Folding Amplitudes into Form Factors: An Antipodal Duality

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We observe that the three-gluon form factor of the chiral part of the stress-tensor multiplet in planar $\mathcal{N} = 4$ super-Yang-Mills theory is dual to the six-gluon MHV amplitude on its parity-preserving surface. Up to a simple variable substitution, the map between these two quantities is given by the antipode operation defined on polylogarithms (as part of their Hopf algebra structure), which acts at symbol level by reversing the order of letters in each term. We provide evidence for this duality through seven loops.

Introduction

In the study of quantum field theory, we occasionally encounter dualities, or relations between seemingly unrelated quantities. One such example is the duality between scattering amplitudes and closed light-like polygonal Wilson loops in planar maximally supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) theory [1–7], and its extension to a triality relating both quantities to a particular kinematic limit of correlation functions of the stress tensor supermultiplet [8–12]. These types of relations provide us with valuable new perspectives on physical quantities, and at times reveal deep and novel types of mathematical structure. In this letter, we present a new weak-weak duality between the maximally-helicity-violating (MHV) three-gluon form factor of the chiral part of the stress tensor supermultiplet in planar $\mathcal{N} = 4$ SYM theory, and a kinematic limit of the six-gluon MHV amplitude in the same theory. This duality holds order-by-order in the 't Hooft coupling $g^2 = \frac{\lambda}{16\pi^2}$ [13].

A great deal is known about both the three-gluon form factor and the six-gluon amplitude. Their infrared structure can be understood to all orders in terms of the Bern-Dixon-Smirnov (BDS) ansatz [14, 15], and each is known to be dual to a polygonal Wilson loop (which, in the case of the form factor, is periodic) [1–7, 16–20]. Moreover, integrability techniques have been leveraged to develop an operator product expansion (OPE) around the near-collinear limit of each quantity [21–33], which has provided useful boundary data for bootstrap approaches, by means of which the amplitude has been computed through seven loops [34–42] and the form factor through eight loops [43, 44]. As will prove important below, both quantities are expressible in terms of multiple polylogarithms. This class of functions comes equipped with a coaction and an associated antipode (or coinverse) [45–51]; see [52] for a review.

In this letter, we show that the antipode relates the three-gluon form factor to the six-gluon amplitude on its parity-preserving surface, up to a simple mapping between their respective kinematic variables. While it is surprising for any direct relation between these quantities to exist, the fact that they are related by the antipode map—which has no clear physical interpretation—is doubly bizarre.[1]

In the remainder of this letter, we provide the full statement of this relation, and present evidence that supports it through seven loops. We also discuss how this duality is consistent with many of the known analytic features of the six-gluon amplitude and three-gluon form factor, and draw out the implications this relation has for the analytic properties of the form factor, and for bootstrapping these quantities at higher loops.

The Duality

Let us first define the specific quantities that enter the amplitude/form factor duality that we find. On the amplitude side, we consider the BDS-like and cosmically-normalized six-point MHV amplitude A_6 [40, 54], which is defined by dividing the full amplitude $\mathcal{A}_6^{\mathrm{MHV}}$ by the BDS-like ansatz $\mathcal{A}_6^{\mathrm{BDS\text{-}like}}$ [55, 56] and a transcendental function $\hat{\rho}$ [54], which is independent of the kinematics but can be perturbatively expanded in the coupling g^2:

$$\mathcal{A}_6^{\mathrm{MHV}} = \mathcal{A}_6^{\mathrm{BDS\text{-}like}} \times \hat{\rho} \times A_6 \,. \tag{1}$$

A_6 is a finite polylogarithmic function[2] of the three dual-conformally-invariant cross ratios

$$\hat{u} = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad \hat{v} = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad \hat{w} = \frac{s_{34}s_{61}}{s_{345}s_{234}}, \tag{2}$$

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[1] At two loops, a different relation between the three-gluon form factor and six-gluon amplitude was observed that did not involve the antipode [15]. However, that relation does not hold beyond two loops [43]. An antipodal relation between one-loop integrals and Aomoto polylogarithms was found in [53].

[2] In particular, A_6 is the function referred to as $\mathcal{E}_{\mathrm{cosmic}}$ in [54], which differs from the $\mathcal{E}$ function defined in [41] by a different choice for the function ρ, which we denote here by $\hat{\rho}$.
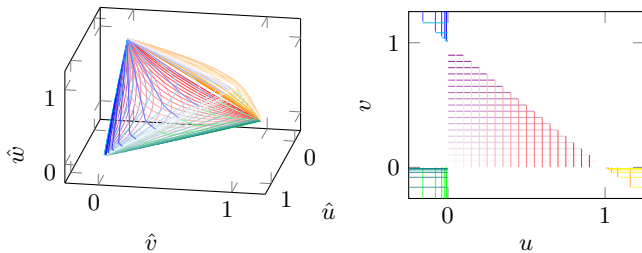
FIG. 1. Correspondence between various lines in the two-parameter three-point form factor space (right) and their images in the three-parameter six-point amplitude kinematic space (left) under the map (7)–(9). The white regions in the right plot map to points outside of the region $0 < \hat{u}, \hat{v}, \hat{w} < 1$ in the left plot.

where $s_{i\ldots k} = (p_i + \cdots + p_k)^2$ are planar Mandelstam invariants. In addition to rational functions of these cross ratios, A_6 depends on the square root of the six-point Gram determinant, which takes the form

$$\Delta = (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w}. \tag{3}$$

Spacetime parity acts on the amplitude through the exchange $\sqrt{\Delta} \to -\sqrt{\Delta}$.

Similarly, we consider a BDS-like and cosmically-normalized version of the three-point MHV form factor $\mathcal{F}_3^{\mathrm{MHV}}$,

$$\mathcal{F}_3^{\mathrm{MHV}} = \mathcal{F}_3^{\mathrm{BDS\text{-}like}} \times \rho \times F_3, \tag{4}$$

where $\mathcal{F}_3^{\mathrm{BDS\text{-}like}}$ was defined in [43] and ρ is related to $\hat{\rho}$ via the cusp anomalous dimension Γ_{cusp}: $\rho = \hat{\rho} \times \exp(-\zeta_2 \Gamma_{\mathrm{cusp}}/2)$. F_3 is a finite polylogarithmic function[3] that depends on three ratios of Mandelstam invariants, which are usually chosen to be

$$u = \frac{s_{12}}{s_{123}}, \quad v = \frac{s_{23}}{s_{123}}, \quad w = \frac{s_{13}}{s_{123}}. \tag{5}$$

We make use of all three variables in order to make manifest the (dihedral) symmetry of the form factor, but only two of these variables are independent due to momentum conservation, which implies that $u + v + w = 1$.

The antipodal duality that we find between these quantities can be expressed as

$$F_3^{(L)}(u, v, w) = S\left(A_6^{(L)}(\hat{u}, \hat{v}, \hat{w})\right)\Big|_{\hat{u}_i \to \hat{u}_i(u,v,w)}, \tag{6}$$

where $F_3^{(L)}$ and $A_6^{(L)}$ denote the $\mathcal{O}(g^{2L})$ contributions to

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[3] The function $\rho \times F_3$ was referred to as $\mathcal{E}$ in [43].
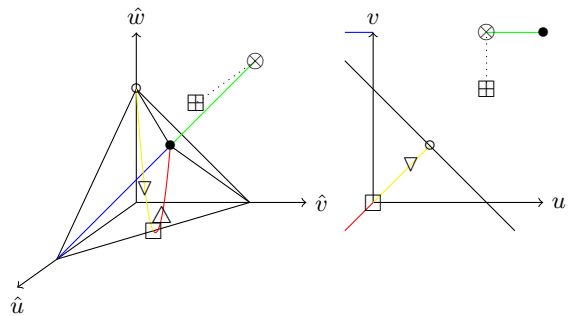

FIG. 2. Schematic correspondence between various interesting points and lines in the two-parameter three-point form factor space (right) and their images in the three-parameter six-point amplitude kinematic space (left) under the map (7)–(9). See Table I for the coordinates of the marked points.

F_3 and A_6, S is the antipode map, and

$$\hat{u}_1 = \hat{u}(u, v, w) = \frac{vw}{(1-v)(1-w)}, \tag{7}$$

$$\hat{u}_2 = \hat{v}(u, v, w) = \frac{uw}{(1-u)(1-w)}, \tag{8}$$

$$\hat{u}_3 = \hat{w}(u, v, w) = \frac{uv}{(1-u)(1-v)}. \tag{9}$$

The antipode map is part of the larger Hopf algebra structure of multiple polylogarithms, which also contains the coproduct and symbol maps [46–50]. The symbol of a polylogarithmic function G is recursively defined via its total differential as

$$dG = \sum_{x \in \mathcal{L}} G^x \, d\ln x \quad \Rightarrow \quad \mathcal{S}(G) = \sum_{x \in \mathcal{L}} \mathcal{S}(G^x) \otimes x, \tag{10}$$

where the set of logarithmic arguments $\mathcal{L}$ is referred to as the symbol alphabet, and each of the functions G^x is also a polylogarithm.

At symbol level, the antipode map simply reverses the order of the letters in every word of the symbol (up to a sign) [45, 57]:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1. \tag{11}$$

However, we find that relation (6) also holds[4] for terms involving transcendental constants, modulo contributions proportional to $i\pi$.

Due to the momentum conservation constraint on the form factor variables, the substitutions (7)–(9) require the amplitude to be evaluated on a two-dimensional surface. In particular, the equation $u + v + w = 1$ gets mapped to the constraint that $\Delta = 0$. Since parity sends $\sqrt{\Delta} \to -\sqrt{\Delta}$, this is the surface on which the parity of the amplitude is preserved.

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[4] Strictly speaking, the antipode map only makes sense on de Rham periods, and as such is not defined on $i\pi$.

	$(\hat{u}, \hat{v}, \hat{w})$	(u, v, w)	functions
$\triangledown$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\sqrt[6]{1}$
$\square$	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(0, 0, 1)$	$\text{Li}_2(\frac{1}{2})$ + logs
$\bullet$	$(1, 1, 1)$	$\lim_{u\to\infty}(u, u, 1-2u)$	MZVs
$\circ$	$(0, 0, 1)$	$(\frac{1}{2}, \frac{1}{2}, 0)$	MZVs + logs
$\triangle$	$(\frac{3}{4}, \frac{3}{4}, \frac{1}{4})$	$(-1, -1, 3)$	$\sqrt[6]{1}$
$\boxplus$	(∞, ∞, ∞)	$(1, 1, -1)$	alternating sums
$\otimes$	$\lim_{\hat{v}\to\infty}(1, \hat{v}, \hat{v})$	$\lim_{v\to\infty}(1, v, -v)$	MZVs
——	$(1, \hat{v}, \hat{v})$	$\lim_{v\to\infty}(u, v, 1-u-v)$	HPL$\{0, 1\}$
——	$(\hat{u}, \hat{u}, (1-2\hat{u})^2)$	$(u, u, 1-2u)$	HPL$\{-1, 0, 1\}$

TABLE I. Kinematic points and lines as well as their images under the map (7)–(9). The points $\triangledown$, $\square$, $\circ$, $\triangle$ and $\boxplus$ all lie on the line $(u, u, 1-2u)$.

We depict the mapping between these kinematical spaces in Figures 1 and 2, and give the translation for various points and lines in Table I. In particular, Figure 2 and Table I show several interesting points and lines on which the multiple polylogarithms simplify to multiple zeta values (MZVs), alternating sums, cyclotomic zeta values [58] including 6th roots of unity [40, 59] (denoted in Table I by $\sqrt[6]{1}$), and harmonic polylogarithms (HPLs) [60]. At two of the points the amplitude and form factor diverge logarithmically. Of the two HPL lines $(1, \hat{v}, \hat{v})$ and $(\hat{u}, \hat{u}, (1-2\hat{u})^2)$, the former is simpler because the HPL index -1 does not appear. In the next section, we will check the duality at some of these points and lines.

As can be seen in Figure 1, the line $v = 0$ is mapped to the point $(\hat{u}, \hat{v}, \hat{w}) = (0, 1, 0)$, while the point $(u, v) = (1, 0)$ is mapped to the line $\hat{u} = 0$, $\hat{v} + \hat{w} = 1$. These relations imply that the duality exchanges soft and collinear limits. This exchange is related to the simplicity of the map (7)–(9) in the OPE parametrization:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}, \tag{12}$$

and

$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1 + T^2},$$

$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))}, \tag{13}$$

see [21, 31]. Namely,

$$\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}, \tag{14}$$

while $\hat{F} = 1$ on the $\Delta = 0$ surface.

Evidence

Since the six-point amplitude and three-point form factor have been computed through seven and eight

loops [41, 43, 44], respectively, we can provide evidence for relation (6) through seven loops. We first do so at symbol level, where the antipode map is given by (11).

The alphabet of symbol letters of the six-point amplitude can be chosen to be

$$\hat{\mathcal{L}} = \{\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{f}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}, \tag{15}$$

where

$$\hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}}, \qquad \hat{b} = \frac{\hat{v}}{\hat{w}\hat{u}}, \qquad \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}},$$

$$\hat{d} = \frac{1 - \hat{u}}{\hat{u}}, \qquad \hat{e} = \frac{1 - \hat{v}}{\hat{v}}, \qquad \hat{f} = \frac{1 - \hat{w}}{\hat{w}}, \tag{16}$$

and the remaining variables invert under parity, for instance $\hat{y}_u \to 1/\hat{y}_u$. Hence, $\hat{y}_u = \hat{y}_v = \hat{y}_w = 1$ on the $\Delta = 0$ surface.[5] An analogous alphabet can be chosen for the three-point form factor, namely

$$\mathcal{L} = \{a, b, c, d, e, f\}, \tag{17}$$

where

$$a = \frac{u}{vw}, \qquad b = \frac{v}{wu}, \qquad c = \frac{w}{uv},$$

$$d = \frac{1 - u}{u}, \qquad e = \frac{1 - v}{v}, \qquad f = \frac{1 - w}{w}. \tag{18}$$

Both $A_6|_{\Delta=0}$ and F_3 are invariant under the same dihedral group D_3, which is generated by

$$\text{cycle: } \{a, b, c, d, e, f\} \to \{b, c, a, e, f, d\} \tag{19}$$

and

$$\text{flip: } \{a, b, c, d, e, f\} \to \{a, c, b, d, f, e\}, \tag{20}$$

and similarly in the hatted letters.

It is easy to check that the map (7)–(9) acts by swapping

$$\sqrt{\hat{a}} \Leftrightarrow d, \qquad \hat{d} \Leftrightarrow a, \tag{21}$$

and all cyclically-related letters in the same way. Notably, this swaps the letters that appear in the first and last entries of the amplitude and form factor. That is, the letters that are allowed to appear in the first entry of the amplitude ($\hat{a}$, $\hat{b}$, and $\hat{c}$) are mapped to the letters that appear in the last entry of the form factor (d, e, and f). Similarly, the first entries of the form factor (a, b and c) are mapped to the last entries of the amplitude ($\hat{d}$, $\hat{e}$, and $\hat{f}$). This combines with the reversal of symbol letters entailed by the antipode map in (6) to maintain the known first and last entry conditions [15, 34, 43, 61] of each quantity.

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[5] Note that this notation differs from [40, 41], where $\hat{d}$, $\hat{e}$, and $\hat{f}$ were referred to as m_u, m_v and m_w.

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

TABLE II. Number of terms in the symbol of $F_3^{(L)}$ as a function of the loop order L.

We now check that relation (6) holds at symbol level. For example, at two loops, the symbols $\mathcal{S}$ of the amplitude and form factor each involve 12 terms, and are given by

$$\mathcal{S}(A_6^{(2)}|_{\Delta=0}) = \frac{1}{2}\,\hat{a}\otimes\hat{a}\otimes\hat{a}\otimes\hat{e}+\hat{a}\otimes\hat{e}\otimes\hat{e}\otimes\hat{e}+\text{dihedral}\,, \quad (22)$$

$$\mathcal{S}(F_3^{(2)}) = 2\,a\otimes a\otimes a\otimes e + 4\,a\otimes e\otimes e\otimes e + \text{dihedral}\,, \quad (23)$$

where we sum over all dihedral images in D_3 generated by (19) and (20). Due to the square root in (21), the two terms in (22) pick up factors of 8 and 2, which are precisely the numerical factors needed to match (23) when the rest of the transformation in (6) is applied. We have similarly checked that the duality holds at symbol level through seven loops, using the expressions provided in the ancillary files of [41] and [43, 44]. Due to the fast growth of the number of terms in the symbol with the loop order (see Table II), this check quickly becomes quite involved, and extremely stringent.

The antipode map is also defined at function level [45, 57] (see also [62] for a discussion of the antipode in the physics literature), and as such we can also check the duality beyond the symbol. The simplest way to do this is to compare the functions at a single point, for instance at the point $\hat{u}=\hat{v}=\hat{w}=1$, which maps to the $u,v\to\infty$ limit of the form factor space. At these points, both functions are real and are known to be expressible in terms of MZVs, which can conveniently be expressed in terms of the so-called f-alphabet [48, 59]; see Table I. Since we do not know how to compute the antipode of $i\pi$, the first nontrivial constants appear at three loops. Through five loops, the amplitude evaluates to [40, (A.3)–(A.5)]

$$A_6^{(3)}(1,1,1) = 0f_{3,3} + \mathcal{O}(\pi^2)\,, \quad (24)$$

$$A_6^{(4)}(1,1,1) = 120f_{3,5} + \mathcal{O}(\pi^2)\,, \quad (25)$$

$$A_6^{(5)}(1,1,1) = -2688f_{3,7} - 1560f_{5,5} + \mathcal{O}(\pi^2)\,, \quad (26)$$

while the form factor evaluates to [43, (5.7)–(5.9)]

$$F_3^{(3)}(\infty,\infty) = 0f_{3,3} + \mathcal{O}(\pi^2)\,, \quad (27)$$

$$F_3^{(4)}(\infty,\infty) = 120f_{5,3} + \mathcal{O}(\pi^2)\,, \quad (28)$$

$$F_3^{(5)}(\infty,\infty) = -2688f_{7,3} - 1560f_{5,5} + \mathcal{O}(\pi^2)\,. \quad (29)$$

Clearly, these values are related to each other by reversing the order of f-alphabet letters.[6] We provide further evidence for the duality at various points up to seven loops in an ancillary file.[7]

We can also check the duality on the line where $\hat{u}=1$ and $\hat{v}=\hat{w}$, where A_6 can be expressed in terms of HPLs with indices $0,1$ and argument $\hat{x}=1-1/\hat{v}$. This line maps via (7)–(9) to the line where $v\to\infty$ (with u fixed), where F_3 can be expressed in terms of the same space of functions, but with the arguments reinterpreted as $x=1-1/u$, and $\hat{x}=1-x$. We have checked that these functions map to each other via relation (6) through seven loops, up to terms proportional to π^2. We then use the duality to predict the eight-loop MHV amplitude on the line $(1,\hat{v},\hat{v})$, modulo π^2 terms, in a second ancillary file. Finally, we have also checked that relation (6) holds at the level of full functions of u and v, up to three loops, finding a complete match up to terms proportional to $i\pi$. While we have not detailed here how the antipode acts in general on multiple polylogarithms, we note that it is conveniently implemented in the MATHEMATICA package PolyLogTools [63].

It would be interesting to find an extension or deformation of relation (6) that also relates the terms proportional to $i\pi$ on both sides of the duality. This is nontrivial, though, for two reasons. The first is that the antipode is not defined on $i\pi$. Second, there is a question of the appropriate Riemann sheets. On its physical sheet, the form factor is real when $0<u,v,w<1$ and complex elsewhere (except when one of these variables is taken to infinity). For the amplitude, with $\hat{u},\hat{v},\hat{w}>0$, we could either be on the Euclidean sheet, or the $2\to4$ physical scattering sheet. Both are problematic: On the Euclidean sheet, the amplitude is real, while the form factor generically has imaginary parts proportional to $i\pi$ (but at least for $u,v,w<1$ both objects are real). On the $2\to4$ physical scattering sheet, the amplitude has imaginary parts, which blow up logarithmically as one approaches the $(1,\hat{v},\hat{v})$ self-crossing line [64], while the dual form factor's imaginary parts vanish there.

Implications

This duality has several interesting implications. The six-point amplitude is known to obey a large set of extended Steinmann relations [40] (or cluster adjacency conditions [65]), which tell us that certain pairs of let-

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[6] More generally, similar to its action on the symbol, the antipode reverses the f-alphabet letters and multiplies each term by $(-1)^m$, where m is the number of letters.

[7] This ancillary file also contains further details on the definitions of $\mathcal{F}_3^{\text{BDS-like}}$, $\mathcal{A}_6^{\text{BDS-like}}$, ρ, and $\hat{\rho}$.

ters never appear in adjacent entries of the symbol:

$$\dots \widehat{\hat{a} \otimes \hat{b}} \dots ,$$
$$\dots \widehat{\hat{a} \otimes \hat{d}} \dots , \quad \dots \widehat{\hat{d} \otimes \hat{a}} \dots , \tag{30}$$

plus all dihedral images. Importantly, these conditions all remain nontrivial and distinct on the $\Delta = 0$ surface, and can be read either backwards or forwards. As such, these constraints are preserved by the antipode, and can be translated directly, via (21), into constraints that should hold for the three-point form factor:

$$\dots \widehat{d \otimes e} \dots ,$$
$$\dots \widehat{a \otimes d} \dots , \quad \dots \widehat{d \otimes a} \dots , \tag{31}$$

plus all dihedral images. The first of these conditions was observed in [43, 66], while the other conditions are new.[8]

More generally, this duality makes it possible to translate knowledge about the functional form of one of these quantities into information about the other. Most obviously, the form factor can simply be 'read off' of the amplitude on the $\Delta = 0$ surface (up to $i\pi$ contributions). Conversely the form factor also provides an enormous amount of boundary data for bootstrapping the amplitude. In fact, we have checked that this information, when combined with parity, is sufficient to uniquely determine the symbol of the amplitude through 5 loops, and through 7 loops when combined also with certain conditions on the final pair of entries and the behavior at the origin [54].

Discussion and Conclusions

In this letter, we have identified a new and unexpected duality in planar $\mathcal{N} = 4$ SYM theory between the three-point form factor of the chiral part of the stress tensor supermultiplet, and a kinematic limit of the six-point MHV amplitude. Amazingly, these quantities are related by the antipode map, which has no clear physical interpretation that we are aware of. In particular, the antipode exchanges the first and last entries of the symbol, which describe the discontinuities and derivatives of these functions, respectively. Thus, the discontinuities of the amplitude seem to be encoded in the derivatives of the form factor, and vice versa!

While we have provided evidence for this duality through seven loops, it would be interesting to find a physical derivation or even a proof of this relation, using for example the non-perturbative integrability-based descriptions of both quantities ([21–30] and [31–33]). At

strong coupling both quantities can also be described via a minimal surface [1, 16] and a corresponding Y-system [17, 67, 68], and it would be interesting to see what relation (6) implies for these formulations.

The three-point form factor is of particular interest due to the principle of maximal transcendentality [69–72], which states that the three-point form factor in $\mathcal{N} = 4$ SYM theory provides the maximally transcendental part of the Higgs-to-three-gluon amplitude in pure Yang-Mills theory in the large-top-mass approximation [15, 73–76].

It would be extremely interesting to see whether a version of the duality we present here exists for higher-point MHV amplitudes and form factors. Since A_{2n} and F_n both exhibit a D_n dihedral symmetry, one might expect these quantities to be related also for $n > 3$. The surface $\Delta = 0$ can be interpreted as 'twisted forward scattering' in which the $2n$ external momenta of the amplitude, $\hat{p}_i$, are related by $\hat{p}_{i+n} = -\hat{p}_i$ for $i = 1, 2, \dots, n$ (for $n = 3$), so that there are only n independent momenta, as in the form factor. This interpretation might give further clues for a generalization to higher n.

Moreover, it would be interesting to see whether a similar duality exists at next-to-MHV and beyond, or for other operators than the chiral part of the stress tensor supermultiplet.

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