THE COHOMOLOGY OF FREE LOOP SPACES OF RANK 2 FLAG MANIFOLDS

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Abstract. By applying Gröbner basis theory to spectral sequences algebras, we develop a new computational methodology applicable to any Leray-Serre spectral sequence for which the cohomology of the base space is the quotient of a finitely generated polynomial algebra. We demonstrate the procedure by deducing the cohomology of the free loop space of flag manifolds, presenting a significant extension over previous knowledge of the topology of free loop spaces. A complete flag manifold is the quotient of a Lie group by its maximal torus. The rank of a flag manifold is the dimension of the maximal torus of the Lie group. The rank 2 complete flag manifolds are $SU(3)/T^2$, $Sp(2)/T^2$, $Spin(4)/T^2$, $Spin(5)/T^2$ and G_2/T^2. In this paper we calculate the cohomology of the free loop space of the rank 2 complete flag manifolds.

1. Introduction

One of the most influential problems in topology and geometry is the study of geodesics on Riemannian manifolds. The geodesic problem refers to finding geodesics connecting two given points of a Riemannian manifold or to finding periodic geodesics, and to giving information regarding their count. The study of geodesics originated with the works of Hadamard [15] and Poincaré [21] and with substantial early contributions by Birkhoff [3], Morse [18], and Lusternik and Schnirelman [16]. The most important offspring of this problem is the development of topological methods in variational calculus, generally referred to as Morse theory [8]. Recently Floer theory developed as a central tool in modern symplectic topology taking its inspiration from the study of geodesics. The geodesic problem also led to the development of computational tools in algebraic topology (spectral sequences), and is connected to the theory of minimal models and to Hochschild and cyclic homology.

One of the most natural starting points in the study of the geodesic problem is the study of spaces of paths and loops on a manifold. In recent years, these spaces have been the object of much interest in topology, symplectic geometry and theoretical physics. The *free loop space* ΛX of a topological space X is defined to be the mapping space $\mathrm{Map}(S^1, X)$, the space of all non-pointed maps from the circle to X.

Given a Riemannian manifold (M, g), the closed geodesics parametrised by S^1 are the critical points of the energy functional

$$E\colon \Lambda M \to \mathbb{R}, \quad E(\gamma) := \frac{1}{2}\int_{S^1} ||\dot{\gamma}(t)||^2 dt.$$

Morse theory applied to the energy functional E gives a description of the loop space ΛM by successive attachments of bundles over the critical submanifolds with rank given by the index of the Hessian d^2E. This allows a grip on the topology of ΛM provided one has enough information on these indices and on the attaching maps. Conversely, knowledge of the topology of ΛM implies existence results for critical points of E.

One of the most powerful results in this direction is due to Gromoll-Meyer [14] who proved that when M is a simply connected closed manifold such that the sequence $\{b_k(\Lambda M)\}, k \geq 0$ of Betti

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numbers of ΛM with coefficients in some field is unbounded, then for any Riemannian metric on M there exist infinitely many geometrically distinct closed geodesics.

A distinctive subspace of ΛM is the *based loops space* $\Omega X = \mathrm{Map}_*(S^1, X)$, the space of all pointed maps from the circle to X. The based loop space functor is an important classical object in algebraic topology and has been well studied. The topology of the free loop space is much less well behaved and it is still only well understood in a handful of examples. In particular, the cohomology of the free loop space spheres, n-dimensional projective spaces and Lie groups.

The starting point for the topological study of a free loop space ΛM is the evaluation fibration

$$\Omega M \to \Lambda M \overset{ev}{\to} M$$

where ev is the evaluation at the origin of a loop, and ΩM is the based loop space, consisting of loops starting and ending at the basepoint of M. This fibration can be used to determine the homotopy groups of ΛM, that is, $\pi_k(\Lambda M) \cong \pi_k(M) \oplus \pi_k(\Omega M)$: the section given by the inclusion of constant loops determines a splitting of the homotopy long exact sequence. However, the situation is very different as far as the homology groups are concerned. It turns out that the Leray-Serre spectral sequence is effective in simple cases (spheres [17, 10]) but of very limited use in general, unless one has additional geometric information about the differentials. In contrast to the evaluation fibration, the path-loop fibration $\Omega M \to PM \to M$ has been successfully used to study ΩM due to it having a contractible total space PM. For example, the author and Terzić [12, 13] calculate the integral Pontryagin homology ring of the flag manifolds and of generalised symmetric spaces. Any successful reasoning in the study of free loop spaces must use specific features of the evaluation fibration.

In this paper we explore the cohomology of the free loop space of homogeneous spaces. In doing so we uncover some surprising combinatorial identities and explicitly compute the cohomology of the free loop space of the flag manifolds of the rank 2 simple Lie groups. The cohomology of the free loop space of Lie groups is easy to calculate as the free loop spaces split as the product $\Lambda G \simeq \Omega G \times G$. The only previously known examples of the free loops of homogeneous spaces were S^n and $\mathbb{C}P^n$ considered by Cohen-Jones-Yan [9]. Despite much interest in free loop spaces, prior to our work, these were the only known examples. Therefore the results presented in this paper greatly expand upon current knowledge. It is worth remarking that the rank 2 flag manifolds represent a significant step forward in complexity form $\mathbb{C}P^n$.

As outlined above, we start by comparing the Leray-Serre spectral sequence for the path-loop fibration (also known as the Wegraum fibration) with the free loop space fibration for which the E_2-terms are isomorphic and there is a mapping of spectral sequences to make the comparison. The analysis of the Wegraum fibration is mitigated by the fact that the projection map is the diagonal mapping on the manifold, inducing the cup product. This idea itself is a classical one. To calculated the differentials, the main problem in any spectral sequences approach, we established deep connections between the free loop spaces of flag manifolds and underlying combinatorics of symmetric polynomials. To completely determine the algebra structure the novel ingredient is an analysis via Gröbner bases, presenting a new method applicable to any Leray-Serre spectral sequence for which the cohomology of the base space is the quotient of a finitely generated polynomial algebra.

A compact connected Lie group is called simple if it is non-abelian, simply connected and has no non-trivial connected normal subgroups. The only compact connected simple Lie groups are $Spin(m)$, $SU(n)$, $Sp(n)$, G_2, F_4, E_6, E_7, E_8 for $n \geq 1$ and $m \geq 2$. In this paper we specialise to rank 2 simple Lie groups $SU(3)$, $Sp(2)$, $Spin(4)$, $Spin(5)$ and G_2. In low dimensions, there are isomorphisms among the classical Lie groups called accidental isomorphisms, identifying certain simple Lie groups of rank 2 such as $Spin(4) \cong SU(2) \times SU(2)$ and $Spin(5) \cong Sp(2)$. Therefore, in this paper we focus on the cohomology of the free loop spaces on $SU(3)/T^2$, $Sp(2)/T^2$ and G_2/T^2.

2. Background

2.1. **Gröbner bases.** Gröbner basis provide us with powerful methods to perform computations in commutative algebra particularly with respect to ideals, although their applications extend far beyond such calculations. In this subsection we briefly describe the Gröbner basis theory to be used later in the paper. For more information and proofs see [1] or [2]. We state all results over Euclidean or principal ideal domain R; in the paper we will only consider the case $R = \mathbb{Z}$ for which all results hold. The theory of Gröbner basis can be generalised to other rings and stronger results can be recovered over a field.

Let R be a ring. Given a finite subset A of $R[x_1, \dots, x_n]$, we denote by $\langle A \rangle$ the ideal generated by elements of A. Fix a monomial ordering on the polynomial ring $R[x_1, \dots, x_n]$. For $f, g, p \in R[x_1, \dots, x_n]$, g is said to be *reduced* from f by p if there exists a term m in f such that the leading term of p divides m and $g = f - m'p$ for some monomial $m' \in R[x_1, \dots, x_n]$.

Let R be a principal ideal domain and let G be a finite subset of $R[x_1, \dots, x_n]$. Then G is a *Gröbner basis* if all elements of $\langle G \rangle$ can be reduced to zero by elements of G.

A set is called *decidable* if for any two elements input, there is an algorithm that can determine whether they are equal. A ring is called *computable* if it is decidable as a set and there is an effectively computable algorithm for addition, multiplication and subtraction in the ring for an input of a pair of elements. A principal ideal domain is called a *computable principal ideal domain* if it is a computable ring, there is an algorithm that can effectively compute whether a given pair of elements is divisible and an extended Euclidean algorithm can be effectively computed. Euclidean domain is a *computable Euclidean domain* if it is a computable ring and there is an algorithm that effectively computes division with remainder.

The integers are a computable Euclidean domain. Moreover, as division with remainder can be applied to construct an extended Euclidean algorithm, so every computable Euclidean domain is also a computable principle ideal domain.

Theorem 2.1 ([2]). *Let R be a computable principal ideal domain and fix a monomial order on $R[x_1, \dots, x_n]$. For any ideal in $R[x_1, \dots, x_n]$, there exists a Gröbner basis. In particular, for finite $A \subseteq R[x_1, \dots, x_n]$ there is an algorithm to obtain a Gröbner basis G such that $\langle G \rangle = \langle A \rangle$.*

The most efficient algorithm is known as the Buchberger algorithm and can easily be implemented by a computer. Over a Euclidean domain a more precise form of reduction is required to replicate some properties of Gröbner basis over a filed.

Definition 2.2. For $f, g, p \in E[x_1, \dots, x_n]$, g is said to be *E-reduced* from f by p to g, if there exists a term t in f such that for some monomial $m = at$ in f the leading term l_p of p divides t with $t = sl_p$ and

$$g = f - qsp,$$

for some non-zero $q \in E$ the quotient of a upon division with unique remainder by l_p.

A Gröbner basis basis G in $E[x_1, \dots, x_n]$ is said to be *reduced* if all polynomials in G cannot be E-reduced by any other polynomial in G.

Theorem 2.3 ([2]). *Let R be a Euclidean domain with unique remainders and let G be a Gröbner basis for the ideal $\langle G \rangle$ in $R[x_1, \dots, x_n]$. Then all elements in $R[x_1, \dots, x_n]$ E-reduce to a unique representative in $R[x_1, \dots, x_n]/\langle G \rangle$.*

In particular, a Gröbner basis can be used to compute the intersection of ideals which we make explicit in the next remark.

Remark 2.4. Let $A = \{a_1, \dots, a_s\}$ and $B = \{b_1, \dots, b_l\}$ be subsets of $R[x_1, \dots, x_n]$. Take a Gröbner basis G of

$$\{ya_1, \dots, ya_t, (1-y)b_1, \dots, (1-y)b_l\}$$

in $R[x_1, \dots, x_n, y]$ using a monomial ordering in which monomials containing y are larger than y free monomials. Then a Gröbner basis of $\langle A \rangle \cap \langle B \rangle$ is given by the elements of G that do not contain y.

2.2. **Cohomology of the complete flag manifolds of simple Lie groups.** A Lie subgroup T of a Lie group G isomorphic to a torus is called maximal if any subtorus containing T coincides with T. Maximal tori are conjugate and cover the Lie group, therefore it is unambiguous to refer to the maximal torus T of G. The homogeneous space G/T, isomorphic regardless of the choice of T, is called the *complete flag manifold* of G. The rank of a Lie group G is the dimension of the maximal torus T.

The cohomology of homogeneous spaces was studied in detail by Borel in [4]. In particular, from Borel's work it is possible to deduce the rational cohomology of G/T.

Theorem 2.5 ([4])**.** *For compact connected Lie group G with maximal torus T,*

$$H^*(G/T;\mathbb{Q}) \cong \frac{H^*(BT;\mathbb{Q})}{\langle \tilde{H}^*(BT;\mathbb{Q})^{W_G}\rangle}$$

where BT is the classifying space of T and W_G is the Weyl group of Lie group G.

In [6] Bott and Samelson, using Morse theory, extended Borel's work by showing that there is no torsion in $H^*(G/T;\mathbb{Z})$. This made it easier to deduced the integral structure of the cohomology of the complete flag manifolds in the cases of $SU(n), Sp(n)$ and G_2. Toda [24] studied the cohomology of homogeneous spaces looking at the mod p cohomology for prime p. In particular, Toda described in a nice form the integral cohomology algebras of the complete flag manifolds in the case of $SO(n)$. Soon after, Toda and Watanabe [25] computed the integral cohomology of the complete flag manifolds of F_4 and E_6. The cohomology of the complete flag manifolds of simple Lie groups were completed by Nakagawa who described the cases for E_7 and E_8 in [19] and [20], respectively.

We recall the cohomology rings of the flag manifolds used in this paper following [4, 7]. The cohomology of the complete flag manifold of the simple Lie group $SU(3)$ is given by

$$H^*(SU(3)/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1,\gamma_2,\gamma_3]}{\langle \sigma_1,\sigma_2,\sigma_3\rangle}$$

where $|\gamma_i| = 2$ for $i = 1,2,3$ and σ_i are the elementary symmetric polynomials of degree i in variables $\gamma_1,\gamma_2,\gamma_3$. To simplify calculations, using $\sigma_1 = \gamma_1+\gamma_2+\gamma_3$ rewrite variable γ_3 as $\gamma_3 = -\gamma_1-\gamma_2$, hence

$$H^*(SU(3)/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1,\gamma_2]}{\langle \sigma_2,\sigma_3\rangle}. \tag{1}$$

Note that we also rewrite $\sigma_2 = \gamma_1^2+\gamma_2^2+\gamma_1\gamma_2$ and $\sigma_3 = \gamma_1^2\gamma_2+\gamma_1\gamma_2^2$.

The cohomology of the complete flag manifold of the simple Lie group $Sp(2)$ is given by

$$H^*(Sp(2)/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1,\gamma_2]}{\langle \sigma_1^2,\sigma_2^2\rangle} \tag{2}$$

where $|\gamma_i| = 2$ for $i = 1,2$ and σ_i^2 denotes elementary symmetric polynomial of degree i in variables γ_1^2,γ_2^2.

The cohomology of the complete flag manifold of the exceptional simple Lie group G_2 is given by

$$H^*(G_2/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1,\gamma_2,\gamma_3,t_3]}{\langle \sigma_1,\sigma_2,\sigma_3-2t_3,t_3^2\rangle}$$

where $|\gamma_i| = 2$ for $i = 1,2,3$, $|t_3| = 6$ and σ_i denotes the elementary symmetric polynomial of degree i in variables $\gamma_1,\gamma_2,\gamma_3$. Again to simplify calculations, using $\sigma_1 = \gamma_1+\gamma_2+\gamma_3$ we rewrite varaiable γ_3 as $\gamma_3 = -\gamma_1-\gamma_2$, hence

$$H^*(G_2/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1,\gamma_2,t_3]}{\langle \sigma_2,\sigma_3-2t_3,t_3^2\rangle}. \tag{3}$$

Similarly, rewrite $\sigma_2 = \gamma_1^2+\gamma_2^2+\gamma_1\gamma_2$ and $\sigma_3 = \gamma_1^2\gamma_2+\gamma_1\gamma_2^2$.

2.3. **Based loop space cohomology of simple Lie groups.** The Hopf algebras of the based loop space of Lie groups were studied by Bott [5]. We recall the results used later in the paper. Recall the integral divided polynomial algebra on variables $x_1, \ldots, x_n$ is given by

$$\Gamma_{\mathbb{Z}}[x_1, \ldots, x_n] = \frac{\mathbb{Z}[(x_i)_1, (x_i)_2, \ldots]}{\langle x_i^k - k!(x_i)_k \rangle}$$

for $1 \leq i \leq n$, $k \geq 1$ where $(x_i)_1 = x_i$. The following results can be obtained from the cohomology of $SU(3)$ and $Sp(2)$ by applying the Leray-Serre spectral sequence to the path-loop fibrations

$$\Omega SU(3) \to PSU(3) \to SU(3) \quad \text{and} \quad \Omega Sp(2) \to PSp(2) \to Sp(2).$$

The integral cohomology of the based loop space of the classical simple Lie group $SU(3)$ is given by

$$H^*(\Omega(SU(3)); \mathbb{Z}) = \Gamma_{\mathbb{Z}}[x_2, x_4] \tag{4}$$

where $|x_2| = 2$ and $|x_4| = 4$. The integral cohomology of the based loop space of the classical simple Lie group $Sp(2)$ is given by

$$H^*(\Omega(Sp(2)); \mathbb{Z}) = \Gamma_{\mathbb{Z}}[x_2, x_6] \tag{5}$$

where $|x_2| = 2$ and $|x_6| = 6$. It is less straightforward to, in a similar way, compute the integral cohomology of ΩG_2.

Proposition 2.6. *The integral cohomology of the based loop space of the exceptional simple Lie group G_2 is given by*

$$H^*(\Omega G_2; \mathbb{Z}) = \frac{\mathbb{Z}[(a_2)_1, (a_2)_2, \ldots]}{\langle a_2^m - (m!/2^{\lfloor \frac{m}{2} \rfloor})(a_2)_m \rangle} \otimes \Gamma_{\mathbb{Z}}[b_{10}]$$

where $m \geq 1$, $(a_2)_1 = a_2$ and $|(a_2)_m| = 2m$, $|b_{10}| = 10$.

Proof. The integral cohomology of G_2 [11] is given by

$$H^*(G_2; \mathbb{Z}) = \frac{\mathbb{Z}[x_3, x_{11}]}{\langle x_3^4,\ x_{11}^2,\ 2x_3^2,\ x_3^2 x_{11} \rangle}$$

where $|x_3| = 3$ and $|x_{11}| = 11$. Since G_2 is simply connected, consider the Leray-Serre spectral sequence $\{E_r, d^r\}$ of the path-loop fibration

$$\Omega G_2 \to PG_2 \to G_2.$$

The spectral sequence $\{E_r, d^r\}$ converges to a trivial algebra and the first non-trivial class in the cohomology of G_2 is generated by x_3 in degree 3. Hence, for dimensional reasons there is an $a_2 \in H^2(\Omega G_2; \mathbb{Z})$ and

$$d^3(a_2) = x_3.$$

For $m \geq 1$, denote by $(a_2)_m$ the additive generator of $H^{2m}(\Omega G_2; \mathbb{Z})$ such that $c(a_2)_m = a_2^m$ for some $c \geq 0$. Element x_3^2 is 2-torsion and $d^3(a_2 x_3) = x_3^2$, so the kernel of $d^3 \colon E_{3,2}^3 \to E_{6,0}^3$ is generated by $2a_2x_3$. Hence, as $d^3(a_2^2) = 2a_2x_3$,

$$(a_2)_2 = a_2^2.$$

As the element $a_2 x_3^2$ is 2-torsion and $d^3((a_2)_2 x_3) = 2a_2x_3^2 = 0$, the kennel of $d^3 \colon E_{3,4}^3 \to E_{6,2}^3$ is generated by $(a_2)_2 x_3$. Hence, as $d^3(a_2^3) = 3(a_2)_2 x_3$,

$$3(a_2)_3 = a_2^3.$$

For $p \geq 1$, $q \geq 2$ even and $r \geq 1$ odd,

$$\ker(d^3)(E_3^{p,2q}) \cong H^*(G_2, \mathbb{Z}) \text{ and } \ker(d^3)(E_3^{p,2r}) \cong E_3^{p,2}.$$

In particular, it can be shown inductively that $\{a_2 = (a_2)_1, (a_2)_2, (a_2)_3, \ldots\}$ generates an algebra

$$\frac{\mathbb{Z}[(a_2)_1, (a_2)_2, \ldots]}{\langle a_2^m - (m!/2^{\lfloor \frac{m}{2} \rfloor})(a_2)_m \rangle}.$$

As the element $(a_2)_{q-1}x_3^2$ is 2-torsion and $d^3((a_2)_q x_3) = (a_2)_{q-1}x_3^2$, the kernel of $d^3\colon E^3_{3,2q} \to E^3_{6,2q-2}$ is generated by $2(a_2)_{q-1}x_3$. As $d^3(a_2^q) = qa_2^{q-1}x_3$, by induction,

$$q!/2^q(a_2)_q = a_2^q.$$

As the element $a_{2r}x_3^2$ is 2-torsion and $d^3((a_2)_{r+1}x_3) = a_r x_3^2$, the kennel $d^3\colon E^3_{3,2r} \to E^3_{6,2r-2}$ is generated by $(a_2)_{r-1}x_3$. Hence, as $d^3(a_2^r) = r(a_2)^{r-1}x_3$, by induction

$$r!/2^{r-1}(a_2)_r = a_2^r.$$

For dimensional reasons, there are no other non-trivial d^3 differentials other than those occurring on multiple of $(a_3)_m$. Hence the only non-trivial elements on the E_4-page divisible by $(a_2)_m$, x_3 or x_{11} for $m \geq 1$ are generated by

$$x_{11},\ (a_2)_t x_2,\ (a_2)_t x_3 x_{11}$$

where $t \geq 1$ is odd, x_{11} is non-torsion, $(a_2)_m x_2$ and $x_3 x_{11}$ are 2-torsion. By [5], there is no torsion in $H^*(\Omega G_2, \mathbb{Z})$. Hence, for dimensional reasons, there is a $b_{10} \in H^*(\Omega G_2; \mathbb{Z})$ and

$$d^5((a_2)_t x_2) = (a_2)_{t-2}x_3x_{11},\ d^9(b_{10}) = a_2x_3,\ d^{11}(2b_{10}) = x_{11}$$

for odd $t \geq 3$. It can be shown that b_{10} generates a divided polynomial algebra in the same way as for the generators of $H^*(\Omega SU(3); \mathbb{Z})$ and $H^*(\Omega Sp(2); \mathbb{Z})$, which completes the proof. $\square$

3. Differentials in the diagonal map spectral sequence

We begin by studying the differentials in the evaluation fibration of a simply connected, simple Lie group G of rank 2 with maximal torus T. The argument is similar to that of [22], in which the cohomology of the free loop spaces of spheres and complex projective spaces are calculated using spectral sequence techniques. However the technical details in the case of the complete flag manifolds is considerably more complex.

For a space X, the fibration $Map(I,X) \to X \times X$ is given by $\alpha \mapsto (\alpha(0), \alpha(1))$. Note that $Map(I,X) \simeq X$. It can be shown that *eval* is a fibration homotopy equivalent to the diagonal map with fibre ΩX. In this section we compute the differentials in the cohomology Leray-Serre spectral sequence of this fibration in the case $X = G$. The aim is to compute $H^*(\Lambda G/T; \mathbb{Z})$. The fibration $eval\colon \Lambda X \to X$ is given by the evaluation of a loop at the base point. The evaluation fibration is also a fibration with fibre ΩX. This fibration is studied in Section 4 by considering a map of fibrations from the evaluation fibration of G to the diagonal fibration and hence the induced map on the spectral sequences. For the rest of this section we consider the fibration

$$\Omega(G/T) \to G/T \xrightarrow{\Delta} G/T \times G/T. \tag{6}$$

By extending the fibration $T \to G \to G/T$, we obtain the homotopy fibration sequence

$$\Omega G \to \Omega(G/T) \to T \to G. \tag{7}$$

It is well known see [23], that the inclusion of the maximal torus into G is null-homotopic. Hence there is a homotopy section $T \to \Omega G$, implying that

$$\Omega(G/T) \simeq \Omega G \times T. \tag{8}$$

All cohomology algebras of spaces in fibration (6) are known. As G/T hence $G/T \times G/T$ are simply connected, the cohomology Leray-Serre spectral sequence of fibration (6), which we denote by $\{\bar{E}_r, \bar{d}^r\}$, converges to $H^*(G/T; \mathbb{Z})$ with an $\bar{E}_2$-page given by $\bar{E}_2^{p,q} = H^p(G/T \times G/T; H^q(\Omega(G/T); \mathbb{Z}))$.

3.1. **Case $SU(3)/T^2$.** When $G = SU(3)$, using decomposition (8), we obtain an algebra isomorphism

$$H^*(\Omega(SU(3)/T^2);\mathbb{Z}) \cong H^*(\Omega(SU(3);\mathbb{Z}) \otimes H^*(T^2;\mathbb{Z}) \cong \Gamma_{\mathbb{Z}}[x_2,x_4] \otimes \Lambda_{\mathbb{Z}}(y_1,y_2),$$

where $|x_2| = 2$, $|x_4| = 4$ and $|y_1| = |y_2| = 1$. Using the cohomology description (1), we set

$$H^*(SU(3)/T^2;\mathbb{Z}) = \tfrac{\mathbb{Z}[\lambda_1,\lambda_2]}{\langle \sigma_2^\lambda,\sigma_3^\lambda\rangle}$$

and

$$H^*(SU(3)/T^2 \times SU(3)/T^2;\mathbb{Z}) = \tfrac{\mathbb{Z}[\alpha_1,\alpha_2]}{\langle \sigma_2^\alpha,\sigma_3^\alpha\rangle} \otimes \tfrac{\mathbb{Z}[\beta_1,\beta_2]}{\langle \sigma_2^\beta,\sigma_3^\beta\rangle}$$

where $|\alpha_i| = |\beta_i| = |\lambda_i| = 2$ for each $i = 1,2$.

The following lemma determines the $\bar{d}^2$ differential on $\bar{E}_2^{*,1}$. We use the alternative basis

$$v_i = \alpha_i - \beta_i \text{ and } u_i = \beta_i$$

for $H^*(SU(3)/T^2 \times SU(3)/T^2;\mathbb{Z})$, where $i = 1,2$.

Lemma 3.1. *In the cohomology Leray-Serre spectral sequence of fibration (6), there is a choice of basis y_1, y_2 such that*

$$\bar{d}^2(y_i) = v_i$$

for each $i = 1,2$.

Proof. For dimensional reasons, $\bar{d}^2$ is the only possible non-zero differential with codomain at any $\bar{E}_*^{2,0}$ and no non-zero differential have domain in any $\bar{E}_*^{2,0}$. As fibration (6) is the diagonal map $\Delta\colon SU(3)/T2 \to SU(3)/T^2 \times SU(3)/T^2$ and the spectral sequence converges to $H^*(SU(3)/T^2)$, the image of $\bar{d}^2\colon \bar{E}_2^{0,1} \to \bar{E}_2^{2,0}$ must be the kernel of the cup product on $H^*(SU(3)/T^2 \times SU(3)/T^2;\mathbb{Z})$, which is generated by v_1, v_2. □

Theorem 3.2. *In the spectral sequence $\{\bar{E}_r, \bar{d}^r\}$, up to class representative and sign in $\bar{E}_2^{2,1}$ and $\bar{E}_2^{4,1}$, the non-trivial differentials are given by*

$$\bar{d}^2(x_2) = y_1v_1 + y_2v_2 + y_1v_2 + 2y_1u_1 + 2y_2u_2 + y_1u_2 + y_2u_1$$

and

$$\bar{d}^4(x_4) = y_1v_1v_2 + y_2v_2v_1 + 2y_1u_1v_2 + 2y_2u_2v_1 + y_1v_1u_2 + y_2v_2u_1 + 2y_1u_1u_2 + 2y_2u_2u_1 + y_1u_2^2 + y_2u_1^2.$$

Proof. All differentials on α_i and β_i are trivial for dimensional reasons. So the only remaining differentials left to determine are those on generators x_2 and x_4. For dimensional reasons, the elements x_2, x_4 cannot be in the image of any differential. By Lemma 3.1, the generators u_1, u_2 must survive to the $\bar{E}_\infty$-page, so generators x_2 and x_4 cannot. The image of $\bar{d}^2\colon \bar{E}_2^{0,2} \to \bar{E}_2^{2,1}$ will be a class in $\bar{E}_2^{2,1}$ in the kernel of $\bar{d}^2$ generated by a single element, not in the image of an element generated by y_i, u_i and v_i alone.

We have $\bar{d}^2(u_i) = \bar{d}^2(v_i) = 0$ and by Lemma 3.1 we may assume that $\bar{d}^2(y_i) = v_i$ for each $i = 1,2$. The non-zero generators in $\bar{E}_2^{2,1}$ can be expressed in the form

$$y_ku_i \text{ or } y_kv_i$$

and the non-zero generators in $\bar{E}_2^{4,1}$ can be expressed in the form

$$y_ku_iu_j,\ y_kv_iu_j \text{ or } y_kv_iv_j$$

for some $1 \le i,j,k \le 2$. Notice that for $1 \le k,i \le 2$

$$\begin{aligned}
&\bar{d}^2(y_1v_1 + y_2v_2 + y_1v_2 + 2y_1u_1 + 2y_2u_2 + y_1u_2 + y_2u_1)\\
&= v_1^2 + v_2^2 + v_1v_2 + 2v_1u_1 + 2v_2u_2 + v_1u_2 + v_2u_1\\
&= (\alpha_1^2 - 2\alpha_1\beta_1 + \beta_1^2) + (\alpha_2^2 - 2\alpha_2\beta_2 + \beta_2^2) + (\alpha_1\alpha_2 - \alpha_1\beta_2 - \alpha_2\beta_1 + \beta_1\beta_2)\\
&\quad + 2(\alpha_1\beta_1 - \beta_1^2) + 2(\alpha_2\beta_2 - \beta_2^2) + (\alpha_1\beta_2 - \beta_1\beta_2) + (\alpha_2\beta_1 - \beta_2\beta_1)\\
&= \alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2 - \beta_1^2 - \beta_2^2 - \beta_1\beta_2 = 0.
\end{aligned}$$

In particular since y_1v_1 is not a term in $\bar{d}^2(y_1y_2) = y_2v_1 - y_1v_2$,

$$y_1v_1 + y_2v_2 + y_1v_2 + 2y_1u_1 + 2y_2u_2 + y_1u_2 + y_2u_1$$

is a generator. Thus since it is the only remaining cycle that can be hit by x_2 in the kernel of $\bar{d}^2$ at $E_2^{2,1}$,

$$\bar{d}^2(x_2) = y_1v_1 + y_2v_2 + y_1v_2 + 2y_1u_1 + 2y_2u_2 + y_1u_2 + y_2u_1$$

up to class representative and sign. Notice that image of $\bar{d}^2(x_2)$ is obtained as the element whose image under $\bar{d}^2$ coincides with the degree 2 generators of the symmetric ideal. Since the choice of degree 2 generator of the symmetric ideal is unique up to sign, so the expression for $\bar{d}^2(x_2)$ is also uniquely determined up to sign.

For dimensional reasons and due to all lower rows except$\bar{E}_4^{*,2}$ and $\bar{E}_4^{*,1}$ being annihilated by the $\bar{d}^2$ differential, the only possible non-zero differential beginning at x_4, is $\bar{d}^4\colon \bar{E}_4^{0,4} \to \bar{E}_4^{4,1}$. The image of the differential $\bar{d}^4$ on x_4 will therefore be a class in $\bar{E}_4^{4,1}$ in the kernel of d^2 generated by a single element, not in the image of the elements generated by y_i, u_i and v_i alone. We see that

$$\begin{aligned}
&\bar{d}^2(y_1v_1v_2 + y_2v_2v_1 + 2y_1u_1v_2 + 2y_2u_2v_1 + y_1v_1u_2 + y_2v_2u_1 + 2y_1u_1u_2 + 2y_2u_2u_1 + y_1u_2^2 + y_2u_1^2)\\
&= v_1^2v_2 + v_2^2v_1 + 2v_1v_2u_1 + 2v_1v_2u_2 + v_1^2u_2 + v_2^2u_1 + 2v_1u_1u_2 + 2v_2u_1u_2 + v_1u_2^2 + v_2u_1^2\\
&= (\alpha_1^2\alpha_2 - \alpha_1^2\beta_2 - 2\alpha_1\alpha_2\beta_1 + 2\alpha_1\beta_1\beta_2 + \alpha_2\beta_1^2 - \beta_1^2\beta_2)\\
&\quad + (\alpha_1\alpha_2^2 - \alpha_2^2\beta_1 - 2\alpha_1\alpha_2\beta_2 + 2\alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2 - \beta_1\beta_2^2)\\
&\quad + 2(\alpha_1\alpha_2\beta_1 - \alpha_1\beta_1\beta_2 - \alpha_2\beta_1^2 + \beta_1^2\beta_2) + 2(\alpha_1\alpha_2\beta_2 - \alpha_1\beta_2^2 - \alpha_2\beta_1\beta_2 + \beta_1\beta_2^2)\\
&\quad + (\alpha_1^2\beta_2 - 2\alpha_1\beta_1\beta_2 + \beta_1^2\beta_2) + (\alpha_2^2\beta_1 - 2\alpha_2\beta_1\beta_2 + \beta_1\beta_2^2)\\
&\quad + 2(\alpha_1\beta_1\beta_2 - \beta_1^2\beta_2) + 2(\alpha_2\beta_1\beta_2 - \beta_1\beta_2^2) + (\alpha_1\beta_2^2 - \beta_1\beta_2^2) + (\alpha_2\beta_1^2 - \beta_1^2\beta_2)\\
&= \alpha_1^2\alpha_2 + \alpha_2^2\alpha_1 - \beta_1^2\beta_2 - \beta_2^2\beta_1 = 0.
\end{aligned}$$

In particular since $y_1v_1u_2$ is not a term in $\bar{d}^2(y_1y_2u_2)$,

$$y_1v_1v_2 + y_2v_2v_1 + 2y_1u_1v_2 + 2y_2u_2v_1 + y_1v_1u_2 + y_2v_2u_1 + 2y_1u_1u_2 + 2y_2u_2u_1 + y_1u_2^2 + y_2u_1^2$$

is a generator. Thus since it is the only remaining cycle that can be hit by x_4 in the kernel of $\bar{d}^2$ at $E_2^{4,1}$,

$$\bar{d}^4(x_4) = y_1v_1v_2 + y_2v_2v_1 + 2y_1u_1v_2 + 2y_2u_2v_1 + y_1v_1u_2 + y_2v_2u_1 + 2y_1u_1u_2 + 2y_2u_2u_1 + y_1u_2^2 + y_2u_1^2$$

up to class representative and sign. Similarly to the $\bar{d}^2$ differential, the image of $\bar{d}^4(x_4)$ is obtained as the element whose image under $\bar{d}^4$ coincides with the degree 3 generators of the symmetric ideal. Since the choice of degree 3 generator of the symmetric ideal is unique up to addition of multiples of the degree 2 generator and sign, so the expression for $\bar{d}^4(x_4)$ is also uniquely determined up to addition of multiples of $\bar{d}^2(x_2)$ and sign. □

Remark 3.3. The elements $(x_2)_m$ and $(x_4)_m$ for each $m \geq 2$ are also generators on the $\bar{E}_2$-page of the spectral sequence $\{\bar{E}_r, \bar{d}^r\}$, arising from the divided polynomial algebra $\Gamma_{\mathbb{Z}}[x_2, x_4]$. We note that the differentials in the spectral sequence are also completely determined on all $(x_2)_m$ and $(x_4)_m$ by Theorem 3.2 in the following way.

Using the relations $x_2^m - m!(x_2)_m$ and the Leibniz rule, it follows by induction that $\bar{d}^2(x_2^m) = m\bar{d}^2(x_2)x_2^{m-1}$ and hence again using the relations

$$\bar{d}^2((x_2)_m) = \bar{d}^2(x_2)(x_2)_{m-1}. \tag{9}$$

Since we know that $\bar{d}^4(x_4)$ must be non-trivial we have that, as $\bar{d}^2(x_4)$ and $\bar{d}^3(x_4)$ must be 0 as the image of $\bar{d}^4(x_4)$ is non-torsion. Due to the fact that there are no torsion elements on the spectral sequence pages $E_i^{*,*}$ and $(x_4)_m$ cannot be in the image of any differential, on successive pages we obtain similarly to equation (9) that $\bar{d}^i((x_4)_m) = \bar{d}^i(x_4)(x_4)_{m-1} = 0$ for $i = 2, 3$. Therefore as

$\bar{d}^i((x_4)_m) = 0$ for all $m \geq 1$ and we can apply the same augments used to derive equation (9) to obtain that

$$\bar{d}^4((x_4)_m) = \bar{d}^4(x_4)(x_4)_{m-1}.$$

3.2. **Case $Sp(2)/T^2$.** Consider now $G = Sp(2)$. By (5), we have

$$H^*(\Omega(Sp(2)/T^2);\mathbb{Z}) \cong \Gamma_{\mathbb{Z}}[x_2, x_6] \otimes \Lambda_{\mathbb{Z}}(y_1, y_2)$$

where $\Gamma_{\mathbb{Z}}[x_2, x_6]$ is the integral divided polynomial algebra on the variables x_2, x_6 with $|x_2| = 2$ and $|x_6| = 6$ and $\Lambda(y_1, y_2)$ is an exterior algebra generated by y_1, y_2 with $|y_1| = |y_2| = 1$. By (2), the cohomology of $Sp(2)/T^2$ is

$$H^*(Sp(2)/T^2);\mathbb{Z}) = \frac{\mathbb{Z}[\lambda_1, \lambda_2]}{\langle \sigma_1^{\lambda^2}, \sigma_2^{\lambda^2} \rangle}$$

and

$$H^*(Sp(2)/T^2 \times Sp(2)/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\alpha_1, \alpha_2]}{\langle \sigma_1^{\alpha^2}, \sigma_2^{\alpha^2} \rangle} \otimes \frac{\mathbb{Z}[\beta_1, \beta_2]}{\langle \sigma_1^{\beta^2}, \sigma_2^{\beta^2} \rangle}$$

for the cohomology of the base space and fibre of fibration (6), where $|\lambda_1| = |\alpha_i| = |\beta_i| = 2$ for $i = 1, 2$. Denote by $\{\bar{E}^r, \bar{d}^r\}$ the cohomology Leray-Serre spectral sequence associated to fibration (6). We again use the alternative basis

$$v_i = \alpha_i - \beta_i \ \text{ and } \ u_i = \beta_i$$

for $i = 1, 2$. For exactly the same reasons as in Lemma 3.1, we get an equivalent lemma in the present case.

Lemma 3.4. *With the notation above, in the cohomology Leray-Serre spectral sequence of fibration (6), there is a choice of basis y_1, y_2 such that*

$$\bar{d}^2(y_i) = v_i$$

for each $i = 1, 2$.

We now prove an equivalent of Theorem 3.2 for $G = Sp(2)$.

Theorem 3.5. *In the spectral sequence $\{E_r, d^r\}$ up to class representative and sign on $\bar{E}^2_{2,1}$ and $\bar{E}^2_{6,1}$, the only non-trivial differentials are given by*

$$\bar{d}^2(x_2) = y_1v_1 + y_2v_2 + 2y_1u_1 + 2y_2u_2$$

and

$$\bar{d}^6(x_6) = y_1v_1^3 + 4y_1v_1^2u_1 + 6y_1v_1u_1^2 + 4y_1u_1^3$$

Proof. All differentials on α_i and β_i are trivial for dimensional reason. So the only remaining differentials left to determine are those on $\langle x_2 \rangle$ and $\langle x_6 \rangle$. For dimensional reasons, the elements x_2, x_6 cannot be in the image of any differential. By Lemma 3.4, the generators u_1, u_2 must survive to the $\bar{E}_\infty$-page, so generators x_2 and x_6 cannot. The image of $\bar{d}^2 \colon \bar{E}_2^{0,2} \to \bar{E}_2^{2,1}$ will be a class in $\bar{E}_2^{2,1}$ in the kernel of $\bar{d}^2$ generated by a single element, not in the image of the elements generated by y_i, u_i and v_i alone.

We have $\bar{d}^2(u_i) = \bar{d}^2(v_i) = 0$ and by Lemma 3.4 we assume that $\bar{d}^2(y_i) = v_i$ for each $i = 1, 2$. The non-zero generators in $\bar{E}_2^{2,1}$ can be expressed in the form

$$y_k u_i \text{ or } y_k v_i$$

and the non-zero generators in $\bar{E}_2^{6,1}$ can be expressed in the form

$$y_k u_{i_1} u_{i_2} u_{i_3},\ y_k v_{i_1} u_{i_2} u_{i_3},\ y_k v_{i_1} v_{i_2} u_{i_3} \text{ or } y_k v_{i_1} v_{i_2} v_{i_3}$$

for some $1 \leq i_2, i_2, i_3, k \leq 2$. Notice that

$$\bar{d}^2(y_k v_i) = v_k^2 = \alpha_k^2 - 2\alpha_k\beta_k + \beta_k^2$$
$$\bar{d}^2(y_k u_k) = v_k u_k = \alpha_k\beta_k - \beta_k^2$$

so

$$\bar{d}^2(y_1v_1 + y_2v_2 + 2y_1u_1 + 2y_2u_2) = \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = 0.$$

In particular since y_1v_1 is not a term in $\bar{d}^2(y_1y_2) = y_2v_1 - y_1v_2$,

$$y_1v_1 + y_2v_2 + 2y_1u_1 + 2y_2u_2$$

is a generator. Thus since it is the only remaining cycle that can be hit by x_2 in the kernel of $\bar{d}^2$ at $E_2^{2,1}$,

$$\bar{d}^2(x_2) = y_1v_1 + y_2v_2 + 2y_1u_1 + 2y_2u_2$$

up to class representative and sign. Notice that image of $\bar{d}^2(x_2)$ is obtained as the element whose image under $\bar{d}^2$ coincides with the degree 2 generators of the symmetric ideal. Since the choice of degree 2 generator of the symmetric ideal is unique up to sign, so the expression for $\bar{d}^2(x_2)$ is also uniquely determined up to sign. Similarly,

$$\bar{d}^2(y_1v_1^3) = v_1^4 = \alpha_1^4 - 4\alpha_1^3\beta_1 + 6\alpha_1^2\beta_1^2 - 4\alpha_1\beta_1^3 + \beta_1^4$$
$$\bar{d}^2(y_1v_1^2u_1) = v_1^3u_1 = \alpha_1^3\beta_1 - 3\alpha_1^2\beta_1^2 + 3\alpha_1\beta_1^3 - \beta_1^4$$
$$\bar{d}^2(y_1v_1u_1^2) = v_1^2u_1^2 = \alpha_1^2\beta_1^2 - 2\alpha_1\beta_1^3 + \beta_1^4$$
$$\bar{d}^2(y_1u_1^3) = v_1u_1^3 = \alpha_1\beta_1^3 - \beta_1^4.$$

Hence

$$\bar{d}^2(y_1v_1^3 + 4y_1v_1^2u_1 + 6y_1v_1u_1^2 + 4y_1u_1^3) = \alpha_1^4 - \beta_1^4 = 0.$$

In particular since $y_1v_1^3$, is not a term in $\bar{d}^2(y_1y_2v_1^2)$,

$$y_1v_1^3 + 4y_1v_1^2u_1 + 6y_1v_1u_1^2 + 4y_1u_1^3$$

is a generator. Thus since it is the only remaining cycle that can be hit by x_6 in the kernel of $\bar{d}^2$ at $E_2^{6,1}$,

$$\bar{d}^6(x_6) = y_1v_1^3 + 4y_1v_1^2u_1 + 6y_1v_1u_1^2 + 4y_1u_1^3$$

up to class representative and sign. Similarly to the $\bar{d}^2$ differential, the image of $\bar{d}^6(x_6)$ is obtained as the element whose image under $\bar{d}^6$ coincides with the degree 4 generators of the symmetric ideal. Since the choice of degree 3 generator of the symmetric ideal is unique up to addition of multiples of the degree 2 generator and sign, so the expression for $\bar{d}^6(x_6)$ is also uniquely determined up to addition of multiples of $\bar{d}^2(x_2)$ and sign. □

Remark 3.6. For the same reasons given in Remark 3.3, the only non-trivial differentials in the spectral sequence $\{\bar{E}_r, \bar{d}^r\}$ on the all generators $(x_2)_m$ and $(x_6)_m$ for $m \geq 1$ arising form the dividend polynomial algebras are $\bar{d}^2$ and $\bar{d}^6$ respectively. Moreover these differentials are determined by

$$\bar{d}^2((x_2)_m) = \bar{d}^2(x_2)(x_2)_{m-1} \text{ and } \bar{d}^6((x_6)_m) = \bar{d}^2(x_6)(x_6)_{m-1}.$$

3.3. **Case G_2/T^2.** Consider now $G = G_2$. Using (3) and Proposition 2.6, in the following argument we use the notation for the cohomology of the base space and fibre in fibration (6)

$$H^*(G_2/T^2 \times G_2/T^2; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha_1, \alpha_2, l_3]}{\langle \sigma_2^\alpha, 2l_3 - \sigma_3^\alpha, l_3^2 \rangle} \otimes \frac{\mathbb{Z}[\beta_1, \beta_2, s_3]}{\langle \sigma_2^\beta, 2s_3 - \sigma_3^\beta, s_3^2 \rangle}$$
$$H^*(\Omega(G_2/T^2); \mathbb{Z}) = \frac{\mathbb{Z}[(a_2)_1, (a_2)_2, \dots]}{\langle a_2^m - (m!/2^{\lfloor \frac{m}{2} \rfloor})(a_2)_m \rangle} \otimes \Gamma_{\mathbb{Z}}[b_{10}] \otimes \Lambda_{\mathbb{Z}}(y_1y_2),$$

where $|\alpha_1| = |\beta_1| = |\alpha_2| = |\beta_2| = 2$, $|y_1| = |y_2| = 1$, $|a_2| = 2$, $|b_{10}| = 10$ and $|l_3| = |s_3| = 6$. Again we use the change of basis $u_i = \beta_i$ and $v_i = \alpha_i - \beta_i$. In addition we also make the change of basis

$$\theta = l_3 - s_3 \text{ and } \psi = l_3.$$

Theorem 3.7. *In the spectral sequence $\{\bar{E}_r, \bar{d}^r\}$ up to class representatives and sign on $\bar{E}^2_{2,1}$ and $\bar{E}^2_{10,1}$, the non-trivial differentials are given by*

$$\bar{d}^2(a_2) = y_1(u_2 + v_2 + 2u_1) + y_2(u_1 + v_1 + 2u_2)$$
$$\bar{d}^4(a_2(y_1(u_2 + v_2 + 2u_1) + y_2(u_1 + v_1 + 2u_2))) = \theta$$

and

$$\bar{d}^{10}(b_{10}) = y_1(\theta v_1^2 + 3\theta v_1 u_1 + 3\theta u_1^2 + 2\psi v_1^2 + 3\psi v_1 u_1 + 3y_1 \psi u_1^2)$$

Proof. Similarly to the previous cases, the differential on y_i is given by

$$\bar{d}^2(y_i) = v_i.$$

The differential on a_2 is obtained as in Theorem 3.2. and is given by

$$\bar{d}^2(a_2) = y_1(u_2 + v_2 + 2u_1) + y_2(u_1 + v_1 + 2u_2)$$

which we denote by ζ.

The reminder of the augment is also similar to the previous cases, however there are two exceptions. Firstly, since

$$\frac{\mathbb{Z}[(a_2)_1, (a_2)_2, \ldots]}{\langle a_2^m - (m!/2^{\lfloor \frac{m}{2} \rfloor})(a_2)_m \rangle}$$

is not a divided polynomials algebra,

$$\bar{d}^2((a_2)_m) = m\frac{2^{\lfloor \frac{m}{2} \rfloor}}{m!}\bar{d}^2(a_2)a_2^{m-1} = \frac{2^{\lfloor \frac{m}{2} \rfloor}}{(m-1)!}\bar{d}^2(a_2)\frac{(m-1)!}{2^{\lfloor \frac{m-1}{2} \rfloor}}(a_2)_{m-1}$$
$$= \frac{2^{\lfloor \frac{m}{2} \rfloor}}{2^{\lfloor \frac{m-1}{2} \rfloor}}(a_2)_{m-1}\bar{d}^2(a_2) = \begin{cases} 2(a_2)_{m-1}\zeta & \text{for } m \text{ even} \\ (a_2)_{m-1}\zeta & \text{for } m \text{ odd} \end{cases}$$

for each $m \geq 1$. Hence the differential is not surejctive and for odd $m \geq 1$, $(a_2)_m\zeta$ multiplicatively generates 2-torsion on the E_3-page. Secondly, the ideal $\langle 2s_3 - \alpha_1^3, 2l_3 - \beta_1^3 \rangle$ does not correspond to any elements of the kernel of $\bar{d}^2$, however due to these relations

$$\bar{d}^2(y_1(v_1^2 + 3v_1u_1 + 3u_1^2)) = 2\theta$$

also multiplicatively generates 2-torsion on the $\bar{E}_3$-page. If $a_2\zeta$ survived to the $\bar{E}_\infty$-page, this would imply that for dimensional reasons after the resolving extension problems there would be torsion class in $H^*(G_2/T^2;\mathbb{Z})$. However there is no torsion in $H^*(G_2/T^2;\mathbb{Z})$, so $a_2\zeta$ must be trivial by the $\bar{E}_\infty$-page. Hence for dimensional reason the only possibility is

$$\bar{d}^4(a_2\zeta) = \theta$$

up to class representative and sign. Since $a_2\zeta$ and θ generate all the 2-torsion, for $r \leq 9$, $\bar{d}^r(b_{10}) = 0$ and there is no torsion by the $\bar{E}_5$-page. In particular, we determine the differentials on b_{10} in the same way as in previous cases. Notice that

$$\bar{d}^2(y_1\theta v_1^2) = v_1^3\theta = 2s_3^2 - 4s_3l_3 - 3s_3\alpha_2^2\beta_1 + 3\alpha_1^2l_3\beta_1 + 3s_3\alpha_1\beta_1^2 - 3\alpha_1l_3\beta_1^2 + 2l_3^2,$$
$$\bar{d}^2(y_1\theta v_1u_1) = \theta v_1^2u_1 = s_3\alpha_1^2\beta_1 - 2s_3\alpha_1\beta_1^2 - \alpha_1^2l_3\beta_1 + 2\alpha_1l_3\beta_1^2 + 2s_3l_3 - 2l_3^2,$$
$$\bar{d}^2(y_1\theta u_1^2) = \theta v_1u_1^2 = s_3\alpha_1\beta_1^2 - 2s_3l_3 - \alpha_1l_3\beta_1^2 + 2l_3^2,$$
$$\bar{d}^2(y_1\psi v_1^2) = v_1^3\psi = 2s_3l_3 - 3\alpha_1^2l_3\beta_1 + 3\alpha_1l_3\beta_1^2 - 2l_3^2,$$
$$\bar{d}^2(y_1\psi v_1u_1) = v_1^2\psi u_1 = \alpha_1^2l_3\beta_1 - 2\alpha_1l_3\beta_1^2 + 2l_3^2,$$
$$\bar{d}^2(y_1\psi u_1^2) = v_1\psi u_1^2 = \alpha_1l_3\beta_1^2 - 2l_3^2.$$

Hence

$$\bar{d}^2(y_1(\theta v_1^2 + 3\theta v_1 u_1 + 3\theta u_1^2 + 2\psi v_1^2 + 3\psi v_1 u_1 + 3\psi u_1^2)) = 2s_3^2 - 2l_3^2 = 0$$

and for the same reasons as in previous cases

$$\bar{d}^{10}(b_{10}) = y_1(\theta v_1^2 + 3\theta v_1 u_1 + 3\theta u_1^2 + 2\psi v_1^2 + 3\psi v_1 u_1 + 3y_1\psi u_1^2)$$

up to class representative and sign. □

Remark 3.8. For similar reasons to those given in Remark 3.3, the only non-trivial differentials in the spectral sequence $\{\bar{E}_r, \bar{d}^r\}$ on generators $(a_2)_m$ and $(b_{10})_m$ for $m \geq 1$ are $\bar{d}^2$ and $\bar{d}^{10}$ respectively. The only difference in this case is that there is some 2-torsion on the $\bar{E}_3$ and $\bar{E}_4$ pages. However this does not effect the augment, as $\bar{d}^3$ is trivial for dimensional reasons and all 2-torsion is either sent to or in the image on another 2-torsion generator under the $\bar{d}^4$ differential. More precisely for odd $m \geq 3$ we have

$$\bar{d}^4((a_2)_m\zeta) = \bar{d}^4(a_2\zeta)(a_2)_{m-2}$$

where $\zeta = y_1(u_2 + v_2 + 2u_1) + y_2(u_1 + v_1 + 2u_2)$. It now follows that the $\bar{d}^2$ and $\bar{d}^{10}$ differentials on $(a_2)_m$ and $(b_{10})_m$ respectively are determined by

$$\bar{d}^2((a_2)_m) = \bar{d}^2(a_2)(a_2)_{m-1}, \text{ and } \bar{d}^{10}((b_{10})_m) = \bar{d}^{10}(x_{10})(x_{10})_{m-1}.$$

4. Differentials in the cohomology Leray-Serre spectral sequence of the evaluation fibration

Throughout the following argument we consider the map ϕ of fibrations between the evaluation fibration of the complete flag manifold G/T and the diagonal fibration given by the following commutative diagram

(10)
$$\begin{array}{ccccc}
\Omega(G/T) & \longrightarrow & \Lambda(G/T) & \xrightarrow{eval} & G/T \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle eval} & & \downarrow{\scriptstyle \Delta} \\
\Omega(G/T) & \longrightarrow & G/T & \xrightarrow{\Delta} & G/T \times G/T.
\end{array}$$

Since we assume that G, hence G/T is simply connected, the cohomology Leray-Serre spectral sequence $\{E_r, d^r\}$ associated with the evaluation fibration converges. Hence ϕ induces a map of spectral sequences $\phi^* : \{\bar{E}_r, \bar{d}^r\} \to \{E_r, d^r\}$. More precisely, for each $r \geq 2$ and $a, b \in \mathbb{Z}$ there is a commutative diagram

(11)
$$\begin{array}{ccc}
\bar{E}_r^{a,b} & \xrightarrow{\bar{d}^r} & \bar{E}_r^{a+r,b-r+1} \\
\downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} \\
E_r^{a,b} & \xrightarrow{d^r} & E_r^{a+r,b-r+1}
\end{array}$$

where ϕ^*, for each r, is the induced map on the homology of the previous page, beginning as the map induced on the tensor on the E_2-pages by the maps $id\colon \Omega(G/T) \to \Omega(G/T)$ and $\Delta\colon G/T \to G/T \times G/T$.

4.1. Case $SU(3)/T^2$.

Let $G = SU(3)$. By (8) and (1), we have

$$H^*(\Omega(SU(3)/T^2); \mathbb{Z}) \cong \Gamma_{\mathbb{Z}}(x_2', x_4') \otimes \Lambda_{\mathbb{Z}}(y_1', y_2')$$

and

$$H^*(SU(3)/T^2; \mathbb{Z}) \cong \frac{\mathbb{Z}[\gamma_1, \gamma_2]}{\langle \sigma_2^\gamma, \sigma_3^\gamma \rangle}$$

where $|y_i'| = 1, |\gamma_i| = 2, |x_{2i}'| = 2i$ for each $1 \leq i \leq 2$. Next we determine all the differentials in $\{E_r, d^r\}$ when $G = SU(3)$.

Theorem 4.1. *The only non-zero differentials on generators of the E_2-page of $\{E_r, d^r\}$ are, up to class representative and sign, given by*

$$d^2(x_2') = y_1'(2\gamma_1 + \gamma_2) + y_2'(\gamma_1 + 2\gamma_2)$$

and

$$d^4(x_4') = y_1'(\gamma_2^2 + 2\gamma_1\gamma_2) + y_2'(\gamma_1^2 + 2\gamma_1\gamma_2).$$

Proof. The identity $id\colon \Omega(SU(3)/T^2) \to \Omega(SU(3)/T^2)$ induces the identity map on cohomology, while the diagonal map $\Delta\colon SU(3)/T^2 \to SU(3)/T^2 \times SU(3)/T^2$ induces the cup product. Hence, by choosing generators in $\{E_2, d^2\}$, we may assume that for $i = 1, 2$

$$\phi^*(y_i) = y_i', \;\; \phi^*(x_i) = x_i' \text{ and } \phi^*(\alpha_i) = \phi^*(\beta_i) = \phi^*(u_i) = \gamma_i$$

Therefore, $\phi^*(v_i) = 0$ for $i = 1, 2$. For dimensional reasons, the only possibly non-zero differential on the generators y_i' in $\{E_r, d^r\}$ is d^2. However using commutative diagram (11) and Lemma 3.1, we have

$$d^2(y_i') = d^2(\phi^*(y_i)) = \phi^*(\bar{d}^2(y_i)) = \phi^*(v_i) = 0.$$

Using commutative diagram (11) and Theorem 3.2, we have up to class representative and sign

$$\begin{aligned} d^2(x_2') &= \phi^*(\bar{d}^2(x_2)) \\ &= \phi^*(y_1v_1 + y_2v_2 + y_1v_2 + 2y_1u_1 + 2y_2u_2 + y_1u_2 + y_2u_1) \\ &= 2y_1'\gamma_1 + 2y_2'\gamma_2 + y_1'\gamma_2 + y_2'\gamma_1 \end{aligned}$$

and

$$\begin{aligned} d^4(x_4') &= \phi^*(\bar{d}^4(x_4)) \\ &= \phi^*(y_1v_1v_2 + y_2v_2v_1 + 2y_1u_1v_2 + 2y_2u_2v_1 + y_1v_1u_2 \\ &\quad + y_2v_2u_1 + 2y_1u_1u_2 + 2y_2u_2u_1 + y_1u_2^2 + y_2u_1^2) \\ &= 2y_2'\gamma_1\gamma_2 + 2y_1'\gamma_1\gamma_2 + y_2'\gamma_1^2 + y_1'\gamma_2^2. \end{aligned}$$

Differentials on the generators γ_i for $i = 1, 2$ are zero for dimensional reasons. □

Remark 4.2. As the restriction $\phi^*\colon \bar{E}_r^{0,*} \to E_r^{0,*}$ is induced by the identity map, we have that $\phi^*((x_i)_m) = (x_i)_m$ for $i = 2, 4$ and for all $m \geq 1$. Therefore using Remark 3.3 it follows that

$$d^2((x_2)_m) = d^2(x_2)(x_2)_{m-1} \text{ and } d^4((x_4)_m) = d^4(x_4)(x_4)_{m-1}.$$

4.2. **Case $Sp(2)/T^2$.** Just as we did in Theorem 4.1, we can now use the results of Theorem 3.5 and diagram (10) to deduce the differentials in the cohomology Leray-Serre spectral sequence $\{E_r, d^r\}$ associated to the evaluation fibration of $Sp(2)/T^2$. For the rest of the section, we denote the cohomology algebras of the base space and fibre of the evaluation fibration by

$$H^*(\Omega(Sp(2)/T^2); \mathbb{Z}) = \Gamma_{\mathbb{Z}}(x_2', x_6') \otimes \Lambda_{\mathbb{Z}}(y_1', y_2')$$

and

$$H^*(Sp(2)/T^2; \mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1, \gamma_2]}{\langle \sigma_1^2, \sigma_2^2 \rangle}$$

where $|{y'}_1| = 1 = |y_2'|$, $|\gamma_1| = 2 = |\gamma_2|$, $|x_2'| = 2$, $|x_6'|$=6 and σ_1^2, σ_2^2 are the elementary symmetric polynomials in variables γ_1^2, γ_2^2.

Theorem 4.3. *The only non-zero differentials on generators of the E_2-page of $\{E_r, d^r\}$ are, up to class representative and sign, given by*

$$d^2(x_2') = 2y_1'\gamma_1 + 2y_2'\gamma_2$$

and

$$d^6(x_6') = 4y_1'\gamma_1^3.$$

Proof. For the same reasons as in the proof of Theorem 4.1, we have for $i = 1, 2$

$$\phi^*(y_i) = y_i',\ \ \phi^*(x_i) = x_i' \ \text{ and } \ \phi^*(\alpha_i) = \gamma_i = \phi^*(\beta_i) = \phi^*(u_i), \ \text{ so } \ \phi^*(v_i) = 0.$$

Hence by the same arguments as in the proof of Theorem 4.1, we have

$$d^r(y_i') = 0 \text{ and } d^r(\gamma_i) = 0$$

and the image of d^r on generators x_2', x_6' is determined by those summands in the image of $\bar{d}^2$ on x_2, x_6 given in Theorem 3.5 containing no v_i, replacing u_i with γ_i and y_i with y_i'. This proves the statement. □

Remark 4.4. As the restriction $\phi^*\colon \bar{E}_r^{0,*} \to E_r^{0,*}$ is induced by the identity map, we have that $\phi^*((x_i)_m) = (x_i)_m$ for $i = 2, 6$ and for all $m \geq 1$. Therefore using Remark 3.6 it follows that

$$d^2((x_2)_m) = d^2(x_2)(x_2)_{m-1} \text{ and } d^6((x_6)_m) = d^6(x_6)(x_6)_{m-1}.$$

4.3. **Case G_2/T^2.** To obtain the differentials in the Leray-Serre spectral sequence of the evaluation fibration of G_2/T^2, as in previous flag manifolds, we consider the map of Leray-Serre spectral sequences induced by (11). In the following argument using (3) and Theorem 2.6 we have

$$H^*(G_2/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1,\gamma_2,t_3]}{\langle \sigma_1^\gamma, \sigma_2^\gamma, 2t_3 - \sigma_3^\gamma, t_3^2\rangle}$$

and

$$H^*(\Omega(G_2/T^2);\mathbb{Z}) = \frac{\mathbb{Z}[(a_2')_1,(a_2')_2,\dots]}{\langle a_2'^m - (m!/2^{\lfloor \frac{m}{2}\rfloor})(a_2')_m\rangle} \otimes \Gamma_{\mathbb{Z}}[b_{10}'] \otimes \Lambda_{\mathbb{Z}}(y_1', y_2')$$

where $|\gamma_1| = |\gamma_2| = 2$, $|y_1| = |y_2| = 1$, $|a_2'| = 2$, $|b_{10}'| = 10$ and σ_i are the elementary symmetric polynomials of degree i in bases $\gamma_1, \gamma_2, \gamma_3$ with $-\gamma_3 = \gamma_1 + \gamma_2$.

Theorem 4.5. *The cohomology Leray-Serre spectral sequence $\{E_r, d^r\}$ associated to the evaluation fibration of G_2/T^2 has, up to class representative, the only non-trivial differentials*

$$d^2(a_2') = y_1'(2\gamma_1 + \gamma_2) + y_2'(\gamma_1 + 2\gamma_2)$$

and

$$d^{10}(b_{10}') = 3y_1't_3\gamma_1^2.$$

Proof. We deduce the differentials in $\{E_r, d^r\}$ using the notation and results of Theorem 3.7. For the same reasons as in the proof of Theorem 4.1, we have

$$\phi^*(y_i) = y_i',\ \ \phi^*(x_i) = x_i' \ \text{ and } \ \phi^*(\alpha_i) = \gamma_i = \phi^*(\beta_i) = \phi^*(u_i), \ \text{ so } \ \phi^*(v_i) = 0.$$

Recall that $\theta = \bar{d}^4(a_2(y_1(u_2 + v_2 + 2u_1) + y_2(u_1 + v_1 + 2u_2)))$ and $\psi = l_3$. Then

$$\phi^*(\theta) = 0 \text{ and } \phi^*(\psi) = t_3.$$

Hence by the same arguments used in the proof of Theorem 4.1, we have

$$d^r(\theta) = 0, d^r(y_i') = 0 \text{ and } d^r(\gamma_i) = 0.$$

Using the results of Theorem 3.7, we deduce the differentials in $\{E_r, d^r\}$. Recall that $\zeta = y_1(u_2 + v_2 + 2u_1) + y_2(u_1 + v_1 + 2u_2)$. Since $\bar{d}^4$ is non-trivial only on $a_2\zeta$, and $\phi^*\bar{d}^4(a_2\zeta) = \phi^*(\theta) = 0$, the differential d^4 is trivial.

The image of d^r on the generators a_2', b_{10}' is determined by those summands in the image of $\bar{d}^2$ on a_2, b_{10} given in Theorem 3.7 containing no v_i or θ, and replacing u_i with γ_i, y_i with y_i' and ϕ with t_3. This gives the result stated in the theorem. □

Remark 4.6. As the restriction $\phi^*\colon \bar{E}_r^{0,*} \to E_r^{0,*}$ is induced by the identity map, we have that $\phi^*((a_2)_m) = (a_2)_m$ and $\phi^*((b_{10})_m) = (b_{10})_m$ for all $m \geq 1$. Therefore using Remark 3.8 it follows that

$$d^2((a_2)_m) = d^2(a_2)(a_2)_{m-1} \text{ and } d^{10}((b_{10})_m) = d^{10}(b_{10})(b_{10})_{m-1}.$$

5. Free loop cohomology of complete flag manifolds of simple Lie groups of rank 2

In this section we calculate the cohomology of the free loop space of all complete flag manifolds arising form simple Lie groups of rank 2.

5.1. Free loop cohomology of $SU(3)/T^2$.

Theorem 5.1. *The integral algebra structure of the E_∞-page of the Leray-Serre spectral sequence associated to the evaluation fibration of $SU(3)/T^2$ is A/I, where*

$$A = \Lambda_{\mathbb{Z}}((x_4)_b\gamma_i,\ (x_4)_b y_i,\ (x_2)_m(x_4)_b y_1y_2,\ (x_2)_m(x_4)_b(y_1(\gamma_1+\gamma_2)-y_2\gamma_2),\ (x_2)_m(x_4)_b(y_2(\gamma_1+\gamma_2)-y_1\gamma_1),\\ (x_2)_m(x_4)_b(2y_2\gamma_1^2+y_1\gamma_1^2),\ (x_2)_m(x_4)_b\gamma_1^2\gamma_2,\ (x_2)_m(x_4)_b\gamma_1^3,\ (x_2)_m(x_4)_b(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2))$$

and

$$I = \langle (x_4)_b((x_2)_1^m - m!(x_2)_m)j,\ ((x_4)_1^m - m!(x_4)_m)k,\ (x_2)_a(x_4)_b(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2),\\ (x_2)_a(x_4)_b\gamma_1^3,\ (x_2)_a(x_4)_b(y_2(\gamma_1+2\gamma_2)+y_1(2\gamma_1+\gamma_2),\ 3(x_2)_a(x_4)_b(y_1\gamma_1\gamma_2+y_2\gamma_2\gamma_2))\rangle$$

where $m\geq 1$, $a,b\geq 0$, $|\gamma_i|=2$, $|y_i|=1$, $|(x_2)_m|=2m$, $|(x_4)_m|=4m$,

$$j\in\{y_1y_2,\ y_1(\gamma_1+\gamma_2)-y_2\gamma_2,\ y_2(\gamma_1+\gamma_2)-y_1\gamma_1,\ 2y_1\gamma_1^2+y_2\gamma_1^2,\ \gamma_1^2+\gamma_2^2+\gamma_1\gamma_2,\ \gamma_1^2\gamma_2,\ \gamma_1^3\},$$

$$k\in\{\gamma_i,\ y_i,\ (x_2)_m y_1y_2,\ (x_2)_m(y_1(\gamma_1+\gamma_2)-y_2\gamma_2),\ (x_2)_m(y_2(\gamma_1+\gamma_2)-y_1\gamma_1),\\ (x_2)_m(2y_2\gamma_1^2+y_1\gamma_1^2),\ (x_2)_m\gamma_1^2\gamma_2,\ (x_2)_m\gamma_1^3,\ (x_2)_m(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2)\}$$

and $1\leq i\leq 2$.

Proof. We consider the cohomology Leray-Serre spectral sequence $\{E_r,d^r\}$ associated to the evaluation fibration of $SU(3)/T^2$ studied in Section 4, that is,

$$\Omega(SU(3)/T^2)\to\Lambda(SU(3)/T^2)\to SU(3)/T^2.$$

By (1), the integral cohomology of the base space $SU(3)/T^2$ is given by

$$\frac{\mathbb{Z}[\gamma_1,\gamma_2]}{\langle\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2,\gamma_1^3\rangle}$$

where $|\gamma_1|=|\gamma_2|=2$. The integral cohomology of the fibre $\Omega(SU(3)/T^2)$ is given by

$$\Lambda_{\mathbb{Z}}(y_1,y_2)\otimes\Gamma_{\mathbb{Z}}[x_2,x_4]$$

where $|y_1|=|y_2|=1$, $|x_2|=2$ and $|x_4|=4$. Additive generators on the E_2-page of the spectral sequence are given by representative elements of the form

$$(x_2)_a(x_4)_bP,\ \ (x_2)_a(x_4)_b y_iP,\ \ (x_2)_a(x_4)_b y_1y_2P$$

where $0\leq a,b$, $1\leq i\leq 2$ and $P\in\mathbb{Z}[\gamma_1,\gamma_2]$ is a monomial of degree at most 3. By Theorem 4.1 and Remark 4.2 , the only non-zero differentials are d^2 and d^4, which are non-zero only on generators x_2 and x_4, respectively. Therefore the spectral sequence converges by the fifth page. The differentials up to sign are given by

$$d^2(x_2)=y_1(2\gamma_1+\gamma_2)+y_2(\gamma_1+2\gamma_2),\quad d^4(x_4)=y_1(\gamma_2^2+2\gamma_1\gamma_2)+y_2(\gamma_1^2+2\gamma_1\gamma_2).$$

However notice that we may write the representative of $d^4(x_4)$ as follows,

$$\begin{aligned} d^4(x_4) &= y_1(\gamma_2^2+2\gamma_1\gamma_2)+y_2(\gamma_1^2+2\gamma_1\gamma_2)+d^2(x_2)(\gamma_1+\gamma_2)\\ &= y_1(2\gamma_1^2+2\gamma_2^2+5\gamma_1\gamma_2)+y_2(2\gamma_1^2+2\gamma_2^2+5\gamma_1\gamma_2)\\ &= 3(y_1\gamma_1\gamma_2+y_2\gamma_1\gamma_2)\end{aligned}$$

where the second equality is given by subtracting elements of the symmetric ideal $2y_i(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2)$ for $i=1,2$. Hence from now on we take

$$d^4(x_4)=3(y_1\gamma_1\gamma_2+y_2\gamma_1\gamma_2).$$

The monomial generators γ_i, x_4, y_i and $(x_2)_m y_1 y_2$ occur in $E_2^{*,0}$ or $E_2^{0,*}$ and are always in the kernel of the differentials, so are algebra generators of the E_∞-page. All relations on the E_5-page coming from the relations in $H^*(\Omega(SU(3)/T^2);\mathbb{Z}) : (x_2)_1^m - m!(x_2)_m$ and $(x_4)_1^m - m!(x_4)_m$, the relations in $H^*(SU(3)/T^2;\mathbb{Z}) : \gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2$ and γ_1^3, are in the image of $d^2 : y_1(\gamma_1 + 2\gamma_2) + y_2(2\gamma_1 + \gamma_2)$ or are in the image of $d^4 : 3(y_1\gamma_1\gamma_2 + y_2\gamma_1\gamma_2)$ hold on the E_∞-page and therefore are in I if they are in A. For this reason, since E^2 generator $(x_2)_m$ and $(x_4)_m$ will not be in A, all generators of I must be considered up to multiple of $(x_2)_a$ and $(x_4)_b$. In addition, we add $((x_2)_1^m - m!(x_2)_m)j$ for j such that $(x_2)_m j$ is a generator, rather than $(x_2)_1^m - m!(x_2)_m$ to I. Similarly we add $((x_4)_1^m - m!(x_4)_m)k$ rather than $(x_4)_1^m - m!(x_4)_m$. It remains to determine all generators of A.

The elements on the E_2-page of the form $(x_2)_a(x_4)_b y_1 y_2 P$, $(x_4)_b y_i P$ and $(x_4)_b P$ are in the kernel of d^2 and generated by $y_1 y_2 (x_2)_m, (x_4)_m, \gamma_1$ and γ_2.

Let $\phi\colon \mathbb{Z}[(x_4)_m,(x_2)_m,\gamma_1,\gamma_2] \to \mathbb{Z}[(x_4)_m,(x_2)_m,y_1,y_2,\gamma_1,\gamma_2]$ be the map defined by d^2 so that the following diagram commutes

$$\begin{array}{ccccc}
\ker(\phi) & \longrightarrow & \mathbb{Z}[(x_4)_m,(x_2)_m,\gamma_1,\gamma_2] & \xrightarrow{\phi} & \mathbb{Z}[(x_4)_m,(x_2)_m,y_1,y_2,\gamma_1,\gamma_2] \\
\downarrow q & & \downarrow q & & \downarrow q \\
\ker(d^2) & \longrightarrow & E_2 & \xrightarrow{d^2} & E_2
\end{array}$$

where q is the quotient map by the symmetric polynomials and the divided polynomial relations.

$$\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2,\ \gamma_1^3,\ (x_2)_1^m - m!(x_2)_m \text{ and } (x_4)_1^m - m!(x_4)_m.$$

By direct calculation, it can be seen that the kernel of ϕ is generated by γ_1, γ_2 and $(x_4)_m$ which are also in the kernel of d^2. Thus the remaining elements of the kennel of d^2 of the form $(x_2)_m(x_4)_b P$ are obtained as elements of the ideal

$$q\phi^{-1}(\langle y_2(2\gamma_2+\gamma_1) + y_1(\gamma_2+2\gamma_1)\rangle \cap \langle \gamma_2^2 + \gamma_2\gamma_1 + \gamma_1^2,\ \gamma_1^3\rangle) \tag{12}$$

where $y_1(2\gamma_2+\gamma_1) + y_1(\gamma_2+2\gamma_1)$ spans $\mathrm{Im}(\phi)$. Note as $(x_4)_b$ has trivial image under d^2 and by Remark 4.2 it is sufficient here just to compute the kernel in the case $m=1$ and $b=0$. Intersection (12) can be computed by Gröbner basis to show it contains no generators with γ_i term lower than degree 4. Hence the generators of the of kernel are of the form $(x_2)_m(x_4)_b P$ where the degree of P is 3. Since it is not yet contained in the set of generators, we add the relation $(x_2)_m(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2)$ as a generator in order to minimize the number of algebra generators required.

Let $\psi\colon \mathbb{Z}[(x_4)_m,(x_2)_m y_1,(x_2)_m y_2,\gamma_1,\gamma_2] \to \mathbb{Z}[(x_4)_m,(x_2)_m,y_1,y_2,\gamma_1,\gamma_2]$ be the map defined by d^2 so that the following diagram commutes

$$\begin{array}{ccccc}
\ker(\psi) & \longrightarrow & \mathbb{Z}[(x_4)_m,(x_2)_m y_1,(x_2)_m y_2,\gamma_1,\gamma_2] & \xrightarrow{\psi} & \mathbb{Z}[(x_4)_m,(x_2)_m,y_1,y_2,\gamma_1,\gamma_2] \\
\downarrow q & & \downarrow q & & \downarrow q \\
\ker(d^2) & \longrightarrow & E_2 & \xrightarrow{d^2} & E_2
\end{array}$$

where q is the quotient map by relations

$$\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2,\ \gamma_1^3,\ (x_2)_1^m - m!(x_2)_m \text{ and } (x_4)_1^m - m!(x_4)_m.$$

By a direct calculation, it can be seen that the kernel of ψ is generated by γ_1, γ_2, $(x_4)_m$. We also note that as d^2 is a differential, the image of d^2 on generators of the form $(x_2)_m(x_4)_b y_i P$, generated by

$$(x_2)_a(x_4)_b(y_2(2\gamma_2+\gamma_1) + y_1(\gamma_2+2\gamma_1))$$

must also be in the kernel of d^2 when restricted to generators of the form $(x_2)_a(x_4)_b y_i P$.

Thus the remaining element of the kernel of d^2 of the form $(x_2)_m(x_4)_b y_i P$ are elements of the ideal

$$q\psi^{-1}(\langle y_1y_2(2\gamma_2+\gamma_1),\ y_1y_2(\gamma_2+2\gamma_1)\rangle \cap \langle \gamma_2^2+\gamma_2\gamma_1+\gamma_1^2,\ \gamma_1^3\rangle) \tag{13}$$

where $y_1y_2(2\gamma_2+\gamma_1)$, $y_1y_2(\gamma_2+2\gamma_1)$ span $\mathrm{Im}(\psi)$. Note as $(x_4)_b$ has trivial image under d^2 and by Remark 4.2 it is sufficient here just to compute the kernel in the case $m=1$ and $b=0$.

Intersection (13) can be computed by Gröbner basis with respect to the lexicographic monomial ordering $y_2>y_1>\gamma_2>\gamma_1$ yielding the ideal

$$\langle y_1y_2(\gamma_2^2+\gamma_2\gamma_1+\gamma_1^2),\ 3y_1y_2\gamma_1^3,\ y_1y_2(\gamma_2\gamma_1^3+2\gamma_1^4)\rangle. \tag{14}$$

Generators of the $q\psi^{-1}$ image of (14) will up to multiple of $(x_2)_m$, generate the kernel of d^2 with elements of the form $(x_2)_m(x_4)_by_iP$. The $q\psi^{-1}$ image of generator $y_1y_2(\gamma_2^2+\gamma_2\gamma_1+\gamma_1^2)$ of (14) is $y_2(\gamma_1+\gamma_2)+y_1\gamma_1$. To write the image of d^2 in terms of the generators we add

$$(x_2)_m(y_2(\gamma_1+\gamma_2)+y_1\gamma_1) \text{ and } (x_2)_m(y_1(\gamma_1+\gamma_2)+y_2\gamma_2)$$

as generators of the algebra. The $q\psi^{-1}$ image of the generator $3y_1y_2\gamma_1^3$ of (14) is $2y_2\gamma_1^2+y_1\gamma_1^2$. Hence we take

$$(x_2)_m(2y_2\gamma_1^2+y_1\gamma_1^2)$$

as generators of the algebra. The $q\psi^{-1}$ image of the generator $y_1y_2(\gamma_2\gamma_1^3+2\gamma_1^4)$ of (14) is trivial. It remains to determine the kernel of the d^4 differential.

We have that

$$d^2(x_2(y_1(2\gamma_1+\gamma_2)))=y_1y_2(2\gamma_1^2+2\gamma_2^2+5\gamma_1\gamma_2)=3y_1y_2\gamma_1\gamma_2=d^4(x_4y_i)$$

where the second equality if give by subtracting the symmetric relation $2y_1y_2(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2)$. Hence y_i multiples of $d^4(x_4)$ are trivial. Since we can obtain form the symmetric relations

$$\gamma_1(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)-\gamma_1^3=\gamma_1^2\gamma_2+\gamma_1\gamma_2^2$$

we have that

$$d^2(x_2(\gamma_1^2+2\gamma_1\gamma_2))=y_1(2\gamma_1^3+3\gamma_1^2\gamma_2+2\gamma_1\gamma_2^2)+y_2(\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2)=3y_1\gamma_1^2\gamma_2=-3y_1\gamma_1\gamma_2^2$$
$$d^2(x_2(\gamma_2^2+2\gamma_1\gamma_2))=y_1(4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+\gamma_2^3)+y_2(2\gamma_1^2\gamma_2+3\gamma_1\gamma_2^2+2\gamma_2^3)=3y_2\gamma_1\gamma_2^2=-3y_2\gamma_1^2\gamma_2.$$

This allows us to easily determine that

$$d^4(x_4\gamma_1)=3(y_1\gamma_1^2\gamma_2+y_2\gamma_1^2\gamma_2)=d^2(x_2(\gamma_1^2-\gamma_2^2))$$
$$d^4(x_4\gamma_2)=3(y_1\gamma_1\gamma_2^2+y_2\gamma_1\gamma_2^2)=d^2(x_2(\gamma_2^2-\gamma_1^2)).$$

Hence γ_i multiples of $d^4(x_4)$ are also trivial and so using Remark 4.2 the only elements on the E^4-page not in the kernel of d^4 are $(x_4)_b$. □

Theorem 5.2. *The cohomology algebra $H^*(\Lambda(SU(3)/T^2);\mathbb{Z})$ is isomorphic as a module to the algebra A/I given in Theorem 5.1. In addition there are no multiplicative extension problem on the sub-algebra generated by γ_1,γ_2, the sub-algebra generated by y_1,y_2 and no multiplicative extension on elements $y_i\gamma_j$ for $1\le i,j\le 2$.*

Proof. Beginning with the structure of the E^∞-page, given in Theorem 5.1, of the Leray-Serre spectral sequence of the evaluation fibration, we first consider the additive extension problems.

Since there is no torsion produced by the divided polynomial relations

$$(x_4)_b((x_2)_1^m-m!(x_2)_m)j \text{ and } ((x_4)_1^m-m!(x_4)_m)k$$

a reduced Gröbner basis of the ideal

$$\langle y_1^2,y_2^2,\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2,\gamma_1^3,y_2(\gamma_1+2\gamma_2)+y_1(2\gamma_1+\gamma_2),y_1y_2(2\gamma_1+\gamma_2),\ y_1y_2(2\gamma_2+\gamma_1),3(y_1\gamma_1\gamma_2+y_2\gamma_2\gamma_2)\rangle$$

has elements with terms containing coefficient 1 or all the coefficients are 3. Hence by Theorem 2.3, all the torsion on the E_∞-page of the spectral sequence is 3-torsion. In order to resolve any additive extension problems we consider the spectral sequence $\{E_r,d^r\}$ over the field of three elements.

None of the generators in the integral spectral sequence are divisible by 3, hence in the modulo 3 spectral sequence all of the integral generators remain non-trivial. In addition when the kernel of d^2 and d^4 at $E_\infty^{p,q}=E_5^{p,q}$ is all of $E_\infty^{p,q}$, the free rank plus torsion rank in the integral spectral sequence must be greater than or equal to the rank in the modulo 3 spectral sequence. So in these cases, the rank in modulo 3 spectral sequence is exactly the free rank plus the torsion rank in the integral case. Hence it remains to consider the cases when kernel of either the d^2 or the d^4 differentials in the cases integral kernel are not the entire domain. By the rank nullity theorem, the rank of the image plus the nullity, the dimension of the kernel, is the dimension of the domain.

When considering the spectral sequence modulo 3, the rank of any differential is the same as in the integral case when the quotient of the preceding kernel by the image contains no torsion. When integral 3-torsion exists, there are generators of the image which are 3 times generators of the kernel. Therefore in the modulo 3 spectral sequence these generators are now generators of the kernel. Hence in the modulo 3 spectral sequence the rank is reduced by the integral torsion rank and the nullity increased by the same number.

Since the modulo 3 spectral sequence has coefficients in a field, there are no extension problems. As the total degrees of the d^2 and the d^4 differentials are -1 and $E_5 = E_\infty$, $\dim(H^i(SU(3)/T^2;\mathbb{Z}_3))$ is the sum of the ranks of total degree i in the integral E_5-page plus the sum of the torsion ranks in total degrees i and $i+1$. Hence, the modulo 3 cohomology algebra is only consistent with the case when all the torsion on the E_∞-page of the spectral sequence is contained in the integral cohomology module. Therefore all additive extension problems are resolved and all the torsion elements in the spectral sequence are present in the integral cohomology.

Now that we have deduced that the module structure of $H^*(\Lambda(SU(3)/T^2);\mathbb{Z})$ is isomorphic to A/I, we have that A/I is an associated graded algebra of $H^*(\Lambda(SU(3)/T^2);\mathbb{Z})$ with respect to multiplication length filtration. Next we study the multiplicative extension problem.

Multiplication on generators γ_1, γ_2 is in the image of the induced map of the evaluation fibration $H^*(SU(3)/T^2;\mathbb{Z}) \to H^*(\Lambda(SU(3)/T^2);\mathbb{Z})$, hence the sub-algebra they generate contains no extension problems. Multiplication on generators y_1, y_2 is a free graded commutative sub-algebra, so contains no extension problems.

Suppose the product $y_i\gamma_j$ for $1 \leq i, j \leq 2$ in A/I contains additional summands in $H^*(\Lambda(SU(3)/T^2);\mathbb{Z})$. For dimensional reasons any summand must have the form $y_{i'}\gamma_{j'}$ for $1 \leq i', j' \leq 2$. However since $\gamma_i, y_i \in H^*(\Lambda(SU(3)/T^2);\mathbb{Z})$ all possible additional summands have the same multiplication length showing that there is no multiplicative extension problem for $y_i\gamma_j$, $1 \leq i, j \leq 2$.

□

5.2. **Free loop cohomology of $Sp(2)/T^2$.**

Theorem 5.3. *The integral algebra structure of the E_∞-page of the Leray-Serre spectral sequence associated to the evaluation fibration of $Sp(2)/T^2$ is A/I, where*

$$\begin{aligned}A = \Lambda_{\mathbb{Z}}(((x_6)_b\gamma_i,\ (x_6)_b y_i,\ (x_2)_m(x_6)_b y_1y_2,\ (x_2)_m(x_6)_b(y_1\gamma_2 - y_2\gamma_1),\\ (x_2)_m(x_6)_b y_2\gamma_1^3,\ (x_2)_m(x_6)_b(y_1\gamma_1 + y_2\gamma_2),\ (x_2)_m(x_6)_b(\gamma_1^2+\gamma_2^2),\\ (x_2)_m(x_6)_b\gamma_1^4,\ (x_2)_m(x_6)_b\gamma_1^3\gamma_2)\end{aligned}$$

and

$$\begin{aligned}I = \langle(((x_2)_1^m - m!(x_2)_m)j,\ ((x_6)_1^m - m!(x_6)_m)k,\\ (x_2)_a(x_6)_b(\gamma_1^2+\gamma_2^2),\ (x_2)_a(x_6)_b\gamma_1^4,\ 2(x_2)_a(x_6)_b(y_1\gamma_1+y_2\gamma_2),\ (x_6)_b 4y_1\gamma_1^3)\rangle\end{aligned}$$

where $i = 1, 2$, $m \geq 1$, $a, b \geq 1$, $|\gamma_i| = 2$, $|(x_2)_m| = 2m$, $|(x_6)_m| = 6m$, $|y_i| = 1$,

$$\begin{aligned}j \in \{(x_6)_b y_1y_2,\ (x_6)_b y_1\gamma_2 - y_2\gamma_1,\ (x_6)_b y_2\gamma_1^3,\\ (x_6)_b y_1\gamma_1 + y_2\gamma_2,\ (x_6)_b\gamma_1^3\gamma_2,\ (x_6)_b\gamma_1^2+\gamma_2^2,\ (x_6)_b\gamma_1^4\}\end{aligned}$$

and

$$\begin{aligned}k \in \{\gamma_i,\ y_i,\ (x_2)_m y_1y_2,\ (x_2)_m(y_1\gamma_2 - y_2\gamma_1),\ (x_2)_m y_2\gamma_1^3,\\ (x_2)_m(y_1\gamma_1+y_2\gamma_2),\ (x_2)_m\gamma_1^3\gamma_2),\ (x_2)_m(\gamma_1^2+\gamma_2^2),\ (x_2)_m\gamma_1^4)\}\end{aligned}$$

for $1 \leq i \leq 2$.

Proof. We consider the cohomology Leray-Serre spectral sequence $\{E_r, d^r\}$ associated to the evaluation fibration of $Sp(2)/T^2$,

$$\Omega(Sp(2)/T^2) \to \Lambda(Sp(2)/T^2) \to Sp(2)/T^2.$$

By (2), the cohomology of the base space $Sp(2)/T^2$ is

$$H^*(Sp(2)/T^2;\mathbb{Z}) = \frac{\mathbb{Z}[\gamma_1,\gamma_2]}{\langle \gamma_1^2+\gamma_2^2,\ \gamma_1^4\rangle}.$$

From (4.2), the cohomology of the fibre $\Omega(Sp(2)/T^2)$ is

$$H^*(\Omega(Sp(2)/T^2);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(y_1,y_2)\otimes\Gamma_{\mathbb{Z}}[x_2,x_6]$$

where $|y_1| = 1 = |y_2|$, $|x_2| = 2$ and $|x_6| = 6$.

The elements on the E_2-page of the spectral sequence are generated additively by the representative elements of the form

$$(x_2)_a(x_6)_bP,\ \ (x_2)_a(x_6)_by_iP,\ \ (x_2)_a(x_6)_by_1y_2P$$

where $0 \leq a, b$, $1 \leq i \leq 2$ and $P \in \mathbb{Z}[\gamma_1,\gamma_2]$ is a monomial of degree at most 4.

By Theorem 4.3 and Remark 4.4 , the only non-zero differentials in $\{E_r, d^r\}$ are d^2 and d^6, which are non-zero only on generators x_2 and x_6 respectively. Hence the spectral sequence converges at the seventh page. The differentials up to sign are given by

$$d^2(x_2) = 2(y_1\gamma_1+y_2\gamma_2),\quad d^4(x_6) = 4y_1\gamma_1^3 \tag{15}$$

The monomial generators γ_i, y_i and $(x_2)_m(x_6)_by_1y_2$ occur in $E_2^{*,0}$ or $E_2^{0,*}$ and are always in the kernel of the differentials, so they are algebra generators of the E_∞-page. All relations on the E^7-page coming from the relations in $H^*(\Omega(Sp(2)/T^2);\mathbb{Z}) : (x_2)_1^m - m!(x_2)_m$ and $(x_6)_1^m - m!(x_6)_m$, the relations in $H^*(Sp(2)/T^2;\mathbb{Z}) : \gamma_1^2+\gamma_2^2$ and γ_1^4 or are either in the image of $d^2 : 2(y_1\gamma_1+y_2\gamma_2)$ or in the image of $d^6 : 4y_1\gamma_1^3$ hold on the E_∞-page and therefore are in I if they are in A. For this reason, since the E^2 generators $(x_2)_m$ and $(x_6)_m$ will not be in A, all the generators of I must be considered up to a multiple of $(x_2)_a$ and $(x_6)_b$. In addition we add $((x_2)_1^m - m!(x_2)_m)j$ and $((x_6)_1^m - m!(x_6)_m)k$ for the j, k such that $(x_2)_mj$ and $(x_6)_mk$ are generators of the algebra. It remains to determine all the generators of A.

Elements on the E_2-page of the form $(x_2)_a(x_6)_by_1y_2P$ are in the kernel of d^2 and generated by $y_1y_2(x_2)_m(x_6)_b, (x_6)_by_i$ and $(x_6)_b\gamma_i$.

Let $\phi\colon \mathbb{Z}[(x_6)_m,(x_2)_m,\gamma_1,\gamma_2] \to \mathbb{Z}[(x_6)_m,(x_2)_m,y_1,y_2,\gamma_1,\gamma_2]$ be the map defined by d^2 so that the following diagram commutes

$$\begin{array}{ccccc}
\ker(\phi) & \longrightarrow & \mathbb{Z}[(x_6)_m,(x_2)_m,\gamma_1,\gamma_2] & \xrightarrow{\ \phi\ } & \mathbb{Z}[(x_6)_m,(x_2)_m,y_1,y_2,\gamma_1,\gamma_2] \\
\downarrow q & & \downarrow q & & \downarrow q \\
\ker(d^2) & \longrightarrow & E_2 & \xrightarrow{\ d^2\ } & E_2
\end{array}$$

where q is the quotient map by the symmetric polynomial and divided polynomial relations

$$\gamma_1^2+\gamma_2^2,\ \gamma_1^4,\ (x_2)_1^m - m!(x_2)_m \text{ and } (x_6)_1^m - m!(x_6)_m.$$

By direct calculation, it can be seen that the kernel of ϕ is generated by γ_1, γ_2 and $(x_6)_m$ which are also in the kernel of d^2. Thus the remaining elements of the kennel of d^2 of the form $(x_2)_m(x_6)_bP$ are elements of the ideal

$$q\phi^{-1}(\langle 2(y_1\gamma_1+y_2\gamma_2)\rangle \cap \langle \gamma_2^2+\gamma_1^2,\ \gamma_1^4\rangle) \tag{16}$$

where $2(y_1\gamma_1+y_2\gamma_2)$ spans $\mathrm{Im}(\phi)$. Note as $(x_6)_b$ has trivial image under d^2 and by Remark 4.2 it is sufficient here just to compute the kernel in the case $m = 1$ and $b = 0$. Intersection (16) can be computed by Gröbner basis. A basis with respect to the lexicographic monomial ordering $y_2 > y_1 > \gamma_1 > \gamma_1$ restricted to the terms containing only a multiple of a single y_i is given by

$$2y_2\gamma_2\gamma_1^2 + y_2\gamma_2^3 + 2y_1\gamma_2^2\gamma_1 + 2y_1\gamma_1^3,\ 2y_2\gamma_2\gamma_1^4 + 2y_1\gamma_1^5,\ 2y_2\gamma_2^2\gamma_1^3 + 2y_1\gamma_2^2\gamma_1^4. \tag{17}$$

The $q\psi^{-1}$ image of (17) will, up to multiple of $(x_2)_m$ and $(x_6)_m$, generate the kernel of d^2 with elements of the form $(x_2)_m(x_4)_bP$. The $q\psi^{-1}$ image of $2y_2\gamma_2\gamma_1^2 + y_2\gamma_2^3 + 2y_1\gamma_2^2\gamma_1 + 2y_1\gamma_1^3$ is trivial. However since it is not yet contained in the generators, we add relation

$$(x_2)_m(x_4)_b(\gamma_1^2 + \gamma_2^2)$$

as a generator in order to minimize the number of algebra generators required. The $q\psi^{-1}$ image of $2y_2\gamma_2^2\gamma_1^3 + 2y_1\gamma_2^2\gamma_1^4$ is $\gamma_2\gamma_1^3$. Hence, we take

$$(x_2)_m, (x_6)_b\gamma_2\gamma_1^3$$

as generators of the algebra. The $q\psi^{-1}$ image of $2y_2\gamma_2\gamma_1^4 + 2y_1\gamma_1^5$ is trivial. Since it is not yet contained in the generators, we add relation

$$(x_2)_m(x_4)_b\gamma_1^4$$

as a generator in order to minimize the number of algebra generators required.

Let $\psi\colon \mathbb{Z}[(x_6)_b, (x_2)_m y_1, (x_2)_m y_2, \gamma_1, \gamma_2] \to \mathbb{Z}[(x_2)_m, (x_6)_b, y_1, y_2, \gamma_1, \gamma_2]$ be the map defined by d^2 so that the following diagram commutes

$$\begin{array}{ccccc}
\ker(\psi) & \longrightarrow & \mathbb{Z}[(x_6)_b, (x_2)_m y_1, (x_2)_m y_2, \gamma_1, \gamma_2] & \xrightarrow{\psi} & \mathbb{Z}[(x_2)_m, (x_6)_b, y_1, y_2, \gamma_1, \gamma_2] \\
\downarrow q & & \downarrow q & & \downarrow q \\
\ker(d^2) & \longrightarrow & E_2 & \xrightarrow{d^2} & E_2
\end{array}$$

where q is the quotient map by relations

$$\gamma_1^2 + \gamma_2^2,\ \gamma_1^4,\ (x_2)_1^m - m!(x_2)_m \text{ and } (x_6)_1^m - m!(x_6)_m.$$

By direct calculation, it can be seen that the kernel of ψ is generated by γ_1, γ_2, $(x_6)_m$. We also note that as d^2 is a differential, the image of d^2 on generators of the form $(x_2)_m(x_6)_b y_i P$, generated by

$$2(x_2)_a(x_6)_b(y_1\gamma_1 + y_2\gamma_2)$$

must also be in the kernel of d^2 when restricted to generators of the form $(x_2)_a(x_6)_b y_i P$.

Thus the remaining element of the kernel of d^2 of the form $(x_2)_m(x_6)_b y_i P$ are elements of the ideal

$$q\psi^{-1}(\langle 2y_1y_2\gamma_1,\ 2y_1y_2\gamma_2\rangle \cap \langle \gamma_2^2 + \gamma_1^2,\ \gamma_1^4\rangle) \tag{18}$$

where $y_1y_2(2\gamma_2 + \gamma_1)$, $y_1y_2(\gamma_2 + 2\gamma_1)$ span $\mathrm{Im}(\psi)$. Note as $(x_6)_b$ has trivial image under d^2 and by Remark 4.2 it is sufficient here just to compute the kernel in the case $m = 1$ and $b = 0$.

It is straightforward to see that intersection (18) is the ideal

$$\langle 2y_1y_2(\gamma_1^2 + \gamma_2^2),\ 2y_1y_2\gamma_1^4\rangle. \tag{19}$$

The generators of the $q\psi^{-1}$ image of (18) will up to multiples of $(x_2)_m$ and $(x_6)_m$, generate the kernel of d^2 with elements of the form $(x_2)_m(x_4)_b y_i P$. The $q\psi^{-1}$ image of generator $2y_1y_2(\gamma_1^2 + \gamma_2^2)$ of (19) is $y_1\gamma_1 - y_2\gamma_2$. Hence we take both

$$(x_2)_m \text{ and } (x_6)_b(y_1\gamma_1 - y_2\gamma_2)$$

as generators of the algebra. The $q\psi^{-1}$ image of the generator $y_1y_2\gamma_1^4$ of (19) is $y_2\gamma_1^3$. Hence we take

$$(x_2)_m, (x_6)_b y_2\gamma_1^3$$

as a generators of the algebra. It remains to consider when the d^6 differential is non-trivial.

Notice that by (15) and Remark 4.4 , on the E^6-page

$$\begin{aligned} d^6((x_6)_m y_1) &= 4(x_6)_{m-1} y_1^2\gamma_1^3\gamma_2 = 0 \\ d^6((x_6)_m y_2) &= 4(x_6)_{m-1} y_1 y_2\gamma_1^4 = 0 \\ d^6((x_6)_m \gamma_1) &= 4(x_6)_{m-1} y_1\gamma_1^4 = 0 \\ \text{and}\ \ d^6((x_6)_m \gamma_2) &= 4(x_6)_{m-1} y_1\gamma_1^3\gamma_2 = 0. \end{aligned}$$

Therefore the only elements on the E^6 page not in the kernel of d^6 are $(x_6)_m$. □

Theorem 5.4. *The cohomology algebra $H^*(\Lambda(Sp(2)/T^2);\mathbb{Z})$ is isomorphic as a module to the algebra A/I given in Theorem 5.3 up to order of 2-torsion.*

Proof. As in the proof of Theorem 5.2 by considering a Gröbner basis of the elements in I it can be seen that the torsion on the E_∞-page of $\{E_r, d^r\}$ is a power of 2, at most 4. Hence we consider the spectral sequence $\{E_r, d^r\}$ over the field of two elements. Since the only non-zero differentials d^2 and d^6 have bidegree $(2,-1)$ and $(6,-5)$ respectively, for exactly the same reasons as for the modulo 3 spectral sequence in Theorem 5.1, all torsion on the E_∞-page survives the additive extension problem over $\mathbb{Z}$. The only remaining additive extension problem is whether the 4-torsion generated by $(x_2)_a y_1\gamma_1^3$ on the E_∞, is 2-torsion or 4-torsion in $H^*(\Lambda(Sp(2)/T^2);\mathbb{Z})$. □

5.3. **Free loop cohomology of G_2/T^2.**

Theorem 5.5. *The integral algebra structure of the E_∞-page of the Leray-Serre spectral sequence associated to the evaluation fibration of G_2/T^2 is A/I, where*

$$\begin{aligned} A = \Lambda_{\mathbb{Z}}(&(b_{10})_l\gamma_i,\ (b_{10})_l t_3,\ (b_{10})_l y_i,\ (a_2)_m(b_{10})_l y_1y_2, \\ &(a_2)_m(b_{10})_l(y_1(\gamma_1+\gamma_2)+y_2\gamma_2),\ (a_2)_m(b_{10})_l(t_3\gamma_1^2(y_1-2y_2)), \\ &(a_2)_m(b_{10})_l(y_1(2\gamma_1+\gamma_2)+y_2(\gamma_1+2\gamma_2)),\ (a_2)_m(b_{10})_l(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2) \\ &(a_2)_m(b_{10})_l(2t_3-\gamma_1^3),\ (a_2)_m(b_{10})_l t_3,\ (a_2)_m(b_{10})_l\gamma_1^3,\ (a_2)_m(b_{10})_l\gamma_1^2\gamma_2) \end{aligned}$$

and

$$\begin{aligned} I = \langle &(a_2^m - m!/2^{\lfloor\frac{m}{2}\rfloor}(a_2)_m)j,\ (b_{10}^m - m!(b_{10})_m)k,\ (a_2)_h(b_{10})_l(2t_3-\gamma_1^3), \\ &(a_2)_h(b_{10})_l(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2),\ (a_2)_h(b_{10})_l t_3^2,\ 2(a_2)_{s+1}(b_{10})_l(y_1(2\gamma_1+\gamma_2)+y_2(\gamma_1+2\gamma_2)), \\ &(a_2)_s(b_{10})_l(y_1(2\gamma_1+\gamma_2)+y_2(\gamma_1+2\gamma_2)),\ 3(a_2)_h(b_{10})_l y_1t_3\gamma_1^2\rangle \end{aligned}$$

where $s,l,h\geq 0$ with s even, $m\geq 1$, $(a_2)_0=1$, $|\gamma_i|=2$, $|y_i|=1$, $|(a_2)_m|=2m$, $|(b_{10})_m|=10m$,

$$\begin{aligned} j\in\{&(b_{10})_l y_1y_2,\ (b_{10})_l(y_1(\gamma_1+\gamma_2)+y_2\gamma_2), \\ &(b_{10})_l(t_3\gamma_1^2(y_1-2y_2)),\ (b_{10})_l\gamma_1^2\gamma_2,\ (b_{10})_l(y_1(2\gamma_1+\gamma_2)+y_2(\gamma+2\gamma_2)), \\ &(b_{10})_l(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2),\ (b_{10})_l(2t_3-\gamma_1^3),\ (b_{10})_l t_3,\ (b_{10})_l\gamma_1^3,\ (b_{10})_l\gamma_1^2\gamma_2\} \end{aligned}$$

and

$$\begin{aligned} k\in\{&\gamma_i,\ y_i,\ t_3,\ (a_2)_s y_1y_2,\ (a_2)_l(y_1(\gamma_1+\gamma_2)+y_2\gamma_2), \\ &(a_2)_l(t_3\gamma_1^2(y_1-2y_2)),\ (a_2)_l\gamma_1^2\gamma_2,\ (a_2)_m(y_1(2\gamma_1+\gamma_2)+y_2(\gamma+2\gamma_2)), \\ &(a_2)_l(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2),\ (a_2)_l(2t_3-\gamma_1^3),\ (a_2)_l t_3,\ (a_2)_l\gamma_1^3,\ (a_2)_l\gamma_1^2\gamma_2\} \end{aligned}$$

for $1\leq i\leq 2$.

Proof. We consider the cohomology Leray-Serre spectral sequence $\{E_r, d^r\}$ associated to the evaluation fibration of G_2/T^2,

$$\Omega(G_2/T^2)\to\Lambda(G_2/T^2)\to G_2/T^2.$$

The cohomology of G_2/T^2 is given by

$$\frac{\mathbb{Z}[\gamma_1,\gamma_2,t_3]}{\langle \gamma_1^2+\gamma_1\gamma_2+\gamma_2^2, 2t_3-\gamma_1^3, t_3^2\rangle}$$

where $|\gamma_1|=|\gamma_2|=2$ and $|t_3|=6$. The cohomology of $\Omega(G_2/T^2)$ is

$$\Lambda(y_1,y_2)\otimes\frac{\mathbb{Z}[(a_2)_1,(a_2)_2,\dots]}{\langle a_2^m-(m!/2^{\lfloor\frac{m}{2}\rfloor})(a_2)_m\rangle}\otimes\Gamma_{\mathbb{Z}}[b_{10}]$$

where $|y_1|=|y_2|=1$, $|a_2|=2$ and $|b_{10}|=10$.

The E_2-page of the spectral sequence is generated additively by the elements of the form

$$(a_2)_h(b_{10})_lP,\ \ (a_2)_h(b_{10})_ly_iP,\ \ (a_2)_h(b_{10})_ly_1y_2P$$

where $0\le l,h,\ 1\le i\le 2$ and $P\in\mathbb{Z}[\gamma_1,\gamma_2,t_3]$ is a monomial of degree at most 6, taking t_3 as monomial of degree 3. By Theorem 4.5 and Remark 4.6 , the only non-zero differentials are d^2 and d^{10}, which are non-zero only on the generators a_2 and b_{10}, respectively. The differentials up to sign are given by

$$d^2(a_2)=y_1(\gamma_1+2\gamma_2)+y_2(2\gamma_1+\gamma_2),\quad d^{10}(b_{10})=3y_1t_3\gamma_1^2. \tag{20}$$

In particular the spectral sequence converges by the E_{11}-page. The monomial generators γ_i, t_3, y_i and $(a_2)_s(b_{10})_ly_1y_2$ occur in $E_2^{*,0}$ or $E_2^{0,*}$ and are always in the kernel of the differentials, so are algebra generators of the E_∞-page. All relations on the E_{11}-page coming from the relations in $H^*(\Omega(G_2/T^2);\mathbb{Z}):a_2^m-m!/2^{\lfloor\frac{m}{2}\rfloor}(a_2)_m$ and $(b_{10})_1^m-m!(b_{10})_m$, the relations in $H^*(G_2/T^2;\mathbb{Z}):$ $\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2, 2t_3-\gamma_1^3$ and t_3^2, or are in the image of $d^2:y_1(\gamma_1+2\gamma_2)+y_2(2\gamma_1+\gamma_2)$ and in the image of $d^{10}:3y_1t_3\gamma_1^2$ hold on the E_∞-page and therefore are in I if they are in A. For this reason, since E_2 generators $(a_2)_m$ and $(b_{10})_m$ will not be in A all generators of I, must be considered up to multiple of $(a_2)_h$ and $(b_{10})_l$. However due to the non-divided polynomial multiplicative structure on $(a_2)_m$, unlike for the previous flag manifolds for $s\ge 1$ even

$$d^2((a_2)_s)=2(a_2)_{s-1}(y_1(\gamma_1+2\gamma_2)+y_2(2\gamma_1+\gamma_2)).$$

Hence, we add generators

$$(a_2)_s(y_1(\gamma_1+2\gamma_2)+y_2(2\gamma_1+\gamma_2))\text{ and }2(a_2)_{s+1}(y_1(\gamma_1+2\gamma_2)+y_2(2\gamma_1+\gamma_2))$$

to I instead of $(a_2)_m(y_1(\gamma_1+2\gamma_2)+y_2(2\gamma_1+\gamma_2))$. In addition, we add $(a_2^m-m!/2^{\lfloor\frac{m}{2}\rfloor}(a_2)_m)j$ and $((b_{10})_1^m-m!(b_{10})_m)k$ for the j,k such that $(a_2)_mj$ and $(b_{10})_mk$ are generators of the algebra. It remains to determine all the generators of A.

The image of the differential and the symmetric ideal generators are similar to the case of the proof of Theorem 5.1. Hence the argument is similar to the proof of Theorem 5.1, we add

$$\begin{aligned}&(a_2)_m(b_{10})_l(y_1(\gamma_1+\gamma_2)+y_2\gamma_2)\ (a_2)_m(b_{10})_l(t_3\gamma_1^2(y_1-2y_2)),\\&(a_2)_m(b_{10})_l(y_1(2\gamma_1+\gamma_2)+y_2(\gamma_1+2\gamma_2)),\ (a_2)_m(b_{10})_l(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)\\&(a_2)_m(b_{10})_l(2t_3-\gamma_1^3),\ (a_2)_m(b_{10})_lt_3,\ (a_2)_m(b_{10})_l\gamma_1^3\text{ and }(a_2)_m(b_{10})_l\gamma_1^2\gamma_2\end{aligned}$$

as gnerators of A. It remains to consider when the d^{10} differential is non-trivial.

Using (20), it can be seen that the images of $b_{10}\gamma_i$ and $b_{10}y_i$ are trivial. Therefore, the only classes on the E^{10} page not in the kernel of d^{10} are $(b_{10})_m$. □

Theorem 5.6. *The cohomology algebra $H^*(\Lambda(G_2/T^2);\mathbb{Z})$ is isomorphic as a module to the algebra A/I given in Theorem 5.5 up to the order of 2-torsion.*

Proof. As in the proof of Theorem 5.2 by considering a Gröbner basis of the elements in I all torsion on the E_∞ is of rank 2, 3, 6 or 12. Most module extension problems are resolved in the same way as previous cases by considering the module 2 and modulo 3 spectral sequences. However this does not determine torsion of rank 12. □

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