Measure-Theoretic Semantics for Quantitative Parity Automata

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Abstract

Quantitative parity automata (QPAs) generalise non-deterministic parity automata (NPAs) by adding weights from a certain semiring to transitions. QPAs run on infinite word/tree-like structures, modelled as coalgebras of a polynomial functor F. They can also arise as certain products between a quantitative model (with branching modelled via the same semiring of quantities, and linear behaviour described by the functor F) and an NPA (modelling a qualitative property of F-coalgebras). We build on recent work on semiring-valued measures to define a way to measure the set of paths through a quantitative branching model which satisfy a qualitative property (captured by an unambiguous NPA running on F-coalgebras). Our main result shows that the notion of *extent* of a QPA (which generalises non-emptiness of an NPA, and is defined as the solution of a nested system of equations) provides an equivalent characterisation of the measure of the accepting paths through the QPA. This result makes recently-developed methods for computing nested fixpoints available for model checking qualitative, linear-time properties against quantitative branching models.

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1 Introduction

When model checking linear-time properties over non-deterministic or probabilistic models, the standard approach is to formalise the property in question as an automaton running over infinite words, and to consider the product of this automaton with the model, in order to answer the questions: *Does there exist a path through the model which conforms to a property automaton?* and *What is the probability of exhibiting a path which conforms to an automaton?* (see e.g. [1][Sections 4.6 and 28.6], [2]). Generalising this approach, we consider state-based system models whose transitions carry weights from a partial semiring. Instances of such systems include non-deterministic systems (with weights from the boolean semiring), probabilistic systems (with weights from the probabilistic semiring), and resource-aware systems (with weights from the tropical semiring). Thus, our work can also answer the following question, using similar automata-based techniques: *What is the minimal amount of resources needed to exhibit a path which conforms to a property automaton?*

In addition to a more general notion of branching, our models also allow a more general notion of path: whereas in existing approaches paths are sequences (of states and transition labels), with each transition resulting in a *single* successor state, here individual transitions can have *finitely-many* successor states, and thus paths can be tree-shaped. This allows us

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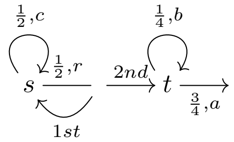
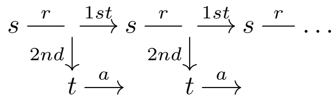
to model systems with *dynamic structure*, as illustrated by the following example:





The model above (left), with initial state s, has standard transitions (labels b, c) which result in a *single* successor state, but also transitions resulting in *two* successors states (label r), or *zero* successor states (label a). One can view this as modelling a probabilistic server which accepts requests (r transition) or carries out other work (c transition), both with probability $\frac{1}{2}$. Following a request, a *new* process is created to deal with the request (state t), and the server itself continues in state s. To model this behaviour, the r transition has *two* successor states; these are ordered, as indicated by the labels on the arrows leading to them. Then, an a-transition models successfully answering a request, while a b-transition models doing other work instead. A possible execution of this system, where the server repeatedly accepts new requests and the newly created processes immediately answer them, is pictured above (right).

We use automata over infinite words (similarly to existing work [1, 2]) but also over infinite trees (given that paths can be tree-shaped), to formalise correctness properties of system executions. Such properties have a *qualitative* interpretation over paths, but also a quantitative interpretation over states in our models. For instance, in the previous example, one might want to formalise (and verify!) the property that every server request is eventually answered. While existing approaches typically use Büchi/Rabin automata to describe ω-regular properties of infinite words [1, 2], here we choose the related formalism of *parity automata* for several reasons: (i) it is as expressive as Büchi/Rabin automata over infinite words, (ii) unlike Büchi automata, they have the full expressive power needed to capture all regular languages of infinite trees [10, 7], and (iii) their acceptance conditions can be described using the solutions of nested systems of equations.

In order to uniformly treat a variety of branching types (with transition weights taken from a semiring) and transition types (linear- or tree-shaped, or a combination), we model systems as *coalgebras*; their type incorporates branching behaviour (described by a monad) and linear behaviour (described by a polynomial endofunctor). We model system executions also as coalgebras (with no branching), and as a result our automata operate on coalgebras.

The question we are concerned with is: *Given a quantitative branching model and a qualitative property of paths, with the latter formalised as a parity automaton, what is the* degree *(e.g. probability/cost) with which the property holds in the quantitative model?* We answer this question in two ways: one which is measure-theoretic and naturally captures the intuition that we are *measuring*, in some generalised sense, the accepting runs of a quantitative automaton (building on results in [4] on semiring-valued measures); and another which is more amenable to computation (using the notion of *extent* from [6]). After defining these two ways of measuring the set of accepting runs of a QPA, our main result establishes their equivalence. The implications of this result are two-fold. On the one hand, the result formally confirms that the notion of extent defined in [6] achieves its intended purpose in key example semirings: it measures the existence of an accepting path in the non-deterministic case; the probability of exhibiting an accepting path in the probabilistic case (and thus instantiates to known results in this case); and the minimal cost required to exhibit an accepting path, in the resource-aware case. On the other hand, since the latter characterisation is in terms of the solution of a nested system of equations, methods for computing such solutions (including those recently developed in [11, 3, 12]) become available for model checking qualitative, linear-time properties against quantitative branching models. In the last part of the paper, we show how the standard automata-based approach to model checking linear-time properties over non-deterministic and probabilistic models [1, 2] generalises to quantitative branching models. We defer computational aspects to future work, as this requires adapting techniques in [3] to our more general notion of system of equations.

At the heart of our main result is a characterisation, due to [15], of the accepting paths of a parity automaton as the solution of a nested system of equations. This allows us to relate, via a semiring-valued measure, the set of accepting paths of a QPA and its extent (also defined as the solution of a system of equations). The proof of this result is non-trivial, partly because semiring-valued measures are not well-behaved w.r.t. intersections.

The paper is structured as follows: Section 2 introduces relevant concepts, including systems of equations and their solutions, qualitative and quantitative parity automata, and semiring-valued measures. Section 3 shows the equivalence of two approaches to measuring accepting runs: via semiring-valued measures and via extents. Next, Section 4 shows how this result can be used to model-check qualitative, linear-time properties against quantitative branching models. Section 5 summarises our contributions and outlines future work.

Related Work. [4] considers quantitative, linear-time fixpoint logics interpreted over the same type of quantitative branching models. Semiring-valued measures are introduced in op. cit., and used to provide a measure-theoretic semantics for these logics. This is then proved equivalent to the original semantics for the logics. However, these logics suffer from limited expressiveness on tree-shaped linear behaviours (they cannot express conjunctions and arbitrary disjunctions). Here we address this limitation, while also taking a more fundamental approach to formalising linear-time properties, namely as automata. Beyond the increased generality, a key difference compared to [4] is that our proofs now exploit a characterisation of the accepting paths of a QPA as the solution of a nested system of equations. Thus, by working at the level of automata, the link between the extent-based semantics and the measure-theoretic semantics becomes conceptually clearer. As added benefit, the move to automata connects our work to existing algorithmic approaches for solving nested systems of equations, thereby paving the way for applications in model checking.

Quantitative verification of weighted systems has been considered in a number of other works, including [8, 9, 14]. Our approach differs from these in that we restrict to *qualitative* properties of paths through a quantitative branching model, and we measure to what degree these hold in such models. One immediate drawback of the increased generality in [8, 9] is that the meaning of quantitative formulas is conceptually less clear, and is defined separately for each model type (namely quantitative transition systems and quantitative Markov chains). The same holds for the model checking algorithms, which are tailored to the underlying semantic model and not generic. In contrast, our quantitative notion of acceptance has an intuitive measure-theoretic description, and our model checking approach (computation of nested extents) is parameterised by the semiring used to model weighted branching.

2 Background

2.1 Nested Systems of Equations

▶ **Definition 1.** *Let $L_0, \ldots, L_n$ be complete lattices. A* nested system of equations E *has the form*

$$\begin{bmatrix} x_0 & =_\nu & f_0(x_0, \ldots, x_n) \\ x_1 & =_\mu & f_1(x_0, \ldots, x_n) \\ \vdots & & \\ x_n & =_\eta & f_n(x_0, \ldots, x_n) \end{bmatrix} \tag{1}$$

where η is either μ, if n is odd, or ν, if n is even, and where for $i \in \{0,\ldots,n\}$, $f_i : L_0 \times \ldots \times L_n \to L_i$ is a monotone function and the variable x_i takes values in the lattice L_i.

For $u_i \in L_i$, we write $E[x_i := u_i]$ for the system of $n-1$ equations obtained by removing the ith equation and substituting x_i by u_i in the remaining equations. We write η_i for either ν or μ, depending on whether i is even or odd. The *solution* of a system of equations is defined similarly to [11, 3].

▶ **Definition 2.** *The* solution $\mathsf{sol}(E)$ *of the nested system of equations E in (1) is defined by induction on the number of equations:*

$$\mathsf{sol}() = ()$$
$$\mathsf{sol}(E) = (\mathsf{sol}(E[x_n := v_n]), v_n), \text{ where } v_n = \eta_n(\lambda x. f_n(\mathsf{sol}(E[x_n := x]), x))$$

In other words, to solve a nested system of equations with variables $x_0,\ldots,x_n$, the system of equations $E[x_n := x]$ is solved by viewing x as a parameter, its solution is substituted in the nth equation, and this equation is then solved to obtain the nth component v_n of the solution of E. The value v_n is finally substituted in the parameterised solutions for $E[x_n := x]$ to obtain solutions for the remaining variables. When solving the ith equation, the greatest, respectively least solution is taken, depending on whether i is even or odd. Given the system of equations in (1), $i \in \{0,\ldots,n\}$ and values $v_k \in L_k$ for $k \in \{i+1,\ldots,n\}$, we write $f_i^{v_{i+1},\ldots,v_n} : L_i \to L_i$ for the map $x \mapsto f_i(\mathsf{sol}(E[x_i := x, x_{i+1} := v_{i+1},\ldots,x_n := v_n]), x, v_{i+1},\ldots,v_n)$.

Sufficient conditions for the existence and uniqueness of the individual fixpoints required in the definition of $\mathsf{sol}(E)$ are provided by Kleene's fixpoint theorem.

▶ **Theorem 3** (Kleene). *Let $\mathsf{Op} : (L,\sqsubseteq) \to (L,\sqsubseteq)$ be a monotone function on a complete lattice. The (transfinite) ascending chain $\mathsf{Op}^\beta(\bot)$, with β ranging over ordinals, is defined by: $\mathsf{Op}^0(\bot) = \bot$, $\mathsf{Op}^{\alpha+1}(\bot) = \mathsf{Op}(\mathsf{Op}^\alpha(\bot))$ for any ordinal α, and $\mathsf{Op}^\alpha(\bot) = \sqcup_{\beta<\alpha}\mathsf{Op}^\beta(\bot)$ for any limit ordinal α. Then, the least fixpoint of Op is $\mathsf{Op}^\gamma(\bot)$ for some ordinal γ. The greatest fixpoint of Op is characterised dually, via the (transfinite) descending chain $\mathsf{Op}^\beta(\top)$.*

▶ Remark 4. Thm. 3 implies that $\eta_i(f_i^{v_{i+1},\ldots,v_n}) \sqsubseteq \eta_i(f_i^{v'_{i+1},\ldots,v'_n})$ if $v_{i+1} \sqsubseteq v'_{i+1},\ldots,v_n \sqsubseteq v'_n$.

2.2 Monads Weighted in Partial Semirings

▶ **Definition 5.** *A* partial commutative monoid *(p.c.m.) $(S,+,0)$ is given by a set S together with a partial operation $+ : S \times S \to S$ and an element $0 \in S$, such that:*

- *$s + 0$ is defined for all $s \in S$ and moreover, $s + 0 = s$,*
- *$(s+t)+u$ is defined if and only if $s+(t+u)$ is defined, and in that case $(s+t)+u = s+(t+u)$,*
- *whenever $s + t$ is defined, so is $t + s$ and moreover, $s + t = t + s$.*

A partial commutative semiring *is a tuple $S := (S,+,0,\bullet,1)$ with $(S,+,0)$ a p.c.m. and $(S,\bullet,1)$ a commutative monoid, with $\bullet$ distributing over sums; that is, for all $s,t,u \in S$, $s\bullet 0 = 0$, and whenever $t+u$ is defined, then so is $s\bullet t+s\bullet u$ and moreover, $s\bullet t+s\bullet u = s\bullet(t+u)$.*

The addition operation of any partial commutative semiring induces a pre-order $\sqsubseteq$ on S:

$$x \sqsubseteq y \quad \text{if and only if} \quad \text{there exists } z \in S \text{ such that } x + z = y \tag{2}$$

for $x,y \in S$. It then follows from the axioms of a partial commutative semiring that $0 \sqsubseteq s$ for all $s \in S$, and that $\sqsubseteq$ is preserved by $+$ and $\bullet$ in each argument (see [5] for details).

▶ **Assumption 6.** *Similarly to [4], we make the following assumptions:*

- *$(S,\sqsubseteq)$ is a complete lattice and has the unit 1 of $\bullet$ as top element;*
- *$+$ preserves joins of increasing countable chains and meets of decreasing countable chains, in each argument;*

- $\bullet$ *preserves both suprema and infima in each argument; moreover, the following holds for all* $A_i \subseteq S$ *with* $i \in \omega$*, whenever* $\sum_{i\in\omega} \inf A_i$ *is defined:*

$$\sum_{i\in\omega} \inf A_i = \inf \{ \sum_{i\in\omega} a_i \mid a_i \in A_i \text{ for } i \in \omega, \sum_{i\in\omega} a_i \text{ is defined} \} \quad (3)$$

The countable (partial) addition operation used in the last condition is defined by $\sum_{i\in\omega} s_i := \sup_{n\in\omega}(s_0 + \ldots + s_n)$. If S is partial, this countable sum is defined iff all sums $s_0 + \ldots + s_n$ with $n \in \omega$ are defined. This definition exploits the fact that $s \sqsubseteq s + t$ for any $s, t \in S$ for which $s + t$ is defined, together with the existence of joins of increasing countable chains.

▶ **Example 7.** As concrete semirings we consider the *boolean semiring* $(\{0, 1\}, \vee, 0, \wedge, 1)$, the partial *probabilistic semiring* $([0, 1], +, 0, *, 1)$, the *tropical semiring* $\mathbb{N}^\infty = (\mathbb{N}^\infty, \min, \infty, +, 0)$ (with $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$) and its bounded variants $S_B = ([0, B] \cup \{\infty\}, \min, \infty, +_B, 0)$ with $B \in \mathbb{N}$, where for $m, n \in [0, B] \cup \{\infty\}$ we have

$$m +_B n = \begin{cases} m + n, & \text{if } m + n \leq B \\ \infty, & \text{otherwise} \end{cases} .$$

The associated orders are $\leq$ on $\{0, 1\}$ and $[0, 1]$, and $\geq$ on $\mathbb{N}^\infty$ and $[0, B] \cup \{\infty\}$. As shown in [4], all these orders satisfy Assumption 6. Note that we allow the semiring $(S, +, 0, \bullet, 1)$ to be partial in order to also cover probabilistic branching.

▶ Remark 8. When the semiring $(S, +, 0, \bullet, 1)$ is partial, we will also consider the total semiring $(S', \oplus, 0, \bullet, 1)$, where $S' = S$ and $\oplus$ is given by

$$s \oplus t = \begin{cases} s + t, & \text{if } s + t \text{ is defined} \\ 1, & \text{otherwise} \end{cases} .$$

It is easy to check that this semiring satisfies Assumption 6 whenever $(S, +, 0, \bullet, 1)$ does. In particular, the induced order is not changed when moving from S to S'.

▶ **Example 9.** The total semiring $([0, 1], \oplus, 0, *, 1)$ associated to the probabilistic semiring has $\oplus : [0, 1] \times [0, 1] \to [0, 1]$ given by addition truncated above at 1.

We use monads weighted in partial semirings to model systems with weighted branching. For a partial semiring satisfying Assumption 6, the monad $(\mathsf{T}_S, \eta, \sqcup)$ is given by

$$\mathsf{T}_S(X) = \{ \varphi : X \to S \mid \mathsf{supp}(\varphi) \text{ is finite}, \sum_{x\in\mathsf{supp}(\varphi)} \varphi(x) \text{ is defined} \},$$

$$\eta_X : X \to \mathsf{T}_S X, \ \eta_X(x)(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases},$$

$$\sqcup_X : \mathsf{T}_S(\mathsf{T}_S X) \to \mathsf{T}_S X, \ \sqcup_X(\Phi)(x) = \sum_{\varphi\in\mathsf{supp}(\Phi)} \Phi(\varphi) \bullet \varphi(x) \text{ for } \Phi \in \mathsf{T}_S(\mathsf{T}_S X) \subseteq S^{(S^X)}$$

where $\mathsf{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ is the *support* of φ. For a function $f : X \to Y$ we put

$$\mathsf{T}_S(f)(\sum_{i\in I} c_i x_i) = \sum_{i\in I} c_i f(x_i)$$

where we use the formal sum notation $\sum_{i\in I} c_i x_i$, with I finite, to denote the element of $\mathsf{T}_S(X)$ mapping $x \in X$ to $(\sum_{j\in J_x} c_j) \in S$ with $J_x = \{i \mid x_i = x\}$, and all $x \notin \{x_i \mid i \in I\}$ to $0 \in S$. Our choice of notation for the monad multiplication avoids unnecessary overloading of the symbol μ, which we use to denote both a least fixpoint and a measure.

2.3 Coalgebras with Branching and their Linear Behaviour

Recall that a *coalgebra for a functor* G (cf. [13]) is a pair (C, γ) with C a set of states and $\gamma : C \to GC$ a *transition map.* A *pointed coalgebra* is a tuple (C, γ, c) with (C, γ) a coalgebra and $c \in C$ a designated state.

We use polynomial functors $F : \mathsf{Set} \to \mathsf{Set}$ of the form $FX = \coprod_{i \in I} X^{j_i}$, with $j_i \in \omega$ for $i \in I$, to describe the structure of individual transitions in a system with branching. We view I as a set of transition *labels*, with j_i the *arity* of transitions labelled by i. Our chosen shape for F allows transitions with finitely-many successors. For $i \in I$, we write $\iota_i : \mathsf{Id}^{j_i} \Rightarrow F$ (with $\mathsf{Id} : \mathsf{Set} \to \mathsf{Set}$ the identity functor) for the canonical injection.

We model quantitative branching systems as *pointed* $(\mathsf{T}_S \circ F)$*-coalgebras*, with $(S, +, 0, \bullet, 1)$ as before and F as above. Such coalgebras have weighted transitions $c \xrightarrow{w,i} (c_1, \ldots, c_{j_i})$ with $w \in S$ the transition weight, $i \in I$ the transition label, and $c_1, \ldots, c_{j_i}$ the successor states. In spite of this potential branching *within* individual transitions, we view the functor F as defining a general notion of *linear* behaviour. (The word *linear* here refers to time!) The elements of the final F-coalgebra thus provide a natural notion of maximal (potentially infinite) trace for our models. The *branching* in our systems is modelled via the monad T_S. Our models thus distinguish between *deadlock* (captured by states with no outgoing transitions) and *successful termination* (captured by transitions labelled by $i \in$I with $j_i = 0$).

▶ **Example 10.** Our model in Section 1 can be viewed as a $(\mathsf{T}_S \circ F)$-coalgebra, with S the probabilistic semiring and $F : \mathsf{Set} \to \mathsf{Set}$ given by $FX = (\{r\} \times X \times X) + (\{b, c\} \times X) + \{a\} \simeq (X \times X) + X + X + 1$. Thus, r-transitions have two successors, b/c-transitions have a single successor, and a-transitions are terminating. In tis case, maximal traces (elements of the final F-coalgebra) can be presented as infinite trees whose nodes are labelled by transitions and have 2, 1 or 0 children, depending on whether they are labelled by r, b/c or a.

Notions of *path* and *path fragment* through a coalgebra with branching are defined below. Informally, a path from a state selects a single transition out of the transitions from that state which have non-zero weight, and continues making similar choices from all successor states of the chosen transition. Thus, a path will typically contain an infinite number of transitions (unless it is terminating). Since paths record the states visited and the transitions taken, they formally correspond to elements of the *final* coalgebra for the functor $C \times F$. Path fragments are similar, except that they contain a finite number of transitions. Technically this means that path fragments correspond to elements of an initial algebra. In order to streamline our presentation we will work with concrete representations of paths and path fragments using trees. We will not formally define trees, but fix some useful notation.

▶ **Notation 11.** *We write $\xi = c(i(\xi_1, \ldots, \xi_{j_i}))$ for the $C \times I$-labelled ranked tree whose root is labelled with $(c, i) \in C \times I$ and whose immediate subtrees are the trees $\xi_1, \ldots, \xi_{j_i}$ where j_i is the arity of the transition label i. Furthermore we write $\xi \rightsquigarrow \xi'$ if $\xi' = \xi_j$ for some $j \in \{1, \ldots, j_i\}$, i.e., $\rightsquigarrow$ denotes the immediate subtree relation.*

▶ **Definition 12.** *Given a set C, a (C-)path is a $C \times I$-labelled ranked tree. The collection of all C-paths will be denoted by Z_C. Let (C, γ) be a $(\mathsf{T}_S \circ F)$-coalgebra. A path $\xi \in Z_C$ is a* path from $c \in C$ in (C, γ) *if ξ has the form $\xi = c(i(\xi_1, \ldots, \xi_{j_i}))$ where for $k \in \{1, \ldots, j_i\}$ we have that ξ_k is a path from some $c_k \in C$ in (C, γ) and where $\gamma(c)(\iota_i(c_1, \ldots, c_{j_i})) \neq 0$.*

To also define path fragments (to be thought of as partial paths, necessarily of finite depth) as labelled trees, we use an additional label $* \notin I$, which we formally treat as a new transition label with arity 0, although its purpose is to indicate the "ends" of a path fragment.

▶ **Definition 13.** *Let (C, γ) be a $(\mathsf{T}_S \circ F)$-coalgebra. A* path fragment from $c \in C$ in (C, γ) *is a $C \times (I \cup \{*\})$-labelled tree $\tau = c(i(\tau_1, \ldots, \tau_{j_i}))$, such that only the leaves of τ can be labelled by $*$, and where for all $k \in \{1, \ldots, j_i\}$ we have that τ_k is a path fragment from $c_k \in C$ in*

(C, γ) *with* $\gamma(c)(\iota_i(c_1, \dots, c_{j_i})) \neq 0$*. Given a path fragment* q*, we will refer to the leaves of* τ *the form* $c(*)$ *as* holes.

Equivalently, $c(*)$ is a path fragment from c, and if τ_k is a path fragment from $c_k \in C$ for all $k \in \{1, \dots, j_i\}$ and $\gamma(c)(\iota_i(c_1, \dots, c_{j_i})) \neq 0$, then $c(i(\tau_1, \dots, \tau_{j_i}))$ is a path fragment from c.

▶ **Definition 14.** *A path fragment* τ *is a* prefix *of a path* ξ *if* ξ *is obtained by replacing each leaf of* τ *of the form* $c(*)$ *by a path from* c*. We write* $\mathsf{pref}(\xi)$ *for the set of prefixes of* ξ*.*

The set of all paths from $c \in C$ in (C, γ) is denoted Paths_c^γ (or simply Paths_c when γ is clear from the context). For a path fragment τ with holes $c_1(*), \dots, c_n(*)$, and sets of paths $A_i \subseteq \mathsf{Paths}_{c_i}$ for $i \in \{1, \dots, n\}$, the set of paths $\tau[A_1/c_1, \dots, A_n/c_n]$ consists of all paths from c obtained by continuing τ with a path in A_i from each hole $c_i(*)$, for $i \in \{1, \dots, n\}$.

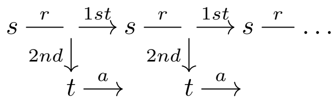
▶ Remark 15. Our definitions of paths and a path fragments are equivalent to those in [4], where paths (respectively path fragments) are defined as elements of the final $C \times F$-coalgebra (Z_C, ζ_C) (the initial $C \times (\{*\} + F)$-algebra (Φ_C, α_C)). In this representation we have

$$\zeta_c(\xi) = (c, \iota_i(\xi_1, \dots, \xi_{j_i})) \quad \text{if } \xi = c(i(\xi_1, \dots, \xi_{j_i})).$$

In what follows, we will use the two definitions interchangibly.

▶ **Example 16.** Below are two paths from s in the $(\mathsf{T}_S \circ F)$-coalgebra from Example 10, depicted as labelled trees:

$$s \xrightarrow{c} s \xrightarrow{c} \dots$$

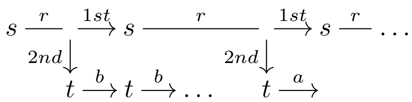


The second path models an execution where requests arrive at each step and are successfully answered in the next step. The path ξ is of the form $\xi = s(r(\xi, \xi'))$ with $\xi' = t(a())$.

A key notion for the semantics of parity automata is that of an *accepting* path. In our setting, where paths are tree-shaped, a path is accepting if all infinite traces through the path satisfy the parity condition. This is formalised in the next definition.

▶ **Definition 17.** *Let* C *be a set and let* $\Omega : C \to \omega$ *be a parity function with finite range. Given a path* $\xi \in Z_C$ *we call an infinite sequence* $\xi_1\xi_2\xi_3 \dots \in (Z_C)^\omega$ *a* trace *through* ξ *if* $\xi = \xi_1$ *and for all* $i \in \mathbb{N}$ *we have* $\xi_i \rightsquigarrow \xi_{i+1}$*. We call a trace* $\xi_1\xi_2\xi_3 \dots \in (Z_C)^\omega$ good *if the maximal parity that occurs infinitely often in* $\Omega(\pi_1(\xi_1))\ \Omega(\pi_1(\xi_2))\ \Omega(\pi_1(\xi_3))\ \dots$ *is even. A path* $\xi \in Z_C$ *is said to be* accepting *if all traces through* ξ *are good.*

▶ **Example 18.** Consider again the coalgebra of Example 10, and let $\Omega(s) = 0$ and $\Omega(t) = 1$. Then, both paths in Example 16 are accepting. On the other hand, the path $\xi_1 \in Z_C$ given by $\xi_1 = s(r(\xi_2, \xi_1'))$ with $\xi_2 = s(r(\xi_1, \xi_2'))$, $\xi_1' = t(b(\xi_1'))$ and $\xi_2' = t(a())$ is not accepting, since e.g. the trace $\xi_1\xi_1'\xi_1' \dots$ is not good. Its corresponding labelled tree is given below:

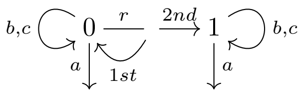


2.4 Qualitative Parity Automata

We use *non-deterministic parity* F*-automata* to describe qualitative properties of paths.

▶ **Definition 19.** *A* non-deterministic parity F-automaton (NPA) (A, α, a_I, Ω) *is given by a pointed* $\mathcal{P}_f \circ F$*-coalgebra* (A, α, a_I) *(with* $\mathcal{P}_f : \mathsf{Set} \to \mathsf{Set}$ *the finite powerset functor) together with a function* $\Omega : A \to \omega$ *with finite range, called a* parity map.

▶ **Example 20.** Let $F : \mathsf{Set} \to \mathsf{Set}$ be as in Example 10. The following NPA, with initial state 0 and state parities identical to the state names, captures the property that each request initiates a *simple* process (second successor of the r-transition) which eventually answers the request. Here, a *simple* process is one whose behaviour does not involve any r-transitions. This constraint is captured by not allowing r-transitions from state 1 of the automaton.



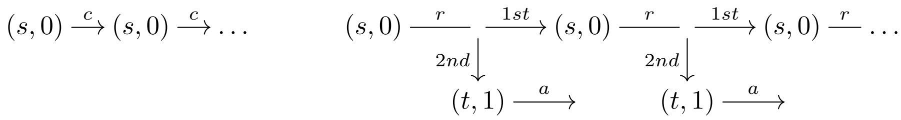
The choice of parities insures that no infinite sequence of b and c transitions is allowed from the second successor of any r-transition, on any accepting run (see below) of this automaton.

A *run* of an NPA on a pointed F-coalgebra records the coalgebra states which the automaton reads, the automaton states visited and the transitions taken.

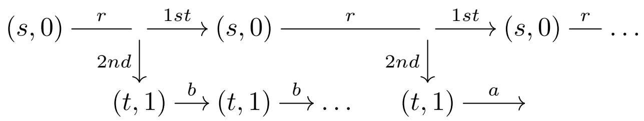
▶ **Definition 21.** *A* run *of an NPA* (A, α, a_I, Ω) *on a pointed* F*-coalgebra* (B, β, b_I) *is a path* $\xi \in Z_{B\times A}$ *of the form* $\xi = (b_I, a_I)(i(\xi_1, \ldots, \xi_{j_i}))$ *such that for each* $\xi' \in Z_{B\times A}$ *reachable from* ξ*, with* $\xi' = (b, a)(k((b_1, a_1)(i_1(\xi^1_1, \ldots, \xi^1_{l_{i_1}})), \ldots, (b_j, a_j)(i_j(\xi^j_1, \ldots, \xi^j_{l_{i_j}}))))$ *we have* $\beta(b) = \iota_k(b_1, \ldots, b_j)$ *and* $\alpha(a) \ni \iota_k(a_1, \ldots, a_j)$ *where* $j = j_k$ *is the arity of* k*.*

A run is accepting *if it is accepting in the sense of Def. 17, w.r.t. the parity function* $\Omega' : B \times A \to \omega$ *given by* $\Omega'(b, a) := \Omega(a)$*. The automaton* (A, α, a_I, Ω) accepts *the pointed* F*-coalgebra* (B, β, b_I) *if there exists an accepting run of* (A, α, a_I, Ω) *on* (B, β, b_I)*.*

▶ **Example 22.** The following are accepting runs of the automaton in Example 20 on the paths in Example 16 (viewed as F-coalgebras):



On the other hand, the following run is not accepting:



Unambiguous automata will play an important role in what follows.

▶ **Definition 23** (Unambiguous parity F-automaton)**.** *A non-deterministic parity* F*-automaton* (A, α, a_I, Ω) *is called* unambiguous *if for each pointed* F*-coalgebra* (B, β, b_I)*, there exists at most one accepting run of* (A, α, a_I, Ω) *on* (B, β, b_I)*.*

Since paths in a $(\mathsf{T}_S \circ F)$-coalgebra (C, γ) carry F-coalgebra structure (see Remark 15), one can consider (accepting) runs of a non-deterministic parity F-automaton on them. The next two sub-sections describe two different ways of *measuring* the set of paths of a pointed $(\mathsf{T}_S \circ F)$-coalgebra which are accepted by a given NPA. Before that, we show how non-deterministic and probabilistic transition systems can be recovered in our framework, and how the associated notion of NPA relates to the standard notion of Büchi automaton.

▶ Remark 24. Let At denote a finite set of atomic propositions. Take $F : \mathsf{Set} \to \mathsf{Set}$ be given by $F = \mathcal{P}(\mathsf{At}) \times \mathsf{Id} \simeq \coprod_{A \subseteq \mathsf{At}} \mathsf{Id}$. Then, non-deterministic (probabilistic) transition systems can be viewed as $(\mathsf{T}_S \circ F)$-coalgebras, with S the boolean (resp. probabilistic) semiring: such transition systems are in one-to-one correspondence with $\mathcal{P}(\mathsf{At}) \times \mathsf{T}_S$-coalgebras, which can be turned into $\mathsf{T}_S \circ (\mathcal{P}(\mathsf{At}) \times \mathsf{Id})$-coalgebras by post-composing the coalgebra maps with the strength map of T_S.Moreover, Büchi automata over the alphabet $\mathcal{P}(\mathsf{At})$ coincide with non-deterministic parity F-automata with $\mathsf{ran}(\Omega) = \{1, 2\}$.

2.5 Quantitative Parity Automata and their Extents

The notion of *ν-extent*, defined next, generalises non-emptiness in non-deterministic coalgebras (existence of a maximal path) to coalgebras with quantitative branching. It assigns, to each coalgebra state, a value in S which "measures" the maximal (completed) paths from it.

▶ **Definition 25** (ν-extent, [6]). *The* ν-extent *of a* $(\mathsf{T}_S \circ F)$*-coalgebra* (C, γ) *is the greatest fixpoint of the operator on* S*-valued predicates on* C*, which takes* $p : C \to S$ *to the composition*

$$C \xrightarrow{\gamma} \mathsf{T}_S FC \xrightarrow{\mathsf{T}_S Fp} \mathsf{T}_S FS \xrightarrow{\mathsf{T}_S(\bullet_F)} \mathsf{T}_S S = \mathsf{T}_S \mathsf{T}_S 1 \xrightarrow{\sqcup_1} \mathsf{T}_S 1 = S$$

where $\bullet_F : FS \to S$ *is given by* $\bullet_F(\iota_i(s_1, \ldots, s_{j_i})) = s_1 \bullet \ldots \bullet s_{j_i}$ *for* $i \in I$*. We write* $\mathsf{ext}^\nu_\gamma : C \to S$ *for the* ν*-extent of* (C, γ)*.*

The operator in Definition 25 expresses that the ν-extent of a state is the weighted sum of the ν-extents of its (structured) successors, where in the case of a *structured* successor (tuple of states resulting from an individual transition), the ν-extents of the states in question are multiplied. We will later use the ν-extent to measure certain sets of paths from a given state of a $(\mathsf{T}_S \circ F)$-coalgebra (C, γ). In particular, the set of *all* paths from $c \in C$ in (C, γ) will have measure $\mathsf{ext}^\nu_\gamma(c)$. This is further motivated by the next example.

▶ **Example 26.** When $S = (\{0, 1\}, \vee, 0, \wedge, 1)$, the ν-extent of a state c in a $(\mathsf{T}_S \circ F)$-coalgebra (C, γ) is 1 iff there exists a maximal path from c in (C, γ). When $S = ([0, 1], +, 0, *, 1)$, the ν-extent of a state measures the probability of not deadlocking; in particular, the ν-extent is always 1 provided that all states of (C, γ) have branching governed by a probability *distribution*. Finally, when $S = (\mathbb{N}^\infty, \min, \infty, +, 0)$, the ν-extent of a state c gives the minimal cost of a maximal path from c in (C, γ).

▶ **Example 27.** Consider the $(\mathsf{T}_S \circ F)$-coalgebra (C, γ) from Example 10, with $C = \{s, t\}$. Its ν-extent $\mathsf{ext}^\nu_\gamma : C \to [0, 1]$ is the greatest solution of the following system of equations (one variable for each state, with x being used for state s and y being used for state t):

$$\begin{bmatrix} x & = & \frac{1}{2} * x + \frac{1}{2} * x * y \\ y & = & \frac{1}{4} * y + \frac{3}{4} \end{bmatrix}$$

This gives $\mathsf{ext}^\nu_\gamma(s) = \mathsf{ext}^\nu_\gamma(t) = 1$. Replacing the probabilistic semiring with the tropical one and assigning weight 0 (the top element of $(S, \sqsubseteq)$) to r and a transitions, and weight 1 to b and c transitions, results in a ν-extent of 0 for both s and t.

Quantitative parity automata generalise NPAs by allowing weighted branching.

▶ **Definition 28** (Quantitative parity automaton, [6]). *A* parity (T_S, F)-automaton*, or simply* quantitative parity automaton (QPA)*,* (D, δ, d_I, Ω) *is given by a pointed* $T_S \circ F$*-coalgebra* (D, δ, d_I) *together with a parity map* $\Omega : D \to \omega$*.*

We will obtain QPAs as products between an unambiguous NPA, representing a qualitative property of pointed F-coalgebras, and a quantitative model. We will then show that such products can be used to measure the degree with which the given property is satisfied in the model (thereby generalising the automata-based approach to model checking non-deterministic and probabilistic systems). This will amount to determining the *nested extent* of the product automaton, to be defined shortly. This generalises the ν-extent of a $(\mathsf{T}_S \circ F)$-coalgebra by also taking into account the state parities. We first define the product automaton.

▶ **Definition 29** (Product automaton). *Let* $(S, +, 0, \bullet, 1)$ *be a total semiring satisfying Assumption 6. Also, let* (A, α, a_I, Ω) *be a NPA and let* (C, γ, c_I) *be a pointed* $(\mathsf{T}_S \circ F)$*-coalgebra.*

The product *of* (C, γ, c_I) *and* (A, α, a_I, Ω) *is the QPA with carrier* $C \times A$*, parity map given by* $\Omega(c, a) = \Omega(a)$ *for* $(c, a) \in C \times A$*, and transition map* $\mathrm{prod}_{\gamma,\alpha}$ *given by:*

$$C \times A \xrightarrow{\gamma\times\alpha} \mathsf{T}_S FC \times \mathcal{P}_f FA \xrightarrow{d_{FC,FA}} \mathsf{T}_S(FC \times FA) \xrightarrow{\langle F\pi_1, F\pi_2\rangle^*} \mathsf{T}_S F(C \times A)$$

where for $X, Y \in \mathsf{Set}$*, the map* $d_{X,Y} : \mathsf{T}_S X \times \mathcal{P}_f Y \to \mathsf{T}_S(X \times Y)$ *is given by:*

$$\mathsf{T}_S X \times \mathcal{P}_f Y \xrightarrow{\mathsf{id}_{\mathsf{T}_S X} \times e_Y} T_S X \times \mathsf{T}_S Y \xrightarrow{\mathsf{dst}_{X,Y}} \mathsf{T}_S(X \times Y) \tag{4}$$

with

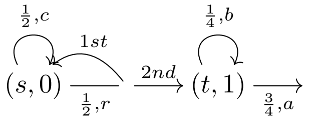
- $e : \mathcal{P}_f \Rightarrow \mathsf{T}_S$ *the embedding of* $\mathcal{P}_f$ *into* T_S*, given by*
  $$e_Y(X)(y) = \begin{cases} 1, & \text{if } y \in X \\ 0, & \text{otherwise} \end{cases}, \quad \text{for } X \in \mathcal{P}_f Y,$$
- $\mathsf{dst}_{X,Y} : \mathsf{T}_S X \times \mathsf{T}_S Y \Rightarrow \mathsf{T}_S(X \times Y)$ *the* double strength *of* T_S*, given by*
  $$\mathsf{dst}_{X,Y}(\varphi, \psi) = \sum_{x\in\mathsf{supp}(\varphi), y\in\mathsf{supp}(\psi)} (\varphi(x) \bullet \psi(y))(x, y), \quad \text{for } \varphi \in \mathsf{T}_S X \text{ and } \psi \in \mathsf{T}_S Y,$$

and where $\langle F\pi_1, F\pi_2\rangle^*$ *is pre-composition with* $\langle F\pi_1, F\pi_2\rangle : F(C \times A) \to FC \times FA$.

We immediately note that the shape of the functor F makes $\langle F\pi_1, F\pi_2\rangle$ injective, and as a result the transition map of the product automaton has finite support.

Transitions in the product automaton thus arise from matching transitions in the $(\mathsf{T}_S \circ F)$-coalgebra and the NPA, with weights inherited from the coalgebra and parities inherited from the NPA; in particular, a coalgebra transition may match more than one NPA transition. The assumption in Definition 29 that $(S, +, 0, \bullet, 1)$ is total ensures that the natural transformation e is well defined. We will explain in Section 4 why this assumption is harmless.

▶ **Example 30.** The product of the coalgebra in Example 10 with the NPA in Example 20 is:



The next lemma characterises paths in a $(\mathsf{T}_S \circ F)$-coalgebra accepted by an unambiguous NPA using the product automaton. It is proved by simply spelling out the relevant definitions.

▶ **Lemma 31.** *Assume* $(S, +, 0, \bullet, 1)$ *is a total semiring. Let* (A, α, a_I, Ω) *be an unambiguous parity automaton and* (C, γ, c_I) *be a pointed* $(\mathsf{T}_S \circ F)$*-coalgebra. There is a one-to-one correspondence between accepting paths from* (c_I, a_I) *in the product of* (A, α, a_I, Ω) *and* (C, γ, c_I)*, and paths from* c_I *in* (C, γ) *accepted by* (A, α, a_I, Ω).

As announced, the notion of *nested extent* of a QPA generalises the ν-extent of a $(\mathsf{T}_S \circ F)$-coalgebra by taking into account the different parities associated to automaton states.

▶ **Definition 32** (Nested extent, [6]). *Let* (D, δ, d_I, Ω) *be a quantitative parity automaton with* $\mathsf{ran}(\Omega) = \{0, \ldots, n\}$*, let* $D_k = \{d \in D \mid \Omega(d) = k\}$*, and let* $\delta_k = \delta \circ \iota_k : D_k \to \mathsf{T}_S FD$ *denote the restriction of* δ *to* D_k *(*$k \in \mathsf{ran}(\Omega)$*). The* extent $\mathsf{ext}_\delta = [\mathsf{ext}_{\delta,0}, \ldots, \mathsf{ext}_{\delta,n}] : D \to S$ of (D, δ, Ω) *is the solution of the following nested system of equations:*

$$\begin{bmatrix} x_0 & =_\nu & \sqcup_1 \circ \mathsf{T}_S(\bullet_F) \circ T_S F[x_0, \ldots, x_n] \circ \delta_0 \\ x_1 & =_\mu & \sqcup_1 \circ \mathsf{T}_S(\bullet_F) \circ T_S F[x_0, \ldots, x_n] \circ \delta_1 \\ & \vdots & \\ x_n & =_\eta & \sqcup_1 \circ \mathsf{T}_S(\bullet_F) \circ T_S F[x_0, \ldots, x_n] \circ \delta_n \end{bmatrix} \tag{5}$$

with $\eta = \mu$ $(= \nu)$ *if* n *is odd (resp. even), variables* x_k $(k \in \mathsf{ran}(\Omega))$ *taking values in the poset* $(S^{D_k}, \sqsubseteq)$ *(and therefore* $[x_0, \ldots, x_n] \in S^D$*), and the rhs of the equation for* x_k *pictured below:*

$$D_k \xrightarrow{\delta_k} \mathsf{T}_S F D \xrightarrow{\mathsf{T}_S F[x_0,\ldots,x_n]} \mathsf{T}_S F S \xrightarrow{\mathsf{T}_S(\bullet_F)} \mathsf{T}_S S = \mathsf{T}_S \mathsf{T}_S 1 \xrightarrow{\sqcup_1} \mathsf{T}_S 1 = S$$

We write $\mathsf{Op}_{\delta,i} : S^{D_0} \times \ldots \times S^{D_n} \to S^{D_i}$ *for the operator used in the rhs of the* i*th equation.*

The existence and uniqueness of a solution for (5) is guaranteed by Kleene's theorem (Thm. 3).

▶ **Example 33.** The nested extent of the product automaton in Example 30 is the solution of the following nested system of equations (where variables x and y are used for the nested extents of states $(s, 0)$, respectively $(t, 1)$):

$$\begin{bmatrix} x & =_\nu & \frac{1}{2} * x + \frac{1}{2} * x * y \\ y & =_\mu & \frac{1}{4} * y + \frac{3}{4} \end{bmatrix}$$

This still gives a nested extent of 1 in each state, essentially because the probability of infinitely-many b-transitions from state $(t, 1)$ is 0.

2.6 Semiring-Valued Measures

We will use *semiring-valued measures* [4] to measure certain sets of paths from a state of a $(\mathsf{T}_S \circ F)$-coalgebra. In particular, we will be able to measure the set of paths accepted by an NPA. Key definitions and results regarding semiring-valued measures are summarised below.

▶ **Definition 34** ([4])**.** *An* S-valued measure *on a* σ*-algebra* $\mathcal{A}$ *is a function* $\mu : \mathcal{A} \to S$ *s.t. (i)* $\mu(\emptyset) = 0$*, and (ii) if* $A_i \in \mathcal{A}$ *for* $i \in \omega$ *are pairwise disjoint, then* $\sum_{i\in\omega} \mu(A_i)$ *is defined and moreover,* $\mu(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} \mu(A_i)$*.*

▶ **Proposition 35** ([4])**.** *Let* $\mu : \mathsf{R} \to S$ *be a measure on a field of sets. Then,* μ *extends to a measure on the* σ*-algebra generated by* R*.*

The proof of the above result defines the resulting measure as

$$\mu^*(A) = \inf\{ \sum_{n\in\omega} \mu(E_n) \mid (E_n \in \mathsf{R})_{n\in\omega} \text{ pairwise disjoint, } A \subseteq \bigcup_{n\in\omega} E_n\}$$

As in [4], we take R to be the field generated by the so-called *cylinder sets*.

▶ **Definition 36** ([4])**.** *Let* (C, γ) *be a* $(\mathsf{T}_S \circ F)$*-coalgebra, and let* $\tau \in \Phi_C$ *be a path fragment from* c *in* (C, γ)*. Its associated* cylinder set *is given by* $\mathsf{Cyl}(\tau) = \{\xi \in \mathsf{Paths}_c \mid \tau \in \mathsf{pref}(\xi)\}$*. A cylinder set* $\mathsf{Cyl}(\tau)$ *is said to* cover *a path* $\xi \in Z_C$ *when* τ *is a prefix of* ξ*. For* $c \in C$*, we let* $\Sigma_c := \{ \mathsf{Cyl}(\tau) \mid \tau$ *is a path fragment from* c *in* $(C, \gamma) \}$*.*

Now given a $(\mathsf{T}_S \circ F)$-coalgebra (C, γ) and $c \in C$, it is shown in [4] that finite unions of pairwise-disjoint elements of Σ_c form a field. Then, an S-valued measure on the generated σ-algebra, denoted by $\mathcal{M}_c$, can be defined from an S-valued measure on Σ_c, using Proposition 35. The natural S-valued measure to consider on cylinder sets is $\mu_\gamma : \Sigma_c \to S$ given by:

1. $\mu_\gamma(\emptyset) = 0$,
2. For τ a path fragment from $c \in C$, $\mu_\gamma(\mathsf{Cyl}(\tau))$ is defined by structural induction on τ:
   a. If $\tau = c(*)$, then $\mu_\gamma(\mathsf{Cyl}(\tau)) = \mathsf{ext}^\nu_\gamma(c)$,
   b. If $\tau = c(i(\tau_1, \ldots, \tau_{j_i}))$ for some $i \in I$ and for path fragments τ_k from $c_k \in C$ for $k \in \{1, \ldots, j_i\}$, then $\mu_\gamma(\mathsf{Cyl}(\tau)) = \gamma(c)(\iota_i(c_1, \ldots, c_{j_i})) \bullet \mu_\gamma(\mathsf{Cyl}(\tau_1)) \bullet \ldots \bullet \mu_\gamma(\mathsf{Cyl}(\tau_{j_i}))$.

Note that the measure of the set Paths_c of *all* paths from c is not 1 (the top element in S) as one might expect, but $\mathsf{ext}^\nu_\gamma(c)$. This is because we consider maximal (completed) paths only, and assigning measure 1 to Paths_c could result in assigning measure 1 to an *empty* set of paths (when there are no completed paths from c, e.g. because c is a deadlock state).

The above $\mu_\gamma : \Sigma_c \to S$ induces an S-valued measure on the ring generated by Σ_c, given by $\mu_\gamma(\bigcup_{i\in\{1,\dots,n\}} C_i) = \sum_{i\in\{1,\dots,n\}} \mu_\gamma(C_i)$ for each pairwise-disjoint family $(C_i)_{i\in\{1,\dots,n\}}$ with $C_i \in \Sigma_c$. The measure $\mu_\gamma : \mathcal{M}_c \to S$ arising from Proposition 35 is then given by

$$\mu_\gamma(A) = \inf\{ \sum_{n\in\omega} \mu_\gamma(C_n) \mid (C_n \in \Sigma_c)_{n\in\omega} \text{ pairwise disjoint, } A \subseteq \bigcup_{n\in\omega} C_n\} \tag{6}$$

▶ **Example 37.** When $S = (\{0,1\}, \vee, 0, \wedge, 1)$, $\mu_\gamma(A) = 0$ iff $A = \emptyset$. When $S = ([0,1], +, 0, *, 1)$, and if only probability *distributions* are used in γ, $\mu_\gamma(A)$ gives the likelihood of exhibiting a path in A. When $S = (\mathbb{N}^\infty, \min, \infty, +, 0)$, $\mu_\gamma(A)$ gives the minimal cost of a path in A.

3 Coincidence of Extents with the Measure-Theoretic Semantics

Throughout this section we fix a quantitative parity automaton (C, γ, c_I, Ω). We will use the measures $\mu_\gamma : \mathcal{M}_c \to S$ with $c \in C$ to link a characterisation of the accepting paths of (C, γ, Ω) (Proposition 39 below) with the definition of extent (Definition 32), thereby proving the equivalence of two different ways of measuring the set of accepting paths of a QPA.

The next result shows that extents are preserved by parity-preserving $(\mathsf{T}_S \circ F)$-coalgebra homomorphisms.

▶ **Proposition 38.** *Let (C, γ, Ω) and (D, δ, Ω) be two quantitative parity automata and let $f : (C, \gamma, \Omega) \to (D, \delta, \Omega)$ be a $(\mathsf{T}_S \circ F)$-coalgebra homomorphism which preserves parities; that is, $\Omega(f(c)) = \Omega(c)$ for $c \in C$. Then, $\mathsf{ext}_\gamma(c) = \mathsf{ext}_\delta(f(c))$ for all $c \in C$.*

To relate the extent of (C, γ, Ω) with the set of *accepting* paths of (C, γ, Ω), we characterise the accepting paths of a QPA as the solution of a nested system of equations. For $i \in \mathsf{ran}(\Omega)$, we let $\mathsf{Paths}_i = \{\xi \in Z_C \mid \exists c \in C.\ \xi \in \mathsf{Paths}_c \text{ and } \Omega(c) = i\}$; that is, Paths_i contains all paths in (C, γ) whose initial state has parity i. The next result is a reformulation of [15, Lemma 4.4]; its proof mirrors that in loc. cit. It is irrelevant that transitions carry weights.

▶ **Proposition 39.** *The accepting paths of a QPA (C, γ, Ω) are the solution of the following nested system of equations, with variables Y_i taking values in the lattice $\mathcal{P}(\mathsf{Paths}_i)$:*

$$\begin{bmatrix} Y_0 & =_\nu & \mathsf{Op}_0(Y_0, \dots, Y_n) \\ Y_1 & =_\mu & \mathsf{Op}_1(Y_0, \dots, Y_n) \\ & \vdots & \\ Y_n & =_\eta & \mathsf{Op}_n(Y_0, \dots, Y_n) \end{bmatrix} \tag{7}$$

where for $k \in \mathsf{ran}(\Omega)$, $\mathsf{Op}_k : \mathcal{P}(\mathsf{Paths}_0) \times \ldots \times \mathcal{P}(\mathsf{Paths}_n) \to \mathcal{P}(\mathsf{Paths}_k)$ is given by

$$\begin{aligned} \mathsf{Op}_k((Y_i)_{i\in\mathsf{ran}(\Omega)}) \ = \{\, \xi \in \mathsf{Paths}_c \mid \xi = c(i(\xi_1, \dots, \xi_{j_i})) \text{ for some } c \in C_k, \\ i \in I \text{ and } \xi_l \in Y_{\Omega(\pi_1(\zeta_C(\xi_l)))} \text{ for } l \in \{1, \dots, j_i\} \,\} \end{aligned}$$

The idea is that the kth component of the solution collects all accepting paths from states with parity k. Now while the domain of the operators $\mathsf{Op}_k : \mathcal{P}(\mathsf{Paths}_0) \times \ldots \times \mathcal{P}(\mathsf{Paths}_n) \to \mathcal{P}(\mathsf{Paths}_k)$ with $i \in \mathsf{ran}(\Omega)$ also includes tuples $(P_0, \dots, P_n)$ with $P_k \cap \mathsf{Paths}_c$ not measurable for some $k \in \mathsf{ran}(\Omega)$ and $c \in C_k$, we will show that only tuples $(P_0, \dots, P_n)$ with $P_k \cap \mathsf{Paths}_c$ measurable for $k \in \mathsf{ran}(\Omega)$ and $c \in C_k$ are involved in the construction of the solution of this system of equations, and the solution itself is measurable in the sense of Definition 40 below.

▶ **Definition 40.** *For $k \in \mathsf{ran}(\Omega)$, we call a set of paths $P \subseteq \mathsf{Paths}_k$* measurable *if $P \cap \mathsf{Paths}_c \in \mathcal{M}_c$ for all $c \in C_k$. We write $\mathcal{M}_k := \{\, P \subseteq \mathsf{Paths}_k \mid P \text{ is measurable} \,\}$, for $k \in \mathsf{ran}(\Omega)$.*

The next result shows that the operators in Proposition 39 restrict to measurable sets and moreover, the solution of the equation system (7) itself consists of measurable sets.

▶ **Proposition 41.** *Let E' be the equation system (7). Then, the following hold:*

1. *For $i \in \mathsf{ran}(\Omega)$ and $P_k \in \mathcal{M}_k$ for $k \in \{i+1,\ldots,n\}$, the operator $\mathsf{Op}_i^{P_{i+1},\ldots,P_n} : \mathsf{Paths}_i \to \mathsf{Paths}_i$ restricts to an operator on $\mathcal{M}_i$.*
2. *E' restricts to an equation system with variables taking values in $\mathcal{M}_i$, whose solution coincides with the solution of E'.*

Proof. For $i \in \mathsf{ran}(\Omega)$, $\mathcal{P}(\mathsf{Paths}_i)$ is a complete lattice. Also, $\mathcal{M}_i \subseteq \mathcal{P}(\mathsf{Paths}_i)$ is a σ-algebra, with countable directed unions / co-directed intersections computed component-wise – recall that each $P \in \mathcal{M}_i$ is a disjoint union of sets $P_c \in \mathcal{M}_c$ with $c \in C_i$. Then, an easy induction on i shows that, if $P_k \in \mathcal{M}_k$ for $k \in \{i+1,\ldots,n\}$, then $\mathsf{Op}_i^{P_{i+1},\ldots,P_n}$ restricts to an operator on $\mathcal{M}_i$ – this is because the least/greatest fixpoints required in the definition of $\mathsf{Op}_i^{P_{i+1},\ldots,P_n}$ are constructed by successively taking limits of ω-chains/ω^{op}-chains of elements of $\mathcal{M}_i$ (see Theorem 3), and the $\mathcal{M}_i$s are closed under countable directed unions / co-directed intersections. As a result, E' restricts to an equation system with variables taking values in $\mathcal{M}_i$, with $i \in \mathsf{ran}(\Omega)$. Moreover, the construction of the solution is the same whether performed in $\mathcal{M}_i$ or in $\mathcal{P}(\mathsf{Paths}_i)$, with $i \in \mathsf{ran}(\Omega)$. This concludes the proof. ◀

We are now ready to state our main result.

▶ **Theorem 42.** *For a quantitative parity automaton (C,γ,Ω) and $c \in C$, we have*
$\mathsf{ext}_\gamma(c) = \mu_\gamma(\{\xi \in \mathsf{Paths}_c \mid \xi \text{ accepting}\})$.

Proof. By Assumption 6, proving the above equality can be reduced to proving two inequalities. These follow from Lemmas 43 and 46, respectively. ◀

▶ **Lemma 43.** *For a quantitative parity automaton (C,γ,Ω) and $c \in C$, we have*
$\mu_\gamma(\{\xi \in \mathsf{Paths}_c \mid \xi \text{ accepting}\}) \sqsubseteq \mathsf{ext}_\gamma(c)$.

Proof. Consider the equation system E in (5), and the restriction of the equation system E' in (7) to *measurable* sets of paths (see Proposition 41). The operators $\mathsf{Op}_i^{P_{i+1},\ldots,P_n}$ and $\mathsf{Op}_{\gamma,i}^{e_{i+1},\ldots,e_n}$ used to define $\mathsf{sol}(E)$ and $\mathsf{sol}(E')$ are given by:

$$\mathsf{Op}_i^{P_{i+1},\ldots,P_n}(Y) = \mathsf{Op}_i(\mathsf{sol}(E[Y_i := Y, Y_{i+1} := P_{i+1},\ldots,Y_n := P_n]), Y, P_{i+1},\ldots,P_n)$$
$$\mathsf{Op}_{\gamma,i}^{e_{i+1},\ldots,e_n}(x) = \mathsf{Op}_{\gamma,i}(\mathsf{sol}(E'[x_i := x, x_{i+1} := e_{i+1},\ldots,x_n := e_n]), x, e_{i+1},\ldots,e_n)$$

We prove the following combined statement by induction on $i \in \mathsf{ran}(\Omega)$:

1. Given $P_j \in \mathcal{M}_j$ and $e_j : C_j \to S$ such that $\mu_\gamma(P_j \cap \mathsf{Paths}_c) \sqsubseteq e_j(c)$ for $c \in C_j$, for $j \in \{i+1,\ldots,n\}$, we have

$$\begin{array}{ccc} \mathcal{M}_i & \xrightarrow{\mathsf{Op}_i^{P_{i+1},\ldots,P_n}} & \mathcal{M}_i \\ {\scriptstyle\mu_\gamma}\downarrow & \sqsupseteq & \downarrow{\scriptstyle\mu_\gamma} \\ S^{C_i} & \xrightarrow[\mathsf{Op}_{\gamma,i}^{e_{i+1},\ldots,e_n}]{} & S^{C_i} \end{array} \tag{8}$$

Here, by slightly abusing notation, we write $\mu_\gamma : \mathcal{M}_i \to S^{C_i}$ for the function taking P_i to the S-valued predicate $e_i : C_i \to S$ given by $e_i(c) = \mu_\gamma(P_i \cap \mathsf{Paths}_c)$ for $c \in C_i$.
2. $\mu_\gamma(\eta_i(\mathsf{Op}_i^{P_{i+1},\ldots,P_n})) \sqsubseteq \eta_i(\mathsf{Op}_{\gamma,i}^{e_{i+1},\ldots,e_n})$, whenever $P_j \in \mathcal{M}_j$ and $e_j : C_j \to S$ are as above, for $j \in \{i+1,\ldots,n\}$.

Since for $i \in \mathsf{ran}(\Omega)$, any $P_i \in \mathcal{M}_i$ is of the form $P_i = \bigcup_{c \in C_i} P_{i,c}$, with $P_{i,c} = P_i \cap \mathsf{Paths}_c$ for $c \in C_i$, it suffices to show that (8) holds when restricted to each $\mathcal{M}_c$ with $c \in C_i$.

- For $i = 0$, the inequality (8) follows from

$$\mu_\gamma(\mathsf{Op}_0^{P_1,\ldots,P_n}(P_{0,c})) = \mathsf{Op}_{\gamma,0}^{\mu_\gamma(P_1),\ldots,\mu_\gamma(P_n)}(\mu_\gamma(P_{0,c})) \sqsubseteq \mathsf{Op}_{\gamma,0}^{e_1,\ldots,e_n}(\mu_\gamma(P_{0,c}))$$

for $P_0 \in \mathcal{M}_0$ and $c \in C_0$. In the above, the equality follows from [4, Proposition 5.12], after noting that $\mathsf{Op}_0^{P_1,\ldots,P_n}(P_{0,c})$ can be written as a finite union of sets of the form

$$\{\, \xi \in \mathsf{Paths}_c \mid \zeta_C(\xi) = (c, \iota_i(\xi_1, \ldots, \xi_{j_i})) \text{ with } \xi_i \in P_{\Omega(\pi_1(\zeta_C(\xi_i))} \text{ for } i \in \{1, \ldots, j_i\} \,\}$$

with $i \in I$. On the other hand, the inequality above follows by Remark 4.
Now let $P_j \in \mathcal{M}_j$ and $e_j : C_j \to S$ be s.t. $\mu_\gamma(P_j \cap \mathsf{Paths}_c) \sqsubseteq e_j(c)$ for $c \in C_j$ and $j \in \{1, \ldots, n\}$. Also, let $P_0 = \nu_0(\mathsf{Op}_0^{P_1,\ldots,P_n})$ and $e_0 = \nu_0(\mathsf{Op}_{\gamma,0}^{e_1,\ldots,e_n})$. We show that $\mu_\gamma(P_0) \sqsubseteq e_0$. We have

$$\mathsf{Op}_{\gamma,0}^{e_1,\ldots,e_n}(\mu_\gamma(P_0)) \overset{(\text{by } (8))}{\sqsupseteq} \mu_\gamma(\mathsf{Op}_0^{P_1,\ldots,P_n}(P_0)) \overset{(P_0 \text{ is a fixpoint of } \mathsf{Op}_0^{P_1,\ldots,P_n})}{=} \mu_\gamma(P_0)$$

and therefore $\mu_\gamma(P_0)$ is a post-fixpoint of $\mathsf{Op}_{\gamma,0}^{e_1,\ldots,e_n}$. Now since e_0 is the greatest post-fixpoint of $\mathsf{Op}_{\gamma,0}^{e_1,\ldots,e_n}$, we immediately obtain $\mu_\gamma(P_0) \sqsubseteq e_0$.

- Now assume that the combined statement holds for all $j < i$, with $0 < i \leq n$. To show that it holds for i, we proceed as in the base case. The inequality (8) follows again using [4, Proposition 5.12], Remark 4, and the induction hypothesis (namely $\mu_\gamma(\eta_j(\mathsf{Op}_j^{P_{j+1},\ldots,P_n})) \sqsubseteq \eta_j(\mathsf{Op}_{\gamma,j}^{e_{j+1},\ldots,e_n})$ for $0 \leq j < i$). To show that $\mu_\gamma(\eta_i(\mathsf{Op}_i^{P_{i+1},\ldots,P_n})) \sqsubseteq \eta_i(\mathsf{Op}_{\gamma,i}^{e_{i+1},\ldots,e_n})$ whenever $P_j \in \mathcal{M}_j$ and $e_j : C_j \to S$ are such that $\mu_\gamma(P_j \cap \mathsf{Paths}_c) \sqsubseteq e_j(c)$ for $c \in C_j$ and $j \in \{i+1, \ldots, n\}$, we distinguish two sub-cases.
  - i is even. In this case the proof is similar to the base case.
  - i is odd. We consider the ordinal-indexed sequence used to obtain the least fixpoint P_i of $\mathsf{Op}_i^{P_{i+1},\ldots,P_n}$. Induction on ordinals together with (8) and the fact that $\mu_\gamma(\bigcup_{i\in\omega} A_i) = \sup_{i\in\omega} \mu_\gamma(A_i)$ for any increasing chain $A_0 \subseteq A_1 \subseteq \ldots$ can be used to show that $\mu_\gamma((\mathsf{Op}_i^{P_{i+1},\ldots,P_n})^\alpha(\emptyset)) \sqsubseteq (\mathsf{Op}_{\gamma,i}^{\mu_\gamma(P_{i+1}),\ldots,\mu_\gamma(P_n)})^\alpha(0)$:
    * For $\alpha = 0$, $\mu_\gamma(\emptyset) = 0 \sqsubseteq 0$.
    * For $\alpha = \beta + 1$, assuming $\mu_\gamma((\mathsf{Op}_i^{P_{i+1},\ldots,P_n})^\beta(\emptyset)) \sqsubseteq (\mathsf{Op}_{\gamma,i}^{\mu_\gamma(P_{i+1}),\ldots,\mu_\gamma(P_n)})^\beta(0)$, we have
    $$\begin{aligned} \mu_\gamma((\mathsf{Op}_i^{P_{i+1},\ldots,P_n})^{\beta+1}(\emptyset)) &\overset{(\text{by } (8))}{\sqsubseteq} (\mathsf{Op}_{\gamma,i}^{\mu_\gamma(P_{i+1}),\ldots,\mu_\gamma(P_n)})(\mu_\gamma((\mathsf{Op}_i^{P_{i+1},\ldots,P_n})^\beta(\emptyset)) \\ &\overset{(\text{I.H.})}{\sqsubseteq} (\mathsf{Op}_{\gamma,i}^{\mu_\gamma(P_{i+1}),\ldots,\mu_\gamma(P_n)})^{\beta+1}(0) \end{aligned}$$
    * For α a limit ordinal, we have
    $$\begin{aligned} \mu_\gamma((\mathsf{Op}_i^{P_{i+1},\ldots,P_n})^\alpha(\emptyset)) &= \sup_{\beta<\alpha} \mu_\gamma((\mathsf{Op}_i^{P_{i+1},\ldots,P_n})^\beta(\emptyset)) \\ \overset{(\text{I.H.})}{\sqsubseteq} \sup_{\beta<\alpha}(\mathsf{Op}_{\gamma,i}^{\mu_\gamma(P_{i+1}),\ldots,\mu_\gamma(P_n)})^\beta(\emptyset)) &\overset{(\text{Remark 4})}{\sqsubseteq} \sup_{\beta<\alpha}(\mathsf{Op}_{\gamma,i}^{e_{i+1},\ldots,e_n})^\beta(\emptyset)) \end{aligned}$$
    The equality above uses that $(\mathsf{Op}_i^{P_{i+1},\ldots,P_n})^\alpha(\emptyset)$ is the union of an increasing *countable* chain.

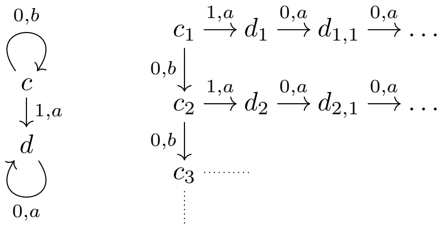
This concludes the proof of $\mu_\gamma(\{\, \xi \in \mathsf{Paths}_c \mid \xi \text{ accepting} \,\}) \sqsubseteq \mathsf{ext}_\gamma(c)$ for $c \in C$. ◀

We note in passing that, although the inequality (8) can be turned into an equality (by strengthening the relationship between the P_js and the e_js), this equality can not be used to prove the inequality $\mathsf{ext}_\gamma(c) \sqsubseteq \mu_\gamma(\{\, \xi \in \mathsf{Paths}_c \mid \xi \text{ accepting} \,\})$ in a similar way (by following the construction of the solutions of the two operators involved), since μ_γ does not behave well w.r.t. countable intersections (see [4, Example 5.10]).

We now turn to proving the second inequality. For this, we will use the so-called *unfolding* of a pointed $(\mathsf{T}_S \circ F)$-coalgebra.

▶ **Definition 44.** *The* unfolding *of a pointed $(\mathsf{T}_S \circ F)$-coalgebra (C, γ, c_I) is the pointed $(\mathsf{T}_S \circ F)$-coalgebra (B, β, b_I), where B contains a copy b_I of the initial state c_I, and for each copy $b \in B$ of some $c \in C$ and each transition $c \xrightarrow{w,i} (c_1, \ldots, c_{j_i})$ in (C, γ), (B, β) contains* *(new) copies* $b_1, \ldots, b_{j_i}$ *of* $c_1, \ldots, c_{j_i}$ *and a transition* $b \xrightarrow{w,i} (b_1, \ldots, b_{j_i})$ *. If* (C, γ, c_I) *is a QPA, the states of* (B, β, b_I) *inherit parities from the corresponding states of* C*.*

▶ **Example 45.** Let $S = (\mathbb{N}^\infty, \min, \infty, +, 0)$ and $F = \{a, b\} \times \mathsf{Id} \simeq \mathsf{Id} + \mathsf{Id}$. The unfolding of the pointed $(\mathsf{T}_S \circ F)$-coalgebra on the left is the infinite tree on the right:



Now to motivate the proof of the next lemma, consider the automaton obtained by putting $\Omega(c) = 1$ and $\Omega(d) = 0$ in the above coalgebra. Then, the states of the unfolding inherit parities from c and d, and one can show that the extent of the unfolding coincides with the extent of the original (pointed) coalgebra; that is, $\mathsf{ext}_\gamma(c) = \mathsf{ext}_\beta(c_1)$. Now recall that $\mu_\gamma(\{\xi \in \mathsf{Paths}_c \mid \xi \text{ accepting}\})$ is given by $\inf\{\mu_\gamma[\mathcal{C}] \mid \mathcal{C}$ is a pairwise-disjoint cylinder set cover for $\{\xi \in \mathsf{Paths}_c \mid \xi \text{ accepting}\}\}$. So to prove that $\mathsf{ext}_\gamma(c) \sqsubseteq \mu_\gamma(\{\xi \in \mathsf{Paths}_c \mid \xi \text{ accepting}\})$, it would suffice to show that $\mathsf{ext}_\gamma(c) \sqsubseteq \mu_\gamma[\mathcal{C}]$ for *every* such cover $\mathcal{C}$. Let us consider, in the above example, one particular cover for $\{\xi \in \mathsf{Paths}_c \mid \xi \text{ accepting}\}$, given by: $C_1 = \mathsf{Cyl}(c(a(d(*))))$, $C_2 = \mathsf{Cyl}(c(b(c(a(d(*))))))$, ... We can use this cover to separate the unfolding of our automaton into a countable number of automata: one automaton $(B^k, \beta_k, b_I^k, \Omega_k)$ for each cylinder set C_k of $\mathcal{C}$, whose paths are precisely the paths in the unfolding covered by C_k (up to a renaming of the states in the unfolding to the original states in C), and one automaton $(B^0, \beta_0, b_I^0, \Omega_0)$ whose paths are those (non-accepting) paths not covered by any $C_k \in \mathcal{C}$:

$$c_1^1 \xrightarrow{1,a} d_1 \xrightarrow{0,a} \cdots$$
$$c_1^2 \xrightarrow{0,b} c_2^1 \xrightarrow{1,a} d_2 \xrightarrow{0,a} \cdots$$
$$c_1^3 \xrightarrow{0,b} c_2^2 \xrightarrow{0,b} c_3^1 \xrightarrow{1,a} d_3 \xrightarrow{0,a} \cdots$$
$$\ldots$$
$$c_1^0 \xrightarrow{0,b} c_2^0 \xrightarrow{0,b} c_3^0 \xrightarrow{0,b} \cdots$$

Then, to prove $\mathsf{ext}_\beta(c_1) \sqsubseteq \mu_\gamma[\mathcal{C}]$ (which would then give $\mathsf{ext}_\beta(c) \sqsubseteq \mu_\gamma[\mathcal{C}]$), it would suffice to prove the following:

- $\mathsf{ext}_\beta(c_1) \sqsubseteq \mathsf{ext}_{\beta_0}(c_1^0) + \sum_{k \in \{1,2,\ldots\}} \mu_{\beta_k}(C'_k)$,
- $\mathsf{ext}_{\beta_0}(c_1^0) = 0$, and
- $\mu_{\beta_k}(C'_k) = \mu_\gamma(C_k)$, where for $k \in \{1, 2, \ldots\}$, the cylinder set C'_k is obtained from the cylinder set C_k by suitably renaming the states which label paths in C_k to states of B^k.

It turns out that all these statements can be proved in general, for any cover $\mathcal{C}$, as shown by (the proof of) the next lemma.

▶ **Lemma 46.** *For a quantitative parity automaton* (C, γ, c_I, Ω)*, we have* $\mathsf{ext}_\gamma(c_I) \sqsubseteq \mu_\gamma(\{\xi \in \mathsf{Paths}_{c_I} \mid \xi \text{ accepting}\})$*.*

Proof (Sketch). We will use the fact that $\mu_\gamma(\{\xi \in \mathsf{Paths}_{c_I} \mid \xi \text{ accepting}\}) = \inf\{\mu_\gamma[\mathcal{C}] \mid \mathcal{C}$ is a pairwise-disjoint cylinder set cover for $\{\xi \in \mathsf{Paths}_{c_I} \mid \xi \text{ accepting}\}\}$. We fix a pairwise-disjoint cylinder set cover $\mathcal{C} = \{C_1, C_2, \ldots\}$ for $\{\xi \in \mathsf{Paths}_{c_I} \mid \xi \text{ accepting}\}$, and prove $\mathsf{ext}_\gamma(c_I) \sqsubseteq \mu_\gamma[\mathcal{C}]$. To this end, we write (B, β, b_I, Ω) for the unfolding of (C, γ, c_I, Ω). Also, for $k \in \{1,2,\ldots\}$, we let $(B^k, \beta_k, b_I^k, \Omega_k)$ denote the part of the automaton (B, β, b_I, Ω) covered by C_k (defined similarly to Example 45). Finally, we let $(B^0, \beta_0, b_I^0, \Omega_0)$ denote the part of the automaton (B, β, b_I, Ω) not covered by any C_k, with $k \in \{1,2,\ldots\}$. (The fact that (B, β, b_I, Ω) is a *tree* unfolding is needed here.) The required inequality is now a consequence of the following three statements:

1. $\mathsf{ext}_\gamma(c_I) = \mathsf{ext}_\beta(b_I)$.
2. If an automaton has *no* accepting paths, then it has extent 0.
3. $\mathsf{ext}_\beta(b_I) \sqsubseteq \mathsf{ext}_{\beta_0}(b_I^0) + \sum_{k\in\{1,2,\ldots\}} \mathsf{ext}^\nu_{\beta_k}(b_I^k)$.

The first statement follows immediately from applying Proposition 38 to the map sending each copy of a state in C to the original state in C. The proof of the second statement, omitted here due to space limitations, uses the computation of extent (see Thm. 3) to construct an accepting path from an automaton state with extent $\neq 0$. The proof of the third statement is by induction on $i \in \mathsf{ran}(\Omega)$ (see below). Then, using all these statements, we have:

$$\mathsf{ext}_\gamma(c_I) = \mathsf{ext}_\beta(b_I) \sqsubseteq \mathsf{ext}_{\beta_0}(b_I^0) + \sum_{k\in\{1,2,\ldots\}} \mathsf{ext}^\nu_{\beta_k}(b_I^k) = \sum_{k\in\{1,2,\ldots\}} \mu_\gamma(C_k) = \mu_\gamma[\mathcal{C}]$$

The second equality above uses the fact that the automaton $(B^0, \beta_0, b_I^0, \Omega_0)$ has no accepting paths (and therefore its extent is 0), together with the fact that, for $k \in \{1,2,\ldots\}$, the automaton $(B^k, \beta_k, b_I^k, \Omega_k)$ contains (copies of) exactly those paths of (C, γ, Ω) which are covered by the cylinder set C_k (and therefore $\mathsf{ext}^\nu_{\beta_k}(b_I^k) = \mu_\gamma(C_k)$). This concludes the proof of the fact that $\mathsf{ext}_\gamma(c_I) \sqsubseteq \mu_\gamma[\mathcal{C}]$. Since this holds for every cover $\mathcal{C}$ for $\mu_\gamma(\{\,\xi \in \mathsf{Paths}_{c_I} \mid \xi \text{ accepting}\,\})$, we now obtain $\mathsf{ext}_\gamma(c_I) \sqsubseteq \mu_\gamma(\{\,\xi \in \mathsf{Paths}_{c_I} \mid \xi \text{ accepting}\,\})$ as required.

It remains to prove the third statement above. Now when the semiring S is partial, although the sum on the rhs of this statement is defined (it is equal to $\mu_\gamma[\mathcal{C}]$), some of the sums appearing later in the proof may not be defined. For this reason, we will interpret these sums in the *total* semiring $(S, \oplus, 0, \bullet, 1)$ (see Remark 8).

We will prove the following more general statement, in $(S, \oplus, 0, \bullet, 1)$:

$$\mathsf{ext}_\beta(b) \sqsubseteq \mathsf{ext}_{\beta_0}(b^0) + \sum_{k\in\{1,2,\ldots\}} \mathsf{ext}^\nu_{\beta_k}(b^k) \tag{9}$$

for each $b \in B$, where for $k \in \{0,1,\ldots\}$, b^k is the copy of b which belongs to (B^k, β_k, Ω_k). For this, we prove by induction on $i \in \mathsf{ran}(\Omega)$ that:

$$(\eta_i(\mathsf{Op}_{\beta,i}^{e_{i+1},\ldots,e_n}))(b) \sqsubseteq (\eta_i(\mathsf{Op}_{\beta_0,i}^{e'_{i+1},\ldots,e'_n}))(b^0) + \sum_{k\in\{1,2,\ldots\}} \mathsf{ext}^\nu_{\beta_k}(b^k) \tag{10}$$

for each $b \in B_i$, whenever $e_j : B_j \to S$, $e_j^0 : B_j^0 \to S$ are such that $e_j \sqsubseteq e_j^0 + \sum_{k\in\{1,2,\ldots\}} (\mathsf{ext}^\nu_{\beta_k} \circ \iota_j)$ for $j \in \{i+1,\ldots,n\}$. In the above, ι_j denotes the inclusion of the set of states with parity j into the entire set of states. We immediately note that (10) holds trivially for those $b \in B_i$ for which the whole of Paths_b is covered by $\mathcal{C}$ – this follows from the definitions of extent and ν-extent, together with the pairwise-disjointness of the cylinder sets in $\mathcal{C}$. Therefore it suffices to show that (10) holds on states some of whose outgoing transitions belong to $(B^0, \beta_0, b_I^0, \Omega)$.

- Consider, first, the case when $i = 0$. Then, induction on ordinals can be used to show that $(\mathsf{Op}_{\beta,i}^{e_{i+1},\ldots,e_n})^\alpha(\top)(b) \sqsubseteq (\mathsf{Op}_{\beta_0,i}^{e'_{i+1},\ldots,e'_n})^\alpha(\top)(b^0) + \sum_{k\in\{1,2,\ldots\}} \mathsf{ext}^\nu_{\beta_k}(b^k)$ holds for all $b \in B_0$ and all ordinals α:
  - For $\alpha = 0$, the statement is trivial (both sides equal $1 \in S$).
  - For $\alpha = \gamma+1$, assume that $(\mathsf{Op}_{\beta,0}^{e_1,\ldots,e_n})^\gamma(\top)(b) \sqsubseteq (\mathsf{Op}_{\beta_0,0}^{e'_1,\ldots,e'_n})^\gamma(\top)(b^0) + \sum_{k\in\{1,2,\ldots\}} \mathsf{ext}^\nu_{\beta_k}(b^k)$

holds for all $b \in B_0$. We then have, for $b \in B_0$:

$$(\mathsf{Op}_{\beta,0}^{e_1,\ldots,e_n})^{\gamma+1}(\top)(b) =$$

(definition of $\mathsf{Op}_{\beta,0}^{e_1,\ldots,e_n}$)

$$\sum_{b \xrightarrow{i,w} b' \in B_0} w \bullet (\mathsf{Op}_{\beta,0}^{e_1,\ldots,e_n})^{\gamma}(\top)(b') + \sum_{b \xrightarrow{i,w} b' \in B_j, j \neq 0} w \bullet e_j(b') \sqsubseteq$$

(I.H., assumption on e_j, e'_j)

$$\sum_{b \xrightarrow{i,w} b' \in B_0} w \bullet \left((\mathsf{Op}_{\beta_0,0}^{e'_1,\ldots,e'_n})^{\gamma}(\top)(b') + \sum_{k \in \{1,2,\ldots\}} \mathsf{ext}^{\nu}_{\beta_k}(b'^k) \right) +$$

$$\sum_{b \xrightarrow{i,w} b' \in B_j, j \neq 0} w \bullet \left(e'_j(b') + \sum_{k \in \{1,2,\ldots\}} \mathsf{ext}^{\nu}_{\beta_k}(b'^k) \right) =$$

(distributivity of $\bullet$ over finite sums, definition of $\mathsf{Op}_{\beta_0,0}^{e'_1,\ldots,e'_n}$ and $\mathsf{ext}^{\nu}_{\beta_k}(b'^k)$)

$$(\mathsf{Op}_{\beta_0,0}^{e'_1,\ldots,e'_n})^{\gamma+1}(\top)(b) + \sum_{k \in \{1,2,\ldots\}} \mathsf{ext}^{\nu}_{\beta_k}(b^k)$$

- For α a limit ordinal, the statement follows from $(\mathsf{Op}_{\beta,0}^{e_{i+1},\ldots,e_n})^{\alpha}(\top)$ and $(\mathsf{Op}_{\beta_0,0}^{e'_{i+1},\ldots,e'_n})^{\alpha}(\top)$ being obtained as infima of decreasing chains.

This then yields the required statement for $i = 0$.

- The induction step is proved similarly, additionally making use of the induction hypothesis.

We have thus proves the inequality (9) in the *total* semiring $(S, \oplus, 0, \bullet, 1)$. This now gives $\mathsf{ext}_{\beta}(b_I) \sqsubseteq \mathsf{ext}_{\beta_0}(b_I^0) + \sum_{k \in \{1,2,\ldots\}} \mathsf{ext}^{\nu}_{\beta_k}(b_I^k)$ in $(S, \oplus, 0, \bullet, 1)$. However, since the sum in the rhs *is* defined in $(S, +, 0, \bullet, 1)$ (it coincides with $\mu_\gamma[\mathcal{C}]$)), it follows that the same inequality also holds in $(S, +, 0, \bullet, 1)$. This concludes the proof. ◀

Theorem 42 yields characterisations of the notion of extent in all our example semirings.

▶ **Example 47.** When $(S, +, 0, \bullet, 1)$ is the boolean semiring, a state in a QPA has extent 0 iff it admits no accepting paths. When $(S, +, 0, \bullet, 1)$ is the probabilistic semiring, the extent of a state measures the likelihood of an accepting path. When $(S, +, 0, \bullet, 1)$ is the tropical semiring, the extent of a state gives the minimal cost of an accepting path from that state.

4 Model Checking Qualitative Properties in Quantitative Models

We now show how to use Theorem 42 to model check qualitative properties captured by F-automata against $(\mathsf{T}_S \circ F)$-coalgebras. When the F-automaton is non-deterministic, its product with a $(\mathsf{T}_S \circ F)$-coalgebra is only defined when the semiring is total. However, even if the product *is* defined, accepting paths through the product are not, in general, in one-to-one correspondence with paths through the coalgebra which conform to the automaton. For this, unambiguity of the automaton is required. This is why in what follows we restrict to qualitative properties captured by *unambiguous* F-automata. We first consider the case when the semiring is total, and then show how to extend our result to a partial semiring.

We instantiate Theorem 42 to the product of an unambiguous NPA (Definition 23) with a $(\mathsf{T}_S \circ F)$-coalgebra in order to prove the following result:

▶ **Theorem 48.** *Assume* $(S,+,0,\bullet,1)$ *is* total. *Let* (A,α,a_I,Ω) *with* $\mathsf{ran}(\Omega)\subseteq\{0,\dots,n\}$ *be an unambiguous automaton, let* (C,γ,c_I) *be a pointed* $(\mathsf{T}_S\circ F)$*-coalgebra, and let* $(D,\delta,(c_I,a_I),\Omega)$ *be the product of* (C,γ,c_I) *and* (A,α,a_I,Ω) *(Definition 29). Then, the extent* $\mathsf{ext}_\delta : D\to S$ *of* $(D,\delta,(c_I,a_I),\Omega)$ *satisfies* $\mu_\gamma(\{\xi\in\mathsf{Paths}^\gamma_{c_I}\mid \xi$ *accepted by* $(A,\alpha,a_I,\Omega)\})=\mathsf{ext}_\delta(c_I,a_I)$*.*

Proof. We have:

$$\mathsf{ext}_\delta(c_I,a_I)=\mu_\delta(\{\,\xi\in\mathsf{Paths}^\delta_{(c_I,a_I)}\mid\xi\text{ acc.}\,\})=\mu_\gamma(\{\xi\in\mathsf{Paths}^\gamma_{c_I}\mid\xi\text{ accepted by }(A,\alpha,a_I,\Omega)\})$$

The first equality follows by Theorem 42, whereas the second equality follows by Lemma 31 and because measuring the sets of paths in question in δ, respectively γ, yields the same result (since weights of δ-transitions are inherited from γ). ◀

Theorem 48 thus states that, assuming that the automaton (A,α,a_I,Ω) is unambiguous, the extent of its product with a model (C,γ,c_I) can be used to compute the measure of the set of paths from c_I which conform to the automaton.

When the semiring S is partial, the product of (C,γ,c_I) and (A,α,a,Ω) is not always a $\mathsf{T}_S\circ F$-automaton. To deal with this, we view (C,γ,c_I) as a $\mathsf{T}_{S'}\circ F$-coalgebra (where $S'=(S,\oplus,0,\bullet,1)$ is as in Remark 8), to which Theorem 42 applies. However, in order to generalise Theorem 48 to *partial* semirings, we must additionally show that the S-valued measure of the set of paths from c in (C,γ) which are accepted by (A,α,a,Ω) coincides with the S'-valued measure of the same set of paths. The next lemma establishes this.

▶ **Lemma 49.** *Let* (C,γ,c_I) *be a pointed* $(\mathsf{T}_S\circ F)$*-coalgebra. Then,* $\mu^S_\gamma(P)=\mu^{S'}_\gamma(P)$ *for any measurable set* P *of paths from* c *in* (C,γ) *(where the superscripts of the resulting measures indicate the semiring these measures are valued into).*

Proof. We have:

$$\begin{aligned}\mu^S_\gamma(P)&\overset{(\text{def. of }\mu^S_\gamma)}{=}\inf\{\sum_{C\in\mathcal{C}}\mu^S_\gamma(C)\mid\mathcal{C}\text{ is a countable, pairwise-disjoint cover for }P\,\}\\&\overset{(*)}{=}\inf\{\sum_{C\in\mathcal{C}}\mu^{S'}_\gamma(C)\mid\mathcal{C}\text{ is a countable, pairwise-disjoint cover for }P\,\}\\&\overset{(\text{def. of }\mu^{S'}_\gamma)}{=}\mu^{S'}_\gamma(P)\end{aligned}$$

The equality $(*)$ above follows from the fact that all sums in the lhs are defined. ◀

Our second main result is now a direct consequence of Theorem 48 and Lemma 49.

▶ **Theorem 50.** *Let* $(S,+,0,\bullet,1)$ *be a partial semiring and let* $(S'=S,\oplus,0,\bullet,1)$ *be as in Remark 8. Let* (A,α,a,Ω) *be an unambiguous* F*-automaton, and let* (C,γ,c_I) *be a pointed* $(\mathsf{T}_S\circ F)$*-coalgebra. Finally, let* (D,δ,d,Ω) *be the product of* $(C,\iota\circ\gamma,c_I)$ *and* (A,α,a,Ω)*. Then, the following holds:* $\mu^S_\gamma(\{\,\xi\in\mathsf{Paths}^\gamma_c\mid\xi$ *accepted by* $(A,\alpha,a,\Omega)\,\}=\mathsf{ext}^{S'}_\delta(c,a)$*.*

In other words, to measure the set of paths in a model (C,γ,c_I) which conform to a qualitative property captured by an unambiguous parity automaton (A,α,a,Ω), one can simply compute the extent of the product automaton, in the extended semiring $S,\oplus,0,\bullet,1)$.

5 Conclusions

We provided a characterisation of the measure of the set of accepting paths of a QPA, as the solution of a nested system of equations. We also showed how to use this characterisation to model check qualitative linear-time properties against quantitative models. Future work will investigate computational results and the expressive power of unambiguous automata, and will use techniques from [3] to approximate nested extents.

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