On Subamalgams of Partially Ordered Monoids

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Abstract

The study of pomonoid amalgams was initiated by Fakhuruddin in the 1980s and subsequently extended by Bulman-Fleming, Sohail and the authors in the 2000s. We further investigate pomonoids amalgams and in particular we consider the concept of subpomonoid amalgams possessing a suitable ordered version of the unitary property. If $[U; T_1, T_2]$ is an *amalgam of subpomonoids* of the amalgam $[U; S_1, S_2]$ we consider the question of whether the free product of the pomonoid amalgam $[U; T_1, T_2]$ is poembeddable in the free product of the pomonoid amalgam $[U; S_1, S_2]$, giving a sufficient condition in terms of *strongly pounitary subpomonoids*.

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1 Introduction and preliminaries

Many of the definitions and concepts introduced in this paper have natural counterparts and are generalisations of similar concepts important in the unordered theory of semigroup amalgams. We refer the reader to the bibliography for more details.
We start with a brief introduction to the category of $S-$posets and introduce the concepts of *pounitary* and *popure* morphisms. Section 2 investigates direct limits and pushouts of $S-$posets with the pounitary property while Section 3 concerns pullpacks. Lastly in section 4 we answer the main problem on amalgams of subpomonoids, the proof of our main rsult follows, in some way, from Renshaw's method of dealing with pure monomorphisms in the unordered case (see [11] for details). However, the construction in the category of pomonoids is more complicated than the unordered case and needs more care.

Free product with amalgamation is an important construction in many areas of algebra and it is reasonable to enquire about sub-structures of these free products. In particular, we would like to know under what conditions a subposemigroup of a free product of posemigroups is itself a free product. Howie [7] showed that, unlike the case for groups, if T_i are subsemigroups of semigroups S_i such that $U \subseteq T_i$ then the amalgamated free product $\prod_U^* T_i$ is not in general a subsemigroup of the amalgamated free product $\prod_U^* S_i$. The authors have previously extended some of Renshaw's subsequent study of this problem, to deal with partially ordered monoids ([1]) and here we further extend his work from [11] and [12].

A *partially ordered monoid* (*pomonoid*) is a monoid S endowed with a partial order relation $\leq$ which is compatible with the binary operation on S in the sense that

$$\forall s, t, u \in S, t \leq u \Rightarrow st \leq su \text{ and } ts \leq us.$$

Recall that if X and Y are posets then a map $f\colon\ X \to Y$ is said to be *monotone* if $\forall x, y \in X, x \leq y \Rightarrow f(x) \leq f(y)$, while it is said to be an *order embedding* if $\forall x, y \in X, x \leq y \Leftrightarrow f(x) \leq f(y)$. It is obvious that any order embedding is one-to-one and monotone. A surjective order embedding is called an *order isomorphism.*

If S and T are pomonoids and A is a non-empty poset then A is called a *right $S-$poset* if A is a right $S-$act and the $S-$action is monotonic in each variable. *Left $S-$posets* are defined dually. Let A and B be $S-$posets. A map $f : A \to B$ is an *$S-$poset morphism* when f is monotonic and an $S-$morphism. The class of $S-$posets and $S-$poset morphisms forms a category in which the monomorphisms are exactly the one-to-one $S-$poset morphisms and the epimorphisms are exactly the onto $S-$poset morphisms [3]. When A is both a left $S-$poset and a right $T-$poset and satisfies $(sa)t = s(at)$ for all $s \in S, a \in A, t \in T$ then A is called an *$(S, T)-$biposet.*

From [3], a *congruence* θ on an $S-$poset A is an $S-$act congruence such that A/θ can be endowed with a suitable partial order making $A \to A/\theta$ is an $S-$poset morphism.
Define the relation $\leq_\theta$ on A by

$$a \leq_\theta b \text{ if and only if there exist } n \geq 1 \text{ and } a_1, a'_1, \ldots, a_n, a'_n \in A, \text{ such that}$$

$$a \leq a_1\ \theta\ a'_1 \leq a_2\ \theta\ a'_2 \leq \cdots \leq a_n\ \theta\ a'_n \leq b.$$

Let R be a binary relation on right $S-$poset A. The *right $S-$poset congruence $\nu(R)$ induced on A by R* is defined as

$$a\ \nu(R)\ b \text{ if and only if } a \leq_{\alpha(R)} b \leq_{\alpha(R)} a$$

where $a\ \alpha(R)\ b$ if and only if either $a = b$ or there exist $n \geq 1$ and $(x_i, x'_i) \in R, s_i \in S$ for $i = 1, \ldots, n$ such that

$$a = x_1 s_1, x'_1 s_1 = x_2 s_2, \ldots, x'_n s_n = b.$$

We shall consider the following order on $A/\nu(R)$

$$a\nu(R) \leq b\nu(R) \text{ if and only if } a \leq_{\alpha(R)} b.$$

The *congruence on A generated by* R is the right $S-$poset congruence $\theta(R) = \nu(R \cup R^{-1})$.

Let S be a pomonoid, A be a right $S-$poset and B be a left $S-$poset. Then the *tensor product* of the $S-$posets A and B is the poset $A \otimes_S B = (A \times B)/\tau$ where τ is the poset congruence generated by

$$H = \{((as, b), (a, sb)) | a \in A, b \in B, s \in S\}.$$

Here $(A \times B)$ is endowed with a poset structure in the obvious way.
For simplicity the tensor product will dnoted by $A \otimes B$. It is known from [14, Theorem 5.2] that the order relation on $A \otimes B$ is defined by:
$a \otimes b \leq a' \otimes b'$ if and only if there exists a scheme such that

$$\begin{array}{rr}
a \leq a_1 s_1 & s_1 b \leq t_1 b_2 \\
a_1 t_1 \leq a_2 s_2 & s_2 b_2 \leq t_2 b_3 \\
\vdots & \vdots \\
a_{n-1} t_{n-1} \leq a_n s_n & s_n b_n \leq t_n b' \\
a_n t_n \leq a', &
\end{array}$$

where $n \geq 1$, $a_1, \ldots, a_n \in A, b_2, \ldots, b_n \in B$ and $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$.

Let λ: $A \to B$ be a left $S-$poset morphism and Y be a right $S-$poset. From [1], if $y \otimes a \leq y' \otimes a'$ in $Y \otimes A$ then $y \otimes \lambda(a) \leq y' \otimes \lambda(a')$ in $Y \otimes B$, while from [13, Corollary 1.3] $y \otimes s \leq y' \otimes s'$ in $Y \otimes_S S$ if and only if $ys \leq y's'$.

Let U be a subpomonoid of the pomonoid S. Recall from [1] that we say

1. U is *upper strongly right pounitary* in S if $v \leq su \Rightarrow s \in U$
2. U is *lower strongly right pounitary* in S if $su' \leq v' \Rightarrow s \in U$
3. U is *strongly right pounitary* in S if $(v \leq su \vee su' \leq v') \Rightarrow s \in U$
4. U is *right pounitary* in S if $v \leq s_1 u_1, s_1 u'_1 \leq s_2 u_2, \ldots s_n u'_n \leq v' \Rightarrow s_1, s_2, \ldots, s_n \in U$
5. U is *right unitary* in S if $su = v \Rightarrow s \in U$,

where $v, v', u, u', u_1, u'_1, \ldots, u_n, u'_n \in U, s, s_1, s_2, \ldots, s_n \in S$. Left-sided and two-sided versions of these conditions are defined similarly. In all cases, the two-sided version is equivalent to the conjunction of the right and left sided version. If both the right and left sided versions hold then we shall omit the right (left) prefix.

We can extend these definitions to $S-$posets as follows (see [1]). Let $f : X \longrightarrow Y$ be a right $U-$poset order embedding. Then

1. f is a *upper strongly right pounitary* (USRPU) if $f(x) \leq yu \Rightarrow y \in \text{im } f$
2. f is a *lower strongly right pounitary* (LSRPU) if $yu \leq f(x) \Rightarrow y \in \text{im } f$

3. f is a *strongly right pounitary* (SRPU) if $(f(x) \leq yu \vee yu' \leq f(x')) \Rightarrow y \in \text{im } f$

4. f is a *right pounitary* (RPU) if

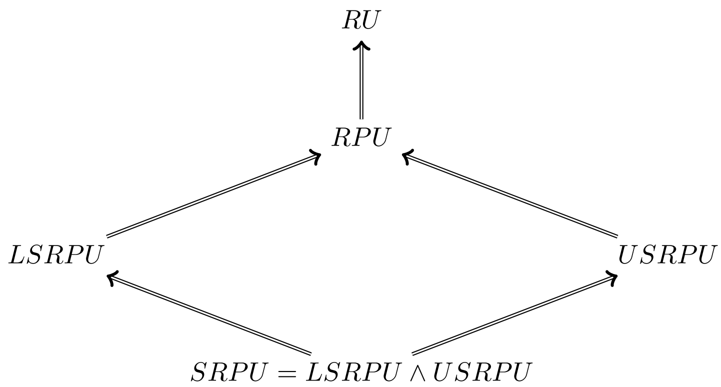
$$f(x) \leq y_1u_1, y_1u_1' \leq y_2u_2, \ldots, y_nu_n' \leq f(x') \Rightarrow y_1, y_2, \ldots, y_n \in \text{im } f$$

5. f is a *right unitary* (RU) if $yu = f(x) \Rightarrow y \in \text{im } f$,

where $u, v, u_1, u_1', \ldots, u_n, u_n' \in U, y, y_1, \ldots, y_n \in Y$, $x \in X$.

Again, the left and two-sided versions of these conditions are defined in similar fashion. The inclusion map $U \longrightarrow S$ is said to be pounitary if U is pounitary in S. From now on, when using these properties, we shall assume that f is an order embedding.

The implications represented by the following diagram are fairly clear.



Lemma 1.1. *Let $f : X \to Y$ and $g : Y \to Z$ be right pounitary. Then $g \circ f$ is also right pounitary.*

Proof. Suppose that $gf(x) \leq z_1u_1, z_1u_1' \leq z_2u_2, \ldots, z_nu_n' \leq gf(x')$ where $z_i \in Z, u_i, u_i' \in U$ and $1 \leq i \leq n$. Since g is pounitary, we have $z_i = g(y_i)$ for some $y_i \in Y$. Hence we may write that $gf(x) \leq g(y_1)u_1, g(y_1)u_1' \leq g(y_2)u_2, \ldots, g(y_n)u_n' \leq gf(x')$. Because g is an order embedding, we find that $f(x) \leq y_1u_1, y_1u_1' \leq y_2u_2, \ldots, y_nu_n' \leq f(x')$. Hence $y_i = f(x_i)$ since f is pounitary. Therefore $z_i = gf(x_i)$ as required. □

The following result will be important later.

Lemma 1.2 ([1, Lemma 3.2])**.** *Let $f : X \longrightarrow Y$ be lower strongly right pounitary and A be a left $U-$poset. If $y \otimes a \leq f(x) \otimes a'$ in $Y \otimes_U A$ then there exists $x' \in X$ such that $y = f(x')$ and indeed $f(x') \otimes a \leq f(x) \otimes a'$ in im $f \otimes A$.*

The proof of the previous result can quite easily be modified to prove the following lemma.

Lemma 1.3. *Let $f : X \longrightarrow Y$ be right pounitary and A a left $U-$poset. If $y \otimes a = f(x) \otimes a'$ in $Y \otimes_U A$ then there exists $x' \in X$ such that $y = f(x')$.*

The following lemma will also prove useful later. Its proof, being straightforward, is omitted.

Lemma 1.4. *Let $f : X \to Y$ and $g : A \to B$ be (resp. lower strongly, upper strongly, strongly) pounitary. Then $f \otimes g : X \otimes A \to Y \otimes B$ is (resp. lower strongly, upper strongly, strongly) pounitary.*

A right $S-$poset order embedding $f : X \to Y$ is called *right popure* (resp. *pure*) if for all left $S-$posets A the map $f \otimes 1 : X \otimes A \to Y \otimes A$ is an order embedding (resp. monomorphism). *Left popure* and *left pure* are defined dually. An $(S, S)-$poset order embedding $f : X \to Y$ is said to be *popure* (resp. *pure*) if for all left $S-$poset A and all right $S-$poset B the map $1 \otimes f \otimes 1 : B \otimes X \otimes A \to B \otimes Y \otimes A$ is an order embedding (resp. monomorphism). It is clear that an $S-$poset order embedding is popure then it is also pure.

Lemma 1.5 (Cf. [1, Lemma 3.3])**.** *Let U be a pomonoid and let $f : X \to Y$ be a right pounitary $U-$poset morphism. Then f is right popure.*

In a similar manner we can prove the following lemma.

Lemma 1.6. *If $f : X \longrightarrow Y$ is $(U, U)-$pounitary then f is popure.*

We say that $A = [U; S_i; \varphi_i]$ is a *pomonoid amalgam* when $\{S_i | i \in I\}$ is a family of pomonoids, U is a pomonoid and $\{\varphi_i : U \to S_i | i \in I\}$ is a family of pomonoid order embeddings. The pomonoid U is called the *core* of the amalgam. We shall often omit reference to φ_i when the context is clear. When there exist a pomonoid W and monomorphisms (resp. order embeddings) $\theta_i : S_i \to W$ such that $\theta_i\varphi_i = \theta_j\varphi_j$ for all $i \neq j$ in I then the amalgam is said to be *weakly embeddable* (resp. *weakly poembeddable*) in W. The amalgam is said to be *strongly embeddable* (resp. *strongly poembeddable*) in W if in addition $\theta_i(S_i) \cap \theta_j(S_i) = \theta_i\varphi_i(U)$.

Recall from [1] that the *free product* of a family $\{S_i : i \in I\}$ of pairwise disjoint posemigroups, $\mathcal{F} = \prod^* S_i$ is the set of non-empty words $a_1 \dots a_n$ with each $a_k \in S_i$ for some $i \in I, 1 \leq k \leq n$ and no two adjacent letters in the same S_i. The binary operation can be defined on this set by

$$(a_1 \dots a_n)(b_1 \dots b_m) = \begin{cases} a_1 \dots a_n b_1 \dots b_m & \text{if } a_n \in S_i, b_1 \in S_j, i \neq j \\ a_1 \dots (a_n b_1) \dots b_m & \text{if } a_n, b_1 \in S_i. \end{cases}$$

The partial order on $\mathcal{F}$ is defined as $a_1 \dots a_r \leq b_1 \dots b_s$ if and only if

1. $r = s$
2. $a_i \leq b_i$ in S_j, for each $a_i, b_i \in S_j$ where $1 \leq i \leq r$ and $1 \leq j \leq n$.

Recall from [1] that the *amalgamated free product of the pomonoid amalgam* $[U; S_1, S_2]$ is $\mathcal{F}/\sigma$ where $\sigma = \nu(R \cup R^{-1})$ is the pomonoid congruence on $\mathcal{F}$ generated by

$$R = \{(\varphi_i(u), \varphi_j(u)) : u \in U\},$$

and $\varphi_i : U \to S_i$ for $i = 1, 2$. It is usually denoted by $S_1 *_U S_2$.

Let X be a subposet of a poset P. We say that X is *convex* if for any $x, y \in X, z \in P$,

$$(x \leq z \leq y) \Rightarrow z \in X.$$

An $S-$poset morphism $f : X \to Y$ is called convex if $\mathrm{im}(f)$ is convex in Y.

2 Direct limits

Fakhruddin was the first to study direct limits in the category of $S-$posets in the 1980s and the reader is directed to [6] for more details.
Let I be a quasi-ordered set (i.e. a set with a reflexive and transitive relation). A *direct system* of right $S-$posets $(X_i, \varphi^i_j)_{i \in I}$ is a collection of right $S-$posets X_i and a collection of $S-$poset morphism φ^i_j: $X_i \to X_j$, $i \leq j$, which satisfy

1. $\varphi_i^i = 1_{X_i}$,
2. $\varphi_k^j \circ \varphi_j^i = \varphi_k^i$ whenever $i \leq j \leq k$.

The *direct limit* of this direct system is an $S-$poset X together with $S-$poset morphisms $\varphi_i\colon X_i \to X$ which satisfy

1. $\varphi_j \circ \varphi_j^i = \varphi_i$ whenever $i \leq j$,
2. if Y is an $S-$poset and $\theta_i\colon X_i \to Y$ is an $S-$poset morphism such that $\theta_j \circ \varphi_j^i = \theta_i$ whenever $i \leq j$, then there exists a unique $S-$poset morphism $\psi\colon X \to Y$ such that $\psi \circ \varphi_i = \theta_i$ for all $i \in I$.

It is easy to show that for any direct system of $S-$posets the direct limit exists and it is unique up to isomorphism.

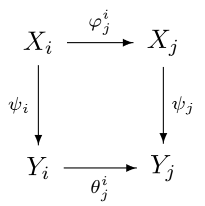
We say that I is *directed* if for all $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$. The first part of the following lemma appears as [2, Proposition 2.5], the other parts are from [1, Lemma 1.3].

Lemma 2.1. *Let (X_i, φ_j^i) be a direct system in the category of $(S,T)-$posets with directed index set and let (X, α_i) be the direct limit of this system. Then*

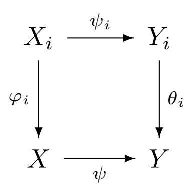
1. *$\varphi_i(x_i) \leq \varphi_j(x_j)$ in X if and only if there exists $k \geq i, j$ such that $\varphi_k^i(x_i) \leq \varphi_k^j(x_j)$;*
2. *the map φ_i is one-to-one if and only if φ_k^i is one to one for all $k \geq i$;*
3. *the map φ_i is an order embedding if and only if φ_k^i is an order embedding for all $k \geq i$.*

We can then easily deduce the following useful result.

Lemma 2.2. (Cf. [10, Lemma I.3.20]) *Let (X_i, φ_j^i) and (Y_i, θ_j^i) be directed systems in $S-$posets, having the same directed index set I and let (X, φ_i) and (Y, θ_i) be the direct limits of these systems respectively. If there exist $S-$poset order embeddings (resp. monomorphism) $\psi_i : X_i \longrightarrow Y_i$ such that the diagram*



commutes whenever $i \leq j$, then there exists an $S-$poset order embedding (resp. monomorphism) $\psi : X \longrightarrow Y$ such that the diagram



commutes for all i.

Lemma 2.3. *Let* (X_i, φ_j^i) *and* (Y_i, θ_j^i) *be directed systems of* $S-$*posets having the same directed index set and let* (X, φ_i) *and* (Y, θ_i) *be the direct limits of these systems respectively. If there exist (upper, lower) strongly right pounitary maps* $\psi_i : X_i \to Y_i$ *such that the diagram*

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_j^i} & X_j \\ {\scriptstyle \psi_i}\downarrow & & \downarrow{\scriptstyle \psi_j} \\ Y_i & \xrightarrow[\theta_j^i]{} & Y_j \end{array}$$

commutes whenever $i \leq j$*, then there exists a (upper, lower) strongly right pounitary map* $\psi : X \to Y$ *such that the diagram*

$$\begin{array}{ccc} X_i & \xrightarrow{\psi_i} & Y_i \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \theta_i} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

commutes for all i*.*

Proof. We prove this result for upper strongly pounitary maps. Let $x \in X$. Notice that $x = \phi_i(x_i)$ for some $i \in I, x_i \in X_i$. Define $\psi : X \to Y$ by $\psi(x) = \theta_i\psi_i(x_i)$. From Lemma 2.2, the map ψ is an order embedding. Suppose that $\psi(x) \leq ys$, where $y = \theta_j(y_j)$ so that $\theta_i\psi_i(x_i) \leq \theta_j(y_j)s$. Since θ_i is upper strongly right pounitary, there exists $y_i \in Y_i$ such that $\theta_i(y_i) = \theta_j(y_j)$. Hence $\psi_i(x_i) \leq y_i s$. Also, since ψ_i is upper strongly right pounitary there exists $x_i' \in X_i$ such that $y_i = \psi_i(x_i')$. Hence $\theta_i\psi_i(x_i') = \theta_i(y_i) = \theta_j(y_j) = y$. So, ψ is upper strongly right pounitary. □

When $I = 2$ the direct limit in the category of $S-$posets is called the *pushout.* Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \beta}\downarrow & & \\ C & & \end{array}$$

From [3] the pushout is isomorphic to the quotient of the coproduct $D = B\dot{\cup}C$ by the $S-$poset congruence ρ generated by

$$R = \{(\alpha(a), \beta(a)) : a \in A\}.$$

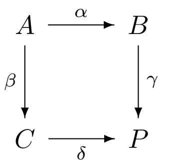
The maps $\gamma : B \to D/\rho$ and $\delta : C \to D/\rho$ are given by $\gamma(b) = b\rho$ and $\delta(c) = c\rho$ respectively. As in the category of $S-$acts, tensor products preserve pushouts [2].

If $f : X \to Y$ is an $S-$poset morphism then we define the relation

$$\overrightarrow{\ker(f)} = \{(x, x') \in X \times X : f(x) \leq f(x')\}.$$

The following result characterises the pushout when α is a monomorphism and β is onto.

Proposition 2.4. *Let*



be a pushout of $S-$posets and let α be a monomorphism and β be onto. Then $P \cong B/\sigma$ where σ is the $S-$poset congruence induced on B by

$$T = \{(\alpha(a), \alpha(a')) : (a, a') \in \overrightarrow{ker\beta}\}.$$

Proof. Let $P' = B/\sigma$. Define $\gamma : B \to P'$ by $\gamma(b) = b\sigma$. Since σ is a congruence then it is clear that γ preserves the $S-$action on B. Also, define $\delta : C \to P'$ by $\delta(c) = \gamma\alpha(a) = \alpha(a)\sigma$, where $c = \beta(a)$ and note that δ is well-defined. It is clear that $\gamma\alpha = \delta\beta$. Now suppose there exists an $S-$poset D and $S-$poset morphisms $\gamma' : B \longrightarrow D$ and $\delta' : C \longrightarrow D$ such that $\gamma'\alpha = \delta'\beta$. We want to show that there exists a unique $S-$poset morphism $\psi : P' \longrightarrow D$ such that $\psi\gamma = \gamma'$ and $\psi\delta = \delta'$. To this end, define $\psi(b\sigma) = \gamma'(b)$. Then $\psi\gamma = \gamma'$ and $\psi\delta\beta(a) = \psi\gamma\alpha(a) = \gamma'\alpha(a) = \delta'\beta(a)$. To show ψ is a well-defined monotonic map suppose that $b\sigma \leq b'\sigma$. Then there exist $n \geq 1$, $b_i, b'_i \in B$ and $1 \leq i \leq n$ such that

$$b \leq b_1\alpha(T)b'_1 \leq b_2\alpha(T)b'_2 \leq \cdots \leq b_n\alpha(T)b'_n \leq b'.$$

We can assume that the number of $\alpha(T)$ terms is minimal. If there are no such terms then $b \leq b'$. Consequently $\gamma'(b) \leq \gamma'(b')$ and so $\psi(b\sigma) \leq \psi(b'\sigma)$. Otherwise, for each i there exists a scheme such as:

$$b_i = \alpha(a_1)s_1, \alpha(a'_1)s_1 = \alpha(a_2)s_2, \ldots, \alpha(a'_m)s_m = b'_i$$

where $(\alpha(a_j), \alpha(a'_j)) \in T$, $s_j \in S$, $1 \leq j \leq m$ and $m \geq 1$. Hence $\beta(a_j s_j) \leq \beta(a'_j s_j)$. Since α is one to one, we have $a'_j s_j = a_{j+1}s_{j+1}$ and so $\beta(a'_j s_j) = \beta(a_{j+1}s_{j+1})$. This gives $\beta(a_1 s_1) \leq \beta(a'_m s_m)$. Now

$$\gamma'(b_i) = \gamma'\alpha(a_1 s_1) = \delta'\beta(a_1 s_1) \leq \delta'\beta(a'_m s_m) = \gamma'\alpha(a'_m s_m) = \gamma'(b'_i).$$

Since γ' is monotonic, we have $\gamma'(b) \leq \gamma'(b')$ and so $\psi(b\sigma) \leq \psi(b'\sigma)$. Hence ψ is monotonic and therefore well-defined. Since γ is an $S-$poset map and σ is an $S-$poset congruence, it follows that ψ preserves the $S-$action on P'. We can then easily check that ψ is unique with respect to this property. □

In the above result when α is strongly pounitary, it easily follows from the definition of σ and the definition of strongly pounitary, that the following conditions hold.

1. $b\sigma \leq b'\sigma$ if and only if $b \leq b'$ or $b = \alpha(a)$, $b' = \alpha(a')$ such that $\beta(a) \leq \beta(a')$;

2. $b\sigma = b'\sigma$ if and only if $b = b'$ or $b = \alpha(a)$, $b' = \alpha(a')$ such that $\beta(a) = \beta(a')$.

Consider the previous pushout diagram where α is strongly pounitary and β is onto. Define

$$R = \{(\alpha(a), \alpha(a')) : (a, a') \in \ker \beta\}.$$

Then it is easy to show that $\rho = R \cup 1_B$ is an $S-$act congruence. This relation is also an $S-$poset congruence since $b \; \rho \; b'$ whenever $b \leq_\rho b' \leq_\rho b$. The order on B/ρ is the compatible order which is given by $b\rho \leq b'\rho$ if and only if $b \leq b'$ or $b = \alpha(a)$, $b' = \alpha(a')$ with $\beta(a) \leq \beta(a')$.

Define the relation

$$\tau_\alpha = \{(b, b') \in B \times B : b \leq b' \text{ or } b \leq \alpha(a), \alpha(a') \leq b' \text{ where } (a, a') \in \overrightarrow{\text{ker}}\,\beta\}.$$

Proposition 2.5. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow[\delta]{} & P \end{array}$$

be a pushout of $S-$posets. If α is an order embedding and β is onto, then $\overrightarrow{ker}\ \gamma = \tau_\alpha$.

Proof. From Proposition 2.4, $P \simeq B/\sigma$. Suppose that $(b, b') \in \overrightarrow{ker}\ \gamma$. Then $b\sigma \leq b'\sigma$ and so there exists $n \geq 1$, $b_i, b'_i \in B$ with $1 \leq i \leq n$ such that

$$b \leq b_1\ \alpha(T)\ b'_1 \leq b_2\ \ldots\ \leq b_n\ \alpha(T)\ b'_n \leq b'.$$

By using a technique similar to the one used in the proof of Proposition 2.4 we can show that for each $b_i\ \alpha(T)\ b'_i$ there are two cases - either $b_i = b'_i$ or $b_i = \alpha(a_i)$, $b'_i = \alpha(a'_i)$ and $\beta(a_i) \leq \beta(a'_i)$. Since α is an order embedding we can easily prove that either $b \leq b'$ or $b \leq \alpha(a_1)$, $\alpha(a_n) \leq b'$ and $\beta(a_1) \leq \beta(a_n)$. Hence $(b, b') \in \tau_\alpha$.

Conversely suppose that $(b, b') \in \tau_\alpha$. Then if $b \leq b'$ it is clear that $(b, b') \in \overrightarrow{ker}\ \gamma$. While if $b \leq \alpha(a)$, $\alpha(a') \leq b'$, where $\beta(a) \leq \beta(a')$, then $\delta\beta(a) \leq \delta\beta(a')$. Hence $\gamma(b) \leq \gamma\alpha(a) = \delta\beta(a)$ $\leq \delta\beta(a') = \gamma\alpha(a') \leq \gamma(b')$ as required. □

When α is $S-$strongly pounitary we have the following special case.

Corollary 2.6. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow[\delta]{} & P \end{array}$$

be a pushout of $S-$posets. If α is strongly pounitary and β is onto then

$$\overrightarrow{ker}\gamma = \{(b, b') : b \leq b' \textit{ or } b = \alpha(a), \alpha(a') = b' \textit{ where } (a, a') \in \overrightarrow{ker}\,\beta\}.$$

3 Pullbacks

The notion of pullback in the category of $S-$posets has been studied by Bulman-Fleming and Mahmoudi [3]. In the category of $S-$acts, pullbacks have a close connection with the study of monoid amalgams, in particular when studying the idea of subamalgams. We shall see later that this is also the case for pomonoid amalgams.

It is known from [3] that the *pullback* of the diagram of $S-$posets

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & D \end{array}$$

exists and is isomorphic to $A \simeq \{(b, c) \in B \times C : \alpha(b) = \beta(c)\}$ where the $S-$poset morphisms $\gamma : A \longrightarrow B$, $\delta : A \longrightarrow C$ are given by $\gamma(b, c) = b$ and $\delta(b, c) = c$ respectively. We shall denote the pullback simply by A.

Proposition 3.1. *Consider the pullback diagram*

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & D \end{array}$$

of $S-$posets. If α satisfies any of the properties of being an order embedding, monomorphism, onto, convex, upper strongly pounitary or lower strongly pounitary, then δ also satisfies the same property.

Proof. Suppose that α is an order embedding and $\delta(b,c) \leq \delta(b',c')$. Then $c \leq c'$ and so $\beta(c) \leq \beta(c')$. In addition since $\alpha\gamma = \beta\delta$, we have $\alpha(b) = \beta(c)$ and $\alpha(b') = \beta(c')$. Hence $\alpha(b) \leq \alpha(b')$ and so $b \leq b'$. Therefore $(b,c) \leq (b',c')$. A similar argument holds if α is a monomorphism.

Suppose now that α is onto. It is clear that for all $c \in C$ there exists $d \in D$ such that $\beta(c) = d$. But since α is onto, there exists $b \in B$ such that $d = \alpha(b)$. Hence $c = \delta(b,c)$.

Suppose next that α is convex and $\delta(b,c) \leq c'' \leq \delta(b',c')$. Then $c \leq c'' \leq c'$ and so $\beta(c) \leq \beta(c'') \leq \beta(c')$. It is known that $\alpha(b) = \beta(c)$ and $\alpha(b') = \beta(c')$. Hence $\alpha(b) \leq \beta(c'') \leq \alpha(b')$. Since α is convex, there exists $b'' \in B$ such that $\alpha(b'') = \beta(c'')$. Therefore $\delta(b'',c'') = c''$. A similar argument holds when α is either upper or lower strongly pounitary. □

Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & D \end{array}$$

of $S-$posets. This diagram is called an *almost pullback* if whenever $\alpha(b) = \beta(c)$ then there exists a unique $a \in A$ such that $\gamma(a) = b$ and $\delta(a) = c$. It is clear that every pullback is an almost pullback.

Proposition 3.2. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & D \end{array}$$

be a commutative diagram in $S-$posets and γ be an $S-$poset order embedding. Then this diagram is an almost pullback if and only if it is a pullback.

Proof. Suppose the diagram is an almost pullback and there exists an $S-$poset P and $S-$poset morphisms $\gamma' : P \to B$ and $\delta' : P \to C$ such that $\alpha\gamma' = \beta\delta'$. Define $\psi : P \to A$ by $\psi(p) = a$ where a is such that $\gamma'(p) = \gamma(a)$ and $\delta'(p) = \delta(a)$. It is clear that a is unique from the definition of almost pullback. Suppose that $p \leq p'$ in P so that $\gamma'(p) \leq \gamma'(p')$. Then $\gamma(a) \leq \gamma(a')$ and so $a \leq a'$. Hence ψ is an $S-$poset morphism. It is clear that $\gamma\psi = \gamma'$ and $\delta\psi = \delta'$ and so A is a pullback. The converse is clear. □

Proposition 3.3. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ C & \xrightarrow{\psi} & D \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\alpha} \\ A & \xrightarrow[\beta]{} & E \end{array}$$

be commutative diagram in $S-$posets where $\varphi\gamma = 1_A$, ψ is an $S-$poset order embedding and $\overrightarrow{ker}\ \alpha \subseteq \tau_\psi$ where

$$\tau_\psi = \{(d,d') : d \leq d' \, or \; d \leq \psi(c), \psi(c') \leq d' \, where \; (c,c') \in \overrightarrow{ker}\ \varphi\}.$$

Then $\alpha\epsilon$ is an $S-$poset order embedding if the following conditions hold

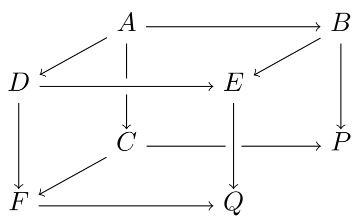
1. *ϵ is an order embedding,*
2. *if $\epsilon(b) \leq \psi(c)$ then there exists $a \in A$ such that $\epsilon(b) \leq \epsilon f(a) \leq \psi(c)$ and if $\psi(c) \leq \epsilon(b)$ there exists $a' \in A$ such that $\psi(c) \leq \epsilon f(a') \leq \epsilon(b)$.*

Proof. Suppose that $\alpha\epsilon(b) \leq \alpha\epsilon(b')$. Then $(\epsilon(b), \epsilon(b')) \in \tau_\psi$ and hence if $\epsilon(b) \leq \epsilon(b')$ then by (1) $b \leq b'$, as required. Otherwise there exist $(c,c') \in \overrightarrow{\ker}\ \varphi$ such that $\epsilon(b) \leq \psi(c)$ and $\psi(c') \leq \epsilon(b')$. From (2) there exist $a, a' \in A$ such that $\epsilon(b) \leq \epsilon f(a) \leq \psi(c)$ and $\psi(c') \leq \epsilon f(a') \leq \epsilon(b')$. Hence $b \leq f(a), f(a') \leq b', \gamma(a) \leq c$ and $c' \leq \gamma(a')$. Now from $\varphi\gamma(a) \leq \varphi(c)$ and $\varphi(c') \leq \varphi\gamma(a')$, we have $\varphi\gamma(a) \leq \varphi(c) \leq \varphi(c') \leq \varphi\gamma(a')$ and so $a \leq a'$. Consequently $b \leq f(a) \leq f(a') \leq b'$ as required. □

Proposition 3.4. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\lambda} \\ C & \xrightarrow[f]{} & P \end{array} \qquad\qquad \begin{array}{ccc} D & \xrightarrow{\delta} & E \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\eta} \\ F & \xrightarrow[g]{} & Q \end{array}$$

be pushouts in the category of $S-$posets and suppose there exist $S-$order embeddings $\varphi : A \to D$, $\theta : B \to E$ and $\epsilon : C \to F$ which makes the following diagram commute.



Also suppose that the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\theta} \\ D & \xrightarrow[\delta]{} & E \end{array}$$

is a pullback. When $\gamma : D \to F$ and $\beta : A \to C$ are both onto and $\alpha : A \to B$ and $\delta : D \to E$ are both strongly pounitary then there exists a unique $S-$order embedding $\zeta : P \to Q$ making the resultant cube commute.

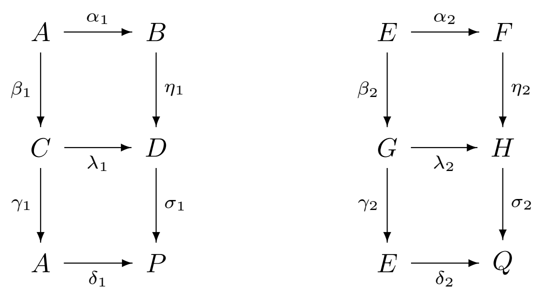
Proof. We have that $P \simeq B/\sigma$, where σ is defined in Proposition 2.4. From the universal property of the pushout P, there exists a unique $S-$poset morphism $\zeta : P \to Q$ such that $\zeta(\lambda(b)) = \eta\, \theta(b)$. From Corollary 2.6 $\overrightarrow{ker}\ \eta = \{(\delta(d), \delta(d')) : (d, d') \in \overrightarrow{ker}\ \gamma\} \cup \overrightarrow{1_E}$. To show ζ is an order embedding suppose $\eta\theta(b) \leq \eta\theta(b')$. Then there are two cases:

1. If $\theta(b) \leq \theta(b')$ then $\lambda(b) \leq \lambda(b')$ as required;
2. Let $\theta(b) = \delta(d)$, $\theta(b') = \delta(d')$ where $\gamma(d) \leq \gamma(d')$. Since the last diagram is a pullback there exist unique $a, a' \in A$ such that $b = \alpha(a)$, $d = \varphi(a)$, $b' = \alpha(a')$ and $d' = \varphi(a')$. This gives $\epsilon\beta(a) = \gamma\varphi(a) \leq \gamma\varphi(a') = \epsilon\beta(a')$. Hence $\beta(a) \leq \beta(a')$ and so $\lambda(b) = f\beta(a) \leq f\beta(a') = \lambda(b')$ as required.

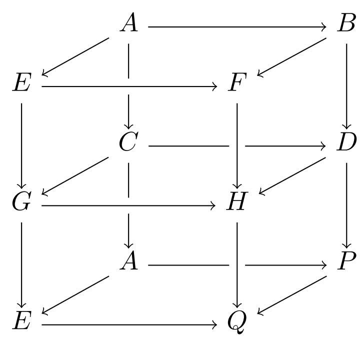
□

The following rather technical result will also prove useful later.

Proposition 3.5. *Let*



be commutative diagrams in the category of $S-$posets where the top squares are pullbacks and the bottom squares are pushouts and $\gamma_1\beta_1 = 1_A$, $\gamma_2\beta_2 = 1_E$. When η_1, η_2 are $S-$poset order embeddings λ_1, λ_2 are strongly pounitary and there exist $S-$poset order embeddings $\varphi : A \to E$, $\theta : C \to G$, $\epsilon : B \to F$, $\psi : D \to H$, $\zeta : P \to Q$ making the following diagram (with the obvious labelling of arrows)



commute, and when the diagrams

are pullbacks then

$$\begin{array}{ccc} B & \xrightarrow{\epsilon} & F \\ {\scriptstyle \sigma_1\eta_1}\downarrow & & \downarrow{\scriptstyle \sigma_2\eta_2} \\ P & \xrightarrow[\zeta]{} & Q \end{array}$$

is also a pullback.

Proof. From Corollary 2.6 it is known that $\overrightarrow{ker}\ \sigma_1 = \{(\lambda_1(c), \lambda(c')) : (c, c') \in \overrightarrow{ker}\ \gamma_1\} \cup \overrightarrow{1_D}$. First, we are aiming to show that $\sigma_1\eta_1$ and $\sigma_2\eta_2$ are order embeddings. We show that Condition (2) of Proposition 3.3 is satisfied. To see this suppose that $\eta_1(b) \leq \lambda_1(c)$. Since λ_1 is strongly pounitary then $\eta_1(b) = \lambda_1(c')$. Also, since the top square is a pullback, there exists a unique $a \in A$ such that $\alpha_1(a) = b$ and $\beta_1(a) = c'$. Hence $\eta_1(b) = \eta_1\alpha_1(a) = \lambda_1\beta_1(a) \leq \lambda_1(c)$ as required. A similar conclusion holds when $\lambda_1(c) \leq \eta_1(b)$. Therefore, from Proposition 3.3, the map $\sigma_1\eta_1$ is an order embedding. In a similar way the map $\sigma_2\eta_2$ is also an order embedding. Now suppose that $\zeta(p) = \sigma_2\eta_2(f)$ where $p \in P$ and $f \in F$. Since γ_1 is onto, by Proposition 2.4, $P \simeq D/\sigma$ where σ is defined as in Proposition 2.4. Hence $p = \sigma_1(d)$ for some $d \in D$. Now, we have $\zeta(p) = \zeta\sigma_1(d) = \sigma_2\psi(d)$ and hence $\sigma_2\psi(d) = \sigma_2\eta_2(f)$. So we have $\sigma_2\psi(d) \leq \sigma_2\eta_2(f)$ and $\sigma_2\eta_2(f) \leq \sigma_2\psi(d)$. From Corollary 2.6 we see that $\overrightarrow{ker}\ \sigma_2 = \{(\lambda_2(g), \lambda_2(g')) : (g, g') \in \overrightarrow{ker}\ \gamma_2\} \cup \overrightarrow{1_H}$. Now there are four cases, as given below.

1. Let $\psi(d) \leq \eta_2(f)$ and $\eta_2(f) \leq \psi(d)$. Then $\psi(d) = \eta_2(f)$. Since the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\epsilon} & F \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_2} \\ D & \xrightarrow[\psi]{} & H \end{array}$$

   is a pullback, there exists a unique $b \in B$ such that $d = \eta_1(b)$ and $f = \epsilon(b)$. Hence $f = \epsilon(b)$ and $p = \sigma_1(d) = \sigma_1\eta_1(b)$.

2. Suppose that $\eta_2(f) \leq \psi(d)$ and $\psi(d) = \lambda_2(g)$, $\eta_2(f) = \lambda_2(g')$ where $(g, g') \in \overrightarrow{ker}\ \gamma_2$. Since $\psi(d) = \lambda_2(g)$ and the diagram

$$\begin{array}{ccc} C & \xrightarrow{\lambda_1} & D \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \psi} \\ G & \xrightarrow[\lambda_2]{} & H \end{array}$$

   is a pullback, there exists a unique $c \in C$ such that $d = \lambda_1(c)$ and $g = \theta(c)$. Suppose that $b = \alpha_1\gamma_1(c)$. Then $\sigma_1\eta_1(b) = \sigma_1\lambda_1\beta_1\gamma_1(c) = \delta_1\gamma_1(c) = \sigma_1\lambda_1(c) = \sigma_1(d) = p$. In addition, $\sigma_2\eta_2\epsilon(b) = \zeta\sigma_1\eta_1(b) = \zeta(p) = \sigma_2\eta_2(f)$ and so $\epsilon(b) = f$. To show that b is unique, suppose there exists $b' \in B$ such that $f = \epsilon(b')$ and $p = \sigma_1\eta_1(b')$. Then $f = \epsilon(b') = \epsilon(b)$ and since ϵ is order embedding it follows that $b = b'$.

3. Assume that $\psi(d) \leq \eta_2(f)$ and $\eta_2(f) = \lambda_2(g_2)$, $\psi(d) = \lambda_2(g_2')$ where $(g_2, g_2') \in \overrightarrow{ker}\ \gamma_2$. By using an argument similar to Case 2 we can show that there exists a unique $b \in B$ such that $f = \epsilon(b)$ and $p = \sigma_1\eta_1(b)$.

4. Let $\psi(d) = \lambda_2(g_1)$, $\eta_2(f) = \lambda_2(g_1')$ where $(g_1, g_1') \in \overrightarrow{ker}\ \gamma_2$ and $\eta_2(f) = \lambda_2(g_2)$, $\psi(d) = \lambda_2(g_2')$ where $(g_2, g_2') \in \overrightarrow{ker}\ \gamma_2$. We have $\psi(d) = \lambda_2(g_1) = \lambda_2(g_2')$ and $\eta_2(f) =$

$\lambda_2(g_1') = \lambda_2(g_2)$. Since λ_2 is an order embedding, we have $g_1 = g_1' = g_2 = g_2'$. So we have $\psi(d) = \lambda_2(g_1)$, $\eta_2(f) = \lambda_2(g_1')$ where $(g_1, g_1') \in ker\ \gamma_2$. Finally using an argument similar to Case 2, we see that there exists a unique $b \in B$ such that $f = \epsilon(b)$ and $p = \sigma_1\eta_1(b)$.

Consequently, in all of the cases the diagram

$$\begin{array}{ccc} B & \xrightarrow{\epsilon} & F \\ {\scriptstyle \sigma_1\eta_1}\downarrow & & \downarrow{\scriptstyle \sigma_2\eta_2} \\ P & \xrightarrow[\zeta]{} & Q \end{array}$$

is an almost pullback and hence by Proposition 3.2 it is a pullback. □

We now examine a connection between pounitary maps and pullbacks.

Proposition 3.6. *Let U be a subpomonoid of a pomonoid T and T be a subpomonoid of a pomonoid S. Suppose also that U is pounitary in S. If A is a $T-$poset, B is an $S-$poset and $f : A \to B$ is $U-$pounitary then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A \otimes_U T & \xrightarrow[f\otimes\gamma]{} & B \otimes_U S \end{array}$$

is a pullback where the arrows $A \to A \otimes_U T$ and $B \to B \otimes_U S$ are defined by $a \mapsto a \otimes 1, b \mapsto b \otimes 1$ respectively.

Proof. Clearly U is pounitary in the pomonoid T. Suppose that $b \otimes 1 = f(a) \otimes t$ in $B \otimes_U S$. Since f is also pounitary, we deduce from Lemma 1.3 that there exists $a' \in A$ such that $b = f(a')$ and that $t \in U$. Consequently $f(a') = b = b1 = f(a)t = f(at)$ and so $a' = at$. Hence the diagram is an almost pullback. It is a pullback by Proposition 3.2 is a pullback. □

Theorem 3.7. *Let $f : X \to Y$ be upper (resp. lower) strongly left pounitary and $g : A \to B$ be lower (resp. upper) strongly right pounitary. Then the commutative diagram*

$$\begin{array}{ccc} A \otimes X & \xrightarrow{g\otimes 1_X} & B \otimes X \\ {\scriptstyle 1_A\otimes f}\downarrow & & \downarrow{\scriptstyle 1_B\otimes f} \\ A \otimes Y & \xrightarrow[g\otimes 1_Y]{} & B \otimes Y \end{array}$$

is a pullback.

Proof. To show that the diagram is an almost pullback suppose that $g(a) \otimes y = b \otimes f(x)$ in $B \otimes Y$. Then from Lemma 1.3 and Lemma 1.5 we see that $y = f(x')$, $b = g(a')$ and $a \otimes x' = a' \otimes x$. Hence the above diagram is an almost pullback. It is a pullback by Proposition 3.2. □

We can also show that tensor products preserve pullbacks in the following sense.

Theorem 3.8. *Let the commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

be a pullback and all its maps be $(S,S)-$pounitary. Then for any right $S-$posets X and left $S-$posets Y the diagram

$$\begin{array}{ccc} X\otimes A\otimes Y & \longrightarrow & X\otimes B\otimes Y \\ \downarrow & & \downarrow \\ X\otimes C\otimes Y & \longrightarrow & X\otimes D\otimes Y \end{array}$$

is a pullback in the category of posets.

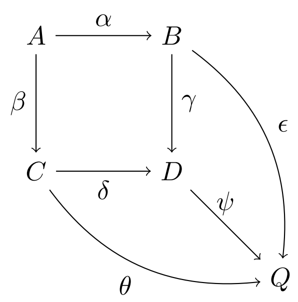
Proof. Suppose that $x\otimes\delta(c)\otimes y = x'\otimes\gamma(b)\otimes y'$ in $X\otimes D\otimes Y$. Using Lemma 1.4 and Lemma 1.3, we can deduce that there exists $b'\in B, c'\in C$ such that $\delta(c)=\gamma(b')$ and $\gamma(b)=\delta(c')$. Since the first diagram is a pullback, there exists unique $a,a'\in A$ such that $\alpha(a)=b$, $\beta(a)=c'$, $\alpha(a')=b'$ and $\beta(a')=c$. Hence $x\otimes c\otimes y = x\otimes\beta(a')\otimes y$ and $x'\otimes b\otimes y' = x'\otimes\alpha(a)\otimes y'$. Since $x\otimes\gamma(b')\otimes y = x\otimes\delta(c)\otimes y = x'\otimes\gamma(b)\otimes y'$, we have by Lemma 1.6 $x\otimes b'\otimes y = x'\otimes b\otimes y'$. So, $x\otimes\alpha(a')\otimes y = x'\otimes\alpha(a)\otimes y'$ and hence, again by Lemma 1.6, we have $x\otimes a'\otimes y = x'\otimes a\otimes y'$. It is clear that $x'\otimes a\otimes y'$ is unique with the required property and consequently the diagram is an almost pullback and therefore a pullback by Proposition 3.2. □

The following result will be useful later.

Lemma 3.9. *Let the commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

be an almost pullback in the category of $S-$posets. In addition suppose that



commutes. If $\psi : D\to Q$ is an $S-$poset order embedding then the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ C & \xrightarrow[\theta]{} & Q \end{array}$$

is an almost pullback.

Proof. Suppose that $\psi : D \to Q$ is order embedding and $\epsilon(b) = \theta(c)$. Then $\psi\gamma(b) = \psi\delta(c)$. Hence $\gamma(b) = \delta(c)$. Now, because

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

is an almost pullback, there exists a unique $a \in A$ such that $\alpha(a) = b$ and $\beta(a) = c$ as required. □

The proof of the following lemma is straightforward and so it is omitted.

Lemma 3.10. *Let*

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\epsilon} & E \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\psi} \\ C & \xrightarrow[\delta]{} & D & \xrightarrow[\theta]{} & F \end{array}$$

be a commutative diagram in the category of $S-$posets and suppose that the left and right hand squares are almost pullbacks. Then the outer square is also an almost pullback.

It is well known that pullbacks in the category of pomonoids exist. In fact it is straightforward to prove that the *pullback* of the diagram of pomonoids and pomonoid morphisms

$$\begin{array}{ccc} & & S \\ & & \downarrow{\scriptstyle\alpha} \\ T & \xrightarrow[\beta]{} & P \end{array}$$

is $Q \simeq \{(s,t) \in S \times T : \alpha(s) = \beta(t)\}$ together with the pomonoid morphisms $\gamma : Q \longrightarrow S$, $\delta : Q \longrightarrow T$ which are defined by $\gamma(s,t) = s$ and $\delta(s,t) = t$ respectively. The next result establishes a connection between the poembeddability property of the pomonoid amalgam and pomonoid pullbacks.
The proof of the following remark is an easy exercise.

Remark 3.11. *The pomonoid amalgam $[U; S_1, S_2]$ is strongly poembeddable (resp. embeddable) in a pomonoid T if and only if there exist pomonoid order embeddings (resp. monomorphisms) $\lambda_i : S_i \to T$ such that*

$$\begin{array}{ccc} U & \xrightarrow{\varphi_1} & S_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\lambda_1} \\ S_2 & \xrightarrow[\lambda_2]{} & T \end{array}$$

commutes and is a pullback.

4 Subpomonoid amalgams

In the unordered case Howie [7] shows that for monoids $U \subseteq T_i \subseteq S_i$ the free product of the monoid amalgam $[U; T_i]_{i\in I}$ is not always embeddable in the free product of the monoid amalgam $[U; S_i]_{i\in I}$. He also proved that when U and T_i are unitary in S_i for all $i \in I$, then such an embedding is possible. Renshaw extended this work and proved that the embedding is also possible when either $U \to T_i$ and $T_i \to S_i$ are both pure [11], or $U \to T_i$ and $T_i \to S_i$ are both perfect [12].

As the trivial order is a partial order relation, Howie's example from [7] can be used to show that embeddability is not always guaranteed in the ordered case. Hence the free product of the pomonoid amalgam $[U; T_i]_{i\in I}$ is not always poembeddable in the free product of the pomonoid amalgam $[U; S_i]_{i\in I}$. We aim to find the conditions under which this poembeddability is possible.

Let $[U; S_i]_{i\in I}$ be a pomonoid amalgam and T_i, $i \in I$, be a family of pomonoids such that $T_i \subseteq S_i$ for all $i \in I$. Then we shall say that the pomonoid amalgam $[U; T_i]_{i\in I}$ is a *subpomonoid amalgam* of the pomonoid amalgam $[U; S_i]_{i\in I}$. We shall restrict our attention to the case of $|I| = 2$.

Let $f : X \to Y$ be right $U-$poset morphism where U is a subpomonoid of a pomonoid S, Y is a right $U-$poset and X is a right $S-$poset. The *free $S-$extension of X and Y* is a right $S-$poset $F = F(S; X, Y)$ with a $U-$poset morphism $g : Y \to F$ such that

1. $h = gf : X \to F$ is an $S-$poset morphism;
2. if there is a right $S-$poset Z and a right $U-$poset morphism $\alpha : Y \to Z$ such that $\beta = \alpha f$ is a right $S-$poset morphism then there exists a unique right $S-$poset morphism $\psi : F \to Z$ such that $\psi g = \alpha$ and $\psi h = \beta$.

From [1] it is known that the free $S-$extension exists in the category of $S-$posets and is unique up to isomorphism. The authors have also shown that it is possible to define the free $S-$extension in terms of pushouts.

Proposition 4.1. *Let U be a subpomonoid of a pomonoid T and T be a subpomonoid of a pomonoid S such that U is strongly pounitary in S and T is strongly pouintary in S. Suppose also that A is a $(U, T)-$poset, B, D are $(U, U)-$posets, C is a $(U, S)-$poset, $\alpha_1 : A \to B$ and $\alpha_2 : C \to D$ are $(U, U)-$poset morphism, $\delta : A \to C$ is a $(U, T)-$poset morphism and $\epsilon : B \to D$ is a $(U, U)-$poset morphism. If α_1, α_2, δ and ϵ are $(U, U)-$strongly pounitary such that*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ C & \xrightarrow[\alpha_2]{} & D \end{array}$$

is a pullback then there exists a $(U, U)-$order embedding $\psi : F(T; A, B) \to F(S; C, D)$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta_1} & F(T; A, B) \\ {\scriptstyle\epsilon}\downarrow & & \downarrow{\scriptstyle\psi} \\ D & \xrightarrow[\beta_2]{} & F(S; C, D) \end{array}$$

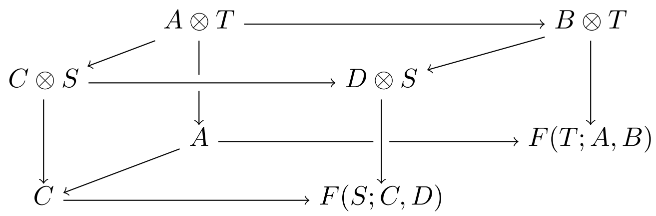
is a pullback, where $\beta_1 : B \to F(T; A, B)$ and $\beta_2 : D \to F(S; C, D)$ are the canonical maps.

Proof. Since U is strongly pounitary in S it follows that U is also strongly pounitary in T. Hence from [1] Theorem 3.5, the maps $\beta_1 : B \to F(T; A, B)$ and $\beta_2 : D \to F(S; C, D)$ are $(U, U)-$strongly pounitary.

Define $\psi : F(T; A, B) \to F(S; C, D)$ by $\psi((b \otimes t)/\sigma_1) = (\epsilon(b) \otimes t)/\sigma_2$, where σ_1 is a $(U, T)-$poset congruence induced on $B \otimes T$ by $R_1 = \{(\alpha_1(a) \otimes t, \alpha_1(a') \otimes t') : at \leq a't'\}$ and σ_2 is a $(U, S)-$poset congruence induced on $D \otimes S$ by $R_2 = \{(\alpha_2(c) \otimes s, \alpha_2(c') \otimes s') : cs \leq c's'\}$. From [1] Lemma 2.3 the $(U, U)-$posets $F(T; A, B)$ and $F(S; C, D)$ are the pushouts of the diagrams

$$\begin{array}{ccc} A \otimes T & \longrightarrow & B \otimes T \\ \downarrow & & \\ A & & \end{array} \qquad\qquad \begin{array}{ccc} C \otimes S & \longrightarrow & D \otimes S \\ \downarrow & & \\ C & & \end{array}$$

respectively. Consequently the diagram



commutes. Hence from Lemma 1.3 and its dual, Lemma 1.5 and Proposition 3.4, the map ψ is an order embedding whenever the following diagram

$$\begin{array}{ccc} A \otimes T & \longrightarrow & B \otimes T \\ \downarrow & & \downarrow \\ C \otimes S & \longrightarrow & D \otimes S \end{array}$$

is a pullback. The above diagram is an almost pullback since from Lemma 3.10 it can be described as follows:

$$\begin{array}{ccc} A \otimes T & \longrightarrow & B \otimes T \\ \downarrow & & \downarrow \\ C \otimes T & \longrightarrow & D \otimes T \\ \downarrow & & \downarrow \\ C \otimes S & \longrightarrow & D \otimes S \end{array}$$

The top square is an almost pullback by the assumption and by Theorem 3.8, and the bottom square is an almost pullback from Theorem 3.7. Hence from Proposition 3.2 it is a pullback. To show that

$$\begin{array}{ccc} B & \xrightarrow{\beta_1} & F(T; A, B) \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \psi} \\ D & \xrightarrow[\beta_2]{} & F(S; C, D) \end{array}$$

is a pullback we use Proposition 3.5 and the fact that the following four diagrams are pullbacks.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \otimes T & \longrightarrow & B \otimes T \end{array} \qquad \begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ C \otimes S & \longrightarrow & D \otimes S \end{array}$$

$$\begin{array}{ccc} A \otimes T & \longrightarrow & B \otimes T \\ \downarrow & & \downarrow \\ C \otimes S & \longrightarrow & D \otimes S \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & D \\ \downarrow & & \downarrow \\ B \otimes T & \longrightarrow & D \otimes S. \end{array}$$

Since $U \to T$ and $U \to S$ are strongly pounitary, it is clear by Theorem 3.7 that the first and second diagrams are pullbacks. That the third diagram is pullback has been shown above. The fourth diagram is a pullback by Proposition 3.6. This completes the proof. $\square$

Let $[U; T_1, T_2]$ be a subpomonoid amalgam of a pomonoid amalgam $[U; S_1, S_2]$ and let U be strongly pounitary in T_i and T_i be strongly pounitary in S_i, $i \in \{1, 2\}$. Using a construction similar to [9] Theorem 1 define a directed system (Y_n, k_n) in the category of $U-$posets as follows:

Let $Y_1 = S_1$, $Y_2 = S_1 \otimes_U S_2$ and $k_1 : Y_1 \to Y_2$ be given by $k_1(s_1) = s_1 \otimes 1$. Define inductively $Y_n = F(S_i; Y_{n-2}, Y_{n-1}) = (Y_{n-1} \otimes_U S_i)/\delta_{n-2}, i \equiv n \pmod 2$, where δ_{n-2} is the S_i-poset congruence induced on $Y_{n-1} \otimes_U S_i$ by

$$V_{n-2} = \{(k_{n-2}(y_{n-2}) \otimes s_i, k_{n-2}(y'_{n-2}) \otimes s'_i) : y_{n-2}s_i \leq y'_{n-2}s'_i\},$$

and let $k_{n-1} : Y_{n-1} \to Y_n$ be the $U-$poset morphism defined by $k_{n-1}(y_{n-1}) = (y_{n-1} \otimes 1)\delta_{n-2}$. Then a typical element of Y_n is

$$y_n = (\ldots((s_1 \otimes s_2 \otimes s_3)\delta_1 \otimes s_4)\delta_2 \otimes \ldots \otimes s_n)\delta_{n-2}.$$

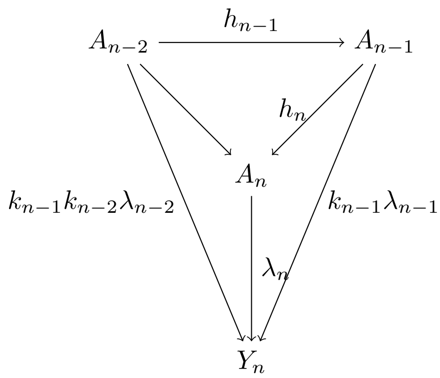
For simplicity, the element y_n of Y_n will be denoted by $[s_1, \ldots, s_n]$ and a typical element of $S_1 *_U S_2$ by $(s_1, \ldots, s_n)$. By a similar argument, define the directed system (A_n, h_n), where

$$A_n = (\ldots(T_1 \otimes_U T_2 \otimes_U T_1)\sigma_1 \otimes_U \cdots \otimes_U T_i)\sigma_{n-2}$$

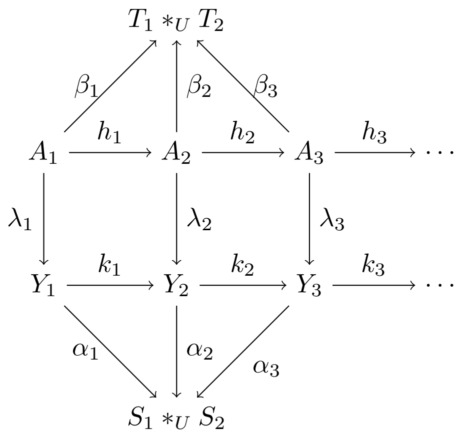
$$\sigma_{n-2} = \{(h_{n-2}(x_{n-2}) \otimes t_i, h_{n-2}(x'_{n-2}) \otimes t'_i) : x_{n-2}t_i \leq x'_{n-2}t'_i\}$$

and $i \equiv n \pmod 2$. For simplicity, a typical element of A_n will be denoted by $[t_1, \ldots, t_n]'$ and a typical element of $T_1 *_U T_2$ by $(t_1, \ldots, t_n)'$. It follows from [1] Theorem 2.7 that the free product of the pomonoid amalgam $[U; S_1, S_2]$, $S_1 *_U S_2$, and the free product of the pomonoid amalgam $[U; T_1, T_2]$, $T_1 *_U T_2$, are direct limits in the category of $U-$posets of these directed systems respectively. From [1] Theorem 3.5 the maps k_n and h_n are $(U, U)-$strongly pounitary order embeddings. In addition from [1] Lemma 3.4 and Theorem 3.6 the maps $\alpha_n : Y_n \longrightarrow S_1 *_U S_2$ and $\beta_n : A_n \longrightarrow T_1 *_U T_2$ are $(U, U)-$strongly pounitary.

Define $\lambda_1 : A_1 \longrightarrow Y_1$ as the inclusion map and $\lambda_2 : A_2 \longrightarrow Y_2$ by $\lambda_2(t_1 \otimes t_2) = t_1 \otimes t_2$, and define the map $\lambda_n : A_n \longrightarrow Y_n$ by $\lambda_n([t_1, t_2, \ldots, t_n]) = [t_1, t_2, \ldots, t_n]'$ for $n \geq 3$. From the uniqueness of the free extension it can be concluded that the map λ_n is the unique T_i-poset morphism which makes the diagram



commute. As a result we get the following commutative diagram.



This includes a map $\psi : T_1 *_U T_2 \longrightarrow S_1 *_U S_2$ such that $\psi\beta_i = \alpha_i\lambda_i$ for all $i \geq 1$, and our main aim is to prove that, under certain conditions, ψ is $U-$strongly pounitary. To this end we notice that from Lemma 2.3, ψ will be $U-$strongly pounitary providing λ_i is $U-$strongly pounitary for each i. We therefore introduce a property that will guarantee that each λ_i is indeed $U-$strongly pounitary.

Let U be a subpomonoid of a pomonoid S and consider the following property

$$\text{for all } s, s' \in S, \text{ if } ss' \in U, \text{then } s, s' \in U. \qquad (*)$$

It is clear that Condition $(*)$ implies that U is unitary.
Let U and V be semigroups and let $S = U\dot{\cup}V\dot{\cup}\{0\}$ be the *0-direct union* of U and V, where the multiplication in U and V is extended to S by setting all other products equal to 0. It is clear that ${}^1U$ is strongly pounitary in ${}^1S$ and satisfies conditions $(*)$.

The amalgamated free product of the amalgam $[U; T_1, T_2]$ is denoted by $T_1 *_U T_2$

Theorem 4.2. *Let $[U; T_1, T_2]$ be a subpomonoid amalgam of a pomonoid amalgam $[U; S_1, S_2]$. If T_i satisfies condition $(*)$ in S_i, U is strongly pounitary in T_i and T_i is strongly pounitary in S_i for $i \in \{1, 2\}$ then the canonical map $T_1 *_U T_2 \to S_1 *_U S_2$ is $(U, U)-$strongly pounitary.*

Proof. We use the above construction and notation of $S_1 *_U S_2$ and $T_1 *_U T_2$ in terms of directed colimits. It is easy to show that λ_1 and λ_2 are $(U, U)-$strongly pounitary. Hence from Theorem 3.7 the diagram

$$\begin{array}{ccc} T_1 & \longrightarrow & T_1 \otimes T_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_1 \otimes T_2 \end{array}$$

is a pullback. Note here that the tensor products are over U and it is clear that the diagram commutes. Since the map $S_1 \otimes T_2 \to S_1 \otimes S_2$ is an order embedding, it follows from Lemma 3.9 and Proposition 3.2 the diagram

$$\begin{array}{ccc} T_1 & \longrightarrow & T_1 \otimes T_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_1 \otimes S_2 \end{array}$$

is also a pullback. Hence from Proposition 4.1 the map $\lambda_3 : A_3 \to Y_3$ is an order embedding and the diagram

$$\begin{array}{ccc} A_2 & \longrightarrow & A_3 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Y_3 \end{array}$$

is a pullback. To show that λ_3 is $(U,U)-$lower strongly pounitary suppose that $v(s_1 \otimes s_2 \otimes s_3)\delta_1 u \le (t_1 \otimes t_2 \otimes t_3)\delta_1$. Then there exist $x_{1j} \otimes x_{2j} \otimes x_{3j}, x'_{1j} \otimes x'_{2j} \otimes x'_{3j} \in S_1 \otimes S_2 \otimes S_1$ such that

$$\begin{aligned} vs_1 \otimes s_2 \otimes s_3 u \le x_{11} \otimes x_{21} \otimes x_{31}\alpha(V_1)x'_{11} \otimes x'_{21} \otimes x'_{31} \le \dots \\ \le x_{1p} \otimes x_{2p} \otimes x_{3p}\alpha(V_1)x'_{1p} \otimes x'_{2p} \otimes x'_{3p} \le t_1 \otimes t_2 \otimes t_3 \end{aligned}$$

where $1 \le j \le p$ and $p \ge 1$. By transitivity of $\le$ on A_3, we can assume without loss of generality that the number of $\alpha(V_1)$ terms is minimal. If there are no such terms then $vs_1 \otimes s_2 \otimes s_3 u \le t_1 \otimes t_2 \otimes t_3$. Hence $s_1, s_3 \in T_1$ and $s_2 \in T_2$ as required. Otherwise, for every $j \in \{1, \dots, p\}$ there exists a scheme

$$\begin{aligned} x_{1j} \otimes x_{2j} \otimes x_{3j} &= v_1 y_{11} \otimes 1 \otimes y_{31} u_1 \\ v_1 y'_{11} \otimes 1 \otimes y'_{31} u_1 &= v_2 y_{12} \otimes 1 \otimes y_{32} u_2 \\ &\vdots \\ v_m y'_{1m} \otimes 1 \otimes y'_{3m} u_m &= x'_{1j} \otimes x'_{2j} \otimes x'_{3j} \end{aligned}$$

where $y_{1l} y_{3l} \le y'_{1l} y'_{3l}$ in S_1, $1 \le l \le m$, and $m \ge 1$. Using Lemma 1.2 it is straightforward to show that $x_{2l}, x'_{2l} \in U$ and $x_{1j}x_{2j}x_{3j} \le x'_{1j}x'_{2j}x'_{3j}$ in S_1. Hence $s_2, t_2 \in U$ and $vs_1s_2s_3u \le t_1t_2t_3$. Since T_1 is strongly pounitary in S_1, we have $s_1s_2s_3 \in T_1$. Therefore, from condition $(*)$ it follows that $s_1, s_3 \in T_1$ and $s_2 \in U$ and so λ_3 is $(U,U)-$lower strongly pounitary. By a similar argument we can show that λ_3 is $(U,U)-$upper strongly pounitary. Hence λ_3 is $(U,U)-$strongly pounitary. From Proposition 4.1 the map $\lambda_4 : A_4 \to Y_4$ is an order embedding and

$$\begin{array}{ccc} A_3 & \longrightarrow & A_4 \\ \downarrow & & \downarrow \\ Y_3 & \longrightarrow & Y_4 \end{array}$$

is a pullback. The result will follow if we show that λ_n is $(U,U)-$strongly pounitary for all n. We proceed inductively. Suppose that λ_{n-1} is $(U,U)-$strongly pounitary and the diagram

$$\begin{array}{ccc} A_{n-2} & \longrightarrow & A_{n-1} \\ \downarrow & & \downarrow \\ Y_{n-2} & \longrightarrow & Y_{n-1} \end{array}$$

is a pullback. Then from Proposition 4.1 the map λ_n is an order embedding. To show λ_n is $(U,U)-$lower strongly pounitary suppose that $v[s_1,\ldots,s_n]'u \leq [t_1,\ldots,t_n]'$. Then there exists a scheme such as that

$$\begin{aligned} &[vs_1,\ldots,s_{n-1}]'\otimes s_n u \leq [x_1,\ldots,x_{n-1}]'\otimes x_n \\ &\qquad\qquad\qquad\qquad \alpha(V_{n-1})[x'_1,\ldots,x'_{n-1}]'\otimes x'_n \leq [t_1,\ldots,t_{n-1}]'\otimes t_n. \end{aligned}$$

We can assume without loss of generality that the number of $\alpha(V_{n-1})$ terms is one. If there are no such terms then $[x_1,\ldots,x_{n-1}]'\otimes x_n = [x'_1,\ldots,x'_{n-1}]'\otimes x'_n$. This gives $[vs_1,\ldots,s_{n-1}]'\otimes s_n u \leq [t_1,\ldots,t_{n-1}]'\otimes t_n$. Hence from the definition of the tensor product and from our assumption, it is clear that $[s_1,\ldots,s_{n-1}]' \in \text{im } \lambda_{n-1}(A_{n-1})$ and $s_n \in T_n$. Otherwise, for $[x_1,\ldots,x_{n-1}]'\otimes x_n \ \alpha(V_{n-1})\ [x'_1,\ldots,x'_{n-1}]'\otimes x'_n$ there exists a scheme

$$\begin{aligned} &[x_1,\ldots,x_{n-1}]'\otimes x_n = v_1[y_1,\ldots,y_{n-2},1]'\otimes y_n u_1 \\ &v_1[y'_1,\ldots,y'_{n-2},1]'\otimes y'_n u_1 = [x'_1,\ldots,x'_{n-1}]'\otimes x'_n \end{aligned}$$

where $[y_1,\ldots,y_{n-2}y_n]' \leq [y'_1,\ldots,y'_{n-2}y'_n]'$. We can also assume without loss of generality that this scheme has minimal length. Since $[x'_1,\ldots,x'_{n-1}]' \in \text{im } \lambda_{n-1}(A_{n-1})$ and $x'_n \in T_n$, it follows from Lemma 1.2 and our inductive assumption that $[y'_1,\ldots,y'_{n-2},1]' \in \text{im } \lambda_{n-1}(A_{n-1})$ and $y'_n \in T_n$. Hence $[y_1,\ldots,y_{n-2}y_n]' \in \text{im } \lambda_{n-1}(A_{n-2})$ and so $y_{n-2}y_n \in T_{n-2} = T_n$. From Condition $(*)$ we get $y_{n-2}, y_n \in T_{n-2} = T_n$. Consequently $[y_1,\ldots,y_{n-2},1]' \in \text{im } \lambda_{n-1}(A_{n-1})$ and $y_n \in T_n$. Therefore $[x_1,\ldots,x_{n-1}]' \in \text{im } \lambda_{n-1}(A_{n-1})$ and $x_n \in T_n$.

By a similar argument we get that $[s_1,\ldots,s_{n-1}]' \in \text{im } \lambda_{n-1}(A_{n-1})$ and $s_n \in T_n$. Therefore λ_n is $(U,U)-$lower strongly pounitary. In a similar manner it can be shown that λ_n is a $(U,U)-$upper strongly pounitary order embedding and hence from Lemma 2.3 ψ is a $(U,U)-$strongly pounitary order embedding. □

From Nasir [8] it is known that the amalgamated free product of a commutative pomonoid amalgam $[U; S_1, S_2]$ is the tensor product $S_1 \otimes_U S_2$.

A right $S-$poset X is called *right poflat* (resp. *flat*) if for every left $S-$poset order embedding $f : A \to B$ the poset morphism $1 \otimes f : X \otimes A \to X \otimes B$ is an order embedding (resp. monomorphism). *Left poflatness* and *left flatness* are defined in a similar manner. A pomonoid S is called *(right, left) absolutely poflat* if all its (left, right) $U-$posets are poflat. Analogous definitions are given for *(right, left) absolutely flat.* It is clear that poflat $S-$posets are flat.

Let $[U; S_1, S_2]$ be a commutative pomonoid amalgam with $[U; T_1, T_2]$ being a subpomonoid amalgam of $[U; S_1, S_2]$ and let S_1 and T_2 be poflat (resp. flat). Since $T_1 \to S_1$ is an order embedding and T_2 is poflat (resp. flat) then the map $T_1 \otimes T_2 \to S_1 \otimes T_2$ is an order embedding (resp. monomorphism). Since $T_2 \to S_2$ is an order embedding and S_1 is poflat (resp. flat) then the map $S_1 \otimes T_2 \to S_1 \otimes S_2$ is an order embedding (resp. monomorphism). Hence the composition $T_1 \otimes T_2 \to S_1 \otimes S_2$ is an order embedding (resp. monomorphism). Therefore we have the following

Proposition 4.3. *Let $[U; S_1, S_2]$ be a commutative pomonoid amalgam with subpomonoid amalgam $[U; T_1, T_2]$ and S_1 and T_2 be poflat (resp. flat). Then the map $T_1 \otimes T_2 \to S_1 \otimes S_2$ is an order embedding (resp. monomorphism).*

The following corollary is then clear.

Corollary 4.4. *Let $[U; S_1, S_2]$ be a commutative pomonoid amalgam with subpomonoid amalgam $[U; T_1, T_2]$. If U is absolutely poflat (resp. flat) then the map $T_1 \otimes T_2 \to S_1 \otimes S_2$ is an order embedding (resp. monomorphism).*

Finally, from the definition of the pounitary, we immediately deduce the following.

Proposition 4.5. *Let $[U; S_1, S_2]$ be a commutative pomonoid amalgam with subpomonoid amalgam $[U; T_1, T_2]$ and T_i be a pounitary in S_i for $i \in \{1, 2\}$. Then the map $T_1 \otimes T_2 \to S_1 \otimes S_2$ is an order embedding.*

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