# The functional $f(R)$ approximation 

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#### Abstract

This article is a review of functional $f(R)$ approximations in the asymptotic safety approach to quantum gravity. It mostly focusses on a formulation that uses a non-adaptive cutoff, resulting in a second order differential equation. This formulation is used as an example to give a detailed explanation for how asymptotic analysis and Sturm-Liouville analysis can be used to uncover some of its most important properties. In particular, if defined appropriately for all values $-\infty<R<\infty$, one can use these methods to establish that there are at most a discrete number of fixed points, that these support a finite number of relevant operators, and that the scaling dimension of high dimension operators is universal up to parametric dependence inherited from the single-metric approximation. Formulations using adaptive cutoffs, are also reviewed, and the main differences are highlighted.


Keywords- Quantum gravity, Renormalization group, Asymptotic safety, $f(R)$ approximation, SturmLiouville, Asymptotic analysis

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## 1 Introduction

One attempted route to a quantum theory of gravity is through the asymptotic safety programme [1-4]. Although quantum gravity based on the Einstein-Hilbert action is plagued by ultraviolet infinities that are perturbatively non-renormalizable (implying the need for an infinite number of coupling constants), a sensible theory of quantum gravity might be recovered if there exists a suitable ultraviolet fixed point [1].

The task is not just that of searching for an ultraviolet fixed point. They must also have the correct properties. Perturbatively renormalizable ones exist for example "Conformal gravity", based on the square of the Weyl tensor, which thus corresponds to a Gaussian ultraviolet fixed point [5]. It is apparently not suitable however, because the theory is not unitary. Suitable unitary fixed points, if they exist, have to be non-perturbative. They must also satisfy phenomenological constraints, for example they have to allow a renormalized trajectory with classical-like behaviour in the infrared, since General Relativity is confirmed by observation across many phenomena and to impressive precision. Of particular relevance for this chapter is that there should be a fixed point with a finite number of relevant directions (otherwise it would be no more predictive than the perturbatively defined theory). Preferably the theory should have only one fixed point, or at least only a finite number (otherwise again we lose predictivity).

Functional RG (renormalization group) equation [6-11] studies, first introduced by Wilson and Wegner many years ago [6, 7] (and called by them the "exact RG"), have flourished into a powerful approach for investigating this possibility. These equations describe the flow of the Wilsonian effective action for some quantum field theory, under changes in an effective cutoff scale $k$. The asymptotic safety literature uses almost exclusively the flow equation for $\Gamma_{k}$ which is, modulo minor details, the Legendre effective action (the generator of one-particle irreducible diagrams) cut off in the infrared by $k$. It was also formulated long ago [9] (in the sharp cutoff limit) and then rediscovered for smooth cutoffs much later in refs. [10, 11]. Following ref. [10], $\Gamma_{k}$ is sometimes called the "effective average action", however in this chapter it will simply be called an effective action.

It is not practical to solve the full functional RG equations exactly. In a situation such as this, where there are no useful small parameters, one can only proceed by considering model approximations. These always proceed from the following observation: Wilsonian effective actions can be written as a sum over operators, where the coefficients are the couplings for these operators and they evolve with the scale $k$.

In fact this sum should be restricted to local operators. This is the requirement of quasi-locality, which comes from the short range nature of the Kadanoff blocking step in Wilsonian RG [6], when implemented in the continuum [12,13]. A related point is that the Wilsonian RG is performed in euclidean signature, so that "short range" has a sensible meaning.

The problem is that for any general solution, this sum is infinite, over all possible local operators allowed by the symmetries (the space of all such couplings being known as "theory space"). However, this motivates the simplest model approximation which is to truncate drastically the infinite dimensional theory space to a
handful of operators. An example is the original truncation studied by Reuter [2]:

$$
\begin{equation*}
\Gamma_{k}\left[g_{\mu \nu}\right]=\int d^{4} x \sqrt{g}\left(u_{0}(k)+u_{1}(k) R\right) \tag{1}
\end{equation*}
$$

which retains only the cosmological constant term and the scalar curvature $R$ term. For obvious reasons this is called the "Einstein-Hilbert truncation". Classically $u_{0}=-\lambda_{c c} /(8 \pi G)$ and $u_{1}=-1 /(16 \pi G)$, where $\lambda_{c c}$ is the cosmological constant and $G$ is Newton's constant, but after quantum corrections these couplings run with $k$ in the functional RG. The minus sign in $u_{1}$ comes from working in euclidean signature.

Apart from RG symmetry, these truncations destroy pretty well all the properties that ought to hold. For example scheme independence (i.e. independence on choice of cutoff, or more generally universality), and modified BRST invariance $[2,14]$ (which encodes diffeomorphism invariance for the quantum field under influence of the cutoff) cannot then be recovered. Furthermore, only by keeping an infinite number of local operators can the non-local long-range nature of the (one-particle irreducible) Green's functions be recovered (see e.g. ref. [15]). One has to trust that by considering ever less restrictive truncations the description gets closer to the truth. There are some examples that go well beyond the Einstein-Hilbert truncation by keeping a large number of operators [16-19]. These are based around polynomial truncations, i.e. where everything is discarded except powers of some suitable local operators, typically the scalar curvature $R$ again, up to some maximum degree. They appear to show convergence, in particular the number of relevant operators is found to be three.

Another approximation in the asymptotic safety literature that is necessary in order to formulate diffeomorphism invariant truncations, such as eqn. (1), conflates the true (quantum) metric with the background metric. It is called the "single metric" or "background field" approximation, and will be described in the next section. It is harder to relax this approximation in any substantive way, although see refs. [20-28] for some approaches.

Whilst very encouraging results are found from multiple studies of such finite order truncations (see e.g. the review [29]), successful implementations of more powerful approximations would build confidence in the scenario. The next step is to keep an infinite number of operators. Arguably the simplest such truncation is to keep a full function $f(R)$, making the ansatz [22-24, 30-43]

$$
\begin{equation*}
\Gamma_{k}[g]=\int d^{4} x \sqrt{g} f_{k}(R) \tag{2}
\end{equation*}
$$

This is the functional $f(R)$ approximation which is the subject of this chapter. It is achieved by specialising to a maximally symmetric background manifold, either a four-sphere or four-hyperboloid.

Closely related approximations have been studied in scalar-tensor [44, 45] and unimodular [46] gravity, and in three space-time dimensions [37]. In fact, the high order finite dimensional truncations [16-19] were developed by taking examples of these $f(R)$ equations and then further approximating to polynomial truncations.

Note that the functional $f(R)$ approximation actually goes beyond keeping a countably infinite number of couplings, the Taylor expansion coefficients $g_{n}=f^{(n)}(0)$, because a priori the large field parts of $f(R)$
contain degrees of freedom that are unrelated to all these $g_{n}$. For example suppose that at large $R$ one finds that $f(R) \approx \exp \left(-a / R^{2}\right)$, where $a>0$ is some parameter. Such an $f(R)$ is in the form of a standard counter-example in mathematical analysis. It has the property that $g_{n}=0$ for all $n$.

As ref. [33] emphasised, the truncation (2) is as close as one can get to the Local Potential Approximation (LPA) [47, 48], a successful approximation for scalar field theory in which only a general potential $V(\varphi)$ is kept for a scalar field $\varphi$ (see e.g. [47-52]). The LPA can be viewed as the start of a systematic derivative expansion [49], in which case this lowest order corresponds to regarding the field $\varphi$ as constant. In rough analogy, an approximation of form (2) may be derived by working on a euclidean signature space of maximal symmetry, where the scalar curvature $R$ is constant. (Typically a four-sphere is chosen.) In particular, techniques that have proved successful in scalar field theory [48-53] have been adapted to this very different context, and used to gain substantial insight [43, 54-57].

The functional truncation (2) still has the problems that were highlighted earlier for its finite dimensional counterparts. However, again one can hope that it is closer to the truth. One hint that this is in fact the case is covered at the end of this chapter. Assuming that the most recent version [43] does have a fixed point solution, then it turns out that operators with high scaling dimension do begin to display universality - unfortunately up to an annoying parameter that remains which is clearly caused by the single-metric approximation.

In this chapter, it will be explained how to construct functional $f(R)$ approximations and how to interpret them. Important properties of formulations that use an adaptive cutoff [22-24,30-41] will be reviewed. These result in third order differential equations, with fixed singularities and problematic asymptotic behaviour. Mostly the chapter will focus on a non-adaptive cutoff formulation [42,43] that results in a second order differential equation, using it as an example to give a detailed exposition of the techniques, especially asymptotic analysis and Sturm-Liouville analysis, that can be used to prove properties of functional $f(R)$ approximations. In particular, if the second order formulation is taken to apply to only one of the two spaces (sphere or hyperboloid), the fixed point solutions form a continuous set and the eigenoperator spectrum is not quantised. However, if these spaces are joined together smoothly (through flat space at their boundary), these methods establish that there are at most a discrete number of fixed points, that the fixed points support a finite number of relevant operators, and yield the result above for operators of high scaling dimension. They do not establish that such fixed points actually exist however. Such a demonstration requires more powerful numerical analysis and/or simpler fixed point formulations [43].

## 2 Flow equations

The starting point is the functional RG flow equation [10, 11]:

$$
\begin{equation*}
\partial_{t} \Gamma_{k}=\frac{1}{2} \operatorname{STr}\left[\left(\Gamma_{k}^{(2)}+\mathcal{R}_{k}\right)^{-1} \partial_{t} \mathcal{R}_{k}\right], \tag{3}
\end{equation*}
$$

where Str is a functional trace over spacetime coordinates and indices that takes into account statistics of the fields. In momentum space this is an integral over one loop momentum. The right hand side is a one-loop integral if $\Gamma_{k}$ is taken to be classical. It includes all higher loops because $\Gamma_{k}$ is actually given by the full (all orders) effective action. $\Gamma_{k}^{(2)}$ (Hessian) is the second variation of the effective action with respect to the fields and $\mathcal{R}_{k}$ is an IR (infrared) cutoff for these fields. The cutoff scale $k$ is related to RG"time" $t$ via $t=\ln (k / \mu)$, where $\mu$ is the standard arbitrary physical energy scale that appears in RG treatments (including in perturbative quantum field theory).

An essential step in Wilsonian RG is to introduce dimensionless variables by multiplication of appropriate powers of the cut-off scale $k$. In the $f(R)$ approximation the appropriate powers are just the canonical (a.k.a. engineering) ones:

$$
\begin{equation*}
\tilde{f}_{k}(\tilde{R}) \equiv \tilde{f}(\tilde{R}, t)=k^{-4} f_{k}\left(k^{2} \tilde{R}\right), \quad \tilde{R}=R / k^{2} \tag{4}
\end{equation*}
$$

In Wilsonian RG, one integrates out modes, starting with the high momentum modes first, by a coarsegraining procedure. Traditionally, after integrating out the modes, one has to rescale the action back to the original UV cut-off of the theory to see how the couplings change. By working with dimensionless quantities this is taken care of automatically.
(From this point onwards it is convenient to drop the tilde denoting dimensionless quantities, unless otherwise specified, but the reader should assume that all the quantities are dimensionless.)

In this way solutions to the flow equation will reveal all the fixed points of the theory, i.e. $t$ independent solutions $f(R, t)=f(R)$. Fixed points are characterized by the number of eigenoperators $v(R)$ (operators of definite scaling dimension) that flow into the fixed point when we increase the cutoff scale $k$. These are called relevant eigenoperators. Conversely irrelevant operators are the ones that flow away from the fixed point.

The terms relevant (irrelevant) are common in the Wilsonian RG literature. Following Weinberg [1], in asymptotic safety literature often they are referred to equivalently as essential (inessential). Eigenoperators whose couplings do not flow (in some approximation or exactly) are called marginal. We do not need to discuss them in this chapter. Exceptionally eigenoperators can appear that are "redundant", corresponding to a change of variables in the theory $[55,58,59]$.

Eigenoperators are found by linearising the flow equations around the fixed point and separating variables:

$$
\begin{equation*}
f_{k}(R)=f(R)+\epsilon v(R) \mathrm{e}^{-\theta t} \tag{5}
\end{equation*}
$$

where $\epsilon$ is a small parameter. This turns the flow equation into an eigenvalue problem where the RG eigenvalue $\theta$ is often called a "critical exponent" in the asymptotic safety literature. From its associated $v(R)$ it can be similarly classified as relevant, irrelevant, marginal or redundant. Thus if $\operatorname{Re} \theta>0$ then it is relevant, whilst if $\operatorname{Re} \theta<0$ it is irrelevant. In statistical physics, non-redundant $\theta$ can be straightforwardly related to experimentally defined and measurable critical exponents, see e.g. [60]. If computed correctly an important property of a non-redundant $\theta$ is that it is universal, which means in particular that its value is independent of the regularisation scheme and the choice of flow equation [58].

As already intimated, one is generally interested in those fixed points that have finitely many relevant operators, because their couplings become the free parameters in the theory, and will have to be fixed by experiments. Thus, theories based around these points are predictive and are safe from UV divergences when $k \rightarrow \infty$. The goal of the asymptotic safety program is to verify if such points exist for gravity, analyse their properties and deduce their consequences, both qualitatively and quantitatively.

Actually, the flow equation (3) requires a significant amount of adaptation to deal with the fact that quantum gravity is a gauge theory. In standard fashion, it therefore requires gauge fixing. This is commonly done by employing the background field method where the full (a.k.a. total) metric $\hat{g}_{\mu \nu}$ is split into a background $g_{\mu \nu}$ plus fluctuations (the quantum field):

$$
\begin{equation*}
\hat{g}_{\mu \nu}=g_{\mu \nu}+h_{\mu \nu} . \tag{6}
\end{equation*}
$$

In common with most of the literature, this chapter will only use a linear split, although other, nonperturbatively better motivated, splits are possible [29,39]. Then the gauge fixing is imposed on the quantum field $h_{\mu \nu}$ in such a way that diffeomorphism invariance of the background metric $g_{\mu \nu}$ is retained:

$$
\begin{equation*}
F_{\mu}=\nabla^{\nu} h_{\mu \nu}-\frac{1}{4} \nabla_{\mu} h_{\nu}^{\nu} \tag{7}
\end{equation*}
$$

where the covariant derivative and raised indices, are defined using the background metric. The process of fixing a gauge, adds the gauge fixing term

$$
\begin{equation*}
\Gamma_{g f}=\frac{1}{2 \alpha} \int d^{4} x \sqrt{g} g^{\mu \nu} F_{\mu} F_{\nu} \tag{8}
\end{equation*}
$$

to the effective action, and leads also to a ghost action. In practice the Landau gauge is chosen: $\alpha \rightarrow 0$. Finally, it proves useful to make a change of variables, this is explained in (12), and this leads to further, auxiliary, fields.

The true solution involves arbitrarily complicated interactions to arbitrarily high order between all these fields, molified only by the symmetries (in particular background field diffeomorphism invariance and modified BRST invariance $[14,61]$ ). The next steps in the approximation drastically truncates all of this [2]. It can be summarised as follows. Only the one-loop contributions from the bilinear ghost and auxiliary field and fluctuation field actions are retained, i.e. on the right hand side of the flow equation (3) only the Hessian from the classical action for these fields is used. The flow of the bit of the effective action that only depends on the background metric, is therefore reproduced correctly at one loop. For the part beyond one loop, the correct Hessian in (3) for the metric,

$$
\begin{equation*}
\frac{\delta^{2} \Gamma_{k}}{\delta h_{\mu \nu}(x) \delta h_{\alpha \beta}(y)}, \tag{9}
\end{equation*}
$$

is replaced by one in which the functional derivatives are with respect to the background field instead:

$$
\begin{equation*}
\frac{\delta^{2} \Gamma_{k}}{\delta g_{\mu \nu}(x) \delta g_{\alpha \beta}(y)} \tag{10}
\end{equation*}
$$

This is the single metric, or background field, approximation. It is almost always applied in asymptotic safety investigations. The review [28] covers exceptions. It should be emphasised that already at one loop
the single metric approximation is not correct, because the dependence of the effective action on $h_{\mu \nu}$ has no direct relation to its dependence on $g_{\mu \nu}$. The replacement above would be correct only if the effective action were a functional of the full metric (6) alone, but that relation is broken at the classical level by the gauge fixing term (8) (and corresponding ghost action). Nevertheless the replacement is attractive as a model, because it leaves us with a flow equation for $\Gamma_{k}[g]$ that depends only on the background metric and in a diffeomorphism invariant way.

Now by choosing the background manifold to be one of maximal symmetry, all diffeomorphism invariants can be related either to the volume or the scalar curvature $R$, which is a constant: $\partial_{\mu} R=0$. In this way the effective action has been reduced to (2): the functional $f(R)$ approximation.

Plugging this with appropriately scaled fields (4) (and coordinates $\tilde{x}^{\mu}=k x^{\mu}$ ), into the flow equation (3), one readily derives the form of the left hand side:

$$
\begin{equation*}
\partial_{t} \Gamma_{k}=\int d^{4} x \sqrt{g}\left[\partial_{t} f_{k}(R)+4 f_{k}(R)-2 R f_{k}^{\prime}(R)\right] \tag{11}
\end{equation*}
$$

The right hand side of the flow equation depends on the detailed way the quantum corrections are handled, which differs between authors [22-24,30-43]. For this we need to compute the second variation of $\Gamma_{k}$ with respect to the fields. First, the gauge fixing term (8) is chosen and the ghost action is derived. Then the transverse traceless (a.k.a. York) decomposition of the metric [62] is used:

$$
\begin{equation*}
h_{\mu \nu}=h_{\mu \nu}^{T}+\nabla_{\mu} \xi_{\nu}+\nabla_{\nu} \xi_{\mu}+\nabla_{\mu} \nabla_{\nu} \sigma+\frac{1}{d} g_{\mu \nu} \bar{h} \tag{12}
\end{equation*}
$$

which separates physical degrees of freedom, viz. $h_{\mu \nu}^{T}$ and $\bar{h}$, from the unphysical ones associated with gauge degrees of freedom, namely $\xi_{\mu}$ and $\sigma$. These fields satisfy

$$
\begin{equation*}
h_{\mu}^{T}=0, \quad \nabla^{\mu} h_{\mu \nu}^{T}=0, \quad \nabla^{\mu} \xi_{\mu}=0, \quad \bar{h}=h-\nabla^{2} \sigma . \tag{13}
\end{equation*}
$$

Expressing $\sqrt{\hat{g}}$ and $\hat{R}$, where the latter is the curvature of the full metric (6), to quadratic order in these fields, the elements of the Hessian can be determined for these components. For example for the physical components one finds

$$
\begin{gather*}
\Gamma_{h_{\mu \nu}^{T} h_{\alpha \beta}^{T}}^{(2)}=-\frac{1}{2}\left[f_{k}^{\prime}(R)\left(-\nabla^{2}+\frac{1}{6} R\right)+\left(f_{k}-\frac{1}{2} R f_{k}^{\prime}\right)\right] \delta^{\mu \nu, \alpha \beta},  \tag{14}\\
\Gamma_{\bar{h} \bar{h}}^{(2)}=\frac{1}{16}\left[9 f_{k}^{\prime \prime}\left(-\nabla^{2}-\frac{R}{3}\right)^{2}+3 f_{k}^{\prime}\left(-\nabla^{2}-\frac{R}{3}\right)-\left(R f_{k}^{\prime}-2 f_{k}\right)\right], \tag{15}
\end{gather*}
$$

where the right hand side is evaluated at $\hat{g}_{\mu \nu}=g_{\mu \nu}$, in preparation for the single metric approximation. We can write these more compactly if we introduce

$$
\begin{equation*}
E_{k}(R)=2 f_{k}(R)-R f_{k}^{\prime}(R) \tag{16}
\end{equation*}
$$

which is the equation of motion that follows from the action (2), and express them instead using the natural Laplacian $\Delta_{s}$ for a spin $s$ component field (on a maximally symmetric background) [33]:

$$
\begin{equation*}
\Delta_{0}=-\nabla^{2}-\frac{R}{3}, \quad \Delta_{1}=-\nabla^{2}-\frac{R}{4}, \quad \Delta_{2}=-\nabla^{2}+\frac{R}{6} \tag{17}
\end{equation*}
$$

A similar decomposition is applied to the ghost action. In the formulation of ref. [33] the contribution to the Hessian coming from the gauge degrees of freedom from the metric and the ghosts cancel each other exactly. Finally, including the contributions of the auxiliary fields that encode the Jacobians due to the transverse traceless decomposition of the metric and the ghost fields, gives the full flow equation (3) in the single metric and functional $f(R)$ approximation:

$$
\begin{equation*}
V\left(\partial_{t} f_{k}(R)+2 E_{k}(R)\right)=\mathcal{T}_{2}+\mathcal{T}_{0}^{\bar{h}}+\mathcal{T}_{1}^{J a c}+\mathcal{T}_{0}^{J a c} \tag{18}
\end{equation*}
$$

where $V=\int d^{4} x \sqrt{g}$ is the volume of the manifold, and the $\mathcal{T}$ objects are the following spacetime traces:

$$
\begin{gather*}
\mathcal{T}_{2}=\operatorname{Tr}\left[\frac{d_{t} \mathcal{R}_{k}^{T}}{-f_{k}^{\prime}(R) \Delta_{2}-E_{k}(R) / 2+2 \mathcal{R}_{k}^{T}}\right]  \tag{19}\\
\mathcal{T}_{0}^{\bar{h}}=\operatorname{Tr}\left[\frac{8 d_{t} \mathcal{R}_{k}^{\bar{h}}}{9 f_{k}^{\prime \prime}(R) \Delta_{0}^{2}+3 f_{k}^{\prime}(R) \Delta_{0}+E_{k}(R)+16 \mathcal{R}_{k}^{\bar{h}}}\right],  \tag{20}\\
\mathcal{T}_{1}^{J a c}=-\frac{1}{2} \operatorname{Tr}\left[\frac{d_{t} \mathcal{R}_{k}^{V}}{\Delta_{1}+\mathcal{R}_{k}^{V}}\right],  \tag{21}\\
\mathcal{T}_{0}^{J a c}=\frac{1}{2} \operatorname{Tr}\left[\frac{d_{t} \mathcal{R}_{S_{1}}^{V}}{\Delta_{0}+R / 3+\mathcal{R}_{k}^{S_{1}}}\right]-\operatorname{Tr}\left[\frac{2 d_{t} \mathcal{R}_{S_{2}}^{V}}{\left(3 \Delta_{0}+R\right) \Delta_{0}+4 \mathcal{R}_{k}^{S_{2}}}\right] . \tag{22}
\end{gather*}
$$

The right hand side of (18) has been subdivided into contributions coming from fields of different spins. The first two come from the physical spin-2 traceless part of the metric and the spin- 0 trace of the metric, as the reader can see by using (14) and (15) in (3). The last two are spin-1 and spin-0 parts coming from field redefinitions.

## 3 Cutoff functions

One place where crucial differences occur between the different implementations is in the choice of cutoff $\mathcal{R}_{k}$. There is quite a lot of freedom as these functions only need to satisfy a few key properties which ensure that they behave like momentum dependent mass terms suppressing low momentum modes:

$$
\begin{equation*}
\lim _{p^{2} \rightarrow 0} \mathcal{R}_{k}\left(p^{2}\right)>0, \quad \lim _{p^{2} \rightarrow \infty} \mathcal{R}_{k}\left(p^{2}\right)=0, \quad \lim _{k \rightarrow 0} \mathcal{R}_{k}\left(p^{2}\right)=0 \tag{23}
\end{equation*}
$$

The first two conditions ensure that we integrate out the UV modes first and ignore the IR modes. The last condition ensures that we are left with the standard definition of the effective action once the cutoff scale is sent to zero.

An apparently attractive strategy is to choose cutoffs that simplify the flow equations as much as possible. "Adaptive cutoffs" are introduced partly with that aim [22-24, 30-41]. They implement the following rule for all appearances of the Laplacian operator $-\nabla^{2}$ :

$$
\begin{equation*}
-\nabla^{2} \mapsto-\nabla^{2}+k^{2} r\left(-\nabla^{2} / k^{2}\right), \tag{24}
\end{equation*}
$$

where $r(z)$ is a cutoff profile function.

Such a choice also seemingly solves an awkward feature of euclidean quantum gravity, which is that the euclidean signature Einstein-Hilbert action (1) has a wrong-sign kinetic term and propagator for $\bar{h}$, the so-called conformal instability [63]. This can be seen in the negative coefficient for $\Delta_{0}$ in (20) in this case. By implementing (24), the cutoff automatically adapts to this wrong sign, so that it continues to modify the propagator in the intended way: by adding a momentum dependent mass term. Indeed if this were not done, the cutoff and kinetic term would have opposite signs, resulting in a singular propagator. However, this trick does not entirely cure the problem since it results in poor asymptotic (large $R$ ) behaviour. This issue will be briefly touched on below and in sec. 4. For further discussion, see refs. [2, 43, 54, 64-66].

Technically the above replacement rule is implemented by setting

$$
\begin{equation*}
\mathcal{R}_{k}^{\phi}=\Gamma_{k}^{(2)}\left[-\nabla^{2}+k^{2} r\left(-\nabla^{2} / k^{2}\right)\right]-\Gamma_{k}^{(2)}\left[-\nabla^{2}\right], \tag{25}
\end{equation*}
$$

for each mode $\phi$, so that the desired effect is created for $\Gamma_{k}^{(2)}\left[-\nabla^{2}\right]+\mathcal{R}_{k}$ in the flow equation (3). Notice that the cutoff function is then of the same form as the Hessian elements themselves and thus now also depends on $f(R)$. This has a particular consequence for the scalar $\bar{h}$ mode, since $\Gamma_{\bar{h} \bar{h}}^{(2)}$ contains $f_{k}^{\prime \prime}(R)$, cf. eqn. (15). It means that plugging this type of cutoff into the flow equation will result in the appearance of $R f_{k}^{\prime \prime \prime}(R)$, due to the presence of $d_{t} \mathcal{R}_{k}^{\bar{h}}$ in the numerator in (20) and the definition (4) of $f_{k}(R)$. This makes the flow equation a third order differential equation, which unfortunately lacks the powerful properties found in a second order formulation (as covered in sec. 4). Furthermore, the factor of $R$ leads to a so-called "fixed singularity" at $R=0$. Third order formulations suffer from further fixed singularities and, as already mentioned, poor asymptotic behaviour, this latter leading to continuous eigenoperator spectra [54]. These problems will be further covered in sec. 4.

When using an adaptive cutoff, the cutoff profile function $r\left(-\nabla^{2} / k^{2}\right)$ is almost always chosen to be the "optimised" profile [67]

$$
\begin{equation*}
r(z)=(1-z) \theta(1-z) . \tag{26}
\end{equation*}
$$

The advantage of using this setup is that $d_{t} \mathcal{R}_{k} \propto \theta\left(1+\nabla^{2} / k^{2}\right)$, and thus the eigenvalues of $-\nabla^{2}$ are restricted to be less than $k^{2}$. This means that in denominators one can simply ignore the $\theta$ and thus $k^{2} r\left(-\nabla^{2} / k^{2}\right) \equiv k^{2}+\nabla^{2}$. Therefore the net effect in denominators is just to replace $\Gamma_{k}^{(2)}\left[-\nabla^{2}\right]$ with $\Gamma_{k}^{(2)}\left[k^{2}\right]$, massively simplifying the computation of spacetime traces.

The second order formulation $[42,43]$ chooses a non-adaptive cutoff function of the form

$$
\begin{equation*}
\mathcal{R}_{k}^{\phi}=k^{m_{\phi}} c_{\phi} r\left(\Delta_{s}+\alpha_{s} R\right) \tag{27}
\end{equation*}
$$

where $s$ is the spin of the mode $\phi, m_{\phi}$ is set such that the cutoff has the same dimension as $\Gamma^{(2)}$ for this mode, and $c_{\phi}$ is a number. In this chapter the $c_{\phi}$ will be taken to be positive for all fields. This is a problem for developing solutions $f_{k}(R)$ that approximate the perturbative quantisation of the Einstein-Hilbert action (1) because the $\bar{h}$ Hessian has the wrong sign there (as noted above). But again the alternative choice $c_{\bar{h}}<0$ leads to poor asymptotic behaviour at large $R$, resulting in a continuous spectrum of eigenoperators [43].

Notice that the cutoffs (27) have been chosen to depend on $\Delta_{s}$, rather than simply the $-\nabla^{2}$ part [33], and furthermore include an "endomorphism", a curvature correction with endomorphism coefficient $\alpha_{s}$ [42]. In refs. $[42,43]$ the traces are computed directly as a sum over modes. The $\alpha_{s}$ are there to ensure that

$$
\begin{equation*}
\Delta_{s}+\alpha_{s} R>0 \tag{28}
\end{equation*}
$$

for all modes, which in turn ensures that they are all integrated out as $k \rightarrow 0$, and that the flow equation does not suffer from fixed singularities. For these non-adaptive cutoffs, the optimised cutoff profile (26) brings no particular advantage. In fact on a sphere the trace is a discrete sum and sharp cutoff profiles would lead to a staircase behaviour [33], with an ill-defined limit as $R \rightarrow 0$. Hence, a smooth (infinitely differentiable) cutoff profile is used, such as [10]

$$
\begin{equation*}
r(z)=\frac{z}{\exp \left(a z^{b}\right)-1}, \quad a>0, b \geq 1 \tag{29}
\end{equation*}
$$

## 4 Flow equations with adaptive cutoff

In those formulations that use an adaptive cutoff, spacetime traces are evaluated using a heat-kernel asymptotic expansion, apart from ref. [33] which uses a direct spectral sum together with a smoothing procedure (to get over the aforementioned staircase problem). As an illustration, the result of the earliest four such formulations [30-32] for the flow of $f \equiv f(R, t)$ on a four-sphere, can be summarised as:

$$
\begin{align*}
& 384 \pi^{2}\left(\partial_{t} f+4 f-2 R f^{\prime}\right)=  \tag{30}\\
& \quad\left[5 R^{2} \theta\left(1-\frac{R}{3}\right)-\left(12+4 R-\frac{61}{90} R^{2}\right)\right]\left[1-\frac{R}{3}\right]^{-1}+\Sigma \\
& +\left[10 R^{2} \theta\left(1-\frac{R}{4}\right)-R^{2} \theta\left(1+\frac{R}{4}\right)-\left(36+6 R-\frac{67}{60} R^{2}\right)\right]\left[1-\frac{R}{4}\right]^{-1} \\
& +\left[\left(\partial_{t} f^{\prime}+2 f^{\prime}-2 R f^{\prime \prime}\right)\left(10-5 R-\frac{271}{36} R^{2}+\frac{7249}{4536} R^{3}\right)+f^{\prime}\left(60-20 R-\frac{271}{18} R^{2}\right)\right]\left[f+f^{\prime}\left(1-\frac{R}{3}\right)\right]^{-1} \\
& +\frac{5 R^{2}}{2}\left[\left(\partial_{t} f^{\prime}+2 f^{\prime}-2 R f^{\prime \prime}\right)\left\{r\left(-\frac{R}{3}\right)+2 r\left(-\frac{R}{6}\right)\right\}+2 f^{\prime} \theta\left(1+\frac{R}{3}\right)+4 f^{\prime} \theta\left(1+\frac{R}{6}\right)\right]\left[f+f^{\prime}\left(1-\frac{R}{3}\right)\right]^{-1} \\
& +\left[\left(\partial_{t} f^{\prime}+2 f^{\prime}-2 R f^{\prime \prime}\right) f^{\prime}\left(6+3 R+\frac{29}{60} R^{2}+\frac{37}{1512} R^{3}\right)\right. \\
& \quad+\left(\partial_{t} f^{\prime \prime}-2 R f^{\prime \prime \prime}\right)\left(27-\frac{91}{20} R^{2}-\frac{29}{30} R^{3}-\frac{181}{3360} R^{4}\right) \\
& \left.\quad+f^{\prime \prime}\left(216-\frac{91}{5} R^{2}-\frac{29}{15} R^{3}\right)+f^{\prime}\left(36+12 R+\frac{29}{30} R^{2}\right)\right]\left[2 f+3 f^{\prime}\left(1-\frac{2}{3} R\right)+9 f^{\prime \prime}\left(1-\frac{R}{3}\right)^{2}\right]^{-1} .
\end{align*}
$$

Here the function $r$ is the optimised cutoff profile (26), which also leads to the appearance of the step functions (a.k.a. Heaviside $\theta$ functions). In ref. [32] the equation is adapted to polynomial truncations only, which means that the step functions are all set to one. The first two lines of the right hand side are independent of $f(R, t)$ and encapsulate the contributions from the ghosts, auxiliaries, $\xi_{\mu}$ and $\sigma$. Here we have introduced the term $\Sigma$. The third and fourth line arises from $h_{\mu \nu}^{T}$, whilst the final ratio is the contribution from $h$. Unphysical modes are isolated differently in these implementations, but the changes can be summarised in the different expressions

$$
\begin{equation*}
\Sigma=0, \quad 10 R^{2} \theta\left(1-\frac{R}{3}\right), \quad-\frac{10 R^{2}\left(R^{2}-20 R+54\right)}{(R-3)(R-4)}, \quad \frac{10(11 R-36)}{(R-3)(R-4)} . \tag{31}
\end{equation*}
$$

The first, third and fourth options are derived in refs. [30,32], whilst the second option comes from ref. [31]. We have suppressed some other details, for more discussion see ref. [54].

Setting $\partial_{t} f=0$ in the above turns this flow equation into the differential equation that must be satisfied by a fixed point $f(R)$. It is a highly non-linear third-order ODE (ordinary differential equation). In the formulation [31], the appearance of the $\theta$ functions, explicitly and in $r$, will result in jumps in $f^{\prime \prime \prime}(R)$ across the point where they switch on or off, but this can be accommodated.

A more important and generic feature is the existence of fixed and moveable singularities. These concepts come from the mathematics of analysis of ODEs. To discuss them it is helpful to cast the fixed point ODE in "normal" form:

$$
\begin{equation*}
f^{\prime \prime \prime}(R)=r h s \tag{32}
\end{equation*}
$$

where $r h s$ (right hand side) contains no $f^{\prime \prime \prime}$ terms. A Taylor expansion about some generic point $R_{p}$ takes the form:

$$
\begin{equation*}
f(R)=f\left(R_{p}\right)+\left(R-R_{p}\right) f^{\prime}\left(R_{p}\right)+\frac{1}{2}\left(R-R_{p}\right)^{2} f^{\prime \prime}\left(R_{p}\right)+\frac{1}{6}\left(R-R_{p}\right)^{3} f^{\prime \prime \prime}\left(R_{p}\right)+\cdots . \tag{33}
\end{equation*}
$$

Since (32) determines the fourth coefficient in terms of the first three, we see that typically (33) provides a series solution depending on three continuous real parameters, here

$$
\begin{equation*}
f\left(R_{p}\right), \quad f^{\prime}\left(R_{p}\right) \quad \text { and } \quad f^{\prime \prime}\left(R_{p}\right), \tag{34}
\end{equation*}
$$

with some finite radius of convergence $\rho$ whose value also depends on these parameters. Therefore the standard mathematical result is recovered that around a generic point $R_{p}$ there is some domain $\mathcal{D}=\left(R_{p}-\right.$ $\left.\rho, R_{p}+\rho\right)$ in which there is a three-parameter set of well-defined solutions. From here one can try to extend the solution to a larger domain, e.g. by matching to a Taylor expansion about another point within $\mathcal{D}$. A typical problem, seen also in the LPA and the derivative expansion [48-50,52,53] and in the second order formulation [42,43], is that eventually, at some point $R=R_{c}$, dependent on the parameters, the denominator of $r h s$ develops a zero, so that as $R \rightarrow R_{c}$, (32) implies

$$
\begin{equation*}
f^{\prime \prime \prime}(R)=2 c /\left(R-R_{c}\right)+\cdots, \tag{35}
\end{equation*}
$$

where $c$ is some constant and the ellipses contains the non-singular part. Integrating this we see that the solution typically ends in a moveable singularity, of form

$$
\begin{equation*}
f(R) \sim c\left(R-R_{c}\right)^{2} \ln \left|R-R_{c}\right| \tag{36}
\end{equation*}
$$

where " $\sim$ " means that less singular parts are neglected.
As already mentioned, fixed point equations derived with adaptive cutoff present another challenge in that they also have fixed singular points $R_{c}$. These correspond in $r h s$ to explicit algebraic poles in $R$, where the domain of interest is $R \geq 0$ since the equations apply to the four-sphere. Whatever the formulation there is always one fixed singularity $R_{c}=0$, which is unavoidable when using an adaptive cutoff as we have
seen $[33,54]$. Different formulations have different numbers and positions for the other fixed singularities (see e.g. the discussion in refs. $[38,57]$ ) but there is always at least one more. Inspecting the example (30), we see that $f^{\prime \prime \prime}$ appears once in the penultimate line in eqn. (30), where it is multiplied by the polynomial

$$
\begin{equation*}
R\left(27-\frac{91}{20} R^{2}-\frac{29}{30} R^{3}-\frac{181}{3360} R^{4}\right) \tag{37}
\end{equation*}
$$

Thus, rearranging the fixed point equation into normal form (32), results in poles from the zeroes of this polynomial. Two of these are in the required domain, namely at $R_{c}=0$ and $R_{c}=2.0065$. There are also two further single poles, at $R_{c}=3$ and $R_{c}=4$, from the first two lines of the right hand side of (30).

As $R$ approaches one of these $R_{c}, f$ will end at a singularity of form (36) unless the $f$-dependent parts in $r h s$ are tuned so as to conspire to cancel the pole. Substituting the Taylor expansion (33), with $R_{p}=R_{c}$, one sees that this requirement forces some generally non-linear combination of $f\left(R_{c}\right), f^{\prime}\left(R_{c}\right)$ and $f^{\prime \prime}\left(R_{c}\right)$ to vanish. Thus, a fixed singularity imposes a constraint on the solution, reducing the number of free parameters by one.

The inevitable fixed singularity at $R_{c}=0$ can thus be seen as restoring consistency since it reduces the three parameter set of solutions to a two parameter set, in agreement in this respect with what is obtained from the non-adaptive-cutoff second order formulation.

Unfortunately, since there are a further three fixed singularities, these equations are overconstrained, and thus there are no fixed point solutions $f(R)$ that are valid over the whole range $R \geq 0$.

However, these fixed singularities are artefacts of the regularisation procedure: it is possible to move them and eliminate most of them. Benedetti and Caravelli were the first to realise this, and we will refer to their version [33] as the "BC" formulation. Before regularisation, the Jacobian trace (21) has a denominator that vanishes if $\Delta_{1}$ vanishes. Likewise the Jacobian trace (22) has a denominator that vanishes when $\Delta_{0}$ vanishes. Recalling the form (17) of the $\Delta_{s}$, and that the net effect of the adaptive optimised cutoff is to replace $-\nabla^{2}$ with $k^{2}$ in the denominator, we see that these contributions give poles $1 /(1-R / 4)$ and $1 /(1-R / 3)$ (after using (4) to scale to dimensionless quantities). These are the poles that are visible in the first two lines of the right hand side of (30).

BC eliminate them by using an endomorphism, namely by using $r\left(\Delta_{s}\right)$ instead of $r\left(-\nabla^{2}\right)$ [33] (a so-called cutoff of type II [32]). Then one is left with the $R_{c}=0$ singularity, and a fixed singularity at some positive $R_{c}$ which is due to the fact that the $\bar{h}$ trace vanishes there [33,54]. These fixed singularities thus reduce $f(R)$ solutions to a one-parameter set.

Now there is still the danger of encountering a moveable singularity (36), and this imposes further restrictions on the remaining parameter. Such a singularity can appear at any value of $R$, and in particular at large $R$ where the equations can then be solved analytically by developing the solution as an asymptotic expansion. In scalar field theory [48-53] and in the second order formulation [43], what is found is that this asymptotic expansion has less than the full number of parameters expected. One can also show that the missing parameters are associated with fast growing perturbations that are incompatible with an asymptotic solution. In this way it is possible to deduce analytically the number of constraints that moveable singularities
are responsible for imposing.
The result for scalar field theory is that the parameters are fixed, typically to a handful of values [48, 49,53], corresponding to a finite set of fixed points, or in special cases a discrete infinity of fixed points [50]. However, there is at this stage also the possibility that there are no fixed point solutions. The actual number of solutions then needs to be determined numerically. ${ }^{1}$ We will see this at work in the second order formulation in sec. 6 where we describe in detail how to find asymptotic solutions $f_{\text {asy }}(R)$.

Unfortunately for third order formulations, asymptotic analysis typically does not find sufficient constraints [57]. For example for the BC formulation, the asymptotic solution turns out to have the maximum three parameters [54]:

$$
\begin{equation*}
f_{a s y}(R)=A R^{2}+R\left\{\frac{3}{2} A+B \cos \ln R^{2}+C \sin \ln R^{2}\right\}+\cdots \tag{38}
\end{equation*}
$$

where the ellipses stand for asymptotic corrections with lower powers of $R$, and the three parameters are restricted only by the inequality:

$$
\begin{equation*}
\frac{121}{20} A^{2}>B^{2}+C^{2} \tag{39}
\end{equation*}
$$

Thus, one still expects to find one-parameter sets (i.e. lines) of global solutions $f(R)$ in this case, and that is exactly what is found by careful numerical analysis [54]. Asymptotic analysis also shows that the BC formulation has continuous eigenoperator spectra. Initially it was suggested that these effects can be attributed to the fact that all eigenoperators are redundant if the equation of motion (16) for the fixed point $f(R)$, has no solution for $R$ in the required range $R \geq 0$ [55]. But it is now clear that the poor behaviour is again associated to the scalar mode $\bar{h}$ [38,54,64], and is one more malign effect of the conformal instability $[54,63,64]$. In fact precisely these problems reappear in the second order formulation if one chooses $c_{\bar{h}}<0$, as already mentioned in sec. 3 .

As emphasised in ref. [57], asymptotic analysis plays three powerful rôles. Firstly, as just sketched and discussed in detail in sec. 6.1, it allows one to deduce the dimension of the solution space. Secondly the asymptotic solution provides a way to validate numerical solutions since if one can integrate out far enough, the numerical solution should match the asymptotic solution, allowing a reliable determination of the asymptotic parameters.

Finally, the asymptotic solution actually contains only the physical part of the fixed point effective action.
To see this, we need to return temporarily to labelling scaled quantities with a tilde, and recall that the effective infrared cutoff $k$ is added by hand such that the physical Legendre effective action is recovered only in the limit that this cutoff $k \rightarrow 0$. This must be done while holding the physical quantities such as $R$ fixed, rather than scaled quantities $\tilde{R}$. In normal field theory, e.g. scalar field theory, the analogous object is the universal scaling equation of state, which for a constant field precisely at the fixed point takes the simple form

$$
\begin{equation*}
V(\varphi)=A \varphi^{d / d_{\varphi}} \tag{40}
\end{equation*}
$$

[^0]where $d$ is the space-time dimension and $d_{\varphi}$ is the full scaling dimension of the field (i.e. incorporating also the anomalous dimension). In the current case we keep fixed the constant background scalar curvature $R$. Thus by (2) and (4), the only physical part of the fixed point action in this approximation is:
\[

$$
\begin{equation*}
\left.f(R)\right|_{\text {phys }}=\lim _{k \rightarrow 0} k^{4} \tilde{f}\left(R / k^{2}\right)=\lim _{k \rightarrow 0} k^{4} \tilde{f}_{\text {asy }}\left(R / k^{2}\right) \tag{41}
\end{equation*}
$$

\]

For example from (38), for the BC formulation one finds:

$$
\begin{equation*}
\left.f(R)\right|_{\mathrm{phys}}=A R^{2} \tag{42}
\end{equation*}
$$

This is invariant under changes of scale as it must be, and is a sensible answer for the scaling equation of state precisely at the fixed point. We will find the same answer from the second order formulation.

We still have the problem that since there are one-parameter sets of fixed-point solutions, $A$ is not fixed. In third order formulations one can use the ability to add endomorphisms to try to patch this up [38] but asymptotic analysis then shows there is actually a whole zoo of possibilities for the scaling equation of state and dimension of the solution space, depending on parameter choices in the endomorphisms [57]. One can also try to extend the solution to negative $R$. This does reduce the solution space of the BC formulation to a discrete set but that set appears to be empty since no numerical solutions were then found [54]. A more careful version of this strategy is also used in the second order formulation.

Actually one can question whether the large $\tilde{R}=R / k^{2}$ regime makes physical sense $[37,38,40]$. The problem arises when the cutoff depends on modified Laplacians, e.g. as in (28), where the endomorphism is added to ensure that the minimum eigenvalue is positive. It is most easily seen if we take a sharp (step function) cutoff profile, and write the minimum eigenvalue as $R \lambda_{\text {min }}$. Then once $k^{2}<R \lambda_{\text {min }}$, i.e. $\tilde{R}>1 / \lambda_{\text {min }}$, there are no more modes to be integrated out. This means that the functional behaviour in this large $\tilde{R}$ regime is meaningless since it is not describing any actual changes. However the physical Legendre effective action is only reached by taking $k \rightarrow 0$, and this argument would appear to imply that such a limit is inherently ill-defined.

In fact this conundrum is another artefact of the single-metric approximation [22]. In reality one should be integrating out over an ensemble of manifolds described by the fluctuating full metric $\hat{g}_{\mu \nu}$. The Wilsonian RG only makes sense when applied to such an ensemble. Then no matter how small $k$ is, there are always manifolds with sufficiently small curvature that their eigenvalues remain to be integrated out. It is possible to repair the single-metric approximation sufficiently in this case by retaining the scale degree of freedom $h_{\mu \nu} \propto g_{\mu \nu}$ in the fluctuation field dependence, and thus regaining an ensemble of manifolds. However the net result of such a repair is the same type of functional RG equations again, but now with a clear explanation for why the large $\tilde{R}$ regime should be trusted [22-24].

We now abandon third order formulations and concentrate on a second order formulation [42,43], which in almost all respects has more promising behaviour.

| Spin S | Eigenvalue $\lambda_{s, n}$ | Multiplicity $D_{n, s}$ |
| :---: | :---: | :---: |
| 0 | $\frac{n(n+3)-4}{12} R$ | $\frac{(n+2)(n+1)(2 n+3)}{6}$ |
| 1 | $\frac{n(n+3)-4}{12} R$ | $\frac{n(n+3)(2 n+3)}{2}$ |
| 2 | $\frac{n(n+3)}{12} R$ | $\frac{5(n+4)(n-1)(2 n+3)}{6}$ |

Table 1: Values of the multiplicities and eigenvalues for evaluating the traces.

## 5 Evaluating traces

In the formulation $[42,43]$ the traces are evaluated by a direct spectral sum. In common with the rest of the literature one chooses a (globally) maximally symmetric background manifold. There are three to choose from: the four sphere $\mathbb{S}^{4}$, which has a finite volume and positive curvature, so the spectrum of the allowed modes form a discrete set that have to be summed over; the hyperboloid $\mathbb{H}^{4}$ which has negative curvature and infinite volume so the spectrum is continuous; and finally flat space $\mathbb{R}^{4}$, which is a limiting case for both of the two previous manifolds when $R \rightarrow 0$. As we will see they all need to be considered. Actually they become smoothly joined together in an ensemble which thus allows the same flow equation to be defined over the entire domain $-\infty<R<\infty$.

### 5.1 Sphere

On the sphere the traces are evaluated using

$$
\begin{equation*}
\operatorname{Tr} W\left(\Delta_{s}\right)=\sum_{n} D_{n, s} W\left(\lambda_{n, s}\right) \tag{43}
\end{equation*}
$$

where $\lambda_{n, s}$ are eigenvalues of the $\Delta_{s}$ defined in (17), and $D_{n, s}$ are their multiplicities. Explicit values are shown in table 1 [33]. There are a few caveats. Not all the modes contribute in the sum, for example vectors satisfying $\nabla_{\mu} \xi_{\nu}+\nabla_{\nu} \xi_{\mu}=0$ and the scalar modes $\sigma=$ constant. Because of this, the tensor mode and the vector mode sums start at $n=2$, the scalar mode of the Jacobian starts at $n=1$ and the $\bar{h}$ mode starts at $n=0$. Now the requirement (28) means that $\lambda_{n, s}+\alpha_{s} R>0$ must be satisfied. For the tensor and vector modes it is sufficient to set $\alpha_{2}=\alpha_{1}=0$, however from table 1 we see that we must have $\alpha_{0}>1 / 3$.

### 5.2 Hyperboloid

As already mentioned, the hyperboloid has a negative curvature, an infinite volume, and a continuous spectrum of eigenvalues. The traces on this manifold are evaluated using [68]

$$
\begin{equation*}
\operatorname{Tr} W\left(\Delta_{s}\right)=\frac{2 s+1}{8 \pi^{2}} \int d^{4} x \sqrt{g}\left(-\frac{R^{2}}{12}\right)^{2} \int_{0}^{\infty} d \lambda\left(\lambda^{2}+\left(s+\frac{1}{2}\right)^{2}\right) \lambda \tanh (\pi \lambda) W\left(\Delta_{\lambda, s}\right) \tag{44}
\end{equation*}
$$

Even though there is now an infinite volume factor in the flow equation (18), this precise factor also appears above, so the equations still make sense once we cancel this factor from both sides. The eigenvalues of the
spectrum are

$$
\begin{equation*}
\Delta_{\lambda, s}=-\frac{R}{12} \lambda^{2}-\beta_{s} R, \quad \text { where } \quad \beta_{0}=\frac{25}{48}, \quad \beta_{1}=\frac{25}{48}, \quad \beta_{2}=\frac{9}{48} . \tag{45}
\end{equation*}
$$

Using the same flow equation, and thus the same endomorphism parameters $\alpha_{s}$, the requirement (28) must again be satisfied. We can still take $\alpha_{2}=\alpha_{1}=0$, but now $\alpha_{0}$ also has an upper bound $\alpha_{0}<25 / 48$.

### 5.3 Flat space

Finally, evaluating traces on flat space can be achieved by taking the limit as $R \rightarrow 0$ from positive or negative side. If we start from the positive side we first make a substitution $p=n \sqrt{R / 12}$ then take $R \rightarrow 0$ while keeping $p$ fixed. All Laplacians then become $\Delta_{n, s} \rightarrow p^{2}$ and $p^{2}$ can be identified as the flat space momentum. Plugging in our choice of the cutoff (27), and performing these substitutions, yields

$$
\begin{align*}
\partial_{t} f_{k}(0)+4 f_{k}(0) & =\frac{1}{8 \pi^{2}} \int_{0}^{\infty} d p p^{3}\left[16 c_{\bar{h}} \frac{2 r\left(p^{2}\right)-p^{2} r^{\prime}\left(p^{2}\right)}{9 f_{k}^{\prime \prime}(0) p^{4}+3 f_{k}^{\prime}(0) p^{2}+2 f_{k}(0)+16 c_{\bar{h}} r\left(p^{2}\right)}\right. \\
& +10 c_{T} \frac{r\left(p^{2}\right)-p^{2} r^{\prime}\left(p^{2}\right)}{-f_{k}^{\prime}(0) p^{2}-f_{k}(0)+2 c_{T} r\left(p^{2}\right)}-3 c_{V} \frac{r\left(p^{2}\right)-p^{2} r^{\prime}\left(p^{2}\right)}{p^{2}+c_{V} r\left(p^{2}\right)} \\
& \left.-4 c_{S_{2}} \frac{2 r\left(p^{2}\right)-p^{2} r^{\prime}\left(p^{2}\right)}{3 p^{4}+4 c_{S_{2}} r\left(p^{2}\right)}+c_{S_{1}} \frac{r\left(p^{2}\right)-p^{2} r^{\prime}\left(p^{2}\right)}{p^{2}+c_{S_{1}} r\left(p^{2}\right)}\right] \tag{46}
\end{align*}
$$

This same equation is arrived at if we take $R \rightarrow 0$ from the negative side by first setting $p=\lambda \sqrt{-R / 12}$ on the hyperboloid and holding $p$ fixed. The form of these equations already give some information about the possible solutions, and can help guide numerical searches [43]. In particular, by inspection, it is clear that there are no fixed singularities, and choices for $f_{k}(0), f_{k}^{\prime}(0)$ and $f_{k}^{\prime \prime}(0)$ can be made that give well defined non-singular integrals.

## 6 Fixed point solutions

The fixed point solution to the flow equation $f_{k}(R)=f(R)$ occurs when $\partial_{t} f_{k}(R)=0$. An advantage of the non-adaptive cutoff is that $\partial_{t} f_{k}(R)$ only appears once on the left hand side of (18), so the fixed point equation is

$$
\begin{equation*}
2 V E(R)=\mathcal{T}_{2}+\mathcal{T}_{0}^{\bar{h}}+\mathcal{T}_{1}^{J a c}+\mathcal{T}_{0}^{J a c} \tag{47}
\end{equation*}
$$

Another crucial advantage is, like (46), inspection of the trace equations (19) - (22) makes clear that there are no fixed singularities any more. The flow equation is non-linear and very hard to work with, so solving the equations exactly is unfeasible. The strategy is to solve analytically for $f(R)$ around $R=0$ as a Taylor expansion and around $R= \pm \infty$ by an asymptotic expansion. Then numerical methods can be used to try to patch in a solution that goes smoothly from the Taylor expansion at $R=0$ to the asymptotic solutions at $R= \pm \infty$.

### 6.1 Asymptotic analysis

We now explain in detail how to develop asymptotic solutions, using these equations as an example. In these large $R$ limits, the equations simplify due to rapidly decaying cutoff profiles $r(z)$. At first sight, it looks like all the traces on the right hand side of the flow equation vanish and one is only left with (16), the equation of motion $E(R)=0$. This is actually true on the hyperboloid and the fixed point solution is therefore the solution of $E(R)=0$ namely

$$
\begin{equation*}
f(R)=A R^{2}, \tag{48}
\end{equation*}
$$

where $A$ is an arbitrary constant. At any finite $R$ this is then accompanied by rapidly decaying corrections as discussed later, cf. eqn. (63).

The story is different on the sphere since upon closer inspection not all of the terms in the sums vanish. There are three such terms, the $n=0$ and $n=1$ components from $\mathcal{T}_{0}^{\bar{h}}$ and the $n=1$ of $\mathcal{T}_{0}^{J a c}$. To see this for the $n=0$ case, note that from table $1, \Delta_{0}=-R / 3$. Thus, using (27), the denominator of this term in the sum (20) is given by

$$
\begin{equation*}
9 f^{\prime \prime}(R) \Delta_{0}^{2}+3 f^{\prime}(R) \Delta_{0}+E(R)+16 \mathcal{R}_{k}^{\bar{h}}=R^{2} f^{\prime \prime}(R)-R f^{\prime}(R)+E(R)+16 k^{4} c_{\phi} r\left(\left[\alpha_{0}-\frac{1}{3}\right] R\right) \tag{49}
\end{equation*}
$$

Now, assuming that the leading asymptotic behaviour is $f(R)=A R^{2}$, we see that the first two terms cancel each other, and likewise $E(R)$ vanishes, so we are left only with the cutoff term in the denominator. Therefore this term takes the form of

$$
\begin{equation*}
\frac{1}{k^{4} r(z)} \frac{d}{d t}\left[k^{4} r(z)\right]=4-2 z \frac{d \ln r(z)}{d z} \tag{50}
\end{equation*}
$$

with $z$ set equal to $z=\left[\alpha_{0}-\frac{1}{3}\right] R$.
Turning to the $n=1$ components, note that from table 1 , both $\Delta_{0}$ and $\Delta_{1}$ vanish for $n=1$. In (20), apart from the cutoff term the whole denominator therefore vanishes (because $E(R)$ vanishes). In (22) it is the second component that has a vanishing denominator apart from the cutoff term. The $S_{1}$ (first) component does not suffer from the same problem because there is also the $+R / 3$ part in the denominator. However, the cutoff dependence is the same for the $n=1$ contributions namely $r\left(\alpha_{0} R\right)$ and the numerical factors are such that these two $n=1$ contributions exactly cancel each other.

Altogether then, effectively the only term on the RHS (right hand side) of the flow equation that does not vanish asymptotically is the $n=0$ component of the $\mathcal{T}_{0}^{\bar{h}}$ trace. This is a problem however, since the $n=0$ component of $\mathcal{T}_{0}^{\bar{h}}$ contributes a term that grows at least as fast as $R^{2}$. This is inconsistent with the fact that the LHS (left hand side) of flow equation has been set to vanish asymptotically. Actually this analysis shows that $f(R)$ grows faster than $R^{2}$. For example in the best-case scenario the RHS $\sim R^{2}$ but that implies $f(R) \sim R^{2} \ln R$ so that the LHS is left with an $E(R) \sim R^{2}$ to balance the contribution from the $n=0$ component of $\mathcal{T}_{0}^{\bar{h}}$.

Therefore we now assume that $f(R)$ actually grows faster than $R^{2}$ at large $R$. But this means we need to check again which terms in the traces have denominators that would vanish without a cutoff. By inspection none of the traces that depend on $f(R)$ can now have this issue. In particular the $n=1$ component of
the $\mathcal{T}_{0}^{\bar{h}}$ trace no longer has a denominator that could vanish, because $E(R)$ no longer vanishes at large $R$, while for the $n=0$ component the $f^{\prime \prime}(R)$ part in the denominator now dominates at large $R$. So the only contribution that survives on the RHS at large $R$, is now the $n=1 S_{2}$ component of $\mathcal{T}_{0}^{\text {Jac }}$.

Keeping just this term it turns out one can solve the fixed point equation in closed form, thus obtaining the correct asymptotic behaviour for general cutoff function $r(z)$. Using the values from table 1 we have that the multiplicity of the $n=1$ component is $D_{1,0}=5$, note that $m_{S_{2}}=4$ and that $1 / V=R^{2} / 384 \pi^{2}$ for the four-sphere. Thus, keeping only this leading term on the RHS of the flow equation, we have

$$
\begin{equation*}
2 f(R)-R f^{\prime}(R)=\frac{R^{2}}{768 \pi^{2}}\left[-10+5 \alpha_{0} R \frac{r^{\prime}\left(\alpha_{0} R\right)}{r\left(\alpha_{0} R\right)}\right] \tag{51}
\end{equation*}
$$

This is exactly soluble. Indeed dividing through by $R^{3}$ it can be rewritten as

$$
\begin{equation*}
-\frac{d}{d R}\left(\frac{f(R)}{R^{2}}\right)=\frac{1}{768 \pi^{2}}\left[-\frac{10}{R}+5 \frac{d}{d R} \ln r\left(\alpha_{0} R\right)\right] \tag{52}
\end{equation*}
$$

which can be immediately integrated to give

$$
\begin{equation*}
f(R)=\frac{5 R^{2}}{768 \pi^{2}} \ln \frac{R^{2}}{r\left(\alpha_{0} R\right)}+A R^{2}+o\left(R^{2}\right) \quad \text { as } \quad R \rightarrow+\infty \tag{53}
\end{equation*}
$$

where we included the integration constant $A$ and finally we noted that terms that grow slower than $R^{2}$ will be generated by iterating this asymptotic solution to higher orders, hence the $o\left(R^{2}\right)$ part. The $\ln r$ term actually dominates, i.e. the large $R$ behaviour is dominated by cutoff-dependent effects. For example using the cutoff (29), gives the first three terms in this series:

$$
\begin{equation*}
f(R)=\frac{5 a \alpha_{0}^{b}}{768 \pi^{2}} R^{2+b}+\frac{5}{768 \pi^{2}} R^{2} \ln R+A R^{2}+\frac{16 c_{\bar{h}}}{5 a b(1+b) \alpha_{0}^{b}}\left(\alpha_{0}-\frac{1}{3}\right) \mathrm{e}^{-a\left(\alpha_{0}-\frac{1}{3}\right)^{b} R^{b}}+\cdots \tag{54}
\end{equation*}
$$

To get the next term in the series, the solution is substituted back into the fixed point equation and the next leading correction is isolated. This leads to the last displayed correction above. It is exponentially decaying and comes from the $n=0$ term in the $\mathcal{T}_{0}^{\bar{h}}$ trace. One finds that other corrections decay faster provided that $\alpha_{0}<\frac{5}{6}+\alpha_{1}$. This is satisfied thanks to the restrictions on the $\alpha_{i}$ parameters discussed in secs. 5.1 and 5.2. Substituting (54) back into the fixed point equation and proceeding similarly one can in principle develop the whole asymptotic series. It is an infinite series of ever faster decaying terms and is indicated by the ellipses. In particular these terms will include a power series in $A$.

At this point we have succeeded in finding consistent asymptotics. $f(R)$ does grow faster than $R^{2}$ on the sphere, as assumed, and using such a form in the RHS of the fixed point equation one can see that the $n=1 S_{2}$ component of $\mathcal{T}_{0}^{J a c}$ dominates at large $R$, which leads back to the above equation.

Recall that the fixed point equation is actually second order. But the asymptotic solutions only have one free parameter $A$, even though there should be two. To find out where the second parameter has gone we linearise about the fixed point $f(R)+\delta f(R)$ and plug it into the flow equation (47) to get

$$
\begin{equation*}
-a_{2}(R) \delta f^{\prime \prime}(R)+a_{1}(R) \delta f^{\prime}(R)+a_{0}(R) \delta f(R)=4 \delta f(R) \tag{55}
\end{equation*}
$$

with

$$
\begin{align*}
a_{2} & =\frac{144 c_{h}}{V} \operatorname{Tr}\left[\frac{\Delta_{0}^{2}\left(2 r\left(\Delta_{0}+\alpha_{0} R\right)-\left(\Delta_{0}+\alpha_{0} R\right) r^{\prime}\left(\Delta_{0}+\alpha_{0} R\right)\right)}{\left(9 f^{\prime \prime}(R) \Delta_{0}^{2}+3 f^{\prime}(R) \Delta_{0}+E(R)+16 c_{h} r\left(\Delta_{0}+\alpha_{0} R\right)\right)^{2}}\right]  \tag{56}\\
a_{1} & =2 R-\frac{16 c_{h}}{V} \operatorname{Tr}\left[\frac{\left(3 \Delta_{0}-R\right)\left(2 r\left(\Delta_{0}+\alpha_{0} R\right)-\left(\Delta_{0}+\alpha_{0} R\right) r^{\prime}\left(\Delta_{0}+\alpha_{0} R\right)\right)}{\left(9 f^{\prime \prime}(R) \Delta_{0}^{2}+3 f^{\prime}(R) \Delta_{0}+E(R)+16 c_{h} r\left(\Delta_{0}+\alpha_{0} R\right)\right)^{2}}\right] \\
& +\frac{2 c_{T}}{V} \operatorname{Tr}\left[\frac{\left(R / 2-\Delta_{2}\right)\left(2 r\left(\Delta_{2}+\alpha_{2} R\right)-\left(\Delta_{2}+\alpha_{2} R\right) r^{\prime}\left(\Delta_{2}+\alpha_{2} R\right)\right)}{\left(-f^{\prime}(R) \Delta_{2}-E(R) / 2+2 c_{T} r\left(\Delta_{2}+\alpha_{2} R\right)\right)^{2}}\right]  \tag{57}\\
a_{0} & =\frac{32 c_{h}}{V} \operatorname{Tr}\left[\frac{\left(2 r\left(\Delta_{0}+\alpha_{0} R\right)-\left(\Delta_{0}+\alpha_{0} R\right) r^{\prime}\left(\Delta_{0}+\alpha_{0} R\right)\right)}{\left(9 f^{\prime \prime}(R) \Delta_{0}^{2}+3 f^{\prime}(R) \Delta_{0}+E(R)+16 c_{h} r\left(\Delta_{0}+\alpha_{0} R\right)\right)^{2}}\right] \\
& +\frac{2 c_{T}}{V} \operatorname{Tr}\left[\frac{\left(2 r\left(\Delta_{2}+\alpha_{2} R\right)-\left(\Delta_{2}+\alpha_{2} R\right) r^{\prime}\left(\Delta_{2}+\alpha_{2} R\right)\right)}{\left(-f^{\prime}(R) \Delta_{2}-E(R) / 2+2 c_{T} r\left(\Delta_{2}+\alpha_{2} R\right)\right)^{2}}\right] \tag{58}
\end{align*}
$$

In the large $R$ limit $a_{1}(R) \sim 2 R$ and $a_{0}$ and $a_{2}$ vanish asymptotically. Then it is tempting to simply set $a_{0}$ and $a_{2}$ to zero to find the asymptotic solution to (55). But if this is done there is only one solution $\delta f(R)=\delta A R^{2}$. In fact this is just the leading term in an asymptotic series which is nothing but what one would derive from (54) by differentiating with respect to $A$. (Recall that the ellipses actually contain a power series in $A$.) This asymptotic solution is an exact series solution to (55) where $a_{0}$ and $a_{2}$ are only involved in constructing the subleading corrections. To find more than the one parameter $\delta A$ in the solution to (55), $\delta f^{\prime \prime}(R)$ cannot be neglected, implying that higher derivatives must dominate over lower ones in the large $R$ limit. Hence, the other solution is one where $\delta f(R)$ can at first be neglected. Then writing (55) as

$$
\begin{equation*}
\frac{d}{d R} \ln \delta f^{\prime}(R)=\frac{a_{1}(R)}{a_{2}(R)} \Longrightarrow \delta f(R)=B \int^{R} d R^{\prime} \exp \int^{R^{\prime}} d R^{\prime \prime} \frac{a_{1}\left(R^{\prime \prime}\right)}{a_{2}\left(R^{\prime \prime}\right)}, \tag{59}
\end{equation*}
$$

where $B$ is the second parameter. For the explicit form, $a_{2}$ is needed. It gets its leading contribution from the same source as the last displayed term in (54). Using the same cutoff choice, (29), asymptotically

$$
\begin{equation*}
a_{2}(R)=\frac{24576 \pi^{2} c_{\bar{h}}}{25 a b(1+b)^{2} \alpha_{0}^{2 b}}\left(\alpha_{0}-\frac{1}{3}\right)^{1+b} R^{1-b} \mathrm{e}^{-a\left(\alpha_{0}-\frac{1}{3}\right)^{b} R^{b}}+\cdots \tag{60}
\end{equation*}
$$

Recalling that $a_{1}=2 R$ to leading order, the integrals can be evaluated by successive integration by parts, as an asymptotic series and where each term is given in closed form.

Since this strategy is used many times in this kind of asymptotic analysis let us sketch it on the indefinite integral:

$$
\begin{equation*}
\int d R G(R) \mathrm{e}^{F(R)}=\frac{G(R)}{F^{\prime}(R)} \mathrm{e}^{F(R)}-\int d R\left(\frac{G(R)}{F^{\prime}(R)}\right)^{\prime} \mathrm{e}^{F(R)} \tag{61}
\end{equation*}
$$

The above equality follows by integration by parts, however if $F(R)$ grows at least as fast as $R$ for large $R$, where $F$ is either sign, and $G(R)$ grows or decays slower than an exponential of $R$, then the integral on the right is subleading compared to the integral on the left. Iterating this identity then evaluates the integral in the large $R$ limit as $\mathrm{e}^{F(R)}$ times an asymptotic series, the first term on the RHS being the leading term.

In this way, using the cutoff (29), the solution (59) on the sphere turns out to be

$$
\begin{equation*}
\delta f(R) \sim B \exp \left\{\frac{12(1+b)^{2} \alpha_{0}^{2 b}}{12288 \pi^{2} c_{\bar{h}}}\left(\alpha_{0}-\frac{1}{3}\right)^{-1-2 b} R \mathrm{e}^{a\left(\alpha_{0}-1 / 3\right)^{b} R^{b}}\right\} \tag{62}
\end{equation*}
$$

The analysis proceeds similarly on the hyperboloid [43]. As $R \rightarrow-\infty$, one finds:

$$
\begin{equation*}
f(R)=A R^{2}+\frac{c_{S 1}}{96 \sqrt{3 \pi a^{3} b^{3}}}\left(\frac{25}{48}-\alpha_{0}\right)^{\frac{5-3 b}{2}}(-R)^{2-\frac{3 b}{2}}\left\{1+O\left(|R|^{-\frac{1}{2}}\right)\right\} \mathrm{e}^{-a\left[\left(\alpha_{0}-\frac{25}{48}\right) R\right]^{b}}+\cdots \tag{63}
\end{equation*}
$$

The correction is again a decaying exponential because $\alpha_{0}$ is restricted to $\alpha_{0}<25 / 48$. All scalar traces (thus also $A$ ) contribute to the $O\left(|R|^{-\frac{1}{2}}\right)$ term, and the ellipses stand for terms with faster decaying exponentials. The asymptotic behaviour of $a_{2}$ turns out now to be:

$$
\begin{equation*}
a_{2}(R)=\frac{4 c_{\bar{h}}}{81 A^{2} \sqrt{3 \pi a b}}\left(\frac{25}{48}-\alpha_{0}\right)^{\frac{5-b}{2}}(-R)^{1-\frac{b}{2}} \mathrm{e}^{-a\left[\left(\alpha_{0}-\frac{25}{48}\right) R\right]^{b}}+\cdots \tag{64}
\end{equation*}
$$

(the ellipses being faster decaying terms). And thus one finds on the hyperboloid

$$
\begin{equation*}
\delta f(R) \sim B \exp \left\{\frac{81 A^{2}}{2 c_{\bar{h}}} \sqrt{\frac{3 \pi}{a b}}\left(\frac{25}{48}-\alpha_{0}\right)^{-\frac{b+5}{2}}(-R)^{1-b / 2} \mathrm{e}^{a\left[\left(\alpha_{0}-25 / 48\right) R\right]^{b}}\right\} \tag{65}
\end{equation*}
$$

However, there is a problem here. Both these solutions (62) and (65) for $\delta f(R)$, are rapidly growing exponentials of an exponential. In the asymptotic regime, these perturbations are no longer small, thus invalidating the initial linearization assumption used to derive them. Therefore, these solutions must be discarded and thus we conclude that the fixed points have only one free parameter on both the sphere and hyperboloid.

These results allow us to draw important conclusions. Each of the asymptotic fixed point solutions, (54) and (63), contribute one constraint on the flow equation. ${ }^{2}$ There are no boundary conditions coming from $R=0$, so we can expect one-parameter sets of fixed point solutions on both $\mathbb{S}^{4}$ and $\mathbb{H}^{4}$. At first sight this is a disappointing result for the asymptotic safety program. However, if we now use $\mathbb{R}^{4}$, eqn. (46), to match smoothly between these solutions then we have two boundary conditions on a second order differential equation, one coming from the sphere and one from the hyperboloid. Thus, there can now only be at most a discrete set of solutions. In the next section more evidence will be presented for why these topologies should be considered smoothly joined together in this way.

### 6.2 Numerical solution

Note that this does not answer the question of whether there is more than one fixed point, or no fixed point at all, or the phenomenologically preferred answer: a single fixed point. As already mentioned, the only way to find out which of these latter possibilities is realised, is to perform an extensive numerical search for such global $f(R)$ solutions. As we have just seen, the asymptotic fixed point solutions (54) and (63), on

[^1]the sphere and hyperboloid respectively, depend on one single parameter, which we called $A$ on both sides. Even if there is a global solution that connects them, the value of $A$ will almost surely be different in the two different topologies. On one of the topologies, one can scan through $A$ at some large value of $R$ and solve the equations backwards towards $R=0$. With the exponential cutoff profile (29), solutions have been found on the sphere this way, in a narrow region around $A=-0.01$, matching to the asymptotic series at $R \sim 10$. On the hyperboloid no solution was found however, although a more comprehensive numerical analysis might find one [43].

## 7 Eigenoperators

So far we have analyzed the flow equation in the $f(R)$ approximation at the fixed point (where $\partial_{t} f(R)=0$ ). Assuming that there is a global solution, we now turn to the question of whether the theory is predictive. This is answered by solving the eigenvalue equation and figuring out how many relevant operators the fixed point solution has. Relevant operators are the ones that fall into the fixed point when increasing the cutoff scale $k$. The number of these operators corresponds to the number of parameters that will have to be fixed experimentally. We now prove that in this second order formulation, if we take the equations to apply simultaneously across the three spaces $\mathbb{S}^{4}, \mathbb{R}^{4}$ and $\mathbb{H}^{4}$, there are a finite number of relevant operators [42]. Plugging (5) into the flow equation (18) we get a second order ordinary differential eigenvalue equation:

$$
\begin{equation*}
-a_{2}(R) v^{\prime \prime}(R)+a_{1}(R) v^{\prime}(R)+a_{0}(R) v(R)=\lambda v(R) \tag{66}
\end{equation*}
$$

where the eigenvalues $\lambda=4-\theta, v(R)$ is the eigenoperator, and the $a_{i}$ 's are given by eqns. ( $56-58$ ).

### 7.1 Asymptotic analysis

The first step is to apply asymptotic analysis to the eigenoperator equation. The procedure closely follows that for the fixed point in sec. 6.1. As already noted there, $a_{0}$ and $a_{2}$ decay exponentially fast and in the large $R$ limit $a_{1} \sim 2 R$. Then the asymptotic form of the eigenvalue equation is:

$$
\begin{equation*}
\lambda v(R)-2 R v^{\prime}(R)=-a_{2}(R) v^{\prime \prime}(R) \tag{67}
\end{equation*}
$$

Starting with the left hand side the solution is

$$
\begin{equation*}
v(R) \propto|R|^{\frac{\lambda}{2}}+\cdots, \tag{68}
\end{equation*}
$$

where the ellipses stand for subleading corrections from the $a_{i}$ 's, in particular from the RHS. The solutions have one parameter, the constant of proportionality. The missing parameter must come from a solution for which $v^{\prime \prime}(R)$ cannot be neglected. But this implies diverging derivatives and thus $v(R)$ can be neglected. The equation is then analogous to what we had before where the second solution is now $v(R) \sim \delta f(R)$ in (62) on the sphere and (65) on the hyperboloid.

Now we ask whether these solutions are actually valid. The linearised solution (5) is meant to describe the RG flow 'close' to the fixed point. For any fixed $\epsilon$, if $|v(R) / f(R)| \rightarrow \infty$ as $R \rightarrow \pm \infty$ that is not necessarily true since linearisation is no longer valid. In this case one can set

$$
\begin{equation*}
f_{k}(R)=f(R)+\epsilon v_{k}(R) \tag{69}
\end{equation*}
$$

and, without linearising, ask for the correct evolution for $v_{k}(R)$ at large $R$. For large negative $R$ the RHS of the flow equation (18) can be neglected. For large positive $R$, the RHS of the flow equation can be neglected except for the $n=1 S_{2}$ component of $\mathcal{T}_{0}^{J a c}$, which however just cancels the contributions from the LHS that grow faster than $R^{2}$ resulting from $f(R)$, cf. (54). Since in fact the $O\left(R^{2}\right)$ part of $f(R)$ also vanishes from the LHS (on both sphere and hyperboloid), in the large $R$ regime one has

$$
\begin{equation*}
\partial_{t} v_{k}(R)-2 R v_{k}^{\prime}(R)+4 v_{k}(R)=o\left(R^{2}\right) \tag{70}
\end{equation*}
$$

Any part of $v_{k}(R)$ growing at least as fast as $R^{2}$ is then easily solved for, and gives mean-field evolution involving some arbitrary function $v$ :

$$
\begin{equation*}
v_{k}(R)=\mathrm{e}^{-4 t} v\left(R \mathrm{e}^{2 t}\right)+o\left(R^{2}\right) \tag{71}
\end{equation*}
$$

It will be the same function $v$ that was introduced in the linearised solution (5) if one requires as boundary condition, $v_{k}(R)=v(R)$ at $k=\mu$. The question that remains is whether the RG evolution (71) is consistent with what we found by linearising.

For the power-law solution (68), linearisation is valid at large $|R|$ if and only if $\lambda \leq 4$. This follows from the hyperboloid fixed point asymptotics (63), the sphere side (54) requiring only the weaker constraint, $\lambda \leq 4+2 b$. On the other hand if $\lambda>4$, one can use the general perturbation (69), finding the solution (71). Substituting the explicit form (68) of the boundary condition, gives:

$$
\begin{equation*}
v_{k}(R)=v(R) \mathrm{e}^{-\theta t}+o\left(R^{2}\right) \tag{72}
\end{equation*}
$$

where $\theta=4-\lambda$, i.e. the linearised solution (5) is reproduced. We conclude that asymptotically, power-law eigenoperators (68) are valid solutions for any $\lambda$. Their $t$ evolution is multiplicative and given by the flow of a conjugate coupling $g(t)=\epsilon \mathrm{e}^{-\theta t}, c f$. (5).

On the other hand, the solutions that behave asymptotically as $v(R) \sim \delta f(R)$, are growing exponentials of exponentials. Linearisation is not valid at large $|R|$, where the $t$ dependence is given instead by (71). Now we cannot separate out the $t$ dependence. Therefore, such perturbations cannot be regarded as eigenoperators evolving multiplicatively.

Excluding them leads to quantisation of the spectrum. This is because the large $R$ dependence (68) provides a boundary condition on both the sphere and the hyperboloid side, and linearity provides a further boundary condition since one can choose a normalisation e.g. $v(0)=1$. These three conditions over-constrain the eigenoperator equation (66) leading to quantisation of $\lambda$, i.e. to a discrete eigenoperator spectrum.

### 7.2 Sturm-Liouville theory

Sturm-Liouville type equations take the form

$$
\begin{equation*}
L v(R)=\lambda w(R) v(R) \tag{73}
\end{equation*}
$$

where $L$ is the self adjoint operator

$$
\begin{equation*}
L=-\frac{d}{d R}\left(p(R) \frac{d}{d R} \cdot\right)+q(R) \tag{74}
\end{equation*}
$$

with $p(R)$ and $q(R)$ being real functions and $w(R)$ also being positive. For the second order formulation, the eigenvalue equation can be put in this form. The properties of these equations will then allow us to draw conclusions about the spectrum of the eigenvalues.

The weight function is defined as

$$
\begin{equation*}
w(R)=\frac{1}{a_{2}(R)} \exp -\int^{R} d R^{\prime} \frac{a_{1}\left(R^{\prime}\right)}{a_{2}\left(R^{\prime}\right)} \tag{75}
\end{equation*}
$$

since, multiplying with the eigenvalue equation (66) and rearranging, casts it in Sturm-Liouville form:

$$
\begin{equation*}
-\left(a_{2}(R) w(R) v^{\prime}(R)\right)^{\prime}+w(R) a_{0}(R) v(R)=\lambda w(R) v(R) \tag{76}
\end{equation*}
$$

Notice that the trace in $a_{2}$ is positive. This is because the cutoff is monotonically decreasing, hence $r^{\prime}(z)<0$ and $r(z)>0$, so the sign of $a_{2}$ depends on $c_{\bar{h}}$, which is positive. This implies that the weight function $w(R)>0$ as required.

Next we check if the operator is self-adjoint. Taking $v=v_{j}(R)$, multiplying by $v_{i}(R)$, and integrating over $R$, gives:

$$
\begin{equation*}
-\int v_{i} L v_{j}=-\int v_{i}\left(a_{2} w v_{j}^{\prime}\right)^{\prime}+\int v_{i} a_{0} w v_{j} . \tag{77}
\end{equation*}
$$

If $L$ is self-adjoint then this should be the same for $j \leftrightarrow i$. The first term on the RHS can be written as

$$
\begin{equation*}
-\int\left[v_{i}\left(a_{2} w v_{j}^{\prime}\right)\right]^{\prime}+\int\left[v_{j}\left(a_{2} w v_{i}^{\prime}\right)\right]^{\prime}-\int\left(a_{2} w v_{i}^{\prime}\right)^{\prime} v_{j} . \tag{78}
\end{equation*}
$$

Thus what is required is that the first two terms above cancel each other. This is automatically satisfied if $R$ is taken to have the full range since $w(R) \rightarrow 0$ exponentially fast as $R \rightarrow \pm \infty$. If the differential equation is restricted to either the four-sphere or four-hyperboloid, there would be a boundary at $R=0$. The weight function does not vanish there and thus the operator $L$ would then not be self-adjoint. This is another powerful hint that the correct treatment is to smoothly join the three topologies together. Note also that none of these equations would make sense if the exponentially growing set of solutions $v(R) \sim \delta f(R)$ are included, where $\delta f(R)$ is given by (62) or (65). From (59) and (61) one can see that actually these $\delta f(R) \sim 1 / w(R)$ and thus such $v(R)$ are not square integrable under the weight function $w(R)$ since $w(R) v^{2}(R) \sim 1 / w(R)$, which diverges at large $R$. Hence, this condition only picks out the correct solutions from the eigenvalue equation and justifies the use of Sturm-Liouville techniques.

Thus, when restricted to perturbations that grow only as a power at large $|R|$, the eigenvalue equation (66) is of Sturm-Liouville type. The consequences for the spectrum of the eigenvalues can be seen by a standard transformation to Liouville normal form. Define a coordinate $x$ as

$$
\begin{equation*}
x=\int_{0}^{R} \frac{1}{\sqrt{a_{2}\left(R^{\prime}\right)}} d R^{\prime} \tag{79}
\end{equation*}
$$

Then $x \rightarrow \pm \infty$ as $R \rightarrow \pm \infty$ because $a_{2}(R)$ vanishes at large $|R|$. Defining the 'wave-function'

$$
\begin{equation*}
\psi(x)=a_{2}^{\frac{1}{4}}(R) w^{\frac{1}{2}}(R) v(R) \tag{80}
\end{equation*}
$$

(66) can be transformed into

$$
\begin{equation*}
-\frac{d^{2} \psi(x)}{d x^{2}}+U(x) \psi(x)=\lambda \psi(x) \tag{81}
\end{equation*}
$$

which is just the one-dimensional Schrödinger equation with energy $\lambda$. The potential turns out to be [42]

$$
\begin{equation*}
U(x)=a_{0}+\frac{a_{1}^{2}}{4 a_{2}}-\frac{a_{1}^{\prime}}{2}+a_{2}^{\prime}\left(\frac{a_{1}}{2 a_{2}}+\frac{3 a_{2}^{\prime}}{16 a_{2}}\right)-\frac{a_{2}^{\prime \prime}}{4} . \tag{82}
\end{equation*}
$$

This potential has no singularities at finite $x$. Asymptotically the term proportional to $a_{1}^{2}$ will dominate for $x \rightarrow \pm \infty$ and thus the potential $U(x) \rightarrow+\infty$. This then implies the following important properties:

1 The eigenvalues $\lambda_{n}$ are discrete, real and non-degenerate.
2 There exists a lowest eigenvalue $\lambda_{0}$ (i.e. bounded from below).
3 The only accumulation point is at infinity.
Asymptotic analysis already showed that the eigenvalues are discrete, but this Sturm-Liouville analysis allows to conclude much more. Now it is straightforward to see that there is a finite number of relevant operators such that $\theta_{n}=4-\lambda_{n} \geq 0$. Indeed this is so because $\lambda_{n} \rightarrow \infty$ as $n \rightarrow \infty$ and because there exists a lowest eigenvalue $\lambda_{0}$.

But these results should be accepted with caution. Recall that to obtain them some severe approximations were used, such as the single metric approximation and the truncation to the function $f(R)$. One way to judge the validity of the results is to check the extent to which they are scheme independent (universal), in particular independent of the choice of cutoff. It turns out that the critical exponents $\theta_{n}$ can be solved for analytically, again by using asymptotic analysis, and this gives a precise way to answer the question of scheme dependence in this regime.

From (60) and (64), the reader can see that the leading contribution to $a_{2}$ takes the following form on both sphere and hyperboloid:

$$
\begin{equation*}
a_{2}(R)=\frac{1}{G^{2}(R)} \mathrm{e}^{-2 F(R)} \tag{83}
\end{equation*}
$$

where $F(R)$ is positive and proportional to $|R|^{b}$ and $G(R)$ goes like a power of $R$. They therefore satisfy the conditions required to use the trick (61) on the equation (79) defining $x$. Then asymptotically

$$
\begin{equation*}
x=\frac{G(R)}{F^{\prime}(R)} \mathrm{e}^{F(R)}+\ldots \tag{84}
\end{equation*}
$$

where the ellipsis stand for multiplicative subleading terms. Alternatively this can be seen by differentiating (79) and (84) with respect to $R$. The potential can then be approximated to leading order as

$$
\begin{equation*}
U(x)=\frac{a_{1}^{2}}{4 a_{2}}=\frac{R^{2}}{a_{2}(R)}=\left[R F^{\prime}(R)\right]^{2} x^{2} \tag{85}
\end{equation*}
$$

Evidently $R F^{\prime}(R)=b F(R)$ and thus, taking logs of (84),

$$
\begin{equation*}
U(x)=(b x \ln |x|)^{2}\left\{1+O\left(\frac{\ln \ln |x|}{\ln |x|}\right)\right\} \quad x \rightarrow \pm \infty \tag{86}
\end{equation*}
$$

where in the equation above the order of the subleading correction is also indicated. (The latter requires taking into account iterations of (61) and the subleading corrections to $a_{2}$.) Using the WKB approximation one can then find the critical exponents for large $n$ [43]:

$$
\begin{equation*}
\theta_{n}=-b(n \ln n)\left\{1+O\left(\frac{\ln \ln n}{\ln n}\right)\right\} \quad \text { as } \quad n \rightarrow \infty \tag{87}
\end{equation*}
$$

The result shows almost a linear dependence on $n$. This much is similar to extensive numerical work done on large polynomial truncations of a third order formulation up to $n \leq 70$ [17]. These authors find nearGaussian scaling dimension. They use an adaptive cutoff so there is no direct comparison, and they use the optimised profile (26) with no free parameters in the cutoff, so universality is not tested in this way. Indeed the scaling dimension should be universal. The leading behaviour of this expression is independent of all parameters in the chosen general family of cutoffs, except one, namely the parameter $b$ in (29). Explicitly, it is independent of $a$ in (29), and of all the $c_{\phi}$ and $\alpha_{i}$. Unfortunately the dependence on $b$ still amounts to strong dependence.

Actually this remaining dependence is an artefact of the single-metric approximation [2]. ${ }^{3}$ We have seen that it comes from the $R^{b}$ dependence of $F(R)$ in (83), equivalently (60) and (64). This in turn arises from the cutoff dependence in eqn. (56) and in particular the cutoff profile's dependence on $R$ (through in fact the lowest eigenvalue). To see that the dependence in (87) is an artefact of the single-metric approximation, imagine for the moment that the single-metric approximation was not made and yet somehow the initial ansatz (2) still made sense. (In reality such a simple ansatz would no longer be possible because diffeomorphism invariance is replaced by BRST invariance for the quantum fields and furthermore it is badly broken, but let us overlook that for the moment.) Now the curvature in it is the full quantum curvature $\hat{R}$, due to the full quantum metric $\hat{g}_{\mu \nu}$ in (6). The trace and the cutoff in (56) come from summing over modes on the background manifold in (3) so they depend on the background curvature $R$. The Hessian in (3) will result in differentiating $f(\hat{R})$ with respect to the fluctuation field $h_{\mu \nu}$ or equivalently differentiating with respect to $\hat{g}_{\mu \nu}$. Thus ultimately the eigenoperator perturbation equation (55) would take the form:

$$
\begin{equation*}
-a_{2}(R, \hat{R}) \delta f^{\prime \prime}(\hat{R})+a_{1}(R, \hat{R}) \delta f^{\prime}(\hat{R})+a_{0}(R, \hat{R}) \delta f(\hat{R})=4 \delta f(\hat{R}) \tag{88}
\end{equation*}
$$

with in particular:

$$
\begin{equation*}
a_{2}=\frac{144 c_{h}}{V} \operatorname{Tr}\left[\frac{\Delta_{0}^{2}\left(2 r\left(\Delta_{0}+\alpha_{0} R\right)-\left(\Delta_{0}+\alpha_{0} R\right) r^{\prime}\left(\Delta_{0}+\alpha_{0} R\right)\right)}{\left(9 f^{\prime \prime}(\hat{R}) \Delta_{0}^{2}+3 f^{\prime}(\hat{R}) \Delta_{0}+E(\hat{R})+16 c_{h} r\left(\Delta_{0}+\alpha_{0} R\right)\right)^{2}}\right] \tag{89}
\end{equation*}
$$

[^2]In deriving (87) one is interested in the large $\hat{R}$ dependence of (88). This depends on the large $\hat{R}$ dependence of the fixed point functional $f(\hat{R})$, and this feeds in to the coefficients $a_{i}(R, \hat{R})$. But there is no $\exp \left(-a \hat{R}^{b}\right)$ dependence because the cutoff profile $r$ depends only on the background curvature $R$, either directly or through the Laplacians whose eigenvalues only depend on the background manifold.

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[^0]:    ${ }^{1}$ Although some may be found analytically, e.g. the Gaussian fixed point, or special cases [40].

[^1]:    ${ }^{2}$ For example at some initial very large $R$ we can set $2 f(R)=R f^{\prime}(R)$ since this boundary condition imposes the leading behaviour (48). Using the subleading corrections we can furnish a more accurate Robin boundary condition at more reasonable values of $R$.

[^2]:    ${ }^{3}$ More generally, single-field approximations are a known source of artefacts [56].

