University of Southampton

Faculty of Engineering and Physical Sciences

Physics and Astronomy

The renormalization group and quantum gravity

by

Alexander V.J. Mitchell

Thesis for the degree of Doctor of Philosophy

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Abstract

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A universally accepted theory of quantum gravity would play a crucial role in a complete understanding of Nature which has so far eluded physicists. This is illustrated by the variety and complexity of the many attempts to produce such a theory. A collection of analyses is presented which investigates a novel, natural approach which combines general relativity with quantum field theory in the framework of the Wilsonian renormalization group whilst respecting the conformal instability.

We investigate the structure of this theory at first order in the coupling and verify that a continuum limit exists. We find that a continuum limit exists only if the theory is defined outside of the diffeomorphism invariant subspace. In the UV, interactions are associated to a coefficient function which is parametrised by an infinite number of fundamental couplings. In the physical limit diffeomorphism invariance is reinstated such that for a suitable choice of these couplings the coefficient functions trivialise. Dynamically generated effective diffeomorphism invariant couplings emerge, in particular Newton's coupling.

This investigation is continued to second order in perturbation theory. For pure quantum gravity, with vanishing cosmological constant, the result of the standard quantisation is recovered. Quantum gravity is renormalizable at second order for kinematic reasons but the structure is shown to hold in general. It may be the case that a continuum limit exists however with a, phenomenologically inconvenient, infinite number of fundamental couplings. However a possible non-perturbative resolution, based on the conformal instability and the parabolic properties of the flow equations, is investigated which would fix higher order effective couplings in terms of Newton's constant and the cosmological constant.

We then explore the properties of these flows with opposite natural direction through a related asymptotic safety problem, studying whether or not they have complete solutions. Finally we conclude with a discussion of the ramifications of this structure and possible applications to outstanding physical problems, in particular we investigate how well this structure is extended to arbitrary space-time dimensions and the physical consequences thereof.

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List of Abbreviations

AdS/CFT	Anti-de Sitter/Conformal Field theory
AS	Asymptotic Safety
BRST	Becchi-Rouet-Stora-Tyutin
CFT	Conformal Field Theory
CIVP	Cauchy Initial Value Problem
CME	Classical Master Equation
EH	Einstein-Hilbert
ERG	Exact Renormalization Group
GFP	Gaussian Fixed Point
GR	General Relativity
IR	Infra Red
LQG	Loop Quantum Gravity
mST	modified Slavnov-Taylor
NGFP	Non-Gaussian Fixed Point
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
QG	Quantum Gravity
QME	Quantum Master Equation
QMF	Quantum Master Functional
RG	Renormalization Group
SM	Standard Model
UV	Ultra Violet

Declaration of Authorship

I, **Alexander V.J. Mitchell**, declare that the thesis, entitled *The renormalization group and quantum gravity*, and the work presented in it are both my own and have been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as references [1–3].

Signed:

Date:

For Mum,
who has been there every step of the way

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Chapter 1

Motivation

A universally accepted unification of gravity with quantum field theory continues to elude physicists. Despite the continued efforts of the physics community there seems to be no general direction to the research and no experimental data to refute or confirm models. There is significant motivation for unifying these two key aspects of modern physics; the expectation is that such a theory would shed light on some of the paramount questions in physics.

These issues include but are not limited to understanding the universe's first moments and the inflationary period, what is the behaviour of space-time beyond a black hole's event horizon and in particular the nature of singularities therein, how can we describe dark energy which makes up some 70% of the energy budget of the universe. Finally and more relevant to this thesis resolving such a unification of gravity with quantum field theory would no doubt resolve major questions regarding how one investigates fundamental physics. Can we continue to use the successful scaffolding of quantum field theory and the ideas of the renormalization group or will Nature demand that we construct something much different in the pursuit of understanding the universe around us?

We begin with an introduction to subjects relevant to this thesis, beginning with the broad concepts of general relativity with a particular focus on the Einstein-Hilbert action and the graviton field. Following this we review the ideas of renormalization and the process of eliminating infinities from our calculation to produce physical results. We then follow these ideas with a broader discussion of the Wilsonian approach to renormalization in the form of the renormalization group with comments on how these ideas relate to the exact renormalization group and asymptotic safety amongst others. It is pertinent to emphasise why gravity is non-renormalizable in this new language and we do so here. We then take a brief pause to examine contemporary approaches to the resolution of this quantum gravity problem, focusing on string theory and loop quantum gravity in particular to give context to the work of this thesis. We then conclude this introductory section with a chapter detailing a review of the 'dilaton portal' which underpins this novel approach to quantum gravity, we also detail the methods of the anti-field approach to Becci-Rouet-Stora-Tyutin (BRST) quantisation and associated subjects.

We begin the analysis of this novel approach to quantum gravity at first order in perturbation theory in chapter 4. This work will underpin the later research discussed in this thesis and is the first step in understanding the structure such a theory demands and the consequences one finds when combining general relativity, the exact renormalization group and a more complete quantisation process.

Chapter 5 follows, here the analysis is progressed to second order in perturbation theory which is a significant step as now aspects of the theory such as the behaviour and existence of couplings in the continuum limit can be more completely investigated. The relationship of the complete theory with the novel behaviour of the dilaton section of the graviton and the ramifications of this are also considered here.

The consequences of this analysis are further elucidated in chapter 6 and investigations are made into a physically significant repercussion of this approach to quantum gravity, namely the existence of contradictory flows for different sectors of the graviton field. This work is more closely related to the field of asymptotic safety and as such links are made to this popular research field as well as an overview of the defining concepts.

There are a significant number of open questions that a theory of quantum gravity would hope to answer, we briefly discuss how such a theory as discussed here can shed light on these problems. In particular we investigate whether this work can be extended to arbitrary space-time dimensions. The idea of this preliminary work stems from a history of suggestions in theoretical physics that extra dimension may play a role, for example the idea of Kaluza-Klein compactification. Other applications of this theory are also discussed here.

Finally we conclude this thesis with comments on the presented results within the context of modern theoretical physics and discuss the current state of this theory and where it may progress in the future.

Chapter 2

Introduction

2.1 General relativity

We begin by outlining the most relevant results of Albert Einstein's magnum opus, General Relativity (GR) [4, 5]. Describing this framework in its entirety as well as the multitude of consequences and results of this work would fill a document many times larger than this thesis. Instead we discuss the key results which are relevant to our work and encourage the reader to review GR for themselves [6, 7].

We begin with the Euclidean Einstein-Hilbert (EH) action of GR

$$S_{EH} = \int d^4x \; \mathcal{L}_{EH} \quad \text{with} \quad \mathcal{L}_{EH} = -2\sqrt{g}R/\kappa^2 \tag{2.1}$$

where $\kappa = \sqrt{32\pi G}$ and G is Newton's constant with $g = det(g_{\mu\nu})$. The Ricci scalar, Ricci tensor and Riemann tensor are given by $R = g^{\mu\nu}R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ with $[\nabla_\mu, \nabla_\nu]v^\lambda = R_{\mu\nu}{}^{\lambda}{}_{\sigma}v^\sigma$. There are a variety of ways to define the metric $g_{\mu\nu}$ which is used to construct $R, R_{\mu\nu}$ and $R^{\alpha}{}_{\mu\rho\nu}$. In this thesis we will typically choose to expand the metric around flat space as

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa H_{\mu\nu} \tag{2.2}$$

where $\delta_{\mu\nu}$ is our flat background and $H_{\mu\nu}$ is our 'complete graviton'. We also define here the *dilaton* φ

$$\varphi = \tfrac{1}{2}H_{\mu\mu} \tag{2.3}$$

which will be of great significance in this thesis, see chapter 3. In this thesis we will be concerning ourselves primarily with pure gravity, that is to say a theory of gravitons and dilatons interacting only between themselves. We will however make comments on the most significant effects one may expect when one extends this theory to include general matter.

GR and the EH action are (conveniently, for physical measurements) valid in

Minkowski signature however as we will be working in the Wilsonian Renormalization Group (RG) framework, see section 2.3, we will be working with a Euclidian signature metric. This is because the Wilsonian RG necessitates quasi-local effective actions constructed from integrating out fluctuations at short distances. As we will be working with a flat background metric $\delta_{\mu\nu}$ there is no physical difference between contravariant and covariant objects, however we will continue to follow the Einstein summation convention out of habit and for legibility, except where it is inconvenient to do so.

When working with this Euclidean signature we see that the Euclidean partition function is less well defined than usual. As (2.1) is unbounded from below (with large positive curvature being the source of the upcoming difficulties) we see that

$$\mathcal{Z} = \int \mathcal{D}g_{\mu\nu} e^{-S_{EH}} = \int \mathcal{D}g_{\mu\nu} e^{2\sqrt{g}R/\kappa^2} \tag{2.4}$$

will fail to converge. Using the expansion (2.2) we find at zeroth order in the coupling κ the Fierz-Pauli action for free gravitons

$$\mathcal{L}_{EH} = \tfrac{1}{2}(\partial_\lambda H_{\mu\nu})^2 - 2(\partial_\lambda \varphi)^2 - (\partial_\mu H_{\mu\nu})^2 + 2\partial_\alpha\varphi\partial_\beta H_{\alpha\beta} \tag{2.5}$$

which using a Feynman - De Donder gauge fixing term $(\partial_\alpha H_{\alpha\beta} - \partial_\beta\varphi)^2$ and splitting the complete graviton into its $SO(4)$ irreducible parts

$$H_{\mu\nu} = h_{\mu\nu} + \tfrac{1}{2}\delta_{\mu\nu}\varphi, \tag{2.6}$$

(2.5) simplifies to

$$\mathcal{L}_{EH} = \tfrac{1}{2}(\partial_\lambda h_{\mu\nu})^2 - \tfrac{1}{2}(\partial_\lambda\varphi)^2. \tag{2.7}$$

The minus sign in front of the dilaton kinetic term will prove to be crucial to a novel approach to Quantum Gravity (QG). We also note that under this expression of the metric and decomposition of the complete graviton (henceforth we refer to $h_{\mu\nu}$ and φ as the graviton and dilaton respectively) we have an overall local rescaling of the metric

$$g_{\mu\nu} = \delta_{\mu\nu}\left(1 + \frac{\kappa}{2}\varphi\right) + \kappa h_{\mu\nu}. \tag{2.8}$$

In this way we can clearly see that the dilaton φ [8, 9] is a a perturbation that leads to an overall local rescaling of the metric and this splitting of the metric makes the conformal instability most easily recognisable in (2.7).

In this thesis we will regularly discuss the symmetry of *diffeomorphism invariance.* In much the same way $SU(3)$ can be regarded as the symmetry of QCD one can regard diffeomorphism invariance as the symmetry of GR. This states that the laws of physics should be invariant under our seemingly arbitrary choice in coordinate frame as there does not exist a priori such a frame or indeed any preferred frame in Nature. That is to say the description of Nature should be invariant from one transformation to another.

Before discussing this in greater detail in chapter 3 as well as the consequences of this negative kinetic term we must first elucidate precisely what renormalization and the renormalization group are in sections 2.2 and 2.3, and in section 2.4 why a satisfying combination of GR with this framework at the perturbative level has so far eluded physicists. We will sketch out some of the more popular attempts at this in section 2.5.

2.2 Renormalization

Renormalization is the systematic process by which the unphysical infinities produced in the mathematics of Quantum Field Theory (QFT) are eliminated such that the calculations reflect the physical reality Nature presents us. These infinities arise due to loops in Feynman diagrams constructed from the 'bare Lagrangian', the Lagrangian that is produced when one considers a set of fields, their kinetic terms and a set of all possible interactions that the symmetries allow. We will briefly review the key results of this field following [10], before focusing our attention on the more complete approach afforded by Kenneth Wilson's renormalization group in section 2.3. We encourage the reader to review this subject as well as the finer points we do not cover here in the many articles and textbooks written covering it [11].

The infinities we are concerned with arise from the integration over the entirety of the momentum space when evaluating Feynman diagrams, these are referred to as Ultra Violet (UV) divergences[1]. In the loops of the Feynman diagrams we will find that evaluating these goes as, schematically for the sake of clarity,

$$\int_0^\Lambda \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2} \;\sim\; \Lambda^2 \tag{2.9}$$

where Λ is a cut-off we introduce to parametrise this divergence, when we set $\Lambda \to \infty$ the above will diverge. As these loops are used to calculate physically observable quantities such as coupling strengths then there is clearly an issue here: these quantities must be finite. There are two key ingredients to remedying this problem, the first is to render such calculations finite. The second is to recognise that the Feynman rules used to produce these diagrams and ergo the resulting calculations are effectively incomplete; it has used the *bare Lagrangian* which has not accounted for the effect of these loop diagrams and to instead re-write these rules using the physical *renormalized Lagrangian.*

Beginning with the former we can introduce a regulator such as the cut-off Λ above, which is then taken to ∞ once the couplings are redefined such that one gets the desired

[1]In many textbooks there will also be mention of so-called Infra Red (IR) divergences that arise in theories with massless particles due to taking the momentum to zero for certain quantities. We do not cover these in detail here.

finite result. This is often referred to as the UV or continuum limit. One may also use processes such as dimensional regularization which do not introduce a cut-off Λ, in this case it was recognised that these divergences occur in four space-time dimensions and so calculations are performed in $d = 4 - \epsilon$ space-time dimensions, with the limit $\epsilon \to 0$ returning us to the Nature we are more familiar with. This is equivalent to taking the $\Lambda \to \infty$ limit above.

The second is resolved via the introduction of *counter terms*, these are introduced order by order to account for the infinities produced by the bare Lagrangian in a process referred to as *perturbative renormalization*,

$$S_\Lambda[\phi] \to S_\Lambda[\phi] + S^{CT}[\phi, \Lambda]. \tag{2.10}$$

As an example take ϕ^4 theory, this has a bare action

$$S_\Lambda[\phi] = \int d^4x \; \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{2.11}$$

and a counter term action

$$S^{CT}[\phi, \Lambda] = \int d^4x \; \tfrac{1}{2}\delta Z\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\delta m^2\phi^2 + \frac{1}{4!}\delta\lambda\phi^4 \tag{2.12}$$

where δZ , δm^2 and $\delta\lambda$ represent the freedom to adjust the couplings in the original action. δZ in particular is the coupling to the kinetic term which is referred to as the *wavefunction renormalization*. These counter term couplings will have Λ-dependency and are representative of the tuning to have the renormalized action reflect reality, this aspect of tuning is elaborated on in the following section. We note that there are many different ways to implement this tuning and perturbative renormalization, depending on the theory considered and the mathematical preferences of the physicists there are benefits and costs to these different *renormalization schemes*. The physics should however be independent of this choice at all times.

This idea of renormalization was at first met with a great deal of criticism and was considered ad hoc. It was not until the work of Kenneth Wilson and the flourishing of the ideas of the renormalization group when this work was underpinned by these more complete, rigorous concepts.

2.3 The renormalization group

We can approach this idea of renormalization and how one best defines a QFT from the fresh perspective that Kenneth Wilson et al [12–14] gave. We will give here a brief summary of the ideas of the Wilsonian Renormalization Group (RG) however we again encourage the reader to discover the finer details of this field for themselves [15–18].

The defining principle behind the RG is the idea of physical scale. More precisely we describe a theory with respect to some length/energy scales. For example one does not need the complexity of some all encompassing theory-of-everything to describe water in a tea cup, instead a simpler low energy *effective* theory such as Navier-Stokes will be more than suitable. Such a theory is fine in this regime however in a high energy scenario it may not be. Continuing with the water analogy Navier-Stokes would not be appropriate for attempting to describe the dynamics of dynamite being detonated in a swimming pool. Nevertheless we can see that one does not need to understand the full complexity of Nature to describe low energy phenomena. Crucially the high energy/small distance behaviour will only make its presence known in certain coefficients in the low energy theory, for example the viscosity of fluids.

This idea of defining a theory at different energy scales[2] lends itself naturally to QFT. In the Wilsonian framework a QFT is defined as an action functional $S[\phi, g_i]$ of the fields ϕ and couplings g_i which is constructed from all possible operators that the symmetries permit. This action depends *a priori* on an infinite number of couplings such as mass parameters, couplings for interaction terms and so on. The functional integral is given by

$$\mathcal{Z} = \int d\phi \ e^{iS[\phi, g_i]} \tag{2.13}$$

and as we saw in section 2.2 we must have a method with which to resolve the continuum limit, that is to say find a way such that every physical observable is not simply ∞. We do so here by introducing cut-offs or regulators which we insert in our kinetic term. These are introduced such that high momentum modes are suppressed[3] in such a way that we can integrate over all momentum modes and get finite results. The choice in cut-off is usually a practical one; some are less physical but provide computational practicality (such as a step function) whilst others are the inverse of this (for example a more subtle exponentially damped one). Invariably though two aspects must be considered: that the physical observables are independent of this choice of cut-off and that introducing the cut-off also brings with it a cut-off scale μ.

We can interpret this cut-off scale μ in several ways. For the purposes of this thesis we will regard it as a momentum above which we begin to suppress modes by modifying the action or the measure, we do so by introducing the aforementioned cut-off function. We can choose μ to match the energy scale of some physical process we are interested in as then we are only concerned with energy scales at or below the region of interest. Critically if one defines a physical quantity $\mathcal{F}(g_i; E)$ where g_i are the couplings of the underlying theory and E is some energy dependence then the idea of the RG is that one can change the cut-off of the theory such that the physics on energy scales below μ remain constant. Our arbitrary choice of the cut-off does not affect the physics we

[2]Henceforth we will use *scales* to refer to energy scales for simplicity. We will also often use UV to refer to a high energy regime and IR to a low energy regime. We note that this usage is relative, an energy may be considered UV in one context and IR in another.

[3]This is also equivalent to introducing a physical lattice spacing of distance Λ^{-1} or in perturbation theory analytically continuing the space-time dimension via dimensional regularization as outlined in section 2.2.

observe. For this to occur the couplings themselves however must change as a function of μ. This consistency of the physical observables can be summed up with the RG equation

$$\mathcal{F}(g_i(\mu); E)_\mu = \mathcal{F}(g_i(\mu'); E)_{\mu'} \tag{2.14}$$

where μ' denotes some new scale.

When one considers the RG we begin in the UV and consider the natural flow to be towards the IR, this idea of a natural direction for the flow where solutions to the forthcoming RG equations are guaranteed to exist will be crucial in understanding this novel approach to QG, in particular in chapter 6. The couplings $g_i(\mu)$ define a *theory space*, an infinite dimensional space where each direction corresponds to one of these couplings, a path through this space is defined by these couplings. This path is the *RG flow*. To ensure (2.14) is satisfied we must integrate out the degrees of freedom between the two cut-off scales μ and μ'. At first glance this would appear to be a monumental task due to the infinitely dimensional space we are working in however there are key properties of the theory space which make this task significantly easier.

The physical observables of a theory are constructed from the action, once the action is well understood everything follows from that. In this RG approach we must then understand how the action changes as a consequence of a change in μ, this leads to the key RG equation.

$$S[Z(\mu)^{1/2}\phi; \mu, g_i(\mu)] = S[Z(\mu')^{1/2}\phi; \mu', g_i(\mu')]. \tag{2.15}$$

Here we consider the generic fields ϕ the action is dependent upon and also recall the wavefunction renormalization $Z(\mu)$.

The actions we consider here are formed of two parts; the kinetic term and a linear combination of operators $\mathcal{O}_i(x)$ which are comprised of fields and their derivatives[4] e.g. ϕ^n,$(\phi^n\partial\phi)^m$, more exotic operators can be constructed such as the tower operator we discuss in chapter 3. The more general expression for the action with a cut-off is the *Wilsonian effective action*, given as

$$S[\phi; \mu, g_i] = \int d^d x \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_i \mu^{d-d_i} g_i \mathcal{O}_i(x)\right] \tag{2.16}$$

where d_i is the classical scaling dimension (also known as the engineering scaling dimension) of $\mathcal{O}_i(x)$. Crucially we have chosen the couplings to be dimensionless by inserting the appropriate power of the cut-off to carry this dimension, this is necessary as it is the value of the coupling relative to the cut-off that will be important to the physical observables. This concept of using scaled variables will also prove to be a crucial tool

[4]The composition of these operators will be limited not only by the continuous internal symmetries of the theory one is considering but also more general considerations such as $\phi \to -\phi$ symmetry as well as Lorentz invariance. As a result we will often eschew Lorentz indices where it is convenient to do so to aid explanation.

in understanding the Wisonian RG.

We now begin to think of the infinitesimal transformations as we change μ, the famous *β-function* of a theory is given by

$$\mu \frac{dg_i(\mu)}{d\mu}. \tag{2.17}$$

This function describes the *running of the couplings*, how they change as we flow along the RG, by integrating out the beta function equations. Since the dimensionful couplings are given by μ^{d-d_i} the β-function always has the form

$$\mu \frac{dg_i}{d\mu} = (d_i - d)g_i + \beta_{g_i}^{quant} \tag{2.18}$$

where the first term on the right hand side stems from the classical scaling dimension of the operators and the second term stems from the non-trivial integrating-out part of the RG transformation. This also corresponds to the first term consisting of only tree level diagrams and the latter, loop level diagrams, which would be more apparent if we re-introduced $\hbar$ when not working in natural units (hence the *quant* label). We also define the *anomalous dimension* of a field ϕ as

$$\gamma_\phi = -\frac{\mu}{2} \frac{d \log Z(\mu)}{d\mu}. \tag{2.19}$$

In this approach to QFTs it is often easier to consider Green's functions of fields as opposed to probabilities of events occuring as the most useful physical observable. Although the latter may be more natural this will often be difficult to calculate in massless theories where the S-matrix must consider long range interactions. The Green's function is given by

$$\langle \phi(x_1)...\phi(x_n) \rangle_{g_i(\mu),\mu} = \frac{\int_\mu [d\phi] \; e^{iS[\phi;\mu,g_i(\mu)]} \; \phi(x_1)...\phi(x_n)}{\int_\mu [d\phi] \; e^{iS[\phi;\mu,g_i(\mu)]}} \tag{2.20}$$

To maintain the consistency of (2.15) we must generalise (2.14) to take account of wavefunction renormalization where again our choice in μ should not affect the physics,

$$Z(\mu)^{-n/2} \langle \phi(x_1)...\phi_n(x_n) \rangle_{g_i(\mu),\mu} = Z(\mu')^{-n/2} \langle \phi(x_1)...\phi_n(x_n) \rangle_{g_i(\mu'),\mu'}. \tag{2.21}$$

We now begin to approach the most important aspects of the RG which will be paramount to this thesis. The most crucial aspects of the RG flows are their UV and IR behaviour, namely what happens when one takes the cut-off to $\mu \to \infty$ and $\mu \to 0$ respectively. We will find that we construct the theory in the UV before flowing naturally[5] down towards the low energy IR, where in reality we are able to make measurements and verify predicted physical observables against experiment. We begin

[5] This premise of a natural direction for the flow stems from the full RG equations which we elucidate in chapter 3.

with a theory of only massive particles and find that as we flow towards the IR the ratio of any mass scales in the theory to the cut-off scale, m/μ, increase ($\mu \to 0$). In this limit, a theory with no mass scales[6] becomes trivial as all physical masses become effectively infinite in comparison to the cut-off and so there is nothing to propagate. We are left with an empty, un-interesting theory.

We must also consider what happens in the UV. We define the *critical surface* as the infinite dimensional sub-space of the theory space for which the mass-gap vanishes, critically the theories in this space will have a non-trivial IR limit where only the massless degrees of freedom survive. As opposed to the former case in the IR limit for these theories the massless particles will remain, all the couplings will flow towards a fixed point of the RG, $g_i(\mu) \to g_i^*$ where the β-functions vanish. We note now this is not to be confused with the Gaussian Fixed Point (GFP) which plays an important role in these concepts and is the fixed point of a free, non-interacting particles. The equation for a fixed point[7] is given by

$$\mu \frac{dg_i}{d\mu}\Big|_{g_j^*} = 0, \tag{2.22}$$

these points in the theory space correspond to Conformal Field Theories (CFTs). We can compare this to (2.17) and see that at the fixed point the β-function is zero, the couplings no longer evolve. These CFTs are a special class of theories with many novel features which are of particular interest to alternative approaches to QG, see section 2.5. They possess only massless particles in addition to having no dimensionful parameters, as a consequence of this they are said to be *scale invariant* and this invariance is promoted to the full group of conformal transformations, hence 'CFTs'. It is in the neighbourhood of these CFTs where we will construct non-trivial theories, taking small perturbations away from the fixed points.

In this neighbourhood, $g_i = g_i^* + \delta g_i$ we can safely linearize the flow

$$\mu \frac{dg_i}{d\mu}\Big|_{g_j^*+\delta g_j} = A_{ij}\delta g_j + \mathcal{O}(\delta g_j^2) \tag{2.23}$$

which in a suitable diagonal basis $\{\delta g_i\}$ which is denoted $\{\sigma_i\}$

$$\mu \frac{d\sigma_i}{d\mu} = (\Delta_i - d)\sigma_i + \mathcal{O}(\sigma^2). \tag{2.24}$$

To linear order the RG flow is then

$$\sigma_i(\mu) = \left(\frac{\mu}{\mu'}\right)^{\Delta_i - d} \sigma_i(\mu'). \tag{2.25}$$

Δ_i is the *scaling* or *conformal* dimension of the operator associated to σ_i, this is not

[6] That is to say there are no massive particles and there is no mass-gap.

[7] A * will typically denote a process or object being evaluated at or in the neighbourhood of the fixed point.

the classical or engineering scaling dimension and the difference

$$\gamma_i = \Delta_i - d_i \tag{2.26}$$

is the anomalous dimension of the operator. This scaling dimension and classical scaling dimension will often differ. For CFTs the Green's functions are covariant under scale transformations which will provide important non-trivial constraints, take for example the two-point Green's functions $\langle\phi(x)\phi(0)\rangle$ which satisfies (2.15)

$$Z(\mu)^{-1}\langle\phi(x)\phi(0)\rangle_{g_i(\mu),\mu} = Z(\mu')^{-1}\langle\phi(x)\phi(0)\rangle_{g_i(\mu'),\mu'}. \tag{2.27}$$

At the fixed point g_i^* we have $Z(\mu) = (\mu'/\mu)^{2\gamma_\phi^*}\, Z(\mu')$ where $\gamma_\phi^* = \gamma_\phi(g_i^*)$. We find via dimensional analysis that

$$\langle\phi(x)\phi(0)\rangle_{g_i^*,\mu} = \mu^{2d_\phi}\mathcal{G}(x,\mu) \tag{2.28}$$

where d_ϕ is the classical scaling dimension of the field ϕ and $\mathcal{G}(x,\mu)$ contains the details of the structure. Upon substituting this into the RG equation (2.27) we find

$$\langle\phi(x)\phi(0)\rangle_{g_i^*,\mu} = \frac{c}{\mu^{2\gamma_\phi^*}x^{2d_\phi+2\gamma_\phi^*}} \propto \frac{1}{x^{2\Delta_\phi^*}} \tag{2.29}$$

where c is some constant. (2.25) then leads to the classification of couplings (and their associated operators) into three distinct groups based on this parameter Δ_ϕ^* in the neighbourhood of the fixed point.

The magnitude of Δ_i leads to these three groups:

- For couplings with $\Delta_i < d$ the flow diverges from the fixed point into the IR (as μ decreases), these couplings are knows as *relevant*
- Couplings with $\Delta_i > d$ the couplings flow into the fixed point and are known as *irrelevant*
- The final group is *marginal* where $\Delta_i = d$. In reality as we go beyond first order there will be slight deviations in the $\mathcal{O}(\sigma^2)$ part of (2.24) and the coupling is then said to be *marginally relevant* or *marginally irrelevant*. It is possible for a coupling to be *truly marginal*, for example in $N = 4$ super-symmetric Yang Mills theories, this implies that the original fixed point is in fact part of a line of fixed points[8]

As we approach the UV where the fixed point is said to exist we find that all particle masses decreases relative to the cut-off μ, in contrast to the behaviour in the IR discussed previously. A theory with a mass gap will approach the critical surface.

[8]In this thesis we will typically discuss only relevant or irrelevant couplings, with the higher order behaviour of marginal couplings assumed to be already resolved in any situation where it is pertinent to the discussion.

Regardless of whether or not the theory has a mass gap it will either diverge off to infinity for some finite μ or approach the fixed point lying on the critical surface as $\mu \to \infty$. If we restrict ourselves to the case of two irrelevant couplings and one relevant as we do in figure 2.1 we can see this structure more clearly. The critical surface is spanned by the irrelevant directions which flow in towards the fixed point and we have the single relevant direction flowing away from this fixed point towards the IR [10].

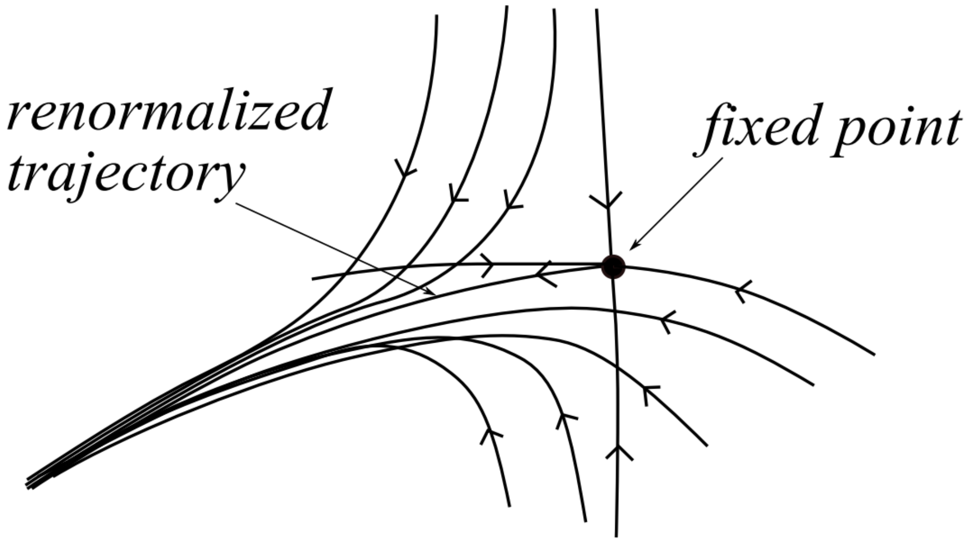

Figure 2.1: The theory space for two irrelevant couplings and one relevant, demonstrating the critical surface defined by the former as well as the renormalized trajectory, emanating from the fixed point. Reproduced from [10].

In this simpler case we also begin to see *universality*. Flows living off of the critical surface naturally focus around the *renormalized trajectory* [19, 20], the path for which there are only relevant couplings.

For a given space-time dimension d there is naturally a finite number of relevant couplings. Consider for example scalar field theory with $d = 4$ which satisfies $\phi \to -\phi$ symmetry, the only (marginally) relevant couplings (ignoring for now the kinetic term) are those associated to the ϕ^2 and ϕ^4 operators. In contrast to that we have the infinite tower of irrelevant couplings associated to ϕ^6, ϕ^8... and $\phi^n(\partial\phi)^m$ type operators. The number of relevant couplings is few and finite compared to the infinite number of irrelevant couplings. This universality is crucial to our fundamental understanding of UV complete QFTs. This means that the behaviour of theories in the IR is controlled by this small number of relevant couplings, hence this focusing effect, as opposed to the infinite number of all possible couplings. This small set of *universality classes* is governed by these relevant couplings. In essence it is this behaviour of the RG that leads to predictivity in QFT.

Finally we can define in a rigorous way the continuum limit of a QFT. As we have discussed, a well defined QFT should not be dependent on our arbitrary choice in μ, this should be taken to infinity whilst maintaining the physics, i.e. keeping the physics below μ the same. This is a very non-trivial issue. As we send $\mu \to \infty$ in practice this means that $g_i(\infty)$ is a fixed point of the RG, in the case that this is a non-trivial theory this is what Weinberg calls *asymptotic safety*. See section 2.5 and chapter 6 for more discussion on this approach to QG. The resulting flow through the theory space, $g_i(\mu)$, is this renormalized trajectory which is defined on all energy scales. As we can see in figure 2.1 the renormalized trajectory lies on the part of theory space where all irrelevant couplings are zero[9]. The intuition here may be that this is an unlikely scenario however universality renders finding such a theory possible.

A theory may be able to live within or near the universality class of the space such that $g_i(\infty)$ is within the domain of attraction of the UV CFT. As figure 2.2 [10] demonstrates when taking the UV limit in a careful way the theory will evolve with a non-zero irrelevant coupling and remain in this domain. The IR physics remains the same if the limit is taken properly, crucially the flow back towards the UV is defined by the number of relevant couplings in the IR. Being in the domain of attraction of the fixed point will mean that we eventually flow back into the fixed point where we have this continuum limit.

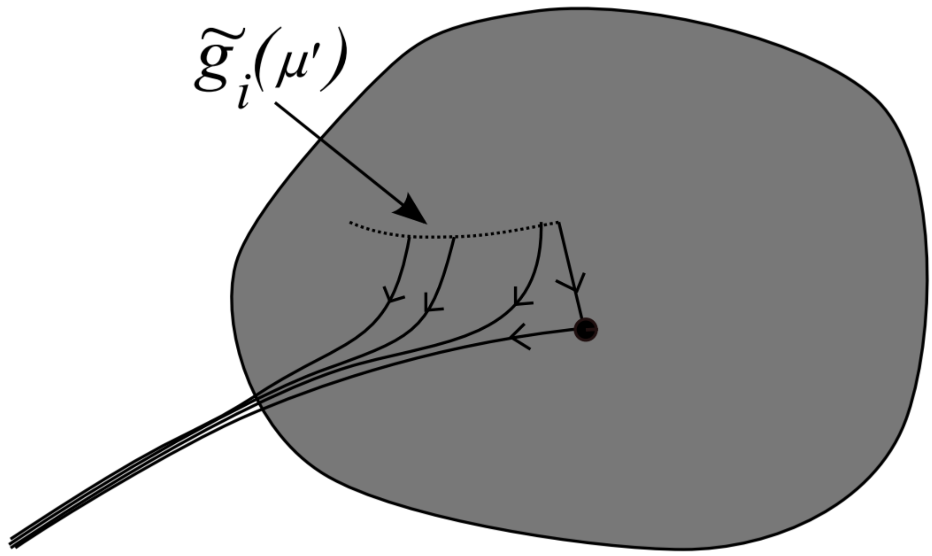

Figure 2.2: The continuum limit of a theory flowing in the domain of attraction of a UV CFT (the shaded area). Reproduced from [10].

The concepts of the RG are crucial to having a well defined QFT with a continuum limit, not only when we concern ourselves with quantum gravity but for all QFTs. It enables us to discuss seriously the ideas of predictivity and with only a finite number of relevant couplings it helps explain how we can construct UV complete theories.

[9] We note that the renormalized trajectory in this illustrative example is a one dimensional line however it will in general be some finite dimensional subspace.

2.4 Gravity as a non-renormalizable theory

Given the subject of this thesis it is worth elucidating in what way a continuum theory of QG fails when approached in a naïve way from the perspective of the Wilsonian RG as well as its perturbative non-renormalizability [21–24]. The failure to produce a well defined theory of QG with a continuum limit ultimately stems from our only coupling, the gravitational coupling κ, being irrelevant about the GFP

$$[\kappa] = -1. \tag{2.30}$$

When one constructs a UV complete theory we do with so with only relevant or marginally relevant couplings. In this way we can tune these parameters such that when we flow back into the UV the flow lands on the critical surface, this is the basis for attraction for the UV fixed point (when flowing from the UV to the IR) from which we construct the theory. As a result we then flow along the renormalized trajectory in a well defined theory. If our only non-zero coupling is irrelevant then when we flow back into the UV we flow away from this fixed point. Those theories that are defined on this critical surface have a non-trivial IR limit where only the massless fields remain, this may be acceptable for gravity but renders a suitable combination of gravity and the Standard Model (SM) of modern particle physics [25] impossible. Crucially also in gravity there are no relevant couplings which can be used to tune this flow such that it is in the neighbourhood of the renormalized trajectory and so eventually flows back into the neighbourhood of the UV fixed point.

We note however that having an infinite number of couplings does not necessarily imply that the theory has no predictive power [26–30]. In particular this is typically true in an effective theory containing negative mass-dimension couplings, which can nevertheless be renormalised order by order in perturbation theory at the expense of introducing new couplings at each new loop order [31]. Gomis and Weinberg showed that such theories can be renormalised in this effective sense, also in the case where gauge invariance (including diffeomorphism invariance) is involved. They did so using BRST methods [32].

As κ is a very fundamental part of gravity, at least in the naïve approach, this seems to put us at an impasse. It would seem then that a theory of QG necessitates either a wholly new approach entirely or a new interpretation of how to describe κ. In this thesis we will explore the latter in great depth however we first comment on approaches to understand QG in the former paradigm.

2.5 Approaches to quantum gravity

We now briefly comment on some of the more popular approaches to resolving this problem of marrying the fundamental ideas of QFT to gravity, we do this to give context to the work discussed in this thesis with respect to the wider field and give a brief glimpse into the state of the art of this research area. This discussion will also help illustrate how this novel approach to QG is far closer to the well understood framework of QFT employed with great success to understand the SM of physics compared to the different approaches being used in the investigation of QG. We will briefly outline the most significant aspects of these approaches and provide references for further reading.

We begin with Asymptotic Safety (AS) [33–40]. We outline this briefly here however the concepts of AS will be very relevant to chapter 6 where there is a more complete introduction. As outlined in section 2.3 a QFT is a dynamic object which transforms at different energy scales which, as one flows from the IR to the UV along the renormalized trajectory (if the theory is properly defined with relevant couplings/operators only), flows into the fixed point. Recall that the GFP is a special case where the fixed point is the free theory with all relevant couplings now zero as the theory is deep in the UV and all irrelevant couplings have already been fixed to be zero. The proposition of AS is that this is not the end of the story. It is possible for non trivial fixed points to exist whose β-functions are zero. Such a theory would be UV complete as one could take the continuum limit and the couplings and associated calculations would be finite. If the non-trivial fixed point was associated to gravity this would be a theory of QG with a well defined continuum limit, that is to say exactly what we are looking for.

Another popular approach to the construction of a theory of gravitons is that of string theory [41–43]. Like many topics in this introduction we could discuss this approach without end, instead we note the broad ideas and encourage the reader to discover this subject for themselves [44, 45]. String theory ultimately stems from treating the different elements of Nature (the fermions and bosons we are familiar with) not as effective point particles but instead as extended objects, initially as one dimensional strings and later to general n-dimensional branes in M-theory [46, 47]. Rather than a variety of different particles there would be instead one fundamental object, the string (and once the theory was more completely understood, in branes). At scales larger than the string its properties such as the mass, charge, spin etc would be dictated by the vibrational modes. Crucially this introduces a massless spin 2 mode which fulfils the role of the graviton. As a consequence string theory would be not only a theory of QG but also a promising candidate for a general complete theory-of-everything as it would include the other particles and forces such as Quantum Electro-Dyanmics (QED) and Quantum Chromo-Dynamics (QCD) one needs to completely describe Nature. Such a theory would also ideally include new physics that is necessary to explain outstanding problems such as dark matter [48], the Muon $g-2$ problem [49] etc . Although this seems tailor made to resolve some of the most interesting outstanding questions in theoretical physics this framework is not without its own problems. The calculations are often prohibitively difficult and the number of vacua which could describe our universe seem negligibly small compared to the total number of possible vacua, the so-called string theory landscape problem [50]. Despite this the field continues to be an exciting area of research for many physicists with many promising avenues.

One such avenue of particular prominence which we note here is the so-called Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [51–53]. This proposes a relationship between theories of gravity in a $d+1$ dimension AdS space and CFTs living on the d dimensional boundary. Crucially calculations that would be difficult to resolve on the gravity AdS side have an equivalent on the CFT side, the calculations here are often significantly more malleable. The correspondence is a strong-weak duality; a strongly coupled theory on one side (again where calculations may be prohibitively difficult) will correspond to a weakly coupled theory on the other side where powerful perturbative techniques can be used. Although not useful for our investigations into QG a consequence of this is that strongly coupled CFT problems can be resolved on the weakly coupled AdS gravity side of the correspondence.

The final approach to QG we discuss here is that of Loop Quantum Gravity (LQG) [54–58] which focuses on the geometric aspects of Einstein's GR and was originally a constrained system quantization of the EH action. In this way space and time are discretized in much the same way energy and spin are in quantum mechanics, from this space-time is itself described as a network of these fundamental 'atoms' of space-time which form large networks. In this way gravity is not described as a fundamental force in the traditional sense but instead as an artefact of interactions within this network. As a result of this granular approach to space-time there is a concept of a minimum length, distances below this have no physical meaning and so this helps resolve the problem of taking the continuum limit.

We have listed here only a handful of the more popular areas of research for finding a theory of QG, the number of approaches we have not been able to discuss here is vast. This great variety in the number of directions in this field of research illustrates that there is no general consensus in the community in which method is the most promising. This may be unfortunate for those of a more phenomenological persuasion who would like concrete predictions, which up to this point the field has been lacking. However, for those who are more interested in the formal construction of these theories there is a great deal to be excited about. With the surface of this vast field of research just barely scratched we now turn our attention to our novel approach to QG. This builds on the topics discussed in this introduction and combines them in a way which is inherently very natural and intuitive.

Chapter 3

The dilaton portal

It is clear following the previous discussions that QG is, as it stands, a problem without a satisfying resolution. In this section we will examine the structure of a theory different to previous attempts to find such a resolution to the QG problem which is however in many respects very similar to the current QFT techniques employed with great success. In the broadest terms this new approach stems from applying this ERG approach to QFT to the EH action, the final crucial ingredient is a more complete quantisation which is necessary when considering the wrong sign kinetic term for φ . When working in the RG we demand Euclidian signature such that $|x - y| \to 0$ as $x \to y$ which yields this famous negative kinetic term for φ (2.7). This issue is typically resolved in Minkowski signature[9] via continuing the functional integral for this sector to the imaginary axis $\varphi \to i\varphi$. We do not apply this ad hoc resolution to the *conformal factor instability* and instead keep it and look to resolve the instability via a more complete quantisation procedure which leads to an exciting new approach to QG.

We begin with the example of a scalar field theory with positive kinetic term, a physical situation that is more typically found in QFTs, to familiarise the reader with the mechanisms of this quantisation procedure. This process has always been present when creating a QFT however it is only due to the presence of this negative kinetic term that we must return to the fundamentals to examine the consequences of this change in sign. We follow this by applying this procedure to the conformal factor instability and this leads in turn to the so-called 'tower operator' which underpins much of the work of this thesis. This operator brings with it a range of physically interesting behaviour however the most paramount of this is the negative scaling dimension which would make it seem almost tailor made to the resolution of the non-renormalizability of gravity as outlined in section 2.4.

We then turn our attention to outlining BRST quantisation and the more powerful anti-field formalism that succeeds it, this will be necessary when gauge fixing our theory and will prove to be a powerful tool, signficantly reducing the number of operators that may be permitted with the new found freedom the novel tower operator permits. We then extend the treatment to the Legendre effective action which is more convenient for calculations and elaborate on how one reinstates diffeomorphism invariance in the physical limit.

3.1 Scalar field theory with positive kinetic term

We illustrate this more complete quantisation procedure with an example of a scalar field with a positive kinetic term, in particular we note that the eigenoperator spectrum is described entirely by a set of orthonormal polynomial interactions that one is more typically familiar with. We begin with an expression for the Wilsonian effective action for some scalar field φ (not to be confused with the dilaton field we describe in more detail in section 3.2 although both are presented here simply as scalar fields). We begin with the Wilsonian effective action [19, 59]

$$S^{\mathrm{tot},\Lambda}[\varphi] = S^{\Lambda}[\varphi] + \frac{1}{2}\varphi \cdot (\Delta^{\Lambda})^{-1} \cdot \varphi \tag{3.1}$$

where

$$\Delta^{\Lambda}(p) := \frac{C^{\Lambda}(p)}{p^2} \tag{3.2}$$

is the massless propagator which is regularised by a smooth UV cut-off profile $C^{\Lambda}(p) := C(p^2/\Lambda^2)$, see figure 3.1. We note for future reference that Δ^{Λ} will always be defined as positive as the above, particularly when considering the negative kinetic term. This UV cut-off profile behaves as follows, for $|p| < \Lambda$, $C^{\Lambda}(p) \approx 1$ and pre-dominantly leaves the modes unaffected, when $|p| > \Lambda$ the modes become suppressed. To ensure the flow is complete we demand that $C(p^2/\Lambda^2)$ is monotonically decreasing such that $C^{\Lambda}(p) \to 1$ for $|p|/\Lambda \to 0$ and for $|p|/\Lambda \to \infty$, $C^{\Lambda}(p) \to 0$ sufficiently fast to ensure all momentum integrals are regulated in the ultraviolet.

The interactions must satisfy the Wilson-Polchinski equation [19, 60]

$$\frac{\partial}{\partial \Lambda} S^{\Lambda}[\varphi] = \tfrac{1}{2} \frac{\delta S^{\Lambda}}{\delta \varphi} \cdot \frac{\partial \Delta^{\Lambda}}{\partial \Lambda} \cdot \frac{\delta S^{\Lambda}}{\delta \varphi} - \tfrac{1}{2} \mathrm{tr}\left[\frac{\partial \Delta^{\Lambda}}{\partial \Lambda} \cdot \frac{\delta^2 S^{\Lambda}}{\delta \varphi \delta \varphi} \right] \tag{3.3}$$

and this in turn is trivially solved by the GFP $S^{\Lambda}(\varphi) = 0$. We then find eigenoperators by linearising around this fixed point

$$\frac{\partial}{\partial \Lambda} \delta S^{\Lambda}[\varphi] = -\tfrac{1}{2} \mathrm{tr}\left[\frac{\partial \Delta^{\Lambda}}{\partial \Lambda} \cdot \frac{\delta^2}{\delta \varphi \delta \varphi} \right] \delta S^{\Lambda}[\varphi] \tag{3.4}$$

We define the scaled variables, denoted by a tilde, and noting the 'RG time' t,

$$x^{\alpha} = \tilde{x}^{\alpha}/\Lambda, \quad \varphi = \Lambda \tilde{\varphi}, \quad V = \Lambda^4 \tilde{V}, \quad t = \ln(\mu/\Lambda). \tag{3.5}$$

This RG time increases in the IR direction, we also re-introduce the arbitrary (although typically chosen at an energy scale we are interested in) energy scale μ. We define

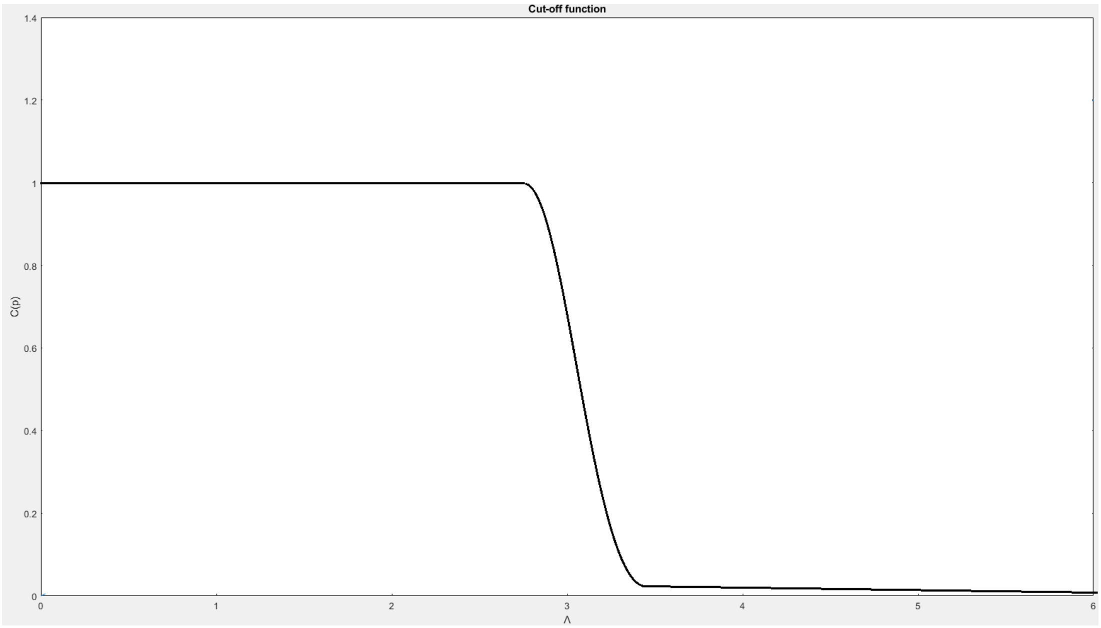

Figure 3.1: A schematic diagram of the UV cut-off function C^Λ, where modes are smoothly suppressed for some $|p| > \Lambda$. Note that $C^\Lambda(p)$ is simply another notation for $C(p^2/\Lambda^2)$, which emphasises that it is a UV cut-off. It should be contrasted with the associated IR cut-off $C_\Lambda(p) = 1 - C^\Lambda(p)$.

$$\delta S^\Lambda = \epsilon \int d^4x \; V(\varphi, t), \tag{3.6}$$

where ϵ is a small parameter. Note that if the linearised effective action $\delta S^\Lambda[\varphi]$ contains only such a potential interaction, this is preserved by the flow equation (3.4), since the only terms generated by the right hand side are tadpole interactions. Such integrals carry no external momentum dependence. Equivalently in position space, they do not generate derivative interactions. To see this in detail substitute (3.6) into the right hand side of (3.4):

$$\begin{aligned}
\mathrm{tr}\left[\frac{\partial \Delta^\Lambda}{\partial \Lambda} \cdot \frac{\delta^2}{\delta\varphi\delta\varphi}\right]\delta S^\Lambda[\varphi] &= \epsilon \int d^4y \; d^4z \; \frac{\partial \Delta^\Lambda(y,z)}{\partial \Lambda} \frac{\delta^2}{\delta\varphi(y)\delta\varphi(z)} \int d^4x \; V(\varphi(x), t) \\
&= \epsilon \int d^4y \; d^4z \; \frac{\partial \Delta^\Lambda(y,z)}{\partial \Lambda} \frac{\delta}{\delta\varphi(y)} V'(\varphi(z), t) \\
&= \epsilon \int d^4y \; d^4z \; \frac{\partial \Delta^\Lambda(y,z)}{\partial \Lambda} \delta(y-z) V''(\varphi(z), t) \\
&= \epsilon \int d^4y \; \frac{\partial \Delta^\Lambda(y,z)}{\partial \Lambda} V''(\varphi(y), t).
\end{aligned} \tag{3.7}$$

Now we note that the $\Delta^\Lambda(y, y)$ is independent of y and given by

$$\Omega_\Lambda := |\langle\varphi(y)\varphi(y)\rangle| = \int \frac{d^4p}{(2\pi)^4}\Delta^\Lambda(p). \tag{3.8}$$

We comment that $f'(\varphi)$ denotes taking the derivative with respect to the field φ for some quantity $f(\varphi)$ in the usual way. Thus substituting (3.7) into (3.4) we simply get

$$\frac{\partial}{\partial\Lambda}V(\varphi,t) = -\tfrac{1}{2}\frac{\partial\Omega_\Lambda}{\partial\Lambda}V''(\varphi,t). \tag{3.9}$$

In particular this step should not be confused with the so-called LPA (Local Potential Approximation) [61] where the restriction to potential interactions is enforced by truncation, i.e. as a result of a crude model approximation introduced by hand.

Eigenoperators are those with a well defined scaling dimension $4-\lambda$ and when expressed in these scaled variables take the form

$$\tilde{V}(\tilde{\varphi},t) = \left(\frac{\mu}{\Lambda}\right)^\lambda \tilde{V}(\tilde{\varphi}). \tag{3.10}$$

The pre-factor is the RG evolution of the scaled coupling $\tilde{g}_\lambda = \epsilon e^{\lambda t}$ at linearised order with the associated dimensionful coupling being $g_\lambda = \epsilon\mu^\lambda$ with operators being relevant, marginal or irrelevant if $\lambda > 0, \lambda = 0, \lambda < 0$ respectively as outlined in section 2.3.

Now we simply substitute $V(\varphi,t) = \Lambda^4\ \tilde{V}(\varphi/\Lambda,t)$ into (3.9) as implied by (3.5), and use (3.10) and (3.12) below, to get:

$$-\lambda\tilde{V}(\tilde{\varphi}) - \tilde{\varphi}\tilde{V}'(\tilde{\varphi}) + 4\tilde{V}(\tilde{\varphi}) = -\frac{\tilde{V}''}{2a^2}. \tag{3.11}$$

By dimensions on substituting $p = \Lambda\tilde{p}$ we see that

$$\Omega_\Lambda = \frac{\Lambda^2}{2a^2} \tag{3.12}$$

where a is a non-universal number dependant on the shape of $C(\tilde{p}^2)$ [41] and defined by

$$\frac{1}{2a^2} = \int \frac{d^4\tilde{p}}{(2\pi)^4}\frac{C(\tilde{p}^2)}{\tilde{p}^2}. \tag{3.13}$$

Note that our cut-off function is positive and must regulate. Thus the integral is positive and finite. This is also true on the left hand side if a is non-vanishing (since $a^2 > 0$ no matter the sign of a). For clarity we choose $a > 0$ from now on.

Let us emphasis again that this is an exact equation for the evolution of the potential interaction linearised around the GFP and has nothing to do with the local potential approximation. Nevertheless within the local potential approximation [61–63] versions of this equation have also appeared often in the literature, see for example references [64–67]

Critically equation (3.11) is of Sturm-Liouville type and its quantised solutions are

Hermite polynomials [68]

$$\mathcal{O}_n(\tilde{\varphi}) = H_n(a\tilde{\varphi})/(2a)^n = \tilde{\varphi}^n - n(n-1)\tilde{\varphi}^{n-2}/4a^2 + ... \tag{3.14}$$

where $\lambda = 4 - n$ and n is a non-negative number. To see this we simply note that the $H_n(x)$ are the quantised solutions of Hermite's (eigenvalue) equation:

$$u''(x) - 2xu'(x) + 2\lambda_H u(x) = 0, \tag{3.15}$$

with $\lambda_H = n$ (see e.g. [69]) This is the same as equation (3.11) if we identify $x = a\tilde{\varphi}$ and $\lambda_H = 4 - \lambda$. The scaling dimension of the operator $\mathcal{O}_n$ is therefore $4 - \lambda = n$ which coincides with the classical scaling dimension $[\varphi^n]$. The lower powers in (3.14) are there to correct for operator mixing as Λ is varied and appear with increasing powers of $\hbar$, they arise from tadpole corrections and are the only quantum corrections remaining at the linearised order.

From the Sturm-Liouville theory [68, 70] we know that the operators $\mathcal{O}_n$ form an orthonormal set :

$$\int_{-\infty}^{\infty} d\tilde{\varphi}\; e^{-a^2\tilde{\varphi}^2}\mathcal{O}_n(\tilde{\varphi})\mathcal{O}_m(\tilde{\varphi}) = \frac{1}{a}\left(\frac{1}{2a^2}\right)^n n!\sqrt{\pi}\delta_{nm} \tag{3.16}$$

which is complete in $\mathfrak{L}_+$, the natural space for Wilsonian interactions around a positive kinetic term. This is a Hilbert space of functions that are square integrable under the Sturm-Liouville measure $e^{-a^2\tilde{\varphi}^2}$ where we place emphasis on the minus sign. As a consequence of this we set the scaled couplings to

$$\tilde{g}_n = \frac{a}{\sqrt{\pi}}\frac{(2a^2)^n}{n!}\int_{-\infty}^{\infty} d\tilde{\varphi} e^{-a^2\tilde{\varphi}^2}\mathcal{O}_n(\tilde{\varphi})\tilde{V}(\tilde{\varphi}), \tag{3.17}$$

the norm-squared of the remainder vanishes as we extend to an infinite series

$$\int_{-\infty}^{\infty} d\tilde{\varphi}\; e^{-a^2\tilde{\varphi}^2}\left(\tilde{V}(\tilde{\varphi}) - \sum_{n=0}^{N}\tilde{g}_n\mathcal{O}_n(\tilde{\varphi})\right)^2 \to\; 0 \;\text{ as }\; N \;\to\; \infty. \tag{3.18}$$

Let us sketch how this follows from the Sturm-Liouville theory. The first step is to put (3.11) in Sturm-Liouville form

$$\tilde{H}\tilde{V}(\tilde{\varphi}) = \frac{d}{d\tilde{\varphi}}\left[p(\tilde{\varphi})\frac{d}{d\tilde{\varphi}}\tilde{V}(\tilde{\varphi})\right] + q(\tilde{\varphi})\tilde{V}(\tilde{\varphi}) = -\lambda\;\omega(\tilde{\varphi})\tilde{V}(\tilde{\varphi}). \tag{3.19}$$

By inspection this can be done by setting the Sturm-Liouville measure to $\omega(\tilde{\varphi}) = e^{-a^2\tilde{\varphi}^2}$, with the other functions proportional to this: $q = -4\omega$ and $p = -\omega/(2a^2)$. In this form it is clear by integration by parts that $\tilde{H}$ is self-adjoint:

$$\int d\tilde{\varphi}\mathcal{O}_1(\tilde{\varphi})\tilde{H}\mathcal{O}_2(\tilde{\varphi}) = \int d\tilde{\varphi}\tilde{H}\mathcal{O}_1(\tilde{\varphi})\mathcal{O}_2(\tilde{\varphi}) \tag{3.20}$$

provided that the perturbation falls off sufficiently fast at large field to drop the boundary terms at $\tilde{\varphi} = \pm\infty$. Let $\mathcal{O}_1(\tilde{\varphi})$ and $\mathcal{O}_2(\tilde{\varphi})$ be two such eigenoperator solutions with eigenvalues λ_1 and λ_2 respectively. Then from (3.19) and (3.20) we have

$$\lambda_2 \int d\tilde{\varphi}\omega(\tilde{\varphi})\mathcal{O}_1(\tilde{\varphi})\mathcal{O}_1(\tilde{\varphi}) = \lambda_1 \int d\tilde{\varphi}\omega(\tilde{\varphi})\mathcal{O}_1(\tilde{\varphi})\mathcal{O}_1(\tilde{\varphi}) \tag{3.21}$$

and this eigenoperators belonging to different eigenvalues are orthogonal under the Sturm-Liouville measure. The integrals make sense here only if the eigenoperators are square-integrable under ω, and this is enough also to ensure (3.20) holds. Given that they are square-integrable, they can also be normalised, and thus we see that the eigenoperators can be chosen to form an orthonormal set. The proof of equation (3.18) is more involved. To do this one uses the Green's function spectrally expanded over the eigenoperators and forms the integral of a square of the solution. For details we refer the reader to the literature [68, 70].

In this sense all perturbations in $\mathfrak{L}_+$ are described by a countable infinity of couplings $\tilde{g}_n$ and their RG evolution is simply given by the RG of these couplings. We form the bare action at some initial energy scale $\Lambda = \Lambda_0$, this is an initial condition for our flow equation (3.3) and we must also choose the bare couplings at this scale $\tilde{g}_0^\lambda := \tilde{g}^\lambda(\Lambda_0)$. We must first define the bare irrelevant couplings. The simplest choice would be to set them all to zero, the more general choice is to set them to some finite non-zero value. The relevant couplings have less freedom as, as in standard in the Wilsonian approach, the structure demands they vanish in the limit $\Lambda_0 \to \infty$. In particular at the linearised level $\tilde{g}_0^\lambda = g^\lambda \Lambda_0^{-\lambda}$ where g^λ is a fixed finite dimension-λ coupling if $\lambda > 0$. It is important to note that as $\Lambda_0 \to \infty$ the linearised approximation for the relevant couplings becomes more and more valid at scales close to the bare scale.

The effective action (3.1) in this way provides us with the bare action and we can therefore study the evolution away from the latter case. We must then begin to construct the apparatus to investigate physical quantities, we do so by first replacing the cut-off C^Λ in (3.2) with a new cut-off

$$C_k^{\Lambda_0}(p) = C^{\Lambda_0}(p) - C^k(p) \tag{3.22}$$

which is regulated both in the UV by Λ_0 and in the IR at a scale k. With this IR cut-off we can then write the more useful Legendre effective action as

$$\Gamma_k^{\mathrm{tot},\Lambda_0}[\varphi] = \Gamma_k^{\Lambda_0}[\varphi] + \frac{1}{2}\varphi \cdot \left(\Delta_k^{\Lambda_0}\right)^{-1} \cdot \varphi \tag{3.23}$$

where

$$\Delta_k^{\Lambda_0} = \Delta^{\Lambda_0} - \Delta^k \tag{3.24}$$

where we note that, up to discarding a field independent part on the right hand side, we have the identity

$$\Gamma^{\Lambda_0}_{\Lambda_0}[\varphi] = S^{\Lambda_0}[\varphi]. \tag{3.25}$$

This provides an initial condition when considering the flow under an IR cut-off, this takes the form

$$\frac{\partial}{\partial k}\Gamma^{\Lambda_0}_k[\varphi] = -\frac{1}{2}\mathrm{tr}\left[\left(1+\Delta^{\Lambda_0}_k\cdot\frac{\delta^2\Gamma^{\Lambda_0}_k}{\delta\varphi\delta\varphi}\right)^{-1}\frac{1}{\Delta^{\Lambda_0}_k}\frac{\partial\Delta^{\Lambda_0}_k}{\partial k}\right] \tag{3.26}$$

where we note that at the GFP the Legendre effective action is simply the field independent part $\Gamma^{\Lambda_0}_k[\varphi] = -\frac{1}{2}\mathrm{tr}\ \ln\Delta^{\Lambda_0}_k$. At the linearised level we have

$$\frac{\partial}{\partial k}\delta\Gamma^{\Lambda_0}_k[\varphi] = -\frac{1}{2}\mathrm{tr}\left[\frac{\partial\Delta^k}{\partial k}\cdot\frac{\delta^2}{\delta\varphi\delta\varphi}\right]\delta\Gamma^{\Lambda_0}_k[\varphi]. \tag{3.27}$$

Now notice that this equation is in fact identical to (3.4), with k now playing the role of the UV cut-off. To get some intuition for why this is so, note that at the linearised level the flow equation becomes insensitive to the UV cut-off Λ_0 and so we can send this to infinity for free. We then note that $\Gamma_\Lambda := \Gamma^\infty_\Lambda$ is related to S^Λ by a Legendre transform [20, 59, 71–73] and now carries the purely quantum one particle irreducible parts of S^Λ. At the linearised level there are only the quantum corrections to consider and so the flow equations coincide. We can make this expression more explicit by defining

$$\delta\Gamma^{\Lambda_0}_k[\varphi] = \epsilon\int d^4x\ V(\varphi(x),k) \tag{3.28}$$

where ϵ denotes a small quantity, this interaction potential will satisfy the eigenoperator equation (3.11) with k replacing Λ in (3.5) and (3.10).

We are free to perturb this solution by adding $g_n\mathcal{O}^{(n)}_{\Lambda_0}(\varphi)$ to the bare action at $k=\Lambda=\Lambda_0$. By inspection we see that in scaled units at the linear order these operators will evolve in a self similar way, in particular

$$\left(\frac{\Lambda_0}{k}\right)^{4-n}\tilde{g}_n(\Lambda_0) = \frac{g_n}{k^{4-n}} = \tilde{g}_n(k) \tag{3.29}$$

and using (3.5) the dimensionful interaction is

$$g_n\mathcal{O}^{(n)}_k(\varphi) = k^4\frac{g_n}{k^{4-n}}\mathcal{O}_n(\varphi/k) = g_n\left(\varphi^n - n(n-1)\frac{k^2}{4a^2}\varphi^{n-2}+...\right) \tag{3.30}$$

i.e.

$$\mathcal{O}^{(n)}_\Lambda(\varphi) = \Lambda^n\mathcal{O}_n(\varphi/\Lambda) = \varphi^n - n(n-1)\frac{\Lambda^2}{4a^2}\varphi^{n-2}+... \tag{3.31}$$

We return the 'physical limit' when $k\to 0$, where we find the universal physical interaction as it appears in the Legendre effective action, namely $\mathcal{O}^{(n)}(\varphi) := \lim_{k\to 0}\mathcal{O}^{(n)}_k(\varphi)$, i.e.

$$g_n \mathcal{O}^{(n)}(\varphi) = g_n \varphi^n \tag{3.32}$$

We can see that for relevant operators this is finite and g_n corresponds to the physical coupling whereas in the UV limit $\Lambda_0 \to \infty$ the irrelevant ones will tend to zero.

Before proceeding to the case of the negative kinetic term we make some final remarks about the positive case. Firstly we note that these arguments can be extended to operators with space-time derivatives, beyond linearised order and investigate further the marginal operators. To do so here would be to mask the key characteristics of this structure and we do not go into further detail here, this standard knowledge is available in the references listed thus far in this section. We can also comment on the non-polynomial solutions to (3.11), these can not be understood using Feynman diagrams as they rely on non-perturbative physics. To summarise these results briefly for the sake of clarity such solutions will not survive as they lie outside of $\mathfrak{L}_+$ and in the large field regime the linearised approximation breaks down. We will find that in the negative kinetic term case these failings of the non-polynomial solutions will have a much different fate, it is only these which can satisfy the eigenoperator equation.

3.2 The tower operator

We now turn our attention to the ramifications of the change in sign of the kinetic term, extending on the analysis of the previous section. We must first generalise expressions such as the flow equation and the Wilsonian effective action. This would initially seem unphysical as the functional integral in the partition function will no longer converge and the momentum cut-off profile will also exacerbate the problem. However, if we are to understand gravity in Wilsonian terms whilst maintaining the conformal instability Nature demands this generalisation. We will see that there are routes out of this seemingly catastrophic impasse.

We begin by replacing (3.1) and (3.23), respectively, with

$$S^{\text{tot},\Lambda}[\varphi] = S^\Lambda[\varphi] - \frac{1}{2}\varphi \cdot (\Delta^\Lambda)^{-1} \cdot \varphi \tag{3.33}$$

$$\Gamma_k^{\text{tot},\Lambda_0}[\varphi] = \Gamma_k^{\Lambda_0}[\varphi] - \frac{1}{2}\varphi \cdot \left(\Delta_k^{\Lambda_0}\right)^{-1} \cdot \varphi. \tag{3.34}$$

We have effectively made the transformation $\Delta \to -\Delta$ and so the flow equations become

$$\frac{\partial}{\partial \Lambda} S^\Lambda[\varphi] = -\frac{1}{2}\frac{\delta S^\Lambda}{\delta \varphi} \cdot \frac{\partial \Delta^\Lambda}{\partial \Lambda} \cdot \frac{\delta S^\Lambda}{\delta \varphi} + \frac{1}{2}\text{tr}\left[\frac{\partial \Delta^\Lambda}{\partial \Lambda} \cdot \frac{\delta^2 S^\Lambda}{\delta\varphi\delta\varphi}\right] \tag{3.35}$$

$$\frac{\partial}{\partial k}\Gamma_k^{\Lambda_0}[\varphi] = -\frac{1}{2}\text{tr}\left[\left(1 - \Delta_k^{\Lambda_0} \cdot \frac{\delta^2 \Gamma_k^{\Lambda_0}}{\delta\varphi\delta\varphi}\right)^{-1} \frac{1}{\Delta_k^{\Lambda_0}}\frac{\partial \Delta_k^{\Lambda_0}}{\partial k}\right] \tag{3.36}$$

$$\frac{\partial}{\partial k}\delta\Gamma_k^{\Lambda_0}[\varphi] = -\frac{1}{2}\mathrm{tr}\left[\frac{\partial\Delta_k^{\Lambda_0}}{\partial k}\cdot\frac{\delta^2}{\delta\varphi\delta\varphi}\right]\delta\Gamma_k^{\Lambda_0}[\varphi] \tag{3.37}$$

It is noted here that this change in sign makes the equations backwards-parabolic which in-turn means the Cauchy Initial Value Problem (CIVP) for flow towards the IR is not well posed, this will have significant consequences which will be discussed in chapter 6. Before this we discuss the above equations and other consequences of this change in sign in more detail, first considering non-derivative interactions at the linear level.

3.2.1 Non-derivative eigenoperators

The linearised flow for the potential is given by

$$\partial_t V(\varphi, t) = -\Omega_\Lambda V''(\varphi, t) \tag{3.38}$$

and can be recast as a heat diffusion equation with a 'time' $T = \Lambda^2$ which naturally runs towards the UV

$$\frac{\partial}{\partial T}V(\varphi, T) = \frac{1}{4a^2}V''(\varphi, T). \tag{3.39}$$

This notion of a 'natural direction' for the flow of these heat equations will continue to be raised several times in this thesis, particularly when constructing non-perturbative arguments in chapter 5 and the effects of it in the findings of chapter 6 [74–76]. Crucially, for a general initial potential $V(\varphi, T)$ a well defined flow exists only in this natural direction, in this case towards the UV. This contrasts to the natural direction of the flow of $h_{\mu\nu}$ towards the IR as a consequence of its positive kinetic term that is more typical in the Wilsonian point of view. Continuing the flow in the 'unnatural' direction *almost* always results in an incomplete, unphysical flow which ends in a singularity however this is not guaranteed for all values of the initial potential $V(\varphi, T)$. Flows in the natural direction, here towards the UV, diffuse out (one can compare this to heat diffusion in the heat equation as the potential smooths out over time) with unnatural flows developing a singularity at some critical time $T = T_p := a^2\Lambda_p^2 > 0$ where a is some non-universal number. We will return to this object Λ_σ in section 3.3 where it will play a significant role in the implementation of diffeomorphism invariance.

Let us expand further on this point to show it also mathematically. The heat equation (3.39) has a Green's function

$$G(\varphi - \varphi_0, T) = \frac{a}{\sqrt{\pi T}}exp\left(-\frac{a^2(\varphi-\varphi_0)^2}{T}\right). \tag{3.40}$$

As can be verified by direct substitution, $V(\varphi, T) = G(\varphi - \varphi_0, T)$ is a solution for all $T > 0$. Since (3.39) is a linear equation we know that therefore also

$$V(\varphi,T)=\int_{-\infty}^{\infty}d\varphi_0\;G(\varphi-\varphi_0,T)\;V_0(\varphi_0), \tag{3.41}$$

is a solution to (3.39) for all $T>0$ and for any $V_0(\varphi_0)$ for which the integral converges. On the other hand one readily confirms that

$$\int_{-\infty}^{\infty}d\varphi_0\;G(\varphi-\varphi_0,T)=1. \tag{3.42}$$

Since $G\to 0$ for all $\varphi-\varphi_0\neq 0$ as $T\to 0^+$ (i.e. so that T is kept positive while taking the limit), it follows that

$$\lim_{T\to 0^+}G(\varphi-\varphi_0,T)=\delta(\varphi-\varphi_0). \tag{3.43}$$

Then we see that (3.41) is a general solution of the heat equation such that the initial heat distribution is $V(\varphi,0)=V_0(\varphi)$. We have thus shown that for any initial heat distribution (that does not grow so fast as to invalidate (3.41)), a solution is guaranteed to exist for all later times $T>0$.
The proof breaks down for flows backwards in time. Not only is the Green's function imaginary for negative T but its exponential is such that it diverges *for all* $\varphi\neq\varphi_0$ as $T\to 0^-$. This reflects the physics of heat flow as we pointed out above: flows in the 'unnatural' direction almost always end in a singularity. That singularity is clearly visible in the behaviour of (3.43). If we insist on trying to push the general solution (3.41) into negative T, i.e. to analytically continue it into the negative T region, it will then in general diverge *everywhere* as T passes through $T=0$ and then becomes complex. This behaviour is clearly unphysical.

Another example is provided by

$$V(\varphi,T)=G(\varphi-\varphi_0,T+T_0). \tag{3.44}$$

This is a solution of the heat equation for all $T>-T_0$. If we set $T_0>0$ then we can flow backwards in time for a finite time, but then we hit the singularity at $T=-T_0$. Using this in place of the Green's function in (3.41), we can construct general solutions which work for all later times but cease to make sense if we try to go earlier than the negative time $T=-T_0$. We will prove in sec. 3.2.2. that this is an inevitable feature: all solutions become complex once they pass through the singularity. So this singular point is a genuine end of the flow for physical (and thus real) solutions.

Given this reversal of the natural direction it would be anticipated that universality would be found in the UV around the GFP rather than in the IR. We will find that the GFP in fact supports eigenoperators of arbitrarily high relevancy, replacing the role of the usual hierarchy of irrelevant operators and that render the theory non-predictive. There will however be significant restrictions on these operators once diffeomorphism invariance has been implemented as well as further constraints we will discuss in chapter

5.

We must now scale our variables to realise the Wilsonian RG, this gives

$$\Lambda\frac{\partial}{\partial\Lambda}\tilde{V}_\Lambda(\tilde{\varphi}) - \tilde{\varphi}\tilde{V}'_\Lambda(\tilde{\varphi}) + 4\tilde{V}_\Lambda(\tilde{\varphi}) = \tilde{V}''_\Lambda(\varphi)/(2a^2) \tag{3.45}$$

and setting $\tilde{V}_\Lambda(\tilde{\varphi}) = e^{\lambda t}\tilde{V}(\tilde{\varphi})$ yields the eigenoperator equation (3.11) except with a crucial plus sign on the right hand side

$$-\lambda\tilde{V}(\tilde{\varphi}) - \tilde{\varphi}\tilde{V}' + 4\tilde{V} = \frac{\tilde{V}''}{2a^2}. \tag{3.46}$$

As a result of this change in sign between the $\tilde{\varphi}\tilde{V}'$ and $\tilde{V}''$ terms for large values of the field there will no longer be exponentially growing solutions. Indeed, keeping only these terms the equation is exactly soluble. The solution is

$$\pm\,\tilde{\varphi}\tilde{V}' = \frac{\tilde{V}}{2a^2} \qquad \Longrightarrow \qquad \tilde{V} = Ae^{\pm a^2\tilde{\varphi}^2} + B, \tag{3.47}$$

where A and B are the integration constants. We see explicitly the exponentially growing solution has turned into an exponentially decaying solution. This analysis is valid if the derivative terms are the most important terms and by inspection for this solution it is indeed true that $\tilde{V}' \gg \tilde{V}$ and $\tilde{V}'' \gg \tilde{V}$ for large $\tilde{\varphi}$. The remaining solution is one in which the derivatives are the least important terms so that now one can neglect $\tilde{V}''$. Doing this the solution is the leading term in (3.48). Again one can verify the assumed behaviour: $\tilde{V}'' \ll \tilde{\varphi}\tilde{V}' \sim \tilde{V}$ for large $\tilde{\varphi}$. Thus, instead the solutions will behave, at worst, as

$$\tilde{V} \propto \tilde{\varphi}^{4-\lambda} + \frac{(4-\lambda)(3-\lambda)}{4a^2}\tilde{\varphi}^{2-\lambda} + O(\tilde{\varphi}^{-\lambda}) \tag{3.48}$$

which is generically an asymptotic series which is also subject to exponentially decaying solutions $\tilde{\varphi}^{\lambda-5}e^{-a^2\tilde{\varphi}^2}$. For $\lambda > 2$ the solutions justify linearisation of the right hand side of (3.36) and are not ruled out by large field analysis however for $\lambda < 2$ the mean field analysis will permit these perturbations since it simply returns the correct multiplicative evolution i.e. $(\Lambda_0/k)^\lambda\ \tilde{V}$. To summarise the large field test rules out none of these solutions.

The general solution of this equation is in terms of a linear combination of two Kummer functions [77]:

$$\tilde{V} = C_1\tilde{\varphi}M\left(\frac{\lambda}{2} - \frac{3}{2}, \frac{3}{2}, -a^2\tilde{\varphi}^2\right) + C_2M\left(\frac{\lambda}{2} - 2, \tfrac{1}{2}, -a^2\tilde{\varphi}^2\right), \tag{3.49}$$

in terms of the Kummer M-function [78] and constants C_i, the first function being odd in $\tilde{\varphi}$ and the second one even in $\tilde{\varphi}$. Here one can arrange for zero coefficient for the asymptotic series in (3.48) on one side of the equation with $\tilde{\varphi} \to \pm\infty$, leaving behind the exponentially decaying corrections. On the other side there will be $\tilde{\varphi} \to \mp\infty$ (note the opposite sign), it will therefore have (3.48) as its asymptotic behaviour. For λ an integer one of the two Kummer functions will degenerate and produce two discrete spectra. At $\lambda = 4 - n$ there are polynomial solutions $\mathcal{O}_n(\tilde{\varphi}) = H_n(ia\varphi)/(2ia)^n$ and for $\lambda = 5 + n$ we see the first glimpse of our novel tower operator. This change in value for λ will seed the solution for the irrelevancy problem in QG. This is an infinite tower of exponentially decaying 'super-relevant' eigenoperators

$$\delta_n(\tilde{\varphi}) := \frac{a}{\sqrt{\pi}} \frac{\partial^n}{\partial \tilde{\varphi}^n} e^{-a^2\tilde{\varphi}^2} = \frac{a}{\sqrt{\pi}} (-a)^n H_n(a\tilde{\varphi}) e^{-a^2\tilde{\varphi}^2}, \qquad \lambda = 5 + n \tag{3.50}$$

where n is a non-negative integer and the dimension of this operator is thus

$$[\delta_n] = 4 - \lambda = -1 - n. \tag{3.51}$$

Such solutions existed for the positive kinetic term case, see (3.11), however they were exponentially growing and thus in the large field limit they did not evolve correctly as was briefly discussed. As we will see shortly the presence of this tower operator is justified further as a consequence of the quantisation process.

The second expression for the tower operator (3.50) follows from the substitution $\tilde{V} \to \tilde{V} e^{-a^2\tilde{\varphi}^2}$ into (3.46) and comparing it to (3.11). The first is found via substituting the Fourier transform

$$\tilde{V}(\tilde{\varphi}) = \int_{-\infty}^{\infty} \frac{d\tilde{\pi}}{2\pi} \tilde{\mathcal{V}}(\tilde{\pi}) e^{i\tilde{\pi}\tilde{\varphi}} \tag{3.52}$$

where $\tilde{\pi} = \pi\Lambda$ is the scaled conjugate momentum, this gives the general solution

$$\tilde{\mathcal{V}}(\tilde{\pi}) = (i\tilde{\pi})^{\lambda-5} \exp(-\frac{\tilde{\pi}^2}{4a^2}). \tag{3.53}$$

This has power law asymptotics (3.48), generated by the singularity at $\tilde{\pi} = 0$ except where the singularity is absent when $\lambda = 5 + n$ where it gives (3.50).

We note that (3.46) is still of Sturm-Liouville type however now the Sturm-Liouville weight function is now $e^{+a^2\tilde{\varphi}^2}$ where we have placed extra emphasis on the positive sign of the exponent which contrasts that of the weight function in the standard positive kinetic term case. We define $\mathfrak{L}_-$ to be the space of square integrable functions under this measure and comment that the polynomials and the continuous spectrum of Kummer functions lie outside this space. The exponentially decaying solutions lie within this space $\mathfrak{L}_-$ and form a complete orthonormal basis for this Hilbert space:

$$\int_{-\infty}^{\infty} d\tilde{\varphi} e^{a^2\tilde{\varphi}^2} \delta_n(\tilde{\varphi}) \delta_m(\tilde{\varphi}) = \frac{a}{\sqrt{\pi}} (2a^2)^n \ n! \ \delta_{nm} \tag{3.54}$$

and using the second equation in (3.50) so that if $\tilde{V}(\tilde{\varphi}) \in \mathfrak{L}_-$ and

$$\tilde{g}_n = \frac{\sqrt{\pi}}{2^n a^{2n+1} n!} \int_{-\infty}^{\infty} d\tilde{\varphi} \ e^{a^2\tilde{\varphi}^2} \delta_n(\tilde{\varphi}) \tilde{V}(\tilde{\varphi}) \tag{3.55}$$

the norm-squared of the remainder vanishes as we extend to the infinite series:

$$\int_{-\infty}^{\infty} d\tilde{\varphi}\ e^{a^2\tilde{\varphi}^2}\left(\tilde{V}(\tilde{\varphi}) - \sum_{n=0}^{\infty}\tilde{g}_n\delta_n(\tilde{\varphi})\right)^2 \to 0 \quad \text{as} \quad N\to\infty. \tag{3.56}$$

3.2.2 Quantisation condition

We now consider the definition of this Hilbert space $\mathfrak{L}_-$ and operators that lie within it or on its boundary as a consequence of this change in sign of the exponent of the Sturm-Liouville weight function. We do not necessarily have to exclude solutions outside of $\mathfrak{L}_-$ by their large field RG properties however we can exclude them by choice and we will also find in any case that in the physical limit they are poorly defined. We are therefore demanding that any interactions lie within $\mathfrak{L}_-$. This is our quantisation condition.

If we consider a finite sum of the basis operators (3.50) then this quantisation condition is clearly respected by the RG at the linear level since the operators evolve multiplicitly . At some initial bare scale $\Lambda = \Lambda_0$, $\delta_n(\tilde{\varphi})$ appears with a sufficiently small coupling $\tilde{g}_n = g_n/\Lambda_0^{5+n}$ and then at some other scale it will be $\tilde{g}_n = g_n/\Lambda^{5+n}$ where g_n is held fixed. We can then switch on an infinite number of couplings and by the quantisation condition we require

$$\tilde{V}_{\Lambda_0}(\tilde{\varphi}) = \sum_{n=0}^{\infty}\tilde{g}_n\delta_n(\tilde{\varphi}) \in \mathfrak{L}_-. \tag{3.57}$$

As before if $\tilde{V}$ is small enough to trust the linear RG then at another scale Λ we simply replace Λ_0 in equation (3.57) with Λ i.e.

$$\tilde{V}_{\Lambda}(\tilde{\varphi}) = \sum_{n=0}^{\infty}\tilde{g}_n\delta_n(\tilde{\varphi}) \in \mathfrak{L}_-. \tag{3.58}$$

We can then use (3.54) to calculate the norm-squared of the evolved potential

$$\int_{-\infty}^{\infty} d\tilde{\varphi}\ e^{a^2\tilde{\varphi}^2}\tilde{V}_{\Lambda}^2(\tilde{\varphi}) = \frac{a}{\Lambda^{10}\sqrt{\pi}}\sum_{n=0}^{\infty} n!\ g_n^2\left(\frac{2a^2}{\Lambda^2}\right)^n, \tag{3.59}$$

by (3.57) we note that series on the right hand side converges for $\Lambda = \Lambda_0$. Crucially we thus see that $\tilde{V}_\Lambda(\tilde{\varphi}) \in \mathfrak{L}_-$ and remains small for all $\Lambda \geq \Lambda_0$. This underpins why we interpret the quantisation condition $\tilde{V}_\Lambda(\tilde{\varphi}) \in \mathfrak{L}_-$ as operating at the bare level. We will find that all the couplings g_n are relevant as a consequence of the super-relevant tower operator and so we set them to be finite at some physical scale and they will therefore parametrise the most general RG trajectory.

These properties ensure that the Wilsonian effective action continues to satisfy the quantisation condition along the entire flow all the way up to $\Lambda \to \infty$. As one would expect in the Wilsonian framework we also find that $\tilde{V}_\Lambda(\tilde{\varphi}) \to 0$ in this continuum limit i.e. it emanates from the GFP and then describes the RG trajectory. Since we can regard this as the continuum limit then we would consider this to be an ultimately fundamental theory, that is to say there is no need to consider this as an effective theory for some microscopic theory. With that being said if the possibility of such a microscopic theory were considered this quantisation condition could shed some light on the form of such a theory.

It is also found that for a generic case the g_n will result in (3.59) having a finite radius of convergence $1/\Lambda = 1/(a\Lambda_\sigma)$ where, by (3.57), $\Lambda_\sigma \leq \Lambda_0$. Then the potential has flowed out of this Hilbert space i.e. $\tilde{V}_\Lambda(\tilde{\varphi}) \notin \mathfrak{L}_-$ for all $\Lambda < a\Lambda_\sigma$. This amplitude suppression scale Λ_σ will be crucial in the implementation of diffeomorphism invariance and a return to standard results in the latter stages of this thesis. In any case once $\tilde{V}_\Lambda(\tilde{\varphi}) \notin \mathfrak{L}_-$ the expansion over the basis (3.50) no longer converges and we no longer have a well defined operator.

There are two possible reasons for why $\tilde{V}_\Lambda(\tilde{\varphi})$ fail and exit $\mathfrak{L}_-$, either $\tilde{V}_\Lambda(\tilde{\varphi})$ has developed its own divergences or it grows too fast such that $\tilde{\varphi}$ the integral in (3.59) no longer converges as $\tilde{\varphi} \to \pm\infty$. In the former case this is because the flow ceases to exist, this can be recognised via a comparison to the reverse heat equation where singularities will cause the flow to end prematurely. The singularity is fatal to the solution: it does *not* exist as a real solution below this critical value $\Lambda = a\Lambda_\sigma$. To see this note that we have just excluded the case where $\tilde{V}_\Lambda(\tilde{\varphi})$ grows too fast. Working with the flow equation written in the form (3.39) we therefore know that

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty} d\varphi \left[V(\varphi,T)\right]^2 = -\frac{1}{2a^2}\int_{-\infty}^{\infty} d\varphi \, \left[V'(\varphi,T)\right]^2, \tag{3.60}$$

Where the right hand side follows by applying (3.39) and integrating by parts using the fact that the boundary terms vanish. This is true since we know from (3.59) that for this solution V vanishes exponentially fast for large φ. If V is real, the right hand side is negative since $(V')^2 \geq 0$. That means as we lower T, the integral on the left hand side must increase. This is consistent with the fact that as T decreases towards the critical scale value $T = T_c = a^2\Lambda_\sigma^2$, the integrals (on both sides) diverge. If V stays real for T below T_c, we see that V must remain singular since the integral of V^2 is already infinite. The only way we can get a finite solution for V once T passes below the critical value is if the right hand side now contributes an infinitely positive part. But that is only possible if V is both complex and divergent.

In the latter case where the integral in (3.59) no longer converges the evolution can still be described by the appropriate flow equation (3.45) or in more general terms (3.35) and (3.36). The flow is first order in Λ and it can be uniquely determined by supplying as boundary conditions the expansion over the basis for any $\Lambda > a\Lambda_\sigma$. At a formal level we can continue to write $\tilde{V}_\Lambda(\tilde{\varphi})$ as an expansion over the basis, even for $\Lambda < a\Lambda_\sigma$. At the linearised level it will continue to be (3.58) since each term sepearately satisfies (3.45). However to achieve this we will need a prescription for resumming this series, to do this we will work in conjugate momentum space. This will be a technique we use on several occasions.

With this quantisation condition understood we can now examine the form of this novel super-relevant tower operator and some of its interesting features. Analogously to the procedure in section 3.1 we identify the dimensionful bare operator $\delta^{(n)}_{\Lambda_0}(\varphi)$ as the conjugate to the dimension $5+n$ unscaled coupling g_n in the bare action. Thus, either directly from its dimension (3.51) or by re-expressing the coupling and re-scaling,

$$\delta^{(n)}_{\Lambda_0}(\varphi) = \delta_n(\varphi/\Lambda_0)/\Lambda_0^{1+n} \tag{3.61}$$

and hence, using $a = \Lambda_0/\sqrt{2\Omega_{\Lambda_0}}$:

$$\delta^{(n)}_{\Lambda_0}(\varphi) := \frac{\partial^n}{\partial\varphi^n}\delta^{(n)}_{\Lambda_0}(\varphi), \quad \text{where } \delta^{(n)}_{\Lambda_0}(\varphi) = \frac{1}{\sqrt{2\pi\Omega_{\Lambda_0}}}\exp\left(-\frac{\varphi^2}{2\Omega_{\Lambda_0}}\right). \tag{3.62}$$

We note now that if one restores $\hbar$ it appears as $\Omega_{\Lambda_0} \propto \hbar\Lambda_0^2$, as a result these operators are said to be *evanescent* in the sense that for a fixed field φ the operators vanish as we reach the continuum limit. Strikingly one can also note that they are non-perturbative in $\hbar$ and appear to have a similar form to instanton [79, 80] or renormalon [81] contributions. Another novel aspect of these operators is that as a consequence of this they are inherently quantum, there is no classical limit that one can take; this suggests that in this framework the classical and quantum aspects of gravity are entirely independent of one another. One may be concerned that these operators are non-perturbative in $\hbar$. One may assume that the problems of being non-perturbative in κ are simply traded in for equally difficult and seemingly rigid problems with this new non-perturbative in $\hbar$ behaviour however once the implementation of diffeomorphism invariance is addressed we will see that such concerns are alleviated, albeit at the cost of a new perspective of the definition of a continuum theory and the summation of Feynman diagrams.

As a consequence of this construction $V(\varphi) = \delta^{(n)}_{\Lambda}(\varphi)$ is a solution of the unscaled flow equations (3.38), we can generalise this solution to the linearised RG as the sum of these with constant, *relevant*, coeffecients g_n

$$V(\varphi,\Lambda) = \sum_{n=0}^{\infty} g_n\delta^{(n)}_{\Lambda}(\varphi). \tag{3.63}$$

This is simply (3.57) in dimensionful terms. We have, by (3.57), found that this will converge for all $\Lambda \geq \Lambda_0$ and this will be true even for an infinite number of non-zero couplings. This general potential will inherit the properties discussed above; it is non-perturbative in $\hbar$ and is also evanescent. We note that this is distinct from the usual 'relevancy' property that the potential tends to zero in the continuum limit.

When we define operators such as this novel tower operator we do so in the UV, around the GFP. There will however be significant effects in the IR i.e. in the physical

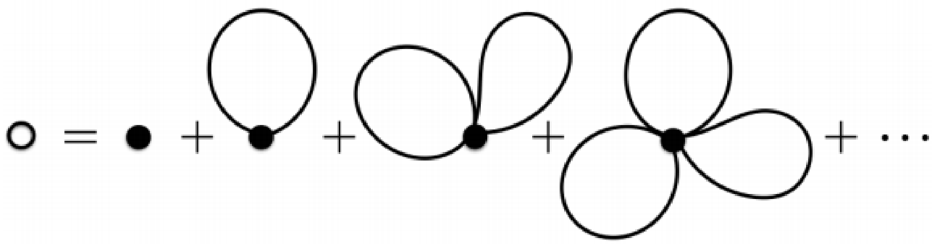

Figure 3.2: The renormalized eigenoperator is the sum of the bare eigenoperator plus its quantum correction, at linearised level.

limit. The scaled eigenoperator is form invariant under the linearised RG and the corresponding dimensionful operator in the IR cut-off Legendre effective action is simply

$$\delta_k^{(n)}(\varphi) = \frac{\partial^n}{\partial\varphi^n}\delta_k^{(0)}(\varphi), \quad \text{where} \;\; \delta_k^{(0)}(\varphi) = \frac{1}{\sqrt{2\pi\Omega_k}} \exp\Bigg(-\frac{\varphi^2}{2\Omega_k}\Bigg). \tag{3.64}$$

We recover the physical limit and the physical operators by taking the IR cut-off k to 0,

$$\lim_{k\to 0} \delta_k^{(n)}(\varphi) = \delta^{(n)}(\varphi). \tag{3.65}$$

We must however remind ourselves that this work is being undertaken in an $\mathbb{R}^4$ spacetime such that we can properly make use of the ERG framework, when extending the treatment to $\mathbb{M}^4$ there will be issues to consider, in particular unitarity and a Fock space with negative norm spaces [82]. When this structure is embedded into gravity however these problems are no longer of concern.

We find that we recover the right hand side of (3.65) in the $\hbar \to 0$ limit, this means that the tower operator is inherently and non-perturbatively quantum; it can not be recovered in the classical limit. To reiterate this means that one can picture the classical and gravity aspects as entirely independent of each other. We can however still understand the renormalization procedure of this eigenoperator in terms of Feynman diagrams. Solutions to (3.37) can be expressed as

$$\int_x \delta_k^{(n)}(\varphi) \;=\; \exp\Bigg(-\frac{1}{2}\mathrm{tr}\Bigg[\Delta_k^{\Lambda_0}\cdot\frac{\delta^2}{\delta\varphi\delta\varphi}\Bigg]\Bigg)\int_x \delta_{\Lambda_0}^{(n)}(\varphi) \tag{3.66}$$

where the expansion of the exponential gives the expected 1PI Feynman diagrams, see figure (3.2).

Here each tadpole propagator is defined as in (3.24) with the correct change in sign as dictated by this more complete treatment. We can also express the bare eigenoperator (3.62) as, using (3.52) and (3.53),

$$\delta^{(n)}_{\Lambda_0}(\varphi) = \exp\left(\tfrac{1}{2}\Omega_{\Lambda_0}\ \frac{\partial^2}{\partial\varphi\partial\varphi}\right)\delta^{(n)}\varphi. \tag{3.67}$$

Then using (3.61) to transform to unscaled variable and taking the Fourier transform we also find

$$\delta^{(n)}_{\Lambda_0}(\varphi) = \int_{-\infty}^{\infty} \frac{d\pi}{2\pi}\ (\pi)^n e^{-\frac{1}{2}\pi^2\Omega_{\Lambda_0}+i\pi\varphi} \tag{3.68}$$

and from this we find as one may expect, after pulling the Ω_{Λ_0} piece outside the integral

$$\int_x \delta^{(n)}_{\Lambda_0}(\varphi) \ = \ \exp\left(\frac{1}{2}\mathrm{tr}\left[\Delta^{\Lambda_0}\cdot\frac{\delta^2}{\delta\varphi\delta\varphi}\right]\right)\int_x \delta^{(n)}_{\Lambda_0}(\varphi). \tag{3.69}$$

Finally we can see that combining this last equation with (3.66) and using (3.24) that the renormalized operator is given by (3.67) with Λ_0 replaced by k, leading to the expression (3.62).

Before concluding this section we comment on the behaviour of the potential when an infinite number of couplings are non-zero as well as for higher order interactions and derivative eigenoperators. In essence the entirety of this structure remains and is flexible enough to permit these new cases with the resulting eigenoperators still satisfying (3.46) and (3.54). We direct the reader to chapters 3 and 4 of [8] for more details on these matters.

3.2.3 Summary of the tower operator

We now briefly summarise the previous section and note the most significant aspects of the tower operator. The paramount feature of this novel operator is this negative scaling dimension $[\delta^{(n)}_{\Lambda}(\varphi)] = -1-n$ which follows from the quantisation condition (3.54) and would seem tailor made to resolve the problems of irrelevancy of interacting graviton operators and couplings which are ultimately the source of what makes creating a theory of QG with a well defined continuum limit so difficult. We see that the tower operator is evanescent, inherently quantum in nature and uniquely satisfies (3.46). The couplings associated to the tower operator are relevant, in contrast to Newton's constant κ.

The discussion thus far however has had a glaring omission; this has not been a theory of gravity. Strictly speaking so far we have been discussing a scalar field theory with a negative kinetic term with some additional aspects. We must respect the symmetry of gravity, diffeomorphism invariance, for this to be the case. We will now discuss how this is achieved as well as associated topics such as how to consistently consider field redefinitions (of which there is a great deal of freedom with gravitons), the methods of BRST symmetry as well as its cohomology and how the addition of exact terms will have implications for the representation of diffeomorphism invariance.

3.3 BRST, QME and the anti-field formalism

We must now concern ourselves with how to implement diffeomorphism invariance into this structure at the quantum level, bearing in mind that the operators are dictated by the ERG. Furthermore the quadratic divergence stemming from Ω_Λ is paramount to the definition of the operators (3.62) and so dimensional regularization is not a suitable method here, instead we must employ a cut-off which breaks gauge invariance which we later restore.

3.3.1 BRST invariance

To implement diffeomorphism invariance whilst simultaneously respecting the RG we will have to solve the Quantum Master Equation (QME) [83–86] as well as the RG equations. To solve this QME we will first have to understand its classical counterpart the Classical Master Equation (CME) and before that we must first understand Becchi-Rouet-Stora-Tyutin (BRST) invariance.

A crucial ingredient of this BRST invariance is a*nilpotent* operator Q[11]. An operator is said to be nilpotent if, upon acting on an object twice, the result is guaranteed to be zero, i.e.

$$Q^2\mathcal{O} = 0 \;\forall\; \mathcal{O}. \tag{3.70}$$

This is often simplified to $Q^2 = 0$. As part of this we can then also define an *exact* object $K = Q\mathcal{O}$ i.e. it is produced via the action of the BRST operator on another object, via the nilpotentcy condition (3.70) then

$$QK = Q(Q\mathcal{O}) = Q^2\mathcal{O} = 0 \tag{3.71}$$

will always hold true. This is often abbreviated to $Q^2 = 0$, that is to say this nilpotentcy should be true in all cases. Objects $\mathcal{C}$ which satisfy

$$Q\mathcal{C} = 0 \tag{3.72}$$

are said to be *closed*. A pertinent example for this thesis which illustrates this is the action of Q_0, the free level part of Q which will be elucidated later, on the graviton $H_{\mu\nu}$ and ghost fields c_μ

$$Q_0 H_{\mu\nu} = \partial c_\mu + \partial c_\nu \tag{3.73}$$

$$Q_0 c_\mu = 0 \tag{3.74}$$

[11] This operator Q will later be referred to as the *BRST charge* however it will continue to act as an operator in this sense. We also note that in a great deal of the literature these operators are accompanied by hats $\hat{Q}$, we eschew this standard practice in this thesis and assume the role of operator is clear in context except when it is pertinent to be particularly explicit.

If the operators are themselves not exact, that is to say $\mathcal{C} \neq Q\mathcal{O}$, they are said to exist in the *cohomology* of Q.

These concepts are more generally associated with the field of differential geometry but are also crucial here. In this context the BRST operator Q generates field transformations. For some action functional of the field ϕ

$$S = \int d^4x \mathcal{L}(\phi) \tag{3.75}$$

we find that the action of Q on ϕ is a local gauge transformation which is proportional to a new field we introduce, the ghost field c_μ. For the symmetry associated to this gauge transformation to be satisfied (which was traditionally non-Abelian gauge theories but is also equally applicable to diffeomorphism invariance) we demand

$$QS = Q \int d^4x \mathcal{L} = 0. \tag{3.76}$$

If (3.76) is true then we say that the symmetry associated to the BRST charge Q is satisfied. There are many further aspects to consider, the first of these is that we also introduce non-physical auxiliary fields B_μ to ensure the action is closed under Q off-shell. We also introduce the anti-ghost $\bar{c}_\mu$ when fixing a gauge to ensure we can calculate Feynman diagrams with a well defined ghost propagator. The ghost itself is also a fermionic field.

3.3.2 QME and the anti-field formalism

We must now consider extensions to the BRST formalism when considering gravity [87, 88], we need to do this as we have significant more freedom in terms of how we parametrise our graviton fields when compared to the gauge fields discussed earlier. We must also consider how to implement the diffeomorphism invariance in this way whilst also respecting the form of the operators as dictated by the Wilsonian RG. To achieve this we will use the QME and its extension the anti-field formalism [87–90] which will combine this implementation of diffeomorphism invariance, considering the freedom of field redefinitions and also respect the Wilsonian RG. We will also see how the anti-field method will significantly restrict the operators we can construct and also how the addition of BRST-exact terms will have important implications for the representation for diffeomorphism invariance.

We begin by defining the QME and the role of the anti-fields. We can maintain renormalizability in the presence of non-Abelian local symmetries provided the CME (also known as the Zinn-Justin equation [91, 92]) for the Legendre effective action is satisfied. We will define the CME shortly, this will form the basis for the QME which is more relevant to the topics discussed here. These equations account for the fact that gauge invariance is realised at the quantum level by demanding BRST invariance [93–96].

The master equations also take into account the deformations under regularisation and renormalization whilst still satisfying this BRST invariance, that is to say our overall structure will not be affected by these deformations provided they are performed in a smooth, well defined way. We remind ourselves of the general, complete form of the BRST transformations on the quantum fields we will be working with, in particular

$$\delta\Phi^A = \epsilon Q\Phi^A \tag{3.77}$$

where ϵ (not to be confused with the small quantity ϵ employed earlier) is a Grassmann number, $\delta\Phi^A$ denotes the transformation of our quantum fields Φ^A and Q is our BRST operator which we have discussed in subsection (3.3.1). Note that equation (3.77) is in general a non-linear transformation since in general Q itself depends on the fields, the full details of this transformation are dictated by (3.88). We note that we will now refer to this 'BRST operator' as the 'BRST charge' as this new anti-field formalism will necessitate an extension to this original BRST charge, this more complete operator will be referred to as the 'BRST operator'. These quantum fields Φ^A contain all of the fields in our theory including the graviton, dilaton and in particular the ghosts and auxiliary fields needed to implement diffeomorphism invariance and realise the BRST invariance off-shell. As before the BRST charge Q acts upon the fields contained within Φ^A, transforming them and we define this charge to act from the left.

We now introduce the anti-fields Φ^*_A. These non-physical fields act as source terms for the BRST transformations of the original field, as part of this we supplement the bare action $\mathcal{S}[\Phi^A]$ such that the total action is

$$S = \mathcal{S}[\Phi^A] - (Q\Phi^A)\Phi^*_A. \tag{3.78}$$

The partition function is then simply

$$\mathcal{Z}[\Phi^*_A] = \int \mathcal{D}\Phi\; e^{-S}. \tag{3.79}$$

We note that the anti-fields Φ^*_A have opposite statistics to the counterpart fields e.g. $h_{\mu\nu}$ is bosonic whereas $h^*_{\mu\nu}$ is fermionic and that we are using the compact Dewitt notation, where the index A runs over all internal indices [97], for the sake of clarity. The contraction of the Dewitt indices indicates summation over internal indices and integration over space-time. Thus for example the last term in (3.78), if we keep only the $h_{\mu\nu}$ component of Φ^A, reads:

$$-(Q\Phi^A)\Phi^*_A \equiv -\int d^4x (Qh_{\mu\nu})(x) h^*_{\mu\nu}(x). \tag{3.80}$$

In this new approach, a gauge symmetry has been successfully incorporated if the functional integral is invariant under (3.77) and this is true only if the QME is satisfied. This is a special case of the Quantum Master Functional (QMF)

$$\mathcal{A} = \frac{1}{2}(S,S) - \Delta S \tag{3.81}$$

where we return the QME if $\mathcal{A} = 0$ i.e.

$$0 = \frac{1}{2}(S,S) - \Delta S. \tag{3.82}$$

The anti-bracket (,) and measure operator Δ are defined, where X and Y are arbitrary functionals, as

$$(X,Y) = \frac{\partial_r X}{\partial \Phi^A}\frac{\partial_l Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A}\frac{\partial_l Y}{\partial \Phi^A} \quad \text{and} \quad \Delta X = (-)^A \frac{\partial_l}{\partial \Phi^A}\frac{\partial_l}{\partial \Phi^*_A} X \tag{3.83}$$

where the A in the exponent of $(-)^A$ returns $(-)^0 = 1$ if A is 0 i.e. it refers to a bosonic field and similarly $(-)^1 = -1$ in the fermionic case. ∂_l and ∂_r refer to partial differentiation from the left and right respectively, the former being the differentiation process one is more familiar with. The standard Einstein summation for matching subscripts and superscripts is also employed.

When acting upon a bosonic functional, such as the action, the measure can be expressed as

$$\Delta S = \frac{\partial_r}{\partial \Phi^*_A}\frac{\partial_l}{\partial \Phi^A} S. \tag{3.84}$$

We note that the anti-bracket is the classical part and is in fact the CME. If $\hbar$ is restored we see that the measure Δ is the quantum part, with regards to the latter we will find that the Wilsonian RG will naturally implement a regularisation such that it is well defined. We can see that the QME follows from

$$\int \mathcal{D}\Phi \; \mathcal{A} e^{-S} \;\; = \;\; \int \mathcal{D}\Phi \; \Delta e^{-S} = 0 \tag{3.85}$$

where the result equals 0 as the second expression is an integral over a total Φ derivative.

We will now review the relationship between this QME structure and the BRST cohomology, in particular how an understanding of one leads to an understanding of the other. We will begin by looking at the free graviton solution of the QME and then perturbing $S + \epsilon\mathcal{O}$ which we find is still a solution where ϵ is some small parameter and $\mathcal{O}$ is a quasi-local operator integrated over space-time. In this context a quasi-local quantity is one that possesses as space-time derivative expansion corresponding to a Taylor expansion in dimensionless momenta p^μ/Λ. This corresponds to the existence of a sensible Kadanoff blocking [98] which is a fundamental ingredient of the Wilsonian RG. Such solutions will deform the BRST algebra allowing us to explore the space of interacting theories which are smoothly connected to the well defined free theory that satisfies the QME.

We begin by substituting the perturbed action into the QME and find that, as one may expect, that the operator must be invariant under the BRST charge Q

$$Q\mathcal{O} = 0 \tag{3.86}$$

and the full BRST transformation is defined as

$$s\mathcal{O} = (S, \mathcal{O}) - \Delta\mathcal{O}. \tag{3.87}$$

We will now define the components of this full BRST operator s. The first part is the BRST charge described in section (3.3.1) which acts on the fields

$$Q\Phi^A = (S, \Phi^A) \tag{3.88}$$

and the second is the part which acts on the anti-fields Φ^*_A which we refer to as the Kozsul-Tate differential [99–101]. We define this to act from the left for consistency,

$$Q^-\Phi^*_A = (S, \Phi^*_A). \tag{3.89}$$

As the measure Δ has not appeared here we can see that these two parts of the full BRST operator form the classical part and satisfy the CME, they are the starting point for the classical BRST cohomology.

We remind ourselves of the nilpotentcy of the BRST operator and express it now in terms of this QME, in general we have

$$s^2\mathcal{O} = (\mathcal{A}, \mathcal{O}) \tag{3.90}$$

and note that the full BRST transformation is nilpotent provided the QME is satisfied. Provided the QME is satisfied by an action S, then BRST-exact operators (3.71) can be expressed as

$$\mathcal{O} = sK = (S, K) - \Delta K \tag{3.91}$$

and are automatically closed under S (3.76). We find that these exact operators are simply re-definitions of our fields and anti-fields, the latter being sources terms for the former

$$\delta\Phi^A = \frac{\partial_l K}{\partial\Phi^*_A}, \quad \delta\Phi^*_A = -\frac{\partial_l K}{\partial\Phi^A} \tag{3.92}$$

with $-\Delta K$ being the Jacobian of the change of variable of the partition function (3.79). We can make this more explicit. If $\mathcal{O}_1, ..., \mathcal{O}_n$ are BRST invariant operators and $\mathcal{O}$ also s-exact then they will have disjointed space-time support i.e.

$$\langle \mathcal{O}\mathcal{O}_1...\mathcal{O}_n\rangle = \langle sK\mathcal{O}_1...\mathcal{O}_n\rangle = -\frac{1}{\mathcal{Z}}\int \mathcal{D}\Phi(K\mathcal{O}_1...\mathcal{O}_n)e^{-S} = 0. \tag{3.93}$$

That is to say that if K generates a change in variables then the solution offers no new physics, it merely re-parametrises the previously understood scenario. We are therefore interested in those operators that are closed under s but not exact, that is to say they exist in the quantum BRST cohomology.

We must now combine these concepts such that the QME is satisfied for the entire flow, in particular we will show that the QME is satisfied at the GFP and for first order perturbations away from the fixed point. We express the QME being satisfied for the entirety of the RG flow as

$$\partial_t \mathcal{A}[S] = 0 \tag{3.94}$$

i.e. if the QME is satisfied at some scale Λ it will remain satisfied along the RG flow, this in turn will determine how the QMF (3.81) is regularised. We begin first by focusing on the well defined free graviton action at the GFP where only the free BRST transformations are defined. The BRST charge and Kozsul-Tate differential can be expressed order by order in the coupling

$$Q = Q_0 + \kappa Q_1 + \frac{1}{2}\kappa^2 Q_2 + ... \tag{3.95}$$

$$Q^- = Q_0^- + \kappa Q_1^- + \frac{1}{2}\kappa^2 Q_2^- + ... \tag{3.96}$$

where we currently focus on the free level however this expansion will be useful for defining the action at first and second order in the coupling. We note that this structure is very general and the BRST charge and Kozsul-Tate differential can be expressed in this way with any pertinent gauge coupling, we use κ here due to the relevance to gravity and diffeomorphism invariance for the ease of understanding. The action can be expressed in a similar way

$$S = S_0 + \kappa S_1 + \frac{1}{2}\kappa^2 S_2 + ... \quad . \tag{3.97}$$

At the free level the BRST charge is given by

$$Q_0 \Phi^A = R^A_{\ B} \Phi^B . \tag{3.98}$$

The free level BRST symmetry is Abelian and diagonal in momentum space and so it is relatively straight forward to regularise it by inserting a momentum cut-off function between the bi-linear terms in the action and between the functional derivatives in (3.83). We can express the action S at the free level as the usual free parts for the fields and the free part of the BRST charge Q_0 and Kozsul-Tate differential Q_0^-

$$S_0 = \mathcal{S}_0 + \mathcal{S}_0^* \tag{3.99}$$

where we note the change in font for the components of the free action on the right hand side of the equation and formally define

$$\mathcal{S}_0^* = -(Q_0\Phi^A)B^\Lambda\Phi_A^* \tag{3.100}$$

where B^Λ is the cut-off function. From the Wilsonian perspective we demand that S_0 is a solution of the flow equation (3.3), as such we convert to dimensionless variables such that we then have the fixed point action with the anti-fields included. The flow equation for the free action then becomes

$$\dot{\mathcal{S}}_0 + \dot{\mathcal{S}}_0^* = -a_0[\mathcal{S}_0.\mathcal{S}_0] + a_0[\mathcal{S}_0^*.\mathcal{S}_0^*] \tag{3.101}$$

Up to a multiplicative factor $a_0[\mathcal{S}_0^*.\mathcal{S}_0^*]$ computes, via BRST invariance,

$$\langle Q_0\Phi^A Q_0\Phi^B\rangle = \langle Q_0(\Phi^A Q_0\Phi^B)\rangle = 0 \tag{3.102}$$

where a_0 and later a_1 stems from an alternative expression for the RG flow equation. We can express the flow equation as

$$\dot{S} = \tfrac{1}{2}\frac{\partial_r S}{\partial\Phi^A}(\triangle^\Lambda)^{AB}\frac{\partial_l\Sigma}{\partial\Phi^B} - \tfrac{1}{2}(\dot{\triangle}^\Lambda)^{AB}\frac{\partial_l}{\partial\Phi^B}\frac{\partial_l}{\partial\Phi^A}\Sigma = a_0[S,\Sigma] - a_1[\Sigma] \tag{3.103}$$

where $\Sigma = S - 2\hat{S}$ and a_0 is the classical piece and symmetric bilinear in its arguments with the the quantum piece a_1 being linear in its arguments. The seed action $\hat{S}$ is chosen to coincide with the GFP action and will re-appear through-out this thesis where it is used in the cohomology of the BRST operator s.

(3.102) would navely result in $\dot{\mathcal{S}}_0^*$ at which point one would conclude $B^\Lambda = 1$ however this is not the full story and would not produce a structure which simultaneously respects the QME and the Wilsonian RG. Expressing the action as (3.97) and substituting into the flow equation we find that first order perturbations satisfy

$$\dot{S}_1 = 2a_0[S_1, S_0 - \hat{S}] - a_1[S_1]. \tag{3.104}$$

From this we can see that there will always be a non-zero anti-field dependence following from the first term on the right hand side where $\hat{S}$ is our seed action. In particular even the the original $S_1 = \mathcal{O}[\Phi]$ eigenoperators from our top terms will develop anti-field dependence which muddies this picture of the anti-fields as source terms to the transformations of the BRST charge. As a result we demand that this seed action agrees with the GFP action i.e.

$$\hat{S} = \mathcal{S}_0 + \mathcal{S}_0^*. \tag{3.105}$$

Substituting in S_0 back into the flow equation yields

$$\dot{\mathcal{S}}_0 + \dot{\mathcal{S}}_0^* = -a_0[\mathcal{S}_0, \mathcal{S}_0] - 2a_0[\mathcal{S}_0, \mathcal{S}_0^*] \tag{3.106}$$

and from this we find that we should set $B^\Lambda(p) = 1/C^\Lambda(p)$, that is to say the free action (3.99) and seed action (3.105) are equal and regularised by inserting this factor of $1/C^\Lambda(p)$ inside all bilinear terms. To find how the QME is regularised we express $e^{-S} = \mu$ and using (3.94) we find

$$\begin{aligned}
\partial_t(\mathcal{A}\mu) &= \dot{\Delta}\mu + \Delta\dot{\mu} \\
&= \dot{\Delta}\mu + \\
&\quad \frac{1}{2}\frac{\partial_r}{\partial\Phi^A}\left[(\dot{\triangle}^\Lambda)^{AB}\frac{\partial_r \Delta S}{\partial\Phi^B}\mu + (\dot{\triangle}^\Lambda)^{BA}\frac{\partial_r\Sigma}{\partial\Phi^B}\Delta\mu + (\dot{\triangle}^\Lambda)^{BA}\left(\frac{\partial_r\Sigma}{\partial\Phi^B}, \mu\right)\right] \\
&= \dot{\Delta}\mu - \frac{\partial_r}{\partial\Phi^A}(\dot{\triangle}^\Lambda)^{BA}\left(\frac{\partial_r\hat{S}}{\partial\Phi^B}, \mu\right) \\
&\quad + \frac{1}{2}\frac{\partial_r}{\partial\Phi^A}\left[(\dot{\triangle}^\Lambda)^{BA}\frac{\partial_r\Sigma}{\partial\Phi^B}\mathcal{A}\mu - (\dot{\triangle}^\Lambda)^{AB}\frac{\partial_r\mathcal{A}}{\partial\Phi^B}\mu\right].
\end{aligned} \tag{3.107}$$

For the QME to be satisfied along the entirety of the RG flow (3.94) we find that the first two terms in the last line of (3.107) must be zero. To achieve this we substitute (3.105) into the second of these terms and expand, the $\mathcal{S}_0^*$ part gives

$$-\frac{1}{2}\left[R^C{}_B(\dot{\triangle}^\Lambda)^{BA} + R^A{}_B(\dot{\triangle}^\Lambda)^{BC}\right]\frac{\partial_l^2\mu}{\partial\Phi^A\partial\Phi^C} \tag{3.108}$$

with the term in square brackets vanishing via linearised BRST invariance. Finally the $\mathcal{S}_0$ part cancels the first term if Δ is regularised by inserting $C^\Lambda(p)$ between its functional derivatives. Finally we pull out the factor of μ and cancel terms using (3.103) and find the more concise expression for (3.107)

$$\dot{\mathcal{A}} = 2a_0[\mathcal{A}, S - \hat{S}] - a_1[\mathcal{A}]. \tag{3.109}$$

This states that the QMF satisfies the flow equation and will continue to satisfy it at all points along the RG flow.

We may now explicitly state the free action with this regularising factor of $1/C^\Lambda(p)$ between bi-linear terms

$$S_0 = \hat{S} = \mathcal{S}_0 + \mathcal{S}_0^* = \frac{1}{2}\Phi^A(\triangle^\Lambda)^{-1}_{AB}\Phi^B - (Q_0\Phi^A)(C^\Lambda)^{-1}\Phi^*_A \tag{3.110}$$

and similarly for the QMF with $C^\Lambda(p)$ between the functional derivatives

$$(X,Y) = \frac{\partial_r X}{\partial \Phi^A} C^\Lambda \frac{\partial_l Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A} C^\Lambda \frac{\partial_l Y}{\partial \Phi^A} \quad \text{and} \quad \Delta X = (-)^A \frac{\partial_l}{\partial \Phi^A} C^\Lambda \frac{\partial_l}{\partial \Phi^*_A} X \quad . \tag{3.111}$$

As a consequence of the introduction of these factors of C^Λ equations (3.88) and (3.89) remain unchanged (the regularisation factors cancel) however the quantum measure term will have this factor in a way which is well defined when acting on local functionals. This measure term Δ will therefore map local functionals to local functionals at the free level. We may substitute (3.97) into (3.87) to find the perturbative expansion of the full BRST operator

$$s = s_0 + \kappa s_1 + \frac{1}{2}\kappa^2 s_2 + ... \tag{3.112}$$

and from this we can study the non-trivial solutions of the free quantum BRST cohomology iteratively to find

$$s_0 S_1 = 0 \quad , \quad s_0 S_2 = -\frac{1}{2}(S_1, S_1) \quad , \quad s_0 S_3 = -(S_1, S_2) \quad , \quad ... \,. \tag{3.113}$$

We note now that it is the free action with the non-zero anti-field dependence (3.99) that is the GFP, the standard part with no anti-fields no longer satisfies the flow equation (3.101) and we have our simplified flow equation

$$\dot{S}_1 = -a_1[S_1]. \tag{3.114}$$

This means that S_1 is a linear combination of eigenoperators with constant coefficients, the couplings, which if we demand these operators span a space of interactions closed under the Wilsonian RG then we weill find these operators will be elements of the Hilbert space $\mathfrak{L}_-$ described in section 3.2.2

We briefly comment that this work can be conducted in either gauge invariant or gauge fixed basis and the results are equivalent. The latter is necessary to define a propagator, in particular for the anti-ghost, when calculating Feynman diagrams.)

3.3.3 Applying the anti-field formalism to quantum gravity

The structures discussed above are very general and have not yet been applied to the case of QG and the implementation of diffeomorphism invariance in a way that consistently respects the Wilsonian RG and containing relevant operators as elements of our Hilbert space $\mathfrak{L}_-$ (which was defined above (3.54)) constructed around the negative measure (3.54). We contrast this to the construction of the Hilbert space $\mathfrak{L}_+$ around the positive measure (as defined below (3.16)), we will later find we must combine these Hilbert spaces when associating the tower operator to monomials of other fields. We now begin applying this structure to QG and uncover more facets of these powerful tools, in particular we will find the anti-field formalism uniquely defines the operators that are diffeomorphism invariant which returns the result one would expect from expanding the Einstein-Hilbert action as well as further investigating the role exact operators play in the representation of diffeomorphism invariance. Additional gradings will be introduced to heavily constrain actions that are diffeomorphism invariant.

We begin by expressing the free Einstein-Hilbert action (2.1), now with the anti-field source terms in a Euclidian signature around flat $\mathbb{R}^d$ with the dilaton $\varphi = \frac{1}{2}H_{\mu\mu}$

$$S_0 = \mathcal{S}_0 + \mathcal{S}_0^* \int d^d x\ \mathcal{L}_0 \tag{3.115}$$

where

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\lambda H_{\mu\nu})^2 - 2(\partial_\lambda \varphi)^2 - (\partial^\mu H_{\mu\nu})^2 + 2\partial^\alpha \varphi \partial^\beta H_{\alpha\beta} - 2\partial_\mu c_\nu H^*_{\mu\nu} \tag{3.116}$$

reminding ourselves that we express the metric as $g_{\mu\nu} = \delta_{\mu\nu} + \kappa H_{\mu\nu}$ and (3.116) should be compared to (2.5). To regularise this properly we insert the factor of $(C^\Lambda)^{-1}$ between bi-linear terms and so re-express (3.115) with

$$\mathcal{L}_0 = \frac{1}{2}H_{\mu\nu}(\triangle^\Lambda)^{-1}_{\mu\nu,\alpha\beta}H_{\alpha\beta} - 2\partial_\mu c_\nu (C^\Lambda)^{-1} H^*_{\mu\nu} \tag{3.117}$$

where $\triangle^{-1}_{\mu\nu,\alpha\beta} = \triangle_{H_{\mu\nu}H_{\alpha\beta}}$ is the differential operator we find from (3.115) via integration by parts, with $(\triangle^\Lambda)^{-1}_{\mu\nu,\alpha\beta} = \triangle^{-1}_{\mu\nu,\alpha\beta}/C^\Lambda$. This is the *minimal gauge invariant basis* which encodes all the properties of the diffeomorphism invariant action and the gauge transformations, we note the *non-minimal gauge invariant basis* now for completeness. This adds the auxiliary field b_μ and the anti-ghost anti-field (where we note that the two 'anti's do not cancel each other out as they do in typical language) $\bar{c}^*_\mu$. Supplementing (3.117) with these terms gives the Lagrangian

$$L_0 = \mathcal{L}_0 + \frac{1}{2\alpha} b_\mu (C^\Lambda)^{-1} b_\mu - i b_\mu (C^\Lambda)^{-1} \bar{c}^*_\mu \tag{3.118}$$

where α is the gauge fixing parameter. As mentioned ealier in section 3.3.2 we can work in either gauge fixed or gauge invariant basis where we will find $\bar{c}_\mu$ dependence at the free level in the former.

Using the anti-bracket we can construct all the non-vanishing actions of the BRST charge, reminding ourselves of some previous results

$$Q_0 H_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu \tag{3.119}$$

$$Q_0 c_\mu = 0 \tag{3.120}$$

$$Q_0 \bar{c}_\mu = i b_\mu \tag{3.121}$$

and Kozsul-Tate differential, namely

$$Q_0^- H^*_{\mu\nu} = -2G^{(1)}_{\mu\nu}, \quad Q_0^- c^*_\nu = -2\partial_\mu H^*_{\mu\nu} \tag{3.122}$$

where $G^{(1)}_{\mu\nu}$ is the linearised Einstein-Tensor

$$G^{(1)}_{\mu\nu} = -R^{(1)}_{\mu\nu} + \tfrac{1}{2}R^{(1)}\delta_{\mu\nu} = \tfrac{1}{2}\Box H_{\mu\nu} - \delta_{\mu\nu}\Box\varphi + \partial^2_{\mu\nu}\varphi + \tfrac{1}{2}\delta_{\mu\nu}\partial^2_{\alpha\beta}H_{\alpha\beta} - \partial_{(\mu}\partial^\alpha H_{\nu)\alpha} \tag{3.123}$$

with the linearised curvatures being

$$\begin{aligned} R^{(1)}_{\mu\alpha\nu\beta} &= -2\partial_{[\nu|}\partial_{[\nu|}H_{\beta]|\alpha]},\ R^{(1)}_{\mu\nu} \\ &= -\partial^2_{\mu\nu}\varphi + \partial_{(\mu}\partial^\alpha H_{\nu)\alpha} - \tfrac{1}{2}\Box H_{\mu\nu},\ R^{(1)} = \partial^2_{\alpha\beta}H_{\alpha\beta} - 2\Box\varphi. \end{aligned} \tag{3.124}$$

Several gradings are introduced which are found to be a powerful tool in restricting the representation of the action. In addition to the usual ghost number and statistics restrictions we can also introduce the anti-ghost number [90]. We define that the action S has ghost number zero, c_μ has ghost number 1 and $H^*_{\mu\nu}$ has ghost number one, these demands lead to table 3.1.

	ϵ	gh #	ag #	dimension
$H_{\mu\nu}$	0	0	0	$(d-2)/2$
c_μ	1	1	0	$(d-2)/2$
$\bar{c}_\mu$	1	-1	1	$(d-2)/2$
b_μ	0	0	1	$d/2$
$H^*_{\mu\nu}$	1	-1	1	$d/2$
c^*_μ	0	-2	2	$d/2$
$\bar{c}^*_\mu$	0	0	0	$d/2$
Q	1	1	0	1
Q^-	1	1	-1	0

Table 3.1: Gradings of the fields and operators, ϵ is the Grassmann grading with 0 (1) being bosonic (fermionic), gh # is the ghost number, ag # is the anti-ghost/anti-field number and we also include the classical scaling dimension. The first two rows of fields are the minimal set of fields we employ in this thesis with the following set the minimal anti-fields and we include $\bar{c}^*_\mu$ for the non-minimal set for completeness. We also include the BRST charge Q and the Kozsul-Tate differential Q^- to illustrate their now definite gradings in this system.

We can see that (X, Y) adds one to the sum of the dimensions of X and Y as well as adding one to the sum of the ghost number of X and Y, hence Q, Q^- and Δ^- do the same. What is very powerful about this technique is that we can then split our actions up by *anti-ghost* number

$$S = \sum_{n=0} S^n. \tag{3.125}$$

This then explains some of our notation, the BRST charge $Q = Q^0$ leaves this grading unaffected, the Kozsul-Tate differential Q^- lowers it by one, hence the minus sign, and the two parts of the measure operator $\Delta = \Delta^- + \Delta^=$ do the same by eliminating the graviton and ghost anti-fields lowering the grading by one and two respectively, i.e.

$$\Delta^- = \frac{\partial}{\partial H_{\mu\nu}} C^\Lambda \frac{\partial_l}{\partial H^*_{\mu\nu}} - \frac{\partial_l}{\partial \bar{c}_\mu} C^\Lambda \frac{\partial}{\partial \bar{c}^*_\mu}, \quad \Delta^= = -\frac{\partial_l}{\partial c_\mu} C^\Lambda \frac{\partial}{\partial c^*_\mu}. \tag{3.126}$$

The full quantum BRST, also known as the BRST operator is now written as

$$\hat{s} = Q + Q^- - \Delta^- - \Delta^= \tag{3.127}$$

and from this we can then split up the cohomology equation $\hat{s}\mathcal{O} = 0$ by this anti-ghost number, which for all $n \geq 0$ (there is no physical meaning for $n < 0$) states that

$$Q\mathcal{O}^n + (Q^- - \Delta^-)\mathcal{O}^{n+1} - \Delta^= \mathcal{O}^{n+2} = 0. \tag{3.128}$$

When we isolate this for each anti-field number we produce the descent equations which leads to an *anti-field cascade*, where the heavily constrained maximal anti-field number if easily found and leads to the lower anti-field number parts, up to the addition of exact pieces

$$Q_0\mathcal{O}^n = 0, \;\; Q_0\mathcal{O}^{n-1} = (\Delta^- - Q_0^-)\mathcal{O}^n, \;\; Q_0\mathcal{O}^{n-2} = (\Delta^- - Q_0^-)\mathcal{O}^{n-1} + \Delta^= \mathcal{O}^n, ... \tag{3.129}$$

We are interested in the solutions that exist in the cohomology of $\hat{s}$ i.e. we do not want solutions of the form $\mathcal{O} = \hat{s}K$, using (3.128) and grading K by anti-ghost number we see that these trivial solutions take the form

$$\mathcal{O}^n = QK^n + (Q^- - \Delta^-)K^{n+1} - \Delta^= K^{n+2} \tag{3.130}$$

which means that K^n are fermionic and have ghost number -1, using (3.128) and (3.130) these exact pieces can be found and eliminated.

We now briefly mention the form of the propagators that will be needed to calculate Feynman diagrams in this thesis, again reminding ourselves that this structure holds in either gauge fixed or gauge invariant basis, and noting that

$$\Delta^{AB} = \langle \Phi^A \Phi^B \rangle \tag{3.131}$$

with

$$\Phi^A(x) = \int \frac{d^dp}{(2\pi)^d} e^{-ip\cdot x}\ \Phi^A(p), \tag{3.132}$$

the propagators are

$$\langle H_{\mu\nu}(p) H_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} + \left(\frac{4}{\alpha} - 2\right)\frac{p_{(\alpha}\delta_{\nu)(\alpha}p_{\beta)}}{p^4} - \frac{1}{d-2}\frac{\delta_{\alpha\beta}\delta_{\mu\nu}}{p^2} \tag{3.133}$$

$$\langle b_\mu(p) H_{\alpha\beta}(-p)\rangle = -\langle H_{\alpha\beta}(p) b_\mu(-p)\rangle = 2\delta_{\mu(\alpha}p_{\beta)}/p^2 \tag{3.134}$$

$$\langle b_\mu(p) b_\nu(-p)\rangle = 0 \tag{3.135}$$

$$\langle c_\mu(p)\bar{c}_\nu(-p)\rangle = -\langle \bar{c}_\mu(p) c_\nu(-p)\rangle = \frac{d_{\mu\nu}}{p^2} \tag{3.136}$$

We can then project the graviton propagator into its irreducible representations

$$\begin{aligned}\langle h_{\mu\nu}(p) h_{\alpha\beta}(-p)\rangle = {} & \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} + \left(\frac{4}{\alpha} - 2\right)\frac{p_{(\alpha}\delta_{\nu)(\alpha}p_{\beta)}}{p^4} \\ & + \frac{1}{d^2}\left(\frac{4}{\alpha} - d - 2\right)\frac{\delta_{\alpha\beta}\delta_{\mu\nu}}{p^2}\frac{2}{d}\left(1 - \frac{2}{\alpha}\right)\frac{\delta_{\alpha\beta}p_\mu p_\nu + p_\alpha p_\beta \delta_{\mu\nu}}{p^4}\end{aligned} \tag{3.137}$$

$$\langle h_{\mu\nu}(p)\varphi(-p)\rangle = \langle \varphi(p) h_{\mu\nu}(-p)\rangle = \left(1 - \frac{2}{\alpha}\right)\left(\frac{\delta_{\mu\nu}}{d} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2} \tag{3.138}$$

$$\langle \varphi(p)\varphi(-p)\rangle = \left(\frac{1}{\alpha} - \frac{d-1}{d-2}\right)\frac{1}{p^2} \tag{3.139}$$

which when specifying to $\alpha = 2$ and $d = 4$ become

$$\langle H_{\mu\nu}(p) H_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} - \tfrac{1}{2}\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} \tag{3.140}$$

$$\langle h_{\mu\nu}(p) h_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu} - \frac{1}{4}\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} \tag{3.141}$$

$$\langle h_{\mu\nu}(p)\varphi(-p)\rangle = \langle \varphi(p) h_{\mu\nu}(-p)\rangle = 0 \tag{3.142}$$

$$\langle \varphi(p)\varphi(-p)\rangle = -\frac{1}{p^2}. \tag{3.143}$$

3.3.4 Legendre effective action and modified Slavnov-Taylor identities

We have thus far described this structure in terms of the Wilsonian effective action for the sake of clarity of explanation, we now re-iterate some of these properties in terms of the renormalized IR cut-off Legendre effective action as this is more useful when performing calculations. BRST invariance is no longer expressed as the as unbroken through the QME but instead through the modified Slavnov-Taylor (mST) identities [102, 103] where off-shell nilpotency at the interacting level is recovered in the $\Lambda \to 0$ where Λ is now this IR cut-off. The free charges remain nilpotent and in using the Legendre effective action there is now direct access to the physical amplitudes

$$\Gamma_{\text{phys}} = \lim_{\Lambda \to 0} \Gamma. \tag{3.144}$$

The flow equation now becomes [59, 71, 72, 104–106]

$$\dot{\Gamma}_I = -\tfrac{1}{2}\mathrm{Str}\left(\dot{\triangle}_\Lambda \, \triangle_\Lambda^{-1} \left[1 + \triangle_\Lambda \Gamma_I^{(2)} \right]^{-1} \right) \tag{3.145}$$

where the over-dot is the usual partial differentiation with respect to the RG time $\partial_t = -\Lambda \partial_\Lambda$ and the mST is

$$\Sigma := \tfrac{1}{2}(\Gamma, \Gamma) - \mathrm{Tr}\left(C^\Lambda \Gamma_{I*}^{(2)} \left[1 + \triangle_\Lambda \Gamma_I^{(2)} \right]^{-1} \right) = 0 \tag{3.146}$$

where $C^\Lambda(p)$ is the UV cut-off function as described in 2.3 and is chosen to satisfy the requirements outlined there. $C_\Lambda(p) = 1 - C^\Lambda(p)$ is its IR counterpart, noting the change in super to sub script and appears in the IR regulated propagators as $\triangle_\Lambda^{AB} = C_\Lambda \triangle^{AB}$.

The two equations are compatible, if the mST is satisfied at an arbitrary energy scale then it will continue to be satisfied along the RG flow. In the physical limit $\Lambda \to 0$ the second term in (3.146) becomes zero (and is found to always remain finite at an arbitrary energy scale) and so in this limit we return the Zinn-Justin equation $\frac{1}{2}(\Gamma, \Gamma) = 0$ and we return to the standard realisation of BRST invariance as outlined earlier. Expanding upon the introduced notation we have $\mathrm{Str}\mathcal{M} = (-)^A \mathcal{M}_A^A$ and $\mathrm{Tr}\mathcal{M} = \mathcal{M}_A^A$ where $(-)^A$ corresponds to the field A being bosonic/fermionic as outlined before. In addition to this we note that ∂_l denotes taking the derivative *from* the left (in the standard way) and ∂_r denotes taking the derivative *from* the right (in the not-so standard way), which can be converted to ∂_l by commuting the derivative through, taking into account the swapping of fermionic fields and the minus sign(s) that will bring. We also set

$$\Gamma_I^{(2)} = \frac{\partial_l}{\partial \Phi^A} \frac{\partial_r}{\partial^B} \Gamma_I, \qquad \left(\Gamma_{I*}^{(2)} \right)^A{}_B = \frac{\partial_l}{\partial \Phi_A^*} \frac{\partial_r}{\partial \Phi^B} \Gamma_I \tag{3.147}$$

where Γ is the *effective average action* [105] part of the IR cut-off Legendre effective action. This is given by

$$\Gamma^{\mathrm{tot}} = \Gamma + \tfrac{1}{2}\Phi^A \mathcal{R}_{AB}\Phi^B, \qquad \triangle^{-1}_{\Lambda\ AB} = \triangle^{-1}_{AB} + \mathcal{R}_{AB} \tag{3.148}$$

where $\mathcal{R}_{AB}$ is the IR cut-off expressed in additive form. As seen with the Wilsonian effective action we split up the Legendre effective action into a free part and an interacting part

$$\Gamma = \Gamma_0 + \Gamma_I, \qquad \Gamma_0 = \tfrac{1}{2}\Phi^A \triangle^{-1}_{AB}\, \Phi^B - (Q_0\Phi^A)\Phi^*_A \tag{3.149}$$

where the BRST charge and also the Kozsul-Tate differential are defined in the same way as for the Wilsonian effective action, namely

$$Q_0\Phi^A := (\Gamma_0, \Phi^A), \qquad Q_0^-\Phi^*_A := (\Gamma_0, \Phi^*_A) \tag{3.150}$$

and similarly expand Γ_I perturbatively in its interactions

$$\Gamma_I = \sum_{n=1}^{\infty} \Gamma_n \epsilon^n/n!. \tag{3.151}$$

At first order the flow equation (3.145) and mST (3.146) become

$$\dot{\Gamma}_1 = \tfrac{1}{2}\mathrm{Str}\dot{\triangle}_\Lambda \Gamma_1^{(2)} \tag{3.152}$$

$$0 = (\Gamma_0, \Gamma_1) - \mathrm{Tr}\Big(C^\Lambda \Gamma^{(2)}_{1*}\Big) = (Q_0 + Q_0^- - \Delta)\Gamma_1 := \hat{s}_0\Gamma_1 \tag{3.153}$$

where the first equation is the Legendre effective action version of the flow equations and is satisfied by the eigenoperators, their RG time derivative is given by the action of the tadpole operator. In the second equation we have the Batalin-Vilkovisky operator as seen before (3.83).

Much of the work in this Legendre frame works identically to that of the Wilsonian frame, in particualar the free action

$$\Gamma_0 = \tfrac{1}{2}(\partial_\lambda H_{\mu\nu})^2 - 2(\partial_\lambda)^2 - (\partial^\mu H_{\mu\nu})^2 + 2\partial^\alpha\varphi\partial^\beta H_{\alpha\beta} - 2\partial_\mu c_\nu H^*_{\mu\nu}, \tag{3.154}$$

non-vanishing free BRST charges

$$Q_0 H_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu, \tag{3.155}$$

the non-vanishing free Kozsul-Tate differentials

$$Q_0^- H^*_{\mu\nu} = -2G^{(1)}_{\mu\nu}, \qquad Q_0^- c^*_\nu = -2\partial_\mu H^*_{\mu\nu}, \tag{3.156}$$

propagators (3.133 - 3.143) and method of switching between gauge fixed and gauge invariant bases is the same. We continue to choose the convenient $\alpha = 2$ Feynman gauge and work in $d = 4$ space-time dimensions for simplicity.

We now make clear a distinction between the interaction terms Γ_1 and our choice of non-trivial $\hat{s}_0$-cohomology representative, $\check{\Gamma}_1$, the former not simply being $\kappa\check{\Gamma}_1$ as in the standard quantisation [107]. So as to respect unitarity and causality we restrict $\check{\Gamma}_1$ to have a maximum of two space-time derivatives, as a consequence it must therefore be a linear combination of a term involving space-time derivatives and a unique non-derivative piece

$$\check{\Gamma}_1 = \check{\Gamma}_1^0 = \varphi. \tag{3.157}$$

The latter is the $\mathcal{O}(\kappa)$ piece of $\sqrt{g}$. The derivative part was found to have a unique expression under this two derivative condition, up to the usual addition of $\hat{s}_0$-exact pieces this is

$$\check{\Gamma}_1^2 = -(c^\mu \partial_\mu c^\nu) c^*_\nu = c_\mu \partial_\nu c_\mu c^*_\nu + Q_0(H_{\mu\nu} c_\mu c^*_\nu) \tag{3.158}$$

where the first bracket term is half the Lie bracket which will aid calculation at higher orders and the addition of an exact piece on the RHS. Using the anti-field cascade as outlined in 3.3.3 we are led to the anti-ghost level 1 and 0 pieces,

$$\check{\Gamma}_1^1 = 2c_\alpha \Gamma^{(1)\alpha}_{\mu\nu} H^*_{\mu\nu} + 2H_{\mu\nu} c_\mu \partial_\alpha H^*_{\alpha\nu} = -(c^\alpha \partial_\alpha H_{\mu\nu} + 2\partial_\mu c^\alpha H_{\alpha\nu}) H^*_{\mu\nu} \tag{3.159}$$

and

$$\begin{aligned}
\check{\Gamma}_1^0 = \ & 2\varphi \partial_\beta H_{\beta\alpha} \partial_\alpha \varphi - 2\varphi (\partial_\alpha \varphi)^2 - 2H_{\alpha\beta} \partial_\gamma H_{\gamma\alpha} \partial_\beta \varphi \\
& + 2H_{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - 2H_{\beta\gamma} \partial_\gamma H_{\alpha\beta} \partial_\alpha \varphi \\
& + \tfrac{1}{2} \varphi (\partial_\gamma H_{\alpha\beta})^2 - H_{\beta\mu} \partial_\gamma H_{\alpha\beta} \partial_\gamma H_{\alpha\mu} + 2H_{\mu\alpha} \partial_\gamma H_{\alpha\beta} \partial_\mu H_{\beta\gamma} \\
& + H_{\beta\mu} \partial_\gamma H_{\alpha\beta} \partial_\alpha H_{\gamma\mu} - \varphi \partial_\gamma H_{\alpha\beta} \partial_\alpha H_{\gamma\beta} - H_{\alpha\beta} \partial_\gamma H_{\alpha\beta} \partial_\mu H_{\mu\gamma} \\
& + 2H_{\alpha\beta} \partial_\gamma H_{\alpha\beta} \partial_\gamma \varphi + \tfrac{7}{2} b \Lambda^4 \varphi \,.
\end{aligned} \tag{3.160}$$

This is the classical three-graviton vertex one finds from expanding Einstein-Hilbert (2.1) under the parametrisation of the metric $g_{\mu\nu} = \delta_{mu\nu} + \kappa H_{\mu\nu}$ plus a quantum correction $\frac{7}{2} b \Lambda^4 \varphi$ where

$$b = \int \frac{d^4 \tilde{p}}{(2\pi)^4} \, C(\tilde{p}^2). \tag{3.161}$$

This quantum correction is generated by the action of the tadpole operator, the RHS of the linearised flow equation (3.152), on this triple graviton vertex. As one would expect all of these terms are of scaling dimension five. We also note the one-loop quantum part which is anti-ghost level zero,

$$\check{\Gamma}_{1\mathrm{q}1} = \check{\Gamma}^0_{1\mathrm{q}1} = \frac{7}{2} b \Lambda^4 \varphi. \tag{3.162}$$

In general (3.152) has solutions, once the conformal factor instability is considered, of the form

$$\delta^{2l+\varepsilon}_{\Lambda}(\varphi)\ \sigma(\partial, \partial\varphi, h, c, \Phi^*) + ... \tag{3.163}$$

where $l \geq 0$ is an integer and $\varepsilon = 0(1)$ according to the even (odd) φ-amplitude parity. We can also express a general solution of (3.152) with $\Gamma_1 = \Gamma(\mu)$ where

$$\Gamma(\mu) = \exp\left(-\tfrac{1}{2} \triangle^{\Lambda\ AB} \frac{\partial^2_l}{\partial\Phi^B \partial\Phi^A} \right)\ \Gamma_{1\mathrm{phys}} = \sum_{\sigma} (\sigma f^{\sigma}_{\Lambda}(\varphi, \mu) + ...) \tag{3.164}$$

which is a linear sum over the eigenoperators (3.163) with the underlying coefficients $g^{\sigma}_{2l+\varepsilon}(\mu)$ which are subsumed into the coefficient functions discussed earlier which we can also express as

$$f^{\sigma}_{\Lambda}(\varphi, \mu) = \int_{-\infty}^{\infty} \frac{d\pi}{2\pi} f^{\sigma}(\pi, \mu) e^{-\frac{\pi^2}{2}\Omega_{\Lambda} + i\pi\varphi},\ \ f^{\sigma}(\pi, \mu) = i^{\varepsilon} \sum_{l=0}^{\infty} (-)^l g^{\sigma}_{2l+\varepsilon}(\mu) \pi^{2n+\varepsilon}. \tag{3.165}$$

It can be shown that the Taylor series of $f^{\sigma}(\pi, \mu)$ (noting the change in argument) converges for all π and decays exponentially for $\pi > 1/\Lambda_{\sigma}$ and so is valid for the entire flow. These points are necessary to elucidate when begin to consider the higher order behaviour of this theory. This infinite tower of underlying coupling for each monomial σ has at first order only relevant couplings with the exception of one marginal coupling. At higher order the quantum corrections introduce higher dimensional monomials σ with infinitely many of their couplings being relevant however the first few are irrelevant and are heavily constrained such that there is a well-defined renormalized trajectory [8, 75]. At the second order, see chapter 5, there are no new marginal couplings, the first order couplings do not run and the irrelevant couplings are determined in terms of the first order couplings.

3.3.5 Implementation of diffeomorphism invariance in the physical limit

We now review how one implements diffeomorphism invariance in this new approach to QG, noting how one initially constructs the action in a way that does not respect this invariance before returning to the invariant subspace in the physical limit.

We summarise this unique implementation of diffeomorphism invariance in figure 3.3, a simplified version of the theory space of our effective actions. The theory is constructed in the UV where the cut-off scale Λ breaks the diffeomorphism invariance,

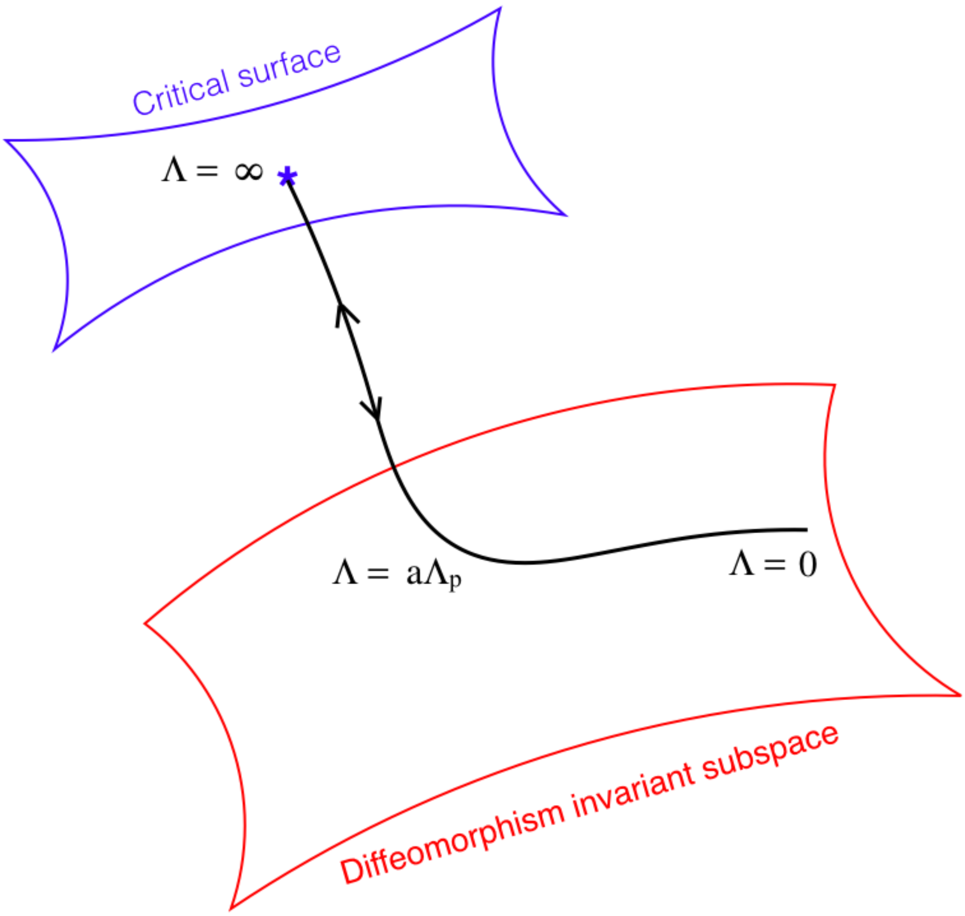

Figure 3.3: The continuum limit is defined in the UV and emanates from the GFP along relevant directions, it then re-enters the diffeomorphism invariant subspace in the physical limit $\Lambda \to 0$, passing below the *amplitude suppression scale* (also referred to as the *amplitude decay scale*) $a\Lambda_\sigma$ where a is a non-universal number.

it is only in the physical limit $\Lambda \to 0$ where, going below a dynamically generated *amplitude suppression scale* $a\Lambda_\sigma$, that the mST is then satisfied. In this subspace the underlying couplings are subsumed into the coefficient functions, effective couplings such as κ and the cosmological constant are recovered and all physical quantities are then guaranteed to be diffeomorphism invariant, for example recovering the EH action (2.1) at first and second order and matching the standard expressions. We can also regard this as equivalent to the limit $\Lambda_\sigma \to \infty$.

This is a unique aspect of this structure which brings with it several considerations, firstly it has always been assumed that the Wilsonian RG properties that define the continuum limit must also respect diffeomorphism invariance. This has led to arguments against the existence of a UV fixed point in QG following from black hole entropy considerations [23, 108] which are not valid in this approach. One can regard the following; QFTs are constructed *on* space-time, classical GR is constructed *of* space-time and in this approach many of the tensions of formulating a theory of QG are resolved as we are constructing the theory *off* space-time.

This is the general behaviour however we can elaborate on this and further aspects will be elucidated in this thesis. To re-iterate we construct the theory with every possible (marginally) relevant underlying bare coupling g_n^σ induced by requiring finite couplings at physical scales, due to the presence of the tower operator we have an infinite number of these. These operators do not respect diffeomorphism invariance and lie in the critical surface in figure 3.3. We then take the physical limit and as a result the coefficient functions, having been chosen careful, then return the effective couplings we are more familiar with. For example at first order we would find

$$\lim_{\Lambda_\sigma \to 0} f^\sigma_\Lambda(\varphi)/\kappa \to 1 \tag{3.166}$$

and in doing so we recover the standard non-trivial solutions to the BRST cohomology equations (3.158), (3.159) and (3.160).

Chapter 4

Perturbatively renormalizable quantum gravity at first order in the coupling

Having outlined the basic structure of this novel approach to QG that follows from combining the Wilsonian RG with the action for free gravitons, whilst considering seriously the wrong sign kinetic term, we must now begin to test if a suitable continuum limit can be found. We begin at the first order in the coupling and re-iterate previously discussed results for clarity before proceeding further. We choose the most general of the coefficient functions that are consistent with the flow equations and verify the universality of the continuum limit. We express the effect of the wrong sign kinetic term on the RG in terms of the Legendre effective action which will aid calculation, following this elaborate on the structure of his coefficient function $f^\sigma_\Lambda(\varphi)$ and how this is parametrised by the truly fundamental underlying couplings g^σ_n. We then further develop the arguments outlined thus far in this thesis, showing how the RG properties determine the form of the dressed interactions and these coefficient functions, this reinforces that this structure follows naturally from the Wilsonian RG, negative kinetic term and the free action for gravitons.

Following these developments we then discuss closed expressions for the tadpole corrections found in the dressed interactions, proving that there exists a dynamically generated amplitude suppression scale Λ_σ that determines the large φ behaviour for the coefficient functions. This is found for all $\Lambda \geq 0$ and also proves that $f^\sigma_\Lambda(\varphi)$ is determined uniquely by this physical limit. As part of this we express the functions in conjugate momentum by an entire function $f^\sigma(\pi)$ whose Taylor expansion coefficients are these underlying couplings g^σ_n.

We then turn our attention to the amplitude suppression scale, showing that it characterises the asymptotic behaviour of these underlying couplings g^σ_n at large n and define what it means for the coefficient functions to trivialise in the large Λ_σ limit, which will play a crucial role in returning standard, consistent results. As discussed in section 3.3.5 these underlying couplings are chosen so as to recover the standard results, re-entering the diffeomorphism invariant subspace at the linearised level. We focus on the simplest case where the coefficient function then tends to a constant however we also discuss that in general it must tend to a Hermite polynomial of degree α whose functional form is then fixed.

In section 4.2.1 we briefly review how to derive new solutions for the coefficient functions from a given one and derive formulae for their underlying couplings. To summarise this is achieved either by multiplying the physical coefficient by a power of φ or differentiating with respect to φ, this will prove to be a useful tool later. Following this aside we characterise the most general form for the coefficient function that trivialise in the large Λ_σ limit, this is most efficiently expressed in terms of the Fourier transform as we have seen previously. In particular we show that $f^\sigma(\pi)$ must tend to the α^{th} derivative of a Dirac δ function. From this we can make two powerful simplifying assumptions. These will still leave us with an infinite dimensional function space flexible enough to encompass the higher order computations, ensuring this structure remains valid there. The first of these assumptions is that the coefficient functions must have definite parity and in the second we insist that at the linearised level the coefficient functions contain only one amplitude suppression scale, leaving a brief discussion of the case where there is a spectrum of suppression scales to the appendix.

With these properties outlined we then have a complete characterisation of $f^\sigma(\pi)$ in terms of both its large and small π behaviour, its normalisation and limiting behaviour of of key integrals in the large amplitude suppression scale limit. Given all this structure we conclude this chapter with the construction of a very general continuum limit to first order and verify that its RG trajectory fulfils the necessary properties of respecting the flow equations and evolving into the diffeomorphism invariant subspace. Finally we discuss the implications of this research and outline where there is room for improvement and open questions, before addressing these concerns in chapter 5.

4.1 Renormalization group properties at the linearised level

To briefly re-iterate, the wrong sign φ propagator

$$\langle \varphi(p)\varphi(-p)\rangle = -\frac{1}{p^2} \tag{4.1}$$

reflects the wrong sign kinetic term for φ in this gauge, which in turn is a reflection of the instability caused by the unboundedness of the Euclidean Einstein-Hilbert action (see [8, 107, 109] for further discussion). The Euclidean partition function is then more than usually ill-defined, which the authors of ref. [9] proposed to solve by analytically continuing the φ integral along the imaginary axis. However this wrong sign does not invalidate the Wilsonian RG flow equations, for example as realised by the Legendre effective action flow equation (3.145), which provide an alternative and anyway more powerful route to defining a continuum limit (see [8, 74, 107, 109] and *e.g.* ref. [20] for further discussion). As shown in refs. [8, 107], the wrong sign then profoundly alters the RG properties that are central to defining such a continuum limit. (For earlier observations see refs. [77, 110].) We review and refine some of those discoveries in this section.

Consider some arbitrary infinitesimal perturbation around the Gaussian fixed point (3.115), whose φ-amplitude dependence[12] is given by $f_\Lambda(\varphi)$. Recalling the wrong sign in the φ propagator (4.1), and using $\dot{C}_\Lambda = -\dot{C}^\Lambda$, the linearised flow equation (3.152) implies that this coefficient function must satisfy

$$\dot{f}_\Lambda(\varphi) = \tfrac{1}{2}\,\dot{\Omega}_\Lambda\, f''_\Lambda(\varphi)\,, \tag{4.2}$$

and

$$\Omega_\Lambda = |\langle\varphi(x)\varphi(x)\rangle| = \int_q \frac{C(q^2/\Lambda^2)}{q^2} = \frac{\Lambda^2}{2a^2} \tag{4.3}$$

is the modulus of the φ tadpole integral regularised by the UV cut-off. Recalling that the now positive sign on the right hand side of this parabolic equation reverses the natural direction of the flow with solutions now only guaranteed in the IR direction. Most importantly, the perturbation can be written as a convergent sum over eigenoperators and their couplings only if the coefficient function is square-integrable under the corresponding Sturm-Liouville measure:

$$\int_{-\infty}^{\infty} d\varphi\ \mathrm{e}^{\varphi^2/2\Omega_\Lambda} f^2_\Lambda(\varphi) < \infty\,, \tag{4.4}$$

where the measure is now a growing exponential. If $f_\Lambda \in \mathfrak{L}_-$, then it can be written as a (typically infinite) linear combination over the operators:

$$\delta^{(n)}_\Lambda(\varphi) := \frac{\partial^n}{\partial\varphi^n}\,\delta^{(0)}_\Lambda(\varphi)\,, \qquad \text{where} \qquad \delta^{(0)}_\Lambda(\varphi) := \frac{1}{\sqrt{2\pi\Omega_\Lambda}}\,\exp\left(-\frac{\varphi^2}{2\Omega_\Lambda}\right) \tag{4.5}$$

(integer $n \geq 0$) with convergence of the sum being in the square-integrable sense under the Sturm-Liouville measure (4.4), under which also the operators are orthonormal:

$$\int_{-\infty}^{\infty} d\varphi\ \mathrm{e}^{\varphi^2/2\Omega_\Lambda}\, \delta^{(n)}_\Lambda(\varphi)\, \delta^{(m)}_\Lambda(\varphi) = \frac{n!}{\Omega_\Lambda^{n+1/2}\sqrt{2\pi}}\,\delta_{nm}\,. \tag{4.6}$$

These $\delta^{(n)}_\Lambda(\varphi)$ are solutions of the linearised flow equation for the coefficient function (4.2), and are nothing but the tower of non-derivative eigenoperators in the φ sector that span $\mathfrak{L}_-$, the general solution of the linearised flow equation in this space being a linear combination of these eigenoperators with constant coefficients, a.k.a. couplings.

[12] *i.e.* its φ dependence other than any dependence through space-time derivatives as in $\partial^m\varphi$

The $\delta^{(n)}_\Lambda(\varphi)$ are all relevant, their scaling dimensions being equal to their engineering dimensions in mass units, namely -1–n. Since $\Omega_\Lambda \propto \hbar$, the $\delta^{(n)}_\Lambda(\varphi)$ are non-perturbative in $\hbar$. It is for this reason that we must develop the theory whilst remaining non-perturbative in $\hbar$. We mention also that they are also evanescent, *i.e.* vanish as $\Lambda \to \infty$, and have the property that the physical operators, gained by sending $\Lambda \to 0$, are $\delta^{(n)}(\varphi)$, the n^{th}-derivatives of the Dirac delta function.

In the $h_{\mu\nu}$ sector and the ghost sector, convergent sums are over eigenoperators that are polynomials in the fields, justifying the usual form of expansion. Altogether, the general eigenoperator can be expressed as [107]

$$\delta^{(n)}_\Lambda(\varphi)\,\sigma(\partial, \partial\varphi, h, c, \Phi^*) + \cdots\,, \tag{4.7}$$

(in gauge invariant minimal basis) where we have displayed the 'top term', σ being a Λ-independent Lorentz invariant monomial involving some or all of the components indicated, in particular the arguments $\partial\varphi, h, c, \Phi^*$ can appear as they are, or differentiated any number of times. If $d_\sigma = [\sigma]$ is its engineering dimension, then the scaling dimension of the corresponding eigenoperator is just the sum of the engineering dimensions, namely $d_\sigma - 1 - n$. Notice that undifferentiated φ does not appear in σ but only in $\delta^{(n)}_\Lambda(\varphi)$. The tadpole operator in the linearised flow equation (3.152) generates a finite number of Λ-dependent UV regulated tadpole corrections involving less fields in σ. These are the terms we indicate with the ellipses. They are formed by attaching the propagators (3.140) – (3.143) (in gauge fixed basis) in all possible ways according to the usual rules of Wick contraction, but excluding φ tadpoles connected only to $\delta^{(n)}_\Lambda(\varphi)$, since these are already accounted for through the flow equation for the coefficient function (4.2).

In fact we can give the general eigenoperator (4.7) in closed form. Note that the linearised flow equation (3.152) implies

$$\dot\Gamma_1 = -\frac{1}{2}\dot\triangle^{\Lambda\,AB}\frac{\partial_l^2}{\partial\Phi^B\partial\Phi^A}\,\Gamma_1\,, \tag{4.8}$$

where $\triangle^{\Lambda\,AB} = C^\Lambda \triangle^{AB}$ is the UV regulated propagator. The solution we need is therefore

$$\exp\left(-\frac{1}{2}\triangle^{\Lambda\,AB}\frac{\partial_l^2}{\partial\Phi^B\partial\Phi^A}\right)\Gamma_{1\,\text{phys}}\,, \qquad \text{where} \quad \Gamma_{1\,\text{phys}} = \sigma\,\delta^{(n)}(\varphi)\,, \tag{4.9}$$

since at $\Lambda = 0$, $\delta^{(n)}_\Lambda(\varphi) = \delta^{(n)}(\varphi)$ and all the tadpole corrections vanish. The exponential operator then just generates all the Wick contractions[13] for the propagator which appears here as $-\triangle^\Lambda$, as illustrated in fig. 3.2. For each functional derivative we can

[13] In particular ghost propagators count an overall $\frac{1}{2}\times(-2) = -1$ through $\langle c\bar c\rangle$ and $\langle \bar c c\rangle$ and statistics.

write by the Leibniz rule

$$\frac{\partial_l}{\partial\Phi^A} = \frac{\partial_l^L}{\partial\Phi^A} + \frac{\partial_l^R}{\partial\Phi^A} \tag{4.10}$$

where ∂^L acts only on the left-hand factor (i.e. it acts on objects on the left, with derivatives being evaluated on those objects only), here σ, and ∂^R acts only on the right-hand factor (i.e. it acts on objects on the right, with derivatives being evaluated on those objects only) [103], here $\delta^{(n)}(\varphi)$. Thus (factoring out $-C^\Lambda$ for later convenience):

$$\frac{1}{2}\triangle^{AB}\frac{\partial_l^2}{\partial\Phi^B\partial\Phi^A} = \frac{1}{2}\triangle^{AB}\frac{\partial_l^{L\,2}}{\partial\Phi^B\partial\Phi^A} + \triangle^{AB}\frac{\partial_l^L}{\partial\Phi^B}\frac{\partial_l^R}{\partial\Phi^A} + \frac{1}{2}\triangle^{AB}\frac{\partial_l^{R\,2}}{\partial\Phi^B\partial\Phi^A}\,. \tag{4.11}$$

The exponential in the eigenoperator solution (4.9) therefore factors into three exponentials. Since $\delta^{(n)}(\varphi)$ only depends on φ, the third exponential collapses to [8]:

$$\exp\left(-\frac{1}{2}\triangle^{\Lambda\,AB}\frac{\partial_l^{R\,2}}{\partial\Phi^B\partial\Phi^A}\right)\delta^{(n)}(\varphi) = \mathrm{e}^{\frac{1}{2}\Omega_\Lambda\partial_\varphi^2}\,\delta^{(n)}(\varphi) = \partial_\varphi^n\int_{-\infty}^{\infty}\frac{d\pi}{2\pi}\,\mathrm{e}^{-\frac{1}{2}\pi^2\Omega_\Lambda+i\pi\varphi} = \delta_\Lambda^{(n)}(\varphi)\,, \tag{4.12}$$

where we used the φ propagator (3.143), giving the tadpole integral (4.3) and derivatives ∂_φ with respect to the amplitude (*i.e.* no longer functional), and expressed the result in conjugate momentum π space, after which the integral evaluates to the expression (4.5) for the $\delta_\Lambda^{(n)}(\varphi)$ operators. Thus the entire eigenoperator can be written as

$$\exp\left(-\triangle^{\Lambda\,\varphi\varphi}\frac{\partial^L}{\partial\varphi}\frac{\partial^R}{\partial\varphi}\right)\left\{\exp\left(-\frac{1}{2}\triangle^{\Lambda\,AB}\frac{\partial_l^2}{\partial\Phi^B\partial\Phi^A}\right)\sigma\right\}\delta_\Lambda^{(n)}(\varphi)\,, \tag{4.13}$$

where the term in braces expresses all the tadpole corrections acting purely on σ, and the left-most term generates φ-propagator (3.143) corrections that attach to both σ and $\delta_\Lambda^{(n)}(\varphi)$ (each such attachment will increase $n\mapsto n{+}1$).

A simple example eigenoperator [107] will prove useful later:

$$-\,\partial_\mu c_\nu H^*_{\mu\nu}\delta_\Lambda^{(n)}(\varphi) + 2b\Lambda^4\delta_\Lambda^{(n)}(\varphi)\,. \tag{4.14}$$

The second term has the ghost tadpole correction to the top monomial $\sigma = -\partial_\mu c_\nu H^*_{\mu\nu}$. (To see this immediately, substitute the $SO(4)$ decomposition (2.6) into the $\hat{s}_0$-exact addition and integrate by parts)

The continuum limit is described by the renormalized trajectory, close to the fixed point, the linearised approximation is justified. The interaction there is therefore expanded only over the marginal and relevant eigenoperators (4.7) with constant couplings g_n^σ whose mass-dimensions

$$[g_n^\sigma] = 4 - (d_\sigma{-}1{-}n) = 5 + n - d_\sigma\,, \tag{4.15}$$

must all be non-negative. Every monomial σ is therefore associated to an infinite tower of operators, which can be subsumed into

$$f^\sigma_\Lambda(\varphi)\,\sigma(\partial,\partial\varphi,h,c,\Phi^*)+\cdots=\exp\left(-\triangle^{\Lambda\,\varphi\varphi}\frac{\partial^L}{\partial\varphi}\frac{\partial^R}{\partial\varphi}\right)\left\{\exp\left(-\frac{1}{2}\triangle^{\Lambda\,AB}\frac{\partial_l^2}{\partial\Phi^B\partial\Phi^A}\right)\sigma\right\}f^\sigma_\Lambda(\varphi)\,, \tag{4.16}$$

where the coefficient function of the top term is given by (at the linearised level)

$$f^\sigma_\Lambda(\varphi)=\sum_{n=n_\sigma}^\infty g^\sigma_n\delta^{(n)}_\Lambda(\varphi)\,, \tag{4.17}$$

and the tadpole corrections are the same as before (now with f^σ_Λ differentiated according to the number of times the left-most operator acts on it). In general all the (marginally) relevant couplings $[g^\sigma_n]\geq 0$ will be needed [8] and thus at the linearised level

$$n_\sigma=\max(0,d_\sigma-5)\,. \tag{4.18}$$

For $d_\sigma\geq 5$, we are thus including the marginal coupling $[g^\sigma_{n_\sigma}]=0$.

The eigenoperators (4.7,4.13) span the complete (Hilbert) space $\mathfrak{L}$ of interactions whose combined amplitude dependence is square integrable under the Sturm-Liouville measure

$$\exp\frac{1}{2\Omega_\Lambda}\left(\varphi^2-h^2_{\mu\nu}-2\,\bar{c}_\mu c_\mu\right)\,. \tag{4.19}$$

At the bare level we require that Γ_I is inside $\mathfrak{L}$, so that expansion over eigenoperators is meaningful. This as the quantisation condition that is thus both natural and necessary for the Wilsonian RG as outlined in section 3.2.2. However, since we will be solving for Γ_I directly in the continuum, our bare cut-off is already sent to infinity. Then this condition is replaced by the requirement that $\Gamma_I\in\mathfrak{L}$ for sufficiently large Λ, where as a consequence we also have $f^\sigma_\Lambda\in\mathfrak{L}_-$.

Recall that we define the amplitude suppression scale $\Lambda_\sigma\geq 0$ to be the smallest scale such that for all $\Lambda>a\Lambda_\sigma$, the coefficient function is inside $\mathfrak{L}_-$. The coefficient function exits $\mathfrak{L}_-$ as Λ falls below $a\Lambda_\sigma$, either because it develops singularities after which the flow to the IR ceases to exist, or because it decays too slowly at large φ.

We need to choose the g^σ_n so that the flow all the way to $\Lambda\to 0$ does exist, so that all modes can be integrated over and so that the physical Legendre effective action (3.144) can be defined. Note that we mean by Γ_{phys} the resulting $\Lambda\to 0$ limit, thus removing the infrared cut-off ($\lim_{\Lambda\to 0}C_\Lambda=0$). The results are not yet physical in terms of properly incorporating diffeomorphism invariance. That requires another limit as we will shortly see.

Since the coefficient function thus exits $\mathfrak{L}_-$ by decaying too slowly, we know from the square-integrability condition (4.4) that asymptotically:

$$f^\sigma_{a\Lambda_\sigma}(\varphi)\propto A_\sigma\,\mathrm{e}^{-\varphi^2/4\Omega_{a\Lambda_\sigma}+o(\varphi^2)}=A_\sigma\,\mathrm{e}^{-\varphi^2/2\Lambda^2_\sigma+o(\varphi^2)}\,, \tag{4.20}$$

for at least one of $\varphi \to \pm\infty$, with the other side decaying at the same rate or faster, where

$$[A_\sigma] = 4 - d_\sigma \tag{4.21}$$

is a dimensionful constant, and $o(\cdots)$ is a dimensionless term of either sign that grows slower than its argument. (Because of the presence of such undetermined terms, the asymptotic formula (4.20) only yields A_σ up to a dimensionless proportionality constant.)

The asymptotic behaviour (4.20) gives us a boundary condition which then fixes the solution of the linearised flow equation (4.2) at large φ. Thus we find (at the linearised level) the asymptotic behaviour for any Λ:

$$f^\sigma_\Lambda(\varphi) \propto A_\sigma \exp\left(-\frac{a^2\varphi^2}{\Lambda^2 + a^2\Lambda_\sigma^2} + o(\varphi^2)\right) \tag{4.22}$$

(on at least one side with the other side being the same rate or faster). From the requirement for square-integrability under the Sturm-Liouville measure, *cf.* (4.4), our definition of Λ_σ is verified: $f^\sigma_\Lambda \in \mathfrak{L}_-$ for all $\Lambda > a\Lambda_\sigma$, while $f^\sigma_\Lambda \notin \mathfrak{L}_-$ for $\Lambda < a\Lambda_\sigma$ (in fact for all such Λ).

Setting $\Lambda = 0$ shows that the physical coefficient function $f^\sigma_{\text{phys}}(\varphi)$, which following [107] we write simply as $f^\sigma(\varphi)$, is characterised by the decay (on at least one side with the other side being the same rate or faster):

$$f^\sigma(\varphi) \propto A_\sigma\, \mathrm{e}^{-\varphi^2/\Lambda_\sigma^2 + o(\varphi^2)}\,. \tag{4.23}$$

It appears as

$$f^\sigma(\varphi)\,\sigma(\partial, \partial\varphi, h, c, \Phi^*) \tag{4.24}$$

in the (physical) Legendre effective action, the regularised tadpole corrections in the $\Lambda > 0$ solution (4.16) having all vanished, since they are all proportional to positive powers of Λ. The asymptotic property for the physical coefficient function (4.23) is the motivation for calling Λ_σ the amplitude suppression scale, or amplitude decay scale [8, 107].

From the linearised flow equation for the coefficient function (4.2), this solution can be written in terms of the Fourier transform over π:

$$f^\sigma_\Lambda(\varphi) = \int_{-\infty}^{\infty} \frac{d\pi}{2\pi}\, \mathfrak{f}^\sigma(\pi)\, \mathrm{e}^{-\frac{\pi^2}{2}\Omega_\Lambda + i\pi\varphi}\,, \tag{4.25}$$

where $\mathfrak{f}^\sigma$ is Λ-independent and is thus the Fourier transform of the physical $f^\sigma(\varphi)$. From the expansion of the coefficient function in terms of $\delta^{(n)}_\Lambda(\varphi)$ operators (4.17) and the Fourier transform expression for these operators (4.12), the couplings are its Taylor expansion coefficients:

$$\mathfrak{f}^\sigma(\pi) = \sum_{n=n_\sigma}^{\infty} g_n^\sigma (i\pi)^n \,. \tag{4.26}$$

Since the asymptotic behaviour of the physical coefficient function (4.23) ensures that the inverse Fourier transform exists for all complex π, $\mathfrak{f}^\sigma$ is an entire holomorphic function (Paley-Wiener theorem)[14]. The asymptotic behaviour of the physical coefficient function (4.23) is reproduced by setting $\mathfrak{f}^\sigma(\pi)$ proportional to

$$A_\sigma \Lambda_\sigma \, \mathrm{e}^{-\pi^2\Lambda_\sigma^2/4+o(\pi^2)} \,, \tag{4.27}$$

which also reproduces the asymptotic behaviour (4.22) at $\Lambda > 0$. However at this stage it needs to be interpreted with care since it captures only the fastest decaying part, corresponding to the slowest decaying behaviour in φ-space. (See app. A.1 for an example. This corrects part of the characterisation given in ref. [107].) It does however control the large-n behaviour of the couplings:

$$g_n^\sigma \propto A_\sigma \left(\frac{\mathrm{e}}{2n}\right)^{\frac{n}{2}} \Lambda_\sigma^{n+1} \, \mathrm{e}^{o(n)} \qquad \text{as} \quad n \to \infty \,, \tag{4.28}$$

where we Taylor expanded the asymptotic formula for the Fourier transform (4.27) and used Stirling's approximation. Indeed from the expansion of the coefficient function in terms of the $\delta_\Lambda^{(n)}(\varphi)$ operators (4.17), square integrability under the Sturm-Liouville measure, as in (4.4), and the orthonormality relations (4.6), we see that

$$\int_{-\infty}^{\infty} \! d\varphi \; \mathrm{e}^{\varphi^2/2\Omega_\Lambda} \, (f_\Lambda^\sigma)^2 = \frac{1}{\sqrt{2\pi}} \sum_{n=n_\sigma}^{\infty} n! \, (g_n^\sigma)^2 / \, \Omega_\Lambda^{n+\frac{1}{2}} < \infty \qquad \text{for} \quad \Lambda > a\Lambda_\sigma \,. \tag{4.29}$$

By its definition, $\Lambda = a\Lambda_\sigma$ marks the radius of convergence, and thus we see that g_n^σ must at large n behave roughly like $\sqrt{\Omega^n_{a\Lambda_\sigma}/n!}$. Using Stirling's approximation we regain the asymptotic formula for the couplings (4.28) (up to sign dependence). This large-n behaviour also verifies that $\mathfrak{f}^\sigma$ is entire.

As mentioned already below (4.2), flows in the φ-sector are guaranteed to exist in the reverse direction, *i.e.* from the IR towards the UV. In particular, the linearised $f_\Lambda^\sigma(\varphi)$ exists for all $\Lambda \geq 0$ and is unique, once the coefficient function at $\Lambda = 0$ is specified, as is also clear from the Fourier integral representation (4.25). Given the asymptotic behaviour for the physical coefficient function (4.23), this is also clear from the Green's function representation:

$$f_\Lambda^\sigma(\varphi) = \int_{-\infty}^{\infty} \! d\varphi_0 \, f^\sigma(\varphi_0) \, \delta_\Lambda^{(0)}(\varphi - \varphi_0) \,. \tag{4.30}$$

[14] Then since $\mathfrak{f}^\sigma$ is also square integrable, the exponential decay part in the Fourier integral solution (4.25) ensures that the Fourier integral converges for all complex φ provided $\Lambda > 0$, and thus that $f^\sigma_{\Lambda>0}(\varphi)$ is also an entire holomorphic function.

It is clear that this is the Green's function representation since it satisfies the linearised flow equation for the coefficient function (4.2) by virtue of the fact that the shifted eigenoperator $\delta^{(0)}_\Lambda(\varphi-\varphi_0)$ does, and returns the boundary condition in the limit $\Lambda\to 0$, since in this limit $\delta^{(0)}_\Lambda(\varphi-\varphi_0)\to\delta(\varphi-\varphi_0)$ [8]. Thus $\delta^{(0)}_\Lambda(\varphi-\varphi_0)$ is in fact the *Heat kernel* for the diffusion equation (4.2). By Taylor expanding $\delta^{(0)}_\Lambda(\varphi-\varphi_0)$ about φ, we recover the expansion of the coefficient function over $\delta^{(n)}_\Lambda(\varphi)$ operators (4.17) (and the series converges for $\Lambda > a\Lambda_\sigma$), and read off a formula for the couplings in terms of the moments of the physical coefficient function [8]:

$$g^\sigma_n = \frac{(-)^n}{n!}\int_{-\infty}^{\infty} d\varphi\, \varphi^n\, f^\sigma(\varphi) \tag{4.31}$$

We see therefore that the general form of the solution is given by specifying the physical coefficient function. At this stage it is subject only to the constraints that it satisfy the asymptotic condition (4.23) and be such that its Taylor expanded Fourier transform (4.26) has vanishing coefficients for $\pi^{n<n_\sigma}$, equivalently that its moments (4.31) vanish for $n<n_\sigma$. Indeed the asymptotic property (4.23) of this $\Lambda=0$ boundary condition, implies the asymptotic solution (4.22) at $\Lambda>0$, which verifies that $\Lambda=a\Lambda_\sigma$ marks the point above which $f^\sigma_\Lambda\in\mathfrak{L}_-$. Substituting the Taylor expansion formula (4.26) for the Fourier transform into the Fourier transform solution (4.25) gives back the expansion of the coefficient function in terms of $\delta^{(n)}_\Lambda(\varphi)$ operators (4.17) which converges for $\Lambda > a\Lambda_\sigma$ and describes a valid renormalized trajectory in the linearised regime.

4.2 Trivialisation in the limit of large amplitude suppression scale

All of the above properties for the linearised solutions are inevitable consequences of respecting the wrong sign kinetic term for the conformal factor φ, while insisting that the Wilsonian RG remains meaningful. However this general form must now be married with the first order BRST constraint (3.153). In ref. [107], it was shown that this is possible only if the coefficient function trivialises in the sense defined below,[15] and we showed that such trivialisations are possible if we now send Λ_σ to infinity. In other words, we can arrange for violations of BRST to be as small as desired by taking sufficiently large Λ_σ. In this way, at first order, we get both the continuum limit and diffeomorphism invariance of the renormalized solution.

In the majority of cases the coefficient function has to become φ-independent, *i.e.*

[15] In the final two paragraphs of section 7.2 of [107] we referred to "non-constant" coefficient functions, where we should have written "non-trivial" as in the current sense.

we need linearised renormalized trajectories that satisfy:

$$f^\sigma_\Lambda(\varphi) \to A_\sigma \qquad \text{as} \quad \Lambda_\sigma \to \infty\,, \tag{4.32}$$

(where we hold Λ, φ and A_σ fixed and finite) such that also its φ-derivatives have a limit, which is thus that they vanish. However if BRST invariance demands a physical vertex of the same dimension but containing an undifferentiated φ^α factor (α a positive integer), then this would appear as

$$\sigma = \varphi^\alpha\, \sigma_\alpha(\partial, \partial\varphi, h, c, \Phi^*) \tag{4.33}$$

in the physical vertex (4.24), where thus the new monomial σ_α has

$$d_{\sigma_\alpha} = d_\sigma - \alpha\,, \tag{4.34}$$

and the φ^α amplitude dependence must be absorbed by the physical coefficient function.

This will correspond to linearised renormalized trajectories satisfying

$$f^{\sigma_\alpha}_\Lambda(\varphi) \to A_\sigma\,(\Lambda/2ia)^\alpha\, H_\alpha(ai\varphi/\Lambda) \qquad \text{as} \quad \Lambda_{\sigma_\alpha} \to \infty\,, \tag{4.35}$$

such that also their φ-derivatives have a limit, where H_α is the α^{th} Hermite polynomial. This follows because

$$(\Lambda/2ia)^\alpha\, H_\alpha(ai\varphi/\Lambda) = \varphi^\alpha + \alpha(\alpha-1)\,\Omega_\Lambda \varphi^{\alpha-2}/2 + \cdots \tag{4.36}$$

is the unique solution of the linearised flow equation for the coefficient function (4.2) with the boundary condition that it just becomes φ^α at $\Lambda = 0$.[16]

Notice that the above conditions (4.35,4.36) actually apply also at $\alpha = 0$, where they just give back the original limit (4.32) as a special case. Since we require the φ-derivatives to have a limit, by l'Hôpital's rule this limit is given by the φ-derivative of the right hand side.

We say that a coefficient function trivialises in the limit of large amplitude suppression scale if it satisfies the limiting condition (4.35) for some α. Since at finite Λ_{σ_α} (with $\sigma = \sigma_\alpha$), the coefficient functions satisfy the asymptotic formula (4.22), they are *non-trivial*, in particular they cannot be polynomial in φ.

From the asymptotic formula for the couplings (4.28) we see that the g^σ_n must diverge in the limit $\Lambda_\sigma \to \infty$. However the vertices are nevertheless well behaved since the coefficient function goes smoothly over to A_σ as in the limiting condition (4.32), or more generally to the finite polynomial in (4.35). What is happening is that the $\Lambda = a\Lambda_\sigma$ boundary, above which $f^\sigma_\Lambda(\varphi)$ enters $\mathfrak{L}_-$, is being sent to ever higher scales. In this sense we are taking a limit towards the boundary of this Hilbert space (and thus

[16]These polynomials are nothing but the eigenoperators in the standard quantisation of a scalar field , analytically continued along the imaginary φ axis , which destroys their Hilbert space properties .

also $\mathfrak{L}$) [107, 109].

Actually, from the asymptotic formula for the couplings (4.28), we can keep g^σ_n perturbative in this limit if we choose A_σ to vanish fast enough with Λ_σ. For example if we set $A_\sigma = a_\sigma\, \mathrm{e}^{-\Lambda_\sigma/\mu}$ for fixed a_σ and μ, then for any finite n, the couplings $g^\sigma_n \to 0$ as $\Lambda_\sigma \to \infty$. Although this means that the coefficient function, and thus the vertex itself, vanishes in the limit, this does not stop us from computing perturbative corrections in the usual way [107], as reviewed in section 4.3. We can also choose A_σ to vanish fast enough to ensure that couplings remain uniformly perturbative (as opposed to pointwise in n as in the above example). From the asymptotic formula for the couplings (4.28) one sees that for large Λ_σ, they first grow with n and then decay once the $n^{-n/2}$ factor dominates. Thus we can estimate the maximum size coupling by differentiating with respect to n and finding the stationary point. We find

$$g^\sigma_{n_{\max}} \propto A_\sigma \Lambda_\sigma\, \mathrm{e}^{\Lambda^2_\sigma/4} \qquad \text{at} \qquad n = n_{\max} = \Lambda^2_\sigma/2\,, \tag{4.37}$$

which implies that we can keep the couplings uniformly perturbative if we set A_σ to vanish faster than $\Lambda^{-1}_\sigma\, \mathrm{e}^{-\Lambda^2_\sigma/4}$. The above result already suggests that it is the large-n g^σ_n couplings that should be important in the limit of large amplitude suppression scale. We will see this more dramatically from a different point of view in ref. [2].

4.2.1 Relations

In this subsection, we pause the main development to explore two rather natural ways for generating new solutions. The first increases α, while the second decreases it. We will see however that the maps are not inverses of each other, but rather when combined generate yet further solutions. This illustrates that there are infinitely many solutions for coefficient functions, with the same trivialisation. The formulae we will derive are then used in the next section to arrive at the general form, in section 4.2.3 and app. A.1 to generate examples with illustrative properties, and in section 4.3 to explain the properties of special limiting cases.

On the one hand, we can convert any solution to flat trivialisation limit (4.32), into one satisfying the polynomial trivialisation limit (4.35), by multiplying the physical coefficient function by φ^α and using the fact that the flow to all $\Lambda > 0$ then exists and is unique. Recalling that we defined $o(\cdots)$ to be dimensionless, we thus identify from the asymptotic formula for the physical coefficient function (4.23):

$$\Lambda_{\sigma_\alpha} = \Lambda_\sigma \qquad \text{and} \qquad A_{\sigma_\alpha} = A_\sigma \Lambda^\alpha_\sigma\,, \tag{4.38}$$

where Λ_{σ_α} is the amplitude suppression scale, and A_{σ_α} the dimensionful constant, in the asymptotic behaviour of the physical coefficient function associated to the new monomial σ_α. Using the Fourier representation of the solution (4.25) at $\Lambda = 0$, and integration by parts, we see that the new physical coefficient function is given by setting:

$$\mathfrak{f}^{\sigma_\alpha}(\pi) = (i\partial_{\pi})^\alpha \mathfrak{f}^\sigma(\pi). \tag{4.39}$$

We confirm that $\mathfrak{f}^{\sigma_\alpha}(\pi)$ thus satisfies the same general Taylor expansion formula (4.26), with

$$n_{\sigma_\alpha} = \max(0, d_{\sigma_\alpha} - 5)\,, \tag{4.40}$$

i.e. defined as in the previous minimum index (4.18), since

$$n_{\sigma_\alpha} = n_\sigma - \alpha = d_\sigma - \alpha - 5 = d_{\sigma_\alpha} - 5\,, \tag{4.41}$$

unless $d_{\sigma_\alpha} < 5$ in which case $n_{\sigma_\alpha} = 0$. Reading off the couplings from the Taylor expansion formula (4.26) and the Fourier transform of the new physical coefficient function (4.39), we have

$$\begin{gathered}\mathfrak{f}^{\sigma'}(\pi) = \sum_{n'=n_{\sigma'}}^{\infty} g^{\sigma'}_{n'}(i\pi)^{n'} \quad \text{where} \\ g^{\sigma_\alpha}_n = (-)^\alpha (n+1)(n+2)\cdots(n+\alpha)\, g^\sigma_{n+\alpha} = (-)^\alpha \frac{(n+\alpha)!}{n!}\, g^\sigma_{n+\alpha}\,.\end{gathered} \tag{4.42}$$

Using this, the asymptotic formula for the couplings (4.28) and the conversion formulae from σ to σ_α (4.38), we confirm that in terms of the appropriate σ_α-labelled quantities, these couplings have the expected limiting behaviour at large n.

On the other hand, thanks to the recurrence relation $H'_\alpha(x) = \alpha H_{\alpha-1}(x)$, one easily verifies that taking the φ-derivative of the polynomial trivialisation (4.35) just maps it to (α times) the $(\alpha-1)^{\text{th}}$ case, as it must since the derivative is still a solution of the flow equation for the coefficient function (4.2) and the result is determined by the physical ($\Lambda \to 0$) limit, in this case $\alpha A_\sigma \varphi^{\alpha-1}$. Of course this does not mean in general that $f^{\sigma_\alpha}_\Lambda{}'(\varphi) = \alpha f^{\sigma_{\alpha-1}}_\Lambda(\varphi)$, since there are infinitely many solutions with these limits. Indeed while f^{σ_α} satisfies the minimum index property (4.40) for each α in general, the coefficient function defined by

$$f^{\sigma'_{\alpha-1}}_\Lambda(\varphi) := \frac{1}{\alpha} f^{\sigma_\alpha}_\Lambda{}'(\varphi) \tag{4.43}$$

is more restricted. From the Fourier transform representation of the solution (4.25) and its Taylor expansion (4.26) we see that it has couplings

$$g^{\sigma'_{\alpha-1}}_n = \frac{1}{\alpha}\, g^{\sigma_\alpha}_{n-1}\,, \tag{4.44}$$

with the lowest n in the sum thus being

$$n_{\sigma'_{\alpha-1}} = \max(1, d_{\sigma_\alpha} - 4) = \max(1, d_{\sigma_{\alpha-1}} - 5)\,, \tag{4.45}$$

4.2.2 Simplifications and general form

In order to check the universal nature of the final result, we want to work with very general solutions for linearised coefficient functions satisfying the required trivialisation constraints (4.32,4.35). These not only determine the form of the interactions at the linearised level, but then contribute at the non-linear level through higher order contributions in the perturbative expansion (3.151). As will become clear [75], the most powerful way to handle these higher order contributions is to express the solutions in conjugate momentum space. Thus we use the fact that the linearised coefficient functions are given by the Fourier transform solution (4.25) via a Λ-independent $\mathfrak{f}^\sigma(\pi)$ which, from its Taylor expansion (4.26) and the discussion below it, we know can be written as an entire function times a π^{n_σ} factor. The flat trivialisation constraint (4.32) is equivalent to

$$\mathfrak{f}^\sigma(\pi) \to 2\pi A_\sigma \, \delta(\pi) \qquad \text{as} \quad \Lambda_\sigma \to \infty\,, \tag{4.46}$$

understood in the usual distributional sense (see also below) while more generally from the polynomial trivialisation constraint (4.35):

$$\mathfrak{f}^{\sigma_\alpha}(\pi) \to 2\pi A_\sigma \, i^\alpha \delta^{(\alpha)}(\pi) \qquad \text{as} \quad \Lambda_\sigma \to \infty\,, \tag{4.47}$$

as we see immediately from the Fourier transform flat trivialisation constraint (4.46) and the map to a Fourier transform for a coefficient function satisfying the polynomial trivialisation constraint (4.39), and which includes the flat one (4.46) as the special case $\alpha = 0$. (From here on for notational simplicity, we use the conversion formulae (4.38) to write $\Lambda_{\sigma_\alpha} = \Lambda_\sigma$.)

These constraints evidently still leave us with a huge (infinite dimensional) function space of renormalized trajectories. We now make two further restrictions that do not result in any significant loss of generality but greatly strengthen and streamline the analysis.

Firstly, we insist that the coefficient functions are of definite parity, *i.e.* even or odd functions of φ. Thus those satisfying the flat trivialisation constraint (4.32) will be even parity, and those satisfying the polynomial trivialisation constraint (4.35) will be even or odd, depending on whether α is even or odd respectively. This also implies the same of $\mathfrak{f}^{\sigma_\alpha}(\pi)$ in the Fourier transform trivialisation constraints (4.46,4.47), and enforces that the asymptotic estimates for the coefficient function and its physical limit (4.22, 4.23) apply for both limits $\varphi \to \pm\infty$. We see from either the expansion of the coefficient function in terms of $\delta^{(n)}_\Lambda(\varphi)$ operators (4.17) or the Taylor expansion of its Fourier transform (4.26), that the couplings $g^{\sigma_\alpha}_n$ will be indexed by an integer of the same parity, and in particular the minimum index (4.40) required in order that the coefficient function represents a linearised renormalized trajectory, actually has this parity, so now n_{σ_α} is the smallest index of the same parity as α such that

$$n_{\sigma_\alpha} \geq \max(0, d_{\sigma_\alpha} - 5)\,. \tag{4.48}$$

Secondly we insist that such linearised solutions contain only one amplitude suppression scale, so that the asymptotic estimate for their Fourier transform (4.27) now genuinely captures their large π behaviour[17]. Then for cases satisfying flat trivialisation (4.46) we have that

$$\mathfrak{f}^\sigma(\pi) = 2\pi A_\sigma\, \Lambda_\sigma\, \bar{\pi}^{n_\sigma}\, \bar{\mathfrak{f}}^\sigma(\bar{\pi}^2)\,, \tag{4.49}$$

where n_σ is even *i.e.* satisfies the minimum index n_{σ_α} formula (4.48) for $\alpha\!=\!0$, $\bar{\pi} = \Lambda_\sigma \pi$ is dimensionless, and $\bar{\mathfrak{f}}^\sigma$ is a dimensionless entire function which from the asymptotic formula for the Fourier transform (4.27) takes the form

$$\bar{\mathfrak{f}}^\sigma(\bar{\pi}^2) = \mathrm{e}^{-\bar{\pi}^2/4 + o(\bar{\pi}^2)}\,, \tag{4.50}$$

at large $\bar{\pi}$. Likewise for general α,

$$\mathfrak{f}^{\sigma_\alpha}(\pi) = 2\pi\, i^\alpha A_\sigma\, \Lambda_\sigma^{\alpha+1}\, \partial^\alpha_{\bar{\pi}} \left[\bar{\pi}^{\bar{n}_{\sigma_\alpha}}\, \bar{\mathfrak{f}}^{\sigma_\alpha}(\bar{\pi}^2)\right]\,, \tag{4.51}$$

where $\bar{\pi}$ has the same definition, and $\bar{\mathfrak{f}}^{\sigma_\alpha}$ is also a dimensionless entire function satisfying the reduced asymptotic formula (4.50). Note that the $\Lambda_\sigma^{\alpha+1}$ factor is fixed by dimensions, *e.g.* using the polynomial trivialisation formula (4.47). Together with A_σ, these factors appear in the same form as cases satisfying flat trivialisation (4.49) if we use the identifications in the conversion formula (4.38).

Note that the parity is carried by $\partial^\alpha_{\bar{\pi}}$, and thus $\bar{n}_{\sigma_\alpha}$ is even. If α is even and $n_{\sigma_\alpha}\!=\!0$ we do not require a separate $\bar{\pi}$ power, likewise if α is odd and $n_{\sigma_\alpha} = 1$ since the $\partial_{\bar{\pi}}$ differentials will generate a Taylor expansion with only odd powers of $\bar{\pi}$. However if the minimum index n_{σ_α} defined in (4.48), is larger than these absolute minima, then the Taylor expansion of the term in square brackets must be such that all powers $\bar{\pi}^{n>\alpha}$ are missing up to the point where we are left with an overall factor of $\bar{\pi}^{n_{\sigma_\alpha}}$ after differentiation by $\partial^\alpha_{\bar{\pi}}$. Without loss of generality we capture this by factoring out this power, leaving behind a function that is still entire. Thus we see that

$$\bar{n}_{\sigma_\alpha} = 0 \quad \text{if} \quad n_{\sigma_\alpha} = \varepsilon\,, \qquad \text{otherwise} \quad \bar{n}_{\sigma_\alpha} = n_{\sigma_\alpha} + \alpha\,, \tag{4.52}$$

where we define $\varepsilon = 0$ or 1 according to whether the coefficient function is even or odd.

The flat trivialisation constraint in Fourier transform space (4.47) is then satisfied (on finite smooth functions) provided that (for $n\!\geq\!0$)

$$\int_{-\infty}^{\infty} \frac{d\pi}{2\pi}\, \frac{(i\pi)^n}{n!}\, \mathfrak{f}^{\sigma_\alpha}(\pi) \to A_\sigma\, \delta_{n\alpha} \qquad \text{as} \quad \Lambda_\sigma \to \infty \tag{4.53}$$

[17]Examples where a spectrum of amplitude suppression scales appear were considered in ref. [107], and are further developed in app. A.1.

(or we get these constraints directly from the physical limit $A_\sigma\,\varphi^\alpha$, by Taylor expanding the Fourier representation (4.25) in φ), and from the general formula for cases satisfying the polynomial trivialisation constraint (4.51) these are in turn satisfied if $\bar{\mathfrak{f}}^{\sigma_\alpha}$ is normalised as

$$\int_{-\infty}^{\infty} d\bar{\pi}\;\bar{\pi}^{\bar{n}_{\sigma_\alpha}}\,\bar{\mathfrak{f}}^{\sigma_\alpha}(\bar{\pi}^2) \;=\; 1\,, \tag{4.54}$$

and provided that for any integer $p>0$, we have

$$\frac{1}{\Lambda_\sigma^{2p}}\int_{-\infty}^{\infty} d\bar{\pi}\;\bar{\pi}^{\bar{n}_{\sigma_\alpha}+2p}\,\bar{\mathfrak{f}}^{\sigma_\alpha}(\bar{\pi}^2) \;\to\; 0\,, \qquad \text{as} \quad \Lambda_\sigma\to\infty\,. \tag{4.55}$$

(These integrals converge for large $\bar{\pi}$ by virtue of the asymptotic formula (4.50).)

At first order in the perturbation theory (3.151), $\bar{\mathfrak{f}}^{\sigma_\alpha}$ can be chosen to be a finite function and independent of Λ_σ, and thus the vanishing limits (4.55) follow trivially. At second order in perturbation theory, we will find that we need linearised coefficient functions for which $\bar{\mathfrak{f}}^{\sigma_\alpha}$ depends on Λ_σ. In the majority of cases we can choose it to tend to a finite function as $\Lambda_\sigma\to\infty$, but exceptionally it will prove useful to allow it to contain terms with coefficients that diverge logarithmically with Λ_σ. Clearly this mild divergence is well within the bounds implied by the vanishing limits (4.55).

Substituting the general formula for cases satisfying flat trivialisation (4.49) into the Fourier transform representation of the solution (4.25) gives

$$f^\sigma_\Lambda(\varphi) = A_\sigma \int_{-\infty}^{\infty} d\bar{\pi}\;\bar{\pi}^{\bar{n}_\sigma}\,\bar{\mathfrak{f}}^\sigma(\bar{\pi}^2)\,\exp\left(-\frac{\bar{\pi}^2}{4}\frac{\Lambda^2}{a^2\Lambda_\sigma^2}+i\bar{\pi}\,\frac{\varphi}{\Lambda_\sigma}\right)\,. \tag{4.56}$$

Using the normalisation limit (4.54) and the vanishing limits (4.55) we thus confirm that flat trivialisation (4.32) is satisfied, and see that at large but finite Λ_σ the remaining dependence is on Λ^2 and φ^2 as dictated (at leading order) by dimensions and parity *viz.* as a Taylor series in $\Lambda^2/\Lambda_\sigma^2$ and $\varphi^2/\Lambda_\sigma^2$, except for those cases at second order where such a Taylor series of corrections will also include a single factor of $\ln(\Lambda_\sigma)$.

Now define the polynomial function $\mathcal{H}_\alpha(\pi,\Omega_\Lambda,\varphi)$ by

$$(-i\partial_\pi)^\alpha\left(\mathrm{e}^{-\frac{\pi^2}{2}\Omega_\Lambda+i\pi\varphi}\right) = \mathcal{H}_\alpha(\pi,\Omega_\Lambda,\varphi)\;\mathrm{e}^{-\frac{\pi^2}{2}\Omega_\Lambda+i\pi\varphi}\,. \tag{4.57}$$

Substituting the Fourier transform polynomial trivialisation constraint (4.47) into the Fourier transform representation of the solution (4.25), integrating by parts, and using the polynomial trivialisation definition in φ-space (4.35), we see that

$$\mathcal{H}_\alpha(0,\Omega_\Lambda,\varphi) = (\Lambda/2ia)^\alpha\,H_\alpha(ai\varphi/\Lambda)\;, \tag{4.58}$$

where the RHS expands as given in the formula for the Hermite polynomial (4.36). Thus substituting the general formula for cases satisfying the polynomial trivialisation

(4.51) into the Fourier transform representation for the solution (4.25), we have that

$$f^{\sigma_\alpha}_\Lambda(\varphi) = A_\sigma \int_{-\infty}^{\infty} d\bar{\pi}\, \bar{\pi}^{\bar{n}_{\sigma_\alpha}}\, \mathcal{H}_\alpha\left(\frac{\bar{\pi}}{\Lambda_\sigma}, \Omega_\Lambda, \varphi\right) \exp\left(-\frac{\bar{\pi}^2}{4}\frac{\Lambda^2}{a^2\Lambda_\sigma^2} + i\bar{\pi}\frac{\varphi}{\Lambda_\sigma}\right). \tag{4.59}$$

Using the normalisation limit (4.54) and the vanishing limits (4.55) we thus confirm that polynomial trivialisation (4.35) is satisfied, and see again that the corrections are dictated by dimensions ($[\mathcal{H}_\alpha]=\alpha$) and parity to be a Taylor series in $\Lambda^2/\Lambda_\sigma^2$ and $\varphi^2/\Lambda_\sigma^2$, except for those cases at second order where these corrections also include a single factor of $\ln(\Lambda_\sigma)$.

We see that the difference between the left and right hand sides in polynomial trivialisation (4.35) is bounded by a term of order $1/\Lambda_\sigma^2$. Furthermore this is true for every relation obtained by differentiating with respect to φ on both sides until the RHS vanishes. At this point successive differentials will bring down further powers of $1/\Lambda_\sigma^2$ from the general finite Λ_σ formula (4.59) via $\varphi^2/\Lambda_\sigma^2$. Thus we have for large Λ_σ:

$$\begin{aligned}
\partial^p_\varphi \left[f^{\sigma_\alpha}_\Lambda(\varphi) - A_\sigma\,(\Lambda/2ia)^\alpha\, H_\alpha(ai\varphi/\Lambda)\right] &= O(1/\Lambda_\sigma^2) &&\text{for} \quad p \le \alpha\,, \\
\partial^p_\varphi f^{\sigma_\alpha}_\Lambda(\varphi) &= O(1/\Lambda_\sigma^{2\lceil\frac{p-\alpha}{2}\rceil}) &&\text{for} \quad p > \alpha\,,
\end{aligned} \tag{4.60}$$

which since this applies for $p = 0$, refines the earlier trivialisation characterisations (4.32,4.35), and where again one should understand that the RHS is corrected by a factor of $\ln(\Lambda_\sigma)$ in some cases at second order.

4.2.3 Examples

For example if there is no $o(\bar{\pi}^2)$ correction in the reduced asymptotic formula (4.50), then the normalisation limit (4.54) fixes the normalisation of the dimensionless entire function so that[18]

$$\bar{\mathfrak{f}}^{\sigma_\alpha}(\bar{\pi}^2) = \frac{\mathrm{e}^{-\bar{\pi}^2/4}}{(\bar{n}_{\sigma_\alpha}-1)!!\, 2^{\frac{\bar{n}_{\sigma_\alpha}}{2}+1}\sqrt{\pi}}\,. \tag{4.61}$$

In the general formula for cases satisfying flat trivialisation (4.49), solutions to flat trivialisation (4.32) that keep all possible couplings, so $n_\sigma = 0$, take the form

$$\mathfrak{f}^\sigma(\pi) = 2\pi A_\sigma\, \Lambda_\sigma\, \bar{\mathfrak{f}}^\sigma(\bar{\pi}^2)\,. \tag{4.62}$$

Using the simplest reduced Fourier transform (4.61) with $\alpha=0$ to generate an explicit example, we have:

$$\bar{\mathfrak{f}}^\sigma(\bar{\pi}^2) = \frac{\mathrm{e}^{-\bar{\pi}^2/4}}{2\sqrt{\pi}}\,, \tag{4.63}$$

[18] In the case $\bar{n}_{\sigma_\alpha} = 0$ one has $(-1)!! = 1$.

which just gives us our previously well-worked specimen :

$$f^{\sigma}_{\Lambda}(\varphi) = \frac{aA_\sigma \Lambda_\sigma}{\sqrt{\Lambda^2 + a^2\Lambda_\sigma^2}}\, \mathrm{e}^{-\frac{a^2\varphi^2}{\Lambda^2+a^2\Lambda_\sigma^2}}\,, \quad f^{\sigma}(\varphi) = A_\sigma\, \mathrm{e}^{-\varphi^2/\Lambda_\sigma^2}\,, \quad g^{\sigma}_{2n} = \frac{\sqrt{\pi}}{n!4^n}\, A_\sigma\, \Lambda_\sigma^{2n+1} \tag{4.64}$$

($n = 0, 1, \cdots$), where the first expression follows from performing the integral in the Fourier transform representation (4.25), the second is its $\Lambda \to 0$ limit, and the couplings follow from the Taylor expansion relation (4.26). Similarly linearised coefficient functions satisfying $f^{\sigma_1}_{\Lambda}(\varphi) \to A_\sigma\, \varphi$, with $n_{\sigma_1} = 1$, have

$$\mathfrak{f}^{\sigma_1}(\pi) = 2\pi i A_\sigma\, \Lambda_\sigma^2\, \partial_{\bar\pi}\, \bar{\mathfrak{f}}^{\sigma_1}(\bar\pi^2) \tag{4.65}$$

from the formula for the general case (4.51) and the reduced minimum index (4.52) with $\alpha = 1$. The explicit example for the simplest reduced Fourier transform (4.61) again gives the special case (4.63), and thus

$$\begin{aligned} f^{\sigma_1}_{\Lambda}(\varphi) = {} & \frac{a^3\Lambda_\sigma^3 A_\sigma}{(\Lambda^2 + a^2\Lambda_\sigma^2)^{3/2}}\, \varphi \\ & \mathrm{e}^{-\frac{a^2\varphi^2}{\Lambda^2+a^2\Lambda_\sigma^2}}\,,\ f^{\sigma_1}(\varphi) = A_\sigma\, \varphi\, \mathrm{e}^{-\varphi^2/\Lambda_\sigma^2}\,,\ g^{\sigma_1}_{2n+1} = -\frac{\sqrt{\pi}}{2}\frac{1}{n!4^n}\, A_\sigma\, \Lambda_\sigma^{2n+3}\,, \end{aligned} \tag{4.66}$$

($n = 0, 1, \cdots$), in agreement with coupling constant mapping formula (4.42) and our previously well-worked specimen (4.64). For $\alpha{=}2$ and $n_{\sigma_2} = 0$ one gets

$$f^{\sigma_2}_{\Lambda}(\varphi) = A_\sigma \left\{ \frac{a^5\Lambda_\sigma^5}{(\Lambda^2 + a^2\Lambda_\sigma^2)^{5/2}}\, \varphi^2 + \frac{a\Lambda_\sigma^3\Lambda^2}{2\,(\Lambda^2 + a^2\Lambda_\sigma^2)^{3/2}} \right\} \mathrm{e}^{-\frac{a^2\varphi^2}{\Lambda^2+a^2\Lambda_\sigma^2}} \tag{4.67}$$

from the simplest reduced Fourier transform (4.61), which gives the physical coefficient function and couplings:

$$f^{\sigma_2}(\varphi) = A_\sigma\, \varphi^2\, \mathrm{e}^{-\varphi^2/\Lambda_\sigma^2}\,, \qquad g^{\sigma_2}_{2n} = \frac{\sqrt{\pi}}{2}\frac{2n{+}1}{n!4^n}\, A_\sigma \Lambda_\sigma^{2n+3} \quad (n = 0, 1, \cdots)\,. \tag{4.68}$$

Its large Λ_σ limit, $f^{\sigma_2}_{\Lambda}(\varphi) \to A_\sigma(\varphi^2{+}\Omega_\Lambda)$, is in agreement with polynomial trivialisation (4.35).

Differentiating the $\alpha{=}1$ example (4.66) with respect to φ:

$$\check{f}^{\sigma}_{\Lambda}(\varphi) = f^{\sigma_1\prime}_{\Lambda}(\varphi) \tag{4.69}$$

gives an alternative example solution for flat trivialisation (4.32):

$$\begin{aligned} \check{f}^{\sigma}_{\Lambda}(\varphi) &= \frac{a^3\Lambda_\sigma^3 A_\sigma}{(\Lambda^2 + a^2\Lambda_\sigma^2)^{3/2}} \left(1 - \frac{2a^2\varphi^2}{\Lambda^2 + a^2\Lambda_\sigma^2}\right) \mathrm{e}^{-\frac{a^2\varphi^2}{\Lambda^2+a^2\Lambda_\sigma^2}}\,, \\ \check{f}^{\sigma}(\varphi) &= A_\sigma \left(1 - \frac{2\varphi^2}{\Lambda_\sigma^2}\right) \mathrm{e}^{-\varphi^2/\Lambda_\sigma^2} \end{aligned} \tag{4.70}$$

as is clear from the large amplitude suppression scale limit. However this solution has $\check{g}^\sigma_0 = 0$ as is immediately clear from integrating the $\check{f}$ relation (4.69) and using the moment relation (4.31). In section 4.2.1 we showed that $n_\sigma = 1$ – or rather $n_\sigma = 2$ since it is even, *cf.* the general n_{σ_α} definition (4.48). Indeed differentiating the Fourier transform representation (4.25), and using the general $\alpha = 1$ Fourier transform (4.65) and the simplest normalised reduced form (4.63), we see that the corresponding $\check{\mathfrak{f}}^\sigma(\pi)$ takes the general form for cases satisfying flat trivialisation (4.49):

$$\check{\mathfrak{f}}^\sigma(\pi) = 2\pi A_\sigma\, \Lambda_\sigma\, \bar{\pi}^2\, \check{\bar{\mathfrak{f}}}^\sigma(\bar{\pi}^2)\,, \qquad \check{g}^\sigma_{2n} = -\frac{2\sqrt{\pi}}{(n-1)!\, 4^n}\, A_\sigma\, \Lambda_\sigma^{2n+1}\,, \tag{4.71}$$

if $\check{\bar{\mathfrak{f}}}^\sigma = \bar{\mathfrak{f}}^\sigma/2$, *cf.* the $\alpha = 1$ example (4.63), in agreement with the simplest reduced Fourier transform (4.61). Expanding in π and using the Taylor expansion formula (4.26) then yields the displayed couplings, in agreement with the coupling constant mapping formula (4.42) (and actually the above formula holds also for $n = 0$ if we interpret $(-1)!$ as the Euler $\Gamma(0) = \infty$). Finally, notice that in all these examples, the approach to the trivialisation limits (4.32,4.35) is as described at the end of section 4.2.2.

4.3 Continuum limit at first order in perturbation theory

We will treat the first order cosmological constant term, associated to its BRST cohomology representative

$$\check{\Gamma}_1 = \check{\Gamma}^0_1 = \varphi \tag{4.72}$$

at the end of this section. The remaining parts of $\check{\Gamma}_1$ were computed to be (3.158), (3.159) and (3.159) as outlined in that section and will provide us with the top monomials σ that we need to construct the derivative part. In order to be supported on the renormalized trajectory, such that Γ_1 is constructed, these σ need to be 'dressed' with coefficient functions $f^\sigma_\Lambda(\varphi)$ as in the general closed formula for the eigenoperator (4.16). In the most general case we should give each top term its own coefficient function. This would provide the most complete test of universality of the continuum limit, however at the expense of carrying around a lot more terms and labels. At sufficiently high order of perturbation theory in the perturbative expansion (3.151), we expect to have to do this because these Γ_1 couplings will then run independently [107]. In fact we will show in ref. [75] that as a consequence of specialising to coefficient functions of definite parity, the Γ_1 couplings do not run at second order but they can be expected to run at third order.

Here it is not necessary to treat the general case, since we will see that the passage to universality is very generic such that it is clear that this will continue to work when we give each top monomial in Γ_1 its own coefficient function. We thus find that for our purposes just two coefficient functions are sufficient for constructing Γ_1, the first of which we label as $f^1_\Lambda(\varphi)$, setting the superscript to $\sigma = 1$ *i.e.* the perturbation level index, and in the second case choose the label $\sigma = 1_1$ as in $\alpha = 1$ trivialisation (4.35) to indicate that $f^{1_1}_\Lambda(\varphi)$ absorbs a factor of φ. Thus $f^1_\Lambda(\varphi)$ is even, while $f^{1_1}_\Lambda(\varphi)$ is odd. Although in principle every vertex can have its own amplitude suppression scale Λ_σ, we will find that we can choose them all to be equal. To make clear that it is independent of σ, we set this common amplitude suppression scale to $\Lambda_\sigma = \Lambda_\partial$ (borrowing the notation already used in [8]).

Now since $\check{\Gamma}_1$ is a dimension $d_1 = 5$ operator, we have by dimensions (4.21) that the dimensionful coefficient $[A_1] = -1$. As the remaining factor in front $\check{\Gamma}_1$, after taking the limit $\Lambda_\partial \to \infty$, we recognise that it is actually $A_1 = \kappa$, where the latter was defined in (2.30), *i.e.* we have

$$f^1_\Lambda(\varphi) \to \kappa\,, \qquad f^{1_1}_\Lambda(\varphi) \to \kappa\,\varphi\,, \qquad \text{as} \quad \Lambda_\partial \to \infty\,, \tag{4.73}$$

where whenever we now write the limit of large amplitude suppression scale, we mean also the more refined regularity properties (4.60), in particular in these cases the limits are reached at least as fast as $1/\Lambda^2_\partial$. We see that Newton's constant therefore arises only as a kind of collective effect of all the renormalizable couplings $\{g^1_{2n}, g^{1_1}_{2n+1}\}$, these latter being responsible for forming the continuum limit. Indeed $A_1 = \kappa$ is not an underlying coupling in its own right but rather appears as the overall proportionality constant when the couplings are expressed in terms of Λ_∂, through their asymptotic formula (4.28).

Examples of such coefficient functions were given in [107] and appear in equations (4.64) and (4.66). We stress however that we are working here with very general solutions for these coefficient functions. From the definition of the minimum index n_{σ_α} (4.48) and the expansion of the coefficient function over the operators $\delta^{(n)}_\Lambda(\varphi)$ (4.17), we have that in general all eigenoperators will be involved:

$$f^1_\Lambda(\varphi) = \sum_{n=0}^{\infty} g^1_{2n}\, \delta^{(2n)}_\Lambda(\varphi)\,, \qquad f^{1_1}_\Lambda(\varphi) = \sum_{n=0}^{\infty} g^{1_1}_{2n+1}\, \delta^{(2n+1)}_\Lambda(\varphi)\,, \tag{4.74}$$

where these sums converge (in the square integrable sense) for $\Lambda > a\Lambda_\partial$. From the general dimension formulae (4.15), and (4.34):

$$[g^1_{2n}] = 2n\,, \qquad [g^{1_1}_{2n+1}] = 2n+2\,. \tag{4.75}$$

Thus all these couplings are relevant, with the exception of g^1_0 which is marginal. Up to second order it does not run [75] and thus behaves as though it is exactly marginal, parametrising a line of fixed points.

From the antighost level two free BRST cohomology representative (3.158), we thus set at anti-ghost level two:

$$\Gamma^2_1 = -c_\nu\, \partial_\nu c_\mu\, c^*_\mu\, f^1_\Lambda(\varphi)\,. \tag{4.76}$$

Since f^1_Λ is taken to satisfy the linearised flow equation for coefficient functions (4.2) and there is no other opportunity to attach tadpoles to (4.76), Γ^2_1 already satisfies the linearised flow equation (3.152), and thus appears correctly as a sum over eigenoperators. Evidently at this anti-ghost level, the linearised mST (3.153) is satisfied in the limit (by the more refined limits (4.60) at least as fast as $1/\Lambda^2_\partial$) since:

$$Q_0\,\Gamma^2_1 = -c_\nu\,\partial_\nu c_\mu\,c^*_\mu\,\partial\!\cdot\! c\,f^{1\prime}_\Lambda(\varphi) \to 0 \qquad \text{as} \quad \Lambda_\partial \to \infty\,, \tag{4.77}$$

and $\Gamma^2_1 \to \kappa\,\check\Gamma^2_1$ then coincides with a legitimate choice in the usual perturbative quantisation.

As discussed above section 4.2.1, if we keep κ fixed in the large amplitude suppression scale limit, all the couplings $\{g^1_{2n}, g^{1_1}_{2n+1}\}$ diverge. As we noted however, we can stay perturbative by requiring instead that κ vanish fast enough. Although this makes the vertex vanish, we can still extract the same results by phrasing the limit more carefully as $\Gamma^2_1/\kappa \to \check\Gamma^2_1$. From here on we will take this phrasing as tacitly understood.[19]

In the anti-ghost level one free BRST cohomology representative (3.159) we need to substitute the $SO(4)$ decomposition (2.6) into the last term to isolate the factor of φ, and thus the dressed anti-ghost level one piece appears as

$$\Gamma^1_1 = -\left(c_\alpha\partial_\alpha H_{\mu\nu} + 2\,\partial_\mu c_\alpha h_{\alpha\nu}\right) H^*_{\mu\nu}\,f^1_\Lambda(\varphi) - \partial_\mu c_\nu H^*_{\mu\nu}\,f^{1_1}_\Lambda(\varphi)\,. \tag{4.78}$$

Added conclusion section to chapter 4, from here until the end of chapter 4 is the new content. This time the result does not yet satisfy the linearised flow equation (3.152) , unlike with the previous choice in ref. [107], because it requires the tadpole correction in the $\hat s_0$-exact eigenoperator

$$\hat s_0(H_{\mu\nu}c_\mu c^*_\nu) = (\partial_\mu c_\nu + \partial_\nu)c_\mu c^*_\nu + 2H_{\mu\nu}c_\mu\partial_\alpha H^*_{\alpha\nu} + 2b\Lambda^4 \tag{4.79}$$

or rather as formulated for the new quantisation in (4.14).[20] In other words the sum over eigenoperators is actually $\Gamma^1_1 + 2b\Lambda^4 f^{1_1}_\Lambda(\varphi)$. Since $\Delta^-\,\Gamma^2_1$ trivially vanishes, the descendant equation (3.129) that relates Γ^2_1 to Γ^1_1 reads:

$$\begin{aligned} Q^-_0\,\Gamma^2_1 + Q_0\,\Gamma^1_1 = {}& -\partial_\mu c_\nu\,\partial\!\cdot\! c\,H^*_{\mu\nu}\left(f^1_\Lambda - f^{1_1\prime}_\Lambda\right) \\ & - 2(c_\alpha\partial_\alpha c_\mu)H^*_{\mu\nu}\partial_\nu\varphi f^{1\prime}_\Lambda - \left(c_\alpha\partial_\alpha H_{\mu\nu} + 2\,\partial_\mu c_\alpha h_{\alpha\nu}\right) H^*_{\mu\nu}\,\partial\!\cdot\! c\,f^{1\prime}_\Lambda\,, \end{aligned} \tag{4.80}$$

where we used the Koszul-Tate charge (3.122) and note from the free BRST transformation (3.119) that

$$Q_0\,h_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu - \tfrac{1}{2}\,\delta_{\mu\nu}\,\partial\!\cdot\! c\,. \tag{4.81}$$

[19] This is in conformity with the reasonable assumption that the expansion in κ is only asymptotic [107]. Then strictly speaking the expansion only anyway makes sense in the $\kappa \to 0$ limit, *i.e.* as Taylor expansion coefficients in κ .

[20] Tadpole contributions from the first term in the dressed anti-ghost level one piece (4.78) all vanish, either because the tadpole integral is odd in momentum or because $h_{\alpha\alpha} = 0$.

It is clear from the first-order coefficient function trivialisation formulae (4.73) that as required $Q_0^- \Gamma_1^2 + Q_0 \Gamma_1^1 \to 0$ (at least as fast as $1/\Lambda_\partial^2$). At the expense of some generality, we could eliminate the first term on the RHS of the descendant equation (3.129) by setting

$$f_\Lambda^1 = f_\Lambda^{1_1\prime} . \tag{4.82}$$

By the minimum index map (4.45) this would also eliminate $\delta_\Lambda^{(0)}(\varphi)$, *i.e.* set $g_0^1 = 0$. We would still be left with the $\Lambda_\partial < \infty$ violations on the second line however.

Finally, extracting the undifferentiated φ pieces from the anti-ghost level zero free BRST cohomology representative (3.160) by using the $SO(4)$ decomposition (2.6), we have as in [107] that the first order graviton interaction is made up of twelve top terms and one tadpole contribution:

$$\begin{aligned}\Gamma_1^0 = \Big(&\frac{1}{4}h_{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - h_{\alpha\beta}\partial_\gamma h_{\gamma\alpha}\partial_\beta\varphi - \frac{1}{2}h_{\gamma\delta}\partial_\gamma h_{\alpha\beta}\partial_\delta h_{\alpha\beta} - h_{\beta\mu}\partial_\gamma h_{\alpha\beta}\partial_\gamma h_{\alpha\mu} \\ &+ 2h_{\mu\alpha}\partial_\gamma h_{\alpha\beta}\partial_\mu h_{\beta\gamma} + h_{\beta\mu}\partial_\gamma h_{\alpha\beta}\partial_\alpha h_{\gamma\mu} - h_{\alpha\beta}\partial_\gamma h_{\alpha\beta}\partial_\mu h_{\mu\gamma} + \frac{1}{2}h_{\alpha\beta}\partial_\gamma h_{\alpha\beta}\partial_\gamma\varphi\Big) f_\Lambda^1 \\ &+ \left(\frac{3}{8}(\partial_\alpha\varphi)^2 - \frac{1}{2}\partial_\beta h_{\beta\alpha}\partial_\alpha\varphi - \frac{1}{4}(\partial_\gamma h_{\alpha\beta})^2 + \frac{1}{2}\partial_\gamma h_{\alpha\beta}\partial_\alpha h_{\gamma\beta}\right) f_\Lambda^{1_1} + \frac{7}{2}b\Lambda^4 f_\Lambda^{1_1} ,\end{aligned} \tag{4.83}$$

except that the tadpole contribution now appears with coefficient $\frac{7}{2} = 2+\frac{3}{2}$. The final descendant equation (3.129) is satisfied in the limit:

$$Q_0 \Gamma_1^0 + \left(Q_0^- - \Delta^-\right)\Gamma_1^1 - \Delta^= \Gamma_1^2 \to 0 , \tag{4.84}$$

at least as fast as $1/\Lambda_\partial^2$, since the individual limits are also reached at least as fast as $1/\Lambda_\partial^2$:

$$\Gamma_1^n \to \kappa \check{\Gamma}_1^n , \qquad \text{as} \quad \Lambda_\partial \to \infty . \tag{4.85}$$

It is straightforward to verify the above descendant equation (4.84) directly. To evaluate *e.g.* $\Delta^- \Gamma_1^1$, one inverts the $SO(4)$ decomposition (2.6) to give $h_{\mu\nu} = H_{\mu\nu} - \frac{1}{4}\delta_{\mu\nu}H_{\alpha\alpha}$ and $\varphi = \frac{1}{2}H_{\mu\mu}$, or recognises that [107]

$$\frac{\partial}{\partial H_{\alpha\beta}} = \frac{\partial h_{\mu\nu}}{\partial H_{\alpha\beta}}\frac{\partial}{\partial h_{\mu\nu}} + \frac{\partial\varphi}{\partial H_{\alpha\beta}}\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial h_{\alpha\beta}} + \frac{1}{2}\delta_{\alpha\beta}\frac{\partial}{\partial\varphi} . \tag{4.86}$$

Note that although these measure terms give contributions proportional to some positive power of Λ, thanks to UV regularisation by C, for example

$$-\Delta^= \Gamma_1^2 = -b\Lambda^4 \partial\cdot c f_\Lambda^1 , \tag{4.87}$$

it does not alter the speed at which they vanish in the limit of large Λ_∂ (as can be verified here by integration by parts).

In the opposing limits there is no sense in which a non-trivial diffeomorphism invariance holds because the dependence on the conformal factor forbids it [107]. For example if $\varphi \gg \Lambda_\partial, \Lambda$, the coefficient functions are no longer given approximately by κ and $\kappa\varphi$, but rather take the exponentially decaying form demanded by the asymptotic formula (4.22).

These statements hold also if we express everything in dimensionless variables using Λ, as needed to clearly see the Wilsonian RG behaviour [8, 20]. Recalling that we write dimensionless variables with a tilde, so *e.g.* $\tilde{q}^\mu = q^\mu/\Lambda$, $\tilde{\varphi} = \varphi/\Lambda$, whilst we write $\delta_n(\tilde{\varphi}) = \Lambda^{1+n}\,\delta^{(n)}_\Lambda(\varphi)$ for the scaled operator [8]. The dimensionless couplings run with Λ according to their mass dimensions (4.75):

$$\tilde{g}^1_{2n}(\Lambda) = g^1_{2n}/\Lambda^{2n}\,, \qquad \tilde{g}^{1_1}_{2n+1}(\Lambda) = g^{1_1}_{2n+1}/\Lambda^{2n+2}\,. \tag{4.88}$$

We thus confirm that Γ approaches the Gaussian fixed point ($g^1_0 = 0$) or more generally the line of fixed points $g^1_0 = \tilde{g}^1_0 \neq 0$, as $\Lambda \to \infty$. In particular all the relevant parts of Γ_1 vanish as negative powers of Λ, with non-trivial $\tilde{\varphi}$ dependent coefficients being the corresponding scaled operator $\delta_{2n+\varepsilon}(\tilde{\varphi})$. In the limit only the marginal contribution $\tilde{f}^1_\Lambda(\tilde{\varphi}) \to g^1_0\,\delta_0(\tilde{\varphi})$ in this sole coefficient function survives (and still carries non-trivial $\tilde{\varphi}$ dependence).

In dimensionful variables, if Λ is much larger than the other scales $\Lambda_\partial, \varphi$, the situation is a little obscured but it is still the case that there is no sense in which a non-trivial diffeomorphism invariance is recovered. The coefficient functions are again dominated by the lowest terms in the expansion (4.74). Using the explicit formulae for the $\delta^{(n)}_\Lambda(\varphi)$ operators (4.5) we have in the current case

$$\begin{aligned}
f^1_\Lambda &= \frac{a}{\Lambda\sqrt{\pi}}\,g^1_0 - \frac{a^3}{\Lambda^3\sqrt{\pi}}\left(g^1_0\varphi^2 + 2g^1_2\right) + \frac{a^5}{2\Lambda^5\sqrt{\pi}}\left(g^1_0\varphi^4 + 12g^1_2\varphi^2 + 24g^1_4\right) + O\left(\frac{1}{\Lambda^7}\right) \\
f^{1_1}_\Lambda &= -\frac{2a^3}{\Lambda^3\sqrt{\pi}}\,g^{1_1}_1\,\varphi + \frac{2a^5}{\Lambda^5\sqrt{\pi}}\left(g^{1_1}_1\varphi^3 + 6g^{1_1}_3\right) + O\left(\frac{1}{\Lambda^7}\right)\,.
\end{aligned} \tag{4.89}$$

The leading terms, and only the leading terms, have the correct φ dependence to allow BRST invariance to be recovered, however with $g^1_0 \neq 0$ they have the wrong ratio. (They should have equal coefficients, but this is impossible at diverging Λ since g^1_0 and $g^{1_1}_1$ must be fixed and finite.) By setting $g^1_2 = g^{1_1}_1$, and $g^1_0 = 0$, (only) the leading terms have both the correct φ dependence and the correct ratio, as in fact would result from the identification (4.82) of the two coefficient functions, *cf.* the coupling constant mapping formula (4.44), although with an effective κ that then vanishes as $\kappa_{\text{eff}} \sim 1/\Lambda^3$. Meanwhile the measure terms in the above descendant formula (4.84) provide divergent obstructions to satisfying $\hat{s}_0\,\Gamma_1 = 0$, if $g^1_0 \neq 0$. Thus evaluating the measure term formula (4.87) tells us that

$$-\,\Delta^{=}\,\Gamma^2_1 = \Lambda\frac{ba^3}{\sqrt{\pi}}\,g^1_0\,\partial\!\cdot\! c\,\varphi^2 + O\left(\frac{1}{\Lambda}\right)\,, \tag{4.90}$$

(dropping total derivative terms), and $\Delta^-\Gamma^1_1$ provides also such a term but with coefficient $-\frac{9}{2}$ and also a $g_0^1 \Lambda (c_\alpha \partial_\alpha \varphi + \partial_\alpha c_\beta h_{\alpha\beta})\varphi$ piece arising from the contribution containing $\Delta^-(H^*_{\mu\nu} f^1_\Lambda)$. Setting $g_0^1 = 0$ removes these divergences but leaves us with subleading terms that violate BRST invariance, as is also true of the subleading terms in the large Λ expansion of the coefficient functions (4.89) in this case.

This completes the demonstration at first order. The result fits the picture we sketched in the Introduction, *cf.* fig. 3.3. In particular for $\Lambda_\partial \gg \Lambda, \varphi$, diffeomorphism invariance holds in the sense that

$$\hat{s}_0 \Gamma_1 = \hat{s}_0 \left(\Gamma_1^2 + \Gamma_1^1 + \Gamma_1^0\right) = O(1/\Lambda_\partial^2)\,. \tag{4.91}$$

This means in particular in the limit $\Lambda_\partial \to \infty$ and the physical limit ($\Lambda \to 0$), we recover diffeomorphism invariance precisely in terms of satisfying the standard Slavnov-Taylor (Zinn-Justin) identities, namely at first order $(Q_0 + Q_0^-)\Gamma_1 = (\Gamma_0, \Gamma_1) = 0$, where we used the general definition of the charges (3.150), the linearised mST (3.153) and noted that from the definition of the measure operator (3.126) that $\Delta \to 0$ as $\Lambda \to 0$.

Finally, we remark that including a cosmological constant is straightforward at first order. We need to dress its BRST cohomology representative (4.72) with its own coefficient function. Since we must absorb the factor of φ, the monomial $\sigma = 1$ is simply the unit operator, whilst we must choose an odd coefficient function $f^{cc}_\Lambda(\varphi)$ with the trivialisation

$$f^{cc}_\Lambda(\varphi) \to \lambda\,, \qquad \text{as} \quad \Lambda_\partial \to \infty\,, \tag{4.92}$$

where $\kappa^2 \lambda/4$ is the standard cosmological constant. At this order we do not need a whole separate odd coefficient function and can by the trivialisation property (4.73) for f^{1_1}, just set $f^{cc}_\Lambda = \lambda f^{1_1}_\Lambda/\kappa$. The linearised mST (3.153) is satisfied in the limit because $Q_0 f^{cc}_\Lambda(\varphi) = \partial{\cdot}c\, f^{cc\,\prime}_\Lambda(\varphi) \to 0$ at least as fast as $1/\Lambda_\partial^2$, as follows by integration by parts and using the refined limits (4.60), or directly by the observation that the first order vertex tends to κ times its free BRST cohomology representatives, *viz.* (4.85). Indeed these properties were already used in proving the invariance (in the limit) of the last term in the anti-ghost level zero part of the first order vertex (4.83).

4.4 Discussion

In this section we discuss further the meaning and implications of this construction and draw out its relation to other approaches. As recalled at the beginning of section 4.1, the Euclidean signature Einstein-Hilbert action is unbounded below. From sign of the action (2.1), the instability is towards manifolds of arbitrarily positive curvature. Whilst this conformal factor instability [9] means that the partition function is not well defined, the Wilsonian exact RG flow equation remains well defined [8, 33], and anyway provides a more powerful route towards constructing the continuum limit. However the wrong sign propagator (3.143) for the conformal factor (φ), has a profound effect on RG properties. Close to the UV Gaussian fixed point, *cf.* fig. 3.3, the requirement that expansion over eigenoperators converges, picks out the Hilbert space $\mathfrak{L}_-$ defined by the Sturm-Liouville measure (4.4), which is spanned by the novel set of eigenoperators (4.5), the $\delta^{(n)}_\Lambda(\varphi)$.

We must emphasise that the requirement that one works within $\mathfrak{L}_-$ (more generally $\mathfrak{L}$ defined by (4.19), when the other fields are included) is crucial for the Wilsonian RG to make sense in an otherwise unrestricted space of functions (of φ). Without this restriction the eigenoperator spectrum degenerates, becoming continuous, and it is no longer possible to unambiguously divide a perturbation into its relevant and irrelevant parts [77]. This problem lay unnoticed until ref. [77] and as yet has only been further addressed in refs. [8, 74, 107, 109, 111]. The reason that it lay undiscovered is primarily because to see this problem of convergence one must work with solutions involving an infinite number of operators (the exact solution being also of this type). However, with few prior exceptions [110, 112, 113], quantum gravity investigations using exact RG flow equations worked within truncations (model ansätze) where only a finite number of operators are retained.

Restricting flows to the diffeomorphism invariant subspace, *cf.* fig. 3.3, might be expected to solve the problem since diffeomorphism invariance at the classical level restricts the functional dependence on the conformal factor to just a few operators at any given order in the derivative expansion. However when carefully analysed, the so-called $f(R)$ approximations [114–124], which are diffeomorphism invariant model ansätze that keep an infinite number of operators, also show the problem that the eigenoperator spectrum degenerates [111, 113], and furthermore it is now clear that the underlying cause is the conformal factor instability [77, 113]. Indeed it was these problems that motivated the studies [77, 125].

Within standard perturbation theory the problem can be ignored, the wrong sign φ propagator (3.143) being apparently harmless. As recalled in section 4.1, the conformal factor instability was identified in ref. [9], where they proposed to solve it by analytically continuing the φ integral along the imaginary axis. This does not alter final perturbative results, but non-perturbatively it is less clear that this treatment makes sense [126]. Some other approaches keep, and seek to cope with, the conformal factor instability (but do not treat the convergence problems whose solution leads uniquely to our proposal). In ref. [127] a model truncation to a finite set of operators, "$-R+R^2$ " gravity, was considered within the non-perturbative asymptotic safety scenario [33]. The right-sign R^2 term stabilises the conformal sector, resulting in an unsuppressed non-perturbative Planckian scale modulated phase which breaks Lorentz symmetry. If physical, this would be phenomenologically challenging [128–130]. A somewhat similar effect is seen in the Causal Dynamical Triangulations approach to quantum gravity [131]. Although a restriction here to a global time foliation leads to an encouraging phase structure, the conformal instability towards a crumpled phase remains, and this programme has yet to succeed in furnishing an acceptable continuum limit [132].

Returning to this chapter, the fact that $[\delta^{(n)}_\Lambda(\varphi)] = -1–n$ form a tower of increasingly relevant operators, implies that all interactions are dressed with coefficient functions $f^\sigma_\Lambda(\varphi)$ which contain an infinite number of relevant underlying couplings, g^σ_n. Close to the Gaussian fixed point, the linearised flow equation (4.2) is justified. Then, as we showed in section 4.1, and also in [8], if $f^\sigma_\Lambda(\varphi) \in \mathfrak{L}_-$, it is guaranteed to remain there at all higher scales. Thus the requirement that for sufficiently high Λ we have $f^\sigma_\Lambda(\varphi) \in \mathfrak{L}_-$, can be seen as a quantisation condition that is both natural and necessary for the Wilsonian RG.

Note that in this step we are relying on the fact that the Cauchy initial value problem itself is well defined in the UV direction [8, 77, 110], *i.e.* the property that the RG flow is guaranteed to exist to all higher scales. This is the reverse direction from normal: another consequence of the wrong sign φ-propagator. However the fact that the well defined flow direction is now opposite to the one defined by integrating out microscopic degrees of freedom, is an example where, even for the Wilsonian RG equation, the wrong sign φ-propagator forces us to reassess some of the usual physical intuition. We emphasise that this property does not alter the fact that the bare action determines, eventually after integration over all momentum modes, and up to universality, the physical effective action. Rather it throws obstacles in the path towards constructing this, that have not been previously encountered or recognised as such. Thus for example for a generic choice of bare coefficient function $f^\sigma_{\Lambda_0}(\varphi)$ at an initial UV scale $\Lambda = \Lambda_0$, the flow to the IR will almost certainly fail at some finite critical scale $0 < \Lambda = \Lambda_{cr} < \Lambda_0$ after which it ceases to exist [8]. Since one is then unable to complete the integration over all modes, the quantum field theory as a physical entity itself ceases to exist in this case [8].

As we saw the coefficient functions that do survive all the way to the IR have a physical limit (4.23) which decays for large φ with some characteristic amplitude suppression scale, Λ_σ. Even for such coefficient functions, if Λ_σ is finite, the complete flow and thus also the physical theory, can cease to exist on sufficiently small and asymmetrical manifolds [8, 74]. Tantalising as this seems [8, 74], in order to recover diffeomorphism invariance we need the coefficient functions to trivialise, *cf.* section 4.2, and in practice this requires taking the limit $\Lambda_\sigma \to \infty$ in the continuum theory [107] (holding everything else fixed). Then the above restrictions on the allowed manifold [8, 74] appear to be ruled out except possibly to rule out manifolds with singularities [107]. The amplitude suppression scale per se should therefore be seen as part of the procedure for forming the continuum limit and not as having direct influence on the physical theory. Nevertheless it is the cross-over scale that matches the RG flow in the diffeomorphism invariant subspace to the upper part of the renormalized trajectory, *cf.* fig. 3.3, and as such plays a role in determining which of these RG flows actually correspond to a valid perturbative continuum limit. It may also leave behind certain finite logarithmic corrections at higher order in perturbation theory [107].

Importantly, notice that the reduction of parameters that takes place on trivialisation (4.73) from the infinitely many underlying couplings (4.74) to the effective coupling

κ (2.30) (Newton's constant) and a cosmological constant at first order is *not* the result of imposing infinitely many relations between these underlying couplings, but rather a dramatic demonstration of universality resulting from the large amplitude suppression scale limit. This reduction of parameters occurs provided only that the underlying couplings are chosen from some loosely specified infinite dimensional domain. Thus $f^1_\Lambda(\varphi)$ is given in general by specifying its Fourier transform as (4.49) (with $n_\sigma = 0$, $A_\sigma = \kappa$ and $\Lambda_\sigma = \Lambda_\partial$, as explained in section 4.3). Similar remarks follow for $f^{1_1}_\Lambda$ following (4.51). These Fourier transforms are proportional to the reduced Fourier transform $\bar{\mathfrak{f}}^\sigma(\bar{\pi}^2)$. As we noted, at first order this latter function can be chosen to be independent of Λ_σ, then the only constraints on it,[21] are that it is a dimensionless entire function, that it has asymptotic behaviour (4.50) as $\bar{\pi} \to \infty$, and that its integral (4.54) is normalised. This still leaves an infinite dimensional function space. In particular any number of underlying couplings (4.26) can still take any value. A key result of this chapter is the demonstration that the same results are then nevertheless recovered [107], thus confirming universality. Indeed it is only the underlying couplings' asymptotic behaviour for large n that is constrained through (4.28), and it is only these values that ultimately influence the physical results, as discussed in deriving their uniform bound (4.37).

This observation was also emphasised at the end of app. A.1 when discussing coefficient functions with a spectrum of amplitude suppression scales. However in the body of the chapter we recognised that we can make three simplifications to the most general case. As explained in section 4.2.2, firstly we can work only with coefficient functions of definite parity, *i.e.* even or odd under $\varphi \mapsto -\varphi$, and secondly with coefficient functions containing only one amplitude suppression scale. Finally in section 4.3, we also recognised that we can set all amplitude suppression scales to a common value $\Lambda_\sigma = \Lambda_\partial$. This still leaves us to choose, for each coefficient function, a reduced Fourier transform function $\bar{\mathfrak{f}}^\sigma(\bar{\pi}^2)$ with its own domain of infinitely many underlying couplings, and thus is more than sufficient again to demonstrate universality of the continuum and large amplitude suppression scale limits.

As we have seen, in the end at first order we are left with just the two effective couplings, Newton's constant and the cosmological constant. A key question [8, 107] is how many (effective) couplings are left once higher order quantum corrections are included. After all, it is at this point operationally, that one meets in standard perturbative quantisation an apparent obstruction to defining quantum gravity since new couplings get introduced to absorb divergences, order by order in perturbation theory. Given the importance of this question, we finish by commenting on this, although we cannot do better than make some remarks, since substantiation requires developments that go well beyond what we report in this part of the thesis. Although in refs. [2, 75] we will establish that this continuum limit can be extended to second order for pure quantum gravity, this does not yet seem enough to settle the above question since, although we

[21] At higher orders the only other constraints are the mild convergence conditions (4.55).

find that the new divergences can be absorbed by wave-function-like renormalization, this is also famously true of pure quantum gravity in its standard quantisation at this order [21]. *A priori* in this quantisation a continuum limit with an infinite number of couplings seems logically consistent [2]. However as we will show, there are indications that the quantisation is more restrictive at higher orders, where the underlying couplings introduced here becoming running couplings [2, 75]. In particular, note again that so far we have been relying on the fact that the renormalized trajectory can be constructed in the φ-sector by flowing upwards from the IR to the UV. At the linearised level this was set out precisely, together with its proof, at the end of section 4.1. At higher orders this kind of 'reverse' flow construction is also key [75]. However the flow in the $h_{\mu\nu}$ (graviton) sector is guaranteed only in the usual direction from the UV to the IR. Put together we are actually dealing with a flow equation that does not have a well-defined Cauchy initial value problem in either direction. In other words, a generic 'initial' effective action will lead to singular flows in *both* directions. This does not mean that there are no solutions (after all we just established one to first order here) but we find [2] that it does appear at higher orders to require solutions to depend ultimately on only the two parameters, Newton's constant and the cosmological constant, which we will now investigate in further detail.

Chapter 5

Perturbatively renormalizable quantum gravity at second order in the coupling

5.1 Introduction

Following the progress made in chapter 4 we now turn out attention to the higher order behaviour and investigate if the structure remains consistent and what the physical ramifications are. We will find that for pure quantum gravity with vanishing cosmological constant to second order in perturbation theory the standard quantisation is recovered. This case is renormalizable at second order due to kinematic reasons however the structure will work in more general cases. A possible conclusion is that gravity does have a genuine consistent continuum limit even though it has an infinite number of underlying couplings. This is phenomenologically difficult and could be regarded as a simple re-phrasing of the standard arguments for the non-renormalizability of QG however we outline a possible non-perturbative mechanism that could offer salvation. This argument is based on the non-parabolic properties of the flow equations and is a consequence of the different natural direction for the flows of the sectors of the graviton which began this approach to QG and would fix all higher order couplings in terms of Newton's constant and the cosmological constant. We also discuss this structure in a general gauge which offers other avenues of exploration.

To elaborate in this chapter we conclude this construction to second order, using what we have discussed thus far as well as insights from [75] where it was shown that the conformal sector maintains this behaviour at second order in the coupling. To summarise those findings it was shown that a well defined renormalized trajectory can be constructed and domains for the underlying couplings can be chosen that the interactions satisfy the trivialisation conditions we have outlined in the large Λ_σ limit which return the conformal sector operators to the diffeomorphism invariant subspace.

We verify that the underlying couplings can be chosen such that the mST is satisfied and by computing the remainder of the renormalized trajectory. We then derive the physical Legendre effective action to $\mathcal{O}(\kappa^2)$ in the $\Lambda \to 0$ limit and this recovers the standard quantisation result at one loop and second order in κ under the $g_{\mu\nu} = \delta_{\mu\nu} + \kappa H_{\mu\nu}$ parametrisation of the metric we have ben using.

The second order renormalized trajectory is non-perturbative in $\hbar$ and involves a sum over tadpoles and melonic Feynman diagrams to all loops [75] however in this crucial large Λ_σ limit where this trivialises occurs it becomes this more familiar one loop, second order in κ calculation. We find that any undetermined parameters which are associated to BRST invariant terms that run logarithmically with Λ are left behind, these being the only place where such ambiguities appear. They are in fact shown the be BRST exact pieces which can therefore be absorbed in a wavefunction like canonical transformation which is the BRST cohomological equivalent of the kinematical accident that pure gravity without cosmological constant is one loop finite in the standard perturbative treatment [21].

In section 5.5 we discuss the implications of these findings. We begin with the observation that once one adds matter or a cosmological constant that the logarithmic running inside the diffeomorphism invariant subspace will no longer be attributable to parametrisation. In the worse case it may be that we are left with an infinite number of these effective couplings which correspond to the infinite number of couplings in the standard perturbative treatment that are added order by order in the number of loops. This would still be a novel result. We would have a theory of QG with a genuine continuum limit that is consistent. It would bare many of the hallmarks of a true continuum theory: it would be controlled by a (potentially infinite) number of (marginally) relevant, renormalizable underlying couplings. This result then demands that we question how one truly defines the continuum limit of QG, despite how distasteful it seems there may indeed be an infinite number of couplings which will not be confronted until there are QG experiments.

This is not the most desirable outcome and in section 5.6 we begin a preliminary investigation into this possible non-perturbative mechanism that would fix the effective couplings in terms of only Newton's constant and the cosmological constant. The parabolic property of the flows equations is crucial here, in particular how the different sectors of the graviton flow in opposite natural directions. The φ sector with its negative kinetic term has flows that are guaranteed to be well-defined only in the UV direction i.e. they are backwards parabolic which was crucial in the earlier developments to the construction of this theory and in this thesis. This backwards parabolic property contrasts with that of the $h_{\mu\nu}$ sector which as flows that are guaranteed well-defined in the IR direction (forwards parabolic) as is more typical of the Wilsonian RG. This parametrisation of the metric is our arbitrary choice and so they can not be treated separately. As a result we are working with partial differential equations whose solutions typically fail regardless of which direction they evolve in.

To remedy this we propose a non-perturbative solution using a simple linearised model where κ is freely variable, as a result the cosmological constant is then the only other coupling which can also be freely variable since perturbations in the higher derivtive couplings would lead to singular trajectories. We follow this with comments on the validity of this proposed solution, listing its pros and cons. We conclude with generalising many of the results in this thesis to a general gauge α, where until we have used the Feynman - De Donder gauge and review further avenues of possible research with this freedom.

5.2 Solving the CME at second order

At intermediate steps we will need

$$\check{\Gamma} = \sum_{n=0}^{\infty} \check{\Gamma}_n \, \kappa^n / n! \, , \qquad (\check{\Gamma}_0 = \Gamma_0) \tag{5.1}$$

where $\check{\Gamma}$ is a local solution of the CME, taking the standard form:

$$\check{\Gamma} = \check{\Gamma}^0 - (Q\Phi^A)\Phi^*_A \, . \tag{5.2}$$

In particular $\check{\Gamma}^0$ and Q then also have such expansions in κ. The CME [83, 84, 91]

$$0 = \frac{1}{2}(\check{\Gamma}, \check{\Gamma}) = (Q\Phi^A)\frac{\partial_l \check{\Gamma}}{\partial \Phi^A} \, , \tag{5.3}$$

just implies the BRST invariance of this action under this *classical* BRST charge Q. The choice of free total quantum BRST cohomology representative (3.158, 3.159, 3.160) was made [1] because Q is then given exactly, *i.e.* has no higher order in κ corrections, provided that the metric is given by the simple linear split $g_{\mu\nu} = \delta_{\mu\nu} + \kappa H_{\mu\nu}$, as we have in this thesis. Indeed using the classical form (5.2), we read from $\check{\Gamma}^2_1$ (3.158) that

$$Qc^\nu = (Q_0 + \kappa \, Q_1) \, c^\nu = \kappa \, c^\mu \partial_\mu c^\nu = \tfrac{1}{2} \, \kappa \, \mathfrak{L}_c \, c^\nu \, , \tag{5.4}$$

expresses exactly the algebra of diffeomorphisms through the Lie derivative $\mathfrak{L}_c$ generated by the vector field κc^μ [107], while from $\check{\Gamma}^1_1$ (3.159) and the $H^*_{\mu\nu}$ part in Γ_0 (3.115) we get exactly the action of diffeomorphisms on the metric, through its Lie derivative:

$$Qg_{\mu\nu} = \kappa(Q_0 + \kappa Q_1)H_{\mu\nu} = 2\kappa \, \partial_{(\mu} c^\alpha g_{\nu)\alpha} + \kappa \, c^\alpha \partial_\alpha g_{\mu\nu} = \kappa \, \mathfrak{L}_c \, g_{\mu\nu} \, . \tag{5.5}$$

In our case, the level zero action, $\check{\Gamma}^0_1 + \check{\Gamma}^0_{1\theta}$, has a classical and one-loop part. Together they must still solve these equations, indeed the CME and Zinn-Justin identities [91] are equivalent algebraically. Thus the one-loop part has an expansion in κ which we write similarly to that for $\check{\Gamma}$ itself (5.1):

$$\check{\Gamma}^0_{1q} = \sum_{n=1}^{\infty} \check{\Gamma}^0_{1qn}\, \kappa^n/n!\,, \tag{5.6}$$

where the $O(\kappa)$ part is $\check{\Gamma}^0_{1q1}$ as already given in (3.162) (and the above now explains the notation). Since this quantum piece is purely level-zero it does not disturb the classical parametrisation (5.2) and thus by the Zinn-Justin identities (5.3) we now get the one-loop identity

$$0 = (\check{\Gamma}, \check{\Gamma}^0_{1q}) = (Q\Phi^A)\frac{\partial_l \check{\Gamma}^0_{1q}}{\partial \Phi^A}\,. \tag{5.7}$$

Using the (now-extended) classical form (5.2) the identities (5.3) follow from nilpotency, $Q^2=0$, and the diffeomorphism invariance of $\check{\Gamma}^0$, while (5.7) expresses the diffeomorphism invariance of $\check{\Gamma}^0_{1q}$. Expanding out the anti-bracket in (5.3) to $O(\kappa^2)$ using the κ expansion of the action (5.1), the absence of classical corrections to Q (5.4,5.5) implies the first two of the following relations (which are readily verified), while the last two relations express the diffeomorphism invariance of $\check{\Gamma}^0$ and $\check{\Gamma}^0_{1q}$ at second order:

$$(\check{\Gamma}^2_1, \check{\Gamma}^2_1) = 0\,, \quad 2\,(\check{\Gamma}^2_1, \check{\Gamma}^1_1) + (\check{\Gamma}^1_1, \check{\Gamma}^1_1) = 0\,, \quad Q_0\, \check{\Gamma}^0_2 = -(\check{\Gamma}^1_1, \check{\Gamma}^0_1)\,, \quad Q_0\, \check{\Gamma}^0_{1q2} = -(\check{\Gamma}^1_1, \check{\Gamma}^0_{1q1})\,. \tag{5.8}$$

Given that $\check{\Gamma}^0_{1q1}$ (3.162) is a Λ-dependent cosmological constant term expanded to first order in κ, while the action for free gravitons (3.115) covariantizes to the Einstein-Hilbert action for which $\check{\Gamma}^0_1$ (3.160) is its first order vertex [107], we know geometrically that all-orders solutions are

$$\check{\Gamma}^0 = -2\sqrt{g}R/\kappa^2\,, \qquad \check{\Gamma}^0_{1q} = \tfrac{7}{2} b \Lambda^4 \sqrt{g}\,, \tag{5.9}$$

where R is the scalar curvature. Expanding (5.9) to $O(\kappa^2)$ we thus find solutions to the last equations in (5.8), namely

$$\begin{aligned}
\check{\Gamma}^0_2 = {} & \varphi^2\Big(\tfrac{1}{4}\partial_\alpha h_{\alpha\beta}\partial_\beta\varphi - \tfrac{3}{16}(\partial_\alpha\varphi)^2 + \tfrac{1}{8}(\partial_\sigma h_{\alpha\beta})^2 - \tfrac{1}{4}\partial_\alpha h_{\beta\sigma}\partial_\beta h_{\alpha\sigma}\Big) + \varphi\Big(h_{\alpha\beta}\partial_\sigma h_{\sigma\alpha}\partial_\beta\varphi \\
& - \tfrac{1}{4}\partial_\mu h^2_{\alpha\beta}\partial_\mu\varphi - \tfrac{1}{4}h_{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + \partial_\alpha h_{\alpha\beta}h_{\mu\nu}\partial_\beta h_{\mu\nu} + \tfrac{1}{2}\partial_\alpha h_{\mu\nu}\partial_\beta h_{\mu\nu}h_{\alpha\beta} - 2\partial_\mu h_{\nu\alpha}\partial_\beta h_{\mu\nu}h_{\alpha\beta} \\
& + \partial_\mu h_{\nu\alpha}\partial_\mu h_{\nu\beta}h_{\alpha\beta} - \partial_\mu h_{\nu\alpha}\partial_\nu h_{\mu\beta}h_{\alpha\beta}\Big) + \tfrac{1}{2}\partial_\sigma h_{\sigma\alpha}h_{\alpha\beta}\partial_\beta h^2_{\mu\nu} + \partial_\sigma h_{\sigma\alpha}\partial_\alpha h_{\beta\mu}h_{\beta\nu}h_{\mu\nu} + \tfrac{1}{4}\partial_\sigma h_{\sigma\alpha}\partial_\alpha\varphi h^2_{\mu\nu} \\
& - \tfrac{1}{8}(\partial_\sigma h_{\alpha\beta})^2 h^2_{\mu\nu} + \tfrac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h_{\alpha\beta}h_{\mu\sigma}h_{\nu\sigma} + \partial_\alpha h_{\beta\mu}\partial_\alpha h_{\beta\nu}h_{\mu\sigma}h_{\nu\sigma} + \partial_\alpha h_{\sigma\mu}\partial_\beta h_{\sigma\nu}h_{\alpha\beta}h_{\mu\nu} \\
& - \partial_\alpha h_{\sigma\mu}\partial_\nu h_{\sigma\beta}h_{\alpha\beta}h_{\mu\nu} - 2\partial_\alpha h_{\beta\mu}\partial_\nu h_{\alpha\beta}h_{\mu\sigma}h_{\nu\sigma} - \tfrac{3}{2}\partial_\mu h_{\nu\sigma}\partial_\sigma h_{\alpha\beta}h_{\alpha\mu}h_{\beta\nu} + \tfrac{1}{2}\partial_\sigma h_{\alpha\beta}\partial_\sigma h_{\mu\nu}h_{\alpha\mu}h_{\beta\nu} \\
& + \tfrac{1}{4}\partial_\sigma h_{\alpha\beta}\partial_\alpha h_{\sigma\beta}h^2_{\mu\nu} + \tfrac{1}{2}h_{\alpha\beta}\partial_\alpha h_{\beta\sigma}\partial_\sigma h^2_{\mu\nu} - \partial_\alpha\varphi\partial_\mu h_{\nu\alpha}h_{\mu\sigma}h_{\nu\sigma} - \partial_\alpha h_{\beta\mu}\partial_\beta h_{\alpha\nu}h_{\mu\sigma}h_{\nu\sigma} \\
& - \tfrac{1}{2}\partial_\alpha h_{\mu\sigma}\partial_\sigma h_{\beta\nu}h_{\alpha\beta}h_{\mu\nu} + \partial_\alpha h_{\alpha\mu}\partial_\nu\varphi h_{\mu\sigma}h_{\nu\sigma} - \tfrac{1}{8}(\partial_\mu h^2_{\alpha\beta})^2 - \tfrac{3}{16}h^2_{\mu\nu}(\partial_\alpha\varphi)^2\,,
\end{aligned} \tag{5.10}$$

which would be awkward to derive working directly with (5.8), and

$$\check{\Gamma}^0_{1q2} = \tfrac{7}{8} b\Lambda^4(\varphi^2 - h^2_{\alpha\beta})\,. \tag{5.11}$$

Note that other all-orders solutions are possible but will differ from (5.9) by addition of further invariants at higher order in κ. At $O(\kappa^2)$ this is precisely the freedom we see in (5.8) to add Q_0-closed terms $\delta\check{\Gamma}^0_2$, which are thus also $\hat{s}_0$-closed, *i.e.* solutions to the linearised mST (3.153). These latter are explored further in section 5.3.

5.3 BRST exact operators

We will see that at second order in perturbation theory (3.151), local $\hat{s}_0$-closed *bilinear* terms,

$$\hat{s}_0\, \delta\check{\Gamma}_2 = 0\,, \tag{5.12}$$

play an important role (*cf.* section 5.4.1). They appear with up to a maximum of four space-time derivatives and as we show now, turn out also to be $\hat{s}_0$-exact. Such $\hat{s}_0$-exact terms just reparametrise the free action and therefore carry no new physics [86, 107]. Indeed if we add an operator $\hat{s}_0 K_2$ to Γ_0 then, from the definitions of the free charges and linearised mST (3.150,3.151) and the form of the anti-bracket (3.83), we see that this corresponds to infinitesimal field and source redefinitions:

$$\delta\Phi^A = \frac{\partial_l K_2}{\partial \Phi^*_A}\,, \qquad \delta\Phi^*_A = -\frac{\partial_l K_2}{\partial \Phi^A}\,, \tag{5.13}$$

with the $-\Delta K_2$ part corresponding to the Jacobian of the change of variables in the partition function [83, 84], regularised by C^Λ [103, 107][22].

Since these local $\hat{s}_0$-closed bilinear terms turn out also to be $\hat{s}_0$-exact, any μ-dependence that they carry, can be eliminated by reparametrisation. This result is the BRST cohomological equivalent of the kinematical accident that pure gravity (without cosmological constant) is one-loop finite in standard quantisation [21], as we will highlight later.

Consider first the following two $\hat{s}_0$-exact solutions:

$$\tfrac{1}{2}\hat{s}_0(H^*_{\mu\nu}H_{\mu\nu}) = \partial_\mu c_\nu H^*_{\mu\nu} - H_{\mu\nu}G^{(1)}_{\mu\nu}\,, \qquad \tfrac{1}{2}\hat{s}_0(c^*_\mu c_\mu) = -\partial_\mu c_\nu H^*_{\mu\nu}\,, \tag{5.14}$$

where we used again the formula for $\hat{s}_0$ (3.153), and the explicit actions for the charges (3.119,3.122) and always discard field independent terms. The last term in the first equation is evidently again the action for free gravitons, while the remaining terms are up to a factor the source term in Γ_0 (3.115). These solutions generate the second order part of wavefunction renormalization $Z_E = 1 + z_E$ $(E = H, c)$, in close correspondence to the case of Yang-Mills [103]:

$$K_2 = \tfrac{1}{2}z_H H^*_{\mu\nu}H_{\mu\nu} + \tfrac{1}{2}z_c\, c^*_\mu c_\mu\,, \tag{5.15}$$

[22] In general it is exact expressions using the *interacting* total BRST charge that correspond to infinitessimal reparametrisations, however since we are interested only in changes at second-order and we are working at this order, only Γ_0 contributes, and not Γ_1.

the full wavefunction renormalization being given by the finite (classical) canonical transformation

$$K = \sum_E Z_E^{\frac{1}{2}} \Phi^*_E \Phi^E_{(r)}\,, \qquad \Phi^E = \frac{\partial_l}{\partial \Phi^*_E} K[\Phi_{(r)}, \Phi^*]\,, \qquad \Phi^*_{(r)E} = \frac{\partial_r}{\partial \Phi^E_{(r)}} K[\Phi_{(r)}, \Phi^*]\,, \tag{5.16}$$

the subscript (r) labelling the renormalized (anti-)fields. This implies that the fields and anti-fields renormalize in opposite directions:

$$H_{\mu\nu} = Z_H^{\frac{1}{2}} H_{(r)\mu\nu}\,,\ H^*_{\mu\nu} = Z_H^{-\frac{1}{2}} H^*_{(r)\mu\nu}\,,\ c_\mu = Z_c^{\frac{1}{2}} c_{(r)\mu}\,,\ c^*_\mu = Z_c^{-\frac{1}{2}} c^*_{(r)\mu}\,. \tag{5.17}$$

However here this is not the whole story, in particular because the reparametrisations that are generated by quantum corrections are more general than this.

Returning to the annihilation condition (5.12), we note that since $\delta\check{\Gamma}_2$ is bilinear and must have ghost number zero overall, it cannot have anti-ghost number larger than one. At lowest order in derivatives, there are only two linearly independent possibilities for the $\delta\check{\Gamma}^1_2$ part, namely $H^*_{\mu\mu}\partial_\alpha c_\alpha$ and $H^*_{\mu\nu}\partial_\mu c_\nu$. The latter option solves (5.12) since it is $\hat{s}_0$-exact; it was treated already (5.14). By inspection (3.119), the former is Q_0-exact, and thus we know that it completes to an $\hat{s}_0$-exact solution

$$\hat{s}_0(\varphi^*\varphi) = \varphi^*\partial\cdot c - R^{(1)}\varphi\,, \tag{5.18}$$

where we have also split the graviton anti-ghost into its $SO(4)$ irreducible parts:

$$H^*_{\mu\nu} = h^*_{\mu\nu} + \tfrac{1}{2}\varphi^*\delta_{\mu\nu}\,, \qquad \varphi^* = \tfrac{1}{2}H^*_{\mu\mu}\,, \tag{5.19}$$

and recalled the standard relation for $G_{\mu\nu}$ (3.123). Comparing to the structure in the previous paragraph, it is evident that (5.18) expresses the fact that the $SO(4)$ irreducible parts can have separate wavefunction renormalizations. The remaining possibility at second order in derivatives, is to have a separate $\delta\check{\Gamma}^0_2$ part, but for it to be annihilated by $\hat{s}_0$ (5.12) it must be invariant under linearised diffeomorphisms (3.119) and the graviton action in (3.115) is the unique such solution at this order in derivatives. Any change in the graviton action normalization is already taken care of by a canonical transformation, being a linear combination of the two $\hat{s}_0$-exact operators in (5.14).

At next order in derivatives there are three linearly independent possibilities for $\delta\check{\Gamma}^1_2$, namely $\varphi^*\Box\partial\cdot c$, $H^*_{\mu\nu}\Box\partial_\mu c_\nu$, and $H^*_{\mu\nu}\partial^3_{\mu\nu\alpha}c_\alpha$. Evidently the first yields a simple generalisation of φ wavefunction renormalization (5.18), while the second two are already $\hat{s}_0$-exact:

$$\hat{s}_0(\varphi^*\Box\varphi) = \varphi^*\Box\partial c - R^{(1)}\Box\varphi\,,\ \tfrac{1}{2}\hat{s}_0(c^*_\mu\Box c_\mu) = H^*_{\mu\nu}\Box\partial_\mu c_\nu\,, \quad -\hat{s}_0(H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi) = H^*_{\mu\nu}\partial^3_{\mu\nu\alpha}c_\alpha\,. \tag{5.20}$$

The remaining possibility is to have a separate $\delta\check{\Gamma}^0_2$ part, now fourth-order in derivatives. Since it must be invariant under linearised diffeomorphisms, it has to be a linear combination of the squares of the linearised curvatures (3.124) (see *e.g.* [133]). By the Gauss-Bonnet identity, only two of these are linearly independent:

$$4(R^{(1)}_{\mu\nu})^2 = (R^{(1)}_{\mu\nu\alpha\beta})^2 + (R^{(1)})^2 \,. \tag{5.21}$$

However it is straightforward to see that they are also $\hat{s}_0$-exact:

$$\begin{aligned}\hat{s}_0(\varphi^* R^{(1)}) &= Q_0^-(\varphi^* R^{(1)}) = -(R^{(1)})^2 \,,\\ \hat{s}_0(H^*_{\mu\nu} R^{(1)}_{\mu\nu}) &= -2G^{(1)}_{\mu\nu} R^{(1)}_{\mu\nu} \;= \tfrac{1}{2}(R^{(1)}_{\mu\nu\alpha\beta})^2 - \tfrac{1}{2}(R^{(1)})^2 \,.\end{aligned} \tag{5.22}$$

This completes the demonstration that the $\hat{s}_0$-cohomology of bilinear $\delta\check{\Gamma}_2$ is trivial up to the fourth order in derivatives.

We note in passing that there are other expressions for the $\hat{s}_0$-exact operators, for example the obvious generalisation of the first equation in wavefunction reparametrisations (5.14):

$$\tfrac{1}{2}\hat{s}_0(H^*_{\mu\nu}\Box H_{\mu\nu}) = \partial_\mu c_\nu \Box H^*_{\mu\nu} - H_{\mu\nu}\Box G^{(1)}_{\mu\nu} = \partial_\mu c_\nu \Box H^*_{\mu\nu} + \tfrac{1}{2}(R^{(1)})^2 - \tfrac{1}{2}(R^{(1)}_{\mu\nu\alpha\beta})^2 \,. \tag{5.23}$$

However these are not linearly independent, *e.g.* the above is a linear combination of the second exact expression in (5.22) and the middle one in (5.20). Stated another way, we have shown that the following action functional is annihilated by $\hat{s}_0$:

$$K_2 = \tfrac{1}{2}H^*_{\mu\nu}\Box H_{\mu\nu} + \tfrac{1}{2}c^*_\mu \Box c_\mu + H^*_{\mu\nu} R^{(1)}_{\mu\nu} = \tfrac{1}{2}\hat{s}_0(c^*_\mu F_\mu) \,. \tag{5.24}$$

This is so because in fact it itself is exact. The appearance of the De Donder gauge fixing functional, $F_\mu = \partial_\nu H_{\nu\mu} - \partial_\mu \varphi$, is here accidental. The most general double-derivative bilinear $\hat{s}_0$-exact K_2 is a linear combination involving the two separate parts of F_μ:

$$K_2 = \hat{s}_0(\alpha c^*_\mu \partial_\mu \varphi + \beta c^*_\mu \partial_\nu H_{\mu\nu}) = \alpha(2H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi + c^*_\mu \partial^2_{\mu\nu} c_\nu) + \beta(H^*_{\mu\nu}\partial^2_{\mu\lambda}H_{\lambda\nu} + c^*_\mu \partial^2_{\mu\nu} c_\nu + c^*_\mu \Box c_\mu) \,, \tag{5.25}$$

where α and β are free parameters. The chosen K_2 s in (5.20) are only $\hat{s}_0$-cohomology representatives determined up to addition of the above expression. Since the above expression is annihilated by $\hat{s}_0$, the canonical transformation (5.13) it generates is actually a (higher-derivative) symmetry of Γ_0:

$$\begin{aligned}\delta H_{\mu\nu} &= 2\partial_{(\mu}\xi_{\nu)} \,, &\quad \text{where} \quad \xi_\mu &= \alpha\partial_\mu\varphi + \beta\partial_\lambda H_{\lambda\nu} \,,\\ \delta c_\mu &= (\alpha+\beta)\partial^2_{\mu\nu} c_\nu + \beta\Box c_\mu \,, &\quad \delta H^*_{\mu\nu} &= -\alpha\delta_{\mu\nu}\partial^2_{\alpha\beta}H^*_{\alpha\beta} - 2\beta\partial_\alpha\partial_{(\mu}H^*_{\nu)\alpha} \,.\end{aligned} \tag{5.26}$$

For the graviton it is just part of linearised diffeomorphism invariance. The significance of the other two transformations is unclear to us. They may not survive into the interacting theory.

5.4 Inside the diffeomorphism invariant subspace

At second order in perturbation theory (3.151), the flow equation (3.152), and mST (3.146), become

$$\dot{\Gamma}_2 - \tfrac{1}{2}\,\mathrm{Str}\,\dot{\triangle}_\Lambda \Gamma_2^{(2)} = -\tfrac{1}{2}\,\mathrm{Str}\,\dot{\triangle}_\Lambda \Gamma_1^{(2)} \triangle_\Lambda \Gamma_1^{(2)}\,, \tag{5.27}$$

$$\hat{s}_0\,\Gamma_2 = -\tfrac{1}{2}\,(\Gamma_1,\Gamma_1) - \mathrm{Tr}\,C^\Lambda\,\Gamma_{1*}^{(2)}\triangle_\Lambda\Gamma_1^{(2)} \cdot \Big(C_\Lambda\,\Gamma_{1*}^{(2)}\triangle_\Lambda\Gamma_1^{(2)}\Big)\,. \tag{5.28}$$

In ref. [75] we constructed the general continuum limit solution to (5.27), *i.e.* the general solution that realises the full renormalized trajectory $\Lambda \geq 0$. It takes the form

$$\Gamma_2 = \tfrac{1}{2}\Big[\,1 + \mathcal{P}_\Lambda - (1+\mathcal{P}_\mu)\,\mathrm{e}^{\mathcal{P}^\mu_\Lambda}\,\Big]\,\Gamma_1\,\Gamma_1 + \Gamma_2(\mu)\,, \tag{5.29}$$

where the first term on the RHS is the particular integral and the last term is the complementary solution. The complementary solution takes exactly the form of the general solution (3.164) to the linearised flow equation, where however μ now has a meaning. It is an arbitrary initial point on the renormalized trajectory, lying in the range $0 < \mu < a\Lambda_\partial$. The particular integral expands out into a sum over melonic Feynman diagrams, the propagators defined through (similarly $\mathcal{P}_\mu$ and $\mathcal{P}_\Lambda$)

$$\mathcal{P}^\mu_\Lambda = \triangle^{\mu\,AB}_\Lambda \,\frac{\partial^L_l}{\partial\Phi^B}\,\frac{\partial^R_l}{\partial\Phi^A}\,. \tag{5.30}$$

They connect the two copies of the first-order solution Γ_1. Importantly the renormalized trajectory solution (5.29) is already finite, all the UV divergences having been absorbed into the relevant underlying second-order couplings, $g^\sigma_{2l+\varepsilon}$, as described in ref. [75].

We now describe the properties of these equations and their solution once the renormalized trajectory has entered the diffeomorphism invariant subspace, *cf.* fig. 3.3. This is equivalent in particular to taking the large Λ_∂ limit. In section 5.4.1 we then provide the detailed solution.

In the large Λ_∂ limit, the limit at first order

$$\Gamma_1 \to \kappa(\check{\Gamma}_1 + \check{\Gamma}_{1\mathrm{q}1}), \quad as\Lambda_\sigma \to \infty \tag{5.31}$$

can be substituted directly into the second-order mST (5.28) and into the particular integral, since these expressions are well defined being both regularised in the IR and the UV and and remain so in this limit [75]. Since $\check{\Gamma}_1$ contains a maximum of three fields, the latter then collapses to a one-loop integral in the sense that the renormalized trajectory (5.29) now reads [75]

$$\Gamma_2 = \Gamma_2(\mu) + \kappa^2\left(I_{2\Lambda} - I_{2\mu}\right)\,, \quad \text{where} \quad I_{2k} = -\tfrac{1}{4}\,\mathrm{Str}\Big[\triangle_k \check{\Gamma}_1^{(2)} \triangle_k \check{\Gamma}_1^{(2)}\Big]\,, \quad (k = \mu, \Lambda)\,. \tag{5.32}$$

In fact I_{2k} is now identical to a one-loop computation in standard quantisation. Although it is built from first-order vertices, which themselves contain a one-loop tadpole contribution $\check{\Gamma}_{1q1}$ (3.162), this latter drops out because it is linear in φ. In a similar way the RHS of the second-order mST (5.28) can be seen to contain all the standard quantisation one-loop contributions and no more.

At this stage the infinite number of underlying couplings have disappeared, leaving behind only κ. Had we chosen to keep a first-order cosmological constant, then it would also appear as an effective coupling. As we will see, similarly to the standard perturbative approach, further effective couplings generically appear order by order in perturbation theory, multiplying covariant higher derivative terms (such as curvature squared terms *etc.*). Here however these effective couplings are collective effects of the infinite number of underlying couplings, and parametrise the remaining freedom in the renormalized trajectory given that it has entered the diffeomorphism invariant subspace, *cf.* fig. 3.3. In our case from here on we can identify the perturbative expansion as being an expansion in κ. Therefore we redefine the second order contribution to be $\kappa^2\,\Gamma_2$ with complementary solution $\kappa^2\,\Gamma_2(\mu)$, so that from here on κ drops out of the equations.

The particular integral is now polynomial in the fields. In this limit we also arrange for $\Gamma_2(\mu)$ to trivialise, *i.e.* become polynomial, as explained in section 3.3.4. We see therefore that from a practical point of view the computation can now proceed in a way which is very close to standard quantisation. We comment further in the Conclusions. We emphasise that the understanding of the result is however very different: in standard quantisation, κ is a fundamental irrelevant coupling and thus there is no interacting continuum limit in the Wilsonian sense [8, 109]. Here the continuum limit is expressed in terms of the infinite number of underlying couplings, which are all (marginally) relevant. It is these latter that get renormalized in this picture, as noted above.

From the perspective of standard quantisation, the large-Λ_∂ limit (5.32) still looks a little peculiar since the particular integral is the difference of two parts: $I_{2\Lambda} - I_{2\mu}$. These parts are IR regulated but separately UV divergent. We can treat them separately by applying some appropriate supplementary regularisation, *e.g.* dimensional regularisation, $d = 4 - 2\epsilon$, as was done in ref. [103]. Furthermore we can subtract their divergences separately using a gauge invariant scheme that is independent of the finite cut-off scale μ or Λ, since such divergences anyway cancel out between the two parts. We will use the $\overline{\mathrm{MS}}$ (modified minimal subtraction) scheme, and thus subtract the terms proportional to $1/\epsilon - \gamma_E + \ln(4\pi)$, where γ_E is the Euler-Mascheroni constant.

Since $\check{\Gamma}_1$ is made of three-point vertices, the particular integral contains only two-point vertices. When derivative-expanded, $I_{2\mu}$ trivially results in polynomial (in the fields) solutions to the linearised flow equation (3.145), because these carry no Λ dependence and the tadpole corrections, where they exist, are field independent and thus –although calculable– discarded since they contain no physics. We can therefore dispense with $I_{2\mu}$ by absorbing it into a redefinition of the complementary solution: $\Gamma_2(\mu) \mapsto \Gamma_2(\mu) + I_{2\mu}$. As further discussed below, this is essentially what we will do except that we will take due account of the fact that $I_{2\Lambda}$ is ambiguous on its own, whereas in fact the difference that appears in the large-Λ_∂ limit (5.32) is finite and well defined.

Emphasising the similarity to the standard perturbative approach we now write:

$$\Gamma_{2\text{cl}} = \Gamma_{2\text{cl}}(\mu) \tag{5.33}$$

$$s_0\Gamma_{2\text{cl}} = -\tfrac{1}{2}\,(\check{\Gamma}_1, \check{\Gamma}_1)\,, \tag{5.34}$$

$$\Gamma_{2q} = \Gamma_{2q}(\mu) + I_{2\Lambda} - I_{2\mu}\,, \tag{5.35}$$

$$s_0\Gamma_{2q} - \Delta\Gamma_{2\text{cl}} = -(\check{\Gamma}_1, \check{\Gamma}_{1q}) - \operatorname{Tr} C^\Lambda\, \check{\Gamma}^{(2)}_{1*} \triangle_\Lambda \check{\Gamma}^{(2)}_1\,. \tag{5.36}$$

Here we have split the solution $\Gamma_2 = \Gamma_{2\text{cl}} + \Gamma_{2q}$ (5.32) to the second-order flow equation, into its classical (5.33) and one-loop (5.35) parts, and similarly split the complementary solution: $\Gamma_2(\mu) = \Gamma_{2\text{cl}}(\mu) + \Gamma_{2q}(\mu)$. We have also split the second-order mST (5.28) into its classical (5.34) and one-loop (5.36) parts, noting by definition of the total free quantum BRST charge (3.153), that $\hat{s}_0 = s_0 - \Delta$, where $s_0 = Q_0 + Q_0^-$ is the classical part, while the measure operator Δ is $O(\hbar)$ [83, 84, 103].

The trivialised complementary solution is just a polynomial (in fields) solution to the linearised flow equation (3.145), so $\Gamma_{2\text{cl}}(\mu)$ is a Λ-independent part, while $\Gamma_{2q}(\mu)$ contains the induced Λ-dependent one-loop tadpole correction plus its own Λ-independent part. In principle (and in general at higher order) there could be higher-loop tadpoles, however we will shortly see that in our case $\Gamma_2(\mu)$ only has a one-loop tadpole, while the one-loop Λ-independent part has no tadpoles. Therefore (5.33)–(5.36) form the complete set of $O(\kappa^2)$ equations in our case.

The classical flow equation (5.33) simply says that $\Gamma_{2\text{cl}}$ must be Λ-independent. If we absorb $I_{2\mu}$ entirely into $\Gamma_{2q}(\mu)$ as discussed above, the remaining three equations (5.34)–(5.36) are then identical to those we would derive in standard quantisation at one loop in this framework [103]. Given that we have defined $I_{2\Lambda}$ using dimensional regularisation and a gauge invariant subtraction scheme such as $\overline{\text{MS}}$, we then find a unique finite solution to these equations, up to the usual arbitrary $\ln \mu_R$ terms appearing after subtracting logarithmic divergences, where the mass scale μ_R arises from dimensionally continued couplings (here $\kappa\mu_R^\epsilon$). The insertion of the cut-off Λ leads to the modified Slavnov-Taylor identity (5.36), but for vertices defined using a gauge invariant scheme such as $\overline{\text{MS}}$, this is still just an identity that is automatically satisfied.

This is however a rather confusing way to arrive at a solution, because in our case ambiguities such as the μ_R-dependence cancel out in the difference $I_{2\Lambda} - I_{2\mu}$, reflecting the fact that the quantum part of our solution (5.35) is actually a well-defined expression. We have instead a mass parameter μ which plays essentially the same role, being the arbitrary initial point on the renormalized trajectory. Indeed like μ_R in the standard approach, physical quantities must ultimately be independent of μ. We therefore choose to absorb all of $I_{2\mu}$ except essentially for exchanging μ_R with μ. As we will see $\overline{\rm MS}$ then amounts to imposing a renormalization condition at $\mu{=}\mu_R$, in the form expected in this framework [103].

The failure point of standard perturbative quantisation is usually seen as stemming from the need to introduce bare couplings to absorb the UV divergences. Since in standard quantisation these multiply new non-trivial BRST cohomology representatives order by order in perturbation theory, new bare couplings are needed at each order. However we do not need direct access to the UV divergences to see the problem. The freedom to change the scheme away from $\overline{\rm MS}$ to some other gauge invariant scheme, is contained in the freedom to add suitable local terms associated to the ambiguities in the finite parts of these divergences. The undetermined parameters that are thus required to parameterise the scheme dependence, are nothing but the new couplings that we know appear at each order in standard quantisation. It is just that phrased this way the required new couplings are finite. Even if we stay within the $\overline{\rm MS}$ scheme, μ_R independence would force the introduction of new finite couplings.

Here the UV divergences have already been absorbed into underlying (non-geometric) second order couplings $g^\sigma_{2l+\varepsilon}$, and the ambiguities in defining the integrals are absent since they cancel out in the difference, $I_{2\Lambda} - I_{2\mu}$. Nevertheless there remains order by order in κ the equivalent freedom. Indeed the requirement that our general second-order solution for the renormalized trajectory (5.29) is independent of the initial point μ, will force the existence of the new effective couplings in the same way.[23] More generally we have the freedom to add a local term to the solution Γ_2 of the second-order flow and mST equations (5.27,5.28), provided that this addition satisfies just their left hand sides, *i.e.* the linear equations (3.145,3.153). In other words it is a change in the complementary solution $\Gamma_2(\mu)$ corresponding to a change in our choice of (quantum) BRST cohomology representative. In particular once we have secured one solution for Γ_2 (*e.g.* using the technique sketched above), we then have all possible solutions since they differ only by such a change in the quantum BRST cohomology representative. Since we already know that $I_{2\Lambda}$ on its own, defined with a suitable gauge invariant scheme, will satisfy the equations, we know that its scheme ambiguities are contained in such changes to the complementary solution.

We therefore have to confront the possibility that, although perturbatively in κ we have a genuine continuum limit (at least to second order as confirmed here), it is of an

[23]Thus also its large-Λ_∂ limit (5.32,5.35). This is so in general even if inconveniently for us, for pure quantum gravity at $O(\kappa^2)$ such additions turn out to be $\hat{s}_0$-exact, as we saw in section 5.3, and therefore can be removed by reparametrising the (anti-)fields. As we noted in section 5.3, this is equivalent to the observations made in ref. [21].

unusual form in that the renormalized trajectory is parametrised by an infinite number of effective couplings. *A priori* there seems to be nothing inconsistent with such a conclusion for quantum gravity, no matter how phenomenologically inconvenient,[24] as we discuss further in section 5.5. However in section 5.6 we uncover hints that the non-polynomial dependence on $h_{\mu\nu}$ required by diffeomorphism invariance should force the BRST cohomology at the non-perturbative level back to be at most two-dimensional, depending only on κ and the cosmological constant.

5.4.1 Vertices at second order

We now fill in the details. We have already noted that (5.33) just says that $\Gamma_{2\mathrm{cl}}$ is Λ-independent. From the first three equations (5.8) derived from the CME, it is clear that the choice we require so as to satisfy the classical BRST invariance (5.34), is

$$\Gamma_{2\mathrm{cl}} = \Gamma_{2\mathrm{cl}}(\mu) = \check{\Gamma}^0_2\,. \tag{5.37}$$

It is therefore actually independent of μ. As anticipated, it only has a one-loop tadpole,

$$\Gamma_{2q}(\mu) \ni \check{\Gamma}^0_{2q2} = \Omega_\Lambda\Big(\tfrac{3}{2}(\partial_\alpha\varphi)^2 - 2\partial_\alpha h_{\alpha\beta}\partial_\beta\varphi - (\partial_\sigma h_{\alpha\beta})^2 + 2(\partial_\alpha h_{\alpha\beta})^2\Big) - \tfrac{3}{4}b\Lambda^4(\varphi^2 + h^2_{\alpha\beta})\,, \tag{5.38}$$

computed using the classical $O(\kappa^2)$ expression (5.10) and the tadpole corrections defined in the general form of the complementary solution (3.164), and labelled using the system introduced in (5.6). (Notice that this involves the trivialisation of $\alpha = 2$ coefficient functions (4.35), as is clear from the top line of the classical $O(\kappa^2)$ expression (5.10), but their tadpole corrections are also joined by $h_{\mu\nu}$-tadpole corrections from the bottom lines in (5.10).)[25] If we had already absorbed $I_{2\mu}$ into $\Gamma_{2q}(\mu)$, (5.38) would actually be the complete solution for $\Gamma_{2q}(\mu)$, being the unique $O(\kappa^2)$ tadpole integral formed from the classical action.

By inspection the particular integral (5.32) and the RHS of the one-loop second-order mST (5.36) can contribute only up to a maximum anti-ghost level two. In fact there is no contribution even at this level, as we now show. In the particular integral this would require attaching two propagators between $\check{\Gamma}^2_1$ (3.158) and $\check{\Gamma}^1_1$ (3.159), or between two copies of $\check{\Gamma}^1_1$ while preserving both anti-fields, but it is not possible to attach the propagators in this way. Since $\check{\Gamma}_{1q}$ only has level zero, the anti-bracket cannot contribute above level zero, whilst there is no correction term at level two in the one-loop second-order mST since this would require $\check{\Gamma}^{2\,(2)}_1$, but there is no way to join this by a propagator to $\check{\Gamma}^{(2)}_{1*}$. Thus all these anti-ghost levels are solved by $\Gamma^{n\geq 2}_{2q} = \Gamma^{n\geq 2}_{2q}(\mu) = 0$.

[24] In the general case, these include couplings for curvature-squared terms, whose sign must be chosen to maintain unitarity, in contrast to the case where quantum gravity would then be renormalizable in standard quantisation [134].

[25] *E.g.* $(\partial_\alpha\varphi)^2$ arises from the second and the last monomial in (5.10) yielding, by (4.36) and (3.141), $-\frac{3}{16}(1-9) = \frac{3}{2}$.

For similar reasons the one-loop second-order mST (5.36) also collapses at anti-ghost level one:

$$Q_0\,\Gamma^1_{2q} = 0\,, \tag{5.39}$$

indeed the correction term now requires $\check{\Gamma}^{1(2)}_1$ with its anti-field intact, but no such contributions are possible. However at this level the particular integral does make a contribution. The integral

$$I^1_{2\Lambda} = i\int_p H^*_{\mu\nu}(p)\,\mathcal{B}^I_{\mu\nu\alpha}(p,\Lambda)\,c_\alpha(-p) \tag{5.40}$$

is a two-point vertex formed from two copies of $\check{\Gamma}^1_1$ (3.159) and fluctuation and ghost propagators (3.140)–(3.143) in the self-energy contribution (5.32). In $d\!=\!4$ dimensions

$$\begin{aligned}\mathcal{B}^I_{\mu\nu\alpha}(p,\Lambda) = -\int_q \frac{C_\Lambda(q)\,C_\Lambda(p\!+\!q)}{q^2}\Big\{\frac{1}{(p\!+\!q)^2}\,[\,\tfrac{3}{2}\,p_\alpha p_{(\mu}q_{\nu)} + \tfrac{3}{2}\,p_\mu p_\nu q_\alpha + 3\,p_{(\mu}q_{\nu)}q_\alpha + p^2 p_{(\mu}\delta_{\nu)\alpha}\\ + 2\,\delta_{\alpha(\mu}p_{\nu)} + 2\,\delta_{\alpha(\mu}q_{\nu)} + \delta_{\mu\nu}q_\alpha\Big\}\,.\end{aligned} \tag{5.41}$$

Choosing the complementary solution to have the same form as (5.40), with kernel $\mathcal{B}^c_{\mu\nu\alpha}(p,\mu)$, Γ^1_{2q} also has this form and is trivially satisfies (5.39). Writing its kernel as $\mathcal{B}_{\mu\nu\alpha}(p,\Lambda)$, we have

$$\mathcal{B}_{\mu\nu\alpha}(p,\Lambda) = \mathcal{B}^c_{\mu\nu\alpha}(p,\mu) + \mathcal{B}^I_{\mu\nu\alpha}(p,\Lambda) - \mathcal{B}^I_{\mu\nu\alpha}(p,\mu)\,. \tag{5.42}$$

The momentum integral (5.41) is a formal expression since it has quadratic and logarithmic divergences. By using dimensional regularisation to define it (using the d-dimensional $\check{\Gamma}_1$ described in section 3.3.4), we automatically subtract the quadratic divergence, and by using the $\overline{\text{MS}}$ scheme we subtract the log divergence leaving just the usual $\ln\mu_R$ ambiguity.[26] Taylor expanding the momentum integral up to cubic order gives:

$$\begin{aligned}(4\pi)^2\,I^1_{2\Lambda} =&\Lambda^2\int_0^\infty\! du\, C(C-2)\,\left[\tfrac{1}{2}\varphi^*\partial\!\cdot\! c - \tfrac{9}{8}\hat{s}_0(c^*_\mu c_\mu)\right]\\ &-\tfrac{1}{2}\varphi^*\Box\partial\!\cdot\! c + \hat{s}_0(\tfrac{1}{4}H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi + \tfrac{5}{16}c^*_\mu\Box c_\mu) + \frac{1}{2}\int_0^\infty\! du\, u\,(C')^2\,\hat{s}_0(H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi - \tfrac{5}{4}c^*_\mu\Box c_\mu)\\ &+\frac{1}{2}\left(\ln\frac{\mu_R^2}{\Lambda^2} + \int_0^1\!\frac{du}{u}\,(1-C)^2 + \int_1^\infty\!\frac{du}{u}\,C(C-2)\right)\hat{s}_0(H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi + \tfrac{3}{4}c^*_\mu\Box c_\mu) + O(\partial^5)\,,\end{aligned} \tag{5.43}$$

Here $C = C(u)$ is the cut-off function, and we recognise amongst these expressions, instances of Ω_Λ (4.3) and b (3.161). The $O(\partial^5)$ and higher terms arise from UV finite integrals (so do not depend on μ_R). The derivation is sketched in app. A.2. As explained earlier, if we had absorbed $I_{2\mu}$ into $\Gamma_{2q}(\mu)$, the remaining level-one part from

[26] If desired, the subtraction can be reinstated since at one loop it always appears with the same coefficient as $\ln\mu_R^2$.

(5.35), $\Gamma^1_{2q}=I^1_{2\Lambda}$, would already be a solution. The Λ-independent $\hat{s}_0$-exact parts could be discarded by changing the choice of $\Gamma_{2q}(\mu)$, but we keep them to match the $\overline{\text{MS}}$ scheme. We only need to recognise that the end result (5.42) must be independent of μ_R. Thus we set the one-loop complementary solution part to

$$\Gamma^1_{2q}(\mu) = i\int_p H^*_{\mu\nu}(p)\,\mathcal{B}^c_{\mu\nu\alpha}(p,\Lambda)\,c_\alpha(-p) = I^1_{2\mu} + Z^1_2(\mu)\,\hat{s}_0(H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi + \tfrac{3}{4}c^*_\mu\Box c_\mu)\,, \quad (5.44)$$

which is independent of μ_R/μ, since this dependence cancels between $I^1_{2\mu}$ and

$$Z^1_2(\mu) = \frac{1}{(4\pi)^2}\ln\frac{\mu}{\mu_R} + z^1_2\,. \quad (5.45)$$

We see that $\kappa^2 Z^1_2(\mu)$ induces a change of BRST cohomology representative at second order, as expected.[27] In this case the change is $\hat{s}_0$-exact and thus amounts to a canonical reparametrisation *cf.* section 5.3, hence Z^1_2 is a wavefunction-like parameter. Its presence ensures that Γ^1_2 is also independent of the initial point μ on the renormalized trajectory, since a change of $\mu \mapsto \alpha\,\mu$ in the total solution $\mathcal{B}_{\mu\nu\alpha}(p,\Lambda)$ (5.42) can be absorbed by a change $\delta Z^1_2 = \delta z^1_2 = -\ln\alpha/(4\pi)^2$. Altogether the one-loop level-one solution (5.35) to the renormalized trajectory is:

$$\begin{aligned}(4\pi)^2\,\Gamma^1_{2q} = {} & \Lambda^2\int_0^\infty\! du\, C(C-2)\,\left[\tfrac{1}{2}\varphi^*\partial\cdot c - \tfrac{9}{8}\hat{s}_0(c^*_\mu c_\mu)\right] \\ & - \tfrac{1}{2}\varphi^*\Box\partial\cdot c + \hat{s}_0(\tfrac{1}{4}H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi + \tfrac{5}{16}c^*_\mu\Box c_\mu) \\ & + \frac{1}{2}\int_0^\infty\! du\, u\,(C')^2\,\hat{s}_0(H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi - \tfrac{5}{4}c^*_\mu\Box c_\mu) \\ & + \frac{1}{2}\left((4\pi)^2 Z^1_2(\Lambda) + \int_0^1\frac{du}{u}\,(1-C)^2 + \int_1^\infty\frac{du}{u}\,C(C-2)\right)\hat{s}_0(H^*_{\mu\nu}\partial^2_{\mu\nu}\varphi + \tfrac{3}{4}c^*_\mu\Box c_\mu) \\ & + O(\partial^5)\,. \end{aligned} \quad (5.46)$$

If we work in scaled variables, where we absorb Λ according to dimensions, the result depends on Λ only indirectly through $Z^1_2(\Lambda)$. The scaled result is thus of self-similar form as expected for a renormalization group trajectory [20]. Renormalization schemes follow from the choice of renormalization condition for Z^1_2. For example, the $\overline{\text{MS}}$ scheme is recovered here with the renormalization condition

$$Z(\mu) = 0 \quad \text{at} \quad \mu = \mu_R\,, \quad (5.47)$$

which sets $z^1_2 = 0$ in (5.45). Evaluating the physical limit, $\mathcal{B}_{\mu\nu\alpha}(p) = \lim_{\Lambda\to 0}\mathcal{B}_{\mu\nu\alpha}(p,\Lambda)$,

[27] In general this would not be clear until we computed the μ_R dependence at all anti-ghost levels, but see (5.53) and the discussion below it.

a standard Feynman integral, we get for the physical vertex in the scheme (5.47)

$$(4\pi)^2\,\mathcal{B}_{\mu\nu\alpha}(p) = \left(\tfrac{3}{4}p^2 p_\mu\delta_{\nu\alpha} - \tfrac{1}{2}p_\mu p_\nu p_\alpha\right)\ln(p^2/\mu^2) + \tfrac{2}{3}p_\mu p_\nu p_\alpha - \tfrac{5}{6}p^2 p_\mu \delta_{\nu\alpha} + \tfrac{1}{6}\delta_{\mu\nu}p^2 p_\alpha\,, \tag{5.48}$$

where the net effect of the choice of complementary solution (5.44) and renormalization condition (5.47) is just to convert μ_R to μ.

At anti-ghost level zero, the one-loop solution (5.35) is now written as

$$\Gamma^0_{2q} = \check{\Gamma}^0_{2q2} + \delta\Gamma^0_{2q}(\mu) + I^0_{2\Lambda} - I^0_{2\mu}\,, \tag{5.49}$$

the first two terms on the RHS being the complementary solution having split off the one-loop tadpole (5.38). Adopting a parallel notation to above we write

$$I^0_{2\Lambda} = \tfrac{1}{2}\int_p H_{\mu\nu}(p)\,\mathcal{A}^I_{\mu\nu\alpha\beta}(p,\Lambda)\,H_{\alpha\beta}(-p)\,. \tag{5.50}$$

Here $\mathcal{A}^I_{\mu\nu\alpha\beta}(p,\Lambda)$ has two contributions: one from using two $\check{\Gamma}^1_1$ vertices joined by ghost propagators and one from two copies of $\check{\Gamma}^0_1$ joined by H propagators. As a formal integral in $d=4$ dimensions, and understood to be symmetrised *i.e.* to be recast as $\mathcal{A}^I_{((\mu\nu)(\alpha\beta))}$, we can write it as:

$$\begin{aligned}
&\mathcal{A}^I_{\mu\nu\alpha\beta}(p,\Lambda) = \\
&\int_q C_\Lambda(q)\,C_\Lambda(p+q)\Bigg\{\frac{-1}{q^2(p+q)^2}\Big[p_\alpha p_\beta p_\mu p_\nu + 2p_\alpha p_\beta p_\mu q_\nu + 2p_\alpha p_\beta q_\mu q_\nu + p_\alpha p_\mu q_\beta q_\nu \\
&+ 2p_\alpha q_\beta q_\mu q_\nu + q_\alpha q_\beta q_\mu q_\nu - p^2\delta_{\alpha\mu}p_\beta p_\nu - \tfrac{1}{2}p^2\delta_{\mu\nu}(p_\alpha p_\beta + 3p_\alpha q_\beta + 3q_\alpha q_\beta) + \tfrac{1}{16}p^4\delta_{\mu\nu}\delta_{\alpha\beta} + \tfrac{1}{2}p^4\delta_{\alpha\mu}\delta_{\beta\nu}\Big] \\
&+ \frac{1}{q^2}\Big[\tfrac{1}{8}p^2\delta_{\alpha\beta}\delta_{\mu\nu} + \tfrac{5}{4}p\cdot q\,\delta_{\alpha\beta}\delta_{\mu\nu} - p\cdot(p+q)\delta_{\alpha\mu}\delta_{\beta\nu} + 2\delta_{\alpha\mu}(p+q)_\beta(p+q)_\nu - \delta_{\mu\nu}(p_\alpha p_\beta + 3p_\alpha q_\beta + q_\alpha q_\beta)\Big] \\
&+ \tfrac{1}{4}\delta_{\alpha\beta}\delta_{\mu\nu}\Bigg\}
\end{aligned} \tag{5.51}$$

Again we define it however using $\overline{\mathrm{MS}}$. Up to $O(\partial^2)$, (5.50) takes the form

$$\begin{aligned}
(4\pi)^2\,I^0_{2\Lambda} &= \Lambda^4\int_0^\infty du\,u\,C(C-2)\ \left[\tfrac{5}{24}h^2_{\mu\nu} + \tfrac{1}{8}\varphi^2\right] \\
&+ \Lambda^2\int_0^\infty du\,C(C-2)\ \left[\tfrac{5}{24}\varphi\partial^2_{\alpha\beta}h_{\alpha\beta} + \tfrac{5}{8}(\partial_\alpha h_{\alpha\beta})^2 - \tfrac{19}{48}(\partial_\gamma h_{\alpha\beta})^2 - \tfrac{5}{32}(\partial_\alpha\varphi)^2\right] \\
&- \Lambda^2\int_0^\infty du\,u^2(C')^2\ \left[\tfrac{1}{12}\varphi\partial^2_{\alpha\beta}h_{\alpha\beta} + \tfrac{1}{8}(\partial_\alpha h_{\alpha\beta})^2 + \tfrac{7}{96}(\partial_\gamma h_{\alpha\beta})^2 + \tfrac{1}{16}(\partial_\alpha\varphi)^2\right] + O(\partial^4)\,.
\end{aligned} \tag{5.52}$$

It is a unique result but acquires dependence on $\ln\mu_R$, which appears amongst the $O(\partial^4)$ terms. (We do not display all these terms because there are rather too many.) Setting $\delta\Gamma^0_{2q}(\mu) = I^0_{2\mu}$, Γ^0_{2q} (5.49) would already be a solution. As before, we choose the complementary solution to be this up to converting the $\ln\mu_R$ dependence to $\ln\mu$

dependence. We find

$$\delta\Gamma^0_{2q}(\mu) = I^0_{2\mu} + Z^0_{2a}(R^{(1)}_{\mu\nu\alpha\beta})^2 + Z^0_{2b}(R^{(1)})^2\,, \tag{5.53}$$

where to one loop,

$$Z^0_{2a}(\mu) = -\frac{61}{120(4\pi)^2}\ln\frac{\mu}{\mu_R} + z^0_{2a}\,, \qquad Z^0_{2b}(\mu) = -\frac{23}{120(4\pi)^2}\ln\frac{\mu}{\mu_R} + z^0_{2b}\,. \tag{5.54}$$

Again the role of these $\kappa^2 Z$s is (also) to ensure that the full solution is actually independent of μ at $O(\kappa^2)$, and ensuring that the scaled result is a self-similar solution [20]. Since the only other $\ln\mu$ part, sitting in $\Gamma^1_{2q}(\mu)$ (5.44), is already $\hat{s}_0$-closed, this addition must be $\hat{s}_0$-closed, which it is by virtue of being invariant under linearised diffeomorphisms. As we saw, (5.22), it is actually $\hat{s}_0$-cohomologically trivial, and thus as a consequence of the Koszul-Tate differential (3.122), vanishes on the free equations of motion (*i.e.* on shell), making the Zs here also wavefunction-like. This is also clear directly, on using the Gauss-Bonnet identity (5.21) [21]. (Note that the coefficients do not agree with those in ref. [21] which are computed in the background field method. The terms only have to agree on-shell, which they do trivially since they both vanish.) Again the $\overline{\text{MS}}$ scheme is recovered by choosing the renormalization condition (5.47). In the physical limit, the tadpole correction (5.38) vanishes, so once more the net effect of our renormalization condition on the choice of complementary solution (5.53) is to swap μ_R for μ. We find for the physical Γ^0_2 two-point vertex (where again we mean this to be recast as $\mathcal{A}_{((\mu\nu)(\alpha\beta))}$):

$$\begin{aligned}(4\pi)^2\,\mathcal{A}_{\mu\nu\alpha\beta}(p) = &\Big(\tfrac{7}{10}p_\alpha p_\beta p_\mu p_\nu - \tfrac{23}{60}p^2\delta_{\alpha\beta}p_\mu p_\nu - \tfrac{61}{60}p^2\delta_{\alpha\mu}p_\beta p_\nu + \tfrac{23}{120}p^4\delta_{\alpha\beta}\delta_{\mu\nu} + \tfrac{61}{120}p^4\delta_{\alpha\mu}\delta_{\beta\nu}\Big)\ln\!\left(\frac{p^2}{\mu^2}\right)\\ &+ \tfrac{19}{75}p_\alpha p_\beta p_\mu p_\nu - \tfrac{1229}{1800}p^2\delta_{\alpha\beta}p_\mu p_\nu - \tfrac{283}{1800}p^2\delta_{\alpha\mu}p_\beta p_\nu + \tfrac{1829}{3600}p^4\delta_{\alpha\beta}\delta_{\mu\nu} + \tfrac{283}{3600}p^4\delta_{\alpha\mu}\delta_{\beta\nu}\,,\end{aligned} \tag{5.55}$$

the quartic on the first line being the same as appears in (5.53,5.54).

Finally, substituting Γ^0_{2q} (5.49) into the one-loop second-order mST (5.36) and using the final equation in the CME relations (5.8) we see that[28]

$$Q_0\left(\Gamma^0_{2q} - \check{\Gamma}^0_{1q2}\right) + Q^-_0\,\Gamma^1_{2q} = -\mathrm{Tr}\,C^\Lambda\,\check{\Gamma}^{(2)}_{1*}\triangle_\Lambda\check{\Gamma}^{(2)}_1\Big|^0\,, \tag{5.56}$$

($\Delta\Gamma_2$ trivially vanishes) where on the RHS we retain only the anti-ghost level zero piece. This last term has three contributions, one with $\check{\Gamma}^2_1$ and $\check{\Gamma}^1_1$ differentiated with respect to c^* and (anti-)ghosts, the other two using $\check{\Gamma}^1_1$ and its H^*, and either a second copy $\check{\Gamma}^1_1$ differentiated with respect to H and $\bar{c}$, or $\check{\Gamma}^0_1$ where the differentials are of

[28] Note that the covariantisation $\check{\Gamma}^0_{1q2}$ thus plays a different role from $\check{\Gamma}^0_{2q2}$.

course both with respect to H. The result is:

$$- \mathrm{Tr}\, C^\Lambda\, \check{\Gamma}^{(2)}_{1*} \triangle_\Lambda \check{\Gamma}^{(2)}_1 \Big|^0 = i\int_p H_{\mu\nu}(p)\, \mathcal{F}_{\mu\nu\alpha}(p,\Lambda)\, c_\alpha(-p)\,, \tag{5.57}$$

where

$$\begin{aligned}\mathcal{F}_{\mu\nu\alpha}(p,\Lambda) = \int_q C_\Lambda(q)\, C^\Lambda(p{+}q)\Big\{&\delta_{\mu\nu}p_\alpha + 3\delta_{\mu\nu}q_\alpha \;+ \frac{1}{q^2}[2q_\mu q_\nu (p{+}q)_\alpha + 4p_\mu p_\nu q_\alpha \\ &- 2p{\cdot}q\,(p{+}q)_{(\mu}\delta_{\nu)\alpha} + p{\cdot}q\,\delta_{\mu\nu}(p{+}q)_\alpha - 4\delta_{\mu\nu}q_\alpha p^2]\Big\}.\end{aligned} \tag{5.58}$$

The above $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}$ vertices are analogous to vertices in Yang-Mills theory, which we labelled similarly in ref. [103]. Note that $\overline{\mathrm{MS}}$ has no effect on $\mathcal{F}_{\mu\nu\alpha}$ or the tadpole integrals, (5.11) and (5.38), since these are already fully regulated by the cut-off functions and thus have no $1/\epsilon$ divergences. Writing $G^{(1)}_{\mu\nu}$ (3.123) in momentum space as

$$G^{(1)}_{\mu\nu}(p) = -G^{(1)}_{\alpha\beta\mu\nu}(p) H_{\alpha\beta}(p)\,, \tag{5.59}$$

we see that (5.56) is a modified Slavnov-Taylor identity for two-point vertices:

$$\mathcal{A}_{\mu\nu\alpha\beta}\, p_\beta + G^{(1)}_{\mu\nu\sigma\lambda}\mathcal{B}_{\sigma\lambda\alpha} = \tfrac{7}{8} b\Lambda^4(\delta_{\mu\nu}p_\alpha - 2p_{(\mu}\delta_{\nu)\alpha}) + \tfrac{1}{2}\mathcal{F}_{\mu\nu\alpha}\,, \tag{5.60}$$

where the first terms on the RHS come from putting $Q_0\,\check{\Gamma}^0_{1q2}$, on the RHS and using the formula for $\check{\Gamma}^0_{1q2}$ (5.11). Note that in the physical limit $\Lambda \to 0$, the above RHS vanishes and this equation becomes the unmodified Slavnov-Taylor identity: it just says that the amplitude $\mathcal{A}$ is gauge invariant on shell, *i.e.* up to terms proportional to the free equation of motion $G^{(1)}_{\mu\nu} = 0$. We have confirmed that the physical vertices, (5.48) and (5.55), do indeed satisfy the physical limit of this equation. This means that if we write the IR cut-off functions in terms of the UV one, $C_\Lambda = 1 - C^\Lambda$, the LHS of the above identity (5.60) can be rewritten as a sum over contributions all of which are UV regulated by C^Λ and thus well defined without further regularisation. Further manipulation similar to those in ref. [103] would then establish that (5.60) holds exactly as an identity between the integrals (5.51,5.41,5.58). In fact by the Bianchi identity, $p_\mu G^{(1)}_{\mu\nu}(p) = 0$, it is apparent that only the last term in the physical $\mathcal{B}$ vertex (5.48) makes a contribution. Therefore the above identity (5.60) states that the part of the physical $\mathcal{A}$ vertex dependent on renormalization conditions, namely the $\ln p^2/\mu^2$ part of (5.55), is transverse, a property we have already established in (5.53). The derivative expansion of $\mathcal{F}$ (5.57) gives:[29]

$$\begin{aligned}-(4\pi)^2\,\mathrm{Tr}\,C^\Lambda\,\check{\Gamma}^{(2)}_{1*}\triangle_\Lambda\check{\Gamma}^{(2)}_1\Big|^0 &= \Lambda^4\int_0^\infty\!\! du\,u\,C(C-2)\,\left[\tfrac{5}{6}h_{\mu\nu}\partial_\mu c_\nu+\tfrac{1}{4}\varphi\partial\cdot c\right]\\ &\quad - b\Lambda^4\left[\tfrac{7}{6}h_{\mu\nu}\partial_\mu c_\nu+\tfrac{15}{4}\varphi\,\partial\cdot c\right]\\ &\quad +\Lambda^2\int_0^\infty\!\! du\,C(C-2)\,\left[\tfrac{1}{3}h_{\mu\nu}\Box\partial_\mu c_\nu+\tfrac{11}{8}\varphi\Box\partial\cdot c-\tfrac{11}{12}h_{\mu\nu}\partial^3_{\mu\nu\alpha}c_\alpha\right]\\ &\quad +\Lambda^2\int_0^\infty\!\! du\,u^2(C')^2\,\left[\tfrac{1}{24}h_{\mu\nu}\partial^3_{\mu\nu\alpha}c_\alpha+\tfrac{13}{24}h_{\mu\nu}\Box\partial_\mu c_\nu\right]+O(\partial^5)\,.\end{aligned}\tag{5.61}$$

We have verified that the one-loop second-order mST identity (5.56) is satisfied up to $O(\partial^3)$ by the derivative expansions (5.43), (5.52) and (5.61) together with the tadpole corrections (5.11,5.38). In particular this confirms explicitly that these tadpole contributions automatically supply required $O(\partial^0)$ and $O(\partial^2)$ terms necessary for satisfying this identity.

5.5 Discussion

We have seen that at second order in perturbation theory the end result is the standard one for the one-particle irreducible effective action at $O(\kappa^2)$, and which is thus a one loop contribution. Since we are dealing with pure quantum gravity at vanishing cosmological constant, the logarithmic running is due to wavefunction-like reparametrisations. This is true in standard quantisation [21] but it is also reflected in the new quantisation. However outside the diffeomorphism invariant subspace these reparametrisations are not purely wavefunction-like but are accompanied by coefficient functions, for example at anti-ghost level zero they will take the form:

$$\delta H_{\mu\nu}=R^{(1)}_{\mu\nu}\,f^a_\Lambda(\varphi,\mu)+\delta_{\mu\nu}\,R^{(1)}\,f^b_\Lambda(\varphi,\mu)\,,\qquad \text{where}\quad f^i_\Lambda(\varphi,\mu)\to c_i\,\kappa^2\ln\mu\quad\text{as}\quad \Lambda_\partial\to\infty\,,\tag{5.62}$$

c_i being numerical constants ($i=a,b$). There are also infinitely many perturbative reparametrisations possible of the form

$$\delta\varphi=f_\Lambda(h_{\mu\nu},\varphi)\,,\tag{5.63}$$

the RHS evidently being made up of Lorentz invariant combinations of $h_{\mu\nu}$. Some combination of these reparametrisations will correspond to redundant operators [135, 136]. It is these kind of reparametrisations that would lead to a demonstration of the quantum equivalence of unimodular gravity and ordinary gravity [107, 137] within this new quantisation.

Notice that the logarithmic running encapsulated in $Z^1_2(\mu)$ (5.45) and $Z^0_{2a,b}(\mu)$ (5.54), is by no means the only logarithmic running in the theory. Infinitely many

[29] Again note that Ω_Λ (4.3) and b (3.161) give alternative expressions for the terms linear in C.

more cases are generated in the derivative expansion of the general solution for the second-order renormalized trajectory (5.29) [75]. However all the other cases vanish as a power of Λ_∂ in the large amplitude suppression scale limit.

It seems clear that once we add matter and/or a cosmological constant, it will no longer be the case that the logarithmic running inside the diffeomorphism invariant subspace is attributable to a reparametrisation. It will have to be attributed to new diffeomorphism-invariant effective couplings. These effective couplings are precisely the same couplings that need to be introduced in standard quantisation [21]. Indeed we still expect to need a complementary solution in the form we gave for $\delta\Gamma^0_{2q}(\mu)$ (5.53), but the curvature-squared terms no longer vanish on the equations of motion since the Einstein tensor is now sourced by the matter stress-energy tensor and/or a term proportional to $g_{\mu\nu}$ in the case of a cosmological constant.

Actually, once inside the diffeomorphism invariant subspace, we are obeying both the flow equation *and* the mST, and therefore the solution must correspond to an RG flow in the standard quantisation. The problem in standard quantisation is that these flows have an infinite number of parameters, new ones appearing at each loop order. In standard quantisation they are identified with renormalized couplings, and the corresponding bare couplings are required to absorb the UV divergences. It is clear that in this standard framework none of these flows can correspond to a genuine perturbative continuum limit in the usual Wilsonian sense, *i.e.* a *renormalized trajectory* emanating from the Gaussian fixed point, since κ is irrelevant. (The same is true of all higher order couplings apart from the curvature squared ones.)

In this new quantisation we have found a solution to this latter problem: we have constructed a genuine perturbative renormalized trajectory. We have demonstrated that it works in perturbation theory, at both first order [1, 107] and now, second order [75]. It emanates from the Gaussian fixed point along relevant directions provided by the underlying (marginally) relevant couplings, $g^\sigma_{2l+\varepsilon}$. It is these couplings that absorb the UV divergences [75]. Once inside the diffeomorphism invariant subspace, this renormalized trajectory must coincide with a subset of the RG flows derived in standard quantisation. The question is which subset. Since we need to send $\Lambda_\partial \to \infty$ in fig. 3.3 to fully recover diffeomorphism invariance, we know at least that these flows must exist all the way to $\Lambda \to \infty$ within the diffeomorphism invariant subspace, even though they will not qualify as part of a perturbative renormalized trajectory inside this subspace.

Once inside the diffeomorphism invariant subspace, the underlying couplings disappear and the trajectory is parametrised by diffeomorphism-invariant effective couplings. One possibility is that there is no restriction: the subset is the whole set, the effective couplings are in one-to-one correspondence with the couplings required in standard quantisation. Devastating as this might be for the general predictivity of the theory, this construction suggests that there is nothing inherently inconsistent with such a scenario.

If this is the outcome, nevertheless the new quantisation provides a different perspective. For example, it is not true that the introduction of these higher order couplings require a loss of unitarity, provided that their signs are chosen to avoid wrong-sign poles in the full propagators. In standard quantisation, the assumption is that once couplings are introduced for the curvature-squared terms for example, these couplings must be part of some 'fundamental' bare action, and thus from the beginning turn the theory into one with higher derivatives even at the free (bilinear) level. Here, the bare action lies outside the diffeomorphism invariant subspace. The higher derivative interactions there must always be accompanied by a $\delta^{(n)}_\Lambda(\varphi)$ operator, and thus cannot alter the kinetic terms. In other words, the bilinear action maintains its two-derivative form [107].

It remains the case that ultimately the perturbative development of the theory is organised in powers of κ and therefore by dimensions, accompanied by increasing numbers of space-time derivatives at higher order. But since we are dealing with a theory with a genuine continuum limit, the fact that perturbation theory breaks down in the regime[30] $\kappa\partial > 1$, just indicates that the theory becomes non-perturbative in this regime and not, as usually interpreted, a signal of breakdown of an effective quantum field theory description.

We see very clearly that it is the logarithmically running terms and their finite part ambiguities, necessarily BRST invariant, that demand the introduction of new couplings order by order in perturbative quantum gravity. In contrast, the power-law Λ dependence is computed unambiguously. Nothing within perturbation theory demands that new couplings be associated to such Λ^{2n} terms (integer $n>0$). Nor is the field dependence associated to Λ^{2n}, closed under BRST, but rather is intimately related to the modifications of the Slavnov-Taylor identities. Thus the problem in quantum gravity is to find the *mechanism*, if there is one, that determines (some or all of) the finite parts associated to the $\ln(\Lambda/\mu)$ terms that appear at the perturbative level. If for example, all these parameters are fixed by such a mechanism, we would be left with only one new parameter at the quantum level, the mass scale that arises by dimensional transmutation from the very existence of the RG (the equivalent to Λ_{QCD} in QCD).

In fact we know that at third order, the first-order couplings will run with Λ [75]. It is conceivable that this running and the required subsequent matching into the diffeomorphism invariant subspace, plays a role in providing this missing mechanism. Below, we discuss another possibility, some hints that this mechanism arises solely from insisting that the RG flow within the diffeomorphism invariant subspace, remains non-singular all the way to $\Lambda\to\infty$. One such well-studied possibility is a non-perturbative (asymptotically safe) UV fixed point [33, 138, 139]. However note that our current construction was born from attempts to solve issues with the degeneration of the fixed points and eigenoperator spectrum that are seen in that scenario if one goes (sufficiently

[30]Here ∂ stands for the typical magnitude of space-time derivatives.

carefully) beyond truncations involving just a finite number of operators (see the final discussions in refs. [1, 77]). As we now explore, a mechanism for fixing the parameters could follow from the same mathematical properties of the partial differential flow equations that lead to these problems in the first place.

5.6 A possible non-perturbative mechanism

In the conformal sector the infinite number of couplings $g^{\sigma}_{2l+\varepsilon}$ lead to a new effect, namely the fact that almost always, even at the linearised level, RG flows towards the IR become singular and then cease to exist [8]. This is very much interwoven into the subsequent development [1, 75, 107]. Indeed it is for this reason that the construction requires the initial point μ for the renormalized trajectory (5.29) to lie below Λ_∂, most of the trajectory then being safely developed from the IR to the UV. This is due to the fact that we are dealing with solutions of a parabolic partial differential equation that are non-polynomial in the amplitude: such solutions are only guaranteed when flowing from the IR to the UV [8].

These comments apply equally well to the $h_{\mu\nu}$ sector however with the crucial difference that there the equation is reverse parabolic, with solutions only guaranteed when flowing from the UV to the IR [8]. The problem is not seen for polynomial linearised solutions, because such solutions are a finite sum of eigenoperators (the Hermite polynomials) [8, 107] with constant coefficients. But diffeomorphism invariance, which is imposed in the IR (inside the diffeomorphism invariant subspace), requires us to use solutions that are non-polynomial in the $h_{\mu\nu}$ amplitude (because the curvature terms require both the metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$). Thus diffeomorphism invariance forces us to consider solutions non-polynomial in $h_{\mu\nu}$, evolving from the IR to the UV. Such solutions almost always fail at some critical scale $\Lambda_{\rm cr}$, before we reach $\Lambda\to\infty$.

In reality, the solution must exist simultaneously in both the $h_{\mu\nu}$ and φ sectors. Consider a solution $\delta\Gamma$ to the linearised flow equation (3.145). Isolating the $h_{\mu\nu}$ and φ amplitude dependence, we can expand $\delta\Gamma$ over monomials $\varsigma_{\mu_1\cdots\mu_n}$:

$$\delta\Gamma = \sum_{\varsigma} \varsigma_{\mu_1\cdots\mu_n}(\partial,\partial\varphi,\partial h,c,\Phi^*)\, f^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}(h_{\alpha\beta},\varphi) + \cdots\,, \tag{5.64}$$

where we suppress Lorentz indices on the arguments in ς and we mean that its (anti-)field arguments can appear as indicated or differentiated any number of times. These new coefficient functions f^{ς}_{Λ} are necessarily non-polynomial in $h_{\alpha\beta}$ and φ for the reasons we have explained. The linearised flow equation (3.145) can be solved exactly using the same integrating factor as in the general solution (3.164). The ellipses in (5.64) refer to the tadpole corrections so formed by attaching propagators to ς either exclusively, or also to $h_{\alpha\beta}$ and φ. Now from the linearised flow equation (3.145), the coefficient functions themselves satisfy the flow equation:

$$\dot{f}^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}(h_{\alpha\beta},\varphi) \;=\; \Omega_\Lambda\left(\frac{\partial^2}{\partial h^2_{\mu\nu}}-\frac{\partial^2}{\partial\varphi^2}\right)f^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}\,. \tag{5.65}$$

Here we clearly see the property that the equation in each sector separately is parabolic, but in opposite directions, and thus in fact the Cauchy initial value problem[31] for such a partial differential equation is not well defined in either direction. This mathematical property is not cured, but only obscured, by using the full non-linear flow equations. We see that we are dealing with novel partial differential equations whose solution typically becomes singular when it is evolved in either direction, *even at the linearised level.* As we have emphasised already for flows towards the IR in the φ sector [8], this does not mean solutions do not exist but rather that the initial conditions must be very special, *i.e.* lie within a heavily restricted subspace. Below we uncover hints that this allows only the cosmological constant and κ ultimately to exist as independent couplings.

Notice that this issue applies only to the fields that are differentiated in the flow equation, *i.e.* to the quantum fields – whose second order differentials together with the RG time derivative make the equations (reverse) parabolic. It does not apply to the anti-fields, nor to background fields if the background field approach is followed. In fact it does not apply to the ghost fields either because these are Grassmann and thus dependence on their amplitude is necessarily polynomial. Therefore the issue only arises for the quantum fluctuation fields $h_{\mu\nu}$ and φ.

To take these arguments a little further, we recall that the finite part ambiguity $\delta\Gamma_{(\ell)}$ that appears at ℓ-loop order, is a local Λ-independent operator, and note that its dimension is

$$[\delta\Gamma_{(\ell)}] = 2(\ell{+}1) \tag{5.66}$$

(*e.g.* as required by dimensions from the factors of κ). We also note that if the mST (3.146) is to be obeyed inside the diffeomorphism invariant subspace, we must have $(\Gamma_0,\delta\Gamma_{(\ell)})=0$ (since all the other parts are at higher loop order, in particular the correction term in the mST carries an extra loop) [103]. In other words, at ℓ-loop order the ambiguous parts $\delta\Gamma_{(\ell)}$ must be invariant under the full classical BRST transformations [103], *cf.* section 5.2, reflecting standard treatments [91, 147, 148]. In particular the level zero part, $\delta\Gamma^0$, must be diffeomorphism invariant, and thus at one loop are curvature-squared terms, as confirmed in $\delta\Gamma^0_{2q}$ (5.53), at two-loop order are κ^2 times curvature cubed, or $\kappa^2 R\nabla^2 R$ type terms, and so forth. They are therefore indeed non-polynomial in $h_{\mu\nu}$ (and also φ as also imposed by the new quantisation).

At loop-order higher than ℓ, where $\delta\Gamma_{(\ell)}$ first appears, $\delta\Gamma_{(\ell)}$ gets altered by the flow equation (3.152) and mST (3.146) in ways that are not straightforward to analyse. If we model the situation by just taking the linearised flow equation (3.145) and imposing

[31]This is a property of the flow under effective cut-off Λ. It has nothing to do with the existence (or otherwise in some approaches [140–146]) of a Cauchy initial value surface in the dynamics of the theory.

$\delta\Gamma = \delta\Gamma_{(\ell)}$ at $\Lambda = 0$, the perturbation will no longer satisfy BRST invariance or the mST once $\Lambda > 0$. However we will be able to see the restrictions that arise from the fact that the flows are typically singular. In close similarity to the solution for the pure-φ coefficient functions (3.165), the partial differential equation (5.65) is solved formally by the Fourier transform:

$$f^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}(h_{\alpha\beta},\varphi) = \int \frac{d^9\pi_{\alpha\beta}\, d\pi}{(2\pi)^{10}}\ \mathfrak{f}^{\varsigma}_{\mu_1\cdots\mu_n}(\pi_{\alpha\beta},\pi)\ \mathrm{e}^{\frac{1}{2}\Omega_\Lambda(\pi^2_{\mu\nu}-\pi^2)+i\pi_{\mu\nu}h_{\mu\nu}+i\pi\varphi}\,, \qquad (5.67)$$

where $\pi_{\mu\nu}$ is traceless, being the momentum conjugate to $h_{\mu\nu}$. That (5.67) is the Fourier form of the solution, can be seen straightforwardly by substitution, and matches the general linearised functional solution (3.164) as one can see by substituting the $\Lambda = 0$ Fourier transform for the physical coefficient function. However for the above to be more than a formal solution to (5.65), we need the Fourier integral to converge. We see that as Λ increases from zero, convergence in the φ sector only improves, since it is weighted by $\mathrm{e}^{-\frac{\pi^2}{2}\Omega_\Lambda}$, reflecting the fact that the Cauchy initial value problem is well defined in this sector for IR$\to$UV [8]. However in the $h_{\mu\nu}$ sector the integral has the exponentially growing weight, $\mathrm{e}^{\frac{\pi^2_{\mu\nu}}{2}\Omega_\Lambda}$. Unless $\mathfrak{f}^\varsigma$ decays faster than an exponential of $\pi^2_{\mu\nu}$ (at fixed π), the solution (5.67) will be singular at some critical scale $\Lambda = \Lambda_{\mathrm{cr}} \geq 0$, above which the flow ceases to exist.

We see therefore that the flows will exist only for carefully chosen parametrisations of the metric in terms of $h_{\mu\nu}$ and φ. Now we show that solutions of the form (5.67) cannot exist simultaneously for all the $\delta\Gamma$ that match diffeomorphism invariant $\delta\Gamma_{(\ell)}$ at $\Lambda = 0$. If we take the Einstein-Hilbert action (5.9) as an example and expand it over monomials as in (5.64), the required strong suppression of high conjugate momenta $\pi_{\mu\nu}$ in $\mathfrak{f}^\varsigma$, means that for the above to be a solution, there must be no rapid variation of the Einstein-Hilbert action under changes in the $h_{\mu\nu}$ amplitude. Obviously, at a minimum we then need a parametrisation that exists for all amplitudes. That is not true of the simple linear split form of $g_{\mu\nu}$ (2.2) which is not positive definite for all $h_{\mu\nu}$ and φ, and for which $g_{\mu\nu}$ is singular at $\kappa\varphi = -2$, and whenever $\kappa h_{\mu\nu}$ has -1 as an eigenvalue. We can cure this by for example parametrising the metric $g_{\mu\nu}$ in terms of an exponential of $\kappa h_\mu{}^\nu$ (considered as a matrix), see *e.g.* [121, 149–152]. Such a parametrisation can also ensure that the square root, in the measure $\sqrt{g}$, does not lead to branch cuts (as also would expressing the metric in terms of a vierbein, since the measure is then its determinant).

This is still not enough to allow a solution in the form (5.67) however. From the already required faster than exponential decay, we see that the mod-squared amplitudes $|\mathfrak{f}^\varsigma_{\mu_1\cdots\mu_n}|^2$ are integrable. Thus by Parseval's theorem, the squared coefficient functions $(f^\varsigma_{\Lambda\mu_1\cdots\mu_n})^2$ must also be integrable over $d^9h_{\alpha\beta}d\varphi$. This in turn implies that the coefficient functions $f^\varsigma_{\Lambda\mu_1\cdots\mu_n}$ must vanish as $h_{\alpha\beta}\to\infty$.[32] Since $\sqrt{g}R \mapsto \alpha\sqrt{g}R$ under scaling

[32]They must decay faster than $1/|h_{\alpha\beta}|^{9/2}$. Given an appropriate choice of $\mathfrak{f}^\varsigma$, one can get a much

$g_{\mu\nu} \mapsto \alpha g_{\mu\nu}$ (where α is some constant), we see that this last condition will hold true for the Einstein-Hilbert action if and only if $g_{\mu\nu}$ itself vanishes in this limit.

A Fourier solution (5.67) for the cosmological constant term, is then not ruled out by this condition, since $\sqrt{g} \mapsto \alpha^2\sqrt{g}$, and thus it will also vanish in the limit $h_{\alpha\beta} \to \infty$. However all the higher derivative terms are then ruled out from having such solutions, since curvature-squared terms go like α^0, while the higher order terms behave as negative powers of α and thus actually diverge in the limit $h_{\alpha\beta} \to \infty$.

Notice that despite the fact that we are modelling using only linearised solutions, the arguments we are making are non-perturbative in κ, because the breakdown in the solutions happens at finite or diverging $\kappa h_{\mu\nu}$. In general the level-zero part satisfies $\delta\Gamma_{(\ell)} \mapsto \alpha^{1-\ell}\delta\Gamma_{(\ell)}$, and thus if these perturbations had to extend to solutions $\delta\Gamma$ of Fourier type (5.67), we would have shown that, despite the apparent freedom to change individually the new effective couplings that appear at each loop order, non-perturbatively in κ the requirement that the renormalized trajectory is non-singular actually rules out all such infinitesimal changes $\delta\Gamma_{(\ell)}$. We would therefore conclude that the only freely variable couplings are in fact κ itself and the cosmological constant.

We cannot quite draw such dramatic conclusions however. The arguments we have presented can only be regarded as hints. Firstly, solutions exist to the linearised flow equations (5.65) that do not fit the assumed Fourier form (5.67). For example solutions polynomial in the graviton can be cast in Fourier space, but $\mathfrak{f}^{\varsigma}$ is then distributional, *viz.* a sum over differentials of $\delta(\pi_{\alpha\beta})$. Another example is provided by the φ part of exponential parametrisation [121, 149–152] which extends to the solution

$$f(\varphi) = \mathrm{e}^{\frac{\kappa}{2}\varphi} \quad \Longrightarrow \quad f_\Lambda(\varphi) = \mathrm{e}^{\frac{\kappa}{2}\varphi + \frac{1}{8}\kappa^2\Omega_\Lambda}\,, \tag{5.68}$$

as can be confirmed by direct substitution in (5.65) or by using the Green's function $\delta^{(0)}_\Lambda(\varphi{-}\varphi_0)$, *cf.* (4.5) [1, 8]. However these are not sufficient to parametrise the Einstein-Hilbert action. In fact finding a parametrisation that can be extended to a solution of the linearised flow equation (5.65), either of Fourier type (5.67) or otherwise, looks challenging.[33] It is even more challenging to find one that also works for the cosmological constant term, and it is not credible that a parametrisation could be found that would also allow solutions for the higher derivative terms $\delta\Gamma_{(\ell)}$. On the contrary, it may be that there is no sensible solution even for the Einstein-Hilbert action alone. Secondly, infinitesimal changes $\delta\Gamma_{(\ell)}$ do not in fact have to satisfy the simple linearised equations (5.65) but operator flow equations that depend on the rest of the effective action:

$$\delta\dot{\Gamma}_{(\ell)} = \tfrac{1}{2}\,\mathrm{Str}\left(\dot{\triangle}_\Lambda\triangle_\Lambda^{-1}\left[1+\triangle_\Lambda\Gamma^{(2)}_I\right]^{-1}\triangle_\Lambda\delta\Gamma^{(2)}_{(\ell)}\left[1+\triangle_\Lambda\Gamma^{(2)}_I\right]^{-1}\right)\,, \tag{5.69}$$

improved estimate by using the method of steepest descents in (5.67).

[33] We did not find a parametrisation of $g_{\mu\nu}$ that leads to $\mathfrak{f}^{\varsigma}$ with decay faster than exponential of $\pi^2_{\mu\nu}$. Approaching from the other direction, nor did we find such $\mathfrak{f}^{\varsigma}$ that then lead to a non-singular $g_{\mu\nu}$.

as follows immediately from perturbing the exact RG flow equation (3.152). However, although these flow equations are much more involved than the simple linearised flow equations (5.65), and are such that they allow solutions that remain compatible with BRST invariance through the (perturbed) mST (3.146), they share with (5.65) the property that their Cauchy initial value problem is not well defined in either direction.

5.7 General gauges

All the results in this chapter, and thus far in this thesis, were derived in Feynman – De Donder gauge. In this section we will show that the structure changes only in an inessential way for a class of gauge conditions. First we recall that a great advantage of using off-shell BRST invariance is that BRST invariant correlators are independent of the choice of gauge [83, 84, 107]. Indeed in terms of the quantum fields, ϕ^A, and the Wilsonian effective action, S, the mST (3.146) is just the QME

$$\tfrac{1}{2}(S,S) - \Delta S = 0\,, \tag{5.70}$$

whose powerful algebraic properties continue to hold exactly despite regularisation by the cut-off function [107]. In more detail, an operator $\mathcal{O}_i$ is (off-shell) BRST invariant if it satisfies

$$s\,\mathcal{O}_i = (S,\mathcal{O}_i) - \Delta\mathcal{O}_i = 0\,, \tag{5.71}$$

while a change of gauge is implemented by adding a BRST exact term $s\,\delta K$ to the action [83, 84]. Then it follows that a BRST invariant correlator is invariant under change of gauge [83, 84, 107]:

$$\delta\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle = -\langle s\delta K\,\mathcal{O}_1\cdots\mathcal{O}_n\rangle = -\langle s\,(\delta K\mathcal{O}_1\cdots\mathcal{O}_n)\rangle = \frac{1}{\mathcal{Z}}\int\!\mathcal{D}\phi\,\Delta\left(\delta K\mathcal{O}_1\cdots\mathcal{O}_n\,\mathrm{e}^{-S}\right) = 0\,. \tag{5.72}$$

($\mathcal{Z}$ is the normalisation of the partition function. In this last step we use algebraic properties of the QME, the disjoint support of the $\mathcal{O}_i$, and the fact that Δ now contains a total functional derivative with respect to ϕ [107].)

The formulation we are using here is entirely equivalent since the two formulations are mapped into each other by the Legendre transform relation [59, 71, 103, 106, 153]:

$$\Gamma_I[\Phi,\Phi^*] = S_I[\phi,\Phi^*] - \tfrac{1}{2}\,(\Phi-\phi)^A\,\triangle^{-1}_{\Lambda\,AB}\,(\Phi-\phi)^B\,, \tag{5.73}$$

where S_I is the corresponding interaction part of the Wilsonian effective action. Even so, since the Φ^A are not BRST invariant operators, Γ will now depend on the choice of gauge in some unilluminating way.[34] However in the physical limit $\Lambda\to 0$, the mST (3.146) becomes the Zinn-Justin equation and is obeyed exactly. Now we can also go

[34] If instead we used this to compute the Schwinger functional of only gauge invariant operators we would still find results that are gauge parameter independent.

on shell and at this point the Φ^A do provide BRST invariant states. Thus the on-shell vertices of the physical Γ are independent of the choice of gauge. These vertices obey the unmodified Slavnov Taylor identities, as we have already seen from (5.60). Indeed we saw that this just tells us that the physical amplitudes are gauge invariant on shell.

While gratifying, these results are hardly unexpected. After all we saw in section 5.4 that the second order equations (5.33) – (5.34) are identical to those we would derive in standard quantisation at one loop in this framework. All this follows provided the flow finishes up inside the diffeomorphism invariant subspace, *i.e.* provided the QME (5.70), equivalently mST (3.146), holds exactly in the infrared. However a central feature of our construction is that the renormalized trajectory lies outside this subspace for $\Lambda > a\Lambda_\partial$. Thus the important question is what happens to this part of the trajectory in other gauges, in particular whether it continues to be supported by a novel tower of relevant operators in the ultraviolet, and whether the trajectory still enters the diffeomorphism invariant subspace in the infrared.

To investigate this we rederive the flow of the upper part of the trajectory in a more general De Donder gauge. This gauge is implemented by using the gauge fixing functional

$$F_\mu = \partial_\nu H_{\nu\mu} - \partial_\mu \varphi \,. \tag{5.74}$$

Although we use the Batalin-Vilkovisky framework [83, 84] to implement off-shell BRST invariance, for the graviton sector the general De Donder gauge amounts to adding to the free graviton action Γ_0, *cf.* (3.115), the term $\frac{1}{2}\alpha F_\mu^2$, where α is the gauge parameter. Up until now we have used Feynman – De Donder gauge, $\alpha = 2$, since such a choice leads to significant simplifications:

$$\Gamma_0|_{\text{Feynman De Donder}} \equiv \tfrac{1}{2}\left(\partial_\lambda h_{\mu\nu}\right)^2 - \tfrac{1}{2}\left(\partial_\lambda \varphi\right)^2 \,, \tag{5.75}$$

in particular decoupling the conformal mode φ from the traceless part $h_{\mu\nu}$. Our construction is built on a succession of results reported in previous papers [1, 8, 74, 75, 107, 109], where also the Feynman – De Donder gauge was used. Therefore we need to go back to the beginning to show how things now change in a more general gauge.

The central observation that led to the new quantisation is that in Euclidean signature, the Einstein-Hilbert action is unbounded from below. This is a gauge invariant statement: the unboundedness is caused by the fact that the action is proportional to the scalar curvature, R, which can take either sign of any magnitude.[35] At the free level Γ_0 still has these problems. In the Feynman De Donder gauge (5.75) this is particularly clear. Before gauge fixing the situation is obscured by linearised gauge invariance $\delta H_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, equivalently linearised BRST (3.119). However the gauge invariant

[35]It is actually positive scalar curvature that is the problem [8, 9].

statement is that the action (3.115) is unbounded below in the following direction

$$2\varphi - \frac{\partial^2_{\alpha\beta}}{\Box} H_{\alpha\beta} = 2\left(1 - \frac{1}{d}\right)\varphi - \partial^2_{\frac{\alpha ta}{\Box}} h_{\alpha\beta} = \frac{1}{-\Box} R^{(1)}\,, (5.76)$$

where we used (3.124). Using different gauge choices we can shift the instability to different modes, but we cannot remove it. Indeed in the Landau gauge limit of the De Donder gauge (5.74), where we insist that $F_\mu = 0$ identically, the conformal mode coincides with this (linearised) gauge invariant quantity:

$$\varphi = \frac{1}{-\Box} R^{(1)}\,. \tag{5.77}$$

By adapting the off-shell BRST Batalin-Vilkovisky framework, what we actually do is first go to the non-minimal gauge invariant basis by adding to the action

$$\frac{1}{2\alpha} b_\mu^2 - i b_\mu \bar{c}_\mu^*\,, \tag{5.78}$$

where $\bar{c}_\mu^*$ is the anti-ghost anti-field and b_μ is the auxiliary field [1, 103, 107]. Adding a BRST exact term involving the gauge fixing fermion [83, 84] $\Psi = \bar{c}_\mu F_\mu$, then induces a canonical transformation to the gauge fixed basis. This map takes us from the gauge invariant and gauge fixes bases

$$H^*_{\mu\nu}|_{\mathrm{gi}} = H^*_{\mu\nu}|_{\mathrm{gf}} + \partial_{(\mu}\bar{c}_{\nu)} - \tfrac{1}{2}\delta_{\mu\nu}\partial \cdot \bar{c}, \tag{5.79}$$

together with [107]

$$\bar{c}_\mu^*\,|_{\mathrm{gi}} = \bar{c}_\mu^*\,|_{\mathrm{gf}} - F_\mu\,. \tag{5.80}$$

Since Ψ does not involve α, these are not affected by the more general gauge. However the shifts allow the kinetic terms to be inverted to give the propagators, now in general gauge α:

$$\langle b_\mu(p)\, H_{\alpha\beta}(-p)\rangle = -\langle H_{\alpha\beta}(p)\, b_\mu(-p)\rangle = 2\,\delta_{\mu(\alpha}p_{\beta)}/p^2\,, \tag{5.81}$$

$$\langle b_\mu(p)\, b_\nu(-p)\rangle = 0\,, \tag{5.82}$$

$$\triangle_{\mu\nu\varphi}(p) := \langle h_{\mu\nu}(p)\,\varphi(-p)\rangle = \langle \varphi(p)\, h_{\mu\nu}(-p)\rangle = \left(1-\frac{2}{\alpha}\right)\left(\frac{\delta_{\mu\nu}}{d}-\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}\,, \tag{5.83}$$

$$\triangle_{\varphi\varphi}(p) := \langle \varphi(p)\,\varphi(-p)\rangle = \left(\frac{1}{\alpha}-\frac{d-1}{d-2}\right)\frac{1}{p^2}\,, \tag{5.84}$$

$$\triangle_{\mu\nu\,\alpha\beta}(p) := \langle h_{\mu\nu}(p)\, h_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} + \left(\frac{4}{\alpha}-2\right)\frac{p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)}}{p^4} \tag{5.85}$$

$$+\frac{1}{d^2}\left(\frac{4}{\alpha}-d-2\right)\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2}$$

$$+\frac{2}{d}\left(1-\frac{2}{\alpha}\right)\frac{\delta_{\alpha\beta}p_\mu p_\nu + p_\alpha p_\beta \delta_{\mu\nu}}{p^4}\,. \tag{5.86}$$

These generalise the Feynman gauge results (3.137)–(3.139). Comparing the new $h_{\mu\nu}$ propagator (5.86) to the old one (3.141), underlines why it is preferable to work in Feynman gauge. Note that b_μ does not actually propagate into itself. The $\frac{1}{2}\alpha F_\mu^2$ term mentioned earlier would be generated by integrating out b_μ after the transformation (5.80). The ghost propagator is not displayed since it is unchanged from (3.136).

Now we recall that to all orders in perturbation theory, no interactions are generated involving the extended basis, b_μ and $\bar{c}^*_\mu$, while the anti-ghost $\bar{c}_\mu$ only appears when one switches to a gauge fixed basis [1, 103]. These statements follow from the fact that the first order interaction Γ_1 can be constructed from the minimal set (see the final paragraphs of section 2 in ref. [1]). We will confirm that this still holds shortly. This means that we can continue to work in minimal gauge invariant basis, switch to the gauge fixed basis only while computing the ghost propagator corrections.

The first step is to solve the linearised flow equation (3.145) to find the eigenoperators, now in general De Donder gauge. Since $h_{\mu\nu}$ and φ now propagate into each other, we need an expansion over monomials with coefficient functions involving both $h_{\mu\nu}$ and φ. In other words we have an expansion which is actually identical to that considered in (5.64):

$$\Gamma_1 = \sum_\varsigma \varsigma_{\mu_1\cdots\mu_n}(\partial,\partial\varphi,\partial h, c,\Phi^*)\, f^\varsigma_{\Lambda\mu_1\cdots\mu_n}(h_{\alpha\beta},\varphi) + \cdots\,. \tag{5.87}$$

Again the ellipses refer to tadpole corrections formed by attaching propagators to ς either exclusively, or also to $h_{\alpha\beta}$ and φ. Once again, the linearised flow equation is solved exactly using the same integrating factor as in the general solution (3.164):

$$\Gamma_1 = \exp\left(-\frac{1}{2}\triangle^{\Lambda\, AB}\frac{\partial_l^2}{\partial\Phi^B\partial\Phi^A}\right)\Gamma_{\text{phys}}\,, \tag{5.88}$$

where $\Gamma_{\rm phys}$ is the $\Lambda\to 0$ limit (3.144). From (3.145), the coefficient functions satisfy the flow equation

$$\dot{f}^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}(h_{\alpha\beta},\varphi) \;=\; \int_p \left(\triangle^{\Lambda}_{\mu\nu\,\alpha\beta}(p)\frac{\partial^2}{\partial h_{\mu\nu}\partial h_{\alpha\beta}} + 2\triangle^{\Lambda}_{\mu\nu\varphi}(p)\frac{\partial^2}{\partial h_{\mu\nu}\partial\varphi} + \triangle^{\Lambda}_{\varphi\varphi}(p)\frac{\partial^2}{\partial\varphi^2}\right) f^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}\,, \tag{5.89}$$

where we set $d=4$ and used the fact that the tadpole integrals have quadratic dependence on Λ. Performing the tadpole integrals we see that the $\varphi\, h_{\mu\nu}$ cross-term vanishes. (It has to because there is no invariant traceless rank two tensor.) Computing the other two we find

$$\dot{f}^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}(h_{\alpha\beta},\varphi) \;=\; \Omega_\Lambda\left[\left(\frac{1}{\alpha}+\frac{1}{2}\right)\frac{\partial^2}{\partial h^2_{\mu\nu}} + \left(\frac{1}{\alpha}-\frac{3}{2}\right)\frac{\partial^2}{\partial\varphi^2}\right] f^{\varsigma}_{\Lambda\mu_1\cdots\mu_n}\,, \tag{5.90}$$

where Ω_Λ was defined in (4.3). Thus the flow of the coefficient function is again parabolic of a simple form in each sector separately. If we choose

$$\alpha > \frac{2}{3} \qquad \text{or} \qquad \alpha < -2\,, \tag{5.91}$$

the sign on the right hand side is negative for φ and positive for $h_{\mu\nu}$ just as before. Therefore we find [107] that the eigenoperators we have to expand in take the same form (4.7) as before. In particular the sum over eigenoperators converges (in the square integrable sense) for otherwise arbitrary coefficient functions, only if the interactions are expanded over polynomials in $h_{\mu\nu}$ times the operators $\delta^{(n)}_\Lambda(\varphi)$. The only difference is that these latter operators now involve a rescaled Ω_Λ:

$$\delta^{(n)}_\Lambda(\varphi) := \frac{\partial^n}{\partial\varphi^n}\,\delta^{(0)}_\Lambda(\varphi)\,, \qquad \text{where} \qquad \delta^{(0)}_\Lambda(\varphi) := \frac{1}{\sqrt{2\pi\Omega^\alpha_\Lambda}}\,\exp\left(-\frac{\varphi^2}{2\Omega^\alpha_\Lambda}\right)\,, \tag{5.92}$$

where

$$\Omega^\alpha_\Lambda = \left(\frac{3}{2}-\frac{1}{\alpha}\right)\Omega_\Lambda\,, \tag{5.93}$$

and the tadpole corrections represented by the ellipses in (4.7) and (5.88) must now be computed with the propagators (5.83) – (5.86). All of the properties of the coefficient functions then go through unchanged. In particular the coefficient functions have an amplitude suppression scale Λ_∂ which can be chosen common to all of them, independent of the monomial, and such that they trivialise in the large amplitude suppression scale limit (4.32,4.35). The BRST representatives (3.158), (3.159) and (3.160) are the same as before since these are computed without gauge fixing (in particular this means they can still be taken to depend only on the minimal set). The same applies to the analysis in secs. 5.2 and 5.3. The only change is to the one-loop tadpole correction (3.162). The second order analysis of the conformal sector [75] also goes through unchanged since this effectively relies on dimensional analysis and general properties of the coefficient functions. This means that once again the renormalized trajectory can smoothly enter the diffeomorphism subspace as discussed in section 5.4. Once inside we have returned to the standard framework where the gauge parameter dependence is absent on shell and well understood off shell, as we already discussed above.

We have thus shown that the structure we have developed changes only in inessential ways for any value of the gauge fixing parameter α in the range (5.91). It is interesting to examine what happens in the gap excluded by the inequalities (5.91). If $\alpha = -2$ (or $\alpha = \frac{2}{3}$), the flow equation (5.92) no longer has a dependence on $h_{\mu\nu}$ (or φ). The eigenoperator equation in this case is no longer of Sturm-Liouville type and some different notion of convergence over eigenoperators would have to be formulated [8, 107]. If $-2 < \alpha < 0$ the sign on the right hand side of (5.92) is negative for both φ and $h_{\mu\nu}$. In this case there is no longer convergence when expanding over polynomials in $h_{\mu\nu}$ [8, 77]. A sensible Wilsonian flow would require generalising the $\delta^{(n)}_{\Lambda}(\varphi)$ to $h_{\mu\nu}$ fields. This opens up new possibilities for quantisation schemes which go beyond the present investigation. The choice $\alpha = 0$ is Landau gauge and is singular; to define it requires taking a limit from $\alpha \neq 0$. Finally if $0 < \alpha < 2/3$, both signs are positive on the right hand side of (5.92). This does not mean we have removed the instability. One can verify that the tadpole integral $\int_p \langle R^{(1)}(p)\, R^{(1)}(-p)\rangle$ is negative and independent of α. The instability remains, but in higher derivative terms. Like unimodular gravity, *cf.* section 5.5 and [107], the consequences of instability for Wilsonian RG are less straightforward to analyse and go beyond the present investigation.

5.8 Summary and Conclusions

In Euclidean signature the Einstein-Hilbert action is unbounded from below. This so-called conformal factor instability [9] means that the partition function for quantum gravity makes no sense without further modification. The authors of ref. [9] proposed to solve this by analytically continuing the conformal factor along the imaginary axis. However the Wilsonian exact RG flow equation still makes sense in the presence of this instability [8, 33] and anyway provides a more powerful route to define the continuum limit. Nevertheless the instability has a profound effect on RG properties. We find that flows close to the Gaussian fixed point, involving otherwise arbitrary functions of the conformal factor amplitude, φ, remain well defined only if expanded over a novel tower of increasingly relevant operators $\delta^{(n)}_{\Lambda}(\varphi)$ ($n = 0, 1, \cdots$) [8]. Everything in the new quantisation just follows from this observation.

The result is the renormalized trajectory sketched in fig. 3.3. Although at first sight this looks like the standard picture for a perturbative continuum limit, an important difference is that the upper part lies outside the diffeomorphism invariant subspace where the corresponding BRST invariance (or rather modified Slavnov-Taylor identities) can be respected. The quantisation is thus defined “off space-time” [1] in the upper part of the renormalized trajectory. In this part, the interactions involve traceless fluctuations $h_{\mu\nu}$ and the conformal factor φ, acting as separate fields. The dynamical metric $g_{\mu\nu}$, which combines these as stipulated by diffeomorphism invariance, only comes together inside the diffeomorphism invariant subspace and does not make sense as a concept outside this subspace.

In ref. [75] we solved for the renormalized trajectory for pure quantum gravity at second order in perturbation theory and showed that, for the underlying coupling constants in appropriate domains, the trivialisation conditions can be satisfied. In this chapter we have shown that it is then indeed possible for the renormalized trajectory to enter the diffeomorphism invariant subspace. We then solved for its subsequent evolution, in particular for the limit $\Lambda \to 0$ where we recover the physical amplitudes. As we saw, the result is equivalent to solving for pure quantum gravity at one loop and $O(\kappa^2)$ in standard perturbation theory. It is not so surprising therefore that we also find that effective parameters are left behind associated to logarithmically running terms at this order, and that for pure quantum gravity these are not physical because they can be absorbed by reparametrisations.

Beyond $O(\kappa^2)$ in pure quantum gravity and/or after including matter or a cosmological constant, it is no longer true in the usual treatment that logarithmic divergences can be absorbed by reparametrisation. Instead they force the introduction of new couplings order by order in the loop expansion. The main question then is whether in this new quantisation one similarly finds that ultimately an infinite number of diffeomorphism invariant effective couplings are required, introduced order by order in perturbation theory. If this is the case, it appears one is left with a genuine entirely consistent continuum theory of perturbative quantum gravity which, unfortunately for its phenomenology, is controlled by an infinite number of couplings.

Actually the precise correspondence, of pure quantum gravity at second order in the new quantisation, to standard quantisation of effective quantum gravity at one-loop and $O(\kappa^2)$, is somewhat of an accident, see below (5.32). The interactions in the upper part of the renormalized trajectory are second order in couplings, but non-perturbatively quantum, and thus involve a sum to all loops over tadpoles and melonic Feynman diagrams. On entering the diffeomorphism invariant subspace, this collapses to something that can be reinterpreted as finite order in $\hbar$. Furthermore at second order, the order in $\hbar$ amounts to one loop in the loop expansion. At higher orders it looks like the large-Λ_∂ limit may differ from the standard solution in that not all contributions perturbative in $\hbar$ are reproduced up to the maximum number of loops that appear. It seems therefore that higher order will imply a finite reordering of the loop-wise expansion, but it is not clear that this has a physical consequence. From third order onwards, the first-order underlying couplings (that parametrise the first order vertices) will run [75]. This may lead to restrictions on matching into the diffeomorphism invariant subspace. On the other hand, since there is no corresponding running of κ in standard quantisation, we expect the running to effectively freeze out on entering the diffeomorphism invariant subspace, as a consequence of the trivialisation conditions.

Finally in section 5.6 we noted that the particular parabolic properties of these flow equations mean that solutions are typically singular when evolved in either the IR or UV directions, once one works with a space of solutions that is non-polynomial in both the quantum fields $h_{\mu\nu}$ and φ. Non-perturbatively in κ the solutions must indeed be non-polynomial in these quantum variables, as forced by diffeomorphism invariance via the mST. We uncovered hints that this property provides a non-perturbative mechanism which fixes the free parameters down to just κ and the cosmological constant. It would appear to be sufficient to have this mechanism at work entirely within the diffeomorphism invariant subspace. Then the theory can be defined after all by working solely within this space. But then the understanding of how the continuum limit is achieved would be very different from the Wilsonian one, since it would not be in terms of a renormalized trajectory emanating from an ultraviolet fixed point.

Chapter 6

Provable properties of asymptotic safety in $f(R)$ approximation

As we have discussed in this thesis there is a great deal to be investigated regarding the different directions of the flows of the different sectors of the graviton. In this chapter we investigate a related problem in AS and show how this relates to the previous results outlined in this thesis with the hope that in addition to shedding light on outstanding problems in AS it will also help illuminate this contradictory flow problem.

The structure of this chapter is as follows. In the first section we review in more detail the structure of AS and note the key results. In the following section, following [154], we set out the form of the flow equation, fixed point equation and eigenoperator equation. We discuss some of the choices to be made in particular for the endomorphism parameters α_s, the choice of sign for the cut-off in the conformal factor sector, and the choice of background manifold. In section 6.2.1 we develop the equations in the case that the latter is a four-sphere, and explain further our choice of exponential cut-off for the common profile $r(z)$. As $R \to 0$, the equations go over to a flat space limit. This is derived and discussed in section 6.2.2, and in particular its implications for Sturm-Liouville theory where the $R = 0$ boundary presents an obstruction. We see that the only sensible option is to continue into the four-dimensional hyperboloid through a kind of smooth topology change, as discussed further also in the Conclusions, section 6.6.

To apply Sturm-Liouville theory we need that the eigenoperators are square integrable under the Sturm-Liouville weight. The question is whether this makes sense in quantum gravity. This leads us naturally into section 6.3 where we derive the asymptotic behaviour of solutions at large R. Separately this allows us to characterise the nature of the fixed points and eigenoperators. First, in section 6.2.3, we explain the setup for the equations on the hyperboloid. In particular we furnish the full constraints on α_0. In section 6.3.1 we derive the large R asymptotic behaviour of a fixed point solution $f(R)$, first on the sphere and then on the hyperboloid. We see that the sphere solution differs from that assumed in [154] and is in fact dominated by cut-off effects as $R \to \infty$. Computed exactly, it ought to be universal, as it is in fact for the hyperboloid. We show that the culprit is the course-graining of certain zero modes (modes with vanishing modified Laplacian) on the sphere.

On both sphere and hyperboloid we see that the asymptotic solution contains only one parameter. Perturbations about this would provide the other parameter but such perturbations are invalid because they grow too fast. Here we find a beautiful connection to Sturm-Liouville theory: asymptotically they coincide with the inverse of the Sturm-Liouville weight, which we derive in this regime on both manifolds. The fact that the asymptotic solutions for $f(R)$ contain only one parameter, allows us to draw an important conclusion: there are at most a discrete set of fixed point solutions.

Section 6.3.2 presents analogous findings for the eigenoperators. The valid solutions are those that grow asymptotically as a power of R, the invalid solutions grow asymptotically like the inverse of the Sturm-Liouville weight. Validity is decided by requiring their RG evolution to be multiplicative in the large R limit. Left only with the power-law solutions, the equations are overconstrained leading to quantised values, λ_n, for their scaling dimensions. It is now immediate to see (in section 6.3.3) that the valid eigenoperators coincide with those that are square-integrable under the Sturm-Liouville weight, justifying the use of Sturm-Liouville analysis.

Thus we have its standard result, stated in section 6.4, which in this context is that the scaling dimensions are real, that there are only a finite number of (marginally) relevant eigenoperators (such that $\lambda_n \leq 4$) and infinitely many irrelevant operators whose scaling dimensions $\lambda_n \to \infty$. By mapping to so-called Liouville normal form, the asymptotic analysis provides us with the large distance behaviour of the corresponding potential. From there by a standard application of WKB analysis, we get the analytical form for the scaling dimension λ_n as a function of n, in the limit $n \to \infty$. This result should be universal. In fact it is independent of all but one of the parameters. We see that the remaining dependence is an artefact of the single-metric approximation.

In section 6.5 we show that the situation changes dramatically if we choose the sign of the cut-off to be negative for the conformal factor sector. The Sturm-Liouville weight now grows asymptotically, fixed points form a continuum, and the eigenoperator spectrum also becomes continuous. We relate this to earlier findings in $f(R)$ approximations with adaptive cut-off and in conformally reduced gravity. We show that we can impose square-integrability under the Sturm-Liouville weight, in which case the valid operators are the ones that decay asymptotically like the inverse Sturm-Liouville weight. We compute their asymptotic scaling dimensions, and we see that these operators are $f(R)$-analogues of the $\delta^{(n)}_k(\varphi)$ eigenoperators pursued in [1, 2, 8, 74, 107, 109] as an alternative quantisation of quantum gravity.

Finally, in section 6.6 we bring these strands together, describe a search for numerical solutions, compare to $f(R)$ approximations with adaptive cut-off, and draw our conclusions.

6.1 Review of asymptotic safety

We briefly outlined the core ideas of Asymptotic Safety (AS) in section 2.5 however we do so in greater detail here, focusing almost entirely on the application of those ideas to QG [34]. Reminding ourselves of the theory space which is constructed by all of the couplings permitted by the symmetries and fields of a theory, with the irrelevant couplings defining the critical surface and the finite number of relevant couplings emanating from the fixed point which follow along the renormalized trajectory. To repeat ourselves for the sake of clarity we are working with actions expressed as

$$S_\Lambda(\phi, g_i) = \sum_i g_i(\Lambda)\mathcal{O}_i(\phi) \tag{6.1}$$

where Λ is our usual cut-off, g_i the couplings and $\mathcal{O}_i$ the operators. We take $\Lambda \to \infty$ in the usual continuum limit and in doing so if we have not prepared the theory properly we would anticipate that some of the couplings $g_i = \Lambda^{d_i}\tilde{g}_i$ (where d_i is the mass dimension of g_i as we have seen previously) would diverge. This is particularly troublesome if the dimensionless coupling $\tilde{g}_i$ diverges as this signals a breakdown of the theory, for example in QED and ϕ^4 theory where this happens for a finite Λ. In the ideal scenario we will find a point where our β-functions all become zero and the couplings no longer run, this is what we find at the GFP where we are left with the free theory. Another possibility is that there also exists a fixed point in the theory space which is not a trivial empty theory but instead one that has dynamics such that the β-functions become zero [155]. As a result these couplings would not diverge and you would have a UV complete theory, this is the primary idea that underpins AS.

There are some further aspects that are worth mentioning, first of all this non-trivial fixed point would lie within the critical surface of the GFP. Ideally this surface would not be infinite dimensional as we would lose predictivity in a slightly different, albeit equally frustrating, way. The fewer dimensions this surface has the better as these would be the undetermined parameters of the theory, found only via experiment.

We now turn our attention to applying this AS idea to gravity, where the necessity of finding UV complete theories is more paramount. When one wants to consider high energy calculations in gravity[36] there are two aspects to consider. The first of these is that the irrelevant coupling $\tilde{G} = GE^2$ where E is the energy scaled considered which will diverge as we take the continuum limit and take derivatives of the energy scale. Secondly is this problem as discussed in section 2.2 where we must introduce counter terms order by order in perturbation theory which must be fixed by experiment, reducing the predictivity of our theory.

A non-trivial fixed point would resolve this first problem and evidence has been found to suggest that this is the case, which we review now. If one evaluates the

[36]There have been many treatments of GR as an effective QFT [156] at low energy where quantum corrections to the Newtonian potential [157].

β-function of G at one loop using perturbation theory one finds

$$\Lambda \frac{d}{d\Lambda} M_{\text{pl}}^2 = c\Lambda^2 \tag{6.2}$$

where M_{pl}^2 is the Planck mass and c is an arbitrary constant. This constant c will be non-universal however crucially the sign of it will be dependent on the number of space-time dimensions [158], with it being positive in 4 dimensions. The β-function for G will then schematically be

$$\Lambda \frac{dG}{d\Lambda} = -16\pi c G^2 \Lambda^2 \tag{6.3}$$

and for $\tilde{G}$

$$\Lambda \frac{d\tilde{G}}{d\Lambda} = 2\tilde{G} - 16\pi c \tilde{G}^2. \tag{6.4}$$

The latter has an IR attractive fixed point at $\tilde{G} = 0$ and importantly a UV attractive non-trivial fixed point at $\tilde{G}_* = \frac{1}{8\pi c}$ provided c is positive which can be calculated [29] and was found to be the case to one loop in the low energy effective theory and is not dependent on a choice in cut-off in any meaningful way. One could ask the question if this treatment is valid beyond the low energy regime and is independent of the choice in regularisation and it was found that this was the case [159–164].

A popular method in AS is to find the β-function for truncations of the action [165–167]

$$\Gamma_\Lambda = \sum_{i=0}^{n} g_i \int d^4x \sqrt{g} R^i \tag{6.5}$$

where it was found that for increasing n [155, 168], the fixed point continues to exist and appears to be stable and the critical surface seems to have a finite number of dimensions which abates any fears of the infinite dimensional, non-predictive case. Before proceeding we note that many studies of AS have used the $f(R)$ approximation above, the results of these are too numerous to list and so we point the reader towards a selection [114–124, 154, 169–171] as well as seminal works and reviews for this field [113, 115, 116, 167, 172].

There is a plethora of other aspects of AS to consider, for example the coupling of gravity to matter and how this impacts the β-function of the standard model couplings [35, 173–176] and the application to cosmology and space-time [177–179]. To review these subjects further here would cloud the purpose of this thesis. With this basic structure outlined we can move forward and discuss in more detail the $f(R)$ equations we will be studying.

6.2 The f(R) equations

As explained in ref. [154], a crucial choice is to take the cut-off profile to be $f(R)$ independent. Although it does not allow the simplifications gained by combining the optimised cut-off function [180] with adaptive cut-off profiles (and thus used almost exclusively in all other studies), it has two advantages. Firstly, the flow equation is then second order in R-derivatives, rather than third order, which is crucial for the proofs. Secondly, also crucial for the proofs (and one would assume also allowing more accurate modelling of the physics - see the further discussion in the conclusions, section 6.6) it ensures that the resulting ODEs for the fixed point solution and eigenoperators are free of fixed singularities. We use the same flow equation formulated in ref. [154], and the same notation except that here we will work exclusively with quantities already scaled by the appropriate power of k to make them dimensionless, thus avoiding the need to signify them with tildes.

The flow equation takes the form of a non-linear partial differential equation for $f_k(R)$ [154]:

$$\partial_t f_k(R) + 2E(R) = \frac{1}{V}\left(\mathcal{T}_2 + \mathcal{T}_0^{\bar{h}} + \mathcal{T}_1^{Jac} + \mathcal{T}_0^{Jac}\right), \tag{6.6}$$

where $E(R)$ happens to be the equation of motion that would be deduced from the action:

$$E = 2f_k(R) - Rf_k'(R)\,. \tag{6.7}$$

Here, V is the volume of space-time (scaled by k^4). The space-time traces are given by:

$$\mathcal{T}_2 = \mathrm{Tr}\left[\frac{\frac{d}{dt}\mathcal{R}_k^T(\Delta_2+\alpha_2 R)}{-f_k'(R)\Delta_2 - E(R)/2 + 2\mathcal{R}_k^T(\Delta_2+\alpha_2 R)}\right], \tag{6.8}$$

$$\mathcal{T}_0^{\bar{h}} = \mathrm{Tr}\left[\frac{8\frac{d}{dt}\mathcal{R}_k^{\bar{h}}(\Delta_0+\alpha_0 R)}{9f_k''(R)\Delta_0^2 + 3f_k'(R)\Delta_0 + E(R) + 16\mathcal{R}_k^{\bar{h}}(\Delta_0+\alpha_0 R)}\right], \tag{6.9}$$

$$\mathcal{T}_1^{Jac} = -\frac{1}{2}\mathrm{Tr}\left[\frac{\frac{d}{dt}\mathcal{R}_k^V(\Delta_1+\alpha_1 R)}{\Delta_1 + \mathcal{R}_k^V(\Delta_1+\alpha_1 R)}\right] \tag{6.10}$$

$$\mathcal{T}_0^{Jac} = \frac{1}{2}\mathrm{Tr}\left[\frac{\frac{d}{dt}\mathcal{R}_k^{S_1}(\Delta_0+\alpha_0 R)}{\Delta_0 + R/3 + \mathcal{R}_k^{S_1}(\Delta_0+\alpha_0 R)}\right] - \mathrm{Tr}\left[\frac{2\frac{d}{dt}\mathcal{R}_k^{S_2}(\Delta_0+\alpha_0 R)}{(3\Delta_0+R)\Delta_0 + 4\mathcal{R}_k^{S_2}(\Delta_0+\alpha_0 R)}\right]. \tag{6.11}$$

As explained in secs. 6.2.1 – 6.2.3, they can be written as sums or integrals over the eigenvalues of the Laplacian operators. The latter are modified to combinations appearing naturally in the space-time traces on a four-sphere [116]:

$$\Delta_s = -\nabla^2 - \beta_s^S R\,, \qquad \text{where} \qquad \beta_0^S = \tfrac{1}{3}\,,\quad \beta_1^S = \tfrac{1}{4}\,,\quad \beta_2^S = -\tfrac{1}{6}\,. \tag{6.12}$$

where a term proportional to R has been added, for scalar, vector, and tensor modes respectively. The cut-off function $r(z)$ must be non-negative monotonic decreasing, and vanishing in the limit $z \to +\infty$. For simplicity the same function is chosen for all field components so that, when scaled by the appropriate power of k, the cut-off profile takes the form

$$\mathcal{R}_k^{\phi} = c_{\phi}\, r(\Delta_s + \alpha_s R)\,, \tag{6.13}$$

where c_ϕ is a free parameter. Note that an additional correction is incorporated, this time with coefficient α_s. These α_s are chosen to ensure that all modes are integrated out as $k \to 0$, *i.e.* such that all modes have positive $\Delta_s + \alpha_s R$. Their value must be determined from knowledge of the spectrum on the appropriate background manifold(s), so we return to this issue later. When written in terms of dimensionless quantities, as is done here, the total differential of the cut-off with respect to t, takes the form

$$\frac{d}{dt}\mathcal{R}_k^{\phi}(z) = c_\phi m_\phi\, r(z) - 2c_\phi\, z r'(z)\,, \tag{6.14}$$

where m_ϕ is the mass-dimension of $\mathcal{R}_k^{\phi}$ (the same dimension as the Hessian it is regularising).

In these equations, ϕ labels the field component. These are metric fluctuation modes, namely the transverse traceless mode ($\phi = T$) and the gauge-invariant trace mode a.k.a. the conformal factor field [9] ($\phi = \bar{h}$), and transverse vector and scalar modes from Jacobians of the field decomposition ($\phi = V, S_1, S_2$). The ghost and longitudinal modes do not appear since they cancel each other in Benedetti's scheme [116].

Actually, choosing $r(z)$ to be the same for all these modes is more than just a question of simplicity. The modes are all either part of the metric itself or directly related to it via the change of variables or via BRST transformations. Although BRST invariance of the quantum field is badly broken in the single metric approximation, it is reasonable to assume that the approximation would be poorer if we chose to regulate the parts in substantially different ways.

The c_ϕ determine the sign of the cut-off terms in the functional integral. If we require convergence of the integral we need $c_\phi > 0$. We insist on this for $\phi = T, V, S_1, S_2$. The situation is less clear however for the conformal factor. At the classical level $f(R) \sim -R$ is just the Einstein-Hilbert action, and in this case the conformal factor has a wrong-sign kinetic term (Hessian). One can see this from the denominator of the $\mathcal{T}_0^{\bar{h}}$ trace, (6.9), where the Hessian would reduce to $\sim -\Delta_0$ in this case. Therefore the trace is non-singular and the Functional RG is well-defined, only for $c_{\bar{h}} < 0$ [8, 33, 77, 125]. At the quantum level and depending on the value of R, the Hessian can be of either sign [181]. Classically the Hessian can also be of either sign if for example one includes a positive R^2 term. (This is the so-called Starobinsky term, a physically acceptable modification of Einstein's gravity. It corresponds to incorporating a "scalaron" [182] at the classical level.)

In the adaptive cut-off scheme the sign adapts so as to always be consistent with the Hessian. In the non-adaptive scheme that we need to use here, we have to make a choice, which will mean that the Functional RG is only applicable in the regime where this choice is consistent. As we will see this choice profoundly influences RG properties. Where we need to decide we will choose $c_{\bar{h}} > 0$, as in ref. [154, 181], which means however that this version of the flow equation does not describe the regime corresponding to perturbative quantisation of the Einstein-Hilbert action. Then at the end of this chapter, in section 6.5, we show what happens if we take $c_{\bar{h}} < 0$ instead.

One small advantage of using a non-adaptive cut-off profile is that, since it does not itself depend on $f_k(R)$, the only occurrence of the RG time derivative acting on $f(R)$ is the one on the LHS of the flow equation (6.6). The fixed point equation for $f_k(R) = f(R)$ is then just given by dropping this term from the LHS, yielding a non-linear second order ordinary differential equation for $f(R)$:

$$2E(R) = \frac{1}{V}\left(\mathcal{T}_2 + \mathcal{T}_0^{\bar{h}} + \mathcal{T}_1^{Jac} + \mathcal{T}_0^{Jac}\right) . \tag{6.15}$$

Linearising around such a fixed point solution, and separating variables,

$$f_k(R) = f(R) + \epsilon\, v(R)\, \mathrm{e}^{-\theta t} , \tag{6.16}$$

(where ϵ is a small parameter) gives a linear second order ordinary differential eigenvalue equation:

$$- a_2(R)\, v''(R) + a_1(R)\, v'(R) + a_0(R)\, v(R) = \lambda\, v(R) , \tag{6.17}$$

where the eigenvalue $\lambda = 4 - \theta$ is the scaling dimension of the eigenoperator $v(R)$ and

$$a_2(R) = \frac{144c_{\bar{h}}}{V}\mathrm{Tr}\left[\frac{\Delta_0^2(2r(\Delta_0+\alpha_0 R) - (\Delta_0+\alpha_0 R)r'(\Delta_0+\alpha_0 R))}{\left\{9f''(R)\Delta_0^2 + 3f'(R)\Delta_0 + E(R) + 16c_{\bar{h}}r(\Delta_0+\alpha_0 R)\right\}^2}\right] \tag{6.18}$$

$$\begin{aligned} a_1(R) = 2R &- \frac{16c_{\bar{h}}}{V}\mathrm{Tr}\left[\frac{(3\Delta_0 - R)(2r(\Delta_0+\alpha_0 R) - (\Delta_0+\alpha_0 R)r'(\Delta_0+\alpha_0 R))}{\left\{9f''(R)\Delta_0^2 + 3f'(R)\Delta_0 + E(R) + 16c_{\bar{h}}r(\Delta_0+\alpha_0 R)\right\}^2}\right] \\ &+ \frac{2c_T}{V}\mathrm{Tr}\left[\frac{(R/2 - \Delta_2)(2r(\Delta_2+\alpha_2 R) - (\Delta_2+\alpha_2 R)r'(\Delta_2+\alpha_2 R))}{\left\{-f'(R)\Delta_2 - E(R)/2 + 2c_T r(\Delta_2+\alpha_2 R)\right\}^2}\right] \end{aligned} \tag{6.19}$$

$$\begin{aligned} a_0(R) = &\frac{32c_{\bar{h}}}{V}\mathrm{Tr}\left[\frac{(2r(\Delta_0+\alpha_0 R) - (\Delta_0+\alpha_0 R)r'(\Delta_0+\alpha_0 R))}{\left\{9f''(R)\Delta_0^2 + 3f'(R)\Delta_0 + E(R) + 16c_{\bar{h}}r(\Delta_0+\alpha_0 R)\right\}^2}\right] \\ &+ \frac{2c_T}{V}\mathrm{Tr}\left[\frac{(2r(\Delta_2+\alpha_2 R) - (\Delta_2+\alpha_2 R)r'(\Delta_2+\alpha_2 R))}{\left\{-f'(R)\Delta_2 - E(R)/2 + 2c_T r(\Delta_2+\alpha_2 R)\right\}^2}\right] . \end{aligned} \tag{6.20}$$

Notice that the trace in $a_2(R)$ is positive thanks to the properties of $r(z)$. This is the reason for the sign in (6.17), since a_2 then has the same sign as $c_{\bar{h}}$ and in particular is positive for our choice $c_{\bar{h}} > 0$. The RG eigenvalue θ is the scaling dimension of the corresponding coupling. It has positive/zero/negative real part if the eigenoperator

$v(R)$ is relevant/marginal/irrelevant.

One of our main goals is to explore the applicability of Sturm-Liouville theory to the eigenoperator equation (6.17) and when applicable, use it to prove properties of the eigenoperator spectrum [154, 183]. The derivation of the traces assumes that the background metric corresponds to a Euclidean-signature space of maximal symmetry. Globally, discrete choices are still possible, for example the real projective space RP^4 ($R > 0$), torii ($R = 0$), and analogous manifolds when $R < 0$. However, we will see that if Sturm-Liouville theory is to be applicable, then the only sensible choice is to incorporate a kind of smooth topology change between $R > 0$, $R = 0$ and $R < 0$ spaces. This is not possible unless we take maximal symmetry to apply also globally, as is standard practice in asymptotic safety approximations. Then $R > 0$ corresponds to the four-sphere, $R = 0$ to $\mathbb{R}^4$, and $R < 0$ to the four-dimensional hyperboloid.

Spin s	Eigenvalue $\lambda_{n,s}$	Multiplicity $D_{n,s}$
0	$\frac{n(n+3)-4}{12}R$	$\frac{(n+2)(n+1)(2n+3)}{6}$
1	$\frac{n(n+3)-4}{12}R$	$\frac{n(n+3)(2n+3)}{2}$
2	$\frac{n(n+3)}{12}R$	$\frac{5(n+4)(n-1)(2n+3)}{6}$

Table 6.1: Multiplicities and eigenvalues for the four-sphere space-time traces for the shifted Laplacians in (6.12). They follow from those for the unshifted Laplacians [184–186] .The sums for $\mathcal{T}_2$ and $\mathcal{T}_1^{Jac}$ begin at $n_\phi = 2$, for $\mathcal{T}_0^{Jac}$ at $n_\phi = 1$, and for $\mathcal{T}_0^{\bar{h}}$ at $n_\phi = 0$ [116].

6.2.1 Sphere

We start on the four-sphere, which was the space-time explicitly treated in [154]. It has space-time volume $V = 384\pi^2/R^2$, and there the space-time traces are sums over the discrete set of eigenvalues of the corresponding Laplacian:

$$\mathrm{Tr}\, W(\Delta_s) = \sum_{n=n_\phi}^{\infty} D_{n,s}\, W(\lambda_{n,s})\,. \tag{6.21}$$

The multiplicities $D_{n,s}$, eigenvalues $\lambda_{n,s}$, and lowest index n_ϕ, are given in table 6.1 (and its caption).

Following [116, 187], we use the transverse-traceless decomposition of the metric fluctuations, given by

$$h_{\mu\nu} = h^T_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma + \frac{1}{d} g_{\mu\nu} \bar{h} \tag{6.22}$$

with the component fields satisfying

$$g^{\mu\nu} h^T_{\mu\nu} = 0, \quad \nabla^\mu h^T_{\mu\nu} = 0, \quad \nabla^\mu \xi_\mu = 0, \quad \bar{h} = h + \Delta\sigma, \quad h = g_{\mu\nu} h^{\mu\nu} \tag{6.23}$$

and $\Delta = -\nabla^2$. And then we have to be careful not to include the fictitious modes in the sum. From our decomposition for the metric fluctuations [120], we see that we should exclude two sets of modes that give no contribution to $h_{\mu\nu}$. First, we should exclude the Killing vectors, satisfying $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. Second, we shoud leave out also the constant scalar modes $\sigma =$ constant. A similar set of modes should be excluded also from the ghosts and auxiliary fields, as these are all fields introduced hand-in-hand with ξ and σ. The only fields for which we retain all the modes are $h^T_{\mu\nu}$ and $\bar{h}$. Note that, differently from [115, 167], we do not exclude the scalar modes corresponding to the conformal Killing vectors $\mathcal{C}_\mu = \nabla_\mu \sigma$, i.e. those scalar modes satisfying $\nabla_\mu \nabla_\nu \sigma = \frac{1}{d} g_{\mu\nu} \sigma$. It is indeed clear that in our decomposition [120] such modes do not contribute to $h_{\mu\nu}$. This can be seen also from the point of view of the ghosts: the ghost modes should be in one-to-one correspondence with the modes of the gauge parameter $(\epsilon^T_\mu, \epsilon)$, and from $\mathcal{L}_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon^T_\nu + \nabla_\nu \epsilon^T_\mu + 2\nabla_\mu \nabla_\nu \epsilon$ it is obvious that there is no reason to exclude the scalar mode ϵ corresponding to the conformal Killing vectors. As a consequence the tensor and vector sums will start at $n = 2$, while all the scalar sums will begin at $n = 1$, except for the $\bar{h}$ mode starting at $n = 0$, as in table 6.1 above.

As is clear from the cut-off profile formula, (6.13), the α_s parameters allow us to shift the action of the cut-off up or down relative to the tower of eigenvalues, so as to ensure all the modes are passed as k is lowered to $k \to 0^+$. Clearly this requires that the lowest mode $\lambda_{n_\phi,s} + \alpha_s R$ is positive. As noted in ref. [154], it is safe to choose $\alpha_2 = 0$ and $\alpha_1 = 0$, but to implement this condition in the physical scalar (a.k.a. conformal factor) sector we need to choose[37] $\alpha_0 > 1/3$.

At this point we recognise the need to specialise to *smooth* (infinitely differentiable) cut-off functions $r(z)$. Given that the eigenvalues are discrete set, proportional to R, cut-off functions that are not smooth, for example the optimised one $r(z) = (1-z)\,\theta(1-z)$ [180], will lead to points of limited differentiability which moreover accumulate as $R \to 0$. It may still be possible to find a suitable *weak* solutions to the fixed point and eigenoperator equations in this circumstance but, given that with a non-adaptive cut-off profile there is no advantage to using the optimised cut-off, there is no point in pursuing this possibility further. In fact one should bear in mind that cut-offs involving the Heaviside θ function have a number of related unpleasant effects[38] that strictly speaking should rule them out as sensible choices, even if these problems are not obvious at current levels of approximation. On the other hand, any smooth profile $r(z)$ will do if it decays sufficiently fast at large z. In our case we only need to guarantee the convergence of the space-time traces above. Later we will specialise to the popular

[37] In this sense the modes are not treated equally. There appears to be no solution that does treat them 'equally' at this level of detail, given constraints that we will also have to satisfy on the hyperboloid, *cf.* eqn. (6.35).

[38] In real space the Kadanoff blocking functions are not truly quasi-local (they have power-law tails) and IR regulated vertices have no Taylor expansion in momentum (derivative expansion) beyond some low order.

choice [105]

$$r(z) = \frac{z}{\exp(az^b) - 1}\,, \qquad a > 0\,, b \geq 1\,. \tag{6.24}$$

For any non-vanishing $R > 0$ the sums are then rapidly convergent. However even if we restrict ourselves to four-spheres, we still need to understand the limiting case $R \to 0^+$, which takes us to the boundary of this set. There, the sums go over to an integral and the equations go over to ones in flat space.

6.2.2 Flat space

This limit can be achieved by setting $p = n\sqrt{R/12}$, and then taking $R \to 0$ whilst keeping p fixed. From table 6.1, it is clear that all the Laplacians $\Delta_{n,s} \to p^2$, *i.e.* go over to their flat space limit where we recognise that p is the flat space momentum. The multiplicities become $p^3(12/R)^{3/2}$ up to a numerical factor, while the cut-off profiles $\mathcal{R}_k^\phi \to c_\phi\, r(p^2)$. Putting all this together gives for the flow equation (6.6),

$$\begin{aligned}\partial_t f_k(0) + 4f_k(0) = \frac{1}{8\pi^2}\int_0^\infty dp\, p^3 \Bigg\{ & 16c_{\bar{h}}\, \frac{2r(p^2) - p^2 r'(p^2)}{9f_k''(0)\, p^4 + 3f_k'(0)\, p^2 + 2f_k(0) + 16c_{\bar{h}} r(p^2)} \\ & + 10c_T\, \frac{r(p^2) - p^2 r'(p^2)}{-f_k'(0)\, p^2 - f_k(0) + 2c_T r(p^2)} - 3c_V\, \frac{r(p^2) - p^2 r'(p^2)}{p^2 + c_V r(p^2)} \\ & + c_{S_1} \frac{r(p^2) - p^2 r'(p^2)}{p^2 + c_{S_1} r(p^2)} - 4c_{S_2}\, \frac{2r(p^2) - p^2 r'(p^2)}{3p^4 + 4c_{S_2} r(p^2)} \Bigg\}\,. \end{aligned} \tag{6.25}$$

For this to be well defined, $f_k(0)$, $f_k'(0)$, and $f_k''(0)$, need to be such that neither denominator vanishes (at some p^2) in its first two terms. This already provides strong constraints if the solution is to exist for all k, which however are soluble locally. The strongest constraints arise if the cut-off function $r(z)$ diverges as $z \to 0$, for example in the cases $b > 1$ in (6.24). In the tensor mode trace, $c_T\, r(p^2) \to +\infty$ as $p \to 0$ and thus the denominator is positive. On the other hand if $f_k'(0)$ is non-vanishing, as $p \to \infty$ the sign of the denominator is given by $-f_k'(0)$. Thus we see that to avoid a singularity we must always have $f_k'(0) \leq 0$ (strictly less than zero is the physically motivated choice since this corresponds to positive Newton's constant at zero momentum). Similarly we see that $f_k(0)$ is bounded above. From the $\bar{h}$ trace we see that since we chose $c_{\bar{h}} > 0$, we must have $f_k''(0) > 0$, while $f_k'(0)$ must also be bounded below by some negative value. If $r(0)$ is finite, for example the case $b = 1$ in (6.24), then other possibilities arise since $f_k'(0)$ can be positive if $f_k(0) > 2c_T\, r(0)$, while the $\bar{h}$ trace would then only require that $f_k''(0) > 0$. These considerations inform numerical searches, which we describe in section 6.6.

For the fixed point solution $f_k(R) = f(R)$, eqn. (6.25) determines $f''(0)$ given boundary conditions $f(0)$ and $f'(0)$ such that all three lie within the bounds above. It then provides us with a Taylor expansion approximant to a putative fixed point solution:

$$f(R) = f(0) + f'(0)R + \frac{1}{2}f''(0)R^2 + o(R^2)\,. \tag{6.26}$$

(In fact taking the expansion further is not straightforward since it then depends on the error in approximating the sums by integrals but these are not captured correctly by Euler-Maclaurin corrections.) With these choices, the eigenoperator equation coefficients have finite limits:

$$a_2(0) = \frac{18c_{\bar{h}}}{\pi^2}\int_0^\infty dp\, p^7 \frac{2r(p^2) - p^2 r'(p^2)}{\{9f''(0)\,p^4 + 3f'(0)\,p^2 + 2f(0) + 16c_{\bar{h}}r(p^2)\}^2}\,, \tag{6.27}$$

$$a_1(0) = \int_0^\infty dp\, p^5 \left\{ \frac{5c_T}{4\pi^2} \frac{r(p^2) - p^2 r'(p^2)}{\{-f'(0)\,p^2 - f(0) + 2c_T r(p^2)\}^2} - \frac{6c_{\bar{h}}}{\pi^2} \frac{2r(p^2) - p^2 r'(p^2)}{\{9f''(0)\,p^4 + 3f'(0)\,p^2 + 2f(0) + 16c_{\bar{h}}r(p^2)\}^2} \right\}, \tag{6.28}$$

$$a_0(0) = \int_0^\infty dp\, p^3 \left\{ \frac{5c_T}{4\pi^2} \frac{r(p^2) - p^2 r'(p^2)}{\{-f'(0)\,p^2 - f(0) + 2c_T r(p^2)\}^2} - \frac{4c_{\bar{h}}}{\pi^2} \frac{2r(p^2) - p^2 r'(p^2)}{\{9f''(0)\,p^4 + 3f'(0)\,p^2 + 2f(0) + 16c_{\bar{h}}r(p^2)\}^2} \right\}, \tag{6.29}$$

the eigenoperator equation itself then just being given by setting $R = 0$ in (6.17). Furthermore $a_2(0)$ is non-vanishing since the integrand is positive definite.

The above implies that the Sturm-Liouville weight function is finite and non-vanishing at $R = 0$:

$$w(R) = \frac{1}{|a_2(R)|}\exp - \int_0^R dR'\, \frac{a_1(R')}{a_2(R')}\,, \tag{6.30}$$

(setting the lower limit in the integral to zero without loss of generality, and taking the modulus in the prefactor so that ω is positive whatever sign of $c_{\bar{h}}$ we choose.) Multiplying the eigenoperator equation (6.17) by the weight function (a.k.a. the Sturm-Liouville measure) we can cast it in Sturm-Liouville form:

$$-\big(a_2(R)w(R)v'(R)\big)' + w(R)a_0(R)v(R) = \lambda w(R)v(R)\,. \tag{6.31}$$

However Sturm-Liouville properties only follow if the differential operator on the LHS is self-adjoint. Taking $v = v_j(R)$, multiplying by $v_i(R)$, and integrating over R, this means in particular that boundary terms must vanish when integrating by parts. We see that we thus require the eigenfunctions to be square integrable under the weight function, and if we work only with fixed topology (here four-spheres), then for any two eigenfunctions $v_i(R)$ and $v_j(R)$, we get from the $R = 0$ boundary:

$$w(0)\,\big(v_i(0)v_j'(0) - v_j(0)v_i'(0)\big) = 0\,, \tag{6.32}$$

the so-called bilinear concomitant. Therefore we would have to either choose all eigenoperators to satisfy Dirichlet boundary conditions ($v_i(0) = 0\ \forall i$), or all eigenoperators to satisfy Neumann boundary conditions ($v_i'(0) = 0\ \forall i$). These conditions lack any physical or other mathematical motivation, in particular in the full theory they cannot be respected beyond linearised order. Our remaining option is to eliminate the $R = 0$ boundary, requiring the solution to extend to all real values of R.

In fact we will get extra motivation for such choices when we analyse the number of fixed point solutions (or rather the dimension of the space of such solutions) in section 6.3. For this latter reason, the $f(R)$ equations of ref. [116] were extended in ref. [113] to all real R, by analytic continuation. Here we do not have the option of analytic continuation if we insist on using the same cut-off function $r(z)$ for all modes. The reason is that $\Delta_{n,s}$, being proportional to R, would change sign. Apart from the cut-off, this makes the denominator in $\mathcal{T}_1^{Jac}$, and in the S_1 term in $\mathcal{T}_0^{Jac}$, change sign, *cf.* (6.10) and (6.11) respectively. By choosing sufficiently large n in the modes in table 6.1, we see that the denominators will then vanish already at small negative R making these traces ill-defined, unless we take $r(z)$ itself to be odd in z (using *e.g.* (6.24) with $b = 2$). However if we do take $r(z)$ odd, then instead the S_2 term in $\mathcal{T}_0^{Jac}$ will diverge already at small negative R by similar arguments. (The other two traces also have their problems but since they involve $f(R)$, the demonstration is more involved.)

6.2.3 Hyperboloid

This leaves us with the remaining alternative, which is to match into the equations on a manifold with $R < 0$. As explained earlier, we take for this the four-dimensional hyperboloid. Here $-\nabla^2$ is positive definite. The volume V is infinite, but the flow equation (6.6) still makes sense since the space-time traces on the RHS trivially contain the same factor [188]:

$$\frac{1}{V}\operatorname{Tr} W(\Delta_s) = \frac{2s+1}{8\pi^2}\left(-\frac{R}{12}\right)^2 \int_0^\infty d\lambda \left(\lambda^2 + \left[s+\tfrac{1}{2}\right]^2\right) \lambda \tanh(\pi\lambda)\, W(\Delta_{\lambda,s})\,; \quad (6.33)$$

the spectrum is now continuous, indexed by λ:

$$\Delta_{\lambda,s} = \left(\lambda^2 + s + \frac{9}{4}\right)\left(-\frac{R}{12}\right) - \beta_s^S R = -\frac{R}{12}\lambda^2 - \beta_s^H R\,, \quad (6.34)$$

where thus

$$\beta_0^H = \frac{25}{48}\,, \qquad \beta_1^H = \frac{25}{48}\,, \qquad \beta_2^H = \frac{9}{48}\,. \quad (6.35)$$

Recalling the reason for the extra endomorphism in the cut-off profiles (6.13), we see that we can continue to set $\alpha_2 = 0$ and $\alpha_1 = 0$ as we wanted for the four-sphere, but the lower bound $\alpha_0 > 1/3$ is now joined by an upper bound $\alpha_0 < 25/48$ [154] so that all modes $\Delta_{\lambda,0} + \alpha_0 R > 0$.

The equations at the $R \to 0^-$ boundary of this set of hyperboloids, are found by setting $p = \lambda\sqrt{-R/12}$ and holding p fixed, so that once again the Laplacian goes over to its flat space expression $\Delta_{\lambda,s} \to p^2$. It is straightforward to verify that the flow equation (6.6) and eigenoperator equation coefficients (6.18)–(6.20) then go over to the flat space expressions (6.25) and (6.27)–(6.29) respectively. Thus we see that the flow, fixed point, and eigenoperator, equations can be smoothly defined over the combined set of all four-spheres, all four-hyperboloids, and $\mathbb{R}^4$. The $R > 0$ and $R < 0$ parts of the solutions can be made to match as Taylor expansions around $R = 0$ up to the second derivative, but not beyond that. In fact the hyperboloid has a straightforward smooth limit[39] whereas $f'''(0)$ on the sphere side depends also on corrections involved in converting the sums over eigenvalues into integrals. In this way we have incorporated a smooth topology change mechanism through these three spaces.

To apply Sturm-Liouville theory, we are left only to establish acceptable behaviour at large R. In particular, as we have already seen, we need the eigenoperators $v(R)$ to be square integrable under the Sturm-Liouville weight $w(R)$. Whilst this condition is natural for Sturm-Liouville theory, and was assumed in ref. [154] for that reason, the question is whether this makes sense in quantum gravity.

6.3 Asymptotic behaviour of solutions at large R

We therefore turn now to the asymptotic behaviour of solutions at large R. This large field analysis also allows us to characterise a number of aspects of the solution space for both fixed points and their eigenoperator spectrum [62, 77, 111, 113, 189–192] and in particular allows us to answer the question above. We will see from section 6.5 that the answer depends very much on the choice of sign for $c_{\bar{h}}$.

6.3.1 Large R dependence of fixed points and how to count them

We start with the asymptotic behaviour of the fixed point solution $f(R)$. We need to know this in order to establish the large R behaviour of the coefficients $a_i(R)$ in the eigenoperator equation (6.17) which in turn will allow us to analyse the asymptotic behaviour of the eigenoperators. However as we will see, it is important also for determining features of the fixed point solution space.

Beginning with the sphere, and given a rapidly decaying cut-off profile $r(z)$, at first sight one can neglect the traces on the RHS of the fixed point equation (2.22) at large R. One would then conclude that $f(R) = AR^2$ plus rapidly decaying corrections [154], for some undetermined coefficient A, this being the solution of just the LHS, $E(R) = 0$. However this is not correct because terms in the traces whose denominator would vanish without a cut-off, yield a contribution on the RHS proportional to

$$\frac{1}{\mathcal{R}^\phi_k(z)} \frac{d}{dt} \mathcal{R}^\phi_k(z) = m_\phi - 2z \frac{d \ln r(z)}{dz} \,, \tag{6.36}$$

[39] Solutions can be straightforwardly developed to all-orders in the Taylor expansion around $R = 0$, with coefficients given by finite integral expressions over p similar to those in (6.25), (6.27)–(6.29).

where we used (6.14). There are three such terms, the $n = 0$, $n = 1$ components from $\mathcal{T}_0^{\bar{h}}$ c.f. (6.9), and the $n = 1$ S_2 (second) component of $\mathcal{T}_0^{Jac}$ c.f. (6.11).

To see this for the $n = 0$ case, note that from table 6.1, $\Delta_0 = -R/3$. Thus the denominator of this term in the sum (6.9) is given by

$$\begin{aligned} 9f''(R)\Delta_0^2 + 3f'(R)\Delta_0 + E(R) + 16\mathcal{R}_k^{\bar{h}}(\Delta_0 + \alpha_0 R) = \\ R^2 f''(R) - Rf'(R) + E(R) + 16\mathcal{R}_k^{\bar{h}}([\alpha_0 - \frac{1}{3}]R). \end{aligned} \tag{6.37}$$

Now, using the assumed leading asymptotic behaviour $f(R) = AR^2$, we see that the first two terms cancel each other, and likewise $E(R)$ vanishes (as already mentioned), so we are left only with the cut-off term in the denominator. Therefore this term takes the form of (6.36) with z set equal to $z = [\alpha_0 - \frac{1}{3}]R$.

Now, turning to the $n = 1$ components, note that from table 6.1, both Δ_0 and Δ_1 vanish for $n = 1$. In (6.9), apart from the cut-off term the whole denominator therefore vanishes (because $E(R)$ vanishes). In (6.11) it is the second component that has a vanishing denominator apart from the cut-off term. (The S_1 (first) component does not suffer from the same problem because there is also the $+R/3$ part in the denominator). Now notice from (6.13), that the cut-off dependence is the same for these $n = 1$ contributions namely $r(\alpha_0 R)$. Furthermore the c_ϕ in (6.12) cancels between numerator and denominator and the numerical factors in front of the cut-off in (6.9) and (6.11) are such that these two $n = 1$ contributions then exactly cancel each other.

Altogether then, effectively the only term on the right hand side of (6.6) that does not vanish asymptotically is the (6.36) type term with $z = [\alpha_0 - \frac{1}{3}]R$ coming from the $n = 0$ component of the $\mathcal{T}_0^{\bar{h}}$ trace. This is a problem however, since $m_{\bar{h}} = 4$ and the second part of (6.36) is necessarily positive because $r(z)$ is a monotonically decreasing. Recalling the factor $1/V = R^2/384\pi^2$ on the right hand side of the fixed point equation (6.15), we see that the $n = 0$ component of $\mathcal{T}_0^{\bar{h}}$ contributes a term that grows at least faster than R^2. For example in the best-case scenario the right hand side goes as $\sim R^2$ but that implies $f(R) \sim R^2 lnR$ so that the left hand side is left with an $E(R) \sim R^2$ to balance the contribution from the $n = 0$ contribution of $\mathcal{T}_0^{\bar{h}}$.

Therefore we now assume that $f(R)$ actually grows faster than R^2 at large R. But this means we need to check again which terms in the traces have denominators that vanish without a cut-off. By inspection none of the traces that depend on $f(R)$ can now have this issue. In particular the $n = 1$ component of the $\mathcal{T}_0^{\bar{h}}$ trace no longer has a denominator that could vanish, because $E(R)$ no longer vanishes at large R, while for the $n = 0$ component the $f''(R)$ part in the denominator now dominates at large R. So the only contribution that survives on the right hand side at large R, is now the $n = 1 S_2$ component of $\mathcal{T}_0^{Jac}$.

Keeping just this term it turns out one can solve the fixed point equation in closed form, thus obtaining the correct asymptotic behaviour for general cut-off function $r(z)$.

To see this we just use (6.36), read off from table 6.1 that the multiplicity of the $n = 1$ S_2 component is $D_{1,0} = 5$, note that $C_{S_2} = 4$ and that $1/V = R^2/384\pi^2$ and thus, keeping only this leading term on the right hand side of the fixed point equation (6.15), we have

$$2f(R) - Rf'(R) = \frac{R^2}{768\pi^2}\left[-10 + 5\alpha_0 R\frac{r'(\alpha_0 R)}{r(\alpha_0 R)}\right]. \tag{6.38}$$

This is exactly soluble. Indeed dividing through by R^3 it can be rewritten as

$$-\frac{d}{dR}\left(\frac{f(R)}{R^2}\right) = \frac{1}{768\pi^2}\left[-\frac{10}{R} + 5\frac{d}{dR} ln r(\alpha_0 R)\right], \tag{6.39}$$

which can be immediately integrated to give

$$f(R) = \frac{5R^2}{768\pi^2} ln \frac{R^2}{r(\alpha_0 R)} + AR^2 + o(R^2) \quad \text{as} \quad R \to +\infty, \tag{6.40}$$

where we included the integration constant A and finally we noted that terms that grow slower than R^2 will be generated by iterating this asymptotic solution to higher orders, hence the $o(R^2)$ part.

As this point we have succeeded in finding consistent asymptotics. $f(R)$ does grow faster than R^2 as per our assumption, and using such a form in the right hand side of the fixed point equation we find the $n = 1 S_2$ component of $\mathcal{T}_0^{Jac}$ dominates at large R, which leads us back to the above equation.

Now notice that in (6.40) the $\ln r$ term actually dominates, *i.e.* the large R behaviour is dominated by cut-off-dependent effects. For example using the cut-off (6.24), *viz.* $r(z) = z/(\exp(az^b) - 1)$ such that $a > 0$, $b \geq 1$, we find:

$$f(R) = \frac{5a\alpha_0^b}{768\pi^2}R^{2+b} + \frac{5}{768\pi^2}R^2 \ln R + AR^2 + \frac{16c_{\bar{h}}}{5ab(1+b)\alpha_0^b}\left(\alpha_0 - \frac{1}{3}\right) \mathrm{e}^{-a\left(\alpha_0 - \frac{1}{3}\right)^b R^b} + \cdots, \tag{6.41}$$

where the ellipses stand for faster decaying terms. Here we adjusted A to absorb a contribution to R^2, and then substituted the solution back into the fixed point equation to isolate the next leading correction. (This exponentially decaying correction comes from the $n = 0$ term in the $\mathcal{T}_0^{\bar{h}}$ trace. All other corrections decay faster provided that $\alpha_0 < \frac{5}{6} + \alpha_1$. This is satisfied thanks to the restrictions imposed below (6.35).)

Recalling that R is the dimensionless version, *i.e.* the physical curvature divided by k^2, we see that the large R limit may be viewed as holding the physical curvature fixed and integrating out all modes by sending $k \to 0$. Therefore it ought to provide us with (an approximation to) the physical Legendre effective action, *i.e.* the universal physical equation of state as a function of R [111]. The cut-off dependence however obstructs any attempt to extract physics from this limit. This problem is not seen in the Local Potential Approximation in scalar field theory, where an approximation to the equation of state can be successfully computed in this way [183], and as we will see it is not a problem on the hyperboloid. Since the issue arises from the fact that the $n = 1$ S_2 modes in the scalar Jacobian have vanishing eigenvalue, it suggests that further research should be done to understand if/how these modes can be better treated on a sphere.

Although the fixed point equation (2.22) is a second order ODE free of fixed singularities, the asymptotic solution we have found contains only the one free parameter: A. It is important to ask where the other parameter has gone. To find out, we linearise the fixed point equation around the asymptotic solution. This just gives the eigenoperator equation (6.17) for a marginal deformation, $\delta f(R) = \epsilon\, v(R)$ *i.e.* such that $\theta = 0$ or equivalently $\lambda = 4$. As a linear second order ODE, it must have two linearly independent solutions. These can be found in the large R limit. Inspecting (6.18) – (6.20), we note that $a_1(R) = 2R$ to leading order, while both $a_2(R)$ and $a_0(R)$ vanish asymptotically. We can therefore neglect a_0 and write

$$4\,\delta f(R) - a_1(R)\,\delta f'(R) = -a_2(R)\,\delta f''(R)\,. \tag{6.42}$$

We know one solution to this already: $\delta f(R) = \delta A R^2 + \cdots$, where the RHS is only involved in supplying one of the subleading corrections. The other solution must thus be such that at leading order, $\delta f''(R)$ cannot be neglected. This tells us higher derivatives dominate over lower derivatives so we know that for the other solution $\delta f(R)$ can instead be neglected (to leading order). The equation is then exactly soluble since it can be rewritten as

$$\frac{d}{dR}\ln\delta f'(R) = \frac{a_1(R)}{a_2(R)} \qquad \Longrightarrow \qquad \delta f(R) = B\int^{R}\! dR'\,\exp\!\int^{R'}\! dR''\,\frac{a_1(R'')}{a_2(R'')}\,, \tag{6.43}$$

where B is the putative missing parameter. For the explicit form we need a_2. It gets its leading contribution from the same source as the leading correction (6.41) to the terms displayed in (6.40). For the same cut-off choice (6.41), we find asymptotically

$$a_2(R) = \frac{24576\pi^2 c_{\bar{h}}}{25ab(1+b)^2\alpha_0^{2b}}\left(\alpha_0 - \frac{1}{3}\right)^{1+b} R^{1-b}\,\mathrm{e}^{-a\left(\alpha_0-\frac{1}{3}\right)^b R^b} + \cdots\,. \tag{6.44}$$

Recalling that $a_1 = 2R$ to leading order, we can evaluate the integrals by successive integration by parts, as an asymptotic series and where each term is given in closed form. Since we will use this strategy many times let us sketch it on the indefinite integral:

$$\int\! dR\, G(R)\,\mathrm{e}^{F(R)} = \frac{G(R)}{F'(R)}\,\mathrm{e}^{F(R)} - \int\! dR\left(\frac{G(R)}{F'(R)}\right)'\mathrm{e}^{F(R)}\,. \tag{6.45}$$

If $F(R)$ grows at least as fast as R for large R, where F is either sign, and $G(R)$ grows or decays slower than an exponential of R, then the integral on the right is subleading compared to the integral on the left. Iterating this identity then evaluates the integral in the large R limit as $\mathrm{e}^{F(R)}$ times an asymptotic series, the first term on the RHS being the leading term.

Using (6.44) this allows us to evaluate the inner integral in (6.43). Its exponential is then the integrand for the outer integral, such that the asymptotic series now provides subleading multiplicative corrections. Up to such corrections, the integrand is actually $1/\omega(R')$, as can be seen from eqn. (6.30). Applying the same integration by parts strategy to the outer integral does not change the leading exponential behaviour, and thus we see that up to subleading multiplicative corrections $\delta f(R) \sim B/\omega(R)$ where we find the Sturm-Liouville weight in the same approximation to be:

$$\omega(R) \sim \exp\left\{-\frac{25(1+b)^2\alpha_0^{2b}}{12288\pi^2 c_{\bar{h}}}\left(\alpha_0-\frac{1}{3}\right)^{-1-2b} R\,\mathrm{e}^{a\left(\alpha_0-\frac{1}{3}\right)^b R^b}\right\}. \tag{6.46}$$

Notice that the sign of $c_{\bar{h}}$ is crucial. Assuming $c_{\bar{h}} > 0$, the linearised perturbation $\delta f(R) \sim B/\omega(R)$ is a rapidly growing exponential of an exponential. Taking the $R \to +\infty$ limit, it is not a small perturbation to our previous result (6.41), no matter how small we choose B, thus invalidating the procedure used to derive it.[40] Evidently it cannot itself satisfy the fixed point equation asymptotically (it would have to solve just the LHS to do that). Therefore there is asymptotically only a one-parameter set of solutions namely (6.41).

The dimension of the fixed point solution space is determined by the asymptotic behaviour [111], thus unless further conditions are imposed we have (some discrete number of) lines of fixed points. Since it is not sustainable to try and impose a condition at $R = 0$, *cf.* the discussion on eigenoperators below (6.32), we need to continue through smooth topology change (as defined at the end of section 6.2) into the hyperboloid side, if we are to reduce the dimension of the fixed point solution space from the current phenomenologically disappointing answer.

Turning to the hyperboloid, the situation is much more straightforward. The assumption that for rapidly decaying cut-off profile $r(z)$ one can neglect the traces (6.8) – (6.11) at large (negative) R, is now correct for the ansatz $f(R) = AR^2$, thanks to the Laplacian eigenvalues (6.34) being bounded below sufficiently by the positive endomorphisms to avoid vanishing denominators, *cf.* (6.35). Since the ansatz solves the LHS of the fixed point equation, it forms the start of the large R asymptotic series solution. The traces provide corrections that decay thanks to the cut-off profiles' dependence on $\Delta_{\lambda,s} + \alpha_s R > (\beta_s^H - \alpha_s)|R|$. From the α and β parameter values, *cf.* (6.35) and below it, we see that

$$0 < \beta_0^H - \alpha_0 < 9/48 = \beta_2^H - \alpha_2 < \beta_1^H - \alpha_1 \tag{6.47}$$

and thus the leading corrections come from the scalar traces $\mathcal{T}_0^{\bar{h}}$ and $\mathcal{T}_0^{Jac}$. From the power of Δ_0 in (6.11) it is the S_1 part that is leading. After some tedious manipulation

[40] It can be understood as the linearised precursor to the solution ending in a (movable) singularity [62, 77, 111, 113, 189–192].

we find:[41]

$$f(R) = AR^2 + \frac{c_{S1}}{96\sqrt{3\pi a^3 b^3}} \left(\frac{25}{48} - \alpha_0\right)^{\frac{5-3b}{2}} (-R)^{2-\frac{3b}{2}} \left\{1 + O\left(|R|^{-\frac{1}{2}}\right)\right\} \mathrm{e}^{-a\left[\left(\alpha_0 - \frac{25}{48}\right)R\right]^b} + \cdots , \tag{6.48}$$

as $R \to -\infty$, all scalar traces (thus also A) contributing to the $O\left(|R|^{-\frac{1}{2}}\right)$ term, and the ellipses standing for terms with faster decaying exponentials. Again we ask where the other parameter has gone. The analysis proceeds in a similar fashion to that on the sphere. We have again the asymptotic perturbed fixed point equation (6.42) except now:

$$a_2(R) = \frac{4c_{\bar{h}}}{81A^2\sqrt{3\pi ab}} \left(\frac{25}{48} - \alpha_0\right)^{\frac{5-b}{2}} (-R)^{1-\frac{b}{2}} \mathrm{e}^{-a\left[\left(\alpha_0 - \frac{25}{48}\right)R\right]^b} + \cdots \tag{6.49}$$

(the ellipses being faster decaying terms). A small perturbation to (6.48) gives (6.43), and thus we have $\delta f(R) \sim B/\omega(R)$ again, except now the Sturm-Liouville weight is

$$\omega(R) \sim \exp\left\{-\frac{81A^2}{2c_{\bar{h}}}\sqrt{\frac{3\pi}{ab}} \left(\frac{25}{48} - \alpha_0\right)^{-\frac{b+5}{2}} (-R)^{1-\frac{b}{2}} \mathrm{e}^{a\left[\left(\alpha_0 - \frac{25}{48}\right)R\right]^b}\right\} \tag{6.50}$$

(where again we neglect also subleading multiplicative terms). As $R \to -\infty$, such a $\delta f(R)$ is a rapidly growing exponential of an exponential, and thus asymptotically we have only the one-parameter set of solutions (6.48).

These results allow us to draw an important conclusion. Each of the hyperboloid and sphere asymptotic solutions impose one constraint.[42] Since we thus have two boundary conditions imposed on a second order ordinary differential equation we have at most a discrete set of solutions. *A priori* this could be no fixed point, or a unique fixed point (the phenomenologically preferred answer), a larger number of fixed points, or a countable infinity of fixed points. As we will see in section 6.5, the conclusion is very different if we choose the conformal factor cut-off to be negative, *i.e.* $c_{\bar{h}} < 0$.

6.3.2 Large R dependence of eigenoperators

Since the eigenoperator equation (6.17) is linear and second order, there are guaranteed to be two independent solutions for any RG eigenvalue λ. Whether they are acceptable or not, crucially depends on their large field behaviour [20, 66, 77, 183, 192]: in particular whether for small but fixed ϵ the exponential dependence in RG time in (6.16) remains valid at large R. In scalar field theories this criterion explains why the correct eigenoperator solutions are the ones with power-law large field behaviour and thus why the RG eigenvalues are quantised [20, 62, 66, 183, 189–192]. It was also applied in

[41] Recall that $\alpha_0 < 25/48$.

[42] *E.g.* $Rf'(R) - 2f(R) = Rf'_{asy}(R) - 2f_{asy}(R)$, for some suitably large R, where f_{asy} is (6.41) for $R > 0$, or (6.48) for $R < 0$, and the RHS has no free parameters since the AR^2 term is cancelled out in this linear combination.

ref. [77] to determine the eigenoperator spectrum around non-trivial fixed points in a conformal truncation to quantum gravity, and in ref. [113] to an $f(R)$ approximation with adaptive cut-off [116].

We have just derived the asymptotic behaviour of $f(R)$ for a fixed point solution to the flow equation (6.6) with non-adaptive cut-off. Substituting this into the corresponding eigenoperator equation (6.17) allows us to determine the large R behaviour of solutions $v(R)$. We will use the above insight to determine which of these solutions are valid.

In fact, since with non-adaptive cut-off, RG time derivatives of $f_k(R)$ appear only the once, as $\partial_t f_k(R)$ on the LHS of the flow equation (6.6), one can immediately read off from the asymptotic form of the perturbed fixed point equation (6.42), the corresponding asymptotic form of the eigenoperator equation (6.17):

$$\lambda\, v(R) - 2R\, v'(R) = -a_2(R)\, v''(R)\,, \tag{6.51}$$

where asymptotically a_2 is given by (6.44) or (6.49) as appropriate. One solution solves just the LHS:

$$v(R) \propto |R|^{\frac{\lambda}{2}} + \cdots\,, \tag{6.52}$$

where the ellipses stand for subleading corrections including those supplied by the RHS, and we note that the solution is determined only up to a constant of proportionality. The other solution must be such that at leading order, $v''(R)$ cannot be neglected. For the same reasons as before, asymptotically the ODE then collapses to (6.43) (with δf replaced by v) and thus these solutions satisfy $v(R) \sim 1/\omega(R)$, with $\omega(R)$ being given by (6.46) and (6.50) on the sphere and hyperboloid respectively.

Now we ask whether these solutions are actually valid. The linearised solution (6.16) is meant to describe the RG flow 'close' to the fixed point. For any fixed ϵ, if $|v(R)/f(R)| \to \infty$ as $R \to \pm\infty$ that is not necessarily true since linearisation is no longer valid. In this case we set

$$f_k(R) = f(R) + \epsilon\, v_k(R)\,, \tag{6.53}$$

and ask for the correct evolution for $v_k(R)$ at large R. We see that for large negative R we can neglect the RHS of the flow equation (6.6). For large positive R we can neglect the RHS of the flow equation except for the $n = 1$ S_2 component of $\mathcal{T}_0^{Jac}$, which however just cancels the contributions from the LHS that grow faster than R^2 resulting from $f(R)$, *cf.* (6.41). Since in fact the $O(R^2)$ part of $f(R)$ also vanishes from the LHS (on both sphere and hyperboloid), we see that in the large R regime we have

$$\partial_t v_k(R) - 2R\, v_k'(R) + 4\, v_k(R) = o(R^2)\,. \tag{6.54}$$

Any part of $v_k(R)$ growing at least as fast as R^2 is then easily solved for, and gives mean-field evolution involving some arbitrary function v:

$$v_k(R) = \mathrm{e}^{-4t}\, v(R\,\mathrm{e}^{2t}) + o(R^2)\,. \tag{6.55}$$

It will be the same function v that we introduced in the linearised solution (6.16) if we require as boundary condition, $v_k(R) = v(R)$ at $k = \mu$. The question that remains is whether the RG evolution (6.55) is consistent with what we were assuming by linearising.

For the power-law solution (6.52), linearisation is valid at large $|R|$ if and only if $\lambda\,4$. This follows from the hyperboloid fixed point asymptotics (6.48), the sphere side (6.41) requiring only the weaker constraint, $\lambda\,4 + 2b$. On the other hand if $\lambda > 4$, we use the general perturbation (6.53), finding the solution (6.55). Substituting the explicit form (6.52) of the boundary condition we get

$$v_k(R) = v(R)\,\mathrm{e}^{-\theta t} + o(R^2)\,, \tag{6.56}$$

where $\theta = 4 - \lambda$, *i.e.* we reproduce the linearised solution (6.16). We conclude that asymptotically, power-law eigenoperators (6.52) are valid solutions for any λ. Their t evolution is multiplicative and given by the flow of a conjugate coupling $g(t) = \epsilon\,\mathrm{e}^{-\theta t}$, *cf.* (6.16).

On the other hand, the solutions that behave asymptotically as $v(R) \sim 1/\omega(R)$, are growing exponentials of exponentials. Linearisation is not valid at large $|R|$, where the t dependence is given instead by (6.55). Now we cannot separate out the t dependence. Therefore such perturbations cannot be regarded as eigenoperators evolving multiplicatively. Excluding them leads to quantisation of the spectrum. The large R dependence (6.52) provides a boundary condition on both the sphere and the hyperboloid side, and linearity provides a further boundary condition since we can choose a normalisation *e.g.* $v(0) = 1$. These three conditions over-constrain the eigenoperator equation (6.17) leading to quantisation of λ, *i.e.* to a discrete eigenoperator spectrum.

Again we will see in section 6.5, that the conclusion is very different if we choose the conformal factor cut-off to be negative, *i.e.* $c_{\bar{h}} < 0$.

6.3.3 Square integrability under the Sturm-Liouville weight

Now we can return to the question posed at the end of section 6.2: whether it makes sense for eigenoperators $v(R)$ to be square-integrable under the Sturm-Liouville weight $w(R)$, *cf.* (6.30), which is the remaining condition that must be satisfied in order for Sturm-Liouville theory to be applicable. We have seen that on both manifolds, $\omega(R)$ is rapidly decaying for large curvature. We saw that the eigenoperator solutions that are actually allowed are the ones that grow as a power, (6.52). Now we see that they are square integrable under this measure. On the other hand the solutions $v(R) \sim 1/\omega(R)$ that we already excluded on physical grounds, satisfy $\omega(R)\,v^2(R) \sim 1/\omega(R)$ which thus diverges at large R. These perturbations are therefore not square integrable under the measure. We conclude that the condition of square-integrability picks out the correct solutions from the eigenoperator equation and that Sturm-Liouville theory is therefore applicable.

Although these formulae have been derived for the specific choice of exponential cut-off (6.24), it is immediate to see that these qualitative properties hold true for a wide range of cut-offs, independent of their details. Indeed the fact that $a_2(R)$ is decaying for large $|R|$ with sign given by $c_{\bar{h}}$, and that $a_1(R) = 2R$ plus decaying terms, is enough to ensure that $\omega(R)$ for $c_{\bar{h}} > 0$ is a rapidly decaying exponential, as follows from its formula (6.30). This behaviour also ensures that $\delta f(R)$, the non-power-law solutions $v(R)$, and $1/\omega(R)$, are all equal up to subleading multiplicative corrections. In section 6.5, we will see that if we choose $c_{\bar{h}} < 0$, these solutions still hold but lead to profoundly different scenarios.

6.4 Liouville normal form

We have seen that Sturm-Liouville theory is (only) applicable to the quantised spectrum of eigenoperators that have power-law asymptotic behaviour in R, given by (6.52), and which we determined already from RG properties were the physical eigenoperators. The consequences of Sturm-Liouville theory for this spectrum can be seen by a standard transformation that takes the linear second order ODE (6.17) to so-called Liouville normal form. For this case we set the coordinate to be (taking $x = 0$ at $R = 0$ without loss of generality):

$$x = \int_0^R \frac{1}{\sqrt{a_2(R')}} dR' \,. \tag{6.57}$$

It is well defined since we have seen that $a_2(R)$ is strictly positive at all finite R. Furthermore since $a_2(R)$ vanishes at large $|R|$ we see that $x \to \pm\infty$ as $R \to \pm\infty$. Then defining the 'wave-function'

$$\psi(x) = a_2^{\frac{1}{4}}(R)\, w^{\frac{1}{2}}(R)\, v(R) \,, \tag{6.58}$$

(6.17) becomes

$$-\frac{d^2\psi(x)}{dx^2} + U(x)\,\psi(x) = \lambda\,\psi(x) \,, \tag{6.59}$$

which is nothing but the time-independent Schrödinger equation at energy λ (and mass $\frac{1}{2}$). This is Liouville normal form. After some manipulation, one finds that the potential is given by [154]:

$$U(x) = a_0 + \frac{a_1^2}{4a_2} - \frac{a_1'}{2} + a_2'\Big(\frac{a_1}{2a_2} + \frac{3a_2'}{16a_2}\Big) - \frac{a_2''}{4} \tag{6.60}$$

(the terms on the RHS being functions of R).

In ref. [154], it was noted that this potential has no singularities at finite x whilst from the asymptotic behaviour of the $a_i(R)$, the second term dominates for $x \to \pm\infty$ such that $U(x) \to +\infty$, leading to the conclusion that there is only a quantised bound-state energy spectrum $\lambda = \lambda_n$ $(n = 0, 1, 2, \cdots)$ bounded from below with the only accumulation point at infinity (following standard analysis of its Green's function, see *e.g.* [193]). In other words there are only a finite number of (marginally) relevant couplings such that $\theta_n = 4 - \lambda_n \geq 0$, and infinitely many irrelevant couplings. These latter have scaling dimensions $\theta_n \to -\infty$ as $n \to \infty$.

There is a hidden assumption here, namely that $\psi(x)$ has appropriate behaviour as $x \to \pm\infty$ for the Schrödinger equation interpretation to make sense. For this, $\psi(x)$ should be either square-integrable, corresponding to a bound state, or correspond to an unbound state such that $\psi(x) = \psi_k(x) \sim \mathrm{e}^{ikx}$ as $x \to \pm\infty$ for some wave-number k. These latter are δ-function normalisable, *i.e.* can be chosen to satisfy $\int_x \psi_k(x)\psi_{k'}(x) = \delta(k - k')$. For this potential these latter solutions do not exist. As we have seen, there are other solutions however, but the missing solutions (which we have rejected on RG grounds) behave asymptotically as $v(R) \sim 1/\omega(R)$. From (6.58), they grow rapidly as $x \to \pm\infty$ (in fact exponentially) so are neither square-integrable nor δ-function normalisable.

On both sphere (6.44) and hyperboloid (6.49), we can write

$$a_2(R) = \frac{1}{G^2(R)}\,\mathrm{e}^{-2F(R)}\,, \tag{6.61}$$

where F and G have the behaviour required for the identity (6.45). Thus we get asymptotically

$$x = \frac{G(R)}{F'(R)}\,\mathrm{e}^{F(R)} + \cdots\,. \tag{6.62}$$

From (6.60) to leading order, we therefore have

$$U(x) = \frac{a_1^2}{4a_2} = \frac{R^2}{a_2(R)} = \left[RF'(R)\right]^2 x^2\,. \tag{6.63}$$

But from (6.44) and (6.49) we see that $RF'(R) = b\,F(R)$. Taking logs of (6.62), we thus find

$$U(x) = (b\,x\ln|x|)^2\left\{1 + O\left(\frac{\ln\ln|x|}{\ln|x|}\right)\right\} \qquad \text{as} \qquad x \to \pm\infty\,. \tag{6.64}$$

It is interesting that the leading large x behaviour of the potential is symmetric about the origin $x = 0$, even though $U(x)$ is surely not. In particular the fact that at large R, $f(R)$ is universal on the hyperboloid but dominated by cut-off effects on the sphere, does not result in different behaviour in the corresponding large x regime of the potential $U(x)$. It is also interesting that this leading behaviour is close to being universal, in that the only cut-off dependence is through the parameter b, the power entering the exponential fall-off form in the cut-off (6.24). Unfortunately this still amounts to strong dependence. Actually this remaining dependence is an artefact of the single-metric approximation [33, 194], one consequence of which is to conflate the background curvature dependence in the cut-off, in particular in F, with that of the quantum field.[43]

From (6.64) we can find for the quantised spectrum the asymptotic behaviour of their scaling dimensions at large n:

$$\theta_n = -b\,(n\ln n)\left\{1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right\} \qquad \text{as} \qquad n \to \infty\,. \tag{6.65}$$

This follows by noting that large values of λ_n closely obey the WKB formula for the Schrödinger equation (6.59):

$$\int_{-x_n}^{x_n} dx\,\sqrt{\lambda_n - U(x)} = (n + \tfrac{1}{2})\pi\,. \tag{6.66}$$

The boundaries of the integral should be the classical turning points, *i.e.* the solutions to $\lambda_n = U(x)$. However up to multiplicative corrections of order $\ln\ln x_n/\ln x_n$ these can be taken to be $\pm x_n$ where at the same level of approximation,

$$\lambda_n = U(\pm x_n) = (b\,x_n\ln x_n)^2\,. \tag{6.67}$$

Substituting this and $x = x_n y$ into (6.66) gives

$$I\,b\,x_n^2\ln x_n^2 = \pi(2n+1)\,, \tag{6.68}$$

where the integral

$$I = \int_{-1}^{1} dy\,\sqrt{1 - \frac{U(yx_n)}{(bx_n\ln x_n)^2}} \;=\; \int_{-1}^{1} dy\,\sqrt{1-y^2} \;=\; \frac{\pi}{2}\,, \tag{6.69}$$

again up to corrections of order $\ln\ln x_n/\ln x_n$. Thus in the large n limit, we can solve (6.68) in terms of the Lambert W function, as $\ln x_n^2 = W(4n/b)$ (using the fact that W satisfies $W(z)\exp W(z) = z$). Substituting this solution into (6.67), using the asymptotic expansion of $W(4n/b)$, and again neglecting multiplicative corrections of order $\ln\ln x_n/\ln x_n$ or smaller, gives (6.65).

Using polynomial truncations taken to very high order, the θ_n were accurately estimated up to $n = 70$ in ref. [195], and found to closely fit $\theta_n \approx 2.91 - 2.042n$. However these were computed in an adaptive cut-off version of the $f(R)$ approximation and using

[43] In reality Γ_k is a functional of both the background metric $g^B_{\mu\nu}$ and the quantum metric $g^Q_{\mu\nu}$. It is $g^Q_{\mu\nu}$ differentials that appear in the fixed point and eigenoperator equations, and thus it is also the behaviour at large $g^Q_{\mu\nu}$ that we are interested in. In a non-adaptive scheme as employed here, cut-off profiles such as (6.13) should in reality not depend on $g^Q_{\mu\nu}$ but only on its field differentials, since the cut-offs are meant to regularise Laplacians for these modes.

optimised cut-off [180]. Given the above strong dependence on cut-off profile we cannot make a sensible comparison, although we note that given the weak dependence of $\ln n$, and ignorance of the neglected corrections, $b \approx 1$ would provide a reasonable match.

6.5 Wrong sign cut-off in the conformal sector

In this section we show what changes if we choose a negative cut-off for the conformal mode, *i.e.* $c_{\bar{h}} < 0$.

In section 6.2.2 we analysed the constraints on $f_k(0)$, $f'_k(0)$, and $f''_k(0)$. For completeness we show how these change with $c_{\bar{h}} < 0$. Recall that the strongest constraints arise if the cut-off function $r(z)$ diverges as $z \to 0$. Then we showed that from the tensor mode trace we must have $f'_k(0) \leq 0$. Now that $c_{\bar{h}} < 0$, from the $\bar{h}$ trace we need $f''_k(0) \leq 0$ to avoid a singularity. The equations are then consistent provided $f_k(0)$ is less than some positive bound. If $r(0)$ is finite then other possibilities again arise for example $f''_k(0) > 0$ is possible provided $f_k(0)$ is sufficiently positive.

Much more interesting is the effect of negative cut-off on the space of fixed points and eigenoperators. Recall that the asymptotic behaviour of the fixed point solutions $f(R)$ is given in the first instance by asymptotic series whose leading term is a power of R: (6.41) in the case of the sphere and (6.48) for the hyperboloid. These contain one parameter A (a different value in general on the sphere or hyperboloid). However to determine the true number of parameters in the asymptotic solution, we study the linear perturbation $\delta f(R)$ to these asymptotic series, and find $\delta f(R) \sim B/\omega(R)$, where ω is the Sturm-Liouville weight and is given by (6.46) or (6.50) on the sphere or hyperboloid respectively. The derivation is still correct if $c_{\bar{h}} < 0$, but the Sturm-Liouville weight is now a rapidly growing exponential of an exponential (on both sides). Thus the perturbation $\delta f(R) \sim B/\omega(R)$ is a rapidly decaying exponential of an exponential. Whatever value of B we choose, asymptotically our assumption that $\delta f(R)$ is much smaller than the series solutions, becomes ever more justified. Therefore asymptotically there is now a full two-parameter set of solutions, being to leading order precisely (6.41) or (6.48) as appropriate, plus $B/\omega(R)$. Now these solutions impose no boundary conditions since at some appropriate large R, $f(R)$ and $f'(R)$ merely fix the values of the two parameters A and B in the appropriate asymptotic solution. Therefore if we have solutions they will be continuous 'planes' of fixed points: two-dimensional sets parametrised by two real free parameters [111].

It had already been noticed in $f(R)$ truncations with adaptive cut-off, that fluctuations from the conformal factor govern the structure of the solutions [113, 118]. We now see that the reason is that it is intimately tied to the way this sector is regularised. For non-adaptive cut-off the choice $c_{\bar{h}} < 0$ is the only one available for the Einstein-Hilbert truncation [33] and for perturbative solution of the flow equation starting from the classical Einstein-Hilbert action (with or without a cosmological constant), but such a wrong-sign kinetic term plus wrong sign cut-off, leads to a continuum of fixed point solutions [77]. The cause is the same as was found in these earlier papers, namely the fact that the fixed point large field asymptotic behaviour has the full set of parameters and thus imposes no boundary conditions. This effect is also seen [113] in one formulation of $f(R)$ with an adaptive cut-off [116] and in some of the asymptotic solutions [111] found in another formulation involving more fixed singularities [118]. It was already suggested in ref. [113] that such a continuum of solutions is a reflection of the conformal mode instability.

Similar conclusions are drawn for the eigenoperator spectrum around any such fixed point. The exponential of exponential solutions $v(R) \sim 1/\omega(R)$, to the eigenoperator equation (6.17), are now exponentially small at large $|R|$ and thus linearisation remains valid. Therefore we now have a continuous spectrum, with degeneracy two, for every value of λ. Again, this effect has been seen before, in the same situations where a continuum of fixed points are found: in a conformal truncation [77], and in $f(R)$ approximation with adaptive cut-off [113].

Note that such a continuous spectrum of eigenoperators is consistent with there being a two-dimensional continuum of fixed points. Indeed the two eigenoperators with $\lambda = 4$ are the exactly marginal operators $v(R) = \delta f(R) \sim R^2$ and $v(R) = \delta f(R) \sim 1/\omega(R)$ (for given sign of R) that move the system infinitesimally from one fixed point to another in this two-dimensional continuum.

The general eigenoperator with scaling dimension λ grows as $|R|^{\frac{\lambda}{2}}$ at large $|R|$, *cf.* (6.52). They are thus not square-integrable under the Sturm-Liouville weight. Although they have conjugate couplings that evolve multiplicatively at the linearised level, and are in this sense physical, we can choose to impose square-integrability as an extra condition. If we do so we exclude the power-law solutions. This amounts to an extra *quantisation condition* that is natural within the Wilsonian RG framework [8]. Indeed without it the Wilsonian RG breaks down because there would be no sense in which an arbitrary linearised perturbation can be broken down uniquely into a convergent series expansion over operators of definite scaling dimension [8, 77, 196]. The remaining solutions $v(R) \sim 1/\omega(R)$ are exponentially decaying for both $R \to +\infty$ and $R \to -\infty$. Since for these, $\omega(R)\, v^2(R) \sim 1/\omega(R)$, they are square-integrable under the Sturm-Liouville weight, and thus form a quantised spectrum.

Their relation to the continuum of fixed points is novel in that it is no longer possible to move to any nearby fixed point by 'switching on' marginal directions. Indeed we have at most one marginal operator now. Generically we will have none.

The $a_2(R)$ coefficient (6.18) changes sign under $c_{\bar{h}} \mapsto -c_{\bar{h}}$, but it still decays exponentially at large $|R|$, as we see from (6.44) and (6.49) for sphere and hyperboloid respectively. We can still transform to Liouville normal form, if we first multiply the eigenoperator equation (6.17) by a minus sign. Then we see that

$$x = \int_0^R \frac{1}{\sqrt{|a_2(R')|}}\, dR' \,, \tag{6.70}$$

is the same transformation as before. The wave-function is now

$$\psi(x) = |a_2|^{\frac{1}{4}}(R)\, w^{\frac{1}{2}}(R)\, v(R)\,, \tag{6.71}$$

whilst the Schrödinger equation now appears as

$$-\frac{d^2\psi(x)}{dx^2} + U(x)\,\psi(x) = -\lambda\,\psi(x)\,, \tag{6.72}$$

i.e. with λ now being minus the energy. The potential U is given by the same formula up to an overall sign, *i.e.*

$$U(x) = -a_0 + \frac{a_1^2}{4|a_2|} + \frac{a_1'}{2} - a_2'\left(\frac{a_1}{2a_2} + \frac{3a_2'}{16a_2}\right) + \frac{a_2''}{4}\,. \tag{6.73}$$

The power-law eigenoperators $v \sim |R|^{\frac{\lambda}{2}}$ are now associated with exponentially growing wave-functions, dominated by the ω dependence in (6.71). From Schrödinger's point of view, they are not acceptable solutions. On the other hand, the solutions $v \sim 1/\omega(R)$ correspond to exponentially decaying $\psi(x)$ and thus bound-state solutions to (6.72).

Since the large R dependence of a_1 and $|a_2|$ is the same as before, we see that the analysis (6.61) – (6.63) goes through unchanged and $U(x)$ has the same large x dependence (6.64) as before. The WKB analysis therefore also goes through unchanged, except that the energies are now $-\lambda_n$. Therefore we see that we have at most a finite number of (marginally) irrelevant operators and an infinite tower of relevant operators, the scaling dimension of the conjugate couplings being

$$\theta_n = b\,(n\ln n)\left\{1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right\} \qquad \text{as} \qquad n\to\infty\,. \tag{6.74}$$

We recognise that these are $f(R)$-approximation analogues of the $\delta^{(n)}_k(\varphi)$ operators introduced in [8] and studied extensively in refs. [1, 2, 74, 107, 109] as elements of a new quantisation of quantum gravity. Indeed the $\delta^{(n)}_k(\varphi)$ operators are eigenoperators appearing in the functional RG when using a wrong-sign cut-off ($c_{\bar{h}} < 0$), where it is needed because the conformal factor field, φ, has wrong-sign kinetic term. The $\delta^{(n)}_k(\varphi)$ span the space of perturbations that are square integrable under an exponentially growing Sturm-Liouville measure, and thus are themselves exponentially decaying at large field. Finally they also form an infinite tower of relevant operators, the scaling dimensions being $\theta_n = 5 + n$.

Note however that the $\delta^{(n)}_k(\varphi)$ are eigenoperators about the Gaussian fixed point, where they are derived exactly, whereas the formula (6.74) applies to the spectrum of square-integrable eigenoperators about any point in the continuum of fixed points in this $f(R)$-approximation. Unlike the φ-versions, the eigenoperator equation has coefficients $a_i(R)$ with non-trivial field dependence. This is responsible for the $\ln n$

dependence in (6.74) while, as we noted in section 6.4, the b dependence appearing in (6.74) is a symptom of the single-metric approximation.

6.6 Summary and Conclusions

We use the $f(R)$ model introduced in ref. [154] where already Sturm-Liouville theory was applied to give a proof that, around any fixed point in such a model, there are a finite number of relevant couplings and an infinite number of irrelevant couplings g_n, these latter having scaling dimensions $\theta_n \to -\infty$ as $n \to \infty$. Note that the scaling dimensions are also proved to be real, in contrast to what is found typically in finite dimensional truncations. In this chapter we scrutinise both the explicit and implicit assumptions that go into this proof, and we combine Sturm-Liouville techniques with asymptotic analysis at large R [111, 113] to find out significantly more about the nature of these fixed points and their eigenoperator spectrum.

Both of these methods can be developed while keeping the cut-off general, which must however be taken to be smooth. In (6.13) we keep general the c_ϕ (the overall size of the cut-off for each field component). As in ref. [154], we set the endomorphism parameters $\alpha_2 = \alpha_1 = 0$, but we keep α_0 general apart from the constraint $1/3 < \alpha_0 < 25/48$ required to ensure that all modes are integrated out in the limit $k \to 0$. We take the same cut-off profile for all field components, since these are all closely tied to the metric either through changes of variables or via BRST invariance. For most of this chapter to be concrete we specialise to the exponential-style cut-off profile [105] (6.24), but we keep its parameters $a > 0$ and $b \geq 1$ general. In particular we are able to determine the asymptotic form of the Sturm-Liouville weight $\omega(R)$ for these cases. It is a rapidly decaying exponential of an exponential *cf.* (6.50) and (6.46) for the hyperboloid and sphere respectively. We show that it is intimately involved in other asymptotic properties, chief amongst them being the detailed form (6.65) of the asymptotic behaviour of the θ_n:

$$\theta_n = -b\,(n \ln n)\left\{1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right\} \qquad \text{as} \qquad n \to \infty\,. \tag{6.75}$$

If computed exactly, these scaling dimensions should be universal. Thus it is gratifying to find that in this model approximation, they are independent of all parameters except one within our general family of cut-offs. It is also encouraging to find that the θ_n have an almost linear dependence on n, since in this respect it is similar to the numerical evidence for near-Gaussian (but complex) dimensions found in ref. [195] for n 70 in an adaptive optimised cut-off version of the $f(R)$ approximation. However the overall dependence on b still amounts to strong residual cut-off dependence, precluding any more meaningful comparison. We saw in section 6.4 that the blame for this lies squarely with the single metric approximation. In fact single field approximations are a known source of artefacts [194].

Sturm-Liouville theory requires the RG eigenvalue equation (6.17) to be second order in R derivatives. This is achieved if and only if we use a non-adaptive cut-off profile. While that leads to the disadvantage of significantly more complicated flow equations compared to those using an adaptive optimised cut-off [180], it does allow us also to ensure that the fixed point ODE has no fixed singularities.

This is an advance on $f(R)$ approximations with adaptive cut-off, where such fixed singularities are endemic. While the fixed singularity at $R = 0$ appears there for a clear physical reason [113, 116], the same is not true for those at $R \neq 0$. These latter fixed singularities can be introduced or shifted to different places, depending on the model [118, 120], but it seems to be impossible to eliminate them entirely [114–124, 154, 169–171]. However, solutions depend sensitively on them, in particular determining whether fixed points exist as global solutions and if so whether they form a continuous set [111, 113].

On the other hand an adaptive cut-off profile has the advantage in that it adapts to the sign of the Hessian. In our case we have to fix the sign of the cut-off via c_ϕ. The Hessian is positive for nearly all field components, requiring $c_\phi > 0$, as would anyway be expected for convergence of the functional integral. However the physical scalar component $\bar{h}$, a.k.a. the conformal factor, is an exception. If we are to describe the regime corresponding to perturbative quantisation of the Einstein-Hilbert term we need to choose $c_{\bar{h}} < 0$ [8, 33, 77, 125]. Otherwise we need to rely on $f_k(R)$ containing higher order terms [181] so that $f_k''(R)$ is positive, *cf.* (6.9) and the discussion in section 6.2 and at the beginning of section 6.2.2. We choose $c_{\bar{h}} > 0$ for the body of this chapter, following ref. [154].

It turns out that on the sphere, we can find the leading asymptotic behaviour of the fixed point solution $f(R)$ in the large R limit for completely general cut-off profile $r(z)$. The result, (6.40), is different from the assumed form in ref. [154]. In fact it is dominated by cut-off effects. For the exponential cut-off it takes the form (6.41). As discussed in section 6.3.1, this limit also ought to be universal, giving the physical equation of state. Here we saw that the blame lies squarely with the course-graining of constant scalar modes in the Jacobian of the change of variables to York decomposition. We saw that this had no effect on the θ_n formula (6.75) however.

The asymptotic solution contains one parameter, A, whereas for a second-order ODE we would expect a general solution to have two. By perturbing around this result we saw that to leading order the other parameter multiplies $\delta f(R) \sim 1/\omega(R)$. Since this perturbation grows more rapidly than $f(R)$, it is not valid asymptotically and thus we see that asymptotically there is only a one-parameter set of fixed point solutions. As discussed in section 6.3.1 if we consider the flow equations as applying only to the sphere, we would then have line(s) of fixed points. This is one motivation for widening the domain of applicability of the flow equations. As discussed in section 6.2.2 nor would we be able to apply Sturm-Liouville theory, the obstruction coming from the existence of an $R = 0$ boundary (where the equations go over to those of flat space).

This provides another motivation. As a final motivation we appeal to the encouraging evidence found in polynomial approximations to $f(R)$ equations [108, 166, 195, 197, 198]. These polynomials probe both signs of R. We saw at the end of section 6.2.2 that if we wish to keep the same cut-off profile for all modes we cannot analytically continue our equations into $R < 0$ however. Instead we match the solution into the equations on the hyperboloid, which also has the property that the equations go over to the flat space ones at its $R = 0$ boundary.

On the hyperboloid the leading asymptotic behaviour is cut-off independent as it should be, being $f(R) \sim AR^2$ (for a typically different A compared to the sphere side). We also provided the leading corrections coming from cut-off terms (6.48), as we did also on the sphere (6.41). Again a perturbation to this solution takes the form $\delta f(R) \sim 1/\omega(R)$ and is thus ruled out. Therefore the asymptotic behaviour as $R \to \pm\infty$ provides two constraints on a global solution for $f(R)$ leading to at most a discrete set of fixed points. This is of course what one would hope to see for asymptotic safety.[44]

In section 6.3.2 we saw that the situation is just as encouraging for the eigenoperators $v(R)$. Since in the eigenoperator equation (6.17), $a_2(R)$ vanishes asymptotically on both the sphere and the hyperboloid (for the explicit formulae see (6.44) and (6.49) respectively), the leading asymptotic behaviour for an eigenoperator is given by $v(R) \propto |R|^{\frac{\lambda}{2}}$, which is again universal, as it should be (if computed exactly). For any RG eigenvalue λ the other solution grows rapidly with $|R|$, satisfying asymptotically $v(R) \sim 1/\omega(R)$ (in agreement with $\delta f(R)$ which corresponds to a putative marginal operator). It is ruled out because it does not evolve multiplicatively under the RG. Since the ODE is linear second order, requiring $v(R) \propto |R|^{\frac{\lambda}{2}}$ overconstrains the equations and leads to quantisation of λ, again as one would hope to see.

Furthermore these 'power-law' eigenoperators are square-integrable under the Sturm-Liouville weight, thus providing the missing justification for using Sturm-Liouville analysis. From general Sturm-Liouville theory, this is already enough to confirm that the eigenoperators $v_n(R)$ form a discrete spectrum and to show that the RG scaling dimensions λ_n, possibly finitely degenerate, have a finite minimum (thus there are a finite number of relevant directions) and form an infinite tower such that (ordering the eigenoperators so λ_n are non-decreasing in n) the $\lambda_n \to \infty$ as $n \to \infty$. The $v_n(R)$ can be chosen to be orthonormal under the Sturm-Liouville weight $\omega(R)$. In fact, the rest of the Sturm-Liouville analysis in ref. [183] can then be straightforwardly taken over to show that arbitrary bare perturbations $\delta f_{k_0}(R)$ (at some UV scale $k = k_0$) will evolve into the space of interactions that can be expanded over the $v_n(R)$ such that the series converges in the square-integrable sense. The map to Liouville normal form, done in section 6.4 and ref. [154], allows us to take this further by computing the large distance behaviour (6.64) of its potential, and from there, by a standard application of WKB analysis, to derive the asymptotic form (6.75) of the $\theta_n = 4 - \lambda_n$ as quoted above.

[44] Note that had we introduced fixed singularities into the $f(R)$ equations we would then have found $f(R)$ to be overconstrained and have no global solutions.

All this is predicated on there actually being a global solution to the fixed point equation (2.22) however. We have searched numerically for such a solution, see figure 6.1 in the case $a = b = 1$, $\alpha_0 = 1/2$ (recall from section 6.2.3 that it has to lie between 1/3 and 25/48) and all the $c_\phi = 1$. We found global solutions on the sphere that asymptote to (6.41) for a small region around $A = -0.01$, starting at $R = 10$ and integrating down to the flat space fixed point equation from (6.25), see figure 6.1 for an example, but we have not been able to find global solutions on the hyperboloid. These are challenging integro-differential (on the sphere-side sum-differential) equations so it is likely that more numerical work is required. This includes exploring other choices of parameters. In fact our solutions on the sphere matched the asymptotic solution (6.41) at $R = 10$, only by choosing to match $f'(R)$ and $f''(R)$ and then computing $f(R)$ from the fixed point equation (rather than the more obvious route of setting $f(R)$ and $f'(R)$ from the asymptotic formula). This indicates that the asymptotic series has not been taken quite far enough for these R values. On the hyperboloid, the asymptotic corrections in (6.48) fall only slowly, so would surely have to go much further to provide a similarly accurate starting point. In fact it would be beneficial to explore simpler equations, if these can be found. An attractive starting point would be to use non-adaptive cut-off together with the exponential parametrisation explored in ref. [119]. Note that if lines of fixed points can be found on both sphere and hyperboloid, there would still have to be a matching point where these $f(R)$ agree to second order in their Taylor expansion (6.26) about $R = 0$, in order to have found a globally defined fixed point.

Finally in section 6.5 we saw that the situation is dramatically different if we choose instead the wrong sign cut-off for the conformal mode: $c_{\bar{h}} < 0$. Perturbing around the asymptotic fixed point solution we still find $\delta f(R) \sim 1/\omega(R)$, but the dependence of the Sturm-Liouville measure on $c_{\bar{h}}$ is such that $\omega(R)$ is now a rapidly growing exponential of an exponential. This means that the perturbation $\delta f(R)$ remains valid asymptotically, and thus the asymptotic solutions have two parameters. They no longer restrict the dimension of the solution space, so fixed points form two-dimensional continuous sets. The alternative asymptotic behaviour for the eigenoperators is also still $v(R) \sim 1/\omega(R)$ but these do now evolve multiplicatively under the RG and thus are also valid solutions. Therefore we have a non-quantised continuous spectrum of RG eigenvalues λ. It is clear that this is mirroring effects previously found [111, 113] in adaptive cut-off $f(R)$ approximations [116, 118], and found [77] in a background-independent version of the so-called conformally reduced gravity [125] where only the conformal factor field is kept. Clearly therefore the culprit for this degeneration is the wrong sign cut-off (which is necessary however if we work with wrong sign kinetic term). In this case, by choosing to keep only interactions square integrable under the Sturm-Liouville measure, the eigenoperator spectrum is again quantised, with $v(R) \sim 1/\omega(R)$ for large R. These form a tower of operators, only finitely many of which are irrelevant, and infinitely many are relevant with dimensions given by minus the θ_n in (6.75). These are the

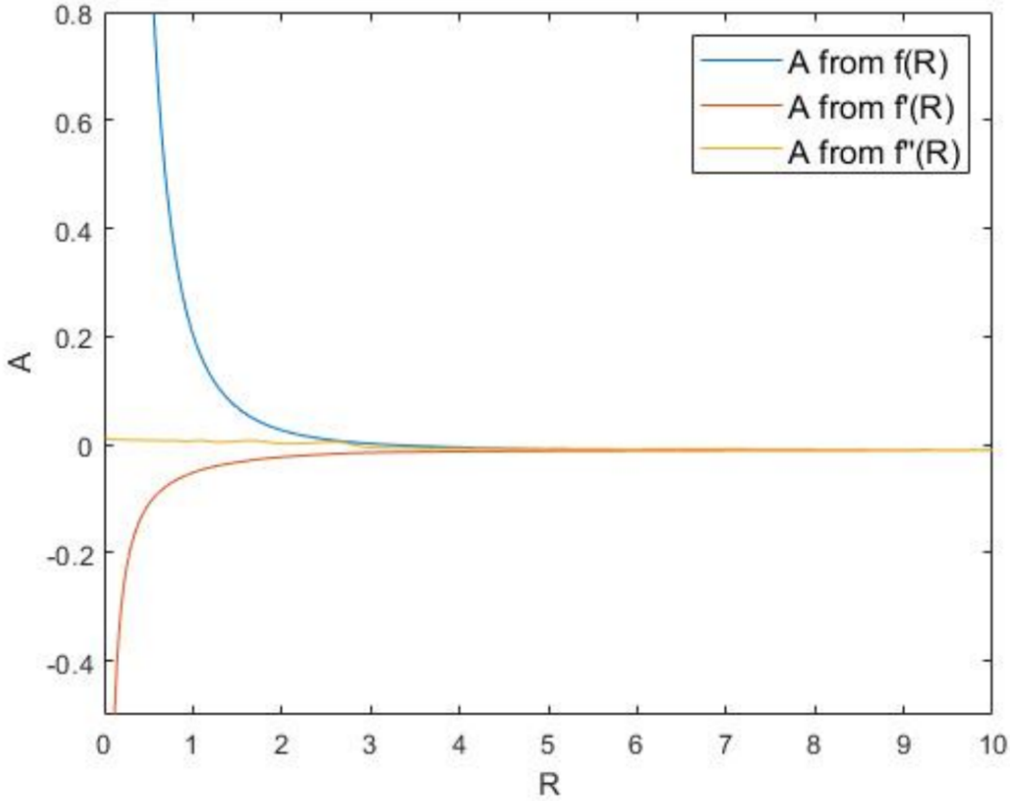

Figure 6.1: The value of the sphere free parameter A in (6.40), deduced by matching the numerical solutions $f(R)$, $f'(R)$ and $f''(R)$ to the corresponding asymptotic formula and solving for A at the different points R. For the asymptotic formula we use (6.41) so as to include the most important subleading corrections (and then differentiate appropriately to get f' and f''). If the numerical solution matches into the asymptotic formula we should find the same value for A for all large enough R and also the same value by matching $f(R)$, $f'(R)$ or $f''(R)$ at these values. We see from the plot that numerically these different determinations do appear to converge, indicating that we have indeed found a numerical solution that matches our asymptotic formula.

$f(R)$-approximation analogues of the $\delta_k^{(n)}(\varphi)$ operators pursued in [1, 2, 8, 74, 107, 109] as an alternative quantisation of quantum gravity.

Chapter 7

Extensions to arbitrary space-time dimensions

In this chapter we outline some preliminary work on an investigation into extending the tower operator (3.62) into arbitrary space-time dimensions. This was motivated primarily by a possible resolution as to why we observe our universe to exist in four space-time dimensions. If this structure seemed to work exclusively in this number of dimensions it would be phenomenologically beneficial to this new approach to QG. As remarked earlier there is also a great history of extending concepts to include extra space-time dimensions, such as Kazula-Klein, string and M theory. This investigation was also motivated by the extra freedom in the existence of $f(R)$ type actions that the tower operator may permit, for example in four dimensional space-times terms such as $R^2_{\alpha\beta\mu\nu}$ are excluded by topological arguments (in particular here by the Gauss-Bonnet theorem [199]) however such terms survive in five and six dimensional space-times. We may be able to create *effective* R^2 and $R^2_{\alpha\beta}$ operators using the new construction outlined in this thesis via quantum corrections which would have far reaching implications, particularly for cosmology.

7.1 The tower operator in arbitrary space-time dimensions

We begin by laying out a glossary of fields, couplings and propagators in arbitrary space-time dimension to facilitate understanding where d is the arbitrary dimension of space-time and the constant α our gauge fixing parameter

$$[(\partial_\alpha H_{\beta\gamma})^2] = d \tag{7.1}$$

$$[H_{\alpha\beta}] = \frac{d-2}{2} \tag{7.2}$$

$$[\kappa] = -\left(\frac{d-2}{2}\right) \tag{7.3}$$

$$[h_{\alpha\beta}] = \frac{d-2}{2} \tag{7.4}$$

$$[\varphi] = \frac{d-2}{2} \tag{7.5}$$

$$[c_\mu] = [\bar{c}_\mu] = \frac{d-2}{2} \tag{7.6}$$

$$[c^*_\mu] = [H^*_{\mu\nu}] = \frac{d}{2} \tag{7.7}$$

and the propagators are, expanding on the general definitions given in section 5, given as

$$\langle H_{\mu\nu}(p)H_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} + \frac{p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)}}{p^4} - \frac{1}{d-2}\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} \tag{7.8}$$

$$\langle c_\mu(p)\bar{c}_\nu(p)\rangle = -\langle \bar{c}_\mu(p)c_\nu(p)\rangle = \frac{\delta_{\mu\nu}}{p^2} \tag{7.9}$$

$$\langle b_\mu(p)H_{\alpha\beta}(-p)\rangle = -\langle H_{\alpha\beta}(p)b_\mu(-p)\rangle = 2\delta_{\mu(\alpha}p_{\beta)}/p^2 \tag{7.10}$$

$$\langle b_\mu(p)b_\nu(-p)\rangle = 0 \tag{7.11}$$

$$\langle h_{\mu\nu}(p)h_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} + \left(\frac{4}{\alpha}-2\right)\frac{p_{(\mu}\delta_{\mu)(\alpha}p_{\beta)}}{p^4} + \frac{1}{d^2}\left(\frac{4}{\alpha}-d-2\right)\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} + \frac{2}{d}\left(1-\frac{2}{\alpha}\right)\frac{\delta_{\alpha\beta}p_\mu p_\nu + p_\alpha p_\beta\delta_{\mu\nu}}{p^4} \tag{7.12}$$

$$\langle h_{\mu\nu}(p)\varphi(-p)\rangle = \langle \varphi(p)h_{\mu\nu}(-p)\rangle = \left(1-\frac{2}{\alpha}\right)\left(\frac{\delta_{\mu\nu}}{d} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2} \tag{7.13}$$

$$\langle \varphi(p)\varphi(-p)\rangle = \left(\frac{1}{\alpha} - \frac{d-1}{d-2}\right)\frac{1}{p^2}. \tag{7.14}$$

Previously the gauge $\alpha = 2$ has been chosen to simplify these expressions, in particular eliminating the $\frac{1}{p^4}$ part of the propagators. In a similar vein we will continue to work in the minimal basis however we note the auxiliary field b_μ above for completeness. The quantisation conditions, in particular the φ part, which led to the tower operator

$\delta_\Lambda^{(n)}(\varphi)$ involved the weights, when extended to general space-time dimension d

$$\exp\left(-\tfrac{1}{2}\frac{h_{\mu\nu}^2}{\hbar\Lambda^{d-2}}\right) \qquad \exp\left(-\tfrac{1}{2}\frac{\varphi^2}{\hbar\Lambda^{d-2}}\right). \tag{7.15}$$

This follows from

$$\Omega_\Lambda = |\langle\varphi(x)\varphi(x)\rangle| = \frac{d}{2(d-2)}\int đ^d p \frac{C^\Lambda(p)}{p^2} \approx \hbar\Lambda^{d-2}, \tag{7.16}$$

as a result there are no new effects in the quantisation procedure outlined in section 3.2.2 and the same general structure emerges from our previous arguments. We note that when extending this treatment to $d = 2$, divergences appear, for example in (7.16), the implications of this require further investigation. We continue in the generalisation of this structure, in particular the tower operator and the associated couplings are now

$$[\delta_\Lambda^{(n)}] = -(1+n)\left(\frac{d-2}{2}\right) \tag{7.17}$$

$$\begin{aligned}[D_\sigma] &= d_\sigma + [\delta_\Lambda^n] \\ &= d_\sigma - (1+n)\left(\frac{d-2}{2}\right)\end{aligned} \tag{7.18}$$

$$\begin{aligned}[g_n^\sigma] &= d - D_\sigma \\ &= d - \left[d_\sigma - (1+n)\left(\frac{d-2}{2}\right)\right].\end{aligned} \tag{7.19}$$

We can then use this generalisation of the structure to see if the operators remain unchanged, in particular we can investigate whether or not the previously found parametrisation of diffeomorphism invariance is still valid, beginning with the anti-field two piece of the first order interaction (we disregard Lorentz indices now for clarity)

$$[\partial c c c^*] = \frac{3}{2}d - 1 \tag{7.20}$$

$$[\partial c H H^*] = \frac{3}{2}d - 1 \tag{7.21}$$

$$[H\partial H\partial H] = \frac{3}{2}d - 1. \tag{7.22}$$

These first order operators are irrelevant and are associated to the zeroth level tower operator, i.e. $n_\sigma = 0$

$$[\delta_\Lambda^{(0)}(\varphi)\ \sigma_1(c, c^*, H, H*, \varphi...)] = -\frac{d-2}{2} + \frac{3}{2}d - 1 = d \tag{7.23}$$

which as we can see means that the tower operator continues to resolve the primary issue of the irrelevancy of the gravitational operators. Extending this to second order in the convenient parametrisation of diffeomorphism invariance where $\mathcal{L}_2^2 = \mathcal{L}_2^1 = 0$, that is to say the anti-field number two and one piece are found to be zero and only the zero level remains. This is the set of operators one would naively find when expanding out the Einstein-Hilbert action.

$$[HH\partial H\partial H] = 2d - 2. \tag{7.24}$$

As these are the second order operators they are now associated to the $n_\sigma = 1$ tower operators which yields

$$[\delta_\Lambda^{(1)}\sigma_2(H, \partial H, ...)] = -\frac{d-2}{2}(1+1) + 2d - 2 = d \tag{7.25}$$

and again the irrelevancy problem is resolved. When finding the explicit actions we also find they remain the exact same in arbitrary dimensions as there is no dependence on d in the coefficients themselves, additionally when finding this via an anti-field cascade there is no dependence on d here either. Important we can also see from the above that n_σ, the minimum number of derivatives of the tower operator needed to ensure the operators remain (marginally) relevant, remains the same which is crucial for this structure being maintained for higher space-time dimensions.

We can now apply this extension to the flow equations and verify that they remain the same, up to some minor generalisations in the power of the scalings. In [107] it was a crucial result that flows would fail and cease to exist for scales below $\Lambda < a\Lambda_p$, does this result still hold for arbitrary d? Following that reference we can, for large φ and for some $a\Lambda_\sigma$ find the expression

$$\begin{aligned} f^\sigma_{a\Lambda_\sigma} &\sim \exp\left(-\frac{a^2\varphi^2}{2a^2\Lambda_\sigma^{2(\frac{d-2}{2})}} \right) \\ &\sim \exp\left(-\frac{\varphi^2}{2\Lambda_\sigma^{(d-2)}} \right) \end{aligned} \tag{7.26}$$

and from this we can then solve flow equations in terms of the conjugate momentum π,

$$f^\sigma_\Lambda(\varphi) = \int_{-\infty}^{\infty} \bar{d}\pi f^\sigma(\pi)\ \ e^{-\frac{\pi^2}{2}\Omega_\Lambda + i\pi\varphi}. \tag{7.27}$$

An effective ansatz for $f^\sigma(\pi)$ is

$$f^\sigma(\pi) \sim e^{-\pi^2\Lambda^{(d-2)/4}} \tag{7.28}$$

$$f^\sigma(\varphi) = e^{-\varphi^2/\Lambda_\sigma^{d-2}} \tag{7.29}$$

and these recover the standard Lagrangian in much the same way as in the $d = 4$ case, in particular for $\sigma = 1$ we can define

$$f^\sigma(\varphi) = \kappa e^{-\varphi^2/\Lambda_\sigma^{d-2}}. \tag{7.30}$$

We also note that the reality condition on the sum over the conjugate momenta expression continues naturally to arbitrary dimensions, namely

$$f^\sigma(\pi) \sum_{n=n_\sigma}^{\infty} g_n^\sigma (i\pi)^n \tag{7.31}$$

demands that all odd couplings are 0. Following [107] we can generalise the expression for these $2m$ couplings

$$g_{2m}^\sigma = \frac{\sqrt{\pi}}{m!4m}\kappa\Lambda_\sigma^{(d-2)m+1}. \tag{7.32}$$

To summarise, the effect of the generalisation to dimension d on the expression f_Λ^σ is a rescaling of any Λ and Λ_σ elements and the general structure, in particular the recovery of κ dependency at first order and the reality condition in (7.31), are maintained. We can therefore find the following general expression which combines these findings,

$$f_\Lambda^\sigma(\varphi) = \frac{\kappa(a\Lambda_\sigma)^{(d-2)/2}}{\sqrt{\Lambda^{d-2} + (a\Lambda_\sigma)^{d-2}}} \exp\left(- \frac{a^2\varphi^2}{\Lambda^{d-2} + (a\Lambda_\sigma)^{d-2}} \right). \tag{7.33}$$

It is also important to note that at first order in the coupling we must also be able to generalise the recovery of terms in the Lagrangian which go as $\kappa\varphi\partial H\partial H$, that is to say there is an undifferentiated φ (note the prime in σ' denotes this type of term) that we must recover via the tower operator. We find that this too is naturally found. We repeat a similar process as above to find

$$f^{\sigma'}(\varphi) = \kappa\varphi e^{-\varphi/\Lambda_\sigma^{d-2}}, \tag{7.34}$$

beginning with $\varphi\delta_\Lambda^{(n)}(\varphi) = -n\delta_\Lambda^{(n-1)} - \Omega_\Lambda\delta_\Lambda^{(n+1)}(\varphi)$ with this final term becoming 0 in the $\Lambda \to 0$ limit. From this we find

$$\begin{aligned} g_{2m+1}^{\sigma'} &= -(2m+2)g_{2m+2}^\sigma|_{\sigma=\sigma'} \\ &= -\tfrac{1}{2}\frac{\sqrt{\pi}}{m!\;4m}\kappa\Lambda_\sigma^{(d-2)m+3} \end{aligned} \tag{7.35}$$

which yields

$$f_\Lambda^{\sigma'}(\varphi) = \frac{\kappa(a\Lambda_\sigma)^{(d-2)/2}}{\sqrt{\Lambda^{d-2}+(a\Lambda_\sigma)^{d-2}}}\exp\left(-\frac{(a\varphi)^{d-2}}{\Lambda^{d-2}+(a\Lambda_\sigma)^{d-2}}\right) - \kappa\Lambda_\sigma^{(d-2)/2}\sqrt{\pi}\delta_\Lambda^0(\varphi) \quad (7.36)$$

which leads to the anticipated generalisation of the coefficients in the physical limit. We do not extend this treatment to higher order in the couplings at this time as the findings of chapter 5 would then have to be included which is beyond the scope of this thesis however this could be an avenue for research in the future.

7.2 Conclusions

To conclude we can see that this extension to arbitrary space-time dimensions happens very easily in this novel framework with much of this framework remaining intact. Although this has been a preliminary investigation it offers insight into the general structure of the theory and what role it may play in describing Nature.

Chapter 8

Conclusions

In this thesis we have reviewed the essential ingredients one must consider when attempting to create a theory of quantum gravity with a well defined continuum limit. Following reviews of general relativity, the renormalization procedure, the Wilsonian ideas of the renormalization group and a brief note on popular attempts to combine gravity with quantum field theory we then discussed the dilaton portal. Combining the above with a more complete quantisation procedure led to the novel tower operator. This has led to the analyses described in this thesis which we now give closing remarks on and finally we speculate on how this line of research may progress and be applied in the future.

In chapter 4 we began by finding the continuum limit at first order in perturbation theory, characterising the most general form of the coefficient functions used to couple to the standard Lagrangians and also verified universality. Crucially this led to recovery of the standard Lagrangian once we have re-entered the diffeomorphism invariant subspace with dependence only on two parameters: Newton's constant and the cosmological constant. This suggested that this structure was very general and robust however there were still open questions; how does this behaviour change at higher orders, is this simply a repackaging of the fortunate kinematic accidents found in the standard quantisation and is there an infinite number of fundamental couplings at higher order?

Following this, in chapter 5 we naturally extended the treatment to second order in perturbation theory to answer these remaining questions. Restricting ourselves to the case where the cosmological constant is set to zero we found that the structure persists to second order and is renormalizable for kinematic reasons, in addition said structure works in general. A consequence of this would be that following the Wilsonian renormalization group this would be a genuine, consistent continuum limit for gravity albeit in a very different way to what one may expect and with a phenomenologically inconvenient infinite number of fundamental couplings. We then suggested a non-perturbative solution which would use the parabolic properties of the flow equations, namely the dilaton and graviton sectors would naturally flow in opposite directions due to the same initial difference in sign of the kinetic terms which sparked this line of inquiry.

A resolution of this poorly posed Cauchy initial value problem could constrain this difficult infinite number of couplings to only Newton's constant and the cosmological constant.

In chapter 6 we investigated related questions in asymptotic safety, namely if complete flows exist for problems similar to that encountered earlier in this thesis. Answers to these questions would be paramount in seeing if the Cauchy initial value problem not being well posed could offer a resolution to this problem of an infinite number of fundamental couplings as we go to higher orders in perturbation theory. In addition to this the role of adaptive cut-offs amongst other general methods in AS were investigated.

In chapter 7 we gave a brief review of some exploratory research into extending this novel tower operator and the the treatment in [107] into arbitrary space-time dimensions with the hope that this could offer fresh insight into several outstanding problems. As we saw this novel tower operator and the associated structure extended very naturally to arbitrary space-time dimension and although a more through investigation was beyond the scope of this work it is our hope that this work expresses the wide range of subjects that may be researched from a new perspective as a consequence of this new and exciting approach to quantum gravity.

To conclude, a theory of quantum gravity with a well defined continuum limit would be one of the crowning achievements in theoretical physics. The expected complexity of such a theory is reflected in the variety of approaches attempted to construct this theory. In this thesis we have discussed the ramifications of combining relatively simple ideas which are generally held to be some of the best tools to hand for the modern theoretical physics community; namely General Relativity, the Wilsonian renormalization group and a complete quantisation procedure. This combination has naturally produced the unique tower operator, which itself has led to a re-examination of how exactly one defines a continuum limit and a UV complete theory. This work has then been explored in the related subject of asymptotic safety and has offered some fresh perspective in that field; how does one suitably find a complete flow for a theory? We hope that this thesis offers an insight into the modern approaches to quantum gravity, the range of techniques employed and how despite the maturity of the subject there are still exciting new discoveries to be made.

Appendix A

Appendix

A.1 Further examples of coefficient functions

A.1.1 Examples with multiple amplitude suppression scales

Here we develop some of the properties of linearised coefficient functions that are constructed from a spectrum of amplitude suppression scales $\gamma_k \Lambda_\sigma$. For example for symmetric coefficient functions satisfying flat trivialisation (4.32), we can take [107]

$$\mathfrak{f}^\sigma(\pi) = A_\sigma \sum_{k=0}^{N} a_k \, \mathfrak{f}(\pi, \gamma_k \Lambda_\sigma) \,, \tag{A.1}$$

where $N \geq \lceil \frac{n_\sigma}{2} \rceil$ will allow us to ensure that couplings $g^\sigma_{2n<n_\sigma}$ vanish, and we define the function

$$\mathfrak{f}(\pi, \bar{\Lambda}) = \sqrt{\pi}\, \bar{\Lambda}\, \mathrm{e}^{-\pi^2 \bar{\Lambda}^2/4} \,, \tag{A.2}$$

which is just the simplest choice of reduced Fourier transform (4.63), where for convenience we have absorbed the factor of $2\pi\Lambda_\sigma$ from the example (4.62). The dimensionless parameters $\gamma_k > 0$ are chosen unequal, and without loss of generality we order them and set the greatest to unity:

$$0 < \gamma_N < \gamma_{N-1} < \cdots < \gamma_0 = 1 \,, \tag{A.3}$$

and the dimensionless coefficients a_k are chosen to satisfy

$$\sum_{k=0}^{N} a_k = 1 \,, \qquad \text{and} \tag{A.4}$$

$$\tag{A.5}$$

Performing the integral in the Fourier transform representation (4.25) we get

$$f^{\sigma}_{\Lambda}(\varphi) = \sum_{k=0}^{N} a_k \, f_{\Lambda}(\varphi, \gamma_k \Lambda_\sigma)\,, \tag{A.6}$$

where $f_\Lambda(\varphi, \gamma_k\Lambda_\sigma)$ is just the $\alpha\!=\!1$ example (4.64) with Λ_σ rescaled by γ_k.

From the definition of the amplitude suppression scale, see above (4.20), we see that f^σ_Λ has overall amplitude suppression scale Λ_σ, corresponding to the maximum one $\gamma_0\Lambda_\sigma = \Lambda_\sigma$. We verify that it also characterises the exponential decay of the physical coefficient function: setting $\Lambda = 0$,

$$f^{\sigma}(\varphi) = A_\sigma \sum_{k=0}^{N} a_k \, \mathrm{e}^{-\varphi^2/\gamma_k^2\Lambda_\sigma^2} \sim \; a_0 \, A_\sigma \, \mathrm{e}^{-\varphi^2/\Lambda_\sigma^2}\,, \tag{A.7}$$

where the last equation holds at large φ. Thus we satisfy the asymptotic formula for the physical coefficient function (4.23), but we have here an example where the asymptotic behaviour is fixed by A_σ only up to an undetermined dimensionless proportionality constant, as already commented below (4.21). Importantly note that the large π behaviour in the sum over a spectrum of amplitude suppression scales (A.1) is however set by the smallest amplitude suppression scale:

$$\mathfrak{f}^{\sigma}(\pi) \sim \sqrt{\pi}\, a_N \, \gamma_N \, A_\sigma \, \Lambda_\sigma \; \mathrm{e}^{-\pi^2\gamma_N^2\Lambda_\sigma^2/4}\,, \tag{A.8}$$

and thus the asymptotic formula for the Fourier transform (4.27) does not hold, hence the comments below it. The couplings in the Taylor expansion of the Fourier transform (4.26) are given by

$$g^{\sigma}_{2n} = \frac{\sqrt{\pi}}{n!4^n}\, A_\sigma \Lambda_\sigma^{2n+1} \sum_{k=0}^{N} a_k \, \gamma_k^{2n+1} \sim \; a_0 \, A_\sigma \frac{\sqrt{\pi}}{n!4^n}\, \Lambda_\sigma^{2n+1}\,, \tag{A.9}$$

and satisfy the constraint that they vanish for $2n < n_\sigma$, thanks to the vanishing summation constraint (A.5). The last equation holds at large n, which thus verifies that the asymptotic formula for couplings (4.28) nevertheless holds, although again we see the presence of an undetermined proportionality. Finally, since $f_\Lambda(\varphi, \gamma_k\Lambda_\sigma) \to 1$ as $\Lambda_\sigma \to \infty$, we have from the sum normalisation constraint (A.4) that flat trivialisation (4.32) is satisfied, while since $\sqrt{\Lambda^2 + a^2\gamma_k^2\Lambda_\sigma^2}$ sets the scale for φ-variation in the components, we see that the flat limit (4.32) is reached at least as fast as $O(1/\gamma_N\Lambda_\sigma)$ and more generally the refined limit (4.60) is satisfied. Notice however that it is the smallest amplitude suppression scale that controls the corrections here.

Since the summation constraints (A.4,A.5) provide $\lceil \frac{n_\sigma}{2} \rceil + 1$ linearly independent conditions on $N + 1 \ge \lceil \frac{n_\sigma}{2} \rceil + 1$ coefficients a_k, they can always be satisfied. By choosing $N > \lceil \frac{n_\sigma}{2} \rceil$ large enough, we can go on to fix the numerical coefficient of finitely many of any of the surviving g^σ_{2n} (with n finite) to any value we wish, including forcing them also to vanish. We also have the freedom to alter couplings through changing the $0 < \gamma_{k>0} < 1$ provided they remain unequal. We see that the flat trivialisation limit (4.32) is independent of the value of any finite set of finite-n couplings or indeed of any finite number of relations between these couplings [75]. Therefore, apart from confirming that we can ensure that $g^\sigma_{2n<n_\sigma} = 0$, the universal information on the couplings is that captured in the large n asymptotic estimate (4.28), which indeed holds for any linearised solution.

For examples satisfying polynomial trivialisation (4.35), we can still use the sum over a spectrum of amplitude suppression scales (A.1), where by the map to a Fourier transform for a coefficient function satisfying the polynomial trivialisation constraint (4.39), we replace $\mathfrak{f}(\uppi, \bar\Lambda)$ with $(i\partial_\uppi)^\alpha\, \mathfrak{f}(\uppi, \bar\Lambda)$ along the lines already discussed in section 4.2.3.

A.1.2 Other examples with only one amplitude suppression scale

As explained in section 4.2.2 we insist in this thesis on using only one amplitude suppression scale, and our examples are all expressible in conjugate momentum space as an exponential decay factor times a polynomial as in secs. 4.2.3. Other examples with only one amplitude suppression scale could be generated, *e.g.*

$$\mathfrak{f}^\sigma(\uppi) = A_\sigma \sum_{k=0}^{N} a_k\, \mathfrak{f}(\uppi, \gamma_k, \Lambda_\sigma)\,, \tag{A.10}$$

for appropriate choices of a_k, where we choose the function to be

$$\mathfrak{f}(\uppi, \gamma, \bar\Lambda) = \sqrt{\pi}\,\bar\Lambda\, \mathrm{e}^{-\left(\uppi^2\bar\Lambda^2+\gamma^2\right)/4} \cosh(\gamma\bar\Lambda\uppi/2)\,, \tag{A.11}$$

corresponding to the physical coefficient function

$$f(\varphi, \gamma, \bar\Lambda) = \mathrm{e}^{-\varphi^2/\bar\Lambda^2} \cos(\gamma\varphi/\bar\Lambda)\,, \tag{A.12}$$

which thus gives the $\Lambda > 0$ solution

$$f_\Lambda(\varphi, \gamma, \bar\Lambda) = \frac{a\bar\Lambda}{\sqrt{\Lambda^2 + a^2\bar\Lambda^2}} \exp\left(-\frac{a^2\varphi^2 + \gamma^2\Lambda^2/4}{\Lambda^2 + a^2\bar\Lambda^2}\right) \cos\left(\frac{a^2\gamma\bar\Lambda\varphi}{\Lambda^2 + a^2\bar\Lambda^2}\right)\,, \tag{A.13}$$

which clearly again has the right limiting properties to satisfy flat trivialisation (4.32) and the refined limits (4.60). These functions have the same amplitude suppression scale $\bar\Lambda$ irrespective of the choice of γ. Further examples can be generated by exchanging cosh with cos in the above, or for odd functions, replacing these with sinh and sine.

A.2 Computing Taylor expanded IR regulated momentum integrals

To compute derivative expansions such as those that appear for level-one (5.43), level-zero (5.52) and in the mST correction term (5.61), we Taylor expand their integrands in the external momentum p_μ. We use the d-dimensional equivalent of the integrands (5.41, 5.51, 5.58) displayed in the thesis, constructed from using the d-dimensional propagators, see chapter 7 attached to the d-dimensional $\check{\Gamma}_1$ described in section 3.3.4. To be concrete we describe how to treat $\mathcal{B}_{\mu\nu\alpha}(p,\Lambda)$ and $\mathcal{A}_{\mu\nu\alpha\beta}(p,\Lambda)$ in the following. We comment on the slight differences for $\mathcal{F}_{\mu\nu\alpha}(p,\Lambda)$ later. The Taylor expansion coefficients involve the integrals

$$\int_q \frac{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2n}}}{q^{2r}}\, \bar{C}(q^2/\Lambda^2)\, \bar{C}^{(m)}(q^2/\Lambda^2)\,, \tag{A.14}$$

for some non-negative integers m, n, r, with normalisation of the measure as in (3.132). Here $\bar{C}(u) = 1 - C(u)$ is the IR cut-off function, and $\bar{C}^{(m)}(u)$ is its m^{th} differential, where $u = q^2/\Lambda^2$. Now d-dimensional rotational invariance ensures that the integral vanishes unless the numerator has even powers of q and moreover it allows us to reduce the latter to a scalar integral using

$$q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2n}} \equiv q^{2n} \prod_{k=1}^{n} \frac{1}{d+2(k-1)} \sum_{\text{pairs}} \delta_{\mu_{\sigma_1}\mu_{\sigma_2}} \delta_{\mu_{\sigma_3}\mu_{\sigma_4}} \cdots \delta_{\mu_{\sigma_{2n-1}}\mu_{\sigma_{2n}}}\,, \tag{A.15}$$

where this formula is valid under the integral, and may be proved by iteration. The sum is over all ways of dividing the $2n$ indices into Kronecker-delta pairs. For these one-loop integrals in $d{=}4{-}2\epsilon$ dimensions, the worst we can get is a $1/\epsilon$ pole, therefore up to terms vanishing as $\epsilon \to 0$,

$$\int_q = \big(1 + \big[1 - \gamma_E + \ln(4\pi/\Lambda^2)\big]\epsilon\big) \int_0^\infty du\, u^{1-\epsilon}\,. \tag{A.16}$$

The integrals are now reduced iteratively using integration by parts on those containing the highest differential $\bar{C}^{(m)}$. Following the philosophy of dimensional regularisation we choose $\epsilon > 0$ large enough such that we can always discard the UV limit (a.k.a. surface term). The IR limit can also be discarded using the same philosophy, choosing $\epsilon < 0$ negative enough.[45] After this we analytically continue ϵ to the neighbourhood of $\epsilon = 0$

[45] At high orders in the derivative expansion this allows us to discard the lower boundary, $\lim_{\epsilon\to 0} u^{-k-\varepsilon}\, \bar{C}^{(m)} \bar{C}^{(n)}$ for any positive integers k, m, n. This could also be assured by choosing C such that it has vanishing Taylor expansion to all orders at $u = 0$ (known as a "bump" function). In practice in the cases dealt with in section 5.4.1 the lower limit can be discarded anyway thanks to the presence of $\bar{C}(u)$ and/or positive integer powers of u.

in the usual way. As a simple but instructive example we thus have the identity

$$\int_0^\infty du\, u^{-\epsilon}\, \bar{C}\frac{d}{du}\bar{C} = \frac{\epsilon}{2}\int_0^\infty du\, u^{-1-\epsilon}\, \bar{C}^2\,. \tag{A.17}$$

At the end of the process, provided at least one of the $\bar{C}$ is differentiated, the integral is in fact both UV and IR regulated by the cut-off function, and thus $\epsilon\to 0$ can be safely taken. The integrals that require more care are those that are only IR regulated which thus take the form

$$\begin{aligned}\int_0^\infty du\, u^{n-\epsilon}\, \bar{C}^2(u) &= \int_0^1 du\, u^{n-\epsilon}\, \bar{C}^2 \;+ \int_1^\infty du\, u^{n-\epsilon}\, \left(\bar{C}^2 - 1\right) \;+ \int_1^\infty du\, u^{n-\epsilon}\,,\\ &= \int_0^1 du\, u^{n}\, \bar{C}^2 \;+ \int_1^\infty du\, u^{n}\, C(C-2) \;- \frac{1}{n+1-\epsilon} + O(\epsilon)\,, \end{aligned} \tag{A.18}$$

for some integer n. Splitting the integral into three parts as in the first line, we see that the first two parts are both IR and UV regulated for any n and thus $\epsilon\to 0$ can be safely taken. The final integral gives the last term on discarding the upper limit.

As a simple example consider the case $n=-1$. This appears on the RHS of (A.17). Substituting (A.18) and taking the limit $\epsilon\to 0$ one finds the answer $\frac{1}{2}$. In this case it is straightforward to derive this directly from (A.17) at $\epsilon=0$, since the LHS is then a total derivative and the answer $\frac{1}{2}$ is recovered from the UV boundary. However applying dimensional regularisation to all cases including the more involved (A.14,A.15) cases, ensures that results are not subject to momentum routing (equivalently surface term) ambiguities.

In (A.18), apart from the case $n=-1$ which, if it has non-vanishing coefficient, is subtracted using $\overline{\text{MS}}$ *cf.* comments above (5.33), the $\epsilon\to 0$ limit of the last term can also now be safely taken. It then just cancels the cut-off-independent contribution in the first integral on the RHS, thus

$$\int_0^\infty du\, u^{n-\epsilon}\, \bar{C}^2(u) = \int_0^\infty du\, u^{n}\, C(C-2) \;+ O(\epsilon)\,, \qquad (n\neq -1)\,, \tag{A.19}$$

which we could have derived directly from substituting $\bar{C}=1-C$, and discarding the cut-off independent piece as would be done as standard in dimensional regularisation (despite the fact that the integral is strictly speaking ill-defined for any ϵ).

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