Littlest Modular Seesaw

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Abstract

We present the first complete model of the Littlest Modular Seesaw, based on two right-handed neutrinos, within the framework of multiple modular symmetries, justifying the use of multiple moduli fields which take their values at 3 specific stabilizers of $\Gamma_4 \simeq S_4$, including a new phenomenological possibility. Using a semi-analytical approach, we perform a χ^2 analysis of each case and show that good agreement with neutrino oscillation data is obtained, including predictive relations between the leptonic mixing angles and the ratio of light neutrino masses, which non-trivially agree with the experimental values. It is noteworthy that in this very predictive setup, the models fit the global fits of the experimental data remarkably well, both with and without the Super-Kamiokande atmospheric data, for both choices of stabilizers. By extending the model to include a weighton and the double cover group $\Gamma_4' \simeq S_4'$, we are able to also account for the hierarchy of the charged leptons using modular symmetries, without altering the neutrino predictions.

PACS number(s): 14.60.Pq, 11.30.Hv, 12.60.Fr
Keywords: Lepton flavour mixing, flavour symmetry

Dedicated to Graham G. Ross

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1 Introduction

The mysterious threefold replication of the fermion generations is one important open issue of the Standard Model (SM) at the heart of the flavour problem. The most promising solution are symmetries that relate the generations, known as family symmetries or flavour symmetries. Recent reviews include [1–3].

The idea of modular invariance [4,5] has been suggested as key ingredient in solutions to the flavour problem [6]. In these promising scenarios, a modular symmetry associated with transformations of a modulus field can lead to very predictive models of flavour. The double covers of the groups have also been used in interesting flavour models [7].

Nevertheless, in order to apply the methodology of residual flavour symmetries, it is relevant to consider all their fixed points or stabilizers [8, 9]: special values for the modulus field where part of the modular transformations are preserved. Furthermore, if multiple residual symmetries are desired, multiple modular symmetries, each with its respective modulus, can be employed - as proposed in [10] and expanded upon in [11–13].

Modular symmetries can further be exploited to explain the mass hierarchy of the fermions by use of an extra field referred to as a weighton [14]. While similar to the Froggatt-Nielsen mechanism, the weighton explicitly relies on modular invariance and does not require extra symmetry.

Another mass hierarchy that is puzzling is the lightness of neutrino masses. Although the type I seesaw mechanism can qualitatively explain the smallness of neutrino masses through the heavy right-handed neutrinos (RHNs), if one doesn't make other assumptions, it contains too many parameters to make any particular predictions for neutrino mass and mixing. The sequential dominance (SD) [15,16] of right-handed neutrinos proposes that the mass spectrum of heavy Majorana neutrinos is strongly hierarchical, i.e. $M_{\rm atm} \ll M_{\rm sol} \ll M_{\rm dec}$, where the lightest RHN with mass $M_{\rm atm}$ is responsible for the atmospheric neutrino mass, that with mass $M_{\rm sol}$ gives the solar neutrino mass, and a third largely decoupled RHN gives a suppressed lightest neutrino mass. It leads to an effective two right-handed neutrino (2RHN) model [17, 18] with a natural explanation for the physical neutrino mass hierarchy, with normal ordering and the lightest neutrino being approximately massless, $m_1 = 0$.

A very predictive minimal seesaw model with two right-handed neutrinos and one texture zero is the so-called constrained sequential dominance (CSD) model [19–28]. The CSD(n) scheme assumes that the two columns of the Dirac neutrino mass matrix are proportional to $(0, 1, -1)$ and $(1, n, 2-n)$ respectively in the RHN diagonal basis, where the parameter n was initially assumed to be a positive integer, but in general may be a real number. For example the CSD(3) (also called Littlest Seesaw model) [21–25], CSD(4) models [26,27] and CSD($-1/2$) [29] can give rise to phenomenologically viable predictions for lepton mixing parameters and the two neutrino mass squared differences Δm^2_{21} and Δm^2_{31}, corresponding to special constrained cases of TM1 lepton mixing. As was observed, modular symmetry remarkably suggests CSD($1-\sqrt{6}$) $\approx$ CSD(-1.45) [8,30], although such a model would require multiple moduli and so far there is no complete model of this kind in the literature.

In this paper, we construct the first complete model of the Littlest Modular Seesaw (LMS), based on CSD($1-\sqrt{6}$) $\approx$ CSD(-1.45), within a consistent framework based on multiple modular symmetries. We also propose a new related possibility based on CSD($1+\sqrt{6}$) $\approx$ CSD(3.45), intermediate between CSD(3) and CSD(4). In each case, three S_4 modular symmetries are introduced, each with their respective modulus field at a distinct stabilizer, leading to three separate residual subgroups, thus dispensing with vacuum alignment mechanisms. The result, in the symmetry basis, is a diagonal charged lepton mass matrix and a LMS scenario of a particular kind. In order to account for the hierarchy of the charged lepton masses, we subsequently introduce a weighton field, where this model is implemented by upgrading the modular symmetries to the respective double covers S_4'. Using a semi-analytical approach, we perform a χ^2 analysis of each case case and show that good agreement with neutrino oscillation data is obtained, for both possible octants of atmospheric angle, including predictive relations between the leptonic mixing angles and the ratio of light neutrino masses, which non-trivially agree with the experimental values. It is noteworthy that in this very predictive setup, all the models fit the experimental data remarkably well, depending on the choice of stablizers and data set, in one case to within approximately 1σ.

In Section 2.1 we present the model with the respective fields and their assignments under the multiple modular symmetries. The charged-lepton structure is shown in Section 2.2, and the neutrino seesaw matrix is shown in sec 2.3. Analytical results for the leptonic mixing angles and the neutrino masses are given in Section 2.4 and a numerical analysis is done in Section 2.5. We conclude in Section 3. Appendix A gives two alternative models where the charged-lepton hierarchies are naturally explained by including a weighton.

2 The Model

2.1 Symmetries and stabilizers

The model we are building features three commuting S_4 modular symmetries, which we label as S_4^A, S_4^B, S_4^C. At low energies, due to the VEVs of fields Φ_{AC} and Φ_{BC}, they are broken down to the diagonal subgroup, as described in [10]. Table 1 contains the transformation properties (representations and modular weights) under the modular symmetries of the fields and of the relevant modular forms.

Field	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
L	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0
e^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1'}$	0	0	-6
μ^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1'}$	0	0	-4
τ^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1'}$	0	0	-2
N_A^c	$\mathbf{1'}$	$\mathbf{1}$	$\mathbf{1}$	-4	0	0
N_B^c	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1}$	0	-2	0
Φ_{AC}	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0
Φ_{BC}	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0	0

Yuk/Mass	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
$Y_e(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3'}$	0	0	6
$Y_\mu(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3'}$	0	0	4
$Y_\tau(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3'}$	0	0	2
$Y_A(\tau_A)$	$\mathbf{3'}$	$\mathbf{1}$	$\mathbf{1}$	4	0	0
$Y_B(\tau_B)$	$\mathbf{1}$	$\mathbf{3'}$	$\mathbf{1}$	0	2	0
$M_A(\tau_A)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	8	0	0
$M_B(\tau_B)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	4	0

Table 1: Transformation properties of fields and modular forms (Yuk/Mass) under the modular symmetries.

These assignments are very similar to those used in [10] [1].

Our goal is to achieve a CSD(3.45) [8] structure from the multiple modular symmetries. To that end, the desired directions of the modular forms are obtained for these representations and weights at specific stabilizers [8–10]. Namely, following the basis of [10], we compute the modular forms:

$$\tau_A = -\frac{3}{2} + \frac{i}{2} : \qquad Y_{\mathbf{3'}}^{(4)}(\tau_A) = (0, -1, 1)\,, \tag{1}$$

[1] We note there is a typo in [10] where RH leptons and the respective modular forms should have primes, as the modular form $Y_\tau(\tau_C)$ (weight 2) only exists as a $3'$.

for one of the Dirac mass matrix columns, and

$$\tau_B = \frac{3}{2} + \frac{i}{2}: \qquad Y^{(2)}_{\mathbf{3'}}(\tau_B) = (1, 1-\sqrt{6}, 1+\sqrt{6}), \tag{2}$$

or

$$\tau_B = -\frac{1}{2} + \frac{i}{2}: \qquad Y^{(2)}_{\mathbf{3'}}(\tau_B) = (1, 1+\sqrt{6}, 1-\sqrt{6}), \tag{3}$$

for the other. These specific modular forms lead to the desired CSD structure. In the same basis, we want to enforce a diagonal structure for the charged-lepton Yukawa coupling matrices. This can be easily achieved through the weights 2, 4, and 6 modular forms transforming as $\mathbf{3'}$, for $\tau_C = \omega$:

$$Y^{(2)}_{\mathbf{3'}}(\tau_C) = (0,1,0) \tag{4a}$$

$$\tau_C = \omega: \qquad Y^{(4)}_{\mathbf{3'}}(\tau_C) = (0,0,1) \tag{4b}$$

$$Y^{(6)}_{\mathbf{3'}}(\tau_C) = (1,0,0) \tag{4c}$$

A subtlety should be noted here. Indeed, for weight 6, there are two independent $\mathbf{3'}$ modular forms, which could spoil the diagonal arrangement of the charged-leptons. Nevertheless, for $\tau = \omega$, one of them vanishes, introducing no further parameters.

With the above fields and assignments, we write the respective lepton sector superpotential as

$$\begin{aligned} w_\ell = & \frac{1}{\Lambda}\left[L\Phi_{AC}Y_A(\tau_A)N^c_A + L\Phi_{BC}Y_B(\tau_B)N^c_B\right]H_u \\ & + \left[LY_e(\tau_C)e^c + LY_\mu(\tau_C)\mu^c + LY_\tau(\tau_C)\tau^c\right]H_d \\ & + \frac{1}{2}M_A(\tau_A)N^c_AN^c_A + \frac{1}{2}M_B(\tau_B)N^c_BN^c_B + M_{AB}(\tau_A,\tau_B)N^c_AN^c_B\,. \end{aligned} \tag{5}$$

2.2 Charged leptons

Expanding the superpotential of Eq. (5), we can find the mass matrices for the fields after the EWSB. Due to the nature of the S_4 tensor products in our chosen basis, and the particular structure chosen for the bi-triplets VEVs, the $\mathbf{3}\otimes\mathbf{3}$ tensor products are non-diagonal:

$$(\mathbf{a}\otimes\mathbf{b})_{\mathbf{1}} = a_1b_1 + a_2b_3 + a_3b_2, \tag{6}$$

$$(\mathbf{a}\otimes\langle\Phi\rangle\otimes\mathbf{b})_{\mathbf{1}} \propto a_1b_1 + a_2b_3 + a_3b_2. \tag{7}$$

Hence, the charged-lepton mass matrix is simply given by

$$M_l = v_d \begin{pmatrix} (Y_e)_1 & (Y_\mu)_1 & (Y_\tau)_1 \\ (Y_e)_3 & (Y_\mu)_3 & (Y_\tau)_3 \\ (Y_e)_2 & (Y_\mu)_2 & (Y_\tau)_2 \end{pmatrix}, \tag{8}$$

where we omit the τ_c dependency for clarity, and v_d stands for $\langle H_d\rangle$. Plugging in the specific shapes of the modular forms given in Eqs. (4a)-(4c) we arrive at a diagonal charged-lepton mass matrix when $\tau_C = \omega$:

$$M_l = v_d \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}. \tag{9}$$

In this model, the hierarchical masses of the charged-leptons are not addressed. In order to naturally deal with this issue, we present two modifications of this model in the Appendix A, where a weighton is responsible for the hierarchy of the masses, without affecting the remaining predictions of the model.

2.3 Neutrinos

We now turn to the Majorana mass terms for the neutrinos, N_A^C and N_B^C. From Table 1, we see that $N_A^C N_A^C$ as well as $N_B^C N_B^C$ are $S_4^A \times S_4^B \times S_4^C$ singlets. As such, we just need to cancel out the weight with a singlet Yukawa form. From [7,31] [2] we see that the Yukawa modular forms of weight 4 do have a singlet representation, needed for the $M_A(\tau_A)$ term. Due to the properties of the modular terms, this implies that there is also a singlet modular form of weight 8, required for $M_B(\tau_B)$. Conversely, as $N_A^C N_B^C$ transforms non-trivially under both S_4^A and under S_4^B, there are no one-dimensional modular forms of weight 2 and the respective term is forbidden by the symmetries, and the RH neutrino mass matrix is diagonal:

$$M_R = \begin{pmatrix} M_A(\tau_A) & 0 \\ 0 & M_B(\tau_B) \end{pmatrix} . \tag{10}$$

Finally, we need to check the shape of the Dirac mass matrices. Given the VEVs for the bi-triplets Φ_{AC}, Φ_{BC}, the tensor products after SSB will mimic those of the usual S_4 (the diagonal S_4 preserved by the bi-triplets symmetry breaking), as explained in [10–13]. This feature is preserved also in the weighton versions of the model, that are using S_4'. The Dirac mass matrix is then given by:

$$M_D = v_u \begin{pmatrix} (Y_A)_1 & (Y_B)_1 \\ (Y_A)_3 & (Y_B)_3 \\ (Y_A)_2 & (Y_B)_2 \end{pmatrix} , \tag{11}$$

where, as usual, v_u denotes the H_u VEV, and the 2×3 structure comes from the CSD with just two RH neutrinos. Choosing specific stabilisers for the two remaining moduli fields, we can achieve a new CSD(3.45) structure with $n = 1 + \sqrt{6}$:

$$M_D = v_u \begin{pmatrix} 0 & b \\ a & b\left(1+\sqrt{6}\right) \\ -a & b\left(1-\sqrt{6}\right) \end{pmatrix} , \qquad \tau_A = -\frac{3}{2} + \frac{i}{2}, \quad \tau_B = \frac{3}{2} + \frac{i}{2}. \tag{12}$$

We can similarly achieve the case CSD(-1.45) with $n = 1 - \sqrt{6}$ already discussed in [8]:

$$M_D = v_u \begin{pmatrix} 0 & b \\ a & b\left(1-\sqrt{6}\right) \\ -a & b\left(1+\sqrt{6}\right) \end{pmatrix} , \qquad \tau_A = -\frac{3}{2} + \frac{i}{2}, \quad \tau_B = -\frac{1}{2} + \frac{i}{2}. \tag{13}$$

The type-I seesaw mechanism will lead to an effective mass matrix for the light neutrinos:

$$m_\nu = M_D \cdot M_R^{-1} \cdot M_D^T = v_u^2 \begin{pmatrix} \frac{b^2}{M_B} & \frac{b^2 n}{M_B} & \frac{b^2(2-n)}{M_B} \\ . & \frac{a^2}{M_A} + \frac{b^2 n^2}{M_B} & -\frac{a^2}{M_A} + \frac{b^2 n(2-n)}{M_B} \\ . & . & \frac{a^2}{M_A} + \frac{b^2(2-n)^2}{M_B} \end{pmatrix} , \tag{14}$$

where $n = 1 + \sqrt{6} \approx 3.45$ or $n = 1 - \sqrt{6} \approx -1.45$.

[2] Although we use a different basis, the assignments of the representations are identical, as can be seen by the weight 2 modular forms. Furthermore, we have explicitly checked that the tensor product of $\left(Y_{\mathbf{3'}}^{(2)} \otimes Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}}$ does not vanish for the relevant τ_A nor any of τ_B. This ensures a non-zero M_A and M_B.

2.4 Analytic results

The effective mass matrix for the light neutrinos can be split into two contributions,

$$m_\nu = \frac{v_u^2}{M_A}|a|^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{v_u^2}{M_B}|b|^2 e^{i\beta} \begin{pmatrix} 1 & n & 2-n \\ n & n^2 & n(2-n) \\ 2-n & n(2-n) & (2-n)^2 \end{pmatrix}. \tag{15}$$

It is worth noting that the above neutrino mass matrix in the diagonal charged lepton mass basis is determined effectively by two real parameters, $m_a = v_u^2 \frac{|a|^2}{M_A}$, $\quad m_b = v_u^2 \frac{|b|^2}{M_B}$, one phase β and a discrete choice of $n = 1 \pm \sqrt{6}$. For a given choice of n, the remaining three real parameters determine all the parameters in the neutrino sector, namely all the neutrino masses and the entire PMNS matrix.

These two terms above can be simultaneously block-diagonalized by the following Tri-bimaximal mixing matrix,

$$\mathcal{U}_{\rm TBM} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \end{pmatrix}, \tag{16}$$

leading to

$$m'_\nu = \mathcal{U}_{\rm TBM}^T \cdot m_\nu \cdot \mathcal{U}_{\rm TBM} = \frac{v_u^2}{M_A}|a|^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \frac{v_u^2}{M_B}|b|^2 e^{i\beta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & \sqrt{6}(n-1) \\ 0 & \sqrt{6}(n-1) & 2(n-1)^2 \end{pmatrix}. \tag{17}$$

We diagonalize the remaining $(2,2)$ block through the matrix

$$\mathcal{U}_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & e^{i\gamma} s_\alpha \\ 0 & -e^{-i\gamma} s_\alpha & c_\alpha \end{pmatrix}, \tag{18}$$

such that

$$\mathcal{U}_\alpha^T \cdot m'_\nu \cdot \mathcal{U}_\alpha = {\rm diag}(0, m_1, m_2). \tag{19}$$

To ensure that m_1, m_2 are real and positive, we use the phase matrix, P_ν:

$$(U_{\rm TBM}\, U_\alpha\, P_\nu)^T\, m_\nu\, (U_{\rm TBM}\, U_\alpha\, P_\nu) = {\rm diag}(0, |m_1|, |m_2|), \tag{20}$$

with

$$U_\nu \equiv (U_{\rm TBM}\, U_\alpha\, P_\nu) = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{c_\alpha}{\sqrt{3}} & e^{i\gamma}\frac{s_\alpha}{\sqrt{3}} \\ \sqrt{\frac{1}{6}} & \frac{c_\alpha}{\sqrt{3}} - e^{-i\gamma}\frac{s_\alpha}{\sqrt{2}} & \frac{c_\alpha}{\sqrt{2}} + e^{i\gamma}\frac{s_\alpha}{\sqrt{3}} \\ \sqrt{\frac{1}{6}} & \frac{c_\alpha}{\sqrt{3}} + e^{-i\gamma}\frac{s_\alpha}{\sqrt{2}} & -\frac{c_\alpha}{\sqrt{2}} + e^{i\gamma}\frac{s_\alpha}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}. \tag{21}$$

As this is effectively a 2×2 diagonalization, it is possible to find analytical relations for α. Namely, by requiring a vanishing $\left(U_\alpha^T m'_\nu U_\alpha\right)_{23}$ element we find [22]:

$$t \equiv \tan 2\alpha = \frac{2y}{z\cos(\varphi - \gamma) - x\cos\gamma}, \tag{22}$$

$$\tan\gamma = \frac{z\sin\varphi}{x + z\cos\varphi}, \qquad \text{with} \quad \varphi = \phi_z - \beta, \tag{23}$$

where we defined

$$m'_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & xe^{i\beta} & ye^{i\beta} \\ 0 & ye^{i\beta} & ze^{i\phi_z} \end{pmatrix}, \tag{24}$$

with

$$x = 3m_b\,, \quad y = \sqrt{6}(n-1)m_b\,, \quad z = |2(m_a + e^{i\beta}(n-1)^2 m_b)|\,, \quad m_a = v_u^2\frac{|a|^2}{M_A}\,, \quad m_b = v_u^2\frac{|b|^2}{M_B}. \tag{25}$$

To relate this to the PMNS matrix in its standard parametrization, we must also take into account the charged-lepton rotation. In our specific realisation, the modular representations of the charged-leptons were chosen in such a way that its mass matrix is already diagonal. As such, the LH rotation is, in general, a diagonal phase matrix

$$U_\ell = \begin{pmatrix} e^{i\delta_e} & 0 & 0 \\ 0 & e^{i\delta_\mu} & 0 \\ 0 & 0 & e^{i\delta_\tau} \end{pmatrix}, \tag{26}$$

which can be used to match the standard parametrization [3]:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \cdot \begin{pmatrix} e^{i\eta_1} & 0 & 0 \\ 0 & e^{i\eta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

which has the measured mixing angles and CP-violating phase, and s_{ij} (c_{ij}) denotes $\sin\theta_{ij}$ ($\cos\theta_{ij}$).

Now, we can relate our Unitary matrix U_ν to U_{PMNS} and find out the relations between the measured neutrino data, and our model's parameters. The resulting relations are

$$\sin\theta_{13} = \frac{\sin\alpha}{\sqrt{3}} = \frac{1}{\sqrt{6}}\sqrt{1 - \sqrt{\frac{1}{1+t^2}}}, \tag{28}$$

$$\tan\theta_{12} = \frac{\cos\alpha}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sqrt{1 - 3\sin^2\theta_{13}}, \tag{29}$$

$$\tan\theta_{23} = \frac{|1+\epsilon_\alpha|}{|1-\epsilon_\alpha|}, \tag{30}$$

where

$$\epsilon_\alpha = \sqrt{\frac{2}{3}}e^{i\gamma}\tan\alpha = \sqrt{\frac{2}{3}}e^{i\gamma}\frac{\sqrt{1+t^2}-1}{t}. \tag{31}$$

Note that the mixing angles depend only on two parameters, with θ_{13} and θ_{12} depending only on t. Since the mixing is unaffected by an overall factor, we can factorise m_b in Eq. (24), leading to

$$m'_\nu = m_b \begin{pmatrix} 0 & 0 & 0 \\ 0 & x'e^{i\beta} & y'e^{i\beta} \\ 0 & y'e^{i\beta} & z'e^{i\phi_z} \end{pmatrix}, \tag{32}$$

[3] Indeed, the RH fields rotate away the possible phases of M_l and, as such, when we write down m_ν we are already in a basis where M_l is diagonal and positive. The LH rotation was used to enforce the reality of a. In general, this won't be the basis where the light neutrino masses are real. U_l is then required to rotate into the standard parametrization basis.

where

$$
\begin{aligned}
x' &= 3, && (33)\\
y' &= \sqrt{6}(n-1), && (34)\\
\phi_z &= \arg\left(\frac{1}{r} + e^{i\beta}(n-1)^2\right), && (35)\\
z' &= \left|2\left(\frac{1}{r} + e^{i\beta}(n-1)^2\right)\right|, && (36)\\
r &= \frac{m_b}{m_a}, && (37)
\end{aligned}
$$

where we note how ϕ_z and z' depend on r and β. For fixed n, the mixing angles themselves will depend solely on r and β.

To obtain the neutrinos masses, we proceed as in [22] by taking the trace and determinant of the hermitian combination $H'_\nu = {m'_\nu}^\dagger m'_\nu$, and equating it to the sum and product of the squared masses, respectively. Given that the LS paradigm forcibly leads to a massless light neutrino and thus, to Normal Ordering, the obtained masses can be readily equated to the Δm^2_{21} and Δm^2_{31} observables. Defining the combinations of parameters, that depend on those of Eqs. (23) and (33)-(37),

$$
\begin{aligned}
\Sigma &\equiv \frac{m_b^2}{2}\left(x'^2 + 2y'^2 + z'^2\right), && (38)\\
\delta M &\equiv \frac{m_b^2}{2}\sqrt{x'^2(4y'^2 - 2z'^2) + x'^4 + 8x'y'^2z'\cos\varphi + 4y'^2z'^2 + z'^4}, && (39)
\end{aligned}
$$

then

$$
\begin{aligned}
\Delta m^2_{21} &= m_2^2 = \Sigma - \delta M, && (40)\\
\Delta m^2_{31} &= m_3^2 = \Sigma + \delta M, && (41)
\end{aligned}
$$

which are functions of r and β, and with the overall factor given by m_b, which cancels out in the ratio. As such, $\Delta m^2_{21}/\Delta m^2_{31}$, the 3 mixing angles, and the CP-phase are all functions of just two effective parameters.

The CP-phase of the PMNS matrix, as well as the physical Majorana phase (since there is one massless neutrino, only η_2 of Eq. (27) is physical[4]) can be extracted through careful combinations of the elements [32], and lead to

$$
\begin{aligned}
\delta_{CP} &= -\arg\left(\text{sign}(t)e^{i\beta}\left(4\left(\sqrt{t^2+1}-1\right) + \left(-2+3e^{2i\gamma}\right)t^2\right)\right), && (42)\\
\eta_2 &= \left(-\gamma - \delta_{CP} - (\phi_3 - \phi_2)\right). && (43)
\end{aligned}
$$

2.5 Numerical analysis

Using the analytical expressions, we plot the allowed experimental ranges for the lepton mixing parameters in the (r, β) plane. We present both the case where $\tau_B = (3+i)/2$ and $\tau_B = (-1+i)/2$, corresponding to the modular forms of Eqs (2) and (3). The results shown correspond to the NuFit 5.1 values [33,34] without SK atmospheric data in Figure 1 and with SK atmospheric data in Figure 2. We reproduce the ranges used in Table 2. In both Figures, the top row displays the 3σ ranges, the bottom row the 1σ ranges, the left column the $n = 1+\sqrt{6}$ case and the right column the $n = 1-\sqrt{6}$ case [5].

Normal Ordering		without SK atmospheric data		with SK atmospheric data	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
	$\sin^2\theta_{12}$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.012}_{-0.012}$	$0.269 \to 0.343$
	$\theta_{12}/^\circ$	$33.44^{+0.77}_{-0.74}$	$31.27 \to 35.86$	$33.45^{+0.77}_{-0.75}$	$31.27 \to 35.87$
	$\sin^2\theta_{23}$	$0.573^{+0.018}_{-0.023}$	$0.405 \to 0.620$	$0.450^{+0.019}_{-0.016}$	$0.408 \to 0.603$
	$\theta_{23}/^\circ$	$49.2^{+1.0}_{-1.3}$	$39.5 \to 52.0$	$42.1^{+1.1}_{-0.9}$	$39.7 \to 50.9$
	$\sin^2\theta_{13}$	$0.02220^{+0.00068}_{-0.00062}$	$0.02034 \to 0.02430$	$0.02240^{+0.00062}_{-0.00062}$	$0.02060 \to 0.02435$
	$\theta_{13}/^\circ$	$8.57^{+0.13}_{-0.12}$	$8.20 \to 8.97$	$8.62^{+0.12}_{-0.12}$	$8.25 \to 8.98$
	$\delta_{\rm CP}/^\circ$	194^{+52}_{-25}	$105 \to 405$	230^{+36}_{-25}	$144 \to 350$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.515^{+0.028}_{-0.028}$	$+2.431 \to +2.599$	$+2.510^{+0.027}_{-0.027}$	$+2.430 \to +2.593$

Table 2: Normal Ordering NuFit 5.1 values [33, 34] for the neutrino observables.

We note the significant differences between the two possibilities $n = 1 + \sqrt{6}$ and $n = 1 - \sqrt{6}$. This corresponds to a change of sign in the effective parameter t, which does not affect the predictions for r, θ_{13}, θ_{12}, but does affect the prediction for θ_{23} and δ. This can be understood as the change of sign corresponds to changing from the tangent to a cotangent in the θ_{23} expression (30), and for δ (42) to adding π.

While qualitatively both possibilities are similarly successful in reproducing the experimental data at 3σ, it is visible from the plots how the 1σ range clearly favours different cases. It is worth emphasising how the new case we are considering is able to fit all observables at 1σ, with the exception of θ_{12}, for which the 1σ contour is just slightly above the intersection of all other observables, which include the very narrow contours from θ_{13} and from the mass ratio. To better quantify this we define

$$\chi^2 = \sum_i \left(\frac{x_i^{\rm pred} - x_i^{\rm exp}}{\sigma_i} \right)^2 \tag{44}$$

and list the respective χ^2 values in Table 3. For the $n = 1 + \sqrt{6}$ case, $\chi^2 = 1.87$ can be obtained.

3 Conclusion

In this paper, we have constructed the first complete model of the Littlest Modular Seesaw (LMS), based on CSD$(1 - \sqrt{6}) \approx$ CSD(-1.45), within a consistent framework based on multiple modular symmetries. We also proposed a new related possibility based on CSD$(1 + \sqrt{6}) \approx$ CSD(3.45). In each case, three S_4 modular symmetries are introduced, each with their respective modulus field at a distinct stabilizer, leading to three separate residual subgroups, thus dispensing with vacuum alignment mechanisms. Of the three moduli, two are responsible implementing the viable Littlest Seesaw leading to Trimaximal 1 mixing, which correlates non-trivially with the observed ratio of neutrino masses. The

[4] This is made clear when computing m_{ee}. Alternatively, we can always rotate ν_1 to absorb η_1, but this will not influence the second and third columns.

[5] The results for $n = 1 - \sqrt{6}$ match the results of [8], as expected.

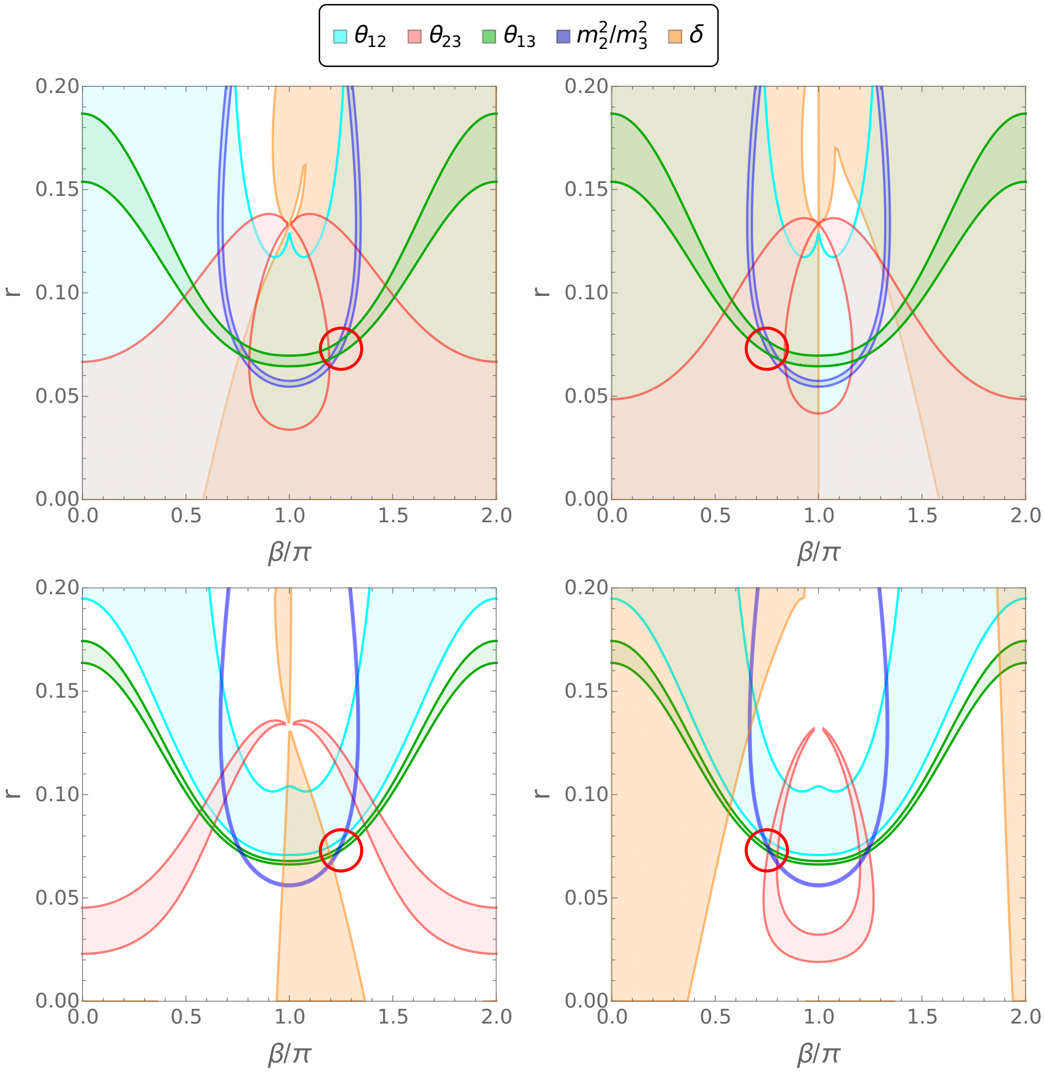

Figure 1: Allowed 3σ (**top**) and 1σ (**bottom**) experimental ranges in the (r, β) plane using NuFit 5.1 values without SK atmospheric data for the $n = 1 + \sqrt{6}$ case (**left**) and for the $n = 1 - \sqrt{6}$ case (**right**). The red circle indicates the best fit region.

remaining modulus guarantees the charged lepton mass matrix is diagonal in the same basis, preserving the predictive power of the model. The result, in the symmetry basis, is a diagonal charged lepton mass matrix and a LMS scenario of a particular kind.

Using a semi-analytical approach, we performed a χ^2 analysis of each case case and showed that good agreement with neutrino oscillation data is obtained, for both possible octants of atmospheric angle, including predictive relations between the leptonic mixing angles and the ratio of light neutrino masses, which non-trivially agree with the experimental values. It is noteworthy that in this very predictive setup, all the models fit the experimental data very well, depending on the choice of stablizers and data set, in one case to within approximately 1σ. This is a remarkable achievement, given that the neutrino mass matrix in the diagonal charged lepton mass basis is determined effectively by two real parameters, m_a, m_b and one phase β together with a discrete choice of $n = 1 \pm \sqrt{6}$. For a given choice of n, the remaining three real parameters determine all the parameters in the neutrino sector, namely all the neutrino masses and the entire PMNS matrix.

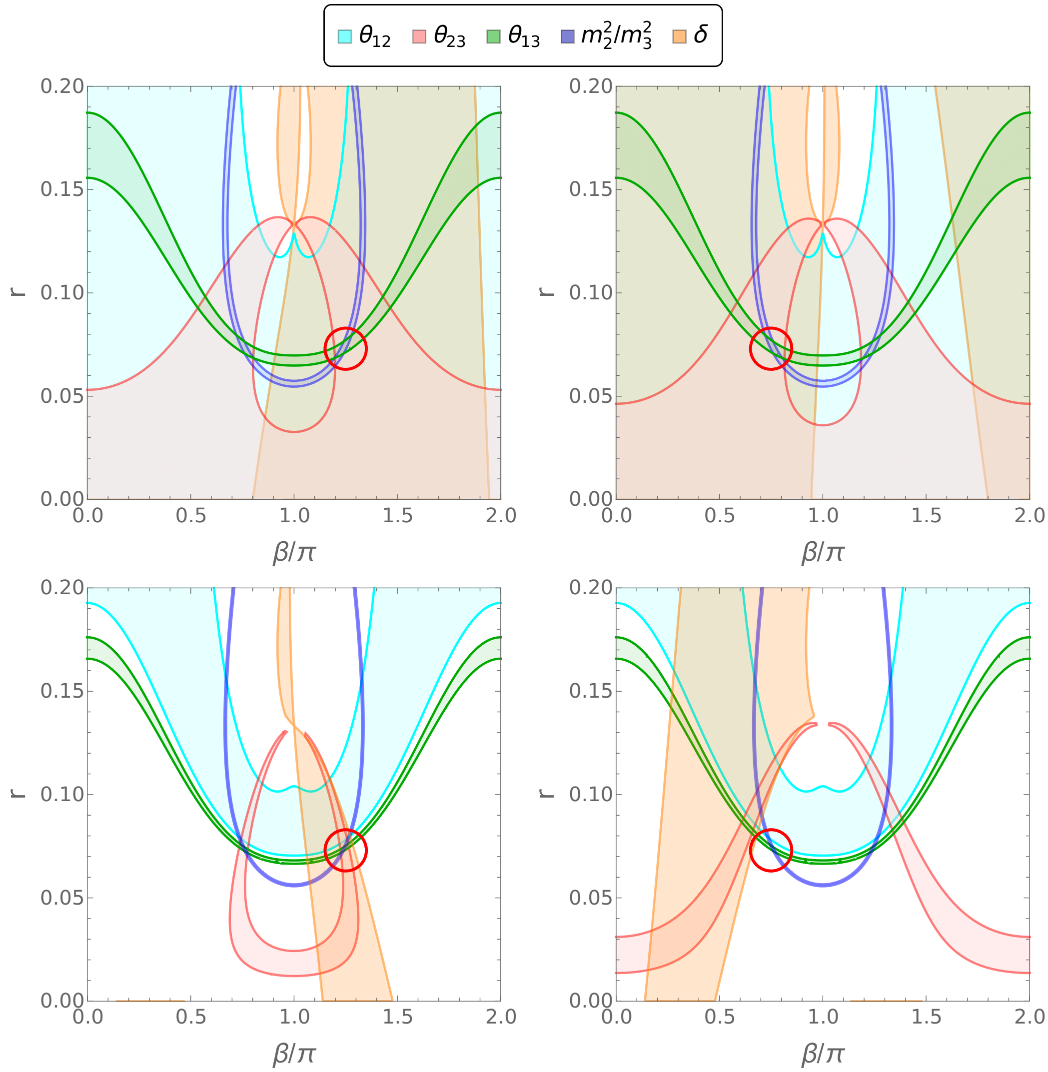

Figure 2: As in Figure 1 but using the NuFit 5.1 values with SK atmospheric data. **Left:** $n = 1 + \sqrt{6}$, **right:** $n = 1 - \sqrt{6}$, **top:** 3σ, **bottom:** 1σ.

By extending the model to include a weighton and the double cover group $\Gamma_4' \simeq S_4'$, we are able to also account for the hierarchy of the charged leptons using modular symmetries, without altering the neutrino predictions.

In summary, we have presented an extremely economical model of leptonic masses and mixing, by combining multiple modular symmetries with the littlest seesaw, and optionally adding a weighton. The latter accounts elegantly for the observed hierarchy of the lepton masses without the need for additional Froggatt-Nielsen style symmetries.

We argue that this is a minimal model of leptonic mixing, as we do not count the moduli as free continuous parameters given that we take them as stabilizers. As such, we have 3 real parameters in the charged lepton sector to fit the 3 masses, 1 real parameter that governs the overall neutrino mass scale, and just 2 effective parameters (the ratio $r = m_b/m_a$ and the phase β) which fit the remaining observables: the neutrino mass ratio, the 3 PMNS mixing angles, the Dirac CP phase and a Majorana phase. The lightest neutrino mass is predicted to be zero and the PMNS phases are predicted in terms of the other observables. Within this predictive setup we are able to fit all the neutrino oscillation data

Goodness of fit against NuFit 5.1 values without SK atmospheric data										
n	χ^2	r	β/π	$m_b/10^{-3}$	$m_2^2/10^{-5}$	$m_3^2/10^{-3}$	θ_{12}	θ_{23}	θ_{13}	δ_{CP}
$1+\sqrt{6}$	29.47	0.076	1.26	2.33	7.19	2.53	34.29	43.06	8.78	262
$1-\sqrt{6}$	4.96	0.073	0.76	2.23	7.45	2.51	34.34	48.26	8.55	284
Goodness of fit against NuFit 5.1 results with SK atmospheric data										
n	χ^2	r	β/π	$m_b/10^{-3}$	$m_2^2/10^{-5}$	$m_3^2/10^{-3}$	θ_{12}	θ_{23}	θ_{13}	δ_{CP}
$1+\sqrt{6}$	1.87	0.074	1.24	2.30	7.42	2.51	34.33	42.03	8.62	257
$1-\sqrt{6}$	25.79	0.077	0.74	2.33	7.15	2.52	34.28	46.76	8.82	277

Table 3: Our χ^2 values for the different cases $n = 1+\sqrt{6}$ and $n = 1-\sqrt{6}$. Note that from Eq. (14) and the definition Eq. (25), the output parameter m_{ee} is directly equal to the input parameters m_b.

to within approximately 1σ.

Acknowledgements

IdMV acknowledges funding from Fundação para a Ciência e a Tecnologia (FCT) through the contract UID/FIS/00777/2020 and was supported in part by FCT through projects CFTP-FCT Unit 777 (UID/FIS/00777/2019), PTDC/FIS-PAR/29436/2017, CERN/FIS-PAR/0004/2019 and CERN/FIS-PAR/0008/2019 which are partially funded through POCTI (FEDER), COMPETE, QREN and EU. The work of ML is funded by FCT Grant No.PD/BD/150488/2019, in the framework of the Doctoral Programme IDPASC-PT. SFK acknowledges the STFC Consolidated Grant ST/L000296/1 and the European Union's Horizon 2020 Research and Innovation programme under Marie Sklodowska-Curie grant agreement HIDDeN European ITN project (H2020-MSCA-ITN-2019//860881-HIDDeN).

A Weighton models

A.1 A minimal weighton model

We now modify the model presented in the main text to include a weighton field ϕ. In order to preserve the features of the previous model (particularly the diagonal charged-lepton mass matrix) we employ S'_4 modular symmetries [7] instead of S_4.

The assignments of the fields under the symmetries are listed in Table 4. Notice that this implementation of the weighton is distinguished from the standard one as the weighton is assigned to non-trivial representations of S'^A_4, S'^B_4, and S'^C_4. Due to this and the representations of the charged leptons, the invariant terms have the desired modular forms Y_τ, Y_μ and Y_e respectively for the field combinations $L\tau^c$, $L\mu^c\phi$ and $Le^c\phi^3$. This is shown (in green) in Table 5, where other possibilites are not invariant.

Since there are no charged leptons with weights under $S'^{A,B}_4$, the charged-leptons Yukawa modular forms must be singlets under $S'^{A,B}_4$ with weight 0 under these symmetries.

By having chosen the weighton to have a positive weight under S'^C_4, there are no additional contributions beyond the leading order ones, as the Yukawa modular forms also have positive weight. An alternative solution, where the weighton has a negative weight under S'^C_4, is presented in Appendix A.2.

Field	$S_4'^A$	$S_4'^B$	$S_4'^C$	$2k_A$	$2k_B$	$2k_C$
L	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0
e^c	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\mathbf{1}'$	0	0	−12
μ^c	$\hat{\mathbf{1}}'$	$\hat{\mathbf{1}}'$	$\mathbf{1}'$	0	0	−6
τ^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	0	0	−2
N_A^c	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{1}$	−4	0	0
N_B^c	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}$	0	−2	0
Φ_{AC}	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0
Φ_{BC}	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0	0
ϕ	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	0	0	+2

Yuk/Mass	$S_4'^A$	$S_4'^B$	$S_4'^C$	$2k_A$	$2k_B$	$2k_C$
$Y_e(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}'$	0	0	6
$Y_\mu(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}'$	0	0	4
$Y_\tau(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}'$	0	0	2
$Y_A(\tau_A)$	$\mathbf{3}'$	$\mathbf{1}$	$\mathbf{1}$	4	0	0
$Y_B(\tau_B)$	$\mathbf{1}$	$\mathbf{3}'$	$\mathbf{1}$	0	2	0
$M_A(\tau_A)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	8	0	0
$M_B(\tau_B)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	4	0

Table 4: Assignments of fields for the weighton version of the model.

	ϕ^0	ϕ^1	ϕ^2	ϕ^3	ϕ^4
Le^c	$(\hat{\mathbf{1}}_0, \hat{\mathbf{1}}_0, \hat{\mathbf{3}}'_{-12})$	$(\mathbf{1}'_0, \mathbf{1}'_0, \mathbf{3}_{-10})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{-8})$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3}'_{-6})$	$(\hat{\mathbf{1}}_0, \hat{\mathbf{1}}_0, \hat{\mathbf{3}}'_{-4})$
$L\mu^c$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{-6})$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3}'_{-4})$	$(\hat{\mathbf{1}}_0, \hat{\mathbf{1}}_0, \hat{\mathbf{3}}'_{-2})$	$(\mathbf{1}'_0, \mathbf{1}'_0, \mathbf{3}_0)$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{+2})$
$L\tau^c$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3}'_{-2})$	$(\hat{\mathbf{1}}_\mathbf{0}, \hat{\mathbf{1}}_\mathbf{0}, \hat{\mathbf{3}}'_0)$	$(\mathbf{1}'_0, \mathbf{1}'_0, \mathbf{3}_{+2})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{+4})$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3}'_{+6})$

Table 5: Irreps of the leptonic tensor products with different powers of the weighton. The invariant combinations are highlighted in green.

A.2 An alternative weighton model

In this subsection we provide an alternative weighton model, that does not require assigning large modular weights to the charged lepton fields.

This allows fields (in particular charged lepton fields) to be assigned as distinct non-trivial singlets of the underlying modular symmetries, as shown in Table 6.

Field	$S_4'^A$	$S_4'^B$	$S_4'^C$	$2k_A$	$2k_B$	$2k_C$
L	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0
e^c	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\mathbf{1}'$	0	0	0
μ^c	$\hat{\mathbf{1}}'$	$\hat{\mathbf{1}}'$	$\mathbf{1}'$	0	0	−2
τ^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	0	0	−2
N_A^c	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{1}$	−4	0	0
N_B^c	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}$	0	−2	0
Φ_{AC}	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0
Φ_{BC}	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0	0
ϕ	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	0	0	−2

Yuk/Mass	$S_4'^A$	$S_4'^B$	$S_4'^C$	$2k_A$	$2k_B$	$2k_C$
$Y_e(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}'$	0	0	6
$Y_\mu(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}'$	0	0	4
$Y_\tau(\tau_C)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}'$	0	0	2
$Y_A(\tau_A)$	$\mathbf{3}'$	$\mathbf{1}$	$\mathbf{1}$	4	0	0
$Y_B(\tau_B)$	$\mathbf{1}$	$\mathbf{3}'$	$\mathbf{1}$	0	2	0
$M_A(\tau_A)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	8	0	0
$M_B(\tau_B)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	4	0

Table 6: Assignments of fields for the alternative weighton version of the model.

Table 7 shows the assignments of the different field combinations and clarifies how the non-trivial singlet choices of the charged leptons allow only one coupling at leading order of powers of ϕ, with the next leading order term appearing only with the insertion of additional ϕ^4 [6]. We estimate this

[6]Since the weighton is charged under $S_4'^C$, and the 1D irreps have at most $\mathbf{r}^4 = \mathbf{1}$, there will always be corrections to

	ϕ^0	ϕ^1	ϕ^2	ϕ^3	ϕ^4
Le^c	$(\hat{\mathbf{1}}_0, \hat{\mathbf{1}}_0, \hat{\mathbf{3}}'_0)$	$(\mathbf{1'}_0, \mathbf{1'}_0, \mathbf{3}_{-2})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{-4})$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3'}_{-6})$	$(\hat{\mathbf{1}}_0, \hat{\mathbf{1}}_0, \hat{\mathbf{3}}'_{-8})$
$L\mu^c$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{-2})$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3'}_{-4})$	$(\hat{\mathbf{1}}_0, \hat{\mathbf{1}}_0, , \hat{\mathbf{3}}'_{-6})$	$(\mathbf{1'}_0, \mathbf{1'}_0, \mathbf{3}_{-8})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{-10})$
$L\tau^c$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3'}_{-2})$	$(\hat{\mathbf{1}}_\mathbf{0}, \hat{\mathbf{1}}_0, \hat{\mathbf{3}}'_{-4})$	$(\mathbf{1'}_0, \mathbf{1'}_0, \mathbf{3}_{-6})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{-8})$	$(\mathbf{1}_0, \mathbf{1}_0, \mathbf{3'}_{-10})$
$L\Phi_{AC}N^c_A$	$(\mathbf{3'}_{-4}, \mathbf{1}_0, \mathbf{1}_0)$	$(\hat{\mathbf{3}}'_{-4}, \hat{\mathbf{1}}_0, \hat{\mathbf{1}}_{-2})$	$(\mathbf{3}_{-4}, \mathbf{1'}_0, \mathbf{1'}_{-4})$	$(\hat{\mathbf{3}}_{-4}, \hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_{-6})$	$(\mathbf{3'}_{-4}, \mathbf{1}_0, \mathbf{1}_{-8})$
$L\Phi_{BC}N^c_B$	$(\mathbf{1}_0, \mathbf{3'}_{-2}, \mathbf{1}_0)$	$(\hat{\mathbf{1}}_0, \hat{\mathbf{3}}'_{-2}, \hat{\mathbf{1}}_{-2})$	$(\mathbf{1'}_0, \mathbf{3}_{-2}, \mathbf{1'}_{-4})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{3}}_{-2}, \hat{\mathbf{1}}'_{-6})$	$(\mathbf{1}_0, \mathbf{3'}_{-2}, \mathbf{1}_{-8})$
$N^c_AN^c_A$	$(\mathbf{1}_{-8}, \mathbf{1}_0, \mathbf{1}_0)$	$(\hat{\mathbf{1}}_{-8}, \hat{\mathbf{1}}_0, \hat{\mathbf{1}}_{-2})$	$(\mathbf{1'}_{-8}, \mathbf{1'}_0, \mathbf{1'}_{-4})$	$(\hat{\mathbf{1}}'_{-8}, \hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_{-6})$	$(\mathbf{1}_{-8}, \mathbf{1}_0, \mathbf{1}_{-8})$
$N^c_BN^c_B$	$(\mathbf{1}_0, \mathbf{1}_{-4}, \mathbf{1}_0)$	$(\hat{\mathbf{1}}_0, \hat{\mathbf{1}}_{-4}, \hat{\mathbf{1}}_{-2})$	$(\mathbf{1'}_0, \mathbf{1'}_{-4}, \mathbf{1'}_{-4})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{1}}'_{-4}, \hat{\mathbf{1}}'_{-6})$	$(\mathbf{1}_0, \mathbf{1}_{-4}, \mathbf{1}_{-8})$
$N^c_AN^c_B$	$(\mathbf{1'}_{-4}, \mathbf{1'}_{-2}, \mathbf{1}_0)$	$\left(\hat{\mathbf{1}}'_{-4}, \hat{\mathbf{1}}'_{-2}, \hat{\mathbf{1}}_{-2}\right)$	$(\mathbf{1}_{-4}, \mathbf{1}_{-2}, \mathbf{1'}_{-4})$	$(\hat{\mathbf{1}}_{-4}, \hat{\mathbf{1}}_{-2}, \hat{\mathbf{1}}'_{-6})$	$(\mathbf{1'}_{-4}, \mathbf{1'}_{-2}, \mathbf{1}_{-8})$
$N^c_A\Phi_{AC}N^c_A$	$(\mathbf{3}_{-8}, \mathbf{1}_0, \mathbf{3}_0)$	$(\hat{\mathbf{3}}_{-8}, \hat{\mathbf{1}}_0, \hat{\mathbf{3}}_{-2})$	$(\mathbf{3'}_{-8}, \mathbf{1'}_0, \mathbf{3'}_{-4})$	$(\hat{\mathbf{3}}'_{-8}, \hat{\mathbf{1}}'_0, \hat{\mathbf{3}}'_{-6})$	$(\mathbf{3}_{-8}, \mathbf{1}_0, \mathbf{3}_{-8})$
$N^c_B\Phi_{AC}N^c_B$	$(\mathbf{3}_0, \mathbf{1}_{-4}, \mathbf{3}_0)$	$(\hat{\mathbf{3}}_0, \hat{\mathbf{1}}_{-4}, \hat{\mathbf{3}}_{-2})$	$(\mathbf{3'}_0, \mathbf{1'}_{-4}, \mathbf{3'}_{-4})$	$(\hat{\mathbf{3}}'_0, \hat{\mathbf{1}}'_{-4}, \hat{\mathbf{3}}'_{-6})$	$(\mathbf{3}_0, \mathbf{1}_{-4}, \mathbf{3}_{-8})$
$N^c_A\Phi_{AC}N^c_B$	$(\mathbf{3'}_{-4}, \mathbf{1'}_{-2}, \mathbf{3}_0)$	$(\hat{\mathbf{3}}'_{-4}, \hat{\mathbf{1}}'_{-2}, \hat{\mathbf{3}}_{-2})$	$(\mathbf{3}_{-4}, \mathbf{1}_{-2}, \mathbf{3'}_{-4})$	$(\hat{\mathbf{3}}_{-4}, \hat{\mathbf{1}}_{-2}, \hat{\mathbf{3}}'_{-6})$	$(\mathbf{3'}_{-4}, \mathbf{1'}_{-2}, \mathbf{3}_{-8})$
$N^c_A\Phi_{BC}N^c_A$	$(\mathbf{1}_{-8}, \mathbf{3}_0, \mathbf{3}_0)$	$(\hat{\mathbf{1}}_{-8}, \hat{\mathbf{3}}_0, \hat{\mathbf{3}}_{-2})$	$(\mathbf{1'}_{-8}, \mathbf{3'}_0, \mathbf{3'}_{-4})$	$(\hat{\mathbf{1}}'_{-8}, \hat{\mathbf{3}}'_0, \hat{\mathbf{3}}'_{-6})$	$(\mathbf{1}_{-8}, \mathbf{3}_0, \mathbf{3}_{-8})$
$N^c_B\Phi_{BC}N^c_B$	$(\mathbf{1}_0, \mathbf{3}_{-4}, \mathbf{3}_0)$	$(\hat{\mathbf{1}}_0, \hat{\mathbf{3}}_{-4}, \hat{\mathbf{3}}_{-2})$	$(\mathbf{1'}_0, \mathbf{3'}_{-4}, \mathbf{3'}_{-4})$	$(\hat{\mathbf{1}}'_0, \hat{\mathbf{3}}'_{-4}, \hat{\mathbf{3}}'_{-6})$	$(\mathbf{1}_0, \mathbf{3}_{-4}, \mathbf{3}_{-8})$
$N^c_A\Phi_{BC}N^c_B$	$(\mathbf{1'}_{-4}, \mathbf{3'}_{-2}, \mathbf{3}_0)$	$(\hat{\mathbf{1}}'_{-4}, \hat{\mathbf{3}}'_{-2}, \hat{\mathbf{3}}_{-2})$	$(\mathbf{1}_{-4}, \mathbf{3}_{-2}, \mathbf{3'}_{-4})$	$(\hat{\mathbf{1}}_{-4}, \hat{\mathbf{3}}_{-2}, \hat{\mathbf{3}}'_{-6})$	$(\mathbf{1'}_{-4}, \mathbf{3'}_{-2}, \mathbf{3}_{-8})$

Table 7: Irreps of the leptonic tensor products with different powers of the weighton following the new assignments. The invariant combinations are highlighted in green.

suppression factor should to be around 10^{-5} by assuming $\mathcal{O}(1)$ couplings for the charged leptons [7].

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the leading terms with 4 more weighton insertions. This is avoided by taking the weighton model of Appendix A.1.

[7] Namely, we take $\langle\phi\rangle/M = \epsilon = 6.5 \times 10^{-2}$, to have $m_\mu \sim 0.92\,\epsilon\, m_\tau$ and $m_e \sim 1.08\,\epsilon^3\, m_\tau$.

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