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One-loop string amplitudes in $\mathrm{AdS}_5\times\mathrm{S}^5$: Mellin space and sphere splitting

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Abstract: We study string corrections to one-loop amplitudes of single-particle operators $\mathcal{O}_p$ in $AdS_5\times S^5$. The tree-level correlators in supergravity enjoy an accidental 10d conformal symmetry. Consequently, one observes a partial degeneracy in the spectrum of anomalous dimensions of double-trace operators and at the same time equality of many different correlators for different external charges $p_{i=1,2,3,4}$. The one-loop contribution is expected to lift such bonus properties, and its precise form can be predicted from tree-level data and consistency with the operator product expansion.

Here we present a closed-form Mellin space formula for $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$ at order $(\alpha')^3$, valid for arbitrary external charges p_i. Our formula makes explicit the lifting of the bonus degeneracy among different correlators through a feature we refer to as 'sphere splitting'. While tree-level Mellin amplitudes come with a single crossing symmetric kernel, which defines the pole structure of the $AdS_5\times S^5$ amplitude, our one-loop amplitude naturally splits the S^5 part into two separate contributions. The amplitude also exhibits a remarkable consistency with the corresponding flat space IIB amplitude through the large p limit.

Keywords: $1/N$ Expansion, AdS-CFT Correspondence, Scattering Amplitudes

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1 Introduction

An ongoing challenge in theoretical physics is to understand the dynamics of quantum gravity and its relation to string theory. Recent advances within the analytic bootstrap program have shown that the study of the AdS/CFT correspondence, beyond the classical bulk approximation, leads to concrete and quantitative results for both quantum gravity and string theory in AdS, see [1, 2] for a review. In this context, the problem of quantum gravity is well posed and the CFT approach is able to tackle AdS gravity with a variety of computational tools. It has the potential to pinpoint the existence of very general structures in the theory, and progressively allow us to move towards a more complete picture.

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The most emblematic and best understood AdS/CFT correspondence is the one between SU(N) $\mathcal{N}=4$ SYM in 4d and type-IIB string theory on $AdS_5\times S^5$. This is the theory we will study in this paper. Our probes for investigating quantum gravity will be four-point amplitudes of single particle operators $\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$. We will study these objects at tree level and one loop in Newton's constant (or equivalently, $1/N^2$ and $1/N^4$ corrections in the CFT) and at the leading orders in string corrections to Einstein gravity, i.e. leading orders in the $\alpha'\sim\lambda^{-\frac{1}{2}}$ expansion, where λ is the 't Hooft coupling.

1.1 Four-point correlators of half-BPS operators

The supergravity multiplet on AdS_5 and the S^5 Kaluza-Klein spectrum are dual to half-BPS single-particle multiplets, whose superconformal primary operators are

$$\mathcal{O}_p(x,y)=y^{R_1}\dots y^{R_p}\,\mathrm{tr}(\phi_{R_1}\dots\phi_{R_p})(x)+\dots \tag{1.1}$$

where ϕ_R are the scalar fields of the $\mathcal{N}=4$ SYM, and the y^R are auxillary null SO(6) vectors used to project onto the traceless symmetric representation. Here the dots refer to multi-trace contributions determined by insuring that the $\mathcal{O}_p$ are orthogonal to any multi-trace operator [3, 15]. The operators $\mathcal{O}_p$ are orthogonal and normalised as follows,[1]

$$\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\rangle=\delta_{p_1p_2}(g_{12})^{p_1}R_{p_1}\,,\qquad R_{p_1}=p_1N^{p_1}+O(N^{p_1-2})\,,\qquad g_{ij}=\frac{y_{ij}^2}{x_{ij}^2}\,. \tag{1.2}$$

where we have defined $y_{12}^2=y_1\cdot y_2$.

We will study four-point functions of the operators $\mathcal{O}_p$. These correlators can be split into a free theory contribution plus a coupling dependent interacting term,

$$\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle=\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\mathrm{free}}+\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\mathrm{int}}\,. \tag{1.3}$$

From the point of view of the Operator Product Expansion (OPE), the free theory piece includes contributions from protected and long operators in any OPE channel, while the interacting term only receives contributions from unprotected (long) multiplets [36].

1.2 The Mellin space representation

It was shown by Mack in [20] that a CFT correlator, when written in Mellin space, acquires a structure of poles and residues which can be nicely put in correspondence with the OPE. For holographic theories in AdS, it was later shown by Penedones [21] that a suitably defined Mellin amplitude shares many similarities with a scattering amplitude, and in fact this Mellin amplitude can be understood as a curved space completion of a flat space scattering amplitude. The flat space scattering amplitude is recovered in the limit of large Mellin variables. For holographic correlators in $AdS_5\times S^5$, it is possible to improve further the Mellin space representation, by considering a double Mellin transform in which the

[1] The full normalisation is $R_p=p^2(p-1)\left[\frac{1}{(N-p+1)_{p-1}}-\frac{1}{(N+1)_{p-1}}\right]^{-1}=p_1N^{p_1}+O(N^{p_1-2})$ [3].

conformal AdS_5 space and the internal S^5 space are treated equally, and moreover crossing symmetry is manifest for both spaces.[2] This double Mellin transform reads,

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\rm int} = p_1p_2p_3p_4 N^{\Sigma-2}\hat{\mathcal{I}} \int d\hat{s}_{ij} \sum_{\check{s}_{ij}} \prod_{i<j} \Big[\frac{x_{ij}^{2\hat{s}_{ij}} y_{ij}^{2\check{s}_{ij}} \Gamma[-\hat{s}_{ij}]}{\Gamma[\check{s}_{ij}+1]} \Big] \mathcal{M}_{\vec{p}}(\hat{s}_{ij},\check{s}_{ij}) \,, \tag{1.4}$$

where $\hat{s}_{ij}$ are AdS Mellin variables and $\check{s}_{ij}$ are sphere Mellin variables subject to the constraints,

$$\textstyle\sum_i \hat{s}_{ij} = -p_j - 2\,, \qquad \sum_i \check{s}_{ij} = p_j - 2\,. \tag{1.5}$$

The factor $\hat{\mathcal{I}}$ in (1.4), which removes two units of conformal/internal weight w.r.t. to the charges p_i, is a consequence of superconformal symmetry [35].[3] It takes the form

$$\hat{\mathcal{I}} = (x_{13}^2 x_{24}^2 y_{13}^2 y_{24}^2)^2\, \tilde{\mathcal{I}}\,, \qquad \tilde{\mathcal{I}} = (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})\,, \tag{1.6}$$

with $x, \bar{x}, y, \bar{y}$ parametrising the cross-ratios,

$$\begin{aligned} x\bar{x} &= U = \tfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad (1-x)(1-\bar{x}) = V = \tfrac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \\ y\bar{y} &= \tilde{U} = \tfrac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}\,, \qquad (1-y)(1-\bar{y}) = \tilde{V} = \tfrac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}\,. \end{aligned} \tag{1.7}$$

Note that $\hat{\mathcal{I}}$ is invariant under permutations of the external operators.

The constraints (1.5) on the Mellin variables can be solved as follows,

$$\begin{aligned} &\hat{s}_{12} + \tfrac{p_1+p_2}{2} = \hat{s}_{34} + \tfrac{p_3+p_4}{2} = \hat{s}, \qquad \check{s}_{12} - \tfrac{p_1+p_2}{2} = \check{s}_{34} - \tfrac{p_3+p_4}{2} = \check{s}\,, \\ &\hat{s}_{14} + \tfrac{p_1+p_4}{2} = \hat{s}_{23} + \tfrac{p_2+p_3}{2} = \hat{t}, \qquad \check{s}_{14} - \tfrac{p_1+p_4}{2} = \check{s}_{23} - \tfrac{p_2+p_3}{2} = \check{t}\,, \\ &\hat{s}_{13} + \tfrac{p_1+p_3}{2} = \hat{s}_{24} + \tfrac{p_2+p_4}{2} = \hat{u}, \qquad \check{s}_{13} - \tfrac{p_1+p_3}{2} = \check{s}_{24} - \tfrac{p_2+p_4}{2} = \check{u}\,, \end{aligned} \tag{1.8}$$

with

$$\hat{s}+\hat{t}+\hat{u} = \Sigma - 2\,, \qquad \check{s}+\check{t}+\check{u} = -\Sigma - 2\,; \qquad \Sigma = \tfrac{p_1+p_2+p_3+p_4}{2}. \tag{1.9}$$

We shall say that the variables $\hat{s}, \check{s}$ define the s-channel and similarly for the other channels. The variables $\hat{s}, \check{s}$ and $\hat{t}, \check{t}$ will be taken to be independent. The integral and the sum in (1.4) only run over the independent variables $\hat{s}, \check{s}$ and $\hat{t}, \check{t}$, and the sum will actually be finite, as we will see. We can further accompany each channel in position and internal space with the following combinations of charges,

$$c_s = \tfrac{p_1+p_2-p_3-p_4}{2}\,; \qquad c_t = \tfrac{p_1+p_4-p_2-p_3}{2}\,; \qquad c_u = \tfrac{p_2+p_4-p_1-p_3}{2} \tag{1.10}$$

The parametrisation of $\vec{p}$ in terms of $\{\Sigma, c_s, c_t, c_u\}$ is invertible, with Σ being an invariant under permutation, and the various triplets $\{\hat{s}, \check{s}, c_s\}$, $\{\hat{t}, \check{t}, c_t\}$, $\{\hat{u}, \check{u}, c_u\}$ transforming into one another under crossing.

[2] This idea was first introduced in [22] and further refined in [29] and [30].

[3] It can also be justified from the fact that only long representations contribute to the OPE decomposition of $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\rm int}$, and these have precisely the $\tilde{\mathcal{I}}$ factor.

It is convenient to rewrite the correlator (1.4) by making manifest the dependence on the cross-ratios, namely

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\rm int} = p_1p_2p_3p_4 N^{\Sigma-2}\,\tilde{\mathcal{I}}\,\frac{\prod_{i<j} g_{ij}^{\frac{p_i+p_j}{2}}}{(g_{13}g_{24})^{\Sigma}}\mathcal{H}_{\vec{p}}(U,V,\tilde{U},\tilde{V}), \tag{1.11}$$

where

$$\mathcal{H}_{\vec{p}}(U,V,\tilde{U},\tilde{V}) = \int d\hat{s}d\hat{t}\sum_{\check{s},\check{t}} U^{\hat{s}}V^{\hat{t}}\tilde{U}^{\check{s}}\tilde{V}^{\check{t}}\,\mathbf{\Gamma}\,\mathcal{M}_{\vec{p}}\,. \tag{1.12}$$

By definition, $\mathbf{\Gamma} = \Gamma_{\tt s}\Gamma_{\tt t}\Gamma_{\tt u}$ with

$$\Gamma_{\tt s} = \frac{\Gamma_{\hat{s}}}{\Gamma_{\check{s}}}\,; \qquad \begin{array}{c} \Gamma_{\hat{s}} = \Gamma[\frac{p_1+p_2}{2}-\hat{s}]\Gamma[\frac{p_3+p_4}{2}-\hat{s}] = \Gamma\left[\frac{\Sigma}{2}\pm\frac{c_s}{2}-\hat{s}\right] \\ \Gamma_{\check{s}} = \Gamma[1+\frac{p_1+p_2}{2}+\check{s}]\Gamma[1+\frac{p_3+p_4}{2}+\check{s}] = \Gamma\left[1+\frac{\Sigma}{2}\pm\frac{c_s}{2}+\check{s}\right]. \end{array} \tag{1.13}$$

Similarly for $\Gamma_{\tt t}$ and $\Gamma_{\tt u}$. The $\pm$ abbreviation stands for taking the product of both Γ with $+$ and $-$ signs. Note, the sum over $\check{s}$ and $\check{t}$ is automatically truncated by the Γ functions in the denominator of $\mathbf{\Gamma}$, thus it is finite. Moreover, $\mathbf{\Gamma}$ has a particular meaning when understood from the point of view of the quantum numbers of the operators exchanged. See section 2.1.1 for a detailed discussion.

Under crossing transformation, the factor $\mathbf{\Gamma}$ is invariant, while for the Mellin amplitude we summarise the result in the following table.

$$\begin{array}{|rcll|} \hline \mathcal{M}_{p_1p_2p_3p_4}(\hat{s},\hat{t},\check{s},\check{t}) & = & \mathcal{M}_{p_2,p_1,p_3,p_4}(\hat{s},\hat{u},\check{s},\check{u}) & = \mathcal{M}_{c_s,+c_u,+c_t}(\hat{s},\hat{u},\check{s},\check{u}) \\ & = & \mathcal{M}_{p_1,p_2,p_4,p_3}(\hat{s},\hat{u},\check{s},\check{u}) & = \mathcal{M}_{c_s,-c_u,-c_t}(\hat{s},\hat{u},\check{s},\check{u}) \\ \hline \mathcal{M}_{p_1p_2p_3p_4}(\hat{s},\hat{t},\check{s},\check{t}) & = & \mathcal{M}_{p_4,p_2,p_3,p_1}(\hat{u},\hat{t},\check{u},\check{t}) & = \mathcal{M}_{c_u,+c_t,+c_s}(\check{u},\hat{t},\check{u},\check{t}) \\ & = & \mathcal{M}_{p_1,p_3,p_2,p_4}(\hat{u},\hat{t},\check{u},\check{t}) & = \mathcal{M}_{-c_u,c_t,-c_s}(\hat{u},\hat{t},\check{u},\check{t}) \\ \hline \mathcal{M}_{p_1p_2p_3p_4}(\hat{s},\hat{t},\check{s},\check{t}) & = & \mathcal{M}_{p_1,p_4,p_3,p_2}(\hat{t},\hat{s},\check{t},\check{s}) & = \mathcal{M}_{c_t,+c_s,+c_u}(\hat{t},\hat{s},\check{t},\check{s}) \\ & = & \mathcal{M}_{p_3,p_2,p_1,p_4}(\hat{t},\hat{s},\check{t},\check{s}) & = \mathcal{M}_{-c_t,-c_s,c_u}(\hat{t},\hat{s},\check{t},\check{s}) \\ \hline \end{array} \tag{1.14}$$

The dependence on Σ is left implicit since Σ does not transform under crossing. It is clear that crossing has simple properties w.r.t. the triplets $\{\hat{s},\check{s},c_s\}$, $\{\hat{t},\check{t},c_t\}$, $\{\hat{u},\check{u},c_u\}$.

1.3 A tree level primer

We will focus in this paper on the quantum regime of $\mathcal{N}=4$ SYM where the theory is dual to classical supergravity (i.e. the regime where we first take N large and then take large 't Hooft coupling λ), and we will study the tree-level and the one-loop contribution to the Mellin amplitude,

$$\mathcal{M} = \mathcal{M}^{(1)} + \frac{1}{N^2}\mathcal{M}^{(2)} + \ldots, \tag{1.15}$$

with both contributions themselves expanded for large λ as follows,

$$\begin{aligned}\mathcal{M}^{(1)} &= \mathcal{M}^{(1,0)} + \lambda^{-\frac{3}{2}}\mathcal{M}^{(1,3)} + \lambda^{-\frac{5}{2}}\mathcal{M}^{(1,5)} + \dots \\ \mathcal{M}^{(2)} &= \lambda^{\frac{1}{2}}\mathcal{M}^{(2,-1)} + \mathcal{M}^{(2,0)} + \lambda^{-1}\mathcal{M}^{(2,2)} + \lambda^{-\frac{3}{2}}\mathcal{M}^{(2,3)} + \dots \end{aligned} \tag{1.16}$$

The large λ expansion is more precisely an expansion in curvature/derivative corrections, thus it is weighted by $\alpha' = \lambda^{-\frac{1}{2}}$.[4] We will be interested in the study of one-loop string contributions, beyond the one-loop supergravity correlators $\mathcal{M}^{(2,0)}$, and specifically we will consider $\mathcal{M}^{(2,3)}$ at $(\alpha')^3$. Below we review and emphasise some important features of the tree level correlators, that will be needed to better appreciate the novelties at one-loop.

The combination of analytic bootstrap techniques and the knowledge about the spectrum of supergravity, which consists of protected half-BPS single-particle states and multi-particle states (but no excited string states), allows one to solve the problem of computing the tree-level contribution to the four-point correlators. The expression for $\mathcal{M}^{(1,0)}$ was given in [4, 5], building on previous work [33].[5] In our notation

$$\mathcal{M}^{(1,0)} = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}\,, \tag{1.17}$$

where the bold variables are given by

$$\mathbf{s} = \hat{s} + \check{s}\,; \qquad \mathbf{t} = \hat{t} + \check{t}\,; \qquad \mathbf{s}+\mathbf{t}+\mathbf{u} = -4\,. \tag{1.18}$$

The appearance of these bold-font variables has a precise meaning [22], which we explain in the next paragraph. The first tree-level $(\alpha')^3$ string correction is given simply by [27]

$$\mathcal{M}^{(1,3)} = 2\zeta_3(\Sigma-1)_3\,. \tag{1.19}$$

Higher order α' corrections come with non trivial polynomials in the Mellin variables and have been studied systematically via the bootstrap programme [23, 24, 27]. To various orders, fully explicit results have been computed in [28–30].

Both $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$ come with interesting properties which are simple to see in our formalism: We would expected $\mathcal{M}^{(1,0)}$ to be function of all variables $\{\hat{s},\hat{t},\check{s},\check{t},p_1,p_2,p_3,p_4\}$ but it happens to depend only on the specific combinations $\mathbf{s} = \hat{s}+\check{s}$ and $\mathbf{t} = \hat{t}+\check{t}$, $\mathbf{u} = -\mathbf{s}-\mathbf{t}-4$. Moreover, it does not depend explicitly on the charges p_i at all. In a similar way, $\mathcal{M}^{(1,3)}$ is just a constant, but for the factor of $(\Sigma-1)_3$ (which is singlet under crossing). The crucial observation is that both $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$ can be understood as 10d objects in the following precise sense. In the case of supergravity, it was shown in [16] that $\mathcal{M}^{(1,0)}$ enjoys a 10d *conformal* symmetry. In the case of tree level α' corrections, the authors of [30] showed that $\mathcal{M}^{(1,3)}$ (and in fact $\mathcal{M}^{(1,i\geq 3)}$) is the dimensional reduction of a 10d contact diagram in $AdS_5\times S^5$. Incidentally, also $\mathcal{M}^{(1,3)}$ can be said to possess 10d conformal symmetry. More generally, since the operators $\mathcal{O}_p(x,y)$ are Kaluza-Klein modes,

[4]The term $\lambda^{\frac{1}{2}}\mathcal{M}^{(2,-1)}$ corresponds to the R^4 counterterm. A term $\lambda^{-\frac{1}{2}}\mathcal{M}^{(2,1)}$ would correspond to the genus one contribution to the modular completion of $\lambda^{-\frac{5}{2}}\mathcal{M}^{(1,5)}$ and it vanishes. The term $\lambda^{-1}\mathcal{M}^{(2,2)}$ corresponds to the genus one contribution to the modular completion of $\mathcal{M}^{(1,6)}$.

[5]See [34] and refs. therein for the gravity counterpart of this result.

one might say that the 10d structure of the tree-level correlators implicitly constrains the way the Mellin amplitude depends on the charges, but even so the cases of $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$ are still very special. We will see how at one-loop such special features are lifted.

Let us note now that $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$ are the "closest" possible to the type-IIB flat space amplitude, where by this we mean here the full *ten-dimensional* amplitude [41]. To see this connection explicitly it is convenient to think about the limit of large $p_i \gg 1$, denoted for simplicity "the large p limit". As it was shown in [22], this limit generalises the flat-space limit a la Penedones, in a way that incorporates also the sphere, and works as follows: Once the interacting correlator $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\rm int}$ is represented in Mellin space, as in (1.4), and sum $\sum_{\check{s},\check{t}}$ is also turned into a contour integral, the limit of large p_i localises the contour integration on a saddle point where all variables $\{\hat{s},\hat{t},\check{s},\check{t}\}$ are large in the same way as the p_i, and quite nicely, they naturally organise themselves into the combinations $\mathbf{s},\mathbf{t}$, with $\mathbf{s}+\mathbf{t}+\mathbf{u}=0$. These in fact become precisely 10d flat space Mandelstam invariants, and the Mellin amplitude becomes a 10d flat space amplitude.[6] A direct consequence of this general statement is that by taking the limit in which the $p_i \gg 1$ and the Mellin variables are large, directly on the Mellin amplitude, the result needs to depend only on $\mathbf{s},\mathbf{t}$, in the same way as the flat space 10d type-IIB amplitude does. By looking at $\mathcal{M}^{(1,0)}$ in (1.17) and $\mathcal{M}^{(1,3)}$ in (1.19) the large p limit amounts to simply "drop the 1". It will be interesting to see how this limit is instead quite non trivial at one-loop. We will do this in section 3.4.

The ultimate goal at tree level would be to resum all the α' corrections to $\mathcal{M}^{(1,0)}$. The resummed $\mathcal{M}^{(1)}$, as function of α', would then give the Virasoro-Shapiro amplitude in $AdS_5\times S^5$, i.e. the generalisation of the well known type IIB flat space amplitude. It is a non-trivial problem because the bootstrap program leaves unfixed a number of ambiguities at each order in α'. Additional input,[7] from supersymmetric localisation [17–19], is already crucial to fix the ambiguities that appear at $(\alpha')^5$ [28].

1.4 Summary of results: One-loop string corrected $AdS_5 \times S^5$

This paper is dedicated to the study of one-loop string contributions beyond the one-loop supergravity correlators $\mathcal{M}^{(2,0)}$ studied in [10]. We will focus on $\mathcal{M}^{(2,3)}$ at $(\alpha')^3$, generalising previous work done in [11, 24–28]. Along the way we will discuss the general picture for higher orders in α'. In the next section we will explain how the bootstrap program at one-loop works. Here we would like to summarise our main results and novelties, compared to tree level and the existing literature.

The CFT data of exchanged two-particle operators at tree level, and the OPE, determine the one-loop gravity amplitude in the following sense.[8] Within certain ranges of twist, which we refer to as Above Window, Window, and Below Window, the tree level OPE carries information about the maximal $\log^2 U$ discontinuity, the $\log U$ discontinuity and the analytic contribution, respectively. The Above Window region contains the $\log^2 U$ discontinuity and is fully determined by the OPE. The Window and the Below Window are

[6] The large p limit is well established in various $AdS\times S$ backgrounds [38, 39].

[7] See [31, 32] for a dispersive sum rule approach.

[8] The full amplitude is determined up to a handful of ambiguities which cannot be fixed by the two-particle bootstrap program.

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finite ranges in the twist and give additional information on the structure of the amplitude. We refer to sections 2.1.1 and 2.1.2 for a detailed description. In the case of $\mathcal{M}^{(2,3)}$, the CFT data comes from $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$, which we reviewed above in (1.17) and (1.19). Let us emphasise that, as for $\mathcal{M}^{(2,0)}$ in supergravity [10], the $\log^2 U$ discontinuity is not enough to bootstrap the one-loop amplitude, and the information coming from Window and Below Window is crucial to obtain the final result.

In order to appreciate the various novelties of one-loop physics, let us begin by noting that the $\mathbf{\Gamma}$ factor used to define the Mellin transform in (1.11) has itself a *bonus property*. It is invariant under *variations* of the charges p_i which swap pairs of values $p_i + p_j$ for i, j in the $\mathtt{s}$, or $\mathtt{t}$, or $\mathtt{u}$ channel. For instance, if we highlight the $\mathtt{s}$ channel in,

$$\mathbf{\Gamma} = \frac{\Gamma\left[\frac{p_1+p_2}{2} - \hat{s}\right]\Gamma\left[\frac{p_3+p_4}{2} - \hat{s}\right]}{\Gamma\left[1+\frac{p_1+p_2}{2} + \check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2} + \check{s}\right]}\Gamma_{\mathtt{t}}\Gamma_{\mathtt{u}} \tag{1.20}$$

this is invariant under variations of the charges such that the values of $p_1 + p_2$ and $p_3 + p_4$ swap, but the other combinations $p_i + p_j$ remain unchanged. Let us emphasise that this property is not crossing symmetry, hence the use of the term *variations*. In terms of the c_s, c_t, c_u parametrisation of the charges, the variations we are discussing just amount to exchange the values $\pm c_s$, see e.g. (1.13). In (1.20) we looked at the $\mathtt{s}$ channel, but of course the same reasoning applies to the other channels.

The bonus property together with the 10d conformal symmetry has notable consequences. To have a concrete and simple example in mind, consider the case of correlators $\vec{p} = (4424)$ and $\vec{p} = (3335)$. These correlators have indeed the same $\mathbf{\Gamma}$, but more importantly, since the Mellin amplitudes $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$ are themselves invariant under the aforementioned variations, the interacting correlators are equal at the corresponding orders in the expansion,

$$\mathcal{H}_{4424}(U, V, \tilde{U}, \tilde{V}) = \mathcal{H}_{3335}(U, V, \tilde{U}, \tilde{V}). \tag{1.21}$$

Recall that $\mathcal{H}_{\vec{p}}$ is introduced in (1.11) and, up to numerical factors dictated by (1.11) and free propagators removed, it is just the interacting part of $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$.

More generally, we shall say that two correlators are degenerate when they have the same values of Σ and $|c_s|, |c_t|, |c_u|$. We will show that whenever two correlators are degenerate in the tree level α' expansion, this degeneracy is lifted at one-loop at the corresponding order in α'. This lift was first discussed in supergravity [10] but its expression at the level of the Mellin amplitude was not yet investigated. In this paper we provide very concrete formulae which exhibit the lift of the degeneracy in the case of $\mathcal{M}^{(2,3)}$, and we believe that analogous formulae will hold at higher orders in α'.

The Mellin amplitude will be written in the following way,

$$\mathcal{M}^{(2,3)}_{\vec{p}} = \left[\mathcal{W}^{(\mathrm{AW})}_{\vec{p}}(\hat{s},\check{s}) + \mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s}) + \mathcal{B}^{(\mathrm{BW})}_{\vec{p}}(\hat{s},\check{s})\right] + \mathtt{crossing}\,, \tag{1.22}$$

where the superscripts indicate which region of OPE data was used to fix the function, i.e. Above Window (AW), Window (W) and Below Window (BW).

The first term in (1.22) is given by

$$\mathcal{W}^{(\mathrm{AW})}_{\vec{p}} = w_{\vec{p}}(\hat{s},\check{s})\,\tilde{\psi}^{(0)}(-\mathbf{s})\,, \tag{1.23}$$

where $w_{\vec{p}}(\hat{s},\check{s})$ is a polynomial and $\tilde{\psi}^{(0)}(-\mathbf{s}) \equiv \psi^{(0)}(-\mathbf{s}) + \gamma_E$ is the digamma function shifted by the Euler constant. This term is entirely determined by the OPE prediction for the $\log^2 U$ discontinuity, corresponding to exchange of two-particle operators Above Window (AW).

The second term in (1.22) represents the novelty of the one-loop function. It takes the form,

$$\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s}) = \sum_{z=0}^{6} \frac{1}{\mathbf{s}-z}\left[(\check{s}+\tfrac{p_1+p_2}{2}-z)_{z+1}\, r^{+}_{\vec{p};z}(\hat{s},\check{s}) + (\check{s}+\tfrac{p_3+p_4}{2}-z)_{z+1}\, r^{-}_{\vec{p};z}(\hat{s},\check{s})\right], \tag{1.24}$$

where crossing implies that $r^{\pm}$ are related to each other, and given in terms of a single function with certain residual symmetry in the charges $\vec{p}$:

$$\begin{cases} r^{+}_{\vec{p}} = r_{\{-c_s,-c_t,c_u\}} \\ r^{-}_{\vec{p}} = r_{\{+c_s,+c_t,c_u\}} \end{cases} ; \qquad r_{\{c_s,c_t,c_u\}} = r_{\{c_s,-c_t,-c_u\}}\,; \quad r_{\{c_s,c_t,c_u\}} = r_{\{c_s,c_u,c_t\}}. \tag{1.25}$$

The function $r_{\vec{p};z}(\hat{s},\check{s})$ is a polynomial, for each z, determined by OPE predictions for the $\log^1 U$ discontinuity in the Window. The poles in z come with the bold font variables, and the structure of poles in $\hat{s},\check{s}$, follows. We will explain this in the next sections around (2.8).

Let us comment on the reason why $\mathcal{R}^{(W)}$ represents a novelty: When we look at $\mathcal{R}^{(W)}$ together with the $\mathbf{\Gamma}$ functions, say we focus on $\Gamma_{\mathbf{s}}$ in the $\mathbf{s}$-channel, the total amplitude undergoes the following split,

$$\frac{\mathcal{R}^{(W)}_{\vec{p}}}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2}+\check{s}\right]}$$
$$= \frac{1}{\mathbf{s}-z}\left[\frac{r^{+}_{\vec{p};z}}{\Gamma\left[-z+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2}+\check{s}\right]} + \frac{r^{-}_{\vec{p};z}}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[-z+\frac{p_3+p_4}{2}+\check{s}\right]}\right] \tag{1.26}$$

In particular the sphere Γ functions (in $\mathbf{\Gamma}$) split into two z-dependent gamma functions, with residues $r^{\pm}$ respectively. Since we will find that $r^{\pm}$ have generic charge dependence, i.e. they depend on c_t and c_u non trivially, it follows that $r^{\pm}$ do not map to each other under variations of charges that leave $\mathbf{\Gamma}$ invariant, and therefore the bonus property, exemplified for instance in (1.21), is lifted. All together we refer to this phenomenon as *sphere splitting*.

At tree level, the following features are well established: 1) the Γ functions of the AdS_5 part and the ones of the S^5 come as a unit block $\mathbf{\Gamma}$, then 2) the Mellin amplitude is a rational/polynomial in the Mellin variables $\{\hat{s},\hat{t},\check{s},\check{t}\}$, and 3) the breaking of the 10d conformal symmetry is manifest only in the Mellin amplitude $\mathcal{M}$ starting from $(\alpha')^{\geq 5}$ as shown in [28, 29]. We see now that the one-loop amplitude in (1.26) accommodates the OPE predictions in the Window, and breaks the accidental symmetry, by changing the organisation of the Γ functions, in particular, splitting into two the Γ functions corresponding to the sphere. This new organisation selects what parts of the Mellin amplitude should actually be considered to be a polynomial, i.e. the $r^{\pm}_{\vec{p}}$.

The third term in (1.22) is found to take the form

$$\mathcal{B}^{(\mathrm{BW})}_{\vec{p}}(\hat{s},\check{s}) = \sum_{z=0}^{2} \frac{\left(\check{s}+\frac{p_1+p_2}{2}-z\right)_{z+1}\left(\check{s}+\frac{p_3+p_4}{2}-z\right)_{z+1}}{\mathbf{s}-z}\, \mathscr{b}_{\vec{p};z}(\hat{s},\check{s})\,, \tag{1.27}$$

where again $\mathscr{b}_{\vec{p}}(\hat{s},\check{s};z)$ are polynomials. This contribution is determined by OPE predictions from the $\log^0 U$ term in the Below Window region.

Remarkably, we find that $\sum_z$ in (1.24) and (1.27) are finite and moreover the number of poles indexed by z is independent of the charges! This feature was already observed in [26] by studying $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle$ in the Window. It is not manifest from the form of the OPE, which instead depends on charges by construction. In fact, in order for this truncation to happen, there is a delicate interplay between $\mathcal{W}^{(\mathrm{AW})}$ and $\mathcal{R}^{(W)}$, and $\mathcal{B}^{(\mathrm{BW})}$.

The function $\mathcal{W}^{(\mathrm{AW})}$ contributes to $\log^2 U$ discontinuity by construction, but also contributes to the $\log^1 U$ discontinuity in the Window and the analytic term $\log^0 U$ in the Below Window regions. Similarly, $\mathcal{R}^{(W)}$ contributes to $\log^1 U$ in the Window region by construction but also to the analytic term $\log^0 U$ in the Below Window region. This cascading behaviour results from our choice to use the bold font variables, $\mathbf{s},\mathbf{t},\mathbf{u}$, in the parametrisation of the poles of $\mathcal{M}$, say $\tilde{\psi}^{(0)}(-\mathbf{s})$ and $\mathbf{s}-z$ in the $\mathbf{s}$ channel. This choice of parametrisation then reveals an additional simplicity: the truncation of the number of poles in z. In particular, we find that $\mathcal{R}^{(W)}$ only contains seven poles $z=0,\ldots 6$ at order $(\alpha')^3$.

We can argue that the use of the bold font variables, $\mathbf{s},\mathbf{t},\mathbf{u}$ in the parametrisation of the poles of $\mathcal{M}$ is natural from the perspective of large p limit [22], i.e. from the expected behaviour of the amplitude when the charges $\vec{p}$ are taken to be large. In the case we are interested in, the crucial observation is that, in the large $\vec{p}$ limit, the $AdS_5\times S^5$ Mellin amplitude asymptotes the flat space amplitude of IIB supergravity, where $\mathbf{s}$ is identified with the corresponding ten-dimensional Mandelstam invariant of the flat space amplitude. Now, the flat space amplitude of the one-loop IIB amplitude has various log contributions, e.g. $\log(-\mathbf{s})$. It is natural that such logs should arise from the digamma in the limit of large $\mathbf{s}$ as $\tilde{\psi}^{(0)}(-\mathbf{s})\to\log(-\mathbf{s})$. It follows that $\mathbf{s}$ is the natural variable entering $\tilde{\psi}^{(0)}(-\mathbf{s})$ in the $AdS_5\times S^5$ Mellin amplitude. More evidence supporting the use of $\mathbf{s},\mathbf{t},\mathbf{u}$, in parametrising the poles of $\mathcal{M}$ also comes from our preliminary investigations on the $(\alpha')^{n+3}$ terms for $n>0$, which show that the number of poles grows with n, but remains finite, and is independent of tree level ambiguities [40].

Following similar logic, the poles of the function $\mathcal{B}^{(\mathrm{BW})}$, are parametrised by $\mathbf{s},\mathbf{t},\mathbf{u}$. This function was previously studied for the single correlator $\langle\mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\rangle$ in [26]. We find that only three poles at $z=0,1,2$ are needed to match the OPE data. Again, this truncation depends on the delicate interplay with both $\mathcal{W}^{(\mathrm{AW})}$ and $\mathcal{R}^{(W)}$. Finally, combining all contributions we are able to explicitly verify consistency with the ten-dimensional flat space limit. Since this property was not used in the detailed construction of the individual contributions, this provides a strong consistency check on the form of our final results.

The rest of the paper is organised as follows: In section 2 we review the bootstrap program for one-loop amplitudes, explaining in particular how the OPE of two-particle operators works. In section 3 we discuss the construction of the one-loop amplitude $\mathcal{M}^{(2,3)}$.

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Emphasis is put on the 'sphere splitting', which is a new effect at one loop. Finally, in section 3.4 we show consistency of the one-loop amplitude with the large p limit and the flat space limit in type IIB string theory.

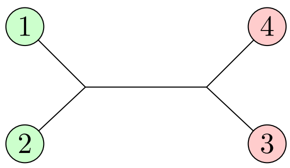
2 The $\mathbf{AdS_5 \times S^5}$ OPE

This section provides preparatory material for the construction of one-loop amplitudes in the large N large 't Hooft coupling regime of $\mathcal{N} = 4$ SYM. It draws mainly from [10] where it was explained how the CFT data collected from tree-level correlators needs to be organised at one-loop, in order to initiate the bootstrap program for correlators with arbitrary external charges $\vec{p}$. There are various important subtleties to be taken into account, that have to do with the relation among 1) the spectrum of two-particle operators, 2) the (super)block decomposition of the correlator, and 3) the arrangement of poles in the Mellin amplitude.[9] Explaining this will be our primary task in this section.

Some readers already familiar with the CFT construction of one-loop amplitudes in $AdS_5 \times S^5$, might wish to go directly to section 3. Before doing so, we would like to emphasise that the OPE of $\mathcal{O}_p \times \mathcal{O}_q$ of the half-BPS operators (1.1) at weak coupling is quite different: it involves not only two-particle (and multi-particle) states but also single trace operators dual to stringy states. These are decoupled in the large 't Hooft coupling limit which we are considering in this paper.[10]

2.1 Tree level OPE

Given four operators $\mathcal{O}_{p_i}(x_i)$ there are three independent ways to make them approach each other in pairs, and use the OPE. To fix conventions, we will always consider the OPE channel where points $x_1 \to x_2$ and $x_4 \to x_3$ approach each other, ordered as illustrated below



(2.1)

For a given set of charges $\vec{p}$, we will need to consider all three OPE channels. Thus we will consider the three possible orderings of $\vec{p}$ (mod $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$ exchange, since this is a symmetry of the OPE).

2.1.1 Mellin space vs quantum numbers

In the Mellin space representation, the interacting correlator is determined by taking residues at the location of certain poles in the Mellin variables. On the other hand, its OPE representation is organised by the quantum numbers of the operators exchanged, i.e. twist,

[9] We will study arbitrary Kaluza-Klein amplitudes, this discussion is far beyond a similar analysis for say the stress-tensor correlator in the $\mathcal{O}_2$ supermultiplet.

[10] A good reference where this important difference can be appreciated is [44]. The authors study the CFT data of long operators of twist 4 spin 0 in [000]. There is only a single double-trace operators, and at strong coupling this is also the only operator in the OPE, but at weak coupling there are four.

spin and $su(4)$ reps. In this section we would like to explain how the two representations are related. The case of twist and spin is perhaps known from the work of Mack [20]. Here we include the sphere part and we will specialise the discussion about twist and spin to the OPE in $\mathcal{N}=4$ SYM.

The interacting part of the correlator (1.11) is written as an expansion in monomials $\tilde{U}^{\check{s}}\tilde{V}^{\check{t}}$ over a triangle in the $(\check{s},\check{t})$ plane given by

$$\check{s} \geq -\min(\tfrac{p_1+p_2}{2},\tfrac{p_3+p_4}{2})\,; \quad \check{t} \geq -\min(\tfrac{p_1+p_4}{2},\tfrac{p_2+p_3}{2})\,; \quad \check{u} \geq -\min(\tfrac{p_1+p_3}{2},\tfrac{p_2+p_4}{2}) \tag{2.2}$$

where $\check{u}=-\Sigma-2-\check{s}-\check{t}$. Finiteness of this sum is due to the sphere Γ functions in the denominator of $\mathbf{\Gamma}$ factor, i.e. outside of (2.2) the factor $1/\Gamma_{\check{s}}\Gamma_{\check{t}}\Gamma_{\check{u}}$ vanishes. The amplitude is thus polynomial in $\tilde{U}$ and $\tilde{V}$. The triangle (2.2) is in correspondence with the $su(4)$ representations $[aba]$ flowing between the common OPEs of $\mathcal{O}_{p_1}(x_1)\times\mathcal{O}_{p_2}(x_2)$ and $\mathcal{O}_{p_3}(x_3)\times\mathcal{O}_{p_4}(x_4)$. We will think of $\check{s}$ as the conjugate variable to 'twist' for the sphere, i.e. b, and $\check{t}$ as the conjugate variable for 'spin' on sphere, i.e. a.

The number of long $su(4)$ channels a correlator contributes to depends only on the charges, and can be accounted by introducing the degree of extremality κ. A nice and fully symmetric expression for κ can be given as follows,

$$\kappa=\min(\tfrac{p_1+p_2}{2},\tfrac{p_3+p_4}{2})+\min(\tfrac{p_1+p_4}{2},\tfrac{p_2+p_3}{2})+\min(\tfrac{p_1+p_3}{2},\tfrac{p_2+p_4}{2})-\Sigma-2 \tag{2.3}$$

The number of long $su(4)$ channels is then $\frac{(\kappa+1)(\kappa+2)}{2}$. The reps $[aba]$ flowing in the OPE instead depend on the orientation of the charges and they are

$$a=0,1,\ldots,\kappa\,; \qquad \begin{array}{l} b=b_{\min},b_{\min}+2,\ldots,b_{\min}+2\kappa \\ b_{\min}=\max(|p_1-p_2|,|p_4-p_3|) \end{array} \tag{2.4}$$

Let us now translate the triangle in $su(4)$ labels into the triangle in Mellin variables $\check{s}$ and $\check{t}$. We shall focus on $\check{s}$ first, since this is relevant for the $(\alpha')^3$ amplitude.

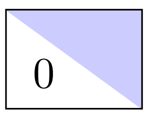
It is almost immediate to see that

$$\begin{array}{l} \check{s}=\check{s}_{\min},\check{s}_{\min}+1,\ldots,\check{s}_{\max} \\ b=b_{\max},b_{\max}-2,\ldots,b_{\min} \end{array}\;; \qquad \left\{\begin{array}{l} \check{s}_{\min}=-\min(\tfrac{p_1+p_2}{2},\tfrac{p_3+p_4}{2}) \\ \check{s}_{\max}=-\max(\tfrac{|p_1-p_2|}{2},\tfrac{|p_4-p_3|}{2})-2 \end{array}\right. \tag{2.5}$$

where $\check{s}$ is max when $\check{t}$ and $\check{u}$ are minimum and $\check{s}_{\max}-\check{s}_{\min}=\kappa$. In sum

$$\frac{b}{2}=-\check{s}-2. \tag{2.6}$$

The change from $su(4)$ harmonics $Y_{[0b0]}$ and monomials $\tilde{U}^{\check{s}}$ is a triangular matrix of the form

$$\begin{bmatrix} Y_{[0,b_{\min},0]} \\ \vdots \\ Y_{[0,b_{\max},0]} \end{bmatrix} =$$

$$\begin{bmatrix} \tilde{U}^{\check{s}_{\max}} \\ \vdots \\ \tilde{U}^{\check{s}_{\min}} \end{bmatrix} \tag{2.7}$$

In particular, the monomial $\tilde{U}^{\check{s}_{\min}}$ is the one and only one contributing to $[0b_{\max}0]$, but as we lower $b\leq b_{\max}$ sequentially a new monomial each time starts contributing. The inclusion

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of $\check{t}$ and a for each $\check{s}$ and b is straightforward at this point. The total range of $\check{t}$ simply covers $\check{t} = \check{t}_{\min}, \check{t}_{\min}+1, \dots, \check{t}_{\max} = \check{t}_{\min} + \kappa$ with $\check{t}_{\min} = -\min(\frac{p_1+p_4}{2}, \frac{p_2+p_3}{2})$. In the same way the total range of a covers $a = 0, 1, \dots, \kappa$.

Now that we have understood how the 4pt Mellin amplitude contributes to the various $su(4)$ channels, we would like to explain how the poles of

$$\mathcal{M}^{(1,0)} = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}\,, \tag{2.8}$$

are in correspondence with contributions in twist for each $su(4)$ channel. This will then lead us to our classification of the three regions Below Window, Window, and Above Window.

The simple pole at $\mathbf{s}+1 = 0$ is equivalently described by $\hat{s} = -1 - \check{s}$. It follows from (2.7) that if we look at $[0b0]$ channels we find

$$\begin{array}{c|lcl}
 & \text{lowest pole from } \mathbf{s}+1=0 & & \\
\hline
[0b_{\max}0] & \hat{s} = -1 - \check{s}_{\min} = -1 + \min(\frac{p_1+p_2}{2}, \frac{p_3+p_4}{2}) & = & \frac{b_{\max}}{2} + 1 \\
\vdots & \vdots & \vdots & \\
[0b_{\min}0] & \hat{s} = -1 - \check{s}_{\max} = +1 + \max(\frac{|p_1-p_2|}{2}, \frac{|p_4-p_3|}{2}) & = & \frac{b_{\min}}{2} + 1 \\
\hline
 & & = & \text{unitarity bound}
\end{array} \tag{2.9}$$

Since there is a triangular transformation between monomials and $su(4)$ harmonics the value of $\hat{s}$ written in the above table is actually the minimum value. Thus for labels $[aba]$ simple poles are given by

$$\hat{s} = [a + \tfrac{b}{2} + 1, \dots, \min(\tfrac{p_1+p_2}{2}, \tfrac{p_3+p_4}{2}) - 1] \tag{2.10}$$

Each simple pole contributes with a $U^{\hat{s}}$. Now, because a long block of twist τ has a leading power in U given by $U^{\tau/2}$, we find that poles in (2.9) imply the presence of a contribution in twist starting at the unitarity bound $\tau = 2a + b + 2$. Then, other simple poles coming from $(\mathbf{s}+1) = 0$ add a *new* contribution in twist up to $\tau = \min(p_1+p_2, p_3+p_4) - 2$. We refer to this region as 'Below Window'.

Another sequence of simple poles contributing to the correlator comes from the $\mathbf{\Gamma}$ factor. We define the Window Region by the sequence of simple poles at

$$\hat{s} = p_{\min}, \dots, p_{\max} - 1\,; \qquad \text{Window Region}, \tag{2.11}$$

where $p_{\min} = \min(\frac{p_1+p_2}{2}, \frac{p_3+p_4}{2})$ and $p_{\max} = \max(\frac{p_1+p_2}{2}, \frac{p_3+p_4}{2})$.

Finally, also from $\mathbf{\Gamma}$, we have an infinite sequence of double poles:

$$\hat{s} \geq p_{\max}\,; \qquad \text{Above Window}. \tag{2.12}$$

These double poles give rise to $\log U$ terms at tree level, upon performing the Mellin integral over $\hat{s}$. We should notice that for $p_1 + p_2 = p_3 + p_4$ the Window is empty. In this case, Above Window simply means above the threshold for exchange of long double traces, as defined in (2.12).

2.1.2 Two-particle operators and OPE data organisation

The operators exchanged in the regions described in (2.10)–(2.11)–(2.12), whose OPE data we are interested in, are two-particle operators of the form

$$\mathcal{O}_{\vec{\tau},(pq)} = \mathcal{O}_p \partial^l \Box^{\frac{1}{2}(\tau-p-q)} \mathcal{O}_q \qquad (p \leq q), \tag{2.13}$$

These operators are degenerate in free theory and will mix in the interacting theory. In the free theory, for fixed free quantum numbers, $\vec{\tau}$, there are as many operators $\mathcal{O}_{(pq)}$ as integer pairs (pq) in a certain rectangle $R_{\vec{\tau}}$ described in [15],

$$\begin{aligned} p &= i + a + 1 + r\,, & q &= i + a + 1 + b - r\,, \\ i &= 1, \ldots, (t-1)\,, & r &= 0, \ldots, (\mu - 1)\,, \end{aligned} \tag{2.14}$$

so that $|R_{\vec{\tau}}| = \mu(t-1)$ with

$$t \equiv \frac{(\tau - b)}{2} - a\,; \qquad \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a + l \text{ even,} \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a + l \text{ odd.} \end{cases} \tag{2.15}$$

Note that long operators have a minimum twist $\tau \geq 2a+b+4$, i.e. one unit above the unitarity bound. Protected two-particle operators are those with twist at the unitarity bound.

We will not discuss further the unmixing problem and we refer to the series of papers [7, 9, 10, 14, 15] for details. The upshot is that, upon resolving the mixing, one is left with a set of true scaling eigenstates $\mathcal{K}_{\vec{\tau}}$ (as many of them as there are pairs (pq) in $R_{\vec{\tau}}$) with dimensions,

$$\Delta_{\mathcal{K}} = \tau + l + \frac{2}{N^2}\eta^{(1)}_{\mathcal{K}} + O(\tfrac{1}{N^4})\,. \tag{2.16}$$

We can similarly expand the three-point couplings $C_{p_i p_j \mathcal{K}_{\vec{\tau}}}$ of the two-particle operators with the external single-particle operators $\mathcal{O}_{p_i}$ and $\mathcal{O}_{p_j}$,

$$C_{p_i p_j \mathcal{K}} = N^{\frac{p_i + p_j}{2}} \left[C^{(0)}_{p_i p_j \mathcal{K}} + \frac{1}{N^2} C^{(1)}_{p_i p_j \mathcal{K}} + \ldots \right] \tag{2.17}$$

Note that $C^{(0)}_{p_i p_j \mathcal{K}_{\vec{\tau}}} = 0$ for $\tau < p_i + p_j$. This follows from the form of the long contribution to disconnected free theory, which is only non-zero in the Above Window region. Because of this, the $\frac{1}{N^2}$ below window region is empty, and it first gets contributions at one-loop. This is better illustrated by figure 1.[11]

Having identified the spectrum we can go ahead and write down the OPE predictions in the large N expansion. The long multiplet contribution to the *full* correlator

[11] Since connected free theory contributes Below Window, the cancellation of Below Window contributions at tree level can be used to fix the tree level correlator [33]. It is actually convenient to always cancel connected free theory contributions, i.e. at all orders in N, and define a minimal interacting correlator whose OPE decomposition is one-to-one with the OPE predictions [10].

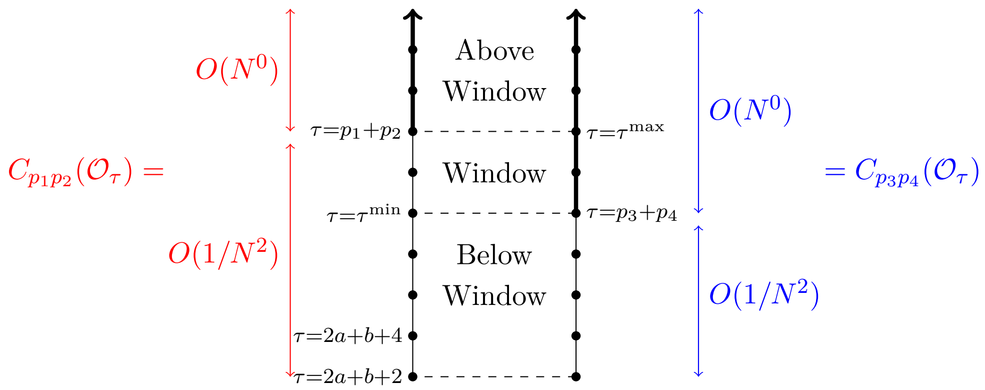


Figure 1. The large N structure of $C_{p_1p_2,\vec{\tau}}C_{p_3p_4,\vec{\tau}}$ for two particle operators $\mathcal{O}_\tau$ in an $su(4)$ representation $[aba]$, and varying twist.

(free+interacting) up to one-loop is given by[12]

$$\begin{aligned}\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{long}} = &\sum_{\vec{\tau}\in\text{AW}} L^{(0)}_{\vec{p};\vec{\tau}}\mathbb{L}^{\vec{p}}_{\vec{\tau}} \\ &+\frac{1}{N^2}\left[\sum_{\vec{\tau}\in W} N^{(1)}_{\vec{p};\vec{\tau}} + \sum_{\vec{\tau}\in\text{AW}} V^{(1)}_{\vec{p};\vec{\tau}}\log(u)\right]\mathbb{L}^{\vec{p}}_{\vec{\tau}}+\dots \\ &+\frac{1}{N^4}\left[\sum_{\vec{\tau}\in\text{BW}} K^{(2)}_{\vec{p};\vec{\tau}} + \sum_{\vec{\tau}\in W} H^{(2)}_{\vec{p};\vec{\tau}}\log(u) + \sum_{\vec{\tau}\in\text{AW}} M^{(2)}_{\vec{p};\vec{\tau}}\log^2(u)\right]\mathbb{L}^{\vec{p}}_{\vec{\tau}} + \dots\end{aligned} \tag{2.18}$$

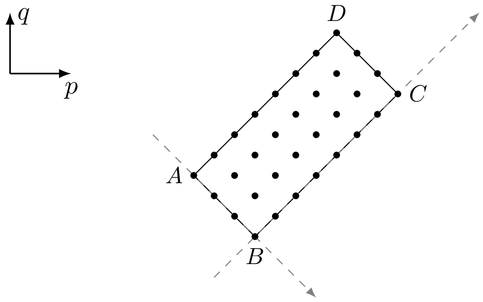
Due to the operator mixing described above, the OPE data is better organised into matrices. Let us package the anomalous dimensions $\eta^{(1)}_{\mathcal{K}}$ into a diagonal $|R_\tau|\times|R_\tau|$ matrix $\boldsymbol{\eta}^{(1)}$. Similarly we can arrange the leading order three-point functions into an $|R_\tau|\times|R_\tau|$ matrix $\mathbf{C}^{(0)}$. Then the OPE implies

$$\begin{aligned}\mathbf{L}^{(0)}_{\vec{\tau}} &= \mathbf{C}^{(0)}\cdot\mathbf{C}^{(0)^T}; &\qquad (\mathbf{L}^{(0)}_{\vec{\tau}})_{(p_1p_2)(p_3p_4)} &= L^{(0)}_{\vec{p},\vec{\tau}} = \sum_{\mathcal{K}} C^{(0)}_{p_1p_2\mathcal{K}}C^{(0)}_{p_3p_4\mathcal{K}}, \\ \mathbf{V}^{(1)}_{\vec{\tau}} &= \mathbf{C}^{(0)}\cdot\boldsymbol{\eta}^{(1)}\cdot\mathbf{C}^{(0)^T}; &\qquad (\mathbf{V}^{(1)}_{\vec{\tau}})_{(p_1p_2)(p_3p_4)} &= V^{(1)}_{\vec{p},\vec{\tau}} = \sum_{\mathcal{K}} C^{(0)}_{p_1p_2\mathcal{K}}\eta^{(1)}_{\mathcal{K}}C^{(0)}_{p_3p_4\mathcal{K}},\end{aligned} \tag{2.19}$$

where both (p_1p_2) and (p_3p_4) belong to $R_{\vec{\tau}}$. The matrices $\mathbf{L}^{(0)}$ and $\mathbf{V}^{(1)}$ are symmetric and are obtained by collecting the CPW expansion, for fixed $\vec{\tau}$, of the correlators $\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$ with varying external charges. The matrix $\mathbf{L}^{(0)}$ gives the (diagonal) CPW coefficients from disconnected free theory, while $\mathbf{V}^{(1)}$ comes from the $\log U$ contribution in the tree-level interacting part. For given $\vec{\tau}$ both matrices have size $|R_{\vec{\tau}}|\times|R_{\vec{\tau}}|$ and are full rank.

A similar organisation principle holds for the subleading three-point couplings $C^{(1)}$ which arise from the window region. This time it is more natural to arrange a vector $(\mathbf{C}^{(1)}_{\vec{\tau}})_{q_1q_2}$, labelled by $\vec{\tau}$ and a fixed pair q_1q_2, such that $q_1+q_2>\tau\geq 2a+b+4$, with the

[12]We have omitted terms with derivatives acting on the blocks as they do not affect the leading logs for each region.



$A=(a+2,a+b+2)$;
$B=(a+1+\mu,a+b+3-\mu)$;
$C=(a+\mu+t-1,a+b+1+t-\mu)$;
$D=(a+t,a+b+t)$;

Figure 2. The set $\mathcal{R}_{\vec{\tau}}$ pictured in the (p,q) plane. With vertical lines indicating operators with the same anomalous dimension see [15] for full details.

vector index running over the operators $\mathcal{K}_{\vec{\tau}}$. From the OPE,

$$\mathbf{N}^{(1)}_{q_1q_2}=\mathbf{C}^{(1)}_{q_1q_2}\cdot\mathbf{C}^{(0)T}\,; \qquad \left[\mathbf{N}^{(1)}_{\vec{\tau},(q_1q_2)}\right]_{(p_3p_4)}=\sum_{\mathcal{K}}C^{(1)}_{q_1q_2\mathcal{K}}C^{(0)}_{p_3p_4\mathcal{K}}\,, \qquad (p_3p_4)\in R_{\vec{\tau}}\,. \tag{2.20}$$

The various entries $\left[\mathbf{N}^{(1)}_{\vec{\tau},(q_1q_2)}\right]_{(p_3p_4)}$ are found from the CPW coefficients of $\langle\mathcal{O}_{q_1}\mathcal{O}_{q_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$ in the Window. In this case it is crucial to consider both the tree-level contribution to $\langle\mathcal{O}_{q_1}\mathcal{O}_{q_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\rm int}$ and connected free theory in the long sector.

The above discussion holds for any value of the 't Hooft coupling λ, but let us recall that we are interested in the regime of large λ and expand the anomalous dimensions and three-point functions accordingly,

$$\begin{aligned}\eta^{(1)}_{\mathcal{K}}&=\eta^{(1,0)}_{\mathcal{K}}&&+\lambda^{-\frac{3}{2}}\eta^{(1,3)}_{\mathcal{K}}&&+\lambda^{-\frac{5}{2}}\eta^{(1,5)}_{\mathcal{K}}+\ldots\,,\\ C^{(0)}_{p_ip_j\mathcal{K}}&=C^{(0,0)}_{p_ip_j\mathcal{K}}&&+\lambda^{-\frac{3}{2}}C^{(0,3)}_{p_ip_j\mathcal{K}}&&+\lambda^{-\frac{5}{2}}C^{(0,5)}_{p_ip_j\mathcal{K}}+\ldots\\ C^{(1)}_{p_ip_j\mathcal{K}}&=C^{(1,0)}_{p_ip_j\mathcal{K}}&&+\lambda^{-\frac{3}{2}}C^{(1,3)}_{p_ip_j\mathcal{K}}&&+\lambda^{-\frac{5}{2}}C^{(1,5)}_{p_ip_j\mathcal{K}}+\ldots\,.\end{aligned}$$

The supergravity contributions to the anomalous dimensions of the operators are given by a very simple formula [15],

$$\eta^{(1,0)}_{\mathcal{K}_{pq}}=-\frac{2M^{(4)}_tM^{(4)}_{t+l+1}}{(\ell_{10}+1)_6} \tag{2.21}$$

where

$$M^{(4)}_t\equiv(t-1)(t+a)(t+a+b+1)(t+2a+b+2) \tag{2.22}$$

$$\ell_{10}=l+2(p-2)-a+\frac{1-(-)^{a+l}}{2}. \tag{2.23}$$

The notation for ℓ_{10} reflects the fact that this quantity can be interpreted as a ten-dimensional spin [16]. For $\mu>1$ and $t>2$ there is a residual degeneracy because ℓ_{10} depends only on p and not on q. This degeneracy is illustrated in figure 2 which gives a sketch of the rectangle $R_{\vec{\tau}}$ with operators of equal anomalous dimension connected by vertical lines. The residual degeneracy is a reflection of the ten-dimensional conformal symmetry described in [16].

At order $\lambda^{-\frac{3}{2}}$ the anomalous dimensions are even simpler [27]. Only operators with $\ell_{10}=0$, i.e. with $a=l=0$, $i=1$, $r=0$ receive an anomalous dimension at this order. In relation to the diagram figure 2, these operators sit at the left-most corner of the rectangle $R_{\vec{\tau}}$ where $(p,q)=(2,2+b)$. Their anomalous dimensions read,

$$\eta^{(1,3)} = -\frac{\zeta_3}{840}\delta_{l,0}\delta_{a,0}M_t^{(4)}M_{t+1}^{(4)}(t-1)_3(t+b+1)_3\,. \tag{2.24}$$

The matrix $\mathbf{L}^{(0)}$ is independent of λ, being derived from the disconnected part of the free theory correlator, hence we can also write $\mathbf{L}^{(0)}=\mathbf{L}^{(0,0)}$. The matrix $\mathbf{V}^{(1)}$ has an expansion for large λ,

$$\mathbf{V}^{(1)} = \mathbf{V}^{(1,0)} + \lambda^{-\frac{3}{2}}\mathbf{V}^{(1,3)} + \lambda^{-\frac{5}{2}}\mathbf{V}^{(1,5)} + \dots\,. \tag{2.25}$$

When we expand the mixing equations (2.19) order by order in $\lambda^{-\frac{1}{2}}$ we find the following equations at leading order,

$$\begin{aligned} \mathbf{C}^{(0,0)}\mathbf{C}^{(0,0)^T} &= \mathbf{L}^{(0,0)}\,, \\ \mathbf{C}^{(0,0)}\boldsymbol{\eta}^{(1,0)}\mathbf{C}^{(0,0)^T} &= \mathbf{V}^{(1,0)}\,. \end{aligned} \tag{2.26}$$

This eigenvalue problem was solved in [14, 15], yielding the double-trace spectrum of anomalous dimensions in supergravity, which exhibit the partial residual degeneracy associated with the hidden conformal symmetry [16]. At order $\lambda^{-\frac{3}{2}}$ we find,

$$\begin{aligned} \mathbf{C}^{(0,3)}\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(0,0)}\mathbf{C}^{(0,3)^T} &= 0\,, \\ \mathbf{C}^{(0,0)}\boldsymbol{\eta}^{(1,3)}\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(0,3)}\boldsymbol{\eta}^{(1,0)}\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(0,0)}\boldsymbol{\eta}^{(1,0)}\mathbf{C}^{(0,3)^T} &= \mathbf{V}^{(1,3)}\,, \end{aligned} \tag{2.27}$$

and at order $\lambda^{-\frac{5}{2}}$,

$$\begin{aligned} \mathbf{C}^{(0,5)}\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(0,0)}\mathbf{C}^{(0,5)^T} &= 0\,, \\ \mathbf{C}^{(0,0)}\boldsymbol{\eta}^{(1,5)}\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(0,5)}\boldsymbol{\eta}^{(1,0)}\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(0,0)}\boldsymbol{\eta}^{(1,0)}\mathbf{C}^{(0,5)^T} &= \mathbf{V}^{(1,5)}\,. \end{aligned} \tag{2.28}$$

In this paper we are mostly concerned with the order $\lambda^{-\frac{3}{2}}$ equations, which in fact yield $\mathbf{C}^{(0,3)}=0$ [27], since it has to be a one-dimensional matrix satisfying (2.27). This simplification will not hold at the next order, i.e. $\mathbf{C}^{(0,5)}\neq 0$ [28]. By expanding the subleading couplings $C^{(1)}$ in $\lambda^{-\frac{1}{2}}$ we have,

$$\begin{aligned} \mathbf{C}^{(1,0)}_{q_1q_2}\cdot\mathbf{C}^{(0,0)^T} &= \mathbf{N}^{(1,0)}_{q_1q_2}\,, \\ \mathbf{C}^{(1,3)}_{q_1q_2}\cdot\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(1,0)}_{q_1q_2}\cdot\mathbf{C}^{(0,3)^T} &= \mathbf{N}^{(1,3)}_{q_1q_2}\,, \\ \mathbf{C}^{(1,5)}_{q_1q_2}\cdot\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(1,0)}_{q_1q_2}\cdot\mathbf{C}^{(0,5)^T} &= \mathbf{N}^{(1,5)}_{q_1q_2}\,. \end{aligned} \tag{2.29}$$

Again note that the order $\lambda^{-\frac{3}{2}}$ equation simplifies because $\mathbf{C}^{(0,3)}=0$.

2.2 One loop OPE

Having reviewed tree level data, we are ready to use it to study one-loop string theory in $AdS_5\times S^5$. This amounts to bootstraping the one-loop correlators by knowing

$$\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{long}} = \ldots + \frac{1}{N^4}\left[\sum_{\vec{\tau}\in\text{BW}}K^{(2)}_{\vec{p};\vec{\tau}} + \sum_{\vec{\tau}\in W}H^{(2)}_{\vec{p};\vec{\tau}}\log(U) + \sum_{\vec{\tau}\in\text{AW}}M^{(2)}_{\vec{p};\vec{\tau}}\log^2(U)\right]\mathbb{L}^{\vec{p}}_{\vec{\tau}} + \ldots$$

i.e. by knowing the values of $M^{(2)}_{\vec{p};\vec{\tau}}$, $H^{(2)}_{\vec{p};\vec{\tau}}$, and $K^{(2)}_{\vec{p};\vec{\tau}}$. Let us address each of these in turn according to power of $\log U$:

- **$\log^2 U$ Above Window.** For fixed quantum numbers $\vec{\tau}$ and charges $\vec{p}$ the OPE prediction can be read off the matrix $\mathbf{M}^{(2)}$ given by

$$\mathbf{M}^{(2)} = \tfrac{1}{2}\,\mathbf{C}^{(0)}\cdot\left(\boldsymbol{\eta}^{(1)}\right)^2\cdot\mathbf{C}^{(0)^T} = \tfrac{1}{2}\,\mathbf{V}^{(1)}\cdot\left(\mathbf{L}^{(0)}\right)^{-1}\cdot\mathbf{V}^{(1)}\,. \tag{2.30}$$

Thus the $\log^2 U$ terms are entirely predicted from the disconnected and tree-level CPW coefficients $\mathbf{L}$ and $\mathbf{V}$.

Expanding the above relation in $\lambda^{-\frac{1}{2}}$ we have

$$\mathbf{M}^{(2,0)} = \tfrac{1}{2}\mathbf{C}^{(0,0)}\cdot\left(\boldsymbol{\eta}^{(1,0)}\right)^2\cdot\mathbf{C}^{(0,0)^T} = \tfrac{1}{2}\mathbf{V}^{(1,0)}\cdot\left(\mathbf{L}^{(0)}\right)^{-1}\cdot\mathbf{V}^{(1,0)}\,, \tag{2.31}$$

$$\begin{aligned}\mathbf{M}^{(2,3)} &= \mathbf{C}^{(0,0)}\cdot\boldsymbol{\eta}^{(1,0)}\cdot\boldsymbol{\eta}^{(1,3)}\cdot\mathbf{C}^{(0,0)^T} + \tfrac{1}{2}\left[\mathbf{C}^{(0,3)}\cdot\boldsymbol{\eta}^{(1,0)}\cdot\boldsymbol{\eta}^{(1,0)}\cdot\mathbf{C}^{(0,0)^T} + \text{tr.}\right]\\ &= \tfrac{1}{2}\left(\mathbf{V}^{(1,3)}\cdot\left(\mathbf{L}^{(0)}\right)^{-1}\cdot\mathbf{V}^{(1,0)} + \mathbf{V}^{(1,0)}\cdot\left(\mathbf{L}^{(0)}\right)^{-1}\cdot\mathbf{V}^{(1,3)}\right)\,.\end{aligned} \tag{2.32}$$

In the second line above we have used $\mathbf{C}^{(0,3)}\mathbf{C}^{(0,0)^T} + \mathbf{C}^{(0,0)}\mathbf{C}^{(0,3)^T} = 0$, and that $\boldsymbol{\eta}^{(1,0)}$ and $\boldsymbol{\eta}^{(1,3)}$ are diagonal and hence commute. The first condition at $(\alpha')^3$ is obvious, since $C^{(0,3)} = 0$. A formula like (2.32) holds at $(\alpha')^5$ upon replacing $\mathbf{V}^{(1,5)}$.

- **$\log^1 U$ in the Window.** In a similar way as above, we arrange the vector $\mathbf{H}^{(2)}_{q_1q_2}$ labelled by a fixed pair of charges (q_1q_2),

$$\mathbf{H}^{(2)}_{q_1q_2} = \mathbf{C}^{(1)}_{q_1q_2}\cdot\boldsymbol{\eta}^{(1)}\cdot\mathbf{C}^{(0)^T} = \mathbf{N}^{(1)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0)}\right)^{-1}\cdot\mathbf{V}^{(1)}\,. \tag{2.33}$$

Recall that $\mathbf{C}^{(1)}_{q_1q_2}$ exists only for operators $\mathcal{K}_{\vec{\tau}}$ such that $\tau < q_1 + q_2$. Recall also that for a correlator $\langle\mathcal{O}_p\mathcal{O}_q\mathcal{O}_p\mathcal{O}_q\rangle$ there is no Window.

If we now expand in $\lambda^{-\frac{1}{2}}$ we find

$$\mathbf{H}^{(2,0)}_{q_1q_2} = \mathbf{N}^{(1,0)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0,0)}\right)^{-1}\cdot\mathbf{V}^{(1,0)}\,, \tag{2.34}$$

$$\mathbf{H}^{(2,3)}_{q_1q_2} = \mathbf{N}^{(1,3)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0,0)}\right)^{-1}\cdot\mathbf{V}^{(1,0)} + \mathbf{N}^{(1,0)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0,0)}\right)^{-1}\cdot\mathbf{V}^{(1,3)}\,. \tag{2.35}$$

which give the OPE predictions in terms of tree level data.

- **$\log^0 U$ Below Window.** Finally, in the below-window region,

$$\mathbf{K}^{(2)}_{q_1q_2q_3q_4} = \mathbf{C}^{(1)}_{q_1q_2}\cdot\mathbf{C}^{(1)\,T}_{q_3q_4} = \mathbf{N}^{(1)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0)}\right)^{-1}\cdot\mathbf{N}^{(1)\,T}_{q_3q_4}\,. \tag{2.36}$$

Once again we can expand in $\lambda^{-\frac{1}{2}}$ to obtain,

$$\mathbf{K}^{(2,0)}_{q_1q_2q_3q_4} = \mathbf{N}^{(1,0)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0,0)}\right)^{-1}\cdot\mathbf{N}^{(1,0)^T}_{q_3q_4}\,, \tag{2.37}$$

$$\mathbf{K}^{(2,3)}_{q_1q_2q_3q_4} = \mathbf{N}^{(1,3)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0,0)}\right)^{-1}\cdot\mathbf{N}^{(1,0)^T}_{q_3q_4} + \mathbf{N}^{(1,0)}_{q_1q_2}\cdot\left(\mathbf{L}^{(0,0)}\right)^{-1}\cdot\mathbf{N}^{(1,3)^T}_{q_3q_4}\,. \tag{2.38}$$

which give the OPE predictions in terms of tree level data. The Below Window predictions are non trivial for the first time only at one-loop!

The double logarithmic behaviour at one-loop in supergravity has been studied extensively [6, 7, 10, 12, 26]. It has been used to make predictions for the form of the one-loop correlator both in position [7, 10, 26] and Mellin space [12, 22]. The relations (2.34) and (2.37) only become relevant for correlators with multiple $su(4)$ channels, i.e. correlators with arbitrary external charges $p_1p_2p_3p_4$. In the next section we shall see this in more details.

2.2.1 New physics at the Window

As pointed out above, when we look at correlators with generic charges the novelties at one-loop are the OPE predictions in- and below Window. The ones in the Window will be particularly important for us and for this reason we illustrate here, with the help of an example, what will be going on.

We already mentioned in section 1.4 the existence of degenerate correlators, i.e. correlators that have the same values of Σ and $|c_s|, |c_t|, |c_u|$, and therefore such that $\mathbf{\Gamma}$ is unchanged. For example, two correlators whose values of $p_1 + p_2$ and $p_3 + p_4$ swap, and the other $p_i + p_j$ are unchanged. At order $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$ these correlators are necessarily proportional to each other since these Mellin amplitudes do not distinguish them.

The example of $\langle \mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\mathcal{O}_5\rangle$ and $\langle \mathcal{O}_4\mathcal{O}_4\mathcal{O}_2\mathcal{O}_4\rangle$ discussed in (1.21) is quite useful to keep in mind. Unpacking the notation for $\mathbf{N}_{q_1q_2}$ in (2.33), we see that the one-loop predictions in the Window now involve

$$\begin{aligned} &\Big[\mathbf{N}^{(1,0)}_{35}\Big]_{(rs)=(24),(33)} \;; \qquad \Big[\mathbf{N}^{(1,0)}_{44}\Big]_{(rs)=(24),(33)}\,, \\ &\Big[\mathbf{N}^{(1,3)}_{35}\Big]_{(rs)=(24),(33)} \;; \qquad \Big[\mathbf{N}^{(1,3)}_{44}\Big]_{(rs)=(24),(33)}\,, \end{aligned} \tag{2.39}$$

where the indices (rs) are running over the rectangle in figure 2 at $\tau = 6$ in $[020]$, i.e. $\{(24),(33)\}$. Note that the list of $\mathbf{N}$ on the left will enter $\langle \mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\mathcal{O}_5\rangle$ at one-loop, and the list of $\mathbf{N}$ on the right will enter $\langle \mathcal{O}_4\mathcal{O}_4\mathcal{O}_2\mathcal{O}_4\rangle$. The purple colored coefficients, come from the very same correlators but at tree level. At tree level these correlators were degenerate and therefore these coefficients are the same. Now at one-loop there are also non-purple coefficients entering the predictions, and these are genuinely distinct. Thus, after matrix multiplication in (2.33), the one-loop OPE prediction will distinguish these correlators. We call this phenomenon "splitting at the Window".

The general statement will be that one-loop OPE predictions in the Window lift the tree-level degeneracy of correlators. This was first noticed in supergravity in [10], and essentially the same mechanism is at work here. In [10] several explicit examples were constructed in position space. Understanding the general pattern in supergravity is however complicated, and the complication comes from the fact that both Window and Below Window predictions are non-trivial at all spins, so it is difficult to find an explicit formula for the one-loop Mellin amplitude for generic external charges.

Our focus in this paper will be on the $(\alpha')^3$ one-loop correlator and thus equations (2.32), (2.35) and (2.38). The advantage of studying the $\mathcal{M}^{(2,3)}$ amplitude, compared to supergravity, stems from the truncation in spin of the spectrum of two-particle operators exchanged. This simplifies the structure of the OPE predictions in the Window and Below Window regions, and will indeed allow us to find an expression for the Mellin amplitude

$\mathcal{M}^{(2,3)}_{\vec{p}}$ for general $\vec{p}$. Given our understanding here, we believe that the same pattern applies at all orders in α', even though computationally it will become a bit more involved.

3 The one loop Mellin amplitude $\boldsymbol{\mathcal{M}^{(2,3)}}$

In this section we translate the structure of the one-loop OPE data into the Mellin space amplitude. We claim that

$$\mathcal{M}^{(2,3)}_{\vec{p}} = \Big[\mathcal{W}^{(\mathrm{AW})}_{\vec{p}}(\hat{s},\check{s}) + \mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s}) + \mathcal{B}^{(\mathrm{BW})}_{\vec{p}}(\hat{s},\check{s})\Big] + \texttt{crossing}\,. \tag{3.1}$$

i.e. $\mathcal{M}^{(2,3)}_{\vec{p}}$ naturally splits into three pieces, described below.

Our starting point will be to use the data from the OPE, at tree level Above Window, to completely fix $\mathcal{W}^{(\mathrm{AW})}$. We will find that

$$\mathcal{W}^{(\mathrm{AW})}_{\vec{p}}(\hat{s},\check{s}) = \tilde{\psi}^{(0)}(-\mathbf{s})\;\, w(\hat{s},\check{s},c_s;\Sigma) \tag{3.2}$$

where w is a determined polynomial, and $\tilde{\psi}^{(0)} = \psi^{(0)} + \gamma_E$ is a digamma function. The latter accounts for the fact that $\mathcal{W}^{\mathrm{AW}}$ has to contribute to triple poles in order to generate a $\log^2 U$ discontinuity, and the $\mathbf{\Gamma}$ factor only gives at most double poles Above Window.[13] The restricted dependence of w on the Mellin variables comes from the fact that the OPE has support only on $a = l = 0$, and therefore w is function of $\mathbf{s}$-type variables only, thus $w(\hat{s},\check{s},c_s;\Sigma)$.

Note: The argument of $\tilde{\psi}^{(0)}$ being $-\mathbf{s}$ implies that contributions from $\mathcal{W}^{(\mathrm{AW})}$ are not restricted to Above Window, but actually start at $\mathbf{s}=0$. As we explained in the previous section, this is one unit above the unitarity bound $\mathbf{s}+1=0$, therefore $\mathcal{W}^{(\mathrm{AW})}$ contributes also to the $\log^1 U$ and $\log^0 U$ discontinuities, respectively, in the Window and Below Window. This fact will play an important role as observed in [26], and explained below.

Next, we will use window OPE data, and also contributions coming from $\mathcal{W}^{(\mathrm{AW})}$, to fix the remainder function in the Window

$$\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s}) = \sum_{z=0}^{6} \frac{\left(\check{s}+\frac{p_1+p_2}{2}-z\right)_{z+1}\; r^{+}_{\vec{p};z}(\hat{s},\check{s}) + \left(\check{s}+\frac{p_3+p_4}{2}-z\right)_{z+1}\; r^{-}_{\vec{p};z}(\hat{s},\check{s})}{\mathbf{s}-z}\,, \tag{3.3}$$

with $r^{\pm}$ are polynomials. We shall call $\mathcal{R}$ a remainder function since it gives the part of the OPE Window predictions not captured by $\mathcal{W}^{(\mathrm{AW})}$. Remarkably, the interplay with $\mathcal{W}^{(\mathrm{AW})}$ will truncate the sum over z to a maximum of seven poles! It will be clear that this function should also be extended to contribute to the Below Window region.

The determination of the function $\mathcal{R}^{(W)}$ is a central result of our investigation, since it characterises the way the *splitting at the Window* described in the section 2.2.1 is implemented in Mellin space. We called this phenomenon "sphere splitting", and it will be discussed in more detail in section 3.2.

[13] Note the formula $\Gamma[-\hat{s}+\frac{p_1+p_2}{2}]\Gamma[-\hat{s}+\frac{p_3+p_4}{2}]\psi(-\mathbf{s})\mathcal{W}^{(\mathrm{AW})}U^{\hat{s}} \to \frac{1}{2}\left[\partial^2_{\hat{s}}\frac{1}{(\hat{s}-\mathbf{n})}\right]\mathcal{W}^{(\mathrm{AW})}U^{\hat{s}}$.

Finally, using Below Window data, as well as contributions now coming from both $\mathcal{W}^{(\mathrm{AW})}(\hat{s},\check{s})$ and $\mathcal{R}^{(W)}(\hat{s},\check{s})$, we fix the last piece of our ansatz[14]

$$\mathcal{B}^{(\mathrm{BW})}_{\vec{p}}(\hat{s},\check{s}) = \sum_{z=0}^{2} \frac{\left(\check{s}+\frac{p_1+p_2}{2}-z\right)_{z+1}\left(\check{s}+\frac{p_3+p_4}{2}-z\right)_{z+1}}{\mathbf{s}-z}\, \mathscr{b}_{\vec{p};z}(\hat{s},\check{s})\,. \tag{3.4}$$

This is also a remainder function, capturing the predictions of the OPE in the Below Window region after contributions from both $\mathcal{W}^{(\mathrm{AW})}(\hat{s},\check{s})$ and $\mathcal{R}^{(W)}(\hat{s},\check{s})$ are taken into account. We find that $\mathcal{B}^{(\mathrm{BW})}(\hat{s},\check{s})$ also truncates, this time to just three poles in z. Again $\mathscr{b}_{\vec{p};z}(\hat{s},\check{s})$ is polynomial.

3.1 Double log discontinuity

The $\log^2 U$ terms at order $\lambda^{-\frac{3}{2}}$ have a very simple form [26]. The reason behind this is that only operators with $\ell_{10}=0$ receive an order $\lambda^{-\frac{3}{2}}$ contribution to their anomalous dimensions. Therefore, in the following expression for $\mathbf{M}^{(2,3)}_{\vec{\tau}}$,

$$[\mathbf{M}^{(2,3)}_{\vec{\tau}}]_{(p_1p_2),(p_3p_4)} = \sum_{\mathcal{K}\in R_{\vec{\tau}}} C^{(0,0)}_{p_1p_2\mathcal{K}}\,\eta^{(1,0)}_{\mathcal{K}}\eta^{(1,3)}_{\mathcal{K}}C^{(0,0)}_{p_3p_4\mathcal{K}}\,. \tag{3.5}$$

only a single operator $\mathcal{K}$ contributes to the sum for a given τ and b! In figure 2 this is the operator labelled by the leftmost corner of the rectangle. If we now insert this expression into the sum over long superconformal blocks to obtain the expansion of the $\log^2 U$ discontinuity, we find an expression that is almost identical to the expansion of the $\log U$ discontinuity at tree level,[15] but for the insertion of a factor of $\eta^{(1,0)}$ restricted to $\ell_{10}=0$. By construction this factor is a number (which includes the value of the denominator of $\eta^{(1,0)}$) multiplying the eigenvalue of a certain eight-order Casimir operator $\Delta^{(8)}$ introduced in [16] in position space, see in particular [26].

In Mellin space, the $\log^2 U$ contribution comes from $\mathbf{\Gamma}\times\tilde{\psi}^{(0)}(-\mathbf{s})$, which is the source of all triple poles. The knowledge of the $\log^2 U$ coefficient will therefore fully fix

$$\mathcal{W}^{(\mathrm{AW})}_{\vec{p}}(\hat{s},\check{s}) = \tilde{\psi}^{(0)}(-\mathbf{s})\;\mathscr{w}(\hat{s},\check{s},c_s;\Sigma). \tag{3.6}$$

From the discussion about the special form of (3.5) we infer that, apart for an overall prefactor, the polynomial $\mathscr{w}$ is given by applying $\Delta^{(8)}$ (rewritten in Mellin space) to 1, where the latter is (up to a numerical factor) the tree level amplitude in the Virasoro Shapiro amplitude at order $(\alpha')^3$.

We have written the full expression of $\mathscr{w}(\hat{s},\check{s},c_s;\Sigma)$ in the supplementary material. Expanded in all variables it has the form

$$\mathscr{w}(\hat{s},\check{s},c_s;\Sigma) = +\frac{(\Sigma-1)_3}{180}\Big(-36c_s^2+9c_s^4+36c_s^2\check{s}+\ldots+8\check{s}\check{s}\hat{s}^3\Sigma^4+2\hat{s}^4\Sigma^4\Big). \tag{3.7}$$

[14] The division between $\mathcal{R}^{(W)}$ and $\mathcal{B}^{(\mathrm{BW})}$ is our choice. We do so because the explicit solution for $\mathcal{R}^{(W)}$ will have nice analytic properties Below Window, see section 3.2.

[15] I.e. $\sum_{\mathcal{K}\in R_{\vec{\tau}}} C^{(0,0)}_{(p_1p_2)\mathcal{K}}\eta^{(1,3)}_{\mathcal{K}}C^{(0,0)}_{p_3p_4\mathcal{K}}$.

A more compact representation can be given by the following double integral,

$$w(\hat{s},\check{s}) = \frac{i}{2\pi}\int_0^\infty \frac{d\alpha}{\alpha}\int_{\mathcal{C}} d\beta\; e^{-\alpha-\beta}\alpha^{\Sigma+2}(-\beta)^{-\Sigma+1}\,\widetilde{w}(\alpha,\beta) \tag{3.8}$$

where $\mathcal{C}$ is the Hankel contour. The α integral is the Γ function integral, and the β integral the reciprocal Γ function integral. Note that the integral in α is nothing but the integral used by Penedones to study flat space limit [21], and the β-integral generalises that to the compact space.

Then, $\widetilde{w}_{\vec{p}}(\alpha,\beta)$ is defined in terms of the following variables,

$$S = \alpha\hat{s} - \beta\check{s}, \qquad \tilde{S} = \alpha\hat{s} + \beta\check{s}, \tag{3.9}$$

by the expression

$$\begin{aligned}
&\widetilde{w}(\alpha,\beta,c_s;\Sigma) = \\
&\quad \frac{1}{90}(S-3\Sigma)(S-2\Sigma)(S-\Sigma)S + \tfrac{\Sigma^2-c_s^2}{20}(2S^2-6S\Sigma+5\Sigma^2-c_s^2) \\
&\quad -\frac{1}{30}\tilde{S}(2S^2-9S\Sigma+11\Sigma^2-3c_s^2) + \frac{1}{180}(S^2-36S\Sigma+36(\Sigma^2-c_s^2)+7\tilde{S}^2).
\end{aligned} \tag{3.10}$$

It is immediate to see that only $c_s^{2\mathbb{N}}$ appear. In fact, since w is a function of $\mathbf{s}$ variables only, the crossing relations,

$$\begin{array}{ccc}
\mathcal{M}_{p_1p_2p_3p_4}(\hat{s},\hat{t},\check{s},\check{t}) & = & \mathcal{M}_{p_4,p_3,p_2,p_1}(\hat{s},\hat{t},\check{s},\check{t}) \\
\| & & \| \\
\mathcal{M}_{c_s,c_t,c_u}(\hat{s},\hat{t},\check{s},\check{t}) & & \mathcal{M}_{-c_s,c_t,-c_u}(\hat{s},\hat{t},\check{s},\check{t})
\end{array} \tag{3.11}$$

implies that w is function of c_s^2, i.e. since there is no c_u dependence the invariance under $c_s \leftrightarrow -c_s$ follows. In particular, as for $\mathbf{\Gamma}$ factor, the polynomial $w(\hat{s},\check{s},c_s;\Sigma)$ has the same bonus property.

Let us emphasise that the OPE does not immediately predict $\tilde{\psi}^{(0)}(-\mathbf{s})$. The Above Window region only requires $\Gamma[\frac{p_1+p_2}{2}-\hat{s}]\Gamma[\frac{p_3+p_4}{2}-\hat{s}]\tilde{\psi}^{(0)}(-\hat{s}+p_{\max})$, since $\hat{s}\geq p_{\max}$ is where the triple poles are. However, the presence of $\tilde{\psi}^{(0)}(-\mathbf{s})$ is strongly motivated by the limit in which the charges p_i are large, and therefore $\mathbf{s}$ is large [22]. As explained already in the Introduction, the key observation is that in this limit $\mathbf{s}$ becomes a 10d Mandelstam invariant and the Mellin amplitude asymptotes the 10d flat space scattering amplitude. In the present case we must recover the $\log(-\mathbf{s})$ of the type IIB flat space amplitude, and the latter comes from $\tilde{\psi}^{(0)}(-\mathbf{s})$ in the limit of large $\mathbf{s}$. We will later show in section 3.4 that in the large p limit we recover exactly the type-IIB flat space amplitude!

It follows from the presence of $\tilde{\psi}^{(0)}(-\mathbf{s})$ that the range of twists where $\mathcal{W}^{(\mathrm{AW})}$ contributes is not restricted to the Above Window region, $\hat{s}\geq p_{\max}$, where $\mathcal{W}^{(\mathrm{AW})}$ naturally lives, but embraces the bigger region bounded from below by the locus $\mathbf{s}=0$. We now understand that $\mathcal{W}^{(\mathrm{AW})}$ contributes to the one-loop correlator starting from the first two-particle operator in the Below Window region at $\tau = 2a+b+4$. Consequently $\mathcal{W}^{(\mathrm{AW})}$ gives contributions that will add to those of $\mathcal{R}^{(W)}$ function and $\mathcal{B}^{(\mathrm{BW})}$ in- and below- Window, respectively. The choice to write the Above Window contribution as described above has the consequence of revealing the truncation of sum over z in the Window contribution to $z = 0, \ldots, 6$, rather than a range growing with the charges.

3.1.1 Flat space relation of the $AdS_5 \times S^5$ amplitude

The fact that the polynomial $w(\hat{s}, \check{s}, c_s; \Sigma)$ can be written in terms of a pre-polynomial (3.10), as in the case of the Virasoro-Shapiro amplitude studied in [29], suggests the following little game: In flat space the part of the $(\alpha')^3$ amplitude at one loop which accompanies $\log(-s)$ is s^4, and the tree-level $(\alpha')^7$ amplitude is proportional to $s^4 + t^4 + u^4$. Therefore, if we assemble

$$\mathcal{M}_7 \equiv w(\hat{s}, \check{s}, c_s; \Sigma) + w(\hat{t}, \check{t}, c_t; \Sigma) + w(\hat{u}, \check{u}, c_u; \Sigma) \tag{3.12}$$

we expect by construction that this quantity has the correct 10d flat space limit. Is there something more? Upon inspection, it turns out that $\mathcal{M}_7$ so constructed is actually the tree-level $(\alpha')^7$ amplitude constructed in [29, 30], up to an overall coefficient and a certain choice of ambiguities! This observation stands at the moment as a curiosity, though quite intriguing. It would be interesting to understand its origin further, and whether or not it generalises beyond this case. We will leave this for a future investigation.

3.2 Window dynamics

The $\log^1 U$ projection of the $(\alpha')^3$ correlator in the Window has the following one-loop OPE expansion, for fixed quantum numbers $\vec{\tau}$,

$$\sum_{\mathcal{K}\in\mathcal{R}_{\vec{\tau}}} C^{(0,0)}_{p_1p_2,\mathcal{K}} \left(\eta^{(1,3)}_{\mathcal{K}} C^{(1,0)}_{p_3p_4,\mathcal{K}} + \eta^{(1,0)}_{\mathcal{K}} C^{(1,3)}_{p_3p_4,\mathcal{K}}\right) + \sum_{\mathcal{K}\in\mathcal{R}_{\vec{\tau}}} C^{(0,0)}_{p_3p_4,\mathcal{K}} \left(\eta^{(1,3)}_{\mathcal{K}} C^{(1,0)}_{p_1p_2,\mathcal{K}} + \eta^{(1,0)}_{\mathcal{K}} C^{(1,3)}_{p_1p_2,\mathcal{K}}\right) \tag{3.13}$$

Recall that there is no Window when $p_1 + p_2 = p_3 + p_4$, see section 2.1.1.

Formula (3.13) contains two terms symmetric under $(p_1, p_2) \leftrightarrow (p_3, p_4)$. However, the two terms never contribute together. This is because the (free theory value of the) twist of the two-particle operator $\mathcal{K}$ is greater equal than $\max(p_1 + p_2, p_3 + p_4)$, and therefore in the Window either $C^{(0,0)}_{p_1p_2,\mathcal{K}} = 0$ or $C^{(0,0)}_{p_3p_4,\mathcal{K}} = 0$. It follows that when we compute the OPE predictions we only have access to one of the two terms at once. Nevertheless, we do expect a final formula for the coefficients which is at the same time symmetric under $(p_1, p_2) \leftrightarrow (p_3, p_4)$ and also analytic in the charges. We can imagine several scenarios of which the simplest one is perhaps the one where a single contribution, say for concreteness the one for $p_1 + p_2 < p_3 + p_4$, is such that upon a natural analytic continuation in the charges, it automatically vanishes when $p_1 + p_2 > p_3 + p_4$, and vice-versa. The final symmetric result will then be the sum of the two contributions. This simple scenario is indeed the one realised by the amplitude.

In the analysis that follows it will prove useful to move between a Mellin space formula of the type,

$$\mathcal{H}_{\vec{p}}(U, V, \tilde{U}, \tilde{V}) = \int d\hat{s} d\hat{t} \sum_{\check{s},\check{t}} U^{\hat{s}} V^{\hat{t}} \tilde{U}^{\check{s}} \tilde{V}^{\check{t}}\, \mathbf{\Gamma}\, \mathcal{M}_{\vec{p}}(\check{s}, \check{t}), \tag{3.14}$$

and a formula where we perform the sum over $\check{s}$ and $\check{t}$ and decompose into the basis of spherical harmonics $Y_{[aba]}$. We can start in the monomial basis, with an expression of the form

$$\int d\hat{s}d\hat{t}\; U^{\hat{s}}V^{\hat{t}}\;\Gamma_{\hat{s}}\Gamma_{\hat{t}}\Gamma_{\hat{u}}\;\sum_{\check{s},\check{t}}\frac{\tilde{U}^{\check{s}}\tilde{V}^{\check{t}}}{\Gamma_{\check{s}}\Gamma_{\check{t}}\Gamma_{\check{u}}}\;\mathcal{F}_{\vec{p}}(\hat{s},\check{s}), \tag{3.15}$$

valid for $\mathcal{F}=\{\mathcal{W}^{(\mathrm{AW})},\mathcal{R}^{(W)},\mathcal{B}^{(\mathrm{BW})}\}$, i.e. each one of the three functions that builds up our amplitude. The sum over $\check{s}$ and $\check{t}$ is finite, as we explained already. Then, upon decomposing into spherical harmonics we can alternatively write,

$$\int d\hat{s}d\hat{t}\; U^{\hat{s}}V^{\hat{t}}\;\Gamma_{\hat{s}}\Gamma_{\hat{t}}\Gamma_{\hat{u}}\;\sum_{[0b0]}Y_{[0b0]}\;\mathcal{F}_{\vec{p}}(\hat{s},b). \tag{3.16}$$

The notation $\mathcal{F}(\hat{s},\check{s})$ and $\mathcal{F}(\hat{s},b)$ will then refer to $\mathcal{F}$ as being written in the monomial basis and spherical harmonic basis, respectively.

As an example, we reproduce an interesting formula for the Above Window function

$$\begin{aligned} w_{\vec{p}}(\hat{s},b) = \frac{(\Sigma-1)_3}{\Gamma_{[0b0]}}\times\frac{1}{15\times 3!}\sum_{i=0}^{4}(-1)^i\binom{4}{i}\left(\hat{s}(\hat{s}+2-i)-\tfrac{b}{2}(\tfrac{b}{2}+2)\right)\\ \times\left(\hat{s}-\frac{b+2}{2}\right)_{3-i}\left(\hat{s}+\frac{b+2}{2}\right)_{3-i}\left(\frac{p_1+p_2}{2}-\hat{s}\right)_i\left(\frac{p_3+p_4}{2}-\hat{s}\right)_i \end{aligned} \tag{3.17}$$

where the analogous factor of $\Gamma_{[aba]}$ function was found in [28].[16]

With either representation, i.e. $\mathcal{F}(\hat{s},\check{s})$ and $\mathcal{F}(\hat{s},b)$, we can perform the $\hat{s}$ and $\hat{t}$ integration, to arrive at an expression in position space which we can then decompose into conformal blocks and match against OPE predictions.

3.2.1 Spherical harmonic basis

We wish to compute first $\mathcal{R}^{W}_{\vec{p}}(\hat{s},b)$ in the spherical harmonic basis. This is derived from matching the $\log^1 U$ projection of the amplitude, i.e.

$$\int d\hat{s}d\hat{t}\; U^{\hat{s}}V^{\hat{t}}\;\Gamma_{\hat{s}}\Gamma_{\hat{t}}\Gamma_{\hat{u}}\;\left(\sum_b Y_{[0b0]}\left[\mathcal{W}^{(\mathrm{AW})}_{\vec{p}}(\hat{s},b)+\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},b)\right]+\texttt{crossing}\right)\Bigg|_{\log^1 U} \tag{3.18}$$

with the prediction from the OPE in the Window. Schematically, we expect

$$\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},b)=\sum_n\frac{\#}{\hat{s}-n}\,;\qquad n\geq\min(\tfrac{p_1+p_2}{2},\tfrac{p_3+p_4}{2}). \tag{3.19}$$

The task will be to find the residues # for the various values of the twist in the Window Region, here labelled by values of $\tau=2n$. In (3.18) we have already restricted the summation to $a=0$, as a consequence of the truncation to spin zero valid at order $(\alpha')^3$.

---

[16]

$$\Gamma^{-1}_{[aba]}=\frac{(\Sigma-2)!b!(b+1)!(2+a+b)}{\Gamma\left[\pm\frac{p_1-p_2}{2}+\frac{b+2}{2}\right]\Gamma[\frac{p_1+p_2}{2}+\frac{b+2}{2}]\Gamma[\frac{p_1+p_2}{2}-\frac{b+2a+2}{2}]\Gamma[\pm\frac{p_3-p_4}{2}+\frac{b+2}{2}]\Gamma[\frac{p_3+p_4}{2}+\frac{b+2}{2}]\Gamma[\frac{p_3+p_4}{2}-\frac{b+2a+2}{2}]}$$

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In practice, say we focus on the $\mathtt{s}$ channel, we will pick double poles in $\hat{s}$ in the Window (this will select the $\log^1 U$ contribution), perform the $\hat{t}$ integral, and obtain a function in position space. This function contains up to $\log^1 V$ and $\log^0 V$ contributions (since there is no $\tilde{\psi}(-\mathbf{t})$ in the $\mathtt{s}$ channel). Note that for given power of U, both $\log^1 V$ and $\log^0 V$ come with a non trivial rational function of V. These contributions are analytic in the small $x, \bar{x}$ expansion, which is the expansion of the blocks we want to match.[17]

As in the case of $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle$ considered in [26], we find that *only five poles* are necessary to fit the OPE predictions.[18] In the sector $p_1+p_2 \leq p_3+p_4$, we find

$$\mathcal{R}^-_{\vec{p}}(\hat{s},b) = \sum_{n\geq \frac{p_1+p_2}{2}} \frac{\gamma_{\vec{p}}(n,b)}{\Gamma\left[5-n+\frac{p_1+p_2}{2}\right]\Gamma\left[\frac{p_3+p_4}{2}-n\right]}\frac{\mathtt{R}^-_{\vec{p}}(n,b)}{(\hat{s}-n)}, \tag{3.20}$$

where the minus superscript stands for $p_1+p_2 \leq p_3+p_4$, then $n = \frac{p_1+p_2}{2}, \frac{p_1+p_2}{2}+1, \ldots$ runs over half-twists, and

$$\gamma_{\vec{p}}(n,b) = \frac{b!(b+1)!(b+2)}{\Gamma\left[\pm\frac{p_{12}}{2}+\frac{b+2}{2}\right]\Gamma\left[\pm\frac{p_{43}}{2}+\frac{b+2}{2}\right]}\frac{\left(\frac{p_1+p_2}{2}-\frac{b+2}{2}\right)_3\left(\frac{p_1+p_2}{2}+\frac{b+2}{2}\right)_3}{\Gamma\left[n-\frac{b+2}{2}\right]\Gamma\left[n+\frac{b+2}{2}\right]} \tag{3.21}$$

and

$$\begin{aligned}\mathtt{R}^-_{\vec{p}}(n,b) = {}& \tfrac{B^3}{15} + \tfrac{(\Sigma^2+14\Sigma-(c_s)^2-10c_s+28-2(c_{tu})^2)B^2}{60} + \ldots \\ & -2n\left[\tfrac{(\Sigma+3)B^2}{15} + \tfrac{(\Sigma^2+7\Sigma-(c_s)^2-c_s+12-(c_{tu})^2(\Sigma+3))B}{30} + \ldots\right]; \qquad B := \tfrac{b(b+4)}{4}.\end{aligned} \tag{3.22}$$

The full expression for $\mathtt{R}^-$ can be found in the supplementary material, it is made of simple polynomials. Note that above we have used the notation $c_{tu} = c_t + c_u$.

The fact that only five poles in $\mathcal{R}^-$ are needed to fit the OPE data is reflected by the factor $1/\Gamma[5+\frac{p_1+p_2}{2}-n]$, which automatically truncates when $n \geq \frac{p_1+p_2}{2}+5$. It implies that beyond the fifth pole, the OPE predictions are fully captured already by the function $\mathcal{W}^{\rm AW}$! This is quite remarkable given that we are evaluating $\mathcal{W}^{\rm AW}$ in the Window. Let us also emphasise that $\mathtt{Res}_{\vec{p}}$ turns out to be only a linear polynomial in n. Considering that we are fitting five poles, this is a non-trivial consistency check of our formula. A closely related formula holds for $\mathcal{R}^+$ in the sector $p_1+p_2 \geq p_3+p_4$. Let us see why:

Above we considered the case $p_1+p_2 < p_3+p_4$ and found $\mathcal{R}^-$, but note that it automatically vanishes when $p_1+p_2 > p_3+p_4$. This is because of $1/\Gamma[\frac{p_3+p_4}{2}-n]$ and the fact that $n \geq \frac{p_1+p_2}{2}$ in the sum. Therefore, we are free to add both contributions in one formula and write the following symmetric and analytic expression

$$\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},b) = \mathcal{R}^-_{\vec{p}}(\hat{s},b) + \mathcal{R}^+_{\vec{p}}(\hat{s},b)\,; \qquad \mathcal{R}^+_{p_1p_2p_3p_4}(\hat{s},b) = \mathcal{R}^-_{p_4p_3p_2p_1}(\hat{s},b) \tag{3.23}$$

[17] It is actually convenient to resum the OPE predictions to exhibit the $\log V$ contribution explicitly, then match. This type of resummation was called a one-variable resummation in [10] see section 4.3.

[18] In $\langle ppqq\rangle$, fix the $su(4)$ channel to start with, and look at the residue of the first pole as function of p and q. The range of p,q is infinite and this gives a p,q dependent polynomial in the numerator and three Γ functions in the denominator. We then vary the $su(4)$ channel, introducing b. By studying in the same way the second pole, the third pole, etc..., we recognise $\Gamma[\frac{p_3+p_4}{2}-n]\Gamma[n-\frac{b+2a+2}{2}]\Gamma[n+\frac{b+2}{2}]$. Thus, even though we can access five values of the twist, i.e. five poles, we can single out $\Gamma[5+\frac{p_1+p_2}{2}-n]$ from looking at B^3 where $B := \frac{b(b+4)}{4}$.

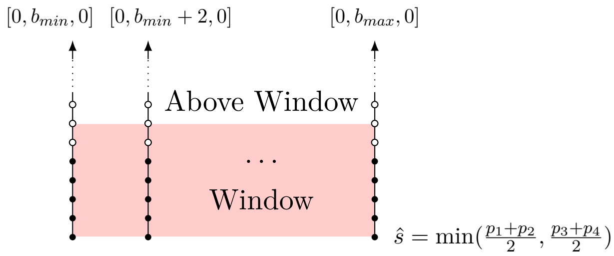

Figure 3. Pole structure of the remainder function in the basis of spherical harmonics. Black dots indicate a non-zero residue. Un-filled dots only receive contributions from $\mathcal{W}^{(\mathrm{AW})}$.

where $\mathcal{R}^+$ is related to $\mathcal{R}^-$ by swapping charges in the appropriate way. Using the c_s, c_t, c_u parametrisation, $\mathcal{R}^+_{\{c_s,c_t,c_u\}}(\hat{s}, b) = \mathcal{R}^-_{\{-c_s,c_t,-c_u\}}(\hat{s}, b)$.

Let us now come back to the splitting at the Window mentioned in section 2.2.1. There, we explained that degenerate correlators at tree level are those correlators with the same values of Σ and $|c_s|, |c_t|, |c_u|$ which therefore are proportional to each other at order $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$, because the bonus property is preserved. We can see from the explicit expressions for $\mathtt{Res}_{\vec{p}}(n, b)$ that at one-loop this is not the case anymore. Coming back to our guiding example of $\vec{p} = (3335)$ and $\vec{p} = (4424)$, we can see that

$$\mathrm{R}^-_{\vec{p}=3335}(n = 3, b = 2) = -\frac{336}{5}\,; \qquad \mathrm{R}^+_{\vec{p}=4424}(n = 3, b = 2) = -\frac{348}{5} \tag{3.24}$$

where the l.h.s. is simply the evaluation of (3.22), while the r.h.s. is obtained from the (3.22) upon $c_s \to -c_s$ and $c_u \to -c_u$. As promised, the one-loop OPE distinguishes these two correlators in the Window. Note instead that if we project the correlators onto the $\log^2 U$ discontinuity Above Window, the corresponding contributions are still degenerate.

Figure 3 illustrates the general structure of poles of the remainder function, as we have obtained it from OPE data. Notice now that $\mathcal{R}^{(W)}(\hat{s}, b)$ can be analytically continued Below Window. Quite nicely, the factor $1/\Gamma\left[n - \frac{b+2}{2}\right]$ ensures vanishing at the unitarity bound, thus giving the correct physical behaviour not just in the Window, where the function was fitted, but also Below Window! We infer that the remainder function can be understood more properly to descend onto the range of twists above the unitarity bound, which means we are free to start the sum in (3.20) from $n \geq \frac{b+4}{2}$. At this point it is clear that the pole structure of $\mathcal{R}^{(W)}$ will be naturally labelled by bold font variables, in complete analogy with the way $\mathcal{W}^{(\mathrm{AW})}$ descends in- and below- Window. Figure 4 illustrates the poles from the latter viewpoint.

To see more clearly the structure of poles in $\mathbf{s}$, from the Window down to the Below Window, let us point out that when we look at the $[0b_{\max}0]$ channel, the locus $\mathbf{s} = 0$ pinpoints the bottom of the window, below there is only the unitarity bound at $\mathbf{s} + 1 = 0$. Lowering $b < b_{\max}$, and looking at the $[0b0]$ channels, the same locus $\mathbf{s} = 0$ enters the below window region depicted in green.

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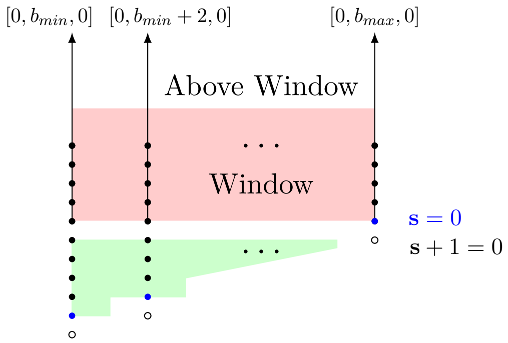

Figure 4. Pole structure of the remainder function after continuation Below Window.

3.2.2 Monomial basis, crossing, and sphere splitting

In this section we will turn the expression $\mathcal{R}^{(W)}(\hat{s}, b)$, written in the basis of harmonics $Y_{[0b0]}$, into the monomials basis $U^{\check{s}}V^{\check{t}}$, yielding the final Mellin amplitude corresponding to the Window region. The result right away is

$$\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s}) = \sum_{z=0}^{6} \frac{1}{\mathbf{s}-z}\left[(\check{s}+\tfrac{p_1+p_2}{2}-z)_{z+1}\ r^{+}_{\vec{p};z}(\hat{s},\check{s}) + (\check{s}+\tfrac{p_3+p_4}{2}-z)_{z+1}\ r^{-}_{\vec{p};z}(\hat{s},\check{s})\right], \quad (3.25)$$

where the notation $r^{\pm}$ refers to the fact that $p_1+p_2=\Sigma+c_s$ and $p_3+p_4=\Sigma-c_s$. The polynomials $r^{\pm}$ are related to each other,

$$r^{+}_{\vec{p}} = r_{\{-c_s,-c_t,c_u\}}\,; \qquad r^{-}_{\vec{p}} = r_{\{+c_s,+c_t,c_u\}}, \quad (3.26)$$

and given in the supplementary material. There are seven of them, since $z=0,\ldots,6$. The first few are simple to write down, see the list in (3.30).

The result for $\mathcal{R}^{(W)}$ in (3.25) has two important properties, which are tied to the splitting at the Window described in the previous section. These properties go together and are 1) the split of $\mathbf{\Gamma}$, and 2) the dependence of r on the charges $\vec{p}$. Both these properties provide the way to distinguish between $p_1+p_2<p_3+p_4$ and $p_1+p_2>p_3+p_4$ at the level of the Mellin amplitude. All together this is the sphere splitting we mentioned in section 1.4.

In more details: Assume $\hat{s}$ belongs to the Window, then $\hat{s}=\min(\frac{p_1+p_2}{2},\frac{p_3+p_4}{2})+n$ for some positive integer, $n\geq 0$. Now, from $\mathbf{s}-z=0$ we obtain the value of $\check{s}$, through the relation $-z+\min(\frac{p_1+p_2}{2},\frac{p_3+p_4}{2})+\check{s}=-n$. Considering that

$$\frac{\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s})}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2}+\check{s}\right]} \rightarrow$$
$$\sum_{z=0}^{6}\frac{1}{\mathbf{s}-z}\left[\frac{r^{+}_{\vec{p};z}}{\Gamma\left[-z+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2}+\check{s}\right]}+\frac{r^{-}_{\vec{p};z}}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma[-z+\frac{p_3+p_4}{2}+\check{s}]}\right] \quad (3.27)$$

we see that only one of the two terms contributes, because

$$\begin{array}{cc} p_1+p_2>p_3+p_4 & p_1+p_2<p_3+p_4 \\ -z+\frac{p_3+p_4}{2}+\check{s}=-n & -z+\frac{p_1+p_2}{2}+\check{s}=-n \\ \text{only} \quad r^+ \quad \text{contributes} & \text{only} \quad r^- \quad \text{contributes} \end{array} \tag{3.28}$$

This splitting is analytic in the arguments of the Γ functions, and therefore the function $r^\pm$, will be polynomial, i.e. there are no absolute value discontinuities. When we change basis, the sum over z truncates to seven poles $z=0,\dots,6!$ and we find

$$r^+_{\vec{p}}=r_{\{-c_s,-c_t,c_u\}}\,; \qquad r^-_{\vec{p}}=r_{\{+c_s,+c_t,c_u\}}, \tag{3.29}$$

where r is given in the supplementary material. Here we will quote for illustration

$$\begin{aligned} r_{\vec{p};6} &= \frac{\left(\check{s}+\frac{\Sigma+c_s}{2}-1\right)\left(\check{s}+\frac{\Sigma+c_s}{2}\right)}{15}, \\ r_{\vec{p};5} &= \frac{\left(\check{s}+\frac{\Sigma+c_s}{2}\right)(-30+11c_s+2c_s^2+9\Sigma+3c_s\Sigma+\Sigma^2+c_{tu}^2+30\check{s}-2\check{s}\Sigma-12\check{s}^2)}{30}, \\ r_{\vec{p};4} &= \frac{30\check{s}^4+10\check{s}^3(\Sigma-9)-5\check{s}(-30+11c_s+2c_s^2+11\Sigma+3c_s\Sigma+\Sigma^2+c_{tu}^2)+\dots}{30}, \\ &\vdots \end{aligned} \tag{3.30}$$

It is now clear how the sphere splitting on the Mellin amplitude achieves the splitting at the Window of section 2.2.1 coming from the OPE. As mentioned already, to break the bonus property of the $\mathbf{\Gamma}$ we need $r_{\{c_s,c_t,c_u\}}$ to depend not only on c_s, but generically on all c_s, c_t, c_u. Considering (3.30) for the s-channel, this indeed has non trivial dependence on c_t and c_u (actually here only on $c_{tu}=c_t+c_u$)[19] and therefore breaks the degeneracy of correlators $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(1,3)}$.

Let us comment further on crossing symmetry, verifying that (3.29) is consistent with crossing, and checking additional symmetries of r. By starting from the following (subset of crossing) relations,

$$\mathcal{R}^{(W)}_{p_1,p_2,p_3,p_4}(\hat{s},\hat{t},\check{s},\check{t})=\mathcal{R}^{(W)}_{p_2,p_1,p_4,p_3}(\hat{s},\hat{t},\check{s},\check{t})=\mathcal{R}^{(W)}_{c_s,-c_t,-c_u}(\hat{s},\hat{t},\check{s},\check{t}) \tag{3.31}$$

$$=\mathcal{R}^{(W)}_{p_4,p_3,p_2,p_1}(\hat{s},\hat{t},\check{s},\check{t})=\mathcal{R}^{(W)}_{-c_s,c_t,-c_u}(\hat{s},\hat{t},\check{s},\check{t}) \tag{3.32}$$

$$=\mathcal{R}^{(W)}_{p_3,p_4,p_1,p_2}(\hat{s},\hat{t},\check{t},\check{s})=\mathcal{R}^{(W)}_{-c_s,-c_t,c_u}(\hat{s},\hat{t},\check{s},\check{t}) \tag{3.33}$$

we find that $r^\pm$ are related to each other, and in fact are given in terms of a single function,

$$r^+_{\vec{p}}=r_{\{-c_s,-c_t,c_u\}}\,; \qquad r^-_{\vec{p}}=r_{\{+c_s,+c_t,c_u\}}. \tag{3.34}$$

Moreover, crossing shows that r has the following residual symmetry[20]

$$r_{\{c_s,c_t,c_u\}}=r_{\{c_s,-c_t,-c_u\}}\,; \qquad r_{\{c_s,c_t,c_u\}}=r_{\{c_s,c_u,c_t\}}. \tag{3.35}$$

[19] Otherwise crossing would imply invariance under $c_s\leftrightarrow -c_s$.

[20] The second property on the l.h.s. does not follow from (3.31)–(3.33). It has to do with exchanging $c_t\leftrightarrow c_u$ and it comes from imposing (1.14) etc., on the full Mellin amplitude

$$\mathcal{R}^{(W)}_{c_s,c_t,c_u}(\hat{s},\check{s})+\mathcal{R}^{(W)}_{c_t,c_u,c_s}(\hat{t},\check{t})+\mathcal{R}^{(W)}_{c_u,c_s,c_t}(\hat{u},\check{u}).$$

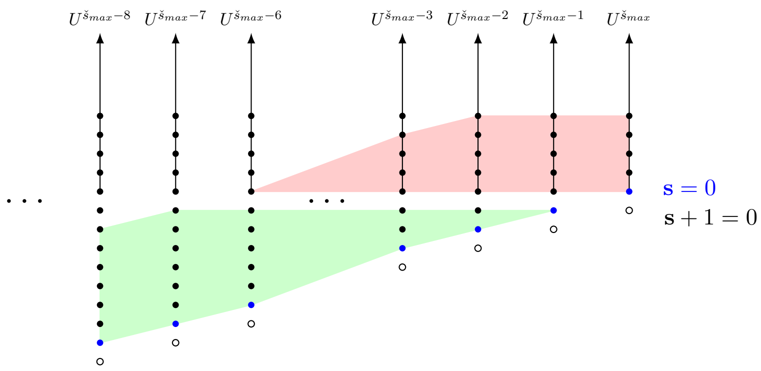

Figure 5. The pole structure of $\mathcal{R}^{(W)}$ in the monomial basis. Red: contributions lying in the window. Green: contributions lying in the Below window.

This is in fact what happens when we look at the explicit expressions of $r^{\pm}_{\vec{p};z}(\hat{s},\check{s})$.

Two final comments.

1) The truncation in the number of poles can be seen to be in one-to-one correspondence with the bound in the degree of the non factorisable polynomial appearing in the numerator of $r_{\vec{p};z}$. Under this logic we cannot have a numerator past $r_{\vec{p};6}$.

2) Notice also that $r_{\vec{p};6}$ and $r_{\vec{p};5}$ vanish in the Window precisely because of the factors $(\check{s}+\frac{p_1+p_2}{2}-1)$ and $(\check{s}+\frac{p_1+p_2}{2})$, thus guaranteeing consistency with the five poles picture in figure 3. The new picture for the poles in the monomial basis is displayed in figure 5.

3.3 Below Window completion

So far we have focussed on the Window region. We also understood that $\mathcal{R}^{(W)}$ has an immediate analytic continuation to Below Window. Now we should ask whether we need or not an additional reminder function Below Window? The answer is affirmative and in fact we need a contribution $\mathcal{B}^{(\mathrm{BW})}$ of the following form,

$$\mathcal{B}^{(\mathrm{BW})}_{\vec{p}}(\hat{s},\check{s})=\sum_{z=0}^{2}\frac{\left(\check{s}+\frac{p_1+p_2}{2}-z\right)_{z+1}\left(\check{s}+\frac{p_3+p_4}{2}-z\right)_{z+1}}{\mathbf{s}-z}\,\mathscr{b}_{\vec{p};z}(\hat{s},\check{s})\,. \tag{3.36}$$

Then, together, the contribution from $\mathcal{B}^{(\mathrm{BW})}$ and the contributions from both $\mathcal{W}^{(\mathrm{AW})}$ and $\mathcal{R}^{(W)}$ Below Window will reproduce the correct OPE prediction.

We find that only poles $z=0,1,2$ receive a Below Window completion. Their explicit form is

$$\begin{aligned}\mathscr{b}_{\vec{p};2}&=\tfrac{4(\check{s}+\Sigma-1)}{15}\,,\\ \mathscr{b}_{\vec{p};1}&=\tfrac{(1-2\check{s}-\Sigma)(4\check{s}^2-4\check{s}(1+2\Sigma)+16-c_s^2+4\Sigma-11\Sigma^2)}{15}\,,\\ \mathscr{b}_{\vec{p};0}&=\tfrac{16\check{s}^5-64\check{s}^4\Sigma-8\check{s}^3(c_s^2-15\Sigma^2+4)+16\check{s}^2\Sigma(c_s^2+26\Sigma^2-29)+\ldots}{60}\,,\end{aligned} \tag{3.37}$$

and can be read in the supplementary material attached to the arXiv.

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Again the truncation to three poles is in one-to-one correspondence with the bound on the degree of the polynomial. It is interesting to note the below window region does not exhibit the sphere splitting. For example in the $\mathtt{s}$-channel these functions depend just on c_s^2 and thus are symmetric under $p_1 + p_2 \leftrightarrow p_3 + p_4$.

With the knowledge of $\mathcal{W}^{(\mathrm{AW})}$, $\mathcal{R}^{(W)}$ and $\mathcal{B}^{(\mathrm{BW})}$ the bootstrap program for $\mathcal{M}^{(2,3)}$ is completed, up to the following ambiguities:

Ambiguities. The ambiguities we can add to the one loop function are tree-like contributions of the form of contributions to the Virasoro-Shapiro tree amplitude and do not spoil the one-loop OPE predictions in- and below- Window. For the case of $\mathcal{M}^{(2,3)}$ we can add the same as the functions corresponding to $\mathcal{M}^{(1,n=3,5,6)}$. The case $\mathcal{M}^{(1,n=7)}$ is simple to exclude since it will contribute to the flat space limit, and there is no such a contribution in flat space. Note that by construction the ambiguities listed above will contribute to the $\log^1(U)$ discontinuity, Above Window, rather than in the Window, and for the analytic part they will contribute, in the Window, rather than Below Window. In other words, they lie on top of what we fixed by the OPE data at one-loop, and therefore contribute with a free coefficient, as far as the bootstrap program is concerned.

3.4 Large p limit and the flat space amplitude

In this section we will compute the so called "large p limit" of the one-loop amplitude found in the previous section, and we will compare it with the type IIB flat space *ten-dimensional* scattering amplitude of [41]. The large p limit, introduced in [22], generalises the usual flat space limit a la Penedones [21] to include the dependence on the sphere. The idea stems from the observation that the regime $p_i \gg 1$ localises the interacting correlator on a saddle point where the Mellin variables are large as the p_i.[21] The saddle point then has the structure of a prefactor coming from $\mathbf{\Gamma}$, and for the rest, it is an evaluation formula for the Mellin amplitude. In particular the $\{\hat{s}, \hat{t}, \check{s}, \check{t}\}$ Mellin variables arrange themselves into the combinations $\mathbf{s}, \mathbf{t}$ with $\mathbf{s} + \mathbf{t} + \mathbf{u} = 0$, and they are in one-to-one correspondence with the ten-dimensional Mandelstam invariants. It follows that in the limit, the Mellin amplitude becomes the type IIB flat space scattering amplitude of variables $\mathbf{s}, \mathbf{t}$.

For what concerns the tree level amplitudes, taking the large p limit is quite a simple process, since these are rational/polynomials, and the result is essentially guaranteed [29]. For what concerns $\mathcal{M}^{(2,3)}$, on the one hand the flat space amplitude is quite a simple amplitude, but on the other hand $\mathcal{M}^{(2,3)}$ involves various pieces and therefore it will be quite an interesting computation to demonstrate the final match.

Let us focus on the $\mathtt{s}$-channel, and introduce for convenience $\tilde{\mathbf{s}} = 2\check{s} + \Sigma$. To take the limit, we simply need to consider

$$\{\mathbf{s}, \tilde{\mathbf{s}}, c_s\} \ \to p\,\{\mathbf{s}, \tilde{\mathbf{s}}, c_s\}\,; \quad \Sigma \to p\Sigma, \,; \qquad p \to \infty \tag{3.38}$$

[21] In particular, since the $\sum_{\check{s},\check{t}}$ becomes an infinite sum in the limit, it is also convenient to turn it into a contour integral [22].

on the various entries in the Mellin amplitude, which we know explicitly as function of $\vec{p}$. Since $\boldsymbol{\Gamma}$ plays a role on its own in the large p saddle point, we have to factor it out for each strata of the amplitude.

The limit on $\mathcal{W}^{\mathrm{AW}}_{\vec{p}}$ is straightforward and reads

$$\lim_{p\to\infty}\frac{\mathcal{W}^{\mathrm{AW}}_{\vec{p}}}{p^8}=\lim_{p\to\infty}\frac{w(\hat{s},\check{s},c_s;\Sigma)\psi(-\mathbf{s})}{p^8}=\frac{\zeta_3}{90}\Sigma^4\mathbf{s}^4\log(-\mathbf{s}) \tag{3.39}$$

This is already the corresponding IIB flat space result [41]. We conclude that for the large p limit to hold the contributions from $\mathcal{R}^{(W)}$ in (3.25) and $\mathcal{B}^{(\mathrm{BW})}$ in (3.36) must give vanishing contributions in the limit.

One can see that the contribution from $\mathcal{B}^{(\mathrm{BW})}$ are subleading w.r.t. to p^8 in the limit. However, each one of the z poles from $\mathcal{R}^{(W)}$ comes with a contribution which is not subleading! In fact, we find the following contributions:

$$\begin{array}{c|c} \text{pole} & \lim\limits_{p\to\infty}\dfrac{\text{contribution from pole}\,(z)}{p^8} \\ \hline \\ z=6 & \frac{(\tilde{\mathbf{s}}-c_s)^2(\tilde{\mathbf{s}}-c_s-2\Sigma)^3((\tilde{\mathbf{s}}-\Sigma-c_{tu})(\tilde{\mathbf{s}}-\Sigma+c_{tu}))^2}{7680\mathbf{s}} \\ \vdots & \vdots \\ z=0 & \frac{(\tilde{\mathbf{s}}-c_s)^7(\tilde{\mathbf{s}}+c_s)^2}{7680\mathbf{s}} \end{array} \tag{3.40}$$

(The full computation can be found in the supplementary material.)

By inspection of (3.40), we find that the leading term of each pole z is as leading as (3.39): It has degree 8 in p. However, when *summing* over all contributions in the second column, the result vanishes! This cancellation is quite remarkable, given that non trivial functions of $\tilde{\mathbf{s}}$ and charges p_i are involved.

We have found that the large p limit of the one-loop amplitude $\mathcal{M}^{(2,3)}$, and the 10d flat space IIB S-matrix, match perfectly. Note that the flat space a la Penedones would instead give a rather trivial result, because the $\sum_{\check{s},\check{t}}$ would not play an immediate role. In fact, only $\mathcal{W}^{\mathrm{AW}}_{\vec{p}}$ projected on the various sphere harmonics would be leading, while the rest of the amplitude would be subleading by construction. This was already observed in [26]. As a byproduct of our analysis here we have given a non-trivial confirmation of the geometric picture associated with the large p limit, as described in [22], and shown the self-consistency of the one-loop bootstrap program.

4 Conclusions

In this paper we studied one-loop stringy corrections to the Mellin amplitude of four single-particle operators $\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$. We started from the $AdS_5\times S^5$ Mellin representation in terms of variables conjugated to cross ratios, $\hat{s},\check{s},\hat{t},\check{t}$, the Mellin amplitude $\mathcal{M}_{\vec{p}}$, and the following kernel of Gamma function,

$$\boldsymbol{\Gamma}=\frac{\Gamma\left[\frac{p_1+p_2}{2}-\hat{s}\right]\Gamma\left[\frac{p_3+p_4}{2}-\hat{s}\right]}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2}+\check{s}\right]}\Gamma_{\mathtt{t}}\Gamma_{\mathtt{u}}, \tag{4.1}$$

see discussion around (1.13). Then, we gave explicit formulas for the leading order $(\alpha')^3$ amplitude at one-loop, in the form,

$$\mathcal{M}^{(2,3)}_{\vec{p}} = \Big[\mathcal{W}^{(\mathrm{AW})}_{\vec{p}}(\hat{s},\check{s}) + \mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s}) + \mathcal{B}^{(\mathrm{BW})}_{\vec{p}}(\hat{s},\check{s})\Big] + \texttt{crossing}\,. \tag{4.2}$$

The various functions $\mathcal{W}^{(\mathrm{AW})}, \mathcal{R}^{(W)}, \mathcal{B}^{(\mathrm{BW})}$, introduced around (1.22), are described in full details in sections 3.1, 3.2, and 3.3, respectively. Each one is bootstrapped from the OPE, and labelled by a corresponding region of twists for the exchange of long two-particle operators. We called these regions Above Window, Window and Below Window, following [10], where the same classification was used to bootstrap the one-loop supergravity amplitude, mainly in position space. The supergravity amplitude is still a rather complicated function, due to the fact that the OPE has support for all spins. The α' corrections have instead finite spin supports and therefore are simpler to deal with. In fact we were able to find explicit formula for the whole (4.2). We believe that the lessons from the study of α' amplitudes are general, and in fact best expressed in Mellin space.

The advantage of using Mellin space stems from the observation that the whole structure of poles in Mellin space can be put in correspondence with the OPE, a result due to Mack [20], which here we have upgraded to $AdS_5 \times S^5$. The pole structure takes into account both $\mathbf{\Gamma}$ and $\mathcal{M}$. The poles of $\mathcal{M}$ at tree level were shown to be captured by bold-font variables [22], i.e. poles given by equations of the form $\mathbf{s}+1=0$, where $\mathbf{s}=\hat{s}+\check{s}$, similarly for $\mathbf{t}$, or $\mathbf{u}$. Quite nicely, this way of parametrising the poles works at one loop, with two important modifications: Poles of $\mathcal{M}$ are now of the form $\psi(-\mathbf{s})$ or $\mathbf{s}-z=0$ for $z=0,\ldots,6$, and the residues, compared to tree level, have generic dependence on the charges $\vec{p}$.

Considering the various contributions to $\mathcal{M}^{(2,3)}$ in (4.2) we have found:

(1) The double logarithmic discontinuity of the one-loop amplitude comes from $\mathcal{W}^{(\mathrm{AW})}$ and is fixed by OPE data Above Window. It takes the form of $\psi(-\mathbf{s})$ times a non-trivial polynomial in the Mellin variables, and for many aspects fits into the discussion of the tree level Virasoro-Shapiro amplitude, as given in [29]. In particular, it can be given as $\Delta^{(8)}$ on a preamplitude. This expectation will be true to all orders in α' and we exemplified the case of $(\alpha')^3$ and $(\alpha')^5$ in the supplementary material.

(2) The structure of $\mathcal{R}^{(W)}$ is the main novelty of the one-loop amplitude. This is determined by OPE data in the Window. This OPE data is the one responsible for lifting a bonus property of the supergravity amplitude, i.e. the fact that certain correlators are equal to each other. Such a degeneracy of amplitudes follows from the special (crossing symmetric) form of $\mathbf{\Gamma}$, and the peculiar charge dependence of the Mellin amplitudes, i.e. no dependence at all in supergravity, and $(\frac{p_1+p_2+p_3+p_4}{2}-1)_3$ at order $(\alpha')^3$, see section 1.3 for further explanations.

The first crucial result is that $\mathcal{R}^{(W)}$ takes the form,

$$\mathcal{R}^{(W)}_{\vec{p}}(\hat{s},\check{s}) = \sum_{z=0}^{6} \frac{1}{\mathbf{s}-z}\Big[(\check{s}+\tfrac{p_1+p_2}{2}-z)_{z+1}\, r^{+}_{\vec{p};z}(\hat{s},\check{s})+(\check{s}+\tfrac{p_3+p_4}{2}-z)_{z+1}\, r^{-}_{\vec{p};z}(\hat{s},\check{s})\Big]\,, \tag{4.3}$$

and comes with two separate contributions, singled out by the z-dependent Pochhammers. This implies that when we look at $\mathcal{R}^{(W)}$ together with the $\mathbf{\Gamma}$ functions, the total amplitude undergoes the following split,

$$\frac{\mathcal{R}^{(W)}_{\vec{p}}}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2}+\check{s}\right]}$$

$$=\frac{1}{\mathbf{s}-z}\left[\frac{r^{+}_{\vec{p};z}}{\Gamma\left[-z+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[1+\frac{p_3+p_4}{2}+\check{s}\right]}+\frac{r^{-}_{\vec{p};z}}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma\left[-z+\frac{p_3+p_4}{2}+\check{s}\right]}\right] \tag{4.4}$$

The polynomials $r^{\pm}$ are related by crossing, $r^{+}_{\vec{p}}=r_{\{-c_s,-c_t,c_u\}}$, $r^{-}_{\vec{p}}=r_{\{+c_s,+c_t,c_u\}}$, and crucially have generic charge dependence. This structure, which all together we called "sphere splitting", follows from the OPE.[22] Our result for $\mathcal{R}^{(W)}$ shows neatly how the OPE structure translates into a structure in Mellin space.

The second crucial result is the use of bold font variables to parametrise the poles of $\mathcal{M}$. For $\mathcal{W}^{(\mathrm{AW})}$, this implies that $\mathcal{W}^{(\mathrm{AW})}$ not only contributes to the OPE Above Window, where it was bootstrapped, but cascades to Window and Below Window regions. Because of this, the OPE predictions in the Window pick the contributions of both $\mathcal{W}^{(\mathrm{AW})}$ and $\mathcal{R}^{(W)}$. In this sense, $\mathcal{R}^{(W)}$ is a remainder function, but precisely for this reason, the sum over z in (4.3) truncates to finitely many poles, $z=0,\dots 6$, rather than depending on the external charges. Quite remarkably, the result for $\mathcal{R}^{(W)}$ can itself be continued Below Window. Thus, with the same logic,

(3) The function $\mathcal{B}^{(\mathrm{BW})}$ is

$$\mathcal{B}^{(\mathrm{BW})}_{\vec{p}}(\hat{s},\check{s})=\sum_{z\geq 0}\frac{\left(\check{s}+\frac{p_1+p_2}{2}-z\right)_{z+1}\left(\check{s}+\frac{p_3+p_4}{2}-z\right)_{z+1}}{\mathbf{s}-z}\,b_{\vec{p};z}(\hat{s},\check{s})\,, \tag{4.5}$$

and is itself a remainder function for $\mathcal{W}^{(\mathrm{AW})}$ and $\mathcal{R}^{(W)}$ in the Below Window region. In sum, the analytic properties of all these functions are quite spectacular: they properly continue from the Above Window down to the Below Window region, and correctly switch off outside the physical range of relevance.

Finally, we studied the 10d flat space limit, using the large p limit of [22]. The consistency with the flat space amplitude of IIB supergravity is again quite remarkable. The limit on $\mathcal{W}^{(\mathrm{AW})}$ is already giving the flat amplitude, and naively, each pole in z from $\mathcal{R}^{(W)}$ adds a non vanishing contribution, non trivial in both $\check{s}$ and the charges. However, upon summing over z all these extra contributions correctly cancel out!

Let us conclude with an outlook. Our main focus has been understanding the structure of the amplitude and the way Mellin space realises the various features of the OPE predictions at one-loop. We focused on the $(\alpha')^3$ contribution, but we believe that the logic behind the

[22] From the OPE viewpoint this is also true at one-loop in supergravity [10].

construction of $\mathcal{M}^{(2,3)}$ is valid to all orders in $(\alpha')^{n+3}$. In particular the sphere splitting is generic, and the number of poles in z increases with n but stays finite! It is only a matter of computational effort to fix the various residues in the Window and Below Window. The one-loop bootstrap program in $AdS_5 \times S^5$ is thus understood at all orders in α', but for the usual ambiguities, which needs additional input to be fixed. It would be interesting to find these extra constraints, either from localisation [17–19], or from sum rules [31, 32]. In this case we would start from a non-perturbative Mellin amplitude [42, 43], and consider the supergravity expansion in search for constraints.

It would be also interesting to understand the "sphere splitting" from a diagrammatic point of view. Perhaps a 10d master amplitude can be found which undergoes the "sphere splitting" onto $\mathcal{R}^{(W)}$ in a natural way. This is partially suggested by the remarkable cancellations that take place when we tested the large p flat space limit in section 3.4, and the fact that poles are parametrised by bold font variables. Perhaps this point of view would give insight on the way to arrange the one-loop supergravity result in a simple form.

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A Useful facts about blocks

In order to make operative the OPE predictions we have to expand the correlator in superconformal blocks. Superconformal blocks depend on the type of multiplet exchanged in the OPE, i.e. protected or long. In a 4pt function protected multiplets all come from free theory contributions, while unprotected (or long) multiplets come from both free theory and interacting contributions. In general we will write,

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle = \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle^{\text{prot.}}_{\text{free}} + \underbrace{\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle^{\text{long}}_{\text{free}} + \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle^{\text{long}}_{\text{int}}}_{\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{long}}}. \tag{A.1}$$

Importantly, some multiplets at the unitarity bound, which therefore are semishort in the free theory, recombine in the interacting theory to become long. This is explained in great detail in [36]. In this paper we have been concerned with long contributions, and long $\mathcal{N}=4$ superblocks, denote by $\mathbb{L}$, are quite simple to deal with, since they have a factorised form (and can be made independent of a parameter called γ in [36] and [37]).

Let us denote the quantum numbers of the exchanged multiplet by $\vec{\tau}=(\tau,l,a,b)$ for a superconformal primary of twist $\tau=\Delta-l$, spin l, and $su(4)$ representation with Dynkin labels $[aba]$. Then,

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle^{\text{long}} = \sum_{\vec{\tau}} c^{\vec{p}}_{\vec{\tau}}\, \mathbb{L}^{\vec{p}}_{\vec{\tau}}\,, \tag{A.2}$$

holds non-perturbatively. The precise form of $\mathbb{L}$ is

$$\mathbb{L}_{\vec{\tau}}^{\vec{p}} = g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{14}}{g_{24}}\right)^{\frac{p_1-p_2}{2}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{p_4-p_3}{2}} \times \prod_{i,j}(x_i - y_j) \times \mathcal{B}_{\tau,l}^{\vec{p}}(x,\bar{x}) Y_{[aba]}^{\vec{p}}(y,\bar{y}) \,. \tag{A.3}$$

The conformal, $\mathcal{B}$, and internal, Y, factors of $\mathbb{L}$ are given by

$$\begin{aligned} \mathcal{B}_{\tau,l}^{\vec{p}}(x,\bar{x}) &= \frac{F_{l+\tau/2+2}^{(\alpha,\beta)}(x) F_{\tau/2+1}^{(\alpha,\beta)}(\bar{x}) - F_{l+\tau/2+2}^{(\alpha,\beta)}(\bar{x}) F_{\tau/2+1}^{(\alpha,\beta)}(x)}{x-\bar{x}} \,, \\ Y_{[aba]}^{\vec{p}}(y,\bar{y}) &= \frac{F_{-b/2}^{(-\alpha,-\beta)}(y) F_{-b/2-a-1}^{(-\alpha,-\beta)}(\bar{y}) - F_{-b/2}^{(-\alpha,-\beta)}(\bar{y}) F_{-b/2-a-1}^{(-\alpha,-\beta)}(y)}{y-\bar{y}} \,, \end{aligned} \tag{A.4}$$

where

$$F_{\rho}^{(\alpha,\beta)}(x) = x^{\rho-1} {}_2F_1(\rho+\alpha, \rho+\beta, 2\rho; x) \,. \tag{A.5}$$

and we have introduced the notation $\alpha = \frac{p_2-p_1}{2}$ and $\beta = \frac{p_3-p_4}{2}$.

In the main text, we considered the non perturbative decomposition (A.1), and we expanded in $1/N$ and α'. Since $\mathcal{N}=4$ SYM in this regime also has a free theory limit, we referred to τ to the free theory twist, and split the dimension into free plus anomalous.

In the block decomposition we could assume that $p_1 \geq p_2$ and $p_4 \geq p_3$. This can be done without loss of generality since the OPE does not change if we exchange $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$. In fact, both crossing transformations behave nicely w.r.t. the blocks. Let us make this statement more precise: From the relation (A.2) consider applying crossing of $1 \leftrightarrow 2$, then

$$\begin{aligned} \langle \mathcal{O}_{p_2}(x_1)\mathcal{O}_{p_1}(x_2)\mathcal{O}_{p_3}(x_3)\mathcal{O}_{p_4}(x_4)\rangle^{\text{long}} &= \textstyle\sum_{\vec{\tau}} c_{\vec{\tau}}^{p_1p_2p_3p_4} \times \mathbb{L}_{\vec{\tau}}^{p_1p_2p_3p_4}\left(\frac{x}{x-1}, \frac{y}{y-1}\right)\Big|_{g_{1i}\leftrightarrow g_{2i}} \\ \| \qquad & \\ \textstyle\sum_{\vec{\tau}} c_{\vec{\tau}}^{p_2p_1p_3p_4} \times \mathbb{L}_{\vec{\tau}}^{p_2p_1p_3p_4}(x,y) & \end{aligned} \tag{A.6}$$

Now, using (one of) the Pfaff identity ${}_2F_1(a,b,c;\frac{x}{x-1}) = (1-x)^b {}_2F_1(c-a,b,c;x)$, the hypergeometric functions entering the long blocks can be shown to match up to signs, and we obtain the relation between $c^{p_2p_1p_3p_4}$ and $c^{p_1p_2p_3p_4}$, namely $c_{\vec{\tau}}^{p_2p_1p_3p_4} = (-1)^{l+a} c_{\vec{\tau}}^{p_1p_2p_3p_4}$ where we used that $\tau + b$ is always even. We can repeat the same argument with $3 \leftrightarrow 4$ by using (the other one of) the Pfaff identity ${}_2F_1(a,b,c;\frac{x}{x-1}) = (1-x)^a {}_2F_1(a,c-b,c;x)$. All together we find,

$$c_{\vec{\tau}}^{\vec{q}} = (-1)^{l+a} c_{\vec{\tau}}^{\vec{p}} \qquad \text{for } \vec{q} = (p_2,p_1,p_3,p_4) \text{ or } \vec{q} = (p_1,p_2,p_4,p_3) \,. \tag{A.7}$$

Of course, by composing the two,

$$c_{\vec{\tau}}^{\vec{q}} = c_{\vec{\tau}}^{\vec{p}} \qquad \text{for } \vec{q} = (p_4,p_3,p_2,p_1) \,. \tag{A.8}$$

This one is also obvious because the cross ratios do not change and the blocks are manifestly invariant under $p_1 \leftrightarrow p_4$ and $p_2 \leftrightarrow p_3$.

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From the above considerations, we expect that the CPW coefficients for $(a+l)$ even and the CPW coefficients for $(a+l)$ odd arise from two distinct analytic functions $c^{\pm}$, namely,

$$c_{\vec{\tau}}^{\vec{p}} = \begin{cases} c^{+}(\vec{p},\vec{\tau}) & (a+l) \text{ even,} \\ c^{-}(\vec{p},\vec{\tau}) & (a+l) \text{ odd.} \end{cases}$$

which respect the symmetry properties (A.7) for all quantum numbers $\vec{\tau}$ and charges $\vec{p}$.

B Combining the weight-two contribution

Note first that we can rewrite the result for $\mathcal{B}^{(\mathrm{BW})}$ using the same split used for $\mathcal{R}^{(W)}$. In this way we can write

$$\mathcal{W}^{(2)}_{\vec{p}}(\hat{s},\check{s}) = \sum_{z\geq 0} \frac{(\check{s}+p_{12}-z)_{z+1}\mathcal{P}^{+}_{\vec{p};z}(\hat{s},\check{s}) + (\check{s}+p_{34}-z)_{z+1}\mathcal{P}^{-}_{\vec{p};z}(\hat{s},\check{s})}{\mathbf{s}-z}\,, \tag{B.1}$$

where $\mathcal{W}^{(2)}_{\vec{p}}(\hat{s},\check{s}) := \mathcal{R}^{(W)}_{\vec{p}} + \mathcal{B}^{(W)}_{\vec{p}}$ and

$$\mathcal{P}^{+}_{\vec{p}} = \mathcal{P}_{\{-c_s,-c_t,c_u\}}\,; \qquad \mathcal{P}^{-}_{\vec{p}} = \mathcal{P}_{\{+c_s,+c_t,c_u\}}, \tag{B.2}$$

The list of functions $\mathcal{P}_{\vec{p};z}$ is given in the supplementary material. Of course only the poles at $z=0,1,2$ are modified by the addition of $\mathcal{B}^{(\mathrm{BW})}$ w.r.t. the contribution of $\mathcal{R}^{(W)}$.

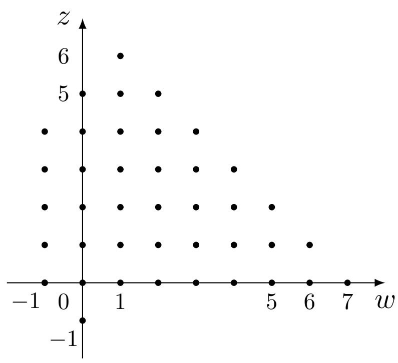
Certain analyticity in z can be made manifest for the $\mathcal{P}^{(z)}_{\vec{p}}$. Assuming we focus on $\mathcal{P}^{-}_{z}$ for concreteness, the idea is that the ratio $\mathcal{P}_z/\Gamma[1+\frac{p_1+p_2}{2}+\check{s}]\Gamma[-z+\frac{p_3+p_4}{2}+\check{s}]$ can be further decomposed by writing $\mathcal{P}_z$ as a polynomial in Pochhammers. This is initially suggested by the special form of $\mathcal{P}_6$ and $\mathcal{P}_5$ which have a partial factorised form of this sort, i.e.

$$\begin{aligned} \mathcal{P}_{\vec{p};6} &= \frac{\left(\check{s}+\frac{\Sigma+c_s}{2}-1\right)\left(\check{s}+\frac{\Sigma+c_s}{2}\right)}{15}\,, \\ \mathcal{P}_{\vec{p};5} &= \frac{\left(\check{s}+\frac{\Sigma+c_s}{2}\right)(-30+11c_s+2c_s^2+9\Sigma+3c_s\Sigma+\Sigma^2+c_{tu}^2+30\check{s}-2\check{s}\Sigma-12\check{s}^2)}{30}\,, \end{aligned} \tag{B.3}$$

In this way we can absorb the $\check{s}$ dependence into the z-independent Γ, and get a structure of the form,

$$\frac{\mathcal{P}_z(\check{s},c_s,c_t,c_u,\Sigma)}{\Gamma\left[1+\frac{p_1+p_2}{2}+\check{s}\right]\Gamma[-z+\frac{p_3+p_4}{2}+\check{s}]} = \sum_{w\geq -1} \frac{\mathcal{P}_{z,w}(c_s,c_t,c_u,\Sigma)}{\Gamma[-z+\frac{p_3+p_4}{2}+\check{s}]\Gamma[-w+\frac{p_1+p_2}{2}+\check{s}]} \tag{B.4}$$

for each value of z. By doing so we find a lattice of points (w,z) of the form



(B.5)

For example, the horizontal axis at $z=0$ corresponds to a polynomial of degree eight in $\tilde{\mathbf{s}}$, and therefore there are nine bullet points, the first of which counts for a degree zero contributions, going with $w=-1$, and the last one for a degree eight contribution, going with $w=7$.

Rearranging the polynomials on the $-45°$ diagonals, we find simple analytic expressions. For illustration,

$$\begin{aligned}
\mathcal{P}_{z,w} &= \tfrac{1}{15}\,\mathtt{Bin}\big[{}^{6}_{z}\big] &&;\quad z+w=7\\
\mathcal{P}_{z,w} &= \tfrac{(6c_s+5\Sigma+3(2z-11))}{15}\,\mathtt{Bin}\big[{}^{5}_{z}\big] &&;\quad z+w=6\\
\mathcal{P}_{z,w} &= \tfrac{-(c_{tu})^2+(c_s)^2(31-6z)+\ldots}{30}\,\mathtt{Bin}\big[{}^{5}_{z}\big] &&;\quad z+w=5\\
&\ \vdots
\end{aligned} \tag{B.6}$$

and so on so forth. The pattern of the binomial $\mathtt{Bin}$ persists, and becomes $\mathtt{Bin}\big[{}^{-1+\#\bullet}_{z}\big]$ where $\#$ counts the number of $\bullet$ on the various diagonals in (B.5). This binomial is always singled out by the fact that some of the top degree terms in $\mathcal{P}_{z,w}$, as function of z, contribute with a constant times such binomial. In the above formulae, these top degree terms are $(c_{tu})^0\otimes\{c_s,\Sigma\}$ when $z+w=6$, and $(c_{tu})^2$ when $z+w=5$.

The analytic structure in (3.22) suggests from the very beginning a sort of analyticity of $\mathcal{P}_z$ in z, and the rearrangement into $\mathcal{P}_{z,w}$ in (B.6), is an example. On the other hand, had we started with a general parametrisation of $\mathcal{P}_z$ as a polynomial in $\check{s}, c_s, c_t, c_u, \Sigma$, and fitted data in- and below- Window, we would have found some free coefficients left. There is indeed a subtlety with a $\mathcal{P}_z$ so constructed, which is the following: Pieces of $\mathcal{P}_z$ cancel out in the sum

$$\frac{\left[\dfrac{\Gamma\left[\check{s}+\frac{p_1+p_2}{2}+1\right]\mathcal{P}^+_z}{\Gamma\left[\check{s}+\frac{p_1+p_2}{2}-z\right]}+\dfrac{\Gamma\left[\check{s}+\frac{p_3+p_4}{2}+1\right]\mathcal{P}^-_z}{\Gamma\left[\check{s}+\frac{p_3+p_4}{2}-z\right]}\right]}{\Gamma\left[\check{s}+\frac{p_1+p_2}{2}+1\right]\Gamma\left[\check{s}+\frac{p_3+p_4}{2}+1\right]}. \tag{B.7}$$

Using that $\Gamma[X+1]/\Gamma[X-z]=(X-z)_{z+1}$ is polynomial in X, and expanding the whole numerator in (B.7), it is simple to see that contributions of the form $c_s^{2\mathbb{N}+1}f(\check{s},c_s^2,c_t^2,c_u^2,\Sigma)$, for any function f, cancel out. We haven't encountered this subtlety in our discussion above because the functions in $\mathcal{R}^{(W)}$, nicely enough, do not have it.

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