I}_5(\ell) &= -V_1V_2T_{3,4,5}\big((\ell\cdot k_2) + s_{21}g^{(1)}_{21} + s_{23}g^{(1)}_{23} + s_{24}g^{(1)}_{24} + s_{25}g^{(1)}_{25}\big) \\ &= -V_1V_2T_{3,4,5}\frac{\partial}{\partial z_2}\mathcal{I}_5(\ell)\,,\end{aligned} \tag{75}$$

where $\mathcal{I}_5(\ell)$ is the Koba-Nielsen factor (57).

The correlator (74) is also homology invariant up to BRST-exact terms, around both A and B cycles as a function of ℓ^m and z_i. In other words, $\mathcal{K}_5(\ell,z_i)$ is an example of a generalized elliptic integrand [63]. To see this note that ℓ^m and $g^{(n)}_{ij}$ are single-valued around A cycles while $\ell^m \to \ell^m - 2\pi i k_j^m$ and $g_{ij}^{(1)} \to -2\pi i$ as z_j is moved around the B cycle (with $\tau \to \tau + 1$). That is, under the action of the monodromy operator (62) we get

$$\begin{aligned} D\mathscr{K}_5(\ell) &= \Omega_1\Big(k_1^m V_1 T^m_{2,3,4,5} + \big[V_{12}T_{3,4,5} + 2 \leftrightarrow 3,4,5\big]\Big) \\ &\quad + \Omega_2\Big(k_2^m V_1 T^m_{2,3,4,5} + V_{21}T_{3,4,5} + \big[V_1 T_{23,4,5} + 3 \leftrightarrow 4,5\big]\Big) + (2 \leftrightarrow 3,4,5)\,, \end{aligned} \tag{76}$$

which can be shown to be BRST-exact [62] as it is BRST closed and a *local* five-point expression.

Six points Similar considerations of the zero-mode structure from the pure spinor prescription together with BRST and homology invariance were used to determine the six-point correlator at genus one in [64]. The result can be written as[4]

$$\begin{aligned} \mathscr{K}_6(\ell, z_i) &= \frac{1}{2} V_1 T^{mn}_{2,3,4,5,6} \mathscr{Z}^{mn}_{1,2,3,4,5,6} \\ &\quad + V_{12} T^m_{3,4,5,6} \mathscr{Z}^m_{12,3,4,5,6} + (2 \leftrightarrow 3,4,5,6) \\ &\quad + V_1 T^m_{23,4,5,6} \mathscr{Z}^m_{1,23,4,5,6} + (2,3|2,3,4,5,6) \\ &\quad + V_{123} T_{4,5,6} \mathscr{Z}_{123,4,5,6} + V_{132} T_{4,5,6} \mathscr{Z}_{132,4,5,6} + (2,3|2,3,4,5,6) \\ &\quad + V_1 T_{234,5,6} \mathscr{Z}_{1,234,5,6} + V_1 T_{243,5,6} \mathscr{Z}_{1,243,5,6} + (2,3,4|2,3,4,5,6) \\ &\quad + \big[\big(V_{12} T_{34,5,6} \mathscr{Z}_{12,34,5,6} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6)\big] \\ &\quad + \big[\big(V_1 T_{2,34,56} \mathscr{Z}_{1,2,34,56} + \mathrm{cyc}(3,4,5)\big) + (2 \leftrightarrow 3,4,5,6)\big]\,, \end{aligned} \tag{77}$$

where the shorthand for the worldsheet functions are

$$\begin{aligned} \mathscr{Z}_{123,4,5,6} &= g_{12}^{(1)} g_{23}^{(1)} + g_{12}^{(2)} + g_{23}^{(2)} - g_{13}^{(2)}\,, \\ \mathscr{Z}_{12,34,5,6} &= g_{12}^{(1)} g_{34}^{(1)} + g_{13}^{(2)} + g_{24}^{(2)} - g_{14}^{(2)} - g_{23}^{(2)}\,, \\ \mathscr{Z}^m_{12,3,4,5,6} &= \ell^m g_{12}^{(1)} + (k_2^m - k_1^m) g_{12}^{(2)} + \big[k_3^m (g_{13}^{(2)} - g_{23}^{(2)}) + (3 \leftrightarrow 4,5,6)\big]\,, \\ \mathscr{Z}^{mn}_{1,2,3,4,5,6} &= \ell^m \ell^n + \big[(k_1^m k_2^n + k_1^n k_2^m) g_{12}^{(2)} + (1,2|1,2,3,4,5,6)\big]\,. \end{aligned} \tag{78}$$

After a lenghty calculation, the correlator (77) was shown to be homology invariant up to vanishing BRST-exact terms therefore constituting a six-point example of a generalized elliptic integrand. The analysis of BRST invariance is more subtle as the six-point open-string correlator at genus one is anomalous before summing over the different worldsheet topologies including the Möbius strip [49, 50]. Since gauge invariance is reflected on BRST invariance, to study anomalous correlators the concept of BRST pseudo invariance was introduced in [70]. The idea is that the BRST variation of pseudo-invariant superfields generate anomalous superfields

[4] As discussed in [64], there is a beautiful Lie-polynomial compact representation of higher-point genus-one correlators which reveals a common structure with genus zero correlators and elucidates the combinatorics of (77). However, as the notation requires concepts such as the decreasing Lyndon factorization of words and Lie polynomials [73] we chose to omit it here for brevity.

$$Y_{A,B,C,D,E} = \frac{1}{2}(\lambda\gamma^m W_A)(\lambda\gamma^n W_B)(\lambda\gamma^p W_C)(W_D\gamma_{mnp}W_E) \tag{79}$$

generalizing the pure-spinor superspace expression found in the six-point anomaly analysis of [28], $(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)(W_5\gamma_{mnp}W_6)$, with parity-odd component expansion

$$\langle(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)(W_5\gamma_{mnp}W_6)\rangle = -\frac{1}{16}\varepsilon_{10}^{m_2n_2\ldots m_6n_6}F^2_{m_2n_2}\ldots F^6_{m_6n_6}. \tag{80}$$

This is captured by the correlator (77) as its BRST variation, after discarding total worldsheet derivatives, is given by

$$Q\mathcal{K}_6(\ell,z_i)\mathcal{I}_6(\ell) = -2\pi i V_1 Y_{2,3,4,5,6}\frac{\partial}{\partial\tau}\log\mathcal{I}_6(\ell)\,. \tag{81}$$

Thus, the BRST variation is a boundary term in moduli space [37] and vanishes due to the anomaly cancellation effect of summing over the different worldsheet topologies when the gauge group is $SO(32)$ [49, 50].

Seven points A seven-point open-string correlator at genus one was also obtained in [64] and can be written using various kinds of generalized elliptic integrands $E^{\cdots}_{\cdots}$ discussed at length in [63]

$$\begin{aligned}
\mathcal{K}_7(\ell,z_i) = {} & \frac{1}{6}V_1 T^{mnp}_{2,3,\ldots,7}E^{mnp}_{1|2,3,\ldots,7} \\
& + \frac{1}{2}V_1 T^{mn}_{23,4,5,6,7}E^{mn}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7) \\
& + \big[V_1 T^m_{234,5,6,7}E^m_{1|234,5,6,7} + V_1 T^m_{243,5,6,7}E^m_{1|243,5,6,7}\big] + (2,3,4|2,3,4,5,6,7) \\
& + \big[V_1 T^m_{23,45,6,7}E^m_{1|23,45,6,7} + \mathrm{cyc}(2,3,4)\big] + (6,7|2,3,4,5,6,7) \\
& + \big[V_1 T_{2345,6,7}E_{1|2345,6,7} + \mathrm{perm}(3,4,5)\big] + (2,3,4,5|2,3,4,5,6,7) \\
& + \big[V_1 T_{234,56,7}E_{1|234,56,7} + V_1 T_{243,56,7}E_{1|243,56,7} + \mathrm{cyc}(5,6,7)\big] + (2,3,4|2,3,4,5,6,7) \\
& + \big[V_1 T_{23,45,67}E_{1|23,45,67} + \mathrm{cyc}(4,5,6)\big] + (3 \leftrightarrow 4,5,6,7) \\
& - V_1 J^m_{2|3,4,5,6,7}E^m_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7) \\
& - V_1 J_{23|4,5,6,7}E_{1|23|4,5,6,7} + (2,3|2,3,4,5,6,7) \\
& - \big[V_1 J_{2|34,5,6,7}E_{1|2|34,5,6,7} + \mathrm{cyc}(2,3,4)\big] + (2,3,4|2,3,4,5,6,7)\,.
\end{aligned} \tag{82}$$

This was shown to be BRST (pseudo) invariant and also homology invariant up to BRST-exact terms and total derivatives in the worldsheet and in moduli space.

2.3.3 One-loop SYM integrands from the cohomology of pure spinor superspace

Another application of the pure spinor formalism and related ideas resulted in expressions for the 1-loop integrands of ten-dimensional SYM theory [65]. The idea is to use the zero-mode structure suggested by the pure spinor prescription, i.e., after removing non-zero modes via OPEs leading to multiparticle superfields, to directly propose SYM 1-loop integrands $A(1,2,\ldots,n|\ell)$ governing the integrated single-trace amplitude via

$$A(1,2,3,\ldots,n) = \int \frac{d^D\ell}{(2\pi)^D} \langle A(1,2,3,\ldots,n|\ell)\rangle \,. \tag{83}$$

More precisely, the 1-loop integrands are expanded in terms of cubic graphs Γ_i

$$A(1,2,3,\ldots,n|\ell) = \sum_{\Gamma_i} \frac{N_i(\ell)}{\prod_k P_{k,i}(\ell)} \,, \tag{84}$$

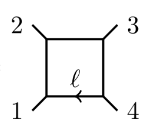
where the sum is over all 1-loop cubic graphs from boxes to n-gons, excluding triangles, bubbles and tadpoles [30]. Note that the superspace numerators $N_i(\ell)$ and the propagators $P_{k,i}(\ell)$ not only depend on the external kinematics but also on the loop momentum ℓ. In proposing the integrand (84), one respects the supersymmetry constraint that the numerators of a p-gon diagram contain at most $p-4$ powers of ℓ. Furthermore, it is not difficult to be convinced that overall BRST invariance of the integrand can be achieved only if each term of $QN_i(\ell)$ has a factor of $P_{k,i}(\ell)$ with $k = 1,2,\ldots,n$. Schematically,

$$QN_i(\ell) = \sum P_{k,i}(\ldots) \,, \tag{85}$$

for some subset of k with the ellipsis representing combinations of (multiparticle) superfields. Integrands up to six points were found in [65] following these lines.

Four point integrand The integrand of the color-ordered amplitude is expressed in terms of a single box with a BRST-invariant numerator:

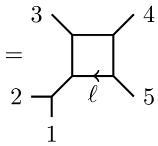
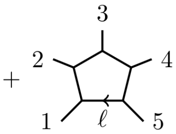
$A(1,2,3,4|\ell) \quad =$



$$= \frac{V_1 T_{2,3,4}}{\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2} \,. \tag{86}$$

This integrand is manifestly BRST invariant using (66) and agrees with the result obtained by Green, Brink and Schwarz [57] from the field-theory limit of string theory.

Five point integrand The integrand of the SYM five-point one-loop amplitude is expanded in terms of five boxes and one pentagon:

$A(1,2,3,4,5|\ell) =$  $+ \text{ cyclic}(12345) +$  (87)

$$= A_{\text{box}}(1,2,3,4,5) + A_{\text{pent}}(1,2,3,4,5|\ell)$$

with the corresponding pure spinor superspace expressions given by

$$A_{\text{box}}(1,2,3,4,5) = \frac{V_{12}T_{3,4,5}}{(k_1+k_2)^2\ell^2(\ell-k_{12})^2(\ell-k_{123})^2(\ell-k_{1234})^2} + \frac{V_1 T_{23,4,5}}{(k_2+k_3)^2\ell^2(\ell-k_1)^2(\ell-k_{123})^2(\ell-k_{1234})^2} + \frac{V_1 T_{2,34,5}}{(k_3+k_4)^2\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{1234})^2} + \frac{V_1 T_{2,3,45}}{(k_4+k_5)^2\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2} + \frac{V_{51}T_{2,3,4}}{(k_1+k_5)^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2(\ell-k_{1234})^2} \quad (88)$$

$$A_{\text{pent}}(1,2,3,4,5|\ell) = \frac{N^{(5)}_{1|2,3,4,5}(\ell)}{\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2(\ell-k_{1234})^2} \quad (89)$$

and pentagon numerator

$$N^{(5)}_{1|2,3,4,5}(\ell) = \ell_m V_1 T^m_{2,3,4,5} + \frac{1}{2}\big[V_{12}T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big] + \frac{1}{2}\big[V_1 T_{23,4,5} + (2,3|2,3,4,5)\big]. \quad (90)$$

To see that this integrand is BRST invariant note that the BRST variation of the local pentagon numerator satisfies the criterion of canceling pentagon propagators as

$$QN^{(5)}_{1|2,3,4,5}(\ell) = \frac{1}{2}V_1V_2T_{3,4,5}\big[(\ell-k_{12})^2 - (\ell-k_1)^2\big] + \frac{1}{2}V_1V_3T_{2,4,5}\big[(\ell-k_{123})^2 - (\ell-k_{12})^2\big] + \frac{1}{2}V_1V_4T_{2,3,5}\big[(\ell-k_{1234})^2 - (\ell-k_{123})^2\big] + \frac{1}{2}V_1V_5T_{2,3,4}\big[\ell^2 - (\ell-k_{1234})^2\big] \quad (91)$$

implies that $QA_{\text{pent}}(1,2,3,4,5|\ell)$ becomes a sum of boxes of the same type as contained in $A_{\text{box}}(1,2,3,4,5)$. In turn, the BRST variation of the boxes cancels the external propagators with external momenta k_i in (88) rather than the internal propagators with loop momentum. Therefore the BRST variation of $QA_{\text{box}}(1,2,3,4,5)$

is still a sum of boxes,

$$
\begin{aligned}
QA_{\text{box}}(1,2,3,4,5) = &\frac{V_1V_2T_{3,4,5}}{2\ell^2(\ell-k_{123})^2(\ell-k_{1234})^2}\left(\frac{1}{(\ell-k_{12})^2}-\frac{1}{(\ell-k_1)^2}\right)\\
&+\frac{V_1V_3T_{2,4,5}}{2\ell^2(\ell-k_1)^2(\ell-k_{1234})^2}\left(\frac{1}{(\ell-k_{123})^2}-\frac{1}{(\ell-k_{12})^2}\right)\\
&+\frac{V_1V_4T_{2,3,5}}{2\ell^2(\ell-k_1)^2(\ell-k_{12})^2}\left(\frac{1}{(\ell-k_{1234})^2}-\frac{1}{(\ell-k_{123})^2}\right)\\
&+\frac{V_1V_5T_{2,3,4}}{2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2}\left(\frac{1}{\ell^2}-\frac{1}{(\ell-k_{1234})^2}\right) .
\end{aligned}
$$

which ultimately cancels the variation of the pentagon (89), leading to an overall BRST invariant five-point one-loop integrand. This example illustrates the mechanism that the BRST variation of a numerator must be engineered to cancel either internal or external propagators in order to achieve overall BRST invariance.

It is worth mentioning that the proposal (88) obtained from BRST considerations alone in 2014 [65] has been derived in 2021 [33] from the field-theory limit of the Kronecker-Eisenstein coefficient functions appearing in the genus-one chiral correlators derived in 2018 [64] using arguments from BRST and homology invariance.

Six point integrand The six point integrand is composed of 21 boxes, 6 pentagons and 1 hexagon

$$A(1,2,\ldots,6|\ell) = A_{\text{box}}(1,2,\ldots,6) + A_{\text{pent}}(1,2,\ldots,6|\ell) + A_{\text{hex}}(1,2,\ldots,6|\ell)\,, \quad (92)$$

whose superspace expressions can be found in [65]. A noteworthy feature of the pure spinor superspace proposal for (92) is that it leads to an anomalous integrated BRST variation

$$\int d^D\ell\, QA(1,2,3,4,5,6|\ell) = -\frac{\pi^5}{240}V_1Y_{2,3,4,5,6}\,, \quad (93)$$

signaling the well-known fact that the ten-dimensional SYM theory is anomalous at one loop, see [65] for more details.

Note that the six-point one-loop integrand was recently derived in [33] in a parameterization satisfying the one-loop color-kinematics duality.

2.3.4 Genus two

After the pioneering genus-two calculation with four massless NS states with the RNS formalism in [40], the pure spinor formalism was used in [23, 22] to extend the computation to the supersymmetric graviton multiplet (see also [24, 68, 45] for explicit component expansions and the overall normalization factor). For five massless closed-string states, the supersymmetric amplitudes were computed in the low-energy approximation including their overall normalization in [48] and later extended to all orders in α' in [41, 42].

The n-point amplitude prescription (27) gives rise to a chiral amplitude $\mathscr{F}_n$ which factorizes into a Koba-Nielsen factor (in conventions where $s_{ij} = k_i \cdot k_j$)

$$\mathscr{I}_n = \exp\Big(\frac{1}{4\pi i}\Omega_{IJ}\ell^I \cdot \ell^J - \sum_{i=1}^{n}(\ell^I \cdot k_i)\int_{z_0}^{z_i}\omega_I + \sum_{i<j}^{n} s_{ij}\ln E(z_i,z_j)\Big) \tag{94}$$

and a chiral correlator $\mathscr{K}_n(\ell^I, z_i)$ carrying the dependence on loop momenta, vertex operator positions and the polarizations and external momenta of the string states. Since the vertex positions will be integrated over the Riemann surface one is free to use chiral correlators which differ by total derivatives as representing the same amplitude under integration by parts (IBP). For instance, the logarithmic derivative of the Koba-Nielsen factor is a primary example of an IBP generator:

$$\partial_{z_1}\ln\mathscr{I}_5 = -(\ell^I\cdot k_1)\omega_I(z_1) + s_{12}\eta_{12} + s_{13}\eta_{13} + s_{14}\eta_{14} + s_{15}\eta_{15}\,. \tag{95}$$

BRST invariance The integration of the zero modes of pure spinor fields together with considerations from group theory to piece together Lorentz-invariant combinations of superfields initially led to the introduction of pure spinor superfield building blocks with four external states in [24]

$$T_{1,2|3,4} = \frac{1}{64}(\lambda\gamma_{mnpqr}\lambda)F_1^{mn}F_2^{pq}\big[F_3^{rs}(\lambda\gamma_s W_4) + F_4^{rs}(\lambda\gamma_s W_3)\big] + (1,2 \leftrightarrow 3,4) \tag{96}$$

Considerations involving the BRST variations of suitable multiparticle superfields allowed an all-multiplicity generalization of (96) to be found in [69]. Using the language of (minimal) pure spinor superspace, these generalizations have the form [69]

$$T_{A,B|C,D} = \frac{1}{64}(\lambda\gamma_{mnpqr}\lambda)F_A^{mn}F_B^{pq}\big[F_C^{rs}(\lambda\gamma_s W_D) + F_D^{rs}(\lambda\gamma_s W_C)\big] + (A,B \leftrightarrow C,D) \tag{97}$$

In addition, starting from the five-point correlator there are additional Lorentz scalar and tensorial building blocks

$$T^m_{1,2,3|4,5} = A_1^m T_{2,3|4,5} + A_2^m T_{1,3|4,5} + A_3^m T_{1,2|4,5} + W^m_{1,2,3|4,5}\,, \tag{98}$$
$$T_{1;2|3|4,5} = \frac{1}{2}\Big((k_1^m + k_2^m - k_3^m)T^m_{1,2,3|4,5} + T_{12,3|4,5} + T_{13,2|4,5} + T_{23,1|4,5}\Big)\,,$$

where $W^m_{1,2,3|4,5}$ is designed in a way as to give the desired BRST variation and symmetry properties to $T^m_{1,2,3|4,5}$, see below. Its explicit form can be found in [69]. The BRST variation of the four-point building block is given by

$$QT_{1,2|3,4} = 0\,, \tag{99}$$

while the BRST variation of the five-point building blocks are given by

$$QT_{12,3|4,5} = s_{12}(V_1T_{2,3|4,5} - V_2T_{1,3|4,5}), \tag{100}$$
$$QT_{1;2|3|4,5} = s_{12}V_1T_{2,3|4,5},$$
$$QT^m_{1,2,3|4,5} = k_1^mV_1T_{2,3|4,5} + k_2^mV_2T_{1,3|4,5} + k_3^mV_3T_{1,2|4,5}.$$

Furthermore, these building blocks satisfy various crucial identities to capture the correct features of the integrand

$$T_{A,B|C,D} = T_{B,A|C,D} = T_{C,D|A,B}, \quad T_{A,B|C,D} + T_{B,C|A,D} + T_{C,A|B,D} = 0 \tag{101}$$
$$T^m_{1,2,3|4,5} = T^m_{(1,2,3)|(4,5)}, \quad \langle T^m_{1,2,3|4,5}\rangle = \langle T^m_{3,4,5|1,2} + T^m_{2,4,5|1,3} + T^m_{1,4,5|2,3}\rangle$$
$$T_{1;2|3|4,5} = T_{1;2|3|5,4}, \quad \langle T_{1;2|3|4,5} + T_{1;2|4|5,3} + T_{1;2|5|3,4}\rangle = 0$$
$$\langle k_3^m(T^m_{1,2,3|4,5} + T^m_{3,4,5|1,2}) - T_{13,2|4,5} - T_{23,1|4,5} + T_{34,5|1,2} + T_{35,4|1,2}\rangle = 0$$
$$k_1^mT^m_{1,2,3|4,5} = T_{2;1|3|4,5} + T_{3;1|2|4,5}$$
$$\langle k_5^mT^m_{1,2,3|4,5}\rangle = \langle T_{1;5|4|2,3} + T_{2;5|4|1,3} + T_{3;5|4|1,2}\rangle$$
$$T_{1;2|3|4,5} - T_{2;1|3|4,5} = T_{12,3|4,5}$$
$$\langle T_{5;1|2|3,4} + T_{5;2|1|3,4} + T_{5;3|4|1,2} + T_{5;4|3|1,2}\rangle = 0$$

where the identities that hold only in the cohomology have been indicated by the pure spinor bracket.

Homology invariance The genus two integrands up to five points can be written in terms of the holomorphic differentials ω_I and loop momenta ℓ_I^m, the prime form $E(z_i,z_j)$, and single derivatives of its logarithm $\partial_i \ln E(z_i,z_j)$. Note that the prime form $E(z,w)$ is holomorphic in z and w, odd under $z \leftrightarrow w$, and has a unique simple zero at $z = w$. Both the loop momentum and the prime form are single valued when z is moved around A_I cycles, but they have non-trivial monodromy around a B_I cycle [39]

$$E(z+B_I,w) = -\exp\left(-i\pi\Omega_{II} - 2\pi i\int_w^z \omega_I\right)E(z,w) \tag{102}$$
$$\partial_z \ln E(z+B_I,w) = \partial_z \ln E(z,w) - 2\pi i\omega_I(z)$$
$$\partial_z \ln E(z,w+B_I) = \partial_z \ln E(z,w) + 2\pi i\omega_I(z)$$
$$\ell_I^m = \ell_I^m - 2\pi i k_i^m$$

In order to avoid clutter in the formulas below, it is convenient to define the genus-two propagator as

$$\eta_{ij} = \frac{\partial}{\partial z_i}\ln E(z_i,z_j). \tag{103}$$

Basis of holomorphic one-forms At genus two the holomorphic one-forms $\omega_I(z_i)$ are labelled by $I = 1,2$ and modular invariance of the string amplitude suggests that they form $SL(2)$ invariant singlets[5], where $\omega_I(z)$ is the (1) of $SL(2)$. At four points, the tensor decomposition $(1)\otimes(1)\otimes(1)\otimes(1) = 2(0)\oplus 3(2)\oplus(4)$ [77] shows that there are two scalars in the decomposition of a four-fold product of $\omega_I(z_i)$. Using the definition

$$\Delta_{ij} = \varepsilon^{IJ}\omega_I(z_i)\omega_J(z_j) \tag{104}$$

the two-dimensional basis of scalars composed of four holomorphic one-forms can be taken in a cyclic arrangement of labels:

$$\Delta_{12}\Delta_{34}, \quad \Delta_{23}\Delta_{41}\,. \tag{105}$$

A third scalar $\Delta_{13}\Delta_{24}$ is not independent as the antisymmetrization over three indices vanish

$$0 = \varepsilon^{I[J}\varepsilon^{KL]}\omega_I(z_1)\omega_J(z_2)\omega_K(z_2)\omega_L(z_4) \longrightarrow \Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} - \Delta_{41}\Delta_{23}\,. \tag{106}$$

At five points, the decomposition $(1)\otimes(1)\otimes(1)\otimes(1)\otimes(1) = 5(1)\oplus 4(3)\oplus(5)$ shows that there exists a five-dimensional basis of a five-fold product of one-forms. These can be taken in a cyclic basis [41][6]

$$\begin{gathered}\omega_I(z_1)\Delta_{23}\Delta_{45}\,, \quad \omega_I(z_2)\Delta_{34}\Delta_{51}\,, \quad \omega_I(z_3)\Delta_{45}\Delta_{12}\,, \\ \omega_I(z_4)\Delta_{51}\Delta_{23}\,, \quad \omega_I(z_5)\Delta_{12}\Delta_{34}\,.\end{gathered} \tag{107}$$

Four points The massless four-point chiral integrand was obtained using the minimal pure spinor formalism in [23] and using the non-minimal formalism in [22]. Luckily, both versions of the formalism imply that the chiral integrand is obtained purely from the zero modes of pure spinor variables. A short analysis of the zero-mode structure of the contributing SYM superfields together with a group theory analysis of $SO(10)$ scalars in pure spinor superspace using a $U(5)$ decomposition of pure spinors implies [23]

$$\mathscr{K}_4 = \langle T_{1,2|3,4}\rangle\Delta_{41}\Delta_{23} + \langle T_{4,1|2,3}\rangle\Delta_{12}\Delta_{34} \tag{108}$$

where $T_{i,j|k,l}$ is the kinematic factor (96) in the minimal pure spinor formalism and Δ_{ij} is defined in (104). It is easy to see that the chiral correlator (108) is BRST closed using (100). Moreover, it is manifestly single valued as it only depends on the vertex positions via Δ_{ij}.

It was shown in [68] via pure spinor BRST cohomology identities that the genus-two kinematic factor (96) is proportional to the four-point tree amplitude (50):

[5] Discussions with Oliver Schlotterer are warmly acknowledged at this point.

[6] An algorithm to arrive at this basis uses the two identities $\Delta_{ij}\Delta_{kl} = -\Delta_{il}\Delta_{kj} - \Delta_{ij}\Delta_{lk}$ and $\omega_I(z_i)\Delta_{jk} = -\omega_I(z_k)\Delta_{ij} - \omega_I(z_j)\Delta_{ki}$ repeatedly until all factors are in cyclic order. It can be shown that the two identities $\omega_I(z_1)\Delta_{24}\Delta_{35} = -\omega_I(z_5)\Delta_{12}\Delta_{34} - \omega_I(z_2)\Delta_{51}\Delta_{34} + \omega_I(z_1)\Delta_{23}\Delta_{45}$ and $\omega_I(z_1)\Delta_{25}\Delta_{34} = -\omega_I(z_5)\Delta_{12}\Delta_{34} - \omega_I(z_2)\Delta_{34}\Delta_{51}$ as well as their permutations are enough to rewrite all products of five one-forms in the basis (107).

$$\langle T_{1,2|3,4}\rangle = s_{12}^2 s_{23} A(1,2,3,4)\,. \tag{109}$$

Five points Several equivalent expressions for the five-point chiral integrand, emphasizing different properties, were given in [41]. For instance,

$$\begin{aligned}\mathscr{K}_5(\ell^I, z_i) = {} & \big[\ell^I_m T^m_{1,2,3|4,5}\Delta_{51}\omega_I(z_2)\Delta_{34} + \text{cycl}(1,2,3,4,5)\big] \\ & + \Big[\eta_{12}\big(T_{1;2|3|4,5}\Delta_{24}\Delta_{35} + T_{1;2|4|3,5}\Delta_{23}\Delta_{45}\big) + (1,2|1,2,3,4,5)\Big] \\ & + \Big[\eta_{21}\big(T_{2;1|3|4,5}\Delta_{14}\Delta_{35} + T_{2;1|4|3,5}\Delta_{13}\Delta_{45}\big) + (1,2|1,2,3,4,5)\Big]\end{aligned} \tag{110}$$

where the notation $+(i,j|1,2,3,4,5)$ means a sum over all ordered choices of i and j from the set $\{1,2,3,4,5\}$ for a total of $\binom{5}{2}$ terms.

BRST invariance Using the BRST variation (100) of the building blocks, the BRST variation of the chiral correlator (110) can be written as

$$\begin{aligned}Q\mathscr{K}_5(\ell^I, z_i) = {} & V_1 T_{5,2|3,4}\Delta_{23}\Delta_{45}(\ell^I\cdot k_1)\omega_I(z_1) \\ & + V_1 T_{2,3|4,5}\Delta_{12}\Delta_{34}(\ell^I\cdot k_1)\omega_I(z_5) \\ & + V_1 T_{2,3|4,5}\Delta_{51}\Delta_{34}(\ell^I\cdot k_1)\omega_I(z_2) \\ & + V_1\big(T_{2,3|4,5}\Delta_{24}\Delta_{35} + T_{2,4|3,5}\Delta_{23}\Delta_{45}\big)s_{12}\eta_{12} \\ & + V_1\big(T_{3,2|4,5}\Delta_{34}\Delta_{25} + T_{3,4|2,5}\Delta_{32}\Delta_{45}\big)s_{13}\eta_{13} \\ & + V_1\big(T_{4,2|3,5}\Delta_{43}\Delta_{25} + T_{4,3|2,5}\Delta_{42}\Delta_{35}\big)s_{14}\eta_{14} \\ & + V_1\big(T_{5,2|3,4}\Delta_{53}\Delta_{24} + T_{5,3|2,4}\Delta_{52}\Delta_{34}\big)s_{15}\eta_{15} + \text{cyc}(1,2,3,4,5)\end{aligned} \tag{111}$$

To see that the terms proportional to V_1 are zero up to a total derivative with respect to z_1, after replacing

$$\Delta_{12}\Delta_{34}(\ell^I\cdot k_1)\omega_I(z_5) = -\Delta_{51}\Delta_{34}(\ell^I\cdot k_1)\omega_I(z_2) - \Delta_{25}\Delta_{34}(\ell^I\cdot k_1)\omega_I(z_1) \tag{112}$$

the terms containing the loop momenta simplify to

$$\begin{aligned}& \big(V_1 T_{5,2|3,4}\Delta_{23}\Delta_{45} - V_1 T_{2,3|4,5}\Delta_{25}\Delta_{34}\big)(\ell^I\cdot k_1)\omega_I(z_1) \cong \\ & \big(V_1 T_{5,2|3,4}\Delta_{23}\Delta_{45} - V_1 T_{2,3|4,5}\Delta_{25}\Delta_{34}\big)(s_{12}\eta_{12} + s_{13}\eta_{13} + s_{14}\eta_{14} + s_{15}\eta_{15})\end{aligned} \tag{113}$$

where the IBP relation (95) $-(\ell^I\cdot k_1)\omega_I(z_1) + s_{12}\eta_{12} + s_{13}\eta_{13} + s_{14}\eta_{14} + s_{15}\eta_{15} \cong 0$ has been used. Plugging this into (111), the terms containing $s_{12}\eta_{12}$ become

$$s_{12}\eta_{12}V_1\Big(T_{2,3|4,5}(\Delta_{24}\Delta_{35} - \Delta_{25}\Delta_{34}) + (T_{2,4|3,5} + T_{2,5|3,4})\Delta_{23}\Delta_{45}\Big) = 0 \tag{114}$$

where we used the kinematic Jacobi identity $T_{2,4|3,5} + T_{2,5|3,4} = -T_{2,3|4,5}$ as well as the worldsheet Jacobi identity $\Delta_{24}\Delta_{35} - \Delta_{25}\Delta_{34} - \Delta_{23}\Delta_{45} = 0$. The analysis of the other terms $s_{1j}\eta_{1j}$ for $j = 3,4,5$ is similar and the vanishing of the full BRST variation (111) follows from the cyclic permutations.

Homology invariance Using the monodromies in (102) one can show that the chiral correlator (110) is single-valued as a function of ℓ_I and z_i. For instance, moving z_1 around the B_I cycle and writing the result in terms of the cyclic basis (107) implies that (110) is single valued around z_1 provided

$$\begin{aligned}\langle k_1^m T^m_{3,4,5|1,2} - T_{3;1|2|4,5} - T_{4;1|2|3,5} - T_{5;1|2|3,4}\rangle = 0 \\ \langle k_1^m T^m_{1,2,3|4,5} + T_{1;4|5|2,3} + T_{1;5|4|2,3} + T_{12,3|4,5} + T_{13,2|4,5}\rangle = 0\end{aligned} \tag{115}$$

which can be verified to be true using the identities in (101). Alternatively, their validity also follows from the fact that these are BRST-closed linear combinations of local building blocks and that the five-point local cohomology is empty.

2.3.5 Genus three

Four points The chiral correlator for four external massless states was determined in [46] up to terms that have no singularities on the worldsheet[7]. It can be written as

$$\begin{aligned}\mathcal{K}_4(\ell) = {} & T^m_{1,4|2|3}\ell^I_m w^I_1 \Delta_{234} + T^m_{2,4|1|3}\ell^I_m w^I_2 \Delta_{134} + T^m_{3,4|1|2}\ell^I_m w^I_3 \Delta_{124} \\ & + T_{12|3|4}\Delta_{234}\,\eta_{12} + T_{13|2|4}\Delta_{324}\,\eta_{13} + T_{14|2|3}\Delta_{423}\,\eta_{14} \\ & + T_{23|1|4}\Delta_{314}\,\eta_{23} + T_{24|1|3}\Delta_{413}\,\eta_{24} + T_{34|1|2}\Delta_{412}\,\eta_{34}\end{aligned} \tag{116}$$

where $\Delta_{ijk} = \varepsilon^{IJK}\omega_I(z_i)\omega_J(z_J)\omega_K(z_k)$ for $I,J,K = 1,2,3$ and η_{ij} is a worldsheet function depending on the genus-three prime form $E(z_i,z_j)$

$$\eta_{ij} = \frac{\partial}{\partial z_i}\ln E(z_i,z_j). \tag{117}$$

The building blocks above depend on non-minimal pure spinor fields. The vectorial building block is constructed as follows

$$T^m_{1234} = L^m_{1342} + L^m_{2341} + \frac{5}{2}S^m_{1234} \tag{118}$$

where $S^m_{1234} = S^{(1)m}_{1234} + S^{(2)m}_{1234} - S^{(2)m}_{1243}$ and

$$\begin{aligned}S^{(1)m}_{1234} = {} & 2(\overline{\lambda}\gamma^m\gamma^{a_1}\lambda)(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\overline{\lambda}\gamma_{m_3n_3p_3}r)(\overline{\lambda}\gamma_{m_4n_4p_4}r)(\overline{\lambda}\gamma_{m_5n_5p_5}r) \\ & \times(\lambda\gamma^{a_2m_1n_1p_1m_3}\lambda)(\lambda\gamma^{a_3m_2n_2p_2m_5}\lambda)(\lambda\gamma^{n_3m_4n_4p_4n_5}\lambda) \\ & \times(W^1\gamma^{a_1a_2a_3}W^2)(\lambda\gamma^{p_3}W^3)(\lambda\gamma^{p_5}W^4)\end{aligned} \tag{119}$$

$$S^{(2)m}_{1234} = 96(\overline{\lambda}\gamma^m\gamma^{m_3}\lambda)(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\overline{\lambda}\gamma_{m_3n_3p_3}r)(\overline{\lambda}\gamma_{m_4n_4p_4}r)(\overline{\lambda}\gamma_{m_5n_5p_5}r)$$

[7] We note a recent [87] conjecture for the full bosonic correlator obtained from matching its field-theory limit with the $\mathcal{N}=8$ integrand in a BCJ parameterization.

$$
\begin{aligned}
&\times(\lambda\gamma^{n_1m_2n_2p_2m_5}\lambda)(\lambda\gamma^{n_3m_4n_4p_4n_5}\lambda)\\
&\times(\lambda\gamma^{n_1}W^1)(\lambda\gamma^{p_1}W^2)(\lambda\gamma^{p_3}W^3)(\lambda\gamma^{p_5}W^4)\\
L^m_{ijkl} &= (\overline{\lambda}\gamma^{abc}r)(\overline{\lambda}\gamma^{def}r)(\overline{\lambda}\gamma^{ghi}r)(\overline{\lambda}\gamma^{mnp}r)(\overline{\lambda}\gamma^{qrs}r)(\overline{\lambda}\gamma^{tuv}r)\\
&\quad\times(\lambda\gamma^{adefm}\lambda)(\lambda\gamma^{bghit}\lambda)(\lambda\gamma^{uqrsn}\lambda)(\lambda\gamma^{c}W_i)(\lambda\gamma^{p}W_j)(\lambda\gamma^{v}W_k)A^m_l.
\end{aligned}
$$

The scalar building block is given by

$$
\begin{aligned}
T_{ij,k,l} = {} & (\overline{\lambda}\gamma^{abc}r)(\overline{\lambda}\gamma^{def}r)(\overline{\lambda}\gamma^{ghi}r)(\overline{\lambda}\gamma^{mnp}r)(\overline{\lambda}\gamma^{qrs}r)(\overline{\lambda}\gamma^{tuv}r)\\
&\times(\lambda\gamma^{adefm}\lambda)(\lambda\gamma^{bghit}\lambda)(\lambda\gamma^{uqrsn}\lambda)(\lambda\gamma^{c}W_{ij})(\lambda\gamma^{p}W_k)(\lambda\gamma^{v}W_l)\,.
\end{aligned}
\tag{120}
$$

The presence of the non-minimal fields $\overline{\lambda}_\alpha, r_\beta$ in these building blocks leads to technical challenges that do not exist when dealing with "minimal" pure spinor superspace expressions. The r_β fields can be straightforwardly converted into superspace derivatives D_β but the handling of $\overline{\lambda}_\alpha$ is not so immediate. But luckily, as proven in the appendix of [46], there exists a procedure to convert an arbitrary non-minimal pure spinor superspace expression containing $\overline{\lambda}^n\lambda^{n+3}$ pure spinors with $n \geq 1$ into an expression in which the $\overline{\lambda}_\alpha$ are contracted with λ^α yielding "minimal" pure spinor superspace expressions with $(\lambda\overline{\lambda})^n\lambda^3$. As the $(\lambda\overline{\lambda})^n$ factor only affects the normalization of the zero-mode integration, one can consider these non-minimal pure spinor superspace expressions more or less in the same footing as their minimal counterparts. It is worth mentioning that there is a proposal for these building blocks directly in minimal pure spinor superspace using higher-mass SYM superfields as [71]

$$
\begin{aligned}
T_{12,3,4} &\equiv \langle(\lambda\gamma_m W^n_{12})(\lambda\gamma_n W^p_{[3})(\lambda\gamma_p W^m_{4]})\rangle\\
T^m_{1234} &\equiv \langle A^m_{(1}T_{2),3,4} + (\lambda\gamma^m W_{(1})L_{2),3,4}\rangle\\
L_{2,3,4} &\equiv \tfrac{1}{3}(\lambda\gamma^n W^q_{[2})(\lambda\gamma^q W^p_3)F^{np}_{4]}\rangle\,,
\end{aligned}
\tag{121}
$$

where $W^{m\alpha}_P$ represents a local superfield of higher-mass dimension as defined in [71]; when $P = i$ is a single letter it reduces to $W^{m\alpha}_i = k^m_i W^\alpha_i$ but when P is a word there are non-trivial contact-term corrections. The component expansion of these building blocks is not exactly the same as their non-minimal counterparts but they yield the same D^6R^4 components as discussed below.

After approximating the Koba-Nielsen factor to one in the low-energy limit and integrating over the volume of moduli space, the holomorphic square of the integrand

$$
\frac{|T_{12,3,4}|^2}{s_{12}} + |T^m_{1234}|^2 + (1,2|1,2,3,4)
\tag{122}
$$

is proportional to the D^6R^4 interaction of type IIB when expanded in bosonic components, regardless of the minimal vs non-minimal representations of the building blocks. One can show that the low-energy contribution (122) is BRST closed, but not the chiral correlator (116). A BRST-closed and single-valued chiral correlator to all orders in α' has since been found [76].

It is worth noting that the computations of [46] were done keeping track of the absolute normalizations coming from the pure spinor prescription with the integration formulas from [47]. As will be reviewed below, these calculations matched the predictions to the D^6R^4 type IIB interaction arising from the S-duality considerations of Green and Vanhove [54].

2.4 Verifying S-duality conjectures

The scattering amplitudes computed with pure spinor formalism have provided an independent check on the S-duality predictions of type IIB interactions.

2.4.1 S-duality and four-point amplitudes

On the one hand, the $SL(2,\mathbb{Z})$-duality prediction for the perturbative four-graviton type IIB effective action in the string frame is given by [51, 52, 53, 54]

$$\begin{aligned} S^{\text{4pt}}_{\text{IIB}} = \int d^{10}x\sqrt{-g}\,\big[&R^4(2\zeta_3 e^{-2\phi}+4\zeta_2)+D^4R^4(2\zeta_5 e^{-2\phi}+\frac{8}{3}\zeta_4 e^{2\phi}) \\ &+D^6R^4(4\zeta_3^2 e^{-2\phi}+8\zeta_2\zeta_3+\frac{48}{5}\zeta_2^2 e^{2\phi}+\frac{8}{9}\zeta_6 e^{4\phi})+\cdots\big], \end{aligned} \tag{123}$$

where the shorthands R^4, D^4R^4 and D^6R^4 denote contractions of covariant derivatives D and Riemann curvature tensors R whose precise structure does not affect the analysis. Factors of $e^{(2g-2)\phi}$ are associated with the genus-g order in string perturbation theory. The key idea of the S-duality analysis was to associate the coefficients of the R^4 interaction with the zero-modes of non-holomorphic Eisenstein series $E_{3/2}(\Phi,\overline{\Phi})$ and those of D^4R^4 with $E_{5/2}(\Phi,\overline{\Phi})$, where

$$\begin{aligned} E_{3/2}(\Phi,\overline{\Phi}) &= 2\zeta_3 e^{-3\phi/2}+4\zeta_2 e^{\phi/2}+\cdots \\ E_{5/2}(\Phi,\overline{\Phi}) &= 2\zeta_5 e^{-5\phi/2}+\frac{8}{3}\zeta_4 e^{3\phi/2}+\cdots \end{aligned} \tag{124}$$

where Φ depends on the complex axio-dilaton field $\Phi = C_0 + ie^{-\phi}$.

On the other hand, the α' expansion of perturbative string scattering amplitude calculations performed with the non-minimal pure spinor formalism with the absolute normalization techniques from [47, 45, 46] for the four-point massless closed string states are given by

$$M^{(0)}_4 = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\kappa^4 e^{-2\lambda}\,2\pi K\tilde{K}\big(\frac{3}{\sigma_3}+2\zeta_3+\zeta_5\sigma_2+\frac{2}{3}\zeta_3^2\sigma_3+\cdots\big)$$

$$
\begin{aligned}
M_4^{(1)} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\kappa^4\Big(\frac{1}{2^4\cdot 3\pi}\Big)K\tilde{K}\big(1+\frac{\zeta_3}{3}\sigma_3+\cdots\big)\\
M_4^{(2)} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\kappa^4 e^{2\lambda}\Big(\frac{1}{2^{10}\cdot 3^3\cdot 5\pi^3}\Big)K\tilde{K}\left(\sigma_2+3\sigma_3+\cdots\right)\\
M_4^{(3)} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\kappa^4 e^{4\lambda}\Big(\frac{1}{2^{15}\cdot 3^6\cdot 5\cdot 7\pi^5}\Big)K\tilde{K}\left(\sigma_3+\cdots\right) \qquad (125)
\end{aligned}
$$

where

$$\sigma_n = \Big(\frac{\alpha'}{2}\Big)^n(s_{12}^n+s_{13}^n+s_{14}^n) \qquad (126)$$

are dimensionless symmetric polynomials of Mandelstam invariants $s_{ij}=(k_i\cdot k_j)$, $e^{(2g-2)\lambda}$ is the string genus-g coupling constant, K is the supersymmetric kinematic factor[8]

$$K = s_{12}s_{23}A^{\rm SYM}(1,2,3,4) \qquad (127)$$

and κ is the normalization of the vertex operators fixed to $\kappa^2 = e^{2\lambda}\pi/\alpha'^2$ by unitarity [48].

It is easy to see the one-to-one correspondence of the genus- and α'-orders in the amplitudes (125) with the curvature couplings in the action (123)

$$e^{(2g-2)\phi}D^{2k}R^4 \leftrightarrow e^{(2g-2)\lambda}K\tilde{K}\sigma_k\,. \qquad (128)$$

Therefore, matching the ratio (genus one)/(genus zero) of the R^4 interactions in the effective action with the corresponding ratio of the amplitudes

$$\frac{4\zeta_2 R^4}{2\zeta_3 e^{-2\phi}R^4} = \frac{e^{2\phi}\pi^2}{3\zeta_3}\,, \qquad \frac{(K\tilde{K})/(2^4 3\pi)}{(K\tilde{K})4\pi\zeta_3 e^{-2\lambda}} = \frac{e^{2\lambda}}{2^6\,3\pi^2\zeta_3} \qquad (129)$$

relates the coupling constants e^ϕ and e^λ,

$$e^{2\lambda} = 2^6\pi^4 e^{2\phi}\;. \qquad (130)$$

D^4R^4 interaction at genus two One can now compare the predicted D^4R^4 interaction terms from the type IIB effective action (123) with the first principles string calculations. Taking the genus-two/genus-zero ratio of the $K\tilde{K}\sigma_2$ term from the amplitudes gives

$$\frac{e^{4\lambda}}{2^{11}3^3 5\pi^4\zeta_5} = \frac{2\pi^4 e^{4\phi}}{3^3 5\zeta_5} \qquad (131)$$

[8] For bosonic external states, note that $-2^3K^{\rm here} = K^{0503}$ from [43] and that $K^{\rm here}\tilde{K}^{\rm here} = \mathscr{K}_4^{(0)}$ from [48]. In addition, the amplitudes in (125) were computed using the tree-level normalization convention encoded by $R^2=\pi^5/2^5$ used in [48] where R is a normalization parameter appearing in the zero-mode measures $[dr]$ and $[ds^I]$. In [46] the normalization $R^2=\sqrt{2}/(2^{16}\pi)$ was chosen, such that the genus g amplitudes A_g^{1308} of [46] are related by $x^{1-g}A_g^{1308} = M_g^{1504}$ to the amplitudes M_g^{1504} of [48] with $x=\sqrt{2}2^{10}\pi^6$ after considering that $K^{1308}\overline{K}^{1308} = 2^6\mathscr{K}_4^{(0)}$. In (127) the bracket notation indicates a trace over the vector indices of the field strength $F_i^{mn} = k_i^m e_i^n - k_i^n e_i^m$.

where we used (130) in the right-hand side. This is the same ratio of the D^4R^4 terms in (123) at the corresponding loop order: $8\zeta_4 e^{4\phi}/(6\zeta_5) = 2\pi^4 e^{4\phi}/(3^3 5\zeta_5)$ as $\zeta_4 = \pi^4/(2\cdot 3^2\cdot 5)$.

D^6R^4 interaction at genus three Similarly, the ratio (genus three)/(genus one) correction $K\tilde{K}\sigma_3$ matches perfectly with the S-duality result in (123). The ratio of the amplitudes is given by $e^{4\lambda}/(2^{11}\cdot 3^4\cdot 5\cdot 7\pi^4\zeta_3)$ while in the effective action it is given by $\zeta_6 e^{4\phi}/(9\zeta_2\zeta_3)$, and these two numbers match after using the conversion (130) and $\zeta_6 = \pi^6/(3^3\cdot 5\cdot 7)$.

D^6R^4 interaction at genus two The coefficient of $K\tilde{K}\sigma_3$ at genus-two was computed in [44] and allowed the comparison between the string scattering amplitude result at genus two with the S-duality prediction in the action (123). The (genus two)/(genus one) ratio of the correction $K\tilde{K}\sigma_3$ is given by $e^{2\lambda}/(2^6 5\pi^2\zeta_3)$ which matches the S-duality (genus two)/(genus one) ratio of the D^6R^4 interaction, given by $6\zeta_2 e^{2\phi}/(5\zeta_3)$ after using the conversion (130) and $\zeta_2 = \pi^2/6$.

2.4.2 S-duality and five-point amplitudes

The pure spinor formalism also allowed to check the S-duality proposals for five graviton interactions as well as four gravitons and one dilaton. For five gravitons the S-duality effective action contains the same ratios appearing in the four graviton action (123); the extension is straightforward with four-curvature corrections $D^{2k}R^4$ followed by a tail of operators $D^{2(k-p)}R^{4+p}$, although there might be novel $D^{2k}R^{\geq 5}$ couplings without a four-field counterpart such as the D^6R^5 interaction at genus one [55]. These S-duality tails such as $(D^4R^4 + D^2R^5)$ are confirmed by the data of the genus-g amplitudes $M_5^{(g)}$ at five points[9] [48]

$$M_5^{(0)} = \left(\frac{\alpha'}{2}\right)\kappa^5 e^{-2\lambda}(2\pi)^2\mathscr{K}_5^{(0)} \tag{132}$$
$$M_5^{(1)}\big|_{\text{IIB}}^{\alpha'^4} = \left(\frac{\alpha'}{2}\right)\frac{\kappa^5}{2^4 3}\mathscr{K}_5^{(0)}\Big|_{\zeta_3}\times\begin{cases} 1 & \text{: five gravitons}\\ -\frac{1}{3} & \text{: four gravitons, one dilaton}\end{cases}$$
$$M_5^{(2)}\big|_{\text{IIB}}^{\alpha'^6} = \left(\frac{\alpha'}{2}\right)\frac{\kappa^5 e^{2\lambda}}{2^9 3^3 5\pi^2}\mathscr{K}_5^{(0)}\Big|_{\zeta_5}\times\begin{cases} 1 & \text{: five gravitons}\\ -\frac{3}{5} & \text{: four gravitons, one dilaton}\end{cases}$$

where the tree-level factor $\mathscr{K}_5^{(0)}$ is given by

$$\mathscr{K}_5^{(0)} = \tilde{A}_{54}^T\cdot S_0\cdot\left[1 + 2\zeta_3\left(\frac{\alpha'}{2}\right)^3 M_3 + 2\zeta_5\left(\frac{\alpha'}{2}\right)^5 M_5 + 2\zeta_3^2\left(\frac{\alpha'}{2}\right)^6 M_3^2 + \mathscr{O}(\alpha'^7)\right]\cdot A_{45}\,, \tag{133}$$

[9] To avoid cluttering, we omit the universal factor of $(2\pi)^{10}\delta^{10}(k)$ from the right-hand side of (132).

where $\tilde{A}_{54}^T$ and A_{45} are two-component vectors of SYM tree-amplitudes

$$\tilde{A}_{54} \equiv \begin{pmatrix} \tilde{A}^{\rm YM}(1,2,3,5,4) \\ \tilde{A}^{\rm YM}(1,3,2,5,4) \end{pmatrix}, \quad A_{45} \equiv \begin{pmatrix} A^{\rm YM}(1,2,3,4,5) \\ A^{\rm YM}(1,3,2,4,5) \end{pmatrix}, \tag{134}$$

S_0 denotes the KLT matrix and the 2×2 matrices M_{2n+1} were introduced in [72].

Since the calculations in the pure spinor formalism are supersymmetric and done exploiting pure spinor superspace, the scattering of any state in the graviton supermultiplet can be systematically obtained once the superspace expression is calculated. As can be seen in (132), the ratios of the string amplitudes depend on the R-symmetry charges of the external type IIB states, as trading one graviton for a dilaton gives the additional factors of $-\frac{1}{3}$ or $-\frac{3}{5}$.

These numbers can be explained by the following argument [56]: scattering processes which violate the R-symmetry of type IIB supergravity are associated with operators which transform with modular weight under S-duality, therefore by modular invariance of the type IIB action, they must be accompanied by modular forms of opposite weights to preserve the modular invariance of the type IIB effective action. These modular forms can be generated as DE_s where D is the modular covariant derivative such that $De^{q\phi} = q \cdot e^{q\phi}$ and E_s a Eisenstein series. For example,

$$\begin{aligned} \mathscr{D}E_{3/2}(\Phi,\overline{\Phi}) &= \Big(-\frac{3}{2}\Big)2\zeta_3 e^{-3\phi/2} + \Big(\frac{1}{2}\Big)4\zeta_2 e^{\phi/2} + \cdots \\ \mathscr{D}E_{5/2}(\Phi,\overline{\Phi}) &= \Big(-\frac{5}{2}\Big)2\zeta_5 e^{-5\phi/2} + \Big(\frac{3}{2}\Big)\frac{8}{3}\zeta_4 e^{3\phi/2} + \cdots . \end{aligned} \tag{135}$$

Thus the ratio between tree-level and higher-genus contributions is deformed by by $-\frac{1}{3}$ and $-\frac{3}{5}$ in cases of $E_{3/2}$ and $E_{5/2}$, suggesting that the type IIB effective action contains the terms

$$\int d^{10}x\sqrt{-g}\big[\phi R^4(-3\zeta_3 e^{-2\phi} + 2\zeta_2) + \phi D^4 R^4(-5\zeta_5 e^{-2\phi} + 4\zeta_4 e^{2\phi})\big] \tag{136}$$

in the string frame[10] and explaining the relative coefficients in the scattering amplitudes (132).

References

[1] N. Berkovits, "Super Poincare covariant quantization of the superstring," JHEP **04**, 018 (2000) doi:10.1088/1126-6708/2000/04/018 [arXiv:hep-th/0001035 [hep-th]].

[2] M. B. Green and J. H. Schwarz, "Covariant Description of Superstrings," Phys. Lett. B **136**, 367-370 (1984) doi:10.1016/0370-2693(84)92021-5

[10] The term ϕR^4 is multiplied by $e^{-\phi/2}$ and $\phi D^4 R^4$ by $e^{\phi/2}$ in going to the string frame.

[3] W. Siegel, "Classical Superstring Mechanics," Nucl. Phys. B **263**, 93-104 (1986) doi:10.1016/0550-3213(86)90029-5
[4] P. S. Howe, "Pure spinors lines in superspace and ten-dimensional supersymmetric theories," Phys. Lett. B **258**, 141-144 (1991) doi:10.1016/0370-2693(91)91221-G
[5] P. S. Howe, "Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions," Phys. Lett. B **273**, 90-94 (1991) doi:10.1016/0370-2693(91)90558-8
[6] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, "Spinorial cohomology and maximally supersymmetric theories," JHEP **02**, 009 (2002) doi:10.1088/1126-6708/2002/02/009 [arXiv:hep-th/0110069 [hep-th]].
[7] N. Berkovits, "Towards covariant quantization of the supermembrane," JHEP **09**, 051 (2002) doi:10.1088/1126-6708/2002/09/051 [arXiv:hep-th/0201151 [hep-th]].
[8] M. Cederwall, "D=11 supergravity with manifest supersymmetry," Mod. Phys. Lett. A **25**, 3201-3212 (2010) doi:10.1142/S0217732310034407 [arXiv:1001.0112 [hep-th]].
[9] M. Cederwall, "Pure spinors in classical and quantum supergravity," [arXiv:2210.06141 [hep-th]].
[10] N. Berkovits, "Untwisting the pure spinor formalism to the RNS and twistor string in a flat and $AdS_5\times S^5$ background," JHEP **06**, 127 (2016) doi:10.1007/JHEP06(2016)127 [arXiv:1604.04617 [hep-th]].
[11] N. Berkovits, "Manifest spacetime supersymmetry and the superstring," JHEP **10**, 162 (2021) doi:10.1007/JHEP10(2021)162 [arXiv:2106.04448 [hep-th]].
[12] N. Berkovits and P. S. Howe, "Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring," Nucl. Phys. B **635**, 75-105 (2002) doi:10.1016/S0550-3213(02)00352-8 [arXiv:hep-th/0112160 [hep-th]].
[13] N. Berkovits, "Quantum consistency of the superstring in AdS(5) x S**5 background," JHEP **03**, 041 (2005) doi:10.1088/1126-6708/2005/03/041 [arXiv:hep-th/0411170 [hep-th]].
[14] N. Berkovits, "Half-BPS vertex operators of the $AdS_5\times S^5$ superstring," JHEP **07**, 084 (2019) doi:10.1007/JHEP07(2019)084 [arXiv:1904.06564 [hep-th]].
[15] L. N. S. Martins, "Type IIB superstring vertex operator from the -8 picture," Nucl. Phys. B **974**, 115638 (2022) doi:10.1016/j.nuclphysb.2021.115638 [arXiv:1912.06498 [hep-th]].
[16] T. Fleury and L. N. S. Martins, "$AdS_5 \times S^5$ supergravity vertex operators," JHEP **07**, 210 (2021) doi:10.1007/JHEP07(2021)210 [arXiv:2104.03333 [hep-th]].
[17] N. Berkovits, "Simplifying and Extending the AdS(5) x S**5 Pure Spinor Formalism," JHEP **09**, 051 (2009) doi:10.1088/1126-6708/2009/09/051 [arXiv:0812.5074 [hep-th]].
[18] N. Berkovits, "Sketching a Proof of the Maldacena Conjecture at Small Radius," JHEP **06**, 111 (2019) doi:10.1007/JHEP06(2019)111 [arXiv:1903.08264 [hep-th]].

[19] E. Witten, "Perturbative gauge theory as a string theory in twistor space," Commun. Math. Phys. **252**, 189-258 (2004) doi:10.1007/s00220-004-1187-3 [arXiv:hep-th/0312171 [hep-th]].

[20] L. Eberhardt, M. R. Gaberdiel and R. Gopakumar, "Deriving the AdS_3/CFT_2 correspondence," JHEP **02**, 136 (2020) doi:10.1007/JHEP02(2020)136 [arXiv:1911.00378 [hep-th]].

[21] M. R. Gaberdiel and R. Gopakumar, "The worldsheet dual of free super Yang-Mills in 4D," JHEP **11**, 129 (2021) doi:10.1007/JHEP11(2021)129 [arXiv:2105.10496 [hep-th]].

[22] N. Berkovits, "Pure spinor formalism as an N=2 topological string," JHEP **10**, 089 (2005) doi:10.1088/1126-6708/2005/10/089 [arXiv:hep-th/0509120 [hep-th]].

[23] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," JHEP **01**, 005 (2006) doi:10.1088/1126-6708/2006/01/005 [arXiv:hep-th/0503197 [hep-th]].

[24] N. Berkovits and C. R. Mafra, "Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms," Phys. Rev. Lett. **96**, 011602 (2006) doi:10.1103/PhysRevLett.96.011602 [arXiv:hep-th/0509234 [hep-th]].

[25] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **09**, 047 (2004) doi:10.1088/1126-6708/2004/09/047 [arXiv:hep-th/0406055 [hep-th]].

[26] N. Berkovits and N. Nekrasov, "Multiloop superstring amplitudes from non-minimal pure spinor formalism," JHEP **12**, 029 (2006) doi:10.1088/1126-6708/2006/12/029 [arXiv:hep-th/0609012 [hep-th]].

[27] N. Berkovits, "Explaining Pure Spinor Superspace," [arXiv:hep-th/0612021 [hep-th]].

[28] N. Berkovits and C. R. Mafra, "Some Superstring Amplitude Computations with the Non-Minimal Pure Spinor Formalism," JHEP **11**, 079 (2006) doi:10.1088/1126-6708/2006/11/079 [arXiv:hep-th/0607187 [hep-th]].

[29] Z. Bern, J. J. M. Carrasco and H. Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. D **78**, 085011 (2008) doi:10.1103/PhysRevD.78.085011 [arXiv:0805.3993 [hep-ph]].

[30] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, "One loop n point gauge theory amplitudes, unitarity and collinear limits," Nucl. Phys. B **425**, 217-260 (1994) doi:10.1016/0550-3213(94)90179-1 [arXiv:hep-ph/9403226 [hep-ph]].

[31] D. Blessenohl, H. Laue, H., "Generalized Jacobi identities," Note di Matematica, 8(1), pp.111-121 (1988) doi:10.1285/i15900932v8n1p111

[32] E. Bridges and C. R. Mafra, "Algorithmic construction of SYM multiparticle superfields in the BCJ gauge," JHEP **10**, 022 (2019) doi:10.1007/JHEP10(2019)022 [arXiv:1906.12252 [hep-th]].

[33] E. Bridges and C. R. Mafra, "Local BCJ numerators for ten-dimensional SYM at one loop," JHEP **07**, 031 (2021) doi:10.1007/JHEP07(2021)031 [arXiv:2102.12943 [hep-th]].

[34] J. Broedel, C. R. Mafra, N. Matthes and O. Schlotterer, "Elliptic multiple zeta values and one-loop superstring amplitudes," JHEP **07**, 112 (2015) doi:10.1007/JHEP07(2015)112 [arXiv:1412.5535 [hep-th]].

[35] J. Broedel, O. Schlotterer, S. Stieberger and T. Terasoma, "All order α'-expansion of superstring trees from the Drinfeld associator," Phys. Rev. D **89**, no.6, 066014 (2014) doi:10.1103/PhysRevD.89.066014 [arXiv:1304.7304 [hep-th]].

[36] Brown, F. and Levin, A., 2011. "Multiple elliptic polylogarithms," [arXiv:1110.6917 [math.NT]].

[37] D. Friedan, E. J. Martinec and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," Nucl. Phys. B **271**, 93-165 (1986) doi:10.1016/0550-3213(86)90356-1

[38] E. D'Hoker and D. H. Phong, "The Geometry of String Perturbation Theory," Rev. Mod. Phys. **60**, 917 (1988) doi:10.1103/RevModPhys.60.917

[39] E. D'Hoker and D. H. Phong, "Conformal Scalar Fields and Chiral Splitting on Superriemann Surfaces," Commun. Math. Phys. **125**, 469 (1989) doi:10.1007/BF01218413

[40] E. D'Hoker and D. H. Phong, "Two-loop superstrings VI: Non-renormalization theorems and the 4-point function," Nucl. Phys. B **715**, 3-90 (2005) doi:10.1016/j.nuclphysb.2005.02.043 [arXiv:hep-th/0501197 [hep-th]].

[41] E. D'Hoker, C. R. Mafra, B. Pioline and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part I. Construction via chiral splitting and pure spinors," JHEP **08**, 135 (2020) doi:10.1007/JHEP08(2020)135 [arXiv:2006.05270 [hep-th]].

[42] E. D'Hoker, C. R. Mafra, B. Pioline and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part II. Low energy expansion and S-duality," JHEP **02**, 139 (2021) doi:10.1007/JHEP02(2021)139 [arXiv:2008.08687 [hep-th]].

[43] E. D'Hoker, M. Gutperle and D. H. Phong, "Two-loop superstrings and S-duality," Nucl. Phys. B **722**, 81-118 (2005) doi:10.1016/j.nuclphysb.2005.06.010 [arXiv:hep-th/0503180 [hep-th]].

[44] E. D'Hoker, M. B. Green, B. Pioline and R. Russo, "Matching the D^6R^4 interaction at two-loops," JHEP **01**, 031 (2015) doi:10.1007/JHEP01(2015)031 [arXiv:1405.6226 [hep-th]].

[45] H. Gomez and C. R. Mafra, "The Overall Coefficient of the Two-loop Superstring Amplitude Using Pure Spinors," JHEP **05**, 017 (2010) doi:10.1007/JHEP05(2010)017 [arXiv:1003.0678 [hep-th]].

[46] H. Gomez and C. R. Mafra, "The closed-string 3-loop amplitude and S-duality," JHEP **10**, 217 (2013) doi:10.1007/JHEP10(2013)217 [arXiv:1308.6567 [hep-th]].

[47] H. Gomez, "One-loop Superstring Amplitude From Integrals on Pure Spinors Space," JHEP **12**, 034 (2009) doi:10.1088/1126-6708/2009/12/034 [arXiv:0910.3405 [hep-th]].

[48] H. Gomez, C. R. Mafra and O. Schlotterer, "Two-loop superstring five-point amplitude and *S*-duality," Phys. Rev. D **93**, no.4, 045030 (2016) doi:10.1103/PhysRevD.93.045030 [arXiv:1504.02759 [hep-th]].

[49] M.B. Green and J.H. Schwarz, "The Hexagon Gauge Anomaly in Type I Superstring Theory," Nucl. Phys. B **255** (1985) 93.

[50] M.B. Green and J.H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," Phys. Lett. B **149** (1984) 117.

[51] M. B. Green and M. Gutperle, "Effects of D instantons," Nucl. Phys. B **498**, 195-227 (1997) doi:10.1016/S0550-3213(97)00269-1 [arXiv:hep-th/9701093 [hep-th]].

[52] M. B. Green, M. Gutperle and P. Vanhove, "One loop in eleven-dimensions," Phys. Lett. B **409**, 177-184 (1997) doi:10.1016/S0370-2693(97)00931-3 [arXiv:hep-th/9706175 [hep-th]].

[53] M. B. Green, H. h. Kwon and P. Vanhove, "Two loops in eleven-dimensions," Phys. Rev. D **61**, 104010 (2000) doi:10.1103/PhysRevD.61.104010 [arXiv:hep-th/9910055 [hep-th]].

[54] M. B. Green and P. Vanhove, "Duality and higher derivative terms in M theory," JHEP **01**, 093 (2006) doi:10.1088/1126-6708/2006/01/093 [arXiv:hep-th/0510027 [hep-th]].

[55] M. B. Green, C. R. Mafra and O. Schlotterer, "Multiparticle one-loop amplitudes and S-duality in closed superstring theory," JHEP **10**, 188 (2013) doi:10.1007/JHEP10(2013)188 [arXiv:1307.3534 [hep-th]].

[56] M. B. Green, M. Gutperle and H. h. Kwon, "Sixteen fermion and related terms in M theory on T**2," Phys. Lett. B **421**, 149-161 (1998) doi:10.1016/S0370-2693(97)01551-7 [arXiv:hep-th/9710151 [hep-th]].

[57] M. B. Green, J. H. Schwarz and L. Brink, "N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories," Nucl. Phys. B **198**, 474-492 (1982) doi:10.1016/0550-3213(82)90336-4

[58] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory. 3. Loops and Renormalization," Nucl. Phys. B **198**, 441-460 (1982) doi:10.1016/0550-3213(82)90334-0

[59] C. R. Mafra, "Towards Field Theory Amplitudes From the Cohomology of Pure Spinor Superspace," JHEP **11**, 096 (2010) doi:10.1007/JHEP11(2010)096 [arXiv:1007.3639 [hep-th]].

[60] C. R. Mafra, O. Schlotterer, S. Stieberger and D. Tsimpis, "A recursive method for SYM n-point tree amplitudes," Phys. Rev. D **83**, 126012 (2011) doi:10.1103/PhysRevD.83.126012 [arXiv:1012.3981 [hep-th]].

[61] C. R. Mafra, O. Schlotterer and S. Stieberger, "Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation," Nucl. Phys. B **873**, 419-460 (2013) doi:10.1016/j.nuclphysb.2013.04.023 [arXiv:1106.2645 [hep-th]].

[62] C. R. Mafra and O. Schlotterer, "Towards the n-point one-loop superstring amplitude. Part I. Pure spinors and superfield kinematics," JHEP **08**, 090 (2019) doi:10.1007/JHEP08(2019)090 [arXiv:1812.10969 [hep-th]].

[63] C. R. Mafra and O. Schlotterer, "Towards the n-point one-loop superstring amplitude. Part II. Worldsheet functions and their duality to kinematics," JHEP **08**, 091 (2019) doi:10.1007/JHEP08(2019)091 [arXiv:1812.10970 [hep-th]].
[64] C. R. Mafra and O. Schlotterer, "Towards the n-point one-loop superstring amplitude. Part III. One-loop correlators and their double-copy structure," JHEP **08**, 092 (2019) doi:10.1007/JHEP08(2019)092 [arXiv:1812.10971 [hep-th]].
[65] C. R. Mafra and O. Schlotterer, "Towards one-loop SYM amplitudes from the pure spinor BRST cohomology," Fortsch. Phys. **63**, no.2, 105-131 (2015) doi:10.1002/prop.201400076 [arXiv:1410.0668 [hep-th]].
[66] C. R. Mafra and O. Schlotterer, "Multiparticle SYM equations of motion and pure spinor BRST blocks," JHEP **07**, 153 (2014) doi:10.1007/JHEP07(2014)153 [arXiv:1404.4986 [hep-th]].
[67] C. R. Mafra and O. Schlotterer, "Double-Copy Structure of One-Loop Open-String Amplitudes," Phys. Rev. Lett. **121**, no.1, 011601 (2018) doi:10.1103/PhysRevLett.121.011601 [arXiv:1711.09104 [hep-th]].
[68] C. R. Mafra, "Pure Spinor Superspace Identities for Massless Four-point Kinematic Factors," JHEP **04**, 093 (2008) doi:10.1088/1126-6708/2008/04/093 [arXiv:0801.0580 [hep-th]].
[69] C. R. Mafra and O. Schlotterer, "Two-loop five-point amplitudes of super Yang-Mills and supergravity in pure spinor superspace," JHEP **10**, 124 (2015) doi:10.1007/JHEP10(2015)124 [arXiv:1505.02746 [hep-th]].
[70] C. R. Mafra and O. Schlotterer, "Cohomology foundations of one-loop amplitudes in pure spinor superspace," [arXiv:1408.3605 [hep-th]].
[71] C. R. Mafra and O. Schlotterer, "Solution to the nonlinear field equations of ten dimensional supersymmetric Yang-Mills theory," Phys. Rev. D **92**, no.6, 066001 (2015) doi:10.1103/PhysRevD.92.066001 [arXiv:1501.05562 [hep-th]].
[72] O. Schlotterer and S. Stieberger, "Motivic Multiple Zeta Values and Superstring Amplitudes," J. Phys. A **46**, 475401 (2013) doi:10.1088/1751-8113/46/47/475401 [arXiv:1205.1516 [hep-th]].
[73] C. Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 1993.
[74] E. P. Verlinde and H. L. Verlinde, "Chiral Bosonization, Determinants and the String Partition Function," Nucl. Phys. B **288**, 357 (1987) doi:10.1016/0550-3213(87)90219-7
[75] E. P. Verlinde and H. L. Verlinde, "Multiloop Calculations in Covariant Superstring Theory," Phys. Lett. B **192**, 95-102 (1987) doi:10.1016/0370-2693(87)91148-8
[76] E. D'Hoker, C. R. Mafra, B. Pioline and O. Schlotterer, unpublished
[77] Van Leeuwen, M.A., Cohen, A.M. and Lisser, B., 1992. LiE: A package for Lie group computations.
[78] J. P. Harnad and S. Shnider, "Constraints and field equations for ten-dimensional superyang-mills theory," Commun. Math. Phys. **106**, 183 (1986) doi:10.1007/BF01454971

[79] E. Witten, "Twistor - Like Transform in Ten-Dimensions," Nucl. Phys. B **266**, 245-264 (1986) doi:10.1016/0550-3213(86)90090-8

[80] W. Siegel, "Superfields in Higher Dimensional Space-time," Phys. Lett. B **80**, 220-223 (1979) doi:10.1016/0370-2693(79)90202-8

[81] A. Neveu and J. Scherk, "Connection between Yang-Mills fields and dual models," Nucl. Phys. B **36**, 155-161 (1972) doi:10.1016/0550-3213(72)90301-X

[82] C.R.Mafra and O.Schlotterer, "Tree-level amplitudes from the pure spinor superstring"

[83] R. Lipinski Jusinskas, "Nilpotency of the b ghost in the non-minimal pure spinor formalism," JHEP **05**, 048 (2013) doi:10.1007/JHEP05(2013)048 [arXiv:1303.3966 [hep-th]].

[84] O. Chandia, "The b Ghost of the Pure Spinor Formalism is Nilpotent," Phys. Lett. B **695**, 312-316 (2011) doi:10.1016/j.physletb.2010.10.058 [arXiv:1008.1778 [hep-th]].

[85] J. M. Figueroa-O'Farrill, "N=2 structures in all string theories," J. Math. Phys. **38**, 5559-5575 (1997) doi:10.1063/1.532151 [arXiv:hep-th/9507145 [hep-th]].

[86] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311-428 (1994) doi:10.1007/BF02099774 [arXiv:hep-th/9309140 [hep-th]].

[87] Y. Geyer, R. Monteiro and R. Stark-Muchão, "Superstring Loop Amplitudes from the Field Theory Limit," Phys. Rev. Lett. **127**, no.21, 211603 (2021) doi:10.1103/PhysRevLett.127.211603 [arXiv:2106.03968 [hep-th]].