

RESEARCH ARTICLE

On double coset separability and the Wilson–Zaleskii property

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Email: aminasyan@gmail.com**Abstract**

A residually finite group G has the Wilson–Zaleskii property if for all finitely generated subgroups $H, K \leq G$, one has $\overline{H} \cap \overline{K} = \overline{H \cap K}$, where the closures are taken in the profinite completion \widehat{G} of G . This property played an important role in several papers, and is usually combined with separability of double cosets. In the present note we show that the Wilson–Zaleskii property is actually enjoyed by every double coset separable group. We also construct an example of a subgroup separable (LERF) group that is not double coset separable and does not have the Wilson–Zaleskii property.

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1 | INTRODUCTION

Every residually finite group G has a natural embedding into its profinite completion \widehat{G} , which is a compact topological group. The topology on \widehat{G} induces the *profinite topology* on G . A subset $S \subseteq G$ is said to be *separable* if it is closed in this topology, that is, $S = \overline{S} \cap G$, where \overline{S} denotes the closure of S in \widehat{G} .

Many residual properties of G can be interpreted in terms of the profinite topology or the embedding of G into \widehat{G} . In establishing various such properties it is often useful to have control over the intersections of images of two subgroups $H, K \in \mathcal{S}$ in finite quotients of G , where \mathcal{S} is a class of subgroups of G (for example, \mathcal{S} could consist of all cyclic subgroups, all abelian subgroups or all

finitely generated subgroups). The best one can hope for is that for all $H, K \in \mathcal{S}$ we have

$$\overline{H} \cap \overline{K} = \overline{H \cap K} \text{ in } \widehat{G} \quad (1)$$

(see Remark 2.2 and Proposition 2.4 below, which explain how this is related to controlling the intersection of the images of H and K in finite quotients of G).

Condition (1) played an important role in the papers of Ribes and Zalesskii [10], Ribes, Segal and Zalesskii [9], Wilson and Zalesskii [13] and Antolín and Jaikin-Zapirain [2], to mention a few. In all of these papers, this condition was established along with (and after) the double coset separability condition, stating that for all $H, K \in \mathcal{S}$

$$HK \text{ is separable in } G. \quad (2)$$

The purpose of this note is to demonstrate that condition (2) implies (1), provided that \mathcal{S} is closed under taking finite index subgroups. More precisely, we prove the following.

Theorem 1.1. *Let H, K be subgroups of a residually finite group G . Then the following are equivalent:*

- (a) *the double coset HK is separable in G and $\overline{H} \cap \overline{K} = \overline{H \cap K}$ in \widehat{G} ;*
- (b) *for every finite index subgroup $L \leq_f G$, with $H \cap K \subseteq L$, the double coset $(H \cap L)K$ is separable in G .*

The above theorem follows from Proposition 2.4 below, which restates condition (1) in terms of finite index subgroups of the group G , and Proposition 3.1, which characterises this restatement in terms of double cosets. Both of these propositions are stated in the general situation of a pro- C topology, where C is a formation of finite groups. In particular, analogues of Theorem 1.1 are also true for the pro- p topology, the pro-soluble topology, etc.

Following [2], we say that a group G has the *Wilson–Zalesskii property* if (1) holds for arbitrary finitely generated subgroups $H, K \leq G$ this property is named after Wilson and Zalesskii, who established it in the case of finitely generated virtually free groups in [13]. We will call a group G *double coset separable* if (2) holds for all finitely generated subgroups $H, K \leq G$.

Corollary 1.2. *Every double coset separable group satisfies the Wilson–Zalesskii property.*

Note that for (virtually) free groups the double coset separability was first proved by Gitik and Rips [3]. This was extended by Niblo [8] to finitely generated Fuchsian groups and fundamental groups of Seifert-fibred 3-manifolds. In [7] the author and Mineh showed that all finitely generated Kleinian groups and limit groups are double coset separable. Hence, by Corollary 1.2, such groups have the Wilson–Zalesskii property. For limit groups this answers a question of Antolín and Jaikin–Zapirain from [2, Subsection 2.2].

More generally, separability of double cosets of ‘convex’ subgroups is known in many non-positively curved groups (see [4, 6, 7, 12]). By combining these results with Theorem 1.1 we gain control over the intersection of such subgroups in finite quotients. Our last corollary describes one such application.

Corollary 1.3. *Let G be a finitely generated group hyperbolic relative to a family of double coset separable subgroups. If every finitely generated relatively quasiconvex subgroup is separable in G , then any two finitely generated relatively quasiconvex subgroups $H, K \leq G$ satisfy (1).*

Proof. Let G be a group from the statement. By [7, Corollary 1.4], the product of two finitely generated relatively quasiconvex subgroups is separable in G . Since a finite index subgroup of a relatively quasiconvex subgroup is again relatively quasiconvex [7, Lemma 5.22], the claim of the corollary follows from Theorem 1.1. \square

We finish this note by constructing, in Section 4, an example of a finitely presented subgroup separable (LERF) group which is not double coset separable and does not have the Wilson–Zalesskii property.

2 | A RESTATEMENT OF CONDITION (1)

Let us fix a formation C of finite groups; in other words, C is a non-empty class of finite groups which is closed under taking quotients and subdirect products (see [11, Section 2.1]).

2.1 | Pro- C topology and completion

In this subsection we summarise basic definitions and properties of pro- C topology and pro- C completions. We refer the reader to [11, Sections 3.1, 3.2] for a detailed exposition.

Given a group G , we define the *pro- C topology* on G by taking the family of normal subgroups $\mathcal{N}_C(G) = \{N \triangleleft G \mid G/N \in C\}$ as a basis of open neighbourhoods of the identity element. A subset $A \subseteq G$ will be called *C -open* if it is open in the pro- C topology on G . *C -closed* and *C -clopen* subsets are defined similarly. We will write $H \leq_o G$ and $N \triangleleft_o G$ to indicate that H is an open subgroup of G and N is an open normal subgroup of G in the pro- C topology. Note that a subgroup $H \leq G$ is C -open if and only if it contains a C -open normal subgroup; and $N \triangleleft G$ is C -open if and only if $G/N \in C$. If $H \leq_o G$ and $X \subseteq G$, then XH and $G \setminus XH$ are both open as unions of cosets modulo H ; thus, XH is a C -clopen subset of G .

We will use $G_{\hat{C}}$ to denote the *pro- C completion* of a group G . Equipped with its pro- C topology, $G_{\hat{C}}$ is a profinite group; in particular, it is compact. The natural homomorphism $G \rightarrow G_{\hat{C}}$ has dense image. This homomorphism is injective if and only if G is *residually- C* , that is, $\bigcap_{N \in \mathcal{N}_C(G)} N = \{1\}$.

2.2 | Tractable intersections

Definition 2.1. Let G be a group and let $H, K \leq G$ be two subgroups. We will say that the intersection $H \cap K$ is *pro- C tractable* in G if for every $M \triangleleft_o G$ there exists $N \triangleleft_o G$ such that $N \subseteq M$ and

$$HN \cap KN \subseteq (H \cap K)M \text{ in } G. \quad (3)$$

Remark 2.2. Note that condition (3) can be restated as $\phi(H) \cap \phi(K) \subseteq \phi(H \cap K)\phi(M)$ in the finite quotient $G/N \in C$, where $\phi : G \rightarrow G/N$ denotes the natural homomorphism.

Remark 2.3. The following observation will be used throughout this note without further justification. If A, B are subsets of a group G and $H' \leq H \leq G$ are subgroups, then

$$AH' \cap BH = (A \cap BH)H' \text{ and } H'A \cap HB = H'(A \cap HB).$$

Proposition 2.4. *For subgroups H, K of a residually- C group G the following are equivalent:*

- (i) *the intersection $H \cap K$ is pro- C tractable in G ;*
- (ii) *$\overline{H \cap K} = \overline{H} \cap \overline{K}$ in $G_{\widehat{C}}$, where \overline{H} denotes the closure of H in the pro- C completion $G_{\widehat{C}}$.*

Proof. Since G is residually- C , we will treat it as a subgroup of $G_{\widehat{C}}$. Note that for an arbitrary $L \triangleleft_o G$ its closure \overline{L} is a clopen subgroup of $G_{\widehat{C}}$ and $\overline{L} \cap G = L$, so that $G_{\widehat{C}}/\overline{L} = G/L$ (see [11, Proposition 3.2.2]). Given any $M \triangleleft_o G$, let $\mathcal{N}_C(M, G) = \{N \triangleleft_o G \mid N \subseteq M\}$ and observe that $\{\overline{N} \mid N \in \mathcal{N}_C(M, G)\}$ is a basis of open neighbourhoods of the identity element in $G_{\widehat{C}}$.

Let us start with showing that (i) implies (ii). Assuming (i), we know that for every $M \triangleleft_o G$ there exists $N \in \mathcal{N}_C(M, G)$ such that (3) holds. After taking closures of both sides we obtain

$$\overline{HN \cap KN} \subseteq \overline{(H \cap K)M} \text{ in } G_{\widehat{C}}. \tag{4}$$

Note that $HN, KN \leq_o G$, so, by [11, Proposition 3.2.2], $\overline{HN \cap KN} = \overline{HN} \cap \overline{KN}$. Clearly $\overline{H} \cap \overline{K} \subseteq \overline{HN} \cap \overline{KN}$ and $\overline{(H \cap K)M} = (H \cap K)\overline{M}$, because \overline{M} is a clopen subgroup of $G_{\widehat{C}}$. Hence, in view of (4), we obtain

$$\overline{H} \cap \overline{K} \subseteq (H \cap K)\overline{M} \text{ in } G_{\widehat{C}}, \text{ for every } M \triangleleft_o G. \tag{5}$$

It is easy to see that $\overline{H \cap K} = \bigcap_{M \triangleleft_o G} (H \cap K)\overline{M}$, because $\mathcal{N}_C(G_{\widehat{C}}) = \{\overline{L} \mid L \in \mathcal{N}_C(G)\}$. Therefore (5) implies that $\overline{H} \cap \overline{K} \subseteq \overline{H \cap K}$. The opposite inclusion is obvious, so (ii) has been established.

We will now prove that (ii) implies (i) (in the case of profinite topology this was done in [2, Corollary 10.4]). Suppose that (ii) holds and $M \triangleleft_o G$ is arbitrary. If (i) is false, then for every $N \in \mathcal{N}_C(M, G)$, we have

$$(HN \cap KN) \setminus (H \cap K)M \neq \emptyset \text{ in } G,$$

hence

$$(\overline{HN \cap KN}) \setminus (H \cap K)\overline{M} \neq \emptyset \text{ in } G_{\widehat{C}}, \text{ for all } N \in \mathcal{N}_C(M, G), \tag{6}$$

where we used the fact that $(H \cap K)\overline{M} \cap G = (H \cap K)(\overline{M} \cap G) = (H \cap K)M$.

The family $\{(\overline{HN \cap KN}) \setminus (H \cap K)\overline{M} \mid N \in \mathcal{N}_C(M, G)\}$ consists of clopen sets in $G_{\widehat{C}}$ and has the finite intersection property by (6) (because the intersection of finitely subgroups from $\mathcal{N}_C(M, G)$ is again in $\mathcal{N}_C(M, G)$). Compactness of $G_{\widehat{C}}$ now implies that

$$\bigcap_{N \in \mathcal{N}_C(M, G)} (\overline{HN \cap KN}) \setminus (H \cap K)\overline{M} \neq \emptyset. \tag{7}$$

Since $\bigcap_{N \in \mathcal{N}_C(M, G)} \overline{HN} = \overline{H}$, $\bigcap_{N \in \mathcal{N}_C(M, G)} \overline{KN} = \overline{K}$ and $\overline{H \cap K} \subseteq (H \cap K)\overline{M}$, (7) demonstrates that $(\overline{H} \cap \overline{K}) \setminus \overline{H \cap K} \neq \emptyset$, contradicting (ii). Thus we have proved that (ii) implies (i). □

3 | CHARACTERISING TRACTABLENESS OF INTERSECTIONS USING DOUBLE COSETS

As before we will work with a fixed formation of finite groups \mathcal{C} . For a subgroup H of a group G the pro- \mathcal{C} -topology on G induces a topology on H (which may, in general, be different from the pro- \mathcal{C} topology of H). We will use $\mathcal{O}_{\mathcal{C}}(H, G)$ to denote the open subgroups of H in this induced topology. In other words,

$$\mathcal{O}_{\mathcal{C}}(H, G) = \{H \cap L \mid L \leq_o G\}.$$

Note that for every $H' \in \mathcal{O}_{\mathcal{C}}(H, G)$, the index $|H : H'|$ is finite because any $L \leq_o G$ has finite index in G .

Proposition 3.1. *Let G be a group with subgroups H, K . Then the following are equivalent:*

- (i) *the double coset HK is \mathcal{C} -closed and the intersection $H \cap K$ is pro- \mathcal{C} tractable in G ;*
- (ii) *for every $H' \in \mathcal{O}_{\mathcal{C}}(H, G)$, with $H \cap K \subseteq H'$, the double coset $H'K$ is \mathcal{C} -closed in G .*

Proof. Let us start with showing that (i) implies (ii). So, assume that (i) is true. Consider any $H' \in \mathcal{O}_{\mathcal{C}}(H, G)$, containing $H \cap K$. Then $H' = H \cap L$, for some $L \leq_o G$, with $H \cap K \subseteq L$. Let $M \triangleleft_o G$ denote the normal core of L (it is \mathcal{C} -open by [11, Lemma 3.1.2]). Since $H \cap K$ is pro- \mathcal{C} tractable, there exists $N \triangleleft_o G$ such that $N \subseteq M$ and

$$HN \cap KN \subseteq (H \cap K)M \subseteq L.$$

Since $NK = KN$, as $N \triangleleft G$, we can conclude that

$$H \cap NK = H \cap KN \subseteq H \cap L = H'.$$

Therefore, we have

$$H'K \subseteq HK \cap H'KN = H'(HK \cap NK) = H'(H \cap NK)K \subseteq H'H'K = H'K,$$

whence $H'K = HK \cap H'KN$ in G . Note that the subset $H'KN$ is \mathcal{C} -clopen in G , as $N \triangleleft_o G$, and the double coset HK is \mathcal{C} -closed by the assumption (i). Thus $H'K$ is \mathcal{C} -closed as the intersection of closed subsets, so (ii) holds.

Now let us assume (ii) and deduce (i). Then the double coset HK is \mathcal{C} -closed in G because $H \in \mathcal{O}_{\mathcal{C}}(H, G)$ and $H \cap K \subseteq H$. Thus, it remains to show that $H \cap K$ is pro- \mathcal{C} tractable in G .

Take any $M \triangleleft_o G$ and set $L = (H \cap K)M \leq_o G$. Then $H' = H \cap L \in \mathcal{O}_{\mathcal{C}}(H, G)$ and we can write $H = \bigsqcup_{i=1}^n H'h_i$, where $h_1 = 1$ and $h_i \in H \setminus H'$, for $i = 2, \dots, n$. Note that $H \cap K \subseteq H'$, by construction, which easily implies that $h_i \notin H'K$, for $i = 2, \dots, n$ (indeed, if $h_i = xy$, where $x \in H'$ and $y \in K$, then $x^{-1}h_i = y \in H \cap K \subseteq H'$, so $h_i \in H'$, whence $i = 1$). By the assumption (ii), the double coset $H'K$ is \mathcal{C} -closed in G ; hence, there exists $N \triangleleft_o G$ such that

$$h_i \notin H'KN, \text{ for } i = 2, \dots, n. \quad (8)$$

After replacing N with $N \cap M$, we can suppose that $N \subseteq M$. Let us show that

$$HN \cap KN \subseteq L = (H \cap K)M.$$

Since $HN \cap KN = (H \cap KN)N$ and $N \subseteq L$, it is enough to check that $H \cap KN \subseteq L$. But, in view of (8), we know that $H'h_i \cap KN = \emptyset$, for $i = 2, \dots, n$, hence $H \cap KN \subseteq H'h_1 = H' \subseteq L$, as required. Therefore $H \cap K$ is pro- C tractable in G and (i) holds. \square

Corollary 3.2. *If H, K are subgroups of a group G , then the following are equivalent:*

- (i) *the double coset HK is C -closed and the intersection $H \cap K$ is pro- C tractable in G ;*
- (ii) *for every $H' \in \mathcal{O}_C(H, G)$, with $H \cap K \subseteq H'$, the double coset $H'K$ is C -closed in G ;*
- (iii) *for every $K' \in \mathcal{O}_C(K, G)$, with $H \cap K \subseteq K'$, the double coset HK' is C -closed in G ;*
- (iv) *for all $H' \in \mathcal{O}_C(H, G)$ and $K' \in \mathcal{O}_C(K, G)$, with $H \cap K = H' \cap K'$, the double coset $H'K'$ is C -closed in G .*

Proof. The equivalence between (i) and (ii) is the subject of Proposition 3.1, and the equivalence between (i) and (iii) follows by symmetry (or because $HK' = (K'H)^{-1}$). Evidently (iv) implies (ii). Conversely, (iv) follows from (ii) and (iii) because

$$H'K \cap HK' = H'(K \cap HK') = H'(K \cap H)K' = H'K',$$

where the last equality is valid since $K \cap H \subseteq H'$. \square

4 | An LERF GROUP WITHOUT THE WILSON-ZALESSKII PROPERTY

Throughout this section we assume that C is the family of all finite groups. In this case the pro- C topology on a group G is the profinite topology, C -open subgroups of G are precisely the finite index subgroups and the C -closed subsets of G are called separable. Recall that G is said to be *ERF* if all subgroups are separable and *LERF* if all finitely generated subgroups are separable.

In this section we show that separability of a double coset HK does not necessarily yield that the intersection $H \cap K$ is profinitely tractable even for finitely generated subgroups H, K of an LERF group G . Our construction is based on examples of Grunewald and Segal from [5].

Let $A = M_2(\mathbb{Z})$ be the additive group of 2×2 matrices with integer entries, and let $H = SL_2(\mathbb{Z})$ act on A by left multiplication. We define the group G as the resulting semidirect product $A \rtimes H = M_2(\mathbb{Z}) \rtimes SL_2(\mathbb{Z})$. Recall that H is finitely generated and virtually free and A is the free abelian group of rank 4; hence, A is ERF and H is LERF, so G is LERF (see [1, Theorem 4]).

Denote by $i \in A$ the identity matrix from $M_2(\mathbb{Z})$ and set $K = iHi^{-1} \leq G$. For any subgroup $F \leq H = SL_2(\mathbb{Z})$ the conjugacy class $i^F = \{fif^{-1} \mid f \in F\} \subseteq A$ is the orbit of the identity matrix under the left action of F , so it consists of matrices from F , but now considered as a subset of $M_2(\mathbb{Z}) = A$. Since the determinant map $\det : A = M_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ is clearly continuous with respect to the profinite topologies on A and \mathbb{Z} , the conjugacy class $i^H = \det^{-1}(\{1\})$ is closed in the profinite topology on A .

Now let us show that the product $i^H H$ is closed in the profinite topology on G . Indeed, suppose that $xy \in G \setminus i^H H$, where $x \in A$ and $y \in H$. Then $x \notin i^H$, so there is $m \in \mathbb{N}$ such that for the finite index characteristic subgroup $A' = M_2(m\mathbb{Z}) \triangleleft A$ we have $x \notin i^H A'$. The latter implies that $xy \notin i^H A' H$. Since $A' H$ is a finite index subgroup of G , we see that $i^H A' H$ is a clopen subset in the profinite topology on G containing $i^H H$ but not containing xy . Thus $i^H H$ is indeed

profininitely closed in G . Note that $i^H H = HiH$; thus, the double coset $HK = (HiH)i^{-1}$ is separable in G .

As Grunewald and Segal observed in [5, Section 5], H contains a finite index free subgroup H' (in fact, $|H : H'| = 36$) such that the orbit of i under the action of H' is not separable in the profinite topology on A (equivalently, H' is not closed in the congruence topology on $H = SL_2(\mathbb{Z})$).

Observe that $H'iH = i^{H'}H$, so $H'iH \cap A = i^{H'}$. Since $i^{H'}$ is not separable in A , it follows that the double cosets $H'iH$ and $H'K = (H'iH)i^{-1}$ cannot be separable in G (this is true because the topology on the subgroup A , induced from the profinite topology on G , is always weaker than the profinite topology on A).

Finally, we note that $H \cap K = \{1\}$ because the H -stabiliser of i is trivial, and every finite index subgroup of H belongs to $\mathcal{O}_c(H, G)$, as G is LERF and H is finitely generated (see, for example, [7, Lemma 4.17]). Thus we have constructed the following example.

Example 4.1. There is an LERF group G (isomorphic to a split extension of \mathbb{Z}^4 by $SL_2(\mathbb{Z})$) and finitely generated subgroups $H, K \leq G$ such that $H \cap K = \{1\}$ and the double coset HK is separable in G , but the double coset $H'K$ is not separable in G , for some finite index subgroup $H' \leq_f H$. We deduce, from Proposition 3.1, that the intersection $H \cap K$ is not profinitely tractable in G , so G does not have the Wilson–Zalesskii property by Proposition 2.4.

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