Information Loss in Volatility Measurement with Flat Price Trading*

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Abstract

A model of financial asset price determination is proposed that incorporates flat trading features into an efficient price process. The model involves the superposition of a Brownian semimartingale process for the efficient price and a Bernoulli process that determines the extent of flat price trading. The approach is related to sticky price modeling and the Calvo pricing mechanism in macroeconomic dynamics. A limit theory for the conventional realized volatility (RV) measure of integrated volatility is developed. The results show that RV is still consistent but has an inflated asymptotic variance that depends on the probability of flat trading. Estimated quarticity is similarly affected, so that both the feasible central limit theorem and the inferential framework suggested in Barndorff-Nielsen and Shephard (2002) remain valid under flat price trading even though there is information loss due to flat trading effects. The results are related to work by Jacod (2018) and Mykland and Zhang (2006) on realized volatility measures with random and intermittent sampling, and to ACD models for irregularly spaced transactions data. Extensions are given to include models with microstructure noise. Some simulation results are reported. Empirical evaluations with tick-by-tick data indicate that the effect of flat trading on the limit theory under microstructure noise is likely to be minor in most cases, thereby affirming the relevance of existing approaches.

Keywords: Bernoulli process, Brownian semimartingale, Calvo pricing, Flat trading, Microstructure noise, Quarticity function, Realized volatility, Stopping times.

JEL classification: C15, G12

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1 Introduction

The expression 'flat trading' refers to situations in market trading where consecutively sampled prices in calendar time take on the same value. The phenomenon of flat pricing is extremely common in stock market trading, affecting almost all traded stocks, especially (but not exclusively) over small time intervals. Possible reasons for flat price trading in data include slow and sluggish trading, order splitting, and price discreteness. An immediate implication of the phenomenon is that both returns and volatility are zero over the flat price subinterval, an outcome that has null probability of occurrence in any model where price behaves like a continuous Brownian semimartingale. This characteristic of the realized data inevitably has implications for the econometric measurement of volatility.

Respect for the form of the data is a fundamental prerequisite of good modeling and the quality of the empirical research that it can generate. The best textbooks in econometrics emphasize the importance of this feature of modeling and applied work. A distinguishing feature of Peter Schmidt's career is the concern that he has placed on the nature of the data in planning and executing his own research, his pedagogy, and his careful assessment of the research of others in his role as a referee and editor. Peter's extensive research on panel data and the manifold properties of such data clearly reflect this concern, as does the large body of his work on production frontier modeling and technical inefficiency. Peter's textbook *Econometrics*, which was written early in his career and published in 1976 (Schmidt, 1976), manifests his respect for precision and brevity in exposition, for clarity in notation, and for the suitability of the model to the data under study.

Consonant with this thematic of Peter Schmidt's research, the present paper seeks to explore some of the implications of the time series property of flat trading in financial data in the context of the use of realized variance (RV) estimates of integrated variance (IV). Part of the task is to develop a model that compounds the presumed semimartingale behavior of underlying efficient market prices with a mechanism that produces periods of flat prices in practical trading. Flat trading is a regular feature of many financial markets, especially for stock price data that is sampled at modest to high frequencies, where it may be regarded as a market microstructure phenomenon arising from discrete trading practices, information arrival in discrete packets, and trading volume effects. Without developing a full microstructure theory, we posit a stochastic mechanism that accords a constant probability of the occurrence of a trading flat over each given subinterval. The formulation leads to the compounding of the efficient price Brownian semimartingale with a Bernoulli process that determines the timing and length of the flat trading periods. The approach is related to sticky price modeling in macroeconomic dynamics and new Keynesian Phillips curve models. In these models, Calvo (1983) pricing is frequently used in which only a fixed share of firms are able to optimize price each period, leading to price stickiness and some time duration between price changes.

Under the flat trading model in this paper, we develop a limit theory for standard econometric estimates of volatility by nonparametric RV measures. It turns out that when we allow for flat trading RV is still consistent, converges to IV, and follows a mixed Gaussian limit theory under standard regularity conditions corresponding to those used in the original work of Barndorff-Nielsen and Shephard (2002). These results generalize the standard theory on empirical quadratic variation estimates. Notably, however, there is some information loss when using RV to do inference about IV due to the presence of flat price effects. This loss takes the form of an increase in the asymptotic variance. The effects are of a sufficient magnitude to be significant in practical applications. For example, if the RV estimate is constructed from 5-minute returns for Alcoa (AA) stock prices on April 5, 1995, the proportion of flat pricing on this day amounts to some 60% of the sample and our results imply that the correct variance quadruples the variance obtained from a semimartingale process without flat pricing.

As with other research efforts on volatility, our interest in the use of RV measures is motivated by the availability of ultra-high frequency data which has made it feasible to measure volatility accurately in a direct nonparametric way. The idea is well explained in earlier work and simply involves the calculation of the sum of squared intra-day returns obtained from observed intra-day prices. The theoretical justification for measuring volatility in this way relies on standard properties of the empirical quadratic variation process for semimartingales (e.g., Protter (2004)), a set up which is commonly assumed for financial asset prices in the literature (see, for example, Andersen et al. (2001b). The main object of interest in this research is the value of IV over a specific time period such as a day. This approach to measuring volatility has attracted a great deal of attention and has led to numerous successful applications — see, for example, Andersen et al. (2001b, 2003, 2005b, 2001a, 2005a); Bandi and Russell (2008); Fleming et al. (2003). For overviews of the literature, see Andersen et al. (2010); Barndorff-Nielsen et al. (2006).

Direct application of empirical quadratic variation limit theory requires that efficient or equilibrium prices be observed. This requirement appears too strong at ultra-high frequencies, such as the tick-by-tick frequency, because of the presence of various market microstructure effects. These market microstructure effects may be regarded as contaminating the efficient price process and may be, albeit somewhat crudely, modeled as noise. Ignoring these effects produces bias and inconsistency in realized volatility estimates. Other institutional elements of importance arise from empirical realities such as thin trading - the fact that the population of market participants is large but nevertheless finite so that the usual (central limit theory) ingredients that underly the determination of infinitesimal Brownian increments is typically absent. Thin trading inevitably arises in situations where trading takes time and there are finite numbers of traders. Accordingly, we must expect to observe some flats in the real data as we move the sampling frequency to zero. This feature in combination with microstructure noise arising from sources such as bid/ask bouncing and misrecording means that the efficient price process is a latent variable that is observed intermittently and with some degree of error.

While maintaining the assumption of martingale-like behavior for efficient prices, the literature has produced three different strands of research on how to deal with microstructure noise in realized volatility calculations with intra-day data. One strand of research is to use all available tick-by-tick data and seek to explicitly model microstructure noise in this fine-grain sampling context. Assumptions about the properties of the microstructure noise are typically made for analytic convenience and include both iid and stationarity conditions. Important contributions to this literature include Zhang et al. (2005); Aït-Sahalia et al. (2005); Barndorff-Nielsen et al. (2008); Da and Xiu (2021).

A second strand of research in the literature is to sample sparsely relative to the available sampling frequency, usually at modest frequencies, of 5 or 10 minute intervals. This approach is motivated by the fact that many sources of microstructure noise (such as bid/ask bounce), which occur in ultra-high frequency data, are mitigated when prices are sampled at these modest frequencies. Correspondingly, it has been argued that these more sparsely sampled prices better approximate the efficient price process, and therefore standard semimartingale theory can be invoked. Under such semimartingale conditions, the consistency of RV was used in Andersen et al.

(2001b) and the asymptotic distribution of RV was developed in Jacod (2018)[1].

In the third strand of the literature, researchers have focused on the finite sample properties of the RV estimates. Here it is argued that the choice of sampling frequency effectively trades off estimation variance against bias. When microstructure noise is explicitly modelled, an "optimal" sampling frequency, which minimizes the mean squared error of the RV estimate, may be calculated. Studies following this approach include Zhou (1996), Hansen and Lunde (2006) and Bandi and Russell (2008).

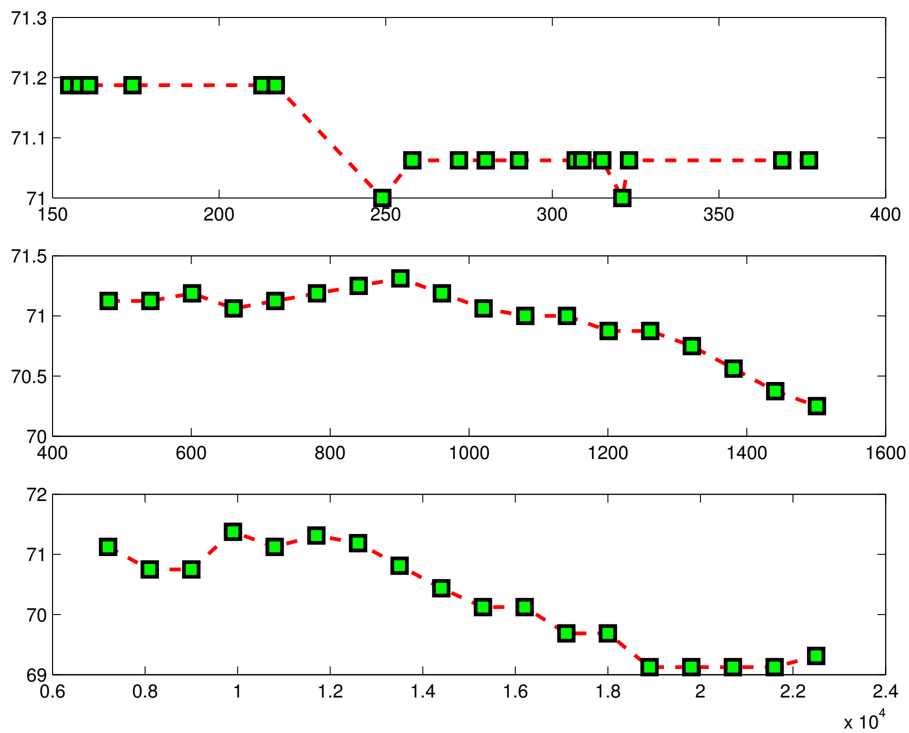

Figure 1: Time series plots of transaction prices for AA on April 5, 2000 at three different frequencies. The horizontal axis is the time stamp (in seconds) since the market opening at 9:30am. The first panel is based on tick-by-tick observations. The second panel is based on data that are sampled every 1 minute. The third panel is based on data that are sampled every 15 minutes. The prices at the 1- and 15-minute frequencies are obtained using the previous tick method. See Hansen and Lunde (2006) for a detailed discussion of the different sampling schemes.

None of the above analyses explicitly models or allows for flat trading in observed prices even though flat trading is a salient feature in actual stock data at most of the frequencies that have been used in this literature, from tick-by-tick data through to 15-minute trading data. Flat trading is a characteristic of both actively traded and inactively traded stocks. To illustrate the former, Figure 1 plots transaction prices for the stock AA from the Dow Jones Industrial Average (DJIA) on the New York Stock Exchange (NYSE) at three different frequencies: tick-by-tick, 1-minute, and 15-minute frequencies on April 5, 2000. Flat trading is obvious at all three frequencies and it becomes a dominant feature in the tick-by-tick data. Table 1 reports the proportions of flat transaction prices for AA when sampling is performed at five different frequencies (1-, 2-, 3-, 4-,

[1] Jacod (2018) is a subsequently published version of a working paper that appeared much earlier and was dated as 1993 in later citations of the paper (Delattre and Jacod, 1997; Barndorff-Nielsen and Shephard, 2002).

5-minute intervals) on the first Wednesday in April from 1993 to 2004. Although flat pricing effects are less pronounced after the decimalization of trading in January 2001, they remain a non-negligible feature of these data. Flat pricing also takes place in tick sampling and in quote data; see, for example, Table 1 in Hansen and Lunde (2006) for the percentages of flat quote prices at the tick-by-tick level for 30 DJIA stocks. Note that AA is a DJIA stock and DJIA stocks are among the most actively traded equities. Flat trading is naturally even more of an issue for less liquid stocks. This feature of trading data deserves attention both in financial modeling and econometric volatility estimation with high frequency data.

Table 1: Proportion of flat trading in AA stock prices

Date	# of ticks	Proportion of flat trading				
		1-min	2-min	3-min	4-min	5-min
April 7, 1993	213	.8793	.8000	.7615	.7041	.6282
April 6, 1994	174	.8897	.8051	.7308	.6429	.6154
April 5, 1995	171	.8872	.7949	.7154	.6633	.6026
April 3, 1996	246	.8795	.7897	.6846	.6327	.5897
April 2, 1997	243	.8179	.6923	.5923	.5612	.4872
April 1, 1998	394	.7846	.6359	.5538	.4898	.4103
April 7, 1999	794	.5513	.4154	.3538	.2245	.2692
April 5, 2000	1007	.4436	.2564	.2231	.2041	.1154
April 4, 2001	1788	.1872	.0872	.0692	.0510	.0128
April 3, 2002	1281	.3872	.2615	.1385	.1224	.0128
April 2, 2003	1986	.3795	.2923	.1769	.1939	.0897
April 7, 2004	3845	.2641	.1590	.1692	.0306	.0513

The contribution of the present paper to these issues relates to both the second strand of the literature on modest frequency sampling and to studies on market microstructure noise. First, the model introduced here extends the models used in Andersen et al. (2001b) and Barndorff-Nielsen and Shephard (2002) by gaining some additional realism in its allowance for flat trading sample paths. Second, we extend the limit theory of RV to the new model, showing that while RV still consistently estimates IV and asymptotically follows a mixed Gaussian law in the presence of flat trading, the asymptotic variance of the RV estimate is inflated, thereby revealing the loss of a substantial amount of information about underlying efficient price volatility in flat trading. Third, we show that the estimated variation of RV based on empirical quarticity is similarly affected by the occurence of trading flats. In consequence, and importantly for empirical research, both the feasible central limit theorem and the inferential framework developed in Barndorff-Nielsen and Shephard (2002) remain valid under flat price trading.

A further contribution of the paper is to relate the specific model used here and associated limit theory to models with sampling at random stopping times considered in Mykland and Zhang (2006) and Jacod (2018). Some new explicit asymptotic results for cases with random sampling are given which show the effects of random duration times between trades on the limit theory for integrated variance and quarticity. A final contribution is to study the effects of random duration times between trades on the limit theory in the presence of microstructure noise of the type studied in Zhang (2006). Importantly, their two time scale estimator is shown to retain its asymptotic properties in the presence of flat trading with only a minor scale change to the asymptotic variance.

The market microstructure literature takes serious account of flat trading data features. For instance, the incidence of daily zero returns has been used to measure illiquidity (Lesmond, 2005; Bekaert et al., 2007); and based on intra-day data, Bandi et al. (2017) proposed to use idle time (IT) and excess idle time (EXIT) to measure the extent of sluggishness in prices. IT is defined as the fraction of a day where the absolute change in prices is smaller than a threshold while EXIT is defined as IT in excess of the spurious sluggishness. These two concepts can therefore be regarded extensions to flat trading. In further related work, Bandi et al. (2020) link the frequency of flat trading to the frequency of small trading volume. Two possible channels can generate flat trading: price discreteness and easy absorption of limited volume without price movement (order splitting). Bandi et al. (2020) found that price discreteness only partly explains flat trading and in consequence introduce a new concept – excess staleness – to measure the percentage of zero returns not due to price discreteness, documenting its empirical significance in real data. While alternative market microstructure features can cause flat trading observed in the data, we do not take any particular stance concerning its source. Our focus instead is to explore the implications of flat trading for statistical inference about RV.

The present work proceeds as follows. After a brief background review we introduce the new flat trading model and develop the corresponding limit theory in Section 2. Section 3 extends the model and relates the limit theory to the work of Jacod (2018) and Mykland and Zhang (2006) where intermittent sampling is employed. Section 4 considers the case of microstructure noise effects. Section 5 reports the results of some Monte Carlo experiments to assess the accuracy of the limit theory in finite samples and some empirical evaluations with tick-by-tick data are performed to assess the effect of flat trading on the limit theory. Section 6 concludes. Proofs are given in the Appendix except for Theorem 3.1 whose proof is in the Online Supplement.

2. A Flat Trading Model and Limit Theory

Let $p^*(t)$ be the logarithm of the efficient price and assume $p^*(t)$ evolves according to a Brownian semimartingale process on a filtered probability space $(\Omega, \mathcal{F}_t, P)$. This assumption is justified by Back (1991) in a frictionless, arbitrage-free economy. As it is typical in the high frequency volatility literature, we further assume that $p^*(t)$ follows the (driftless) diffusion

$$dp^*(t) = \sigma(t)dB(t), \tag{1.1}$$

where $B(t)$ is a standard Brownian motion and $\sigma(t)$ is an $\mathcal{F}_t$- measurable càdlàg volatility process. The quantity of interest is $IV = \int_0^1 \sigma^2(t)dt$, the IV of $p^*(t)$ over a certain unit time period, say a day. The integral may be defined as the limit of the empirical quadratic variation

$$IV = \text{plim}_{h\to 0} \sum_{i=1}^{m} [p^*_{i,m} - p^*_{i-1,m}]^2, \tag{1.2}$$

where $p^*_{i,m} = p^*(t_{i,m})$, $0 = t_{0,m} < t_{1,m} < \cdots < t_{m,m} = 1$ is a sequence of deterministic partitions of $[0, 1]$, and $h_{1,m} = \sup_i |t_{i,m} - t_{i-1,m}|$ is the grid size. Since we are interested in intermittent sampling of the process, it is also useful to define notation for intermittent grid sizes $h_{\ell,m} = \sup_i |t_{i,m} - t_{i-\ell,m}|$ for $\ell \geq 1$. Sometimes, it is convenient to assume that the partition involves a simple grid of equi-spaced points $\{t_{i,m} = \frac{i}{m} : i = 0, ..., m\}$ and then $h_{1,m} = \frac{1}{m}$, and $h_{\ell,m} = \frac{\ell}{m}$. The theory of quadratic variation allows the time spacing in (1.2) to be stochastic and depend on the filtration, thereby allowing the times of trade to have some dependence on equilibrium prices.

The limiting value IV in (1.2) is a (unit time period) segment of the quadratic variation process of p^*. The sample counterpart is the empirical quadratic variation

$$\sum_{i=1}^{m} [p^*_{i,m} - p^*_{i-1,m}]^2 := RV^{(m)}(p^*),$$

which is now commonly referred to as RV in financial economics.

Since $RV^{(m)}(p^*) \overset{p}{\to} IV$ as $h_{1,m} \to 0$ (e.g., Protter (2004)), RV is a natural candidate for estimating IV, motivating the recent interest in this approach to volatility measurement. To quantify the statistical difference between RV and IV, Barndorff-Nielsen and Shephard (2002) used the limit theory

$$\sqrt{m}\left[RV^{(m)}(p^*) - IV\right] \overset{d}{\to} \mathcal{MN}\left(0, 2\int_0^1 \sigma^4(t)dt\right), \tag{1.3}$$

where $\mathcal{MN}$ signifies mixed normality. A feasible version of this limit involves the estimation of the quarticity functional $\int_0^1 \sigma^4(t)dt$ using empirical quarticity. Barndorff-Nielsen and Shephard (2002) obtained the following result

$$\frac{RV^{(m)}(p^*) - IV}{\sqrt{\frac{2}{3}\sum_{i=1}^{m}[p^*_{i,m} - p^*_{i-1,m}]^4}} \overset{d}{\to} \mathcal{N}(0,1), \tag{1.4}$$

which is convenient for use in inference.

These asymptotic results all require knowledge of the log-efficient price, $p^*_{i,m}$. At ultra high frequencies market microstructure effects challenge this requirement, contaminating observations with microstructure noise so that the actual price data $p_{i,m} = p(t_{i,m})$ differs from $p^*_{i,m}$ and $RV^{(m)}(p) \neq RV^{(m)}(p^*)$. To mitigate such market microstructure effects, Andersen et al. (2001b,a); Barndorff-Nielsen and Shephard (2002) suggested sampling sparsely, say at five minute intervals, so that the accumulative effects of noise are less important and $p_{i,m}$ is treated the same as $p^*_{i,m}$. Andersen et al. (2001b) justified the choice of five minute intervals using the signature plot, a graphical device used to assess the degree of bias caused by market microstructure effects at different sampling frequencies. Signature plots typically suggest that RV is more severely biased when the sampling frequency increases but stabilizes at modest frequencies (such as 5 minutes or so). This observation has prompted researchers to view the observed price as a good approximation to the efficient price and has the same semimartingale characteristics at these modest frequencies.

The impact of market microstructure noise has also been examined in the more specific analytic framework

$$p(t) = p^*(t) + u(t), \tag{1.5}$$

where $u(t)$ is microstructure noise. Most studies assume that the noise process $u(t)$ and price process $p^*(t)$ are independent. However, there are many different proposals in the literature about how to model the noise process and how to treat the presence of noise in the estimation of IV. Some studies (e.g., Zhou (1996); Bandi and Russell (2008); Zhang et al. (2005)) assume a pure noise structure for $u(t)$. Some other studies (e.g. Hansen and Lunde (2006); Aït-Sahalia et al. (2011); Da and Xiu (2021)) assume $u(t)$ is covariance stationary.

Neither pure noise nor covariance stationary microstructure effects explain flat trading. In fact, when the efficient price follows a Brownian semimartingale as in (1.1), then during periods of flat trading prices the microstructure noise effect completely offsets the efficient price fluctuations to produce a sustained flat transactions price. The noise process therefore inherits the same local martingale-like behavior of the efficient price process over this subinterval. Inspection of trading data such as that shown in Fig. 1 shows that while sampling at modest frequencies reduces the effects of flat trading. To assess the effects of flat pricing on RV asymptotics, we propose to build a model that directly incorporates flat trading features.

We follow the existing literature and assume that the efficient price process $p^*(t)$ follows (1.1). This specification implies that, for any $t_i \in [0,1]$, $p^*_{i,m}$ has the local martingale structure $p^*_{i,m} = \sum_{j=1}^{i} \varepsilon_{j,m}$, where $\varepsilon_{j,m} = \int_{t_{j-1,m}}^{t_{j,m}} \sigma(s)dB(s)$.

The new model adds a simple Bernoulli process to (1.1) to determine the trading price

$$p_{i,m} = \begin{cases} p^*_{i,m} & \text{if } \xi_i = 1 \\ p_{i-1,m} & \text{if } \xi_i = 0 \end{cases}, \tag{1.6}$$

where ξ_i is a Bernoulli sequence independent of p^* with $E(\xi_i = 1) = \pi$, and $p_{0,m} = p^*_{0,m} = O_p(1)$. Thus, while $p^*_{i,m}$ follows an underlying local martingale in the background, the observed price compounds this efficient process with an independent Bernoulli sequence that determines whether flat trading occurs in the price realization . Whenever $p_{i,m} \neq p_{i-1,m}$, the realization follows the efficient price and we observe $p^*_{i,m}$. Otherwise, flat trading occurs. In that event, the microstructure noise effect completely offsets the efficient price movement over the subinterval in which flat trading occurs.

We can think of p as a time changed process arising from the equilibrium price p^* where the time change is random and discontinuous, depending on the realizations of the Bernoulli sequence ξ. The model may therefore be linked to other literatures, particularly the important paper by Jacod (2018) which allows for random stopping times, the work of Mykland and Zhang (2006) which allows for deterministic intermittent sampling, and the work by Engle and Russell (1998) on ACD modeling of duration. Some of the connections with this literature are explored in Section 3, where the model (1.6) is generalized to allow for random stopping times which may depend on past prices.

In other related work, Delattre and Jacod (1997) introduced round-off to the measurement of a stochastic process and examined the impact of this type of measurement error on the limt theory for certain functionals of the process. In such cases, the efficient price $p^*(t)$ is effectively rounded off to a grid of values determined by a parameter α_m, say, so that at t_i we observe $p^*_{\alpha_m}(t_i) = \alpha_m [p^*(t_i)/\alpha_m]$ where $[\cdot]$ signifies the integer part of the argument. Thus, $p^*_{\alpha_m}(t_i)$ is measured in increments of α_m. Importantly, this model incorporates the institutional feature of the market that trading takes place at well defined increments. In practice, of course, these increments (like 1/8 cent or decimalization) change over time, which is a further institutional complication. Delattre and Jacod (1997) show that the effects of this type of roundoff on the limit theory are subtle and depend on the rate at which $\alpha_m \to 0$ as $m \to \infty$, inconsistencies arising in the estimation of volatilities, for example, when the roundoff parameter $\alpha_m \to 0$ too slowly. Round-off effects of this type will induce some flat trading in the observed process when the efficient price wanders around a particular level over successive observations. Duration of flats is then completely determined by the efficient price process and the mechanics of the roundoff process.

In the model (1.6), the duration of flat trading is determined by the number of successive zero draws in the Bernoulli sequence ξ_i. In units of the sampling grid, the induced duration K is therefore random and unbounded. In particular, it is known (e.g. Schilling (1990)) that the maximum run time, $\bar{K}_m$, for a sequence of identical Bernoulli draws in a sample of size m has mean $E\left(\bar{K}_m\right) = O\left(\log_{1/\pi}\left\{m\left(1-\pi\right)\right\}\right) = O\left(\log\left\{m\left(1-\pi\right)\right\}/\log\frac{1}{\pi}\right)$ and variance $\mathrm{Var}(\bar{K}_m) = \frac{\pi^2}{6\log^2\left(\frac{1}{\pi}\right)}$. It follows that $\bar{K}_m = O_p\left(\log m\right)$. So the induced duration K of trading flats in (1.6) is unbounded but at most $O_p\left(\log m\right)$ as $m \to \infty$. In cases where the sampling grid is equispaced with $t_{i,m} = \frac{i}{m}$, it follows that the calendar duration time is K/m and therefore $o_p\left(1\right)$ as $m \to \infty$. In the case of non-equispaced grid $\left\{t_{i,m}\right\}$, the maximum calendar duration time on the grid is $h_{\bar{K}_m,m} = \sup_i\left|t_{i,m} - t_{i-\bar{K}_m,m}\right|$.

The model (1.6) may be construed as producing sticky price effects analogous to those in Calvo (1983) pricing schemes. In those schemes, monopolistic power enables some fraction (the Calvo share) of firms to set prices, so that at any point in time there is a probability of sticky pricing and a corresponding probability of a price change. Similar mechanisms appear in other pricing models. For instance, information may reach only a fraction of traders as in the Mankiw and Reis (2002) wholesale pricing model or there may be "rational inattention" in the sense that certain traders may update their information irregularly due to various costs of information gathering or other frictions, as in the pricing model of Reis (2006).

In a similar fashion, the present model (1.6) allows for flat trading with a constant probability of $1-\pi$, so that there is a positive probability of flat trading at each point on the temporal grid when $\pi \in [0,1)$. When $\pi = 1$, $p_{i,m} = p^*_{i,m}$ almost surely and the model reduces to the earlier model of Andersen et al. (2001b,a); Barndorff-Nielsen and Shephard (2002). If $p_{i-1,m} = p^*_{i-1,m}$ and $p_{i,m} = p_{i-1,m}$, then $p_{i,m} - p^*_{i,m} = p^*_{i-1,m} - p^*_{i,m} = -\varepsilon_{i,m}$. So the new model allows for noise in the observed price and the noise depends on the efficient price. The noise can be interpreted as a discrete price effect, according to which the realized price changes only when the information content is strong enough. Eventually, of course, the observed price will change and follow the efficient price provided $\pi > 0$. One consequence of the specification is that when noise occurs in the model it takes the form $p_{i,m} - p^*_{i,m} = -\varepsilon_{i,m} = -\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)dB(s)$ and is therefore negatively correlated with the efficient price process. Negative correlation between microstructure noise and the efficient price has been empirically documented page 132 of Hansen and Lunde (2006). However, since $p_{i,m} = p^*_{i,m}$ when $p_{i,m} \neq p_{i-1,m}$, the present model eliminates noise effects when the price changes. Thus, this particular model may be more appropriate at modest frequencies rather than at ultra-high frequencies such as second-by-second frequency. The model is extended in Sections 3 and 4 to allow for more general sampling schemes and microstructure noise, making it better suited to data at very high frequencies.

We remark that the parameter π is assumed to remain fixed as $m \to \infty$, so that the no matter how finely the data are observed there is always a positive probability of a flat when $\pi < 1$. This assumption ensures that, although there is a positive probability of flat trading, the length of the flats will inevitably shorten as m increases because the flats are measured in increments of $1/m$ for an equi-spaced grid. In this respect, the model clearly differs from data such as that observed in Figure 1 where no matter how frequently we make observations the flats remain fixed in size. Thus, the model abstracts from the reality in the data, and this abstraction from the data applies in the same way for all the other approaches discussed above because the data is typically of this form. We further remark that if the model were to accommodate fixed periods of flat trading as the sampling frequency $m \to \infty$ then it would not be possible to consistently estimate the integrated volatility $\int_0^1 \sigma^2(t)dt$ of $p^*(t)$ because there would be fixed subperiods of the interval $[0,1]$ in which the integrated volatility is inconsistently set to zero to accord with trading flats in the observed price $p(t)$. Similar problems would arise with all other approaches in this case because integrated volatility is not identified over these subperiods.

The following result confirms that the compound model preserves the martingale property for trading prices.

Theorem 2.1 (Martingale Property): *If $p^*(t)$ follows (1.1) with $\mathrm{E}\left(\int_0^r \sigma^2(t)dt\right) < \infty$ for all $r \in [0,\infty)$, and the trading price $p(t)$ follows (1.6) with $\pi \in (0,1]$, then $\{p_{i,m}\}$ is a martingale with $E(p_{i,m}|\mathcal{F}_{i-1,m}) = p_{i-1,m}$ and the natural filtration $\mathcal{F}_{i,m} = \sigma(p_{i,m}, p_{i-1,m}, \cdots)$.*

Theorem 2.1 allows us to show that under flat trading RV consistently estimates IV, just as in the standard limit theory for empirical quadratic variation (e.g., Jacod and Shiryaev (1987); Andersen et al. (2001b)). In fact, the result is covered by standard limit theory because the maximum duration of flat trading is $h_{\bar{K}_m,m}$, which for an equi-spaced grid is of order $O_p(\log m/m)$ and therefore tends to zero. Theorem 2.3 derives the corresponding central limit theorem (CLT) for RV and Theorem 2.4 provides a feasible version of the CLT for inference about IV using an empirical quarticity estimate. For the CLT results it is convenient to assume that the discrete sampling grid is equi-spaced, so that $\{t_{i,m} = \frac{i}{m} : i = 0, ..., m\}$. This requirement fits in with earlier conditions used in Barndorff-Nielsen and Shephard (2002) on RV limit theory without flat trading, and is relaxed in Section 3.

Theorem 2.2 (Consistency): *If $\pi \in (0,1]$* and if, for some $\delta > 0$, $h_{m^\delta,m} \to 0$ *as $m \to \infty$*, then *$RV^{(m)}(p) \xrightarrow{p} IV$.*

Theorem 2.3 (Infeasible CLT): *Assume the observation grid is equispaced with $\{t_{i,m} = \frac{i}{m} : i = 0, ..., m\}$. If $\pi \in (0,1]$, then as $m \to \infty$*

$$\sqrt{m}\left[RV^{(m)}(p) - IV\right] \xrightarrow{d} \mathcal{MN}\left(0, \frac{4-2\pi}{\pi}\int_0^1 \sigma^4(t)dt\right). \quad (1.7)$$

Table 2: Ratio of asymptotic variance with flats to that without flat

π	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\frac{2-\pi}{\pi}$	19.00	9.00	5.67	4.00	3.00	2.33	1.86	1.50	1.22	1

When $\pi = 0.5$, some 50% of the data involves flat trading and the asymptotic variance in (1.7) is three times as large as when $\pi = 1$. This magnitude seems to be in line with what has been documented empirically in Hansen and Lunde (2006, p. 137). Table 2 shows the ratio of the asymptotic variance to the case where there is no flat trading for various values of π and Figure 2 plots this nonlinear relationship. As π becomes small, the ratio blows up rapidly.

Result (1.7) holds even when flats are removed from the sample. This is because the empirical quadratic variation is unaffected by the presence of flat trading periods. Hence removing flat prices from data does not reduce the asymptotic variance or change the limit theory. In effect, the limit result shows that, when trading which does not reflect the true efficient price occurs, the asymptotic variance of the RV estimate increases proportionately. That is, when there is flat price trading there is less information about the efficient price $p^*(t)$, and the asymptotic theory reflects this reduction in information by an inflation of the variance. In effect, there is a reduced sample size due to flat trading.

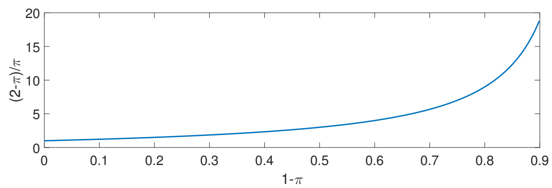

Figure 2: Ratio of the asymptotic variance under flat trading to that with no flat trading ($\pi = 1$).

To use (1.7) in practice the asymptotic variance must be estimated, which involves estimating the integrated quarticity functional $\int_0^1 \sigma^4(t)dt$. Following Barndorff-Nielsen and Shephard (2002), integrated quarticity can be estimated consistently and used in a feasible CLT that is suitable for inference about IV.

Lemma 2.4: *Under the conditions of Theorem 2.3, as $m \to \infty$*

$$\frac{\pi}{6-3\pi} m \sum_{i=1}^{m} [p_{i,m} - p_{i-1,m}]^4 \xrightarrow{p} \int_0^1 \sigma^4(t)dt. \tag{1.8}$$

When $\pi = 1$, $\frac{\pi}{6-3\pi} = \frac{1}{3}$ and result (1.8) is identical to that of Barndorff-Nielsen and Shephard (2002), as expected.

Theorem 2.5 (Feasible CLT): *Under the conditions of Theorem 2.3, as $m \to \infty$*

$$\sqrt{\frac{3}{2}} \frac{\left(RV^{(m)}(p) - IV\right)}{\sqrt{\sum_{i=1}^{m} [p(t_{i,m}) - p(t_{i-1,m})]^4}} \xrightarrow{d} \mathcal{N}(0,1). \tag{1.9}$$

Interestingly, the standardization in the feasible CLT (1.9) does not depend on π and the feasible CLT result is therefore the same as that given in Barndorff-Nielsen and Shephard (2002) for the case where there is no flat trading ($\pi = 1$). In effect, the quantity involving π appears as a factor $\frac{2-\pi}{\pi}$ in the asymptotic variance and the estimated quarticity functional and is therefore scaled out in the feasible CLT. Nonetheless, the effects of flat trading are implicitly embodied in the feasible CLT since they are carried in the empirical measure $\sum_{i=1}^{m} [p(t_{i,m}) - p(t_{i-1,m})]^4$, which is correspondingly reduced by periods of flat pricing. Thus, the asymptotic inferential apparatus of Barndorff-Nielsen and Shephard (2002) continues to hold under the present model where flat trading is manifest.

3. Stopping Time Models

In model (1.6) it is assumed that $p^*(t)$ is intermittently observed according to the realizations of a Bernoulli sequence. More generally, we may assume that p^* is generated by (1.1) and observed at random times $\{\tau_j : j = 0, ..., J_m\}$ determined according to a scheme in which the increments take the form

$$\tau_j - \tau_{j-1} = \frac{D_j}{m}, \quad \tau_0 = 0, \tag{1.10}$$

where D_j is a strictly stationary and ergodic sequence of nonnegative random variables for which $\mathrm{E}D_j = \mu_D > 0$ and $\mathrm{E}D_j^2 = \omega_D^2$. The sequence D_j measures the duration between observations (in units of the interval m^{-1}) and may be (partly) dependent on past prices. In the model (1.6), the integer process $D_j = 1 + L_j \geq 1$ is determined by the outcome of a sequence of independent Bernoulli draws ξ_i at each of the points $t_i = \frac{i}{m}$ of the original grid, those draws representing whether the efficient price process p^* is observed ($\xi_i = 1$) or not ($\xi_i = 0$) at this point. The integer L_j is then the number of successive zero draws in the sequence prior to τ_j. The terminal quantity J_m is such that $p^*(\tau_{J_m})$ is the last p^* observed on the grid $\{p^*(t_{i,m}) : i = 1, ..., m\}$.[2]

In order to develop an asymptotic theory for this more general case, we need more specific conditions on the sequence D_j in (1.10). We also want to consider the more general case where the duration sequence may form an array and depend on m, in which case we use the notation $\{D_{m,j}\}$ in place of $\{D_j\}$.

Assumption D1

(i) *$\{D_j\}_{j\geq 1}$ is a strictly stationary and ergodic sequence of nonnegative random variables with finite mean $\mathrm{E}D_j = \mu_D > 0$ and second moment $\mathrm{E}D_j^2 = \omega_D^2$. Partial sums of the centred values $D_j - \mu_D$ satisfy the functional law*

$$m^{-1/2} \sum_{j\leq [m\cdot]} \{D_j - \mu_D\} \Rightarrow V(\cdot)$$

for some Brownian motion V with variance Ω_D.

(ii) $\max_{j\leq m} D_j = o_p(\sqrt{m})$ *as $m \to \infty$.*

Assumption D2

(i) *$\{D_{m,j}\}_{j\geq 1}$ is an array of nonnegative random variables whose conditional first and second moments satisfy*

$$\mathrm{E}\left(D_{m,[ms]} | \mathcal{F}_{\tau_{[ms]-1}}\right) \to_p \mu_D(s), \quad \mathrm{E}\left(D^2_{m,[ms]} | \mathcal{F}_{\tau_{[ms]-1}}\right) \to_p \omega_D^2(s), \tag{1.11}$$

as $m \to \infty$, where $\mu_D(s)$ and $\omega_D^2(s)$ are nonnegative càdlàg processes for $s \in [0, 1]$. Partial sums of the centred values $D_{m,j} - \mathrm{E}\left(D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right)$ satisfy the functional law

$$m^{-1/2} \sum_{j\leq [m\cdot]} \left\{D_{m,j} - \mathrm{E}\left(D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right)\right\} \Rightarrow V(\cdot)$$

for some Brownian motion V with variance Ω_D.

[2] Concerns about how to calculate RV based on business time sampling and transaction time sampling have received a great deal of attention in the RV literature; see Hansen and Lunde (2006); Oomen (2006). In particular, business time sampling relates to stopping time approaches and the connection is achieved via the Dubins-Schwarz Theorem. See Yu and Phillips (2001) for an application of the Dubins-Schwarz Theorem to estimate continuous-time models.

(ii) $\max_{j\leq m} D_{m,j} = o_p\left(\sqrt{m}\right)$ *as* $m \to \infty$.

D1(i) involves standard stationarity, moment and partial sum functional law conditions on the sequence D_j. In D2(i) the convergence of the conditional moments in (1.11) is satisfied for certain autoregressive conditional duration (ACD) models, as we discuss below. Condition (ii) involves the relative stability of the maxima of a stationary sequence and is quite weak. For instance, for Gaussian sequences when the m'th autocovariance decays according to the rate $o\left(m^{-1}\right)$ then it is known (Berman (1962)) that $\max_{j\leq m} D_{m,j} = O_p\left(\sqrt{\log m}\right)$. Primitive conditions for (ii) are also available in the literature (e.g. Naveau (2003)).

Theorem 3.1 *Suppose* p^* *is observed at random times* τ_j *determined as in (1.10) with* D_j *satisfying Assumption D1. Then as* $m \to \infty$

$$\sum_{j=1}^{J_m} \left[p^*\left(\tau_j\right) - p^*\left(\tau_{j-1}\right)\right]^2 \to_p \int_0^1 \sigma^2(t)dt, \tag{1.12}$$

and

$$\sqrt{m}\left\{\sum_{j=1}^{J_m} \left[p^*\left(\tau_j\right) - p^*\left(\tau_{j-1}\right)\right]^2 - \int_0^1 \sigma^2(t)dt\right\} \Rightarrow \mathcal{MN}\left(0, 2\frac{\omega_D^2}{\mu_D}\int_0^1 \sigma^4(t)dt\right) \tag{1.13}$$

If the D_j *satisfy Assumption D2, then (1.12) continues to hold as* $m \to \infty$ *and (1.13) is replaced by*

$$\sqrt{m}\left\{\sum_{j=1}^{J_m} \left[p^*\left(\tau_j\right) - p^*\left(\tau_{j-1}\right)\right]^2 - \int_0^1 \sigma^2(t)dt\right\} \Rightarrow \mathcal{MN}\left(0, 2\int_0^1 \sigma^4(t)\frac{\omega_D^2\left(t\right)}{\mu_D\left(t\right)}dt\right) \tag{1.14}$$

The corresponding self normalized quantity has the following limit under D2 as $m \to \infty$

$$\sqrt{\frac{3}{2}}\frac{\sum_{j=1}^{J_m} \left[p^*\left(\tau_j\right) - p^*\left(\tau_{j-1}\right)\right]^2 - \int_0^1 \sigma^2(t)dt}{\left\{\sum_{j=1}^{J_m} \left[p^*\left(\tau_j\right) - p^*\left(\tau_{j-1}\right)\right]^4\right\}^{1/2}} \Rightarrow \mathcal{N}\left(0,1\right). \tag{1.15}$$

Example 1: Mykland and Zhang (2006) To illustrate, suppose the sequence D_j is non random and the increments $\tau_j - \tau_{j-1} = D_j/m$ are such that

$$m^{-1}\sum_{j=1}^{m} D_j^2 \to \omega_D^2, \quad m^{-1}\sum_{j=1}^{m} D_j \to \mu_D,$$

and

$$m^{-1}\sum_{j=1}^{J_m} D_j = 1 + O\left(\frac{\log m}{m}\right),$$

so that there are J_m increments over the interval $[0,1]$, as earlier. This type of intermittent deterministic sampling is covered in recent work by Mykland and Zhang (2006)[3]. Set $\Delta\tau_j =$

[3] Mykland and Zhang (2006) assume the sampling points τ_j to be deterministic but note later in their paper that the scheme covers the case where the sampling points are random but independent of the observed process. The stopping time scheme (1.10) is random and allows for dependence on past prices.

$\tau_j - \tau_{j-1}$ and, using the notation of Mykland and Zhang (2006) for the average interval length, we find that

$$\overline{\Delta\tau}^{(m)} = \frac{\sum_{j=1}^{J_m} D_j}{mJ_m} = \frac{\mu_D}{m} + o\left(1\right),$$

and

$$\begin{aligned} m^{-1} \sum_{j:\tau_j \le t} D_j^2 &= m^{-1} \sum_j D_j^2 1_{\left\{m^{-1}\sum_{i=1}^j D_j \le t\right\}} \\ &= m^{-1} \sum_j D_j^2 1_{\left\{m^{-1} j \mathrm{E} D_i \le t\right\}} + o\left(1\right) \\ &= m^{-1} \sum_{j \le mt/\mu_D} D_j^2 + o\left(1\right) \\ &= m^{-1} \mathrm{E} D_i^2 \frac{mt}{\mu_D} + o\left(1\right) \to \frac{\omega_D^2}{\mu_D} t. \end{aligned}$$

Then, again in the notation of Mykland and Zhang (2006), we have

$$H_{(m)}\left(t\right) = \frac{\sum_{j:\tau_j \le t} \left(\Delta\tau_j\right)^2}{\overline{\Delta\tau}^{(m)}} = \frac{m^{-2}\sum_{j:\tau_j\le t} D_j^2}{\frac{\mu_D}{m} + o\left(1\right)} \to \frac{\omega_D^2}{\mu_D^2} t := H(t).$$

Under these conditions of deterministic sampling and noting that $H'\left(t\right) = \frac{\omega_D^2}{\mu_D^2}$, Proposition 1 of Mykland and Zhang (2006) gives the following limit theory

$$\left(\overline{\Delta\tau}^{(m)}\right)^{-1/2} \left\{ \sum_{j=1}^{J_m} [p^*\left(\tau_j\right) - p^*\left(\tau_{j-1}\right)]^2 - \int_0^1 \sigma^2(t)dt \right\} \Rightarrow \mathcal{MN}\left(0, 2\frac{\omega_D^2}{\mu_D^2} \int_0^1 \sigma^4(t)dt\right),$$

which, since $\overline{\Delta\tau}^{(m)} = \mu_K/m + o\left(1\right)$, conforms to result (1.13) above.

In their development Mykland and Zhang (2006) impose the following additional condition (Assumption A(i) in their paper) on the intermittent sampling sequence

$$\frac{\delta^m}{\overline{\Delta\tau}^{(m)}} = O\left(1\right), \quad \text{where } \delta^m = \max_{j\le m} \frac{D_j}{m}. \tag{1.16}$$

Allowing for random intermittent sampling with duration D_j/m between observations, condition (1.16) is equivalent to

$$\frac{\max_{j\le m} D_j}{\mathrm{E} D_j} = O_p\left(1\right),$$

which does not hold in many interesting cases. For example, in the model (1.6) where sampling is determined by a simple Bernoulli scheme, we have

$$\frac{\max_{j\le m} D_j}{\mathrm{E} D_j} = O_p\left(\log m\right).$$

Thus, Assumption A(i) in Mykland and Zhang (2006) holds only for deterministic sequences or bounded random sequences $\{D_j\}$ and excludes the model (1.6), although as shown above in Theorem 3.1 the limit theory does extend to this case so condition (1.16) is, in fact, not necessary.

Observe that for the model (1.6) we have[4] $\mu_D = \mathrm{E}(1+L_i) = \frac{1}{\pi}$ and $\omega_D^2 = \mathrm{E}(1+L_i)^2 = \frac{2-\pi}{\pi^2}$, so that

$$2\frac{\omega_D^2}{\mu_D} = \frac{4-2\pi}{\pi}, \tag{1.17}$$

corresponding to the scale factor in the asymptotic variance in (1.7).

Example 2: Jacod (2018) For a second illustration, we note that the sampling scheme (1.10) fits into the same stopping time framework as that used in Jacod (2018). The associated empirical measures that appear in Jacod's paper can be worked out as follows using Jacod' notation:

$$\begin{aligned}
\mu_m[0,t] \quad &: \quad = \frac{1}{m}\sum_j \mathbf{1}_{\{\tau_j < t\}} = \frac{1}{m}\sum_j \mathbf{1}_{\left\{m^{-1}\sum_{\ell \le j} D_\ell < t\right\}} \\
&= \frac{1}{m}\sum_j \mathbf{1}_{\left\{m^{-1}\sum_{\ell\le j}\mathrm{E}D_\ell + O_p\left(m^{-1/2}\right) < t\right\}} \\
&= \frac{1}{m}\sum_{j \le [mt]/\mu_D} 1 + o_p(1) \to_p t/\mu_D := \mu[0,t], \quad \text{say; and} \qquad (1.18)\\
\mu_m^*[0,t] \quad &: \quad = \frac{\sqrt{\mu_D}}{m}\sum_{j\le[mt]} \mathbf{1}\{\tau_j < t\} \to_p t/\sqrt{\mu_D} := \mu^*[0,t], \quad \text{say.}
\end{aligned}$$

Note that $(\mu^*[0,t]/t)^2 = (\mu[0,t]/t)$, thereby satisfying result (3.2) of Jacod (2018). More generally under D2, the limit measure[5] is $\mu[0,t] = \int_0^t \mu_D(s)^{-1}ds$, so that $\mu(dt) = dt/\mu_D(t)$.

The limit theory in Section 6 of Jacod (2018) applies in the current context but to standardized price increments. In particular, equations (3.8), (6.5) and Theorem 6.1 of Jacod (2018) lead to the following limit theory for the empirical quadratic variation of the standardized increments $\{p^*(\tau_j) - p^*(\tau_{j-1})\}/(\tau_j - \tau_{j-1})^{1/2}$

$$\frac{1}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j) - p^*(\tau_{j-1})\}^2}{\tau_j - \tau_{j-1}} \to_p \int_0^1 \sigma^2(t)\mu(dt). \tag{1.19}$$

Correspondingly,

$$\frac{\mu_D}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j) - p^*(\tau_{j-1})\}^2}{\tau_j - \tau_{j-1}} \to_p \int_0^1 \sigma^2(t)dt.$$

In the case of D1, the limit is $\frac{1}{\mu_D}\int_0^1 \sigma^2(t)dt$ and under D2 the limit is $\int_0^1 \sigma^2(t)\frac{dt}{\mu_D(t)}$. A proof is

[4]Note that the values $L_i = 0,1,2,\ldots$, correspond to realizations $\xi_i = 1$, $\{\xi_i = 1, \xi_{i-1} = 0\}$, $\{\xi_i = 1, \xi_{i-1} = 0, \xi_{i-2} = 0\}$ with respective probabilities π, $\pi(1-\pi)$, $\pi(1-\pi)^2$, ...

[5]In this case we have $\left\{m^{-1}\sum_{\ell\le j} D_{m,\ell} < t\right\}$, which under D2 is asymptotically equivalent to $\left\{\int_0^{j/m}\mu_D(s)\,ds < t\right\}$. The measure $\mu[0,t]$ is then given by $\mu[0,t] = r(t)$ where $\int_0^r \mu_D(s)\,ds = t$ so that $\mu_D(r)\,dr = dt$.

given in the Online Supplement, where it is also shown that

$$\sqrt{m}\left\{\frac{1}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j)-p^*(\tau_{j-1})\}^2}{m(\tau_j-\tau_{j-1})}-\int_0^1\sigma^2(t)\mu(dt)\right\}\Rightarrow\mathcal{MN}\left(0,\int_0^1\sigma^4(t)\frac{\omega_D^2(t)+\mu_D(t)^2}{\mu_D(t)^2}\mu(dt)\right), \tag{1.20}$$

which after restandardization in the case of D1 becomes

$$\sqrt{m}\left\{\frac{\mu_D}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j)-p^*(\tau_{j-1})\}^2}{\tau_j-\tau_{j-1}}-\int_0^1\sigma^2(t)dt\right\}\Rightarrow\mathcal{MN}\left(0,\frac{\omega_D^2+\mu_D{}^2}{\mu_D}\int_0^1\sigma^4(t)dt\right) \tag{1.21}$$

Result (1.19) is compatible with (1.12). The different weight factors reflect the use of standardized price increments in the realized variance. In this case, the empirical quadratic variation converges to a weighted version of integrated volatility where the weight function depends on the mean duration process of flats $\mu_D(t)$.

The central limit theory (1.20) is similar to (1.14) but has a different weighting function in the conditional variance. As shown in the Appendix, the limit distribution stems from a martingale term that involves both the numerator and the denominator of the standardized squared price increments. Interestingly, because of the standardization of the price increments the weight function in the conditional variance of (1.21) is

$$\frac{\omega_D^2(t)+\mu_D(t)^2}{\mu_D(t)}=\frac{2\omega_D^2(t)}{\mu_D(t)}-\frac{\omega_D^2(t)-\mu_D(t)^2}{\mu_D(t)}<\frac{2\omega_D^2(t)}{\mu_D(t)},$$

showing that the variance in the limit distribution (1.20) is less than that of (1.14).

In the special case where D1 applies, we may restandardize by μ_D, as shown in (1.21), to produce a consistent estimator of IV. In the case of model (1.6) we have the decomposition

$$\frac{\omega_D^2+\mu_D^2}{\mu_D}=\frac{4-2\pi}{\pi}-\frac{1-\pi}{\pi}=\frac{3-\pi}{\pi}, \tag{1.22}$$

showing the variance inflation from flat trading on the estimation of IV using standardized squared increments is less than it is in the case of squared increments in the model (1.6).

Of course, in (1.21) the IV estimator depends on scaling by μ_D. Further, since mean duration can be consistently estimated by $\hat{\mu}_D=J_m^{-1}\sum_{j=1}^{J_m}D_j$, we may construct the feasible estimator

$$\frac{\hat{\mu}_D}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j)-p^*(\tau_{j-1})\}^2}{\tau_j-\tau_{j-1}}\to_p\int_0^1\sigma^2(t)dt. \tag{1.23}$$

From Assumption D2(i), we have $\sqrt{m}(\hat{\mu}_D-\mu_D)\Rightarrow\mathcal{N}(0,\Omega_D)$ where the variance depends on the sampling properties of the array D_{mj} including autocorrelation and heterogeneity. The limit distribution of the feasible estimator (1.23) therefore involves additional terms arising from the use of the fitted mean $\hat{\mu}_D$.

Finally, as shown in the Online Supplement, under Assumption D2 we have the following limit theory for empirical quarticity in the case of standardized increments

$$\frac{1}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j)-p^*(\tau_{j-1})\}^4}{(\tau_j-\tau_{j-1})^2}\to_p\ 3\int_0^1\sigma^4(t)\frac{1}{\mu_D(t)}dt, \tag{1.24}$$

which reduces in the case Assumption D1 to

$$\frac{\mu_D}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j)-p^*(\tau_{j-1})\}^4}{(\tau_j-\tau_{j-1})^2}\to_p\ 3\int_0^1\sigma^4(t)dt.$$

Example 3: Relation to the ACD Model Engle and Russell (1998) introduced autoregressive conditional duration (ACD) formulations to model the duration between transaction times. In such models the conditional expectation $\mu_{D_{m,j}}=\mathrm{E}\{(D_{m,j}-D_{m,j-1})\,|\mathcal{F}_{\tau_{j-1}}\}$ of the intra trade time is a measurable function of past durations. In a GARCH(1,1) framework, we have

$$\mu_{D_{m,j}}=\delta_m+\alpha_m D_{m,j-1}+\beta_m\mu_{D_{m,j-1}}, \tag{1.25}$$

where we allow the coefficients $(\omega_m,\alpha_m,\beta_m)$ to depend on m because of the array structure of $D_{m,j}$. Models of this type have been studied by Ghysels and Jasiak (1998) and Engle (2000) among many others.

Nelson (1990) showed that under certain conditions on the coefficients in (1.25) GARCH models of the form (1.25) satisfy in the limit as $m\to\infty$ a diffusion equation

$$d\mu_D(s)=(\delta-\theta\mu_D(s))\,ds+\alpha dW(s)\,, \tag{1.26}$$

where W is a standard Brownian motion. In particular, in an appropriately defined and enlarged probability space and for the parameterization

$$\delta_m=\frac{\delta}{m},\alpha_m=\frac{\alpha}{\sqrt{m}},\quad\text{and } 1-\alpha_m-\beta_m=\frac{\theta}{m}$$

we have the convergence $\mu_{D_{[ms]}}\to_p\mu_D(s)\,,$ where the limit is a stochastic process satisfying (1.26). This type of limit theory has also been studied by Corradi (2000) and Boswijk (2001, 2005). Thus, the stopping time model with duration array $D_{m,j}$ satisfying condition D2 subsumes certain ACD specifications and condition (1.11) may be regarded as version of the associated convergence of the discrete conditional expectation to a continuous process.

4. Microstructure Noise Effects

Microstructure noise in the form of additive errors may be incorporated into the framework (1.6) by setting

$$p_{i,m}=\begin{cases}p^*_{i,m}+u_{i,m} & \text{if }\xi_i=1\\ p_{i-1,m} & \text{if }\xi_i=0\end{cases}, \tag{1.27}$$

so that when prices do change a noise component $u_{i,m}$ perturbs the observed price. Similarly, adding noise to the more general formulation in Section 3 leads to a process p that is observed at stopping times τ_j according to the scheme

$$\begin{aligned} p(\tau_j) &= p^*(\tau_j) + u_{\tau_j}, \\ \tau_j - \tau_{j-1} &= \frac{D_{m,j}}{m}, \quad \tau_0 = 0, \end{aligned}$$

where the durations $D_{m,j}$ satisfy Assumption D2. Various assumptions about the noise process are possible, but for the purposes here it will be sufficient to assume that u_{τ_j} is *iid* with zero mean, variance σ_u^2 and finite fourth moment.

The effects of this noise may be taken into account using methods such as the use of different time scales (Zhang et al. (2005)) or kernel smoothing techniques (Hansen and Lunde (2006); Aït-Sahalia et al. (2011); Barndorff-Nielsen et al. (2008)). We will illustrate by considering the two time scale approach developed in Zhang et al. (2005). Our approach differs from the existing literature in that all flats are retained whereas flats are removed in other analysis. The motivation arises from the desire to utilize all available high frequency data as suggested in Aït-Sahalia et al. (2005).

Let the empirical quadratic variation of a process Y_t on the full grid $\mathcal{G}_m = \{\tau_j : j = 0, ...J_m\}$ be denoted

$$[Y]_t^{\mathcal{G}_m} = \sum_{\tau_j \in \mathcal{G}_m, \tau_j \le t} \left(Y_{\tau_j} - Y_{\tau_{j-1}}\right)^2, \text{ with } [Y]^{\mathcal{G}_m} = [Y]_1^{\mathcal{G}_m},$$

with similar definitions for the empirical quadratic covariation between Y_t and another process X_t, i.e.,

$$[Y, X]_t^{\mathcal{G}_m} = \sum_{\tau_i \in \mathcal{G}_m, \tau_j \le t} \left(Y_{\tau_j} - Y_{\tau_{j-1}}\right)\left(X_{\tau_j} - X_{\tau_{j-1}}\right).$$

Then

$$[p]^{\mathcal{G}_m} = [p^*]^{\mathcal{G}_m} + 2\left[p^*, u\right]^{\mathcal{G}_m} + [u]^{\mathcal{G}_m} = [p^*]^{\mathcal{G}_m} + 2J_m\sigma_u^2 + O_p\left(J_m^{1/2}\right). \tag{1.28}$$

Following Zhang et al. (2005), we construct a two time scale estimator of IV using $[p]^{\mathcal{G}_m}$ and an average estimator using K nonoverlapping grids

$$\mathcal{G}_m^k = \{\tau_{k-1}, \tau_{k-1+K}, \tau_{k-1+2K}, ..., \tau_{k-1+m_k K}\}$$

involving every K'th observation in $\mathcal{G}_m$ for $k = 1, ..., K$, and where m_k is the integer for which $\tau_{k-1+m_k K}$ is the last element in $\mathcal{G}_m^k$. The average estimator has the explicit form

$$[p]^{(avg)} = K^{-1}\sum_{k=1}^{K} [p]^{\mathcal{G}_m^k}, \tag{1.29}$$

where

$$[p]^{\mathcal{G}_m^k} = \sum_{s=1}^{m_j} [p(\tau_{k-1+sK}) - p\left(\tau_{k-1+(s-1)K}\right)]^2, \quad k = 1, ..., K. \tag{1.30}$$

Define

$$\bar{m} = K^{-1}\sum_{k=1}^{K} m_k = \frac{J_m - K + 1}{K}. \tag{1.31}$$

As in Zhang et al. (2005), we require $K \to \infty$ and $m/K \to \infty$ as $m \to \infty$. It follows that

$K/J_m \to 0$. We also need to modify D(ii) to take account of the time scale induced by the new grids $\mathcal{G}_m^k$.

Assumption D2

(ii)′ $\max_{j\leq m} D_{m,j} = o_p\left(m^{1/3}\right)$.

The two time scale estimator of Zhang et al. (2005) is

$$\widehat{[p^*]} = [p]^{(avg)} - \frac{\bar{m}}{J_m}[p]^{\mathcal{G}_m} \tag{1.32}$$

and uses the two different time scales in computation, one based on the subgrids $\mathcal{G}_m^k$ and the other on the full grid $\mathcal{G}_m$. The limit theory for $\widehat{[p^*]}$ follows.

Theorem 4.1 *Suppose p^* is observed at random times τ_j determined as in (1.10) with $D_{m,j}$ satisfying Assumption D2 (i) and (ii)′. Let $K = cm^{2/3}$ for some constant c. Then as $m \to \infty$*

$$m^{1/6}\left\{\widehat{[p^*]} - \int_0^1 \sigma^2(t)dt\right\} \Rightarrow \left\{\frac{8}{c^2}\sigma_u^4 + c\eta^2\right\}^{1/2} \mathcal{N}(0,1), \tag{1.33}$$

where

$$\eta^2 = \frac{4\left\{\int_0^1 (1-s)^2 \mu_D(1-s)\,ds\right\}}{\int_0^1 \mu_D(s)\,ds}\int_0^1 \sigma^4(t)\,dt. \tag{1.34}$$

This result corresponds with the limit theory in Zhang et al. (2005) and the only difference occurs in the definition of η^2, which directly involves the mean duration effects arising from the process $\mu_D(s)$. In particular, the effect of flat trading (or random durations in intermittent sampling) on the limit theory of the two time scale estimator of Zhang et al. (2005) arises through the conditional mean functional of duration, $\mu_D(s)$, in the limiting variance. Interestingly, as compared with (1.14), when there is no microstructure noise, the conditional second moment of duration, $\omega_D^2(s)$, does not affect the limit distribution.

When $\mu_D(s) = \mu_D$ *a.s.* so that conditional mean duration is constant throughout the period of observation, (1.34) reduces to

$$\eta^2 = 4\int_0^1 (1-s)^2\,ds \int_0^1 \sigma^4(t)\,dt = \frac{4}{3}\int_0^1 \sigma^4(t)\,dt, \tag{1.35}$$

corresponding to equation (50) in Zhang et al. (2005), as obtained for the case of equispaced observations. This shows that the limit theory of Zhang et al. (2005) remains correct in the presence of flat trading of the Bernoulli type.

The random coefficient in (1.34) depends on the average conditional mean duration which may be estimated by

$$\overline{D} = J_m^{-1}\sum_{j=1}^{J_m} D_{m,j} = J_m^{-1}\sum_{j=1}^{J_m} \mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right) + o_p(1) \to_p \int_0^1 \mu_D(s)\,ds.$$

In view of Assumption D2 we then have the consistent estimate

$$\hat{h}_m = \frac{4}{\overline{D}J_m}\sum_{j=1}^{J_m}\left(1-\frac{j}{J_m}\right)^2 D_{m,J_m-j} \to_p \frac{4\left\{\int_0^1 (1-s)^2 \mu_D(1-s)\, ds\right\}}{\int_0^1 \mu_D(s)\, ds}. \tag{1.36}$$

Some other aspects of the estimation of η^2 and the use of the result in empirical application are discussed in Zhang et al. (2005).

In a similar way, it is possible to examine the effects of flat trading on other consistent estimators of integrated volatility in the presence of microstructure noise. In particular, the multiple time scale estimator of Zhang (2006) and the realized kernel method of Barndorff-Nielsen et al. (2008) may be treated similarly and have the advantage of the faster rate of convergence $m^{1/4}$.

5. Simulations and Empirical Applications

The simulation study in this section considers two models – the Brownian motion model and Heston's stochastic volatility model. Using the Brownian motion model we check the accuracy of the CLT (1.7) and the CLT (1.21). Using the stochastic volatility model we check the accuracy of the CLT (1.9). Data are simulated over a day so that $t_{0,m} = 0$ and $t_{m,m} = 1$. A day is assumed to have 6.5 hours and 23,400 seconds. In the experimental design we set $m \in \{39, 78, 130, 195, 390\}$, which values correspond respectively to frequencies of 10, 5, 3, 2, 1 minutes. The empirical application assesses the practical impact of flat trading on the CLT (1.33) using the two datasets that were discussed in the Introduction and three other datasets.

5.1 The Brownian motion model

This subsection reports simulations from a simple Brownian motion model where volatility is a known constant ($\sigma^2(t) = 1$) so that

$$dp^*(t) = dB(t). \tag{1.37}$$

This formulation allows us to assess the accuracy of CLT (1.7) $\int_0^1 \sigma^4(t)dt = 1$ and then the asymptotic variance in (1.7) is simply $\frac{4-2\pi}{\pi}$.

Table 3 shows both the asymptotic and finite sample simulated variances of the statistic $\sqrt{m}\left[RV^{(m)}(p) - IV\right]$ based on 5,000 replications for various combinations of π and m. The asymptotic formula is clearly very accurate except for very small values of π. The effect of flat trading on the asymptotic variance is dramatic, producing a three fold increase in variance when $\pi = 0.5$.

Table 3: Asymptotic and finite sample variances

π	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Asymp. variance $\frac{4-2\pi}{\pi}$	38	18	11.33	8	6	4.67	3.71	3	2.44	2
Asymp variance $\pi = 1$	2	2	2	2	2	2	2	2	2	2
Variance ($m = 39$)	24.31	14.84	10.18	7.61	5.82	4.56	3.74	3.06	2.51	2.04
Variance ($m = 78$)	30.38	16.56	10.88	7.71	5.79	4.61	3.70	3.02	2.49	2.02
Variance ($m = 130$)	33.47	17.15	11.16	7.85	5.88	4.66	3.71	3.00	2.45	2.02
Variance ($m = 195$)	34.39	17.06	10.87	7.74	5.95	4.69	3.74	3.04	2.45	2.02
Variance ($m = 390$)	36.32	17.41	11.03	7.92	6.03	4.74	3.76	3.04	2.47	2.03

We also examine the accuracy of the CLT (1.21) for the restandardized quantity. With the Bernoulli process model, the asymptotic variance of the restandardized quantity is

$$\int_0^1 \sigma^4(t) \frac{\omega_D^2(t) + \mu_D(t)^2}{\mu_D(t)} dt = \frac{\omega_D^2 + \mu_D(t)^2}{\mu_D} = \frac{3-\pi}{\mu}.$$

Table 4 shows both the asymptotic and finite sample simulated variances of the statistic

$$\sqrt{m} \left\{ \mu_D \sum_{j=1}^{J_m} \frac{\{p^*(\tau_j) - p^*(\tau_{j-1})\}^2}{\tau_j - \tau_{j-1}} - \int_0^1 \sigma^2(t) dt \right\}$$

with $\mu_D = \frac{1}{\pi}$, based on 5,000 replications for various values of π and $m = 39$. The asymptotic formula is clearly very accurate in all cases and even when m is as small as 39. Although not reported here, we have also found smaller bias in the restandardized quantity.

Table 4: Asymptotic and finite sample variances

π	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Asymp. variance $\frac{3-\pi}{\pi}$	29	14	9	6.5	5	4	3.2857	2.75	2.44	2
Variance ($m = 39$)	28.7	13.97	8.90	6.54	5.04	3.96	3.24	2.66	2.25	1.96

5.2 A stochastic volatility model

In this subsection, price data is simulated from Heston's stochastic volatility model with volatility following a square root model (Heston (1993)):

$$\begin{cases} dp^*(t) &= \sigma(t)dB_1(t), \\ d\sigma^2(t) &= \kappa(\mu - \sigma^2(t))dt + \eta\sigma(t)dB_2(t). \end{cases} \tag{1.38}$$

Feller (1951) showed that the density of $\sigma^2(t+h)$ conditional on $\sigma^2(t)$ is $ce^{-u-v}(v/u)^{q/2}I_q(2(uv)^{1/2})$ and the marginal density of $\sigma^2(t)$ is $w_1^{w_2}\sigma^{2(w_2-1)}e^{-w_1\sigma^2}/\Gamma(w_2)$, where $c = 2\kappa/(\eta^2(1-e^{-\kappa h}))$, $u = c\sigma^2(t)e^{-\kappa h}$, $v = c\sigma^2(t+h)$, $q = 2\kappa\mu/\eta^2 - 1$, $w_1 = 2\kappa/\eta^2$, $w_2 = 2\kappa\mu/\eta^2$, and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The conditional density together with the marginal density are used for data simulation. The parameters in the model are set at $\kappa = 0.01, \mu = 1$ and $\eta = 0.05$.

Table 5: Asymptotic and finite sample variances

π	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Asymptotic variance	1	1	1	1	1	1	1	1	1
Variance ($m = 39$)	15.06	5.76	4.61	2.15	1.90	1.62	1.45	1.21	1.16
Variance ($m = 78$)	5.85	2.71	2.05	1.54	1.50	1.35	1.26	1.16	1.11
Variance ($m = 130$)	3.17	1.95	1.54	1.31	1.29	1.28	1.21	1.17	1.11
Variance ($m = 195$)	1.89	1.52	1.27	1.20	1.20	1.20	1.14	1.12	1.08
Variance ($m = 390$)	1.37	1.35	1.21	1.20	1.15	1.16	1.14	1.12	1.08

The aim of the experiment is to assess the accuracy of the empirical quarticity formula in the feasible CLT (1.9). Table 5 gives the Monte Carlo results. In particular, we report the variance of the standardized statistic $\sqrt{\frac{3}{2}}\frac{(RV^{(m)}(p)-IV)}{\sqrt{\sum_{i=1}^{m}[p(t_{i,m})-p(t_{i-1,m})]^4}}$ from 1,000 replications, for various combinations of π and m, shown against the asymptotic variance of unity.

Some conclusions can be drawn from the table. First, the asymptotic theory clearly works better for large π and large m. This is unsurprising because larger values of π imply fewer flat price trading periods and therefore larger effective sample sizes. Second, the asymptotic theory does eventually work well even for small π, but needs larger values of m to provide a good approximation. The main reason for these effects is that it is more difficult to estimate the integrated quarticity than the integrated volatility. This corroborates existing findings in the literature on realized volatility without flat pricing. Table 4 shows that these effects are exacerbated when there is flat trading, especially when π is small, because of the smaller effective sample size.

5.3 Empirical Evaluation

The limit theory in Section 4 under microstructure noise can be used to evaluate the empirical effect of flat trading in real data applications. Here we consider the empirical impact of flat trading using the asymptotics given in Theorem 4.1 and (1.33). Using the tick-by-tick transaction data for AA on April 2, 1997 and on April 1, 1998 that was discussed earlier, and for three other companies traded in the NYSE (3M, GE and Boeing) on April 2, 1997, we numerically compare the coefficient η^2 given in (1.34) with that of (1.35).[6] This involves a simple comparison of a consistent estimate of $4\int_0^1 (1-s)^2 \mu_D(1-s)\,ds/\int_0^1 \mu_D(s)\,ds$ with the quantity 4/3 that applies in the case of equispaced sampling and regular allocation to subgrids.

Table 6 reports the consistent estimate $\hat{h}_m = \frac{4}{\bar{D}J_m}\sum_{j=1}^{J_m}\left(1-\frac{j}{J_m}\right)^2 D_{m,J_m-j}$ of the duration parameter $4\int_0^1 (1-s)^2 \mu_D(1-s)\,ds/\int_0^1 \mu_D(s)\,ds$ together with the effective sample sizes for several stocks, including the AA data. It is interesting that while flats are a dominating feature of the tick data, the estimates $\hat{h}_m$ differ little from the factor 4/3 in all these cases, suggesting that the implied variances are very close to each other and that the limit theory of Zhang et al. (2005) should be a good approximation in practical work with similar data of this type. As observed with these data, most ticks do not involve price changes. So if the flats are removed prior to econometric analysis, a much smaller number of prices are retained, and the effective rate of convergence is accordingly reduced.

Table 6: Scale Effects in the CLT (1.33) for NYSE Stocks

Date	Stock	$\hat{h}$	Total m	Effective m
April 1, 1998	AA	1.3078	394	117
April 2, 1997	AA	1.2840	243	74
April 2, 1997	3M	1.2691	328	124
April 2, 1997	General Electric	1.4251	1222	370
April 2, 1997	Boeing	1.3295	796	225

6. Conclusion

When trading does not reflect the efficient price because of flat trading effects, the variance of the RV estimate of integrated volatility increases because we have correspondingly less information

[6]The datasets and dates are arbitrarily selected but are illustrative of heavily traded stocks.

about the efficient price than the number of observations might indicate. Of course, the same conclusion holds when the flats in the trading price are simply ignored and the previous tick method of constructing the RV estimate is employed. Furthermore, since fitted quarticity is similarly affected by flat trading, the framework suggested in Barndorff-Nielsen and Shephard (2002) for inference about RV remains valid. These conclusions are intuitively obvious in that the operationally useful data for IV estimation are simply those observations that reflect the underlying efficient price.

Both the Bernoulli and stopping time/duration models may be interpreted as embodying some noise effects – flat trades imply that past values are observed rather than the current efficient price, so that the offset from the efficient price is a form of noise. When other sources of microstructure noise are present, such as disturbances or measurement errors that occur when the observed price changes, multiple time scale or realized kernel estimators may be used, as have been recommended in the recent literature. The analysis of the two time scale estimator given here shows that these approaches to remove the effects of noise remain valid after appropriate adjustment to the limiting variance. Empirical evaluations with tick-by-tick data indicate that the effect on the limit theory of flat trading is likely to be minor in most cases.

Appendix

Proof of Theorem 2.1: The specification of $p(t)$ implies that

$$p_{i,m} = p^*_{i,m}\xi_{i,m} + p_{i-1,m}(1 - \xi_{i,m}), \tag{1.39}$$

$$p_{i,m} - p_{i-1,m} = (p^*_{i,m} - p_{i-1,m})\xi_{i,m}, \tag{1.40}$$

and

$$p^*_{i,m} - p_{i,m} = (p^*_{i,m} - p_{i-1,m})(1 - \xi_{i,m}). \tag{1.41}$$

Taking conditional expectations of both sides of equation (1.39), we have

$$\begin{aligned} E(p_{i,m}|\mathcal{F}_{i-1,m}) &= E(p^*_{i,m}|\mathcal{F}_{i-1,m})\pi + p_{i-1,m}(1-\pi) \\ &= E(p^*_{i-1,m}|\mathcal{F}_{i-1,m})\pi + p_{i-1,m}(1-\pi) \end{aligned} \tag{1.42}$$

To compute $E(p^*_{i-1,m}|\mathcal{F}_{i-1,m})$, note that if $p_{i-1,m} \neq p_{i-2,m}$, then $p_{i-1,m} = p^*_{i-1,m}$ and hence $E(p^*_{i-1,m}|\mathcal{F}_{i-1,m}) = p_{i-1,m}$. If $p_{i-1,m} = p_{i-2,m}$ but $p_{i-2,m} \neq p_{i-3,m}$, then $p_{i-2,m} = p^*_{i-2,m}$, $p^*_{i-1,m} = p^*_{i-2,m} + \varepsilon_{i-1,m}$, and $\mathcal{F}_{i-1,m} = \mathcal{F}_{i-2,m}$. Hence $E(p^*_{i-1,m}|\mathcal{F}_{i-1,m}) = E(p^*_{i-2,m} + \varepsilon_{i-1,m}|\mathcal{F}_{i-2,m}) = p_{i-2,m} = p_{i-1,m}$. Similarly, if $p_{i-1,m} = \cdots = p_{i-K,m}$ but $p_{i-K,m} \neq p_{i-K=1,m}$, then $p_{i-K,m} = p^*_{i-K,m}$, $p^*_{i-1,m} = p^*_{i-K,m} + \varepsilon_{i-K+1,m} + \cdots + \varepsilon_{i-1,m}$ and $\mathcal{F}_{i-1,m} = \cdots = \mathcal{F}_{i-K,m}$. Hence $E(p^*_{i-1,m}|\mathcal{F}_{i-1,m}) = E(p^*_{i-K,m} + \varepsilon_{i-K+1,m} + \cdots + \varepsilon_{i-1,m}|\mathcal{F}_{i-K,m}) = p_{i-K,m} = p_{i-1,m}$. In general, we have $E(p^*_{i-1,m}|\mathcal{F}_{i-1,m}) = p_{i-1,m}$,and so $E(p_{i,m}|\mathcal{F}_{i-1,m}) = p_{i-1,m}$, as required.

Proof of Theorem 2.2: Let K_i be the run time of flat trading prior to $t_{i,m}$. As discussed in the paper, the maximum run time, $\bar{K}_m$, for a sequence of identical Bernoulli draws in a sample of size m has mean $E\left(\bar{K}_m\right) = O\left(\log_{1/\pi}\left\{m\left(1-\pi\right)\right\}\right) = O\left(\frac{\log\{m(1-\pi)\}}{\log\frac{1}{\pi}}\right)$ and variance $Var(\bar{K}_m) = \frac{\pi^2}{6\log^2\left(\frac{1}{\pi}\right)}$. It follows that

$$\bar{K}_m = O_p\left(\log m\right), \tag{1.43}$$

and so each K_i is at most $O_p\left(\log m\right)$. It follows that in equi-spaced sampling when $t_{i,m} = \frac{i}{m}$, we have a maximum grid size $h_{\bar{K}_m,m} = \frac{\bar{K}_m}{m} = O_p\left(\frac{\log m}{m}\right) \to 0$ as $m \to \infty$. In the case of a general grid $\{t_{i,m}\}$, the maximum grid size is bounded as follows for large m

$$h_{\bar{K}_m,m} = \sup_i \left|t_{i,m} - t_{i-\bar{K}_m,m}\right| \leq \sup_i \left|t_{i,m} - t_{i-m^\delta,m}\right| \quad \text{for some } \delta > 0.$$

Hence, $h_{\bar{K}_m,m} \to 0$ if $h_{m^\delta,m} \to 0$ as $m \to \infty$ for some $\delta > 0$. In both cases, therefore, the grid size tends to zero and standard quadratic variation theory ensures that

$$\sum_{i=1}^{m} [p_{i,m} - p_{i-1,m}]^2 \to_p \int_0^1 \sigma^2(t)\, dt. \tag{1.44}$$

The result may be proved directly by writing the left side of (1.44) in terms of the empirical quadratic variation of the efficient price $RV^{(m)}(p^*) = \sum_{i=1}^m (p^*_{i,m} - p^*_{i-1,m})^2$ and showing that the error converges in probability to zero. The derivation is useful in later arguments so it is given here. In particular, from equation (1.40), we have

$$\begin{aligned}
\sum_{i=1}^{m} [p_{i,m} - p_{i-1,m}]^2 &= \sum_{i=1}^{m} (p^*_{i,m} - p_{i-1,m})^2 \xi^2_{i,m} \qquad (1.45)\\
&= \sum_{i=1}^{m} (p^*_{i,m} - p_{i-1,m})^2 E[\xi^2_{i,m}] + \sum_{i=1}^{m} (p^*_{i,m} - p_{i-1,m})^2 \left(\xi^2_{i,m} - E[\xi^2_{i,m}]\right)\\
&= \pi \sum_{i=1}^{m} (p^*_{i,m} - p_{i-1,m})^2 + \sum_{i=1}^{m} (p^*_{i,m} - p_{i-1,m})^2 \left(\xi^2_{i,m} - E[\xi^2_{i,m}]\right).
\end{aligned}$$

Write the sum $\sum_{i=1}^m (p^*_{i,m} - p_{i-1,m})^2$ in the first term above as follows

$$\begin{aligned}
\sum_{i=1}^{m} (p^*_{i,m} - p_{i-1,m})^2 &= \sum_{i=1}^{m} (p^*_{i,m} - p^*_{i-1,m} + p^*_{i-1,m} - p_{i-1,m})^2\\
&= RV^{(m)}(p^*) + 2\sum_{i=1}^{m} (p^*_{i,m} - p^*_{i-1,m})\left(p^*_{i-1,m} - p_{i-1,m}\right) + \sum_{i=1}^{m} (p^*_{i-1,m} - p_{i-1,m})^2\\
&= RV^{(m)}(p^*) + 2\sum_{i=1}^{m} \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) + \sum_{i=1}^{m} (p^*_{i-1,m} - p_{i-2,m})^2 (1 - \xi_{i-1,m})^2\\
&= RV^{(m)}(p^*) + 2\sum_{i=1}^{m} \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) + \sum_{i=1}^{m} (p^*_{i-1,m} - p_{i-2,m})^2 (1 - \pi) \qquad (1.46)\\
&\quad + \sum_{i=1}^{m} (p^*_{i-1,m} - p_{i-2,m})^2 \left\{(1 - \xi_{i-1,m})^2 - (1 - \pi)\right\}.
\end{aligned}$$

Set $A_m = \sum_{i=1}^m (p^*_{i,m} - p_{i-1,m})^2$. So $A_{m-1} = A_m - (p^*_{m,m} - p_{m-1,m})^2$. Substituting out A_{m-1} in equation (1.46) gives

$$A_m = RV^{(m)}(p^*) + 2\sum_{i=1}^{m} \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) + A_m(1 - \pi)$$

$$-(1-\pi)(p^*_{m,m}-p_{m-1,m})^2+\sum_{i=1}^{m}(p^*_{i-1,m}-p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2-(1-\pi)\right\}.$$

Hence

$$A_m = \frac{1}{\pi}RV^{(m)}(p^*)+\frac{2}{\pi}\sum_{i=1}^{m}\varepsilon_{i,m}\left(p^*_{i-1,m}-p_{i-1,m}\right) \tag{1.47}$$
$$-\frac{1-\pi}{\pi}(p^*_{m,m}-p_{m-1,m})^2+\frac{1}{\pi}\sum_{i=1}^{m}(p^*_{i-1,m}-p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2-(1-\pi)\right\}.$$

Substituting (1.47) into (1.45) we have

$$\begin{aligned}\sum_{i=1}^{m}[p_{i,m}-p_{i-1,m}]^2 &= RV^{(m)}(p^*)+2\sum_{i=1}^{m}\varepsilon_{i,m}\left(p^*_{i-1,m}-p_{i-1,m}\right)-(1-\pi)(p^*_{m,m}-p_{m-1,m})^2\\ &\quad+\sum_{i=1}^{m}(p^*_{i-1,m}-p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2-(1-\pi)\right\}\\ &\quad+\sum_{i=1}^{m}(p^*_{i,m}-p_{i-1,m})^2\left(\xi^2_{i,m}-\pi\right)\\ &= RV^{(m)}(p^*)+\underbrace{2\sum_{i=1}^{m}\varepsilon_{i,m}\left(p^*_{i-1,m}-p_{i-1,m}\right)}_{A}\underbrace{-(1-\pi)(p^*_{m,m}-p_{m-1,m})^2}_{B}\\ &\quad\underbrace{-2\sum_{i=1}^{m}(p^*_{i,m}-p_{i-1,m})^2\xi_{i,m}(1-\xi_{i,m})}_{C}\\ &= RV^{(m)}(p^*)+A+B+C. \end{aligned} \tag{1.48}$$

Since $\xi_{i,m}$ is a Bernoulli variable, $\xi_{i,m}(1-\xi_{i,m})=0$ *a.s.*, and so $C=0$. Consider term B. Note that for some duration $K_m\geq 1$ and for which at most $K_m=O_p(\log m)$ we have

$$p^*_{m,m}-p_{m-1,m}=p^*_{m,m}-p^*_{m-K_m,m}=\int_{t_{m-K_m,m}}^{1}\sigma(s)dB(s)=O_p\left(\sqrt{h_{\bar{K}_m,m}}\right),$$

so that

$$B=-(1-\pi)(p^*_{m,m}-p_{m-1,m})^2=-(1-\pi)O_p\left(h_{\bar{K}_m,m}\right)=o_p(1), \tag{1.49}$$

For term A, note that $\varepsilon_{i,m}=\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)dB(s)$ and

$$p^*_{i-1,m}-p_{i-1,m}=(p^*_{i-1,m}-p_{i-2,m})(1-\xi_{i-1,m})=(p^*_{i-1,m}-p^*_{i-K_{i-1},m})(1-\xi_{i-1,m}), \tag{1.50}$$

for some $K_{i-1}\geq 2$ and where K_{i-1} is at most of $O_p(\log m)$. Then

$$\sum_{i=1}^{m}\varepsilon_{i,m}\left(p^*_{i-1,m}-p_{i-1,m}\right)=\sum_{i=1}^{m}\varepsilon_{i,m}(p^*_{i-1,m}-p^*_{i-K_{i-1},m})(1-\xi_{i-1,m}).$$

Now $\varepsilon_{i,m}$ is independent of $\xi_{i-1,m}$, and $E(\varepsilon_{i,m}) = 0$ and $Var(\varepsilon_{i,m}) \to 0$, as $m \to \infty$, because $h_{\bar{K}_m,m} = o(1)$, while $\sum_{i=1}^{m}(p^*_{i-1,m} - p^*_{i-K_{i-1},m})^2(1-\xi_{i-1,m})^2$ is bounded as $m \to \infty$. It follows that $A = o_p(1)$. Thus, $\sum_{i=1}^{m}[p_{i,m} - p_{i-1,m}]^2 = RV^{(m)}(p^*) + o_p(1)$, as required.

Proof Theorem 2.3: From (1.48) we have

$$\begin{aligned}
\sqrt{m}\left\{\sum_{i=1}^{m}[p_{i,m} - p_{i-1,m}]^2 - IV\right\} &= \sqrt{m}\left\{RV^{(m)}(p^*) - IV\right\} + \sqrt{m}2\sum_{i=1}^{m}\varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) \\
&\quad -\sqrt{m}(1-\pi)(p^*_{m,m} - p_{m-1,m})^2 \\
&\quad +\sqrt{m}\sum_{i=1}^{m}(p^*_{i-1,m} - p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2 - (1-\pi)\right\} \\
&\quad +\sqrt{m}\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2\left(\xi^2_{i,m} - \pi\right) \\
&= \sqrt{m}\left\{RV^{(m)}(p^*) - IV\right\} + \sqrt{m}A + \sqrt{m}B + \sqrt{m}C \\
&= \sqrt{m}\left\{RV^{(m)}(p^*) - IV\right\} + \sqrt{m}A + \sqrt{m}B, \qquad (1.51)
\end{aligned}$$

since $C = 0$, $a.s.$. Standard theory (Jacod (2018); Barndorff-Nielsen and Shephard (2002); Barndorff-Nielsen et al. (2006)) gives the CLT

$$\sqrt{m}\left\{RV^{(m)}(p^*) - IV\right\} \xrightarrow{d} \mathcal{N}\left(0, 2\int_0^1 \sigma^4(t)dt\right), \qquad (1.52)$$

stably as $m \to \infty$. We now study the asymptotic behavior of $\sqrt{m}A$, and $\sqrt{m}B$.

For term $\sqrt{m}A$, from (1.50) we get

$$\begin{aligned}
\sqrt{m}A &= \sqrt{m}2\sum_{i=1}^{m}\varepsilon_{i,m}(p^*_{i-1,m} - p^*_{i-K_{i-1},m})(1-\xi_{i-1,m}) \\
&= 2\sqrt{m}\sum_{i=1}^{m}\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)dB(s)(p^*_{i-1,m} - p^*_{i-K_{i-1},m})(1-\xi_{i-1,m}) \\
&= 2\sqrt{m}\sum_{i=1}^{m}\nu_{i,m}(p^*_{i-1,m} - p^*_{i-K_{i-1},m}),
\end{aligned}$$

where $\nu_{i,m} = \int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)dB(s)(1-\xi_{i-1,m})$ is uncorrelated with $(p^*_{i-1,m} - p^*_{i-K_{i-1},m})$, because of the martingale property, and has mean 0 and conditional variance $(1-\pi)\, m\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)^2 ds$. So $\sqrt{m}A$ is a martingale with conditional variance

$$m\,(1-\pi)\sum_{i=1}^{m}(p^*_{i-1,m} - p^*_{i-K_{i-1},m})^2\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)^2 ds.$$

By stochastic Taylor series expansion we have

$$p^*_{i-1,m} - p^*_{i-K_{i-1},m} = \int_{t_{i-K_{i-1},m}}^{t_{i-1,m}}\sigma(s)dB(s)$$

$$
\begin{aligned}
&= \left\{\sigma\left(t_{i-K_{i-1},m}\right)+O_p\left(\sqrt{K_{i-1}h}\right)\right\}\left(B\left(t_{i-1,m}\right)-B\left(t_{i-K_{i-1},m}\right)\right)\\
&= \sigma\left(t_{i-K_{i-1},m}\right)\left(B\left(t_{i-1,m}\right)-B\left(t_{i-K_{i-1},m}\right)\right)+O_p\left(K_{i-1}h\right)\\
&= \sigma\left(\frac{i-1}{m}\right)\left(B\left(\frac{i-1}{m}\right)-B\left(\frac{i-K_{i-1}}{m}\right)\right)+O_p\left(\frac{K_{i-1}}{m}\right) \qquad (1.53)
\end{aligned}
$$

on the equispaced grid $\{t_{i,m}=\frac{i}{m}: i=0,...,m\}$ with $h=\frac{1}{m}$. Then

$$
\begin{aligned}
& m\sum_{i=1}^{m}(p^*_{i-1,m}-p^*_{i-K_{i-1},m})^2\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)^2ds\\
=\;& m\sum_{i=1}^{m}\left\{\sigma\left(\frac{i-1}{m}\right)\left(B\left(\frac{i-1}{m}\right)-B\left(\frac{i-K_{i-1}}{m}\right)\right)+O_p\left(\frac{K_{i-1}}{m}\right)\right\}^2\\
& \times\left\{\sigma\left(\frac{i-1}{m}\right)^2+O_p\left(\frac{1}{\sqrt{m}}\right)\right\}\frac{1}{m}\\
=\;& \sum_{i=1}^{m}\left\{\sigma\left(\frac{i-1}{m}\right)^4\left(B\left(\frac{i-1}{m}\right)-B\left(\frac{i-K_{i-1}}{m}\right)\right)^2\left[1+O_p\left(\frac{K_{i-1}}{m}\right)\right]\right\}\\
=\;& \sum_{i=1}^{m}\left\{\sigma\left(\frac{i-1}{m}\right)^4\frac{K_{i-1}-1}{m}\left[1+O_p\left(\frac{K_{i-1}}{m}\right)\right]\right\}\\
\to_p\;& \left(\int_0^1\sigma^4(t)dt\right)E(K_{i-1}-1).
\end{aligned}
$$

It follows by the martingale central limit theorem (Hall and Heyde, 1980, Theorem 3.2) that

$$
\sqrt{m}A \xrightarrow{d} A_\infty =_d 2\times\mathcal{MN}\left(0,(1-\pi)\left(\int_0^1\sigma^4(t)dt\right)E(K_{i-1}-1)\right),
$$

stably in the sense that $\{Z,\sqrt{m}A\}\xrightarrow{d}\{Z,A_\infty\}$ for $Z=\int_0^1\sigma^4(t)dt$. Observe that

$$
K_{i-1}-2=\begin{cases}0 & \text{with probability } \pi\\ 1 & \text{with probability } \pi(1-\pi)\\ 2 & \text{with probability } \pi(1-\pi)^2\\ \vdots & \end{cases},
$$

so that $E(K_{i-1}-2)=\pi(1-\pi)+2\pi(1-\pi)^2+\cdots=\frac{1-\pi}{\pi}$, which implies that $E(K_{i-1}-1)=\frac{1}{\pi}$. Thus

$$
\sqrt{m}A \xrightarrow{d} \mathcal{MN}\left(0,4\frac{1-\pi}{\pi}\int_0^1\sigma^4(t)dt\right), \qquad (1.54)
$$

stably.

Next consider term $\sqrt{m}B$. From (1.49) we have

$$
\sqrt{m}B=-\sqrt{m}O_p\left(\frac{\log m}{m}\right)(1-\pi)=o_p(1). \qquad (1.55)
$$

Observe that the components of $\sqrt{m}A$ involve the product $\varepsilon_{i,m}(p^*_{i-1,m}-p^*_{i-K_{i-1},m})(1-\xi_{i-1,m})$

whereas $\sqrt{m}\left\{RV^{(m)}(p^*)-IV\right\}$ is a centred quadratic in the increments $\varepsilon_{i,m}=p^*_{i,m}-p^*_{i-1,m}$, so that $\sqrt{m}A$ and $\sqrt{m}\left\{RV^{(m)}(p^*)-IV\right\}$ are asymptotically uncorrelated. By combining (1.54) and (1.52), it follows that

$$\sqrt{m}\left\{RV^{(m)}(p)-IV\right\}\xrightarrow{d}\mathcal{MN}\left(0,\frac{4-2\pi}{\pi}\int_0^1\sigma^4(t)dt\right),$$

giving the required result.

Proof of Lemma 2.4: From equation (1.40), we have

$$m\sum_{i=1}^m[p_{i,m}-p_{i-1,m}]^4=\pi m\sum_{i=1}^m(p^*_{i,m}-p_{i-1,m})^4+m\sum_{i=1}^m(p^*_{i,m}-p_{i-1,m})^4\left(\xi^4_{i,m}-E[\xi^4_{i,m}]\right). \quad (1.56)$$

Consider $\sum_{i=1}^m(p^*_{i,m}-p_{i-1,m})^4$ in the first term, giving

$$\begin{aligned}\sum_{i=1}^m(p^*_{i,m}-p_{i-1,m})^4 &= \sum(p^*_{i,m}-p^*_{i-1,m}+p^*_{i-1,m}-p_{i-1,m})^4\\ &= \sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^4+4\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^3\left(p^*_{i-1,m}-p_{i-1,m}\right)\\ &\quad+6\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^2\left(p^*_{i-1,m}-p_{i-1,m}\right)^2\\ &\quad+4\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)\left(p^*_{i-1,m}-p_{i-1,m}\right)^3+\sum_{i=1}^m\left(p^*_{i-1,m}-p_{i-1,m}\right)^4\end{aligned}$$

$$\begin{aligned}&= \sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^4+4\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^3\left(p^*_{i-1,m}-p_{i-1,m}\right)\\ &\quad+6\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^2\left(p^*_{i-1,m}-p_{i-1,m}\right)^2\\ &\quad+4\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)\left(p^*_{i-1,m}-p_{i-1,m}\right)^3+m\sum_{i=1}^m\left(p^*_{i-1,m}-p_{i-2,m}\right)^4(1-\pi)\\ &\quad+\sum_{i=1}^m\left(p^*_{i-1,m}-p_{i-2,m}\right)^4((1-\xi_{i-1,m})^4-(1-\pi)).\end{aligned} \quad (1.57)$$

Set $B_m=\sum_{i=1}^m(p^*_{i,m}-p_{i-1,m})^4$. So $B_{m-1}=B_m-(p^*_{m,m}-p_{m-1,m})^4$. Substituting out B_{m-1} in equation (1.57) and solving for B_m, we get

$$\begin{aligned}B_m &= \frac{1}{\pi}\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^4+\frac{4}{\pi}\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^3\left(p^*_{i-1,m}-p_{i-1,m}\right) \quad (1.58)\\ &\quad+\frac{6}{\pi}\sum_{i=1}^m\left(p^*_{i,m}-p^*_{i-1,m}\right)^2\left(p^*_{i-1,m}-p_{i-1,m}\right)^2\end{aligned}$$

$$+\frac{4}{\pi}\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)\left(p^*_{i-1,m}-p_{i-1,m}\right)^3$$
$$-\frac{1-\pi}{\pi}(p^*_{m,m}-p_{m-1,m})^2+\frac{1}{\pi}\sum_{i=1}^{m}(p^*_{i-1,m}-p_{i-2,m})^4\left\{(1-\xi_{i-1,m})^4-(1-\pi)\right\}.$$

Substituting (1.58) into (1.56) we have

$$\begin{aligned}
m\sum[p_{i,m}-p_{i-1,m}]^4 &= m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)^4+4m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)^3\left(p^*_{i-1,m}-p_{i-1,m}\right)\\
&\quad +6m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)^2\left(p^*_{i-1,m}-p_{i-1,m}\right)^2\\
&\quad +4m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)\left(p^*_{i-1,m}-p_{i-1,m}\right)^3-(1-\pi)(p^*_{m,m}-p_{m-1,m})^2\\
&\quad +m\sum(p^*_{i-1,m}-p_{i-2,m})^4\left\{(1-\xi_{i-1,m})^4-(1-\pi)\right\}\\
&\quad +m\sum_{i=1}^{m}(p^*_{i,m}-p_{i-1,m})^4\left(\xi^4_{i,m}-\pi\right)\\
&= m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)^4+\underbrace{4m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)^3\left(p^*_{i-1,m}-p_{i-1,m}\right)}_{A}\\
&\quad \underbrace{+6m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)^2\left(p^*_{i-1,m}-p_{i-1,m}\right)^2}_{B}\\
&\quad \underbrace{+4m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)\left(p^*_{i-1,m}-p_{i-1,m}\right)^3}_{C}\underbrace{-(1-\pi)m(p^*_{m,m}-p_{m-1,m})^4}_{D}\\
&\quad \underbrace{+m\sum(p^*_{i,m}-p_{i-1,m})^4(1-\xi_{i,m})^2\xi^2_{i,m}}_{E}\\
&= m\sum_{i=1}^{m}\left(p^*_{i,m}-p^*_{i-1,m}\right)^4+A+B+C+D+E. \qquad (1.59)
\end{aligned}$$

As in (1.53) we have

$$\begin{aligned}
p^*_{i,m}-p^*_{i-1,m} &= \sigma\left(t_{i-1,m}\right)\left(B\left(t_{i,m}\right)-B\left(t_{i,m}\right)\right)+O_p\left(\frac{1}{m}\right)\\
&= \sigma\left(t_{i-1,m}\right)\frac{\epsilon_{i,m}}{\sqrt{m}}+O_p\left(\frac{1}{m}\right), \qquad (1.60)
\end{aligned}$$

where $\epsilon_{i,m}$ is *iid* $\mathcal{N}(0,1)$. Hence

$$\begin{aligned} m\sum_{i=1}^{m}\left(p_{i,m}^{*}-p_{i-1,m}^{*}\right)^{4} &= \sum_{i=1}^{m}\sigma\left(t_{i-1,m}\right)^{4}\frac{\epsilon_{i,m}^{4}}{m}+O_{p}\left(\frac{1}{m^{1/2}}\right) \\ &= \sum_{i=1}^{m}\sigma\left(t_{i-1,m}\right)^{4}\frac{E\left(\epsilon_{i,m}^{4}\right)}{m}+O_{p}\left(\frac{1}{m^{1/2}}\right) \\ &\to_{p} 3\int_{0}^{1}\sigma^{4}(t)dt. \end{aligned} \tag{1.61}$$

Hence

$$\frac{2}{3}m\sum_{i=1}^{m}\left(p_{i,m}^{*}-p_{i-1,m}^{*}\right)^{4}\to_{p}2\int_{0}^{1}\sigma^{4}(t)dt. \tag{1.62}$$

This corresponds with the result obtained in Barndorff-Nielsen and Shephard (2002).

We now consider the limit behavior of terms A,B,C,D, and E. First, for term E, since $\left(1-\xi_{i,m}\right)^{2}\xi_{i,m}^{2}=0$ almost surely, $E=0$. Second, for term D, note that

$$D=-(1-\pi)m(p_{m,m}^{*}-p_{m-1,m})^{4}=-(1-\pi)m\times O_{p}\left(\frac{\log^{2}m}{m^{2}}\right)=o_{p}(1).$$

Next, consider term A, viz.,

$$\begin{aligned} & m\sum_{i=1}^{m}\left(p_{i,m}^{*}-p_{i-1,m}^{*}\right)^{3}\left(p_{i-1,m}^{*}-p_{i-1,m}\right) \\ = \;& m\sum_{i=1}^{m}\left(p_{i,m}^{*}-p_{i-1,m}^{*}\right)^{3}\left(p_{i-1,m}^{*}-p_{i-2,m}\right)\left(1-\xi_{i-1,m}\right) \\ = \;& m\sum_{i=1}^{m}\left\{\sigma\left(t_{i-1,m}\right)\frac{\epsilon_{i,m}}{\sqrt{m}}+O_{p}\left(\frac{1}{m}\right)\right\}^{3}\left(p_{i-1,m}^{*}-p_{i-K_{i-1},m}^{*}\right)\left(1-\xi_{i-1,m}\right) \\ & \text{for some } K_{i-1}=O_{p}(\log m) \\ = \;& \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\sigma^{3}\left(t_{i-1,m}\right)\epsilon_{i,m}^{3}\left(p_{i-1,m}^{*}-p_{i-K_{i-1},m}^{*}\right)\left(1-\xi_{i-1,m}\right)+o_{p}\left(1\right). \end{aligned} \tag{1.63}$$

The component $\sigma^{3}\left(t_{i-1,m}\right)\epsilon_{i,m}^{3}\left(p_{i-1,m}^{*}-p_{i-K_{i-1},m}^{*}\right)\left(1-\xi_{i-1,m}\right)$ in the sum (1.63) has mean zero and conditional variance $E\left[\epsilon_{i,m}^{6}\right]\left(1-\pi\right)\sigma^{6}\left(t_{i-1,m}\right)\left(p_{i-1,m}^{*}-p_{i-K_{i-1},m}^{*}\right)^{2}=O_{p}\left(\frac{\log m}{m}\right)$ since from (1.53)

$$\begin{aligned} p_{i-1,m}^{*}-p_{i-K_{i-1},m}^{*} &= \sigma\left(\frac{i-1}{m}\right)\left(B\left(\frac{i-1}{m}\right)-B\left(\frac{i-K_{i-1}}{m}\right)\right)+O_{p}\left(\frac{K_{i-1}}{m}\right) \\ &= \sigma\left(\frac{i-1}{m}\right)\frac{\eta_{K_{i-1}}}{\sqrt{m}}+O_{p}\left(\frac{K_{i-1}}{m}\right)=O_{p}\left(\sqrt{\frac{\log m}{m}}\right), \end{aligned} \tag{1.64}$$

where

$$\eta_{K_{i-1}} := B\left(\frac{i-1}{m}\right) - B\left(\frac{i-K_{i-1}}{m}\right) = O_p\left(\sqrt{\log m}\right).$$

It follows that

$$E\left\{\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\sigma^3\left(t_{i-1,m}\right)\epsilon_{i,m}^3\left(p_{i-1,m}^* - p_{i-K_{i-1},m}^*\right)\left(1-\xi_{i-1,m}\right)\right\}^2 = O_p\left(\frac{\log m}{m}\right),$$

and so $A = o_p(1)$. Similarly, for term C, $4m\sum\left(p_{i,m}^* - p_{i-1,m}^*\right)\left(p_{i-1,m}^* - p_{i-1,m}\right)^3$ is $o_p(1)$.

Next, consider term B. Using (1.40) and (1.64), we have

$$\begin{aligned}
& m\sum_{i=1}^{m}\left(p_{i,m}^* - p_{i-1,m}^*\right)^2\left(p_{i-1,m}^* - p_{i-1,m}\right)^2 \\
= \;& m\sum_{i=1}^{m}\left(p_{i,m}^* - p_{i-1,m}^*\right)^2\left(p_{i-1,m}^* - p_{i-2,m}\right)^2\left(1-\xi_{i-1,m}\right)^2 \\
= \;& m\sum_{i=1}^{m}\left(p_{i,m}^* - p_{i-1,m}^*\right)^2\left(p_{i-1,m}^* - p_{i-K_{i-1},m}^*\right)^2\left(1-\xi_{i-1,m}\right)^2 \text{ for some } K_{i-1} = O_p(\log m) \\
= \;& m\sum_{i=1}^{m}\left\{\sigma\left(t_{i-1,m}\right)\frac{\epsilon_{i,m}}{\sqrt{m}} + O_p\left(\frac{1}{m}\right)\right\}^2\left(\sigma\left(t_{i-1,m}\right)\frac{\eta_{K_{i-1}}}{\sqrt{m}} + O_p\left(\frac{\log m}{m}\right)\right)^2\left(1-\xi_{i,m}\right)^2 \\
= \;& \frac{1}{m}\sum_{i=1}^{m}\sigma^2(t_{i,m})\epsilon_{i,m}^2\sigma\left(t_{i-1,m}\right)^2\eta_{K_{i-1}}^2\left(1-\xi_{i-1,m}\right)^2 + O_p\left(\sqrt{\frac{\log m}{m}}\right) \\
= \;& \frac{1}{m}\sum_{i=1}^{m}\sigma^4(t_{i,m})E\left[\epsilon_{i,m}^2\eta_{K_{i-1}}^2\left(1-\xi_{i-1,m}\right)^2\right] \\
& +\frac{1}{m}\sum_{i=1}^{m}\sigma^4(t_{i,m})\left\{\epsilon_{i,m}^2\eta_{K_{i-1}}^2\left(1-\xi_{i-1,m}\right)^2 - E\left[\epsilon_{i,m}^2\eta_{K_{i-1}}^2\left(1-\xi_{i-1,m}\right)^2\right]\right\} \\
& +O_p\left(\sqrt{\frac{\log m}{m}}\right) \\
= \;& \frac{1}{m}\sum_{i=1}^{m}\sigma^4\left(t_{i,m}\right)(1-\pi)E(K_{i-1}-1) + O_p\left(\frac{\log m}{\sqrt{m}}\right) \\
\to_p \;& \left(\int_0^1 \sigma^4(t)dt\right)(1-\pi)E(K_{i-1}-1) = \frac{(1-\pi)}{\pi}\int_0^1 \sigma^4(t)dt,
\end{aligned} \tag{1.65}$$

since $\epsilon_{i,m}$, $\eta_{K_{i-1}}$, and $\xi_{i-1,m}$ are independent. Therefore,

$$\begin{aligned}
m\sum\left[p_{i,m} - p_{i-1,m}\right]^4 &= 3\int_0^1 \sigma^4(t)dt + 6\frac{(1-\pi)}{\pi}\int_0^1 \sigma^4(t)dt + o_p(1) \\
&= \frac{6-3\pi}{\pi}\int_0^1 \sigma^4(t)dt + o_p(1),
\end{aligned}$$

leading to the required result.

Proof Theorem 2.5: This follows directly from Lemma 2.4 and Theorem 2.3.

Proof Theorem 3.1: This proof is given in the Online Supplement to the paper.

Proof Theorem 4.1: As in Zhang et al. (2005) we set $K = cm^{2/3}$ for some constant $c > 0$. Then

$$\frac{K}{\bar{m}} = \frac{K^2}{m} = c^2 m^{1/3}.$$

The two time scale measures involving the efficient price series are denoted by $[p^*]^{(avg)}$ and $[p^*]^{\mathcal{G}_m}$, and the corresponding measures using the actual price series are denoted by $[p]^{(avg)}$ and $[p]^{\mathcal{G}_m}$.

The two time scale estimator is $\widehat{[p^*]} = [p]^{(avg)} - \frac{\bar{m}}{J_m}[p]^{\mathcal{G}_m}$. Decompose the estimation error as

$$\widehat{[p^*]} - \int_0^1 \sigma^2(t)dt = \left(\widehat{[p^*]} - [p^*]^{(avg)}\right) + \left([p^*]^{(avg)} - \int_0^1 \sigma^2(t)dt\right). \tag{1.66}$$

From the proof of Theorem A.1 and equation (56) in Zhang et al. (2005), we have

$$\sqrt{\frac{K}{\bar{m}}}\left\{\widehat{[p^*]} - [p^*]^{(avg)}\right\} \Rightarrow \mathcal{N}\left(0, 8\sigma_u^4\right), \tag{1.67}$$

and their proof applies in the present case with only minor notational changes to account for random τ_j sampling under Assumption D2 (ii).

The second component of (1.66) is the discretization effect and is the same as that in Zhang et al. (2005), after allowing for the fact that we have random stopping times and adjusting accordingly, as we do below. Theorems 2 and 3 of Zhang et al. (2005) give the following limit theory

$$\left(\frac{m}{K}\right)^{1/2}\left([p^*]^{(avg)} - \int_0^1 \sigma^2(t)dt\right) \Rightarrow \mathcal{MN}\left(0, \eta^2\right), \tag{1.68}$$

where η^2 is the probability limit of the quantity

$$\eta_m^2 = \sum_{j=1}^{J_m} \sigma^4(\tau_j) h_j (\tau_j - \tau_{j-1}), \tag{1.69}$$

where

$$h_j = \frac{4}{K\overline{\Delta\tau}} \sum_{i=1}^{(K-1)\wedge j} \left(1 - \frac{i}{K}\right)^2 (\tau_{j-i} - \tau_{j-i-1}), \tag{1.70}$$

and in the present case $m\overline{\Delta\tau} = J_m^{-1}\sum_{j=1}^{J_m} D_{m,j}$. The limiting forms of h_j and the random (variance) quantity η^2 are evaluated below.

We first observe that result (1.68) relies on the following representation, given in the proof of Theorem 2 of Zhang et al. (2005),

$$[p^*]^{(avg)} - [p^*]^{\mathcal{G}_m} = 2\sum_{j=1}^{J_m-1}(p^*(\tau_j) - p^*(\tau_{j-1}))\sum_{i=1}^{(K-1)\wedge j}\left(1 - \frac{i}{K}\right)(p^*(\tau_{j-i}) - p^*(\tau_{j-i-1})) + O_p\left(\frac{K}{m}\right)$$

whose leading term is

$$2\sum_{j=1}^{J_m-1}\sigma\left(\tau_{j-1}\right)\left(B\left(\tau_j\right)-B\left(\tau_{j-1}\right)\right)\sum_{i=1}^{(K-1)\wedge j}\left(1-\frac{i}{K}\right)\sigma\left(\tau_{j-i}\right)\left(B\left(\tau_{j-i}\right)-B\left(\tau_{j-i-1}\right)\right)\times$$
$$\left\{1+O_p\left(\frac{\max_{j\leq m}D_{m,j}}{m}\right)\right\}.$$

Now

$$2\sum_{j=1}^{J_m-1}\sigma\left(\tau_{j-1}\right)\left(B\left(\tau_j\right)-B\left(\tau_{j-1}\right)\right)\sum_{i=1}^{(K-1)\wedge j}\left(1-\frac{i}{K}\right)\sigma\left(\tau_{j-i}\right)\left(B\left(\tau_{j-i}\right)-B\left(\tau_{j-i-1}\right)\right)$$

is a local martingale with conditional variance whose leading term, as in the proof of Theorem 2 of Zhang et al. (2005), is

$$\begin{aligned}
& 4\sum_{j=1}^{J_m-1}\sigma\left(\tau_{j-1}\right)^2\left(\tau_j-\tau_{j-1}\right)\chi_{1j}^2\sum_{i=1}^{(K-1)\wedge j}\left(1-\frac{i}{K}\right)^2\sigma\left(\tau_{j-i}\right)^2\left(\tau_{j-i}-\tau_{j-i-1}\right)\chi_{1j-i}^2 \\
= & 4\sum_{j=1}^{J_m-1}\sigma\left(\tau_{j-1}\right)^2\left(\tau_j-\tau_{j-1}\right)\sum_{i=1}^{(K-1)\wedge j}\left(1-\frac{i}{K}\right)^2\sigma\left(\tau_{j-i}\right)^2\left(\tau_{j-i}-\tau_{j-i-1}\right)\mathrm{E}\left(\chi_{1j}^2\right)\mathrm{E}\left(\chi_{1j-1}^2\right)+o_p\left(1\right) \\
= & 4\sum_{j=1}^{J_m-1}\sigma\left(\tau_{j-1}\right)^4\left(\tau_j-\tau_{j-1}\right)\sum_{i=1}^{(K-1)\wedge j}\left(1-\frac{i}{K}\right)^2\left(\tau_{j-i}-\tau_{j-i-1}\right)\left\{1+O_p\left(\frac{K\max_{j\leq m}D_{m,j}}{m}\right)\right\}+o_p\left(1\right) \\
= & K\overline{\Delta\tau}\sum_{j=1}^{J_m}\sigma^4\left(\tau_j\right)h_j\left(\tau_j-\tau_{j-1}\right)\left\{1+o_p\left(1\right)\right\}=K\overline{\Delta\tau}\left\{\eta_m^2+o_p\left(1\right)\right\},
\end{aligned}$$

thereby giving (1.69) by martingale central limit theory since, under D2(ii)$'$ and $K=cm^{2/3}$,

$$\frac{K\max_{j\leq m}D_{m,j}}{m}=o_p\left(1\right).$$

To find an explicit expression for the limit quantity η^2 in (1.68) we proceed as follows. First observe that for the present model

$$m\overline{\Delta\tau}=J_m^{-1}\sum_{j=1}^{J_m}D_{m,j}=J_m^{-1}\sum_{j=1}^{J_m}\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)+o_p\left(1\right)\to_p\int_0^1\mu_D\left(s\right)ds,$$

so that for $r\geq 0$

$$\begin{aligned}
h_{j=[Kr]} &= \frac{4m}{K\int_0^1\mu_D\left(s\right)ds}\sum_{i=1}^{(K-1)\wedge[Kr]}\left(1-\frac{i}{K}\right)^2\left(\tau_{[Kr]-i}-\tau_{[Kr]-i-1}\right)\left\{1+o_p\left(1\right)\right\} \\
&= \frac{4}{K\int_0^1\mu_D\left(s\right)ds}\sum_{i=1}^{(K-1)\wedge[Kr]}\left(1-\frac{i}{K}\right)^2D_{m,[Kr]-i}\left\{1+o_p\left(1\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{K\int_0^1 \mu_D(s)\,ds} \sum_{i=1}^{(K-1)\wedge[Kr]} \left(1-\frac{i}{K}\right)^2 \mathrm{E}\left(D_{m,[Kr]-i}|\mathcal{F}_{\tau_{[Kr]-i}}\right)\{1+o_p(1)\} \\
&\to_p \frac{4\left\{\int_0^{1\wedge r}(1-s)^2\mu_D(1\wedge r-s)\,ds\right\}}{\int_0^1 \mu_D(s)\,ds}.
\end{aligned}
$$

Since $K/J_m \to 0$, it follows that $[J_m t] > K$ as $m\to\infty$ for all $t\in(0,1]$. Hence, for all $t\in(0,1]$,

$$
h_{j=[J_m t]} = \frac{4}{K\overline{\Delta\tau}} \sum_{i=1}^{(K-1)\wedge[J_m t]} \left(1-\frac{i}{K}\right)^2 \left(\tau_{[J_m t]-i}-\tau_{[J_m t]-i-1}\right) \to_p \frac{4\left\{\int_0^1 (1-s)^2\mu_D(1-s)\,ds\right\}}{\int_0^1 \mu_D(s)\,ds},
$$

and then

$$
\eta_m^2 = \sum_{j=1}^{J_m} \sigma^4(\tau_j)\,h_j(\tau_j-\tau_{j-1}) \to_p \frac{4\left\{\int_0^1(1-s)^2\mu_D(1-s)\,ds\right\}}{\int_0^1\mu_D(s)\,ds}\int_0^1 \sigma^4(t)\,dt.
$$

Observe that when $\mu_D(s)=\mu_D\ \ a.s.$, the right side of the last expression simplifies to

$$
4\left\{\int_0^1(1-s)^2\,ds\right\}\int_0^1\sigma^4(t)\,dt = \frac{4}{3}\int_0^1\sigma^4(t)\,dt,
$$

as given in Zhang et al. (2005) equation (5). It therefore follows that

$$
\left(\frac{m}{K}\right)^{1/2}\left([p^*]^{(avg)}-\int_0^1\sigma^2(t)dt\right) \Rightarrow \mathcal{MN}\left(0,\frac{4\left\{\int_0^1(1-s)^2\mu_D(1-s)\,ds\right\}}{\int_0^1\mu_D(s)\,ds}\int_0^1\sigma^4(t)\,dt\right). \tag{1.71}
$$

Next, consider (1.67). Since $\widehat{[p^*]} = [p]^{(avg)} - \frac{\bar{m}}{J_m}[p]^{\mathcal{G}_m}$, we have

$$
\begin{aligned}
&\sqrt{\frac{K}{\bar{m}}}\left\{\widehat{[p^*]}-[p^*]^{(avg)}\right\} \\
&= \sqrt{\frac{K}{\bar{m}}}\left\{[p]^{(avg)}-[p^*]^{(avg)}-\frac{\bar{m}}{J_m}[p]^{\mathcal{G}_m}\right\} \\
&= \sqrt{\frac{K}{\bar{m}}}\left\{[p]^{(avg)}-[p^*]^{(avg)}-2\bar{m}\sigma_u^2\right\} \\
&\quad -2\sqrt{K\bar{m}}\left\{J_m^{-1}[p]^{\mathcal{G}_m}-\sigma_u^2\right\} \\
&\Rightarrow \mathcal{N}\left(0,8\sigma_u^4\right),
\end{aligned} \tag{1.72}
$$

as in Theorem A.1 and equation (56) of Zhang et al. (2005). Combining (1.71) and (1.72) as in Theorem 4 of Zhang et al. (2005) and using $K=cm^{2/3}$ for some constant $c>0$, so that $\sqrt{K/\bar{m}} = K/m^{1/2} = cm^{1/6}$ and $\sqrt{m/K} = m^{1/6}/c^{1/2}$, we then have the limit theory

$$
m^{1/6}\left\{\widehat{[p^*]}-\int_0^1\sigma^2(t)dt\right\} \Rightarrow \left\{\frac{8}{c^2}\sigma_u^4+c\eta^2\right\}^{1/2}\mathcal{N}(0,1),
$$

as required.

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Online Supplement to: "Information Loss in Volatility Measurement with Flat Price Trading"*

by

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This Online Supplement provides proofs of some of the results in the main paper. These include the proof of Theorem 3.1 and additional results given in equations (1.19), (1.20) and (1.24) of the paper.

Proof Theorem 3.1: We prove the results under Assumption D2 and deduce those under D1. Since p^* is generated by (1.1), $\sigma(t)$ is càdlàg, and $\tau_j - \tau_{j-1} = D_{m,j}/m$, we have for large enough m under D2(ii)

$$\begin{aligned} p^*(\tau_j) - p^*(\tau_{j-1}) &= \int_{\tau_{j-1}}^{\tau_j} \sigma(t)\, dB(t) = \sigma(\tau_{j-1})[B(\tau_j) - B(\tau_{j-1})]\left\{1 + O_p\left(\frac{\max_{j\le m} D_{m,j}}{m}\right)\right\} \\ &= \sigma(\tau_{j-1})[B(\tau_j) - B(\tau_{j-1})]\left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\}, \end{aligned} \tag{1.73}$$

uniformly in $j \le m$. Then

$$\begin{aligned} \sum_{j=1}^{J_m}[p^*(\tau_j) - p^*(\tau_{j-1})]^2 &= \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})[B(\tau_j) - B(\tau_{j-1})]^2 \left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\} \\ &= \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})[\tau_j - \tau_{j-1}]\chi^2_{1j} \left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\} \\ &= \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})[\tau_j - \tau_{j-1}]\mathrm{E}\left(\chi^2_{1j}\right) \\ &\quad + \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})[\tau_j - \tau_{j-1}]\left\{\chi^2_{1j} - 1\right\} + o_p\left(\frac{1}{\sqrt{m}}\right) \\ &\to_p \int_0^1 \sigma^2(t)dt, \end{aligned} \tag{1.74}$$

where $\left\{\chi^2_{1j}\right\}$ is an independent sequence of chi squared variates with unit degrees of freedom and where the second term of (1.74) is $O_p\left(m^{-1/2}\right)$, as shown below, thereby giving (1.12).

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To prove (1.14), start by noting that

$$\left|\int_{\tau_{J_m}}^{1} \sigma^2(t)dt\right| \leq (1-\tau_{J_m}) \max_{t\in[\tau_{J_m},1]} \sigma^2(t) \leq \frac{\max_{j\leq m+1} D_{m,j}}{m} \max_{t\in[\tau_{J_m},1]} \sigma^2(t) = o_p\left(\frac{1}{\sqrt{m}}\right).$$

Then

$$\begin{aligned}
&\sum_{j=1}^{J_m}[p^*(\tau_j) - p^*(\tau_{j-1})]^2 - \int_0^1 \sigma^2(t)dt \\
&= \sum_{j=1}^{J_m}\left\{[p^*(\tau_j) - p^*(\tau_{j-1})]^2 - \int_{\tau_{j-1}}^{\tau_j} \sigma^2(t)dt\right\} - \int_{\tau_{J_m}}^{1} \sigma^2(t)dt \\
&= \sum_{j=1}^{J_m}\sigma^2(\tau_{j-1})[\tau_j - \tau_{j-1}]\left(\chi^2_{1j} - 1\right) + o_p\left(\frac{1}{\sqrt{m}}\right) \\
&= \sum_{j=1}^{J_m}\sigma^2(\tau_{j-1})\frac{D_{m,j}}{m}\left(\chi^2_{1j} - 1\right) + o_p\left(\frac{1}{\sqrt{m}}\right), \qquad (1.75)
\end{aligned}$$

and so

$$\begin{aligned}
&\sqrt{m}\left\{\sum_{j=1}^{J_m}[p^*(\tau_j) - p^*(\tau_{j-1})]^2 - \int_0^1 \sigma^2(t)dt\right\} \\
&= \sum_{j=1}^{J_m}\sigma^2(\tau_{j-1})\frac{D_{m,j}}{\sqrt{m}}\left(\chi^2_{1j} - 1\right) + o_p(1). \qquad (1.76)
\end{aligned}$$

The first term of (1.76) is a martingale with conditional variance

$$\begin{aligned}
2\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})\frac{\mathrm{E}\left(D^2_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{m} &= 2\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})\frac{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{m}\frac{\mathrm{E}\left(D^2_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)} \qquad (1.77)\\
&= 2\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})\frac{D_{m,j}}{m}\frac{\mathrm{E}\left(D^2_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)} \\
&\quad -2\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})\frac{D_{m,j} - \mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{m}\frac{\mathrm{E}\left(D^2_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)} \\
&= 2\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})[\tau_j - \tau_{j-1}]\frac{\mathrm{E}\left(D^2_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)} + o_p(1), \qquad (1.78)
\end{aligned}$$

since $D_{m,j} - \mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)$ is a martingale difference array and the second term of (1.78) is therefore $o_p(1)$ by the martingale central limit theorem. By Assumption D2 we have

$$\mathrm{E}\left(D_{m,[ms]}|\mathcal{F}_{\tau_{[ms]-1}}\right) \to_p \mu_D(s), \quad \mathrm{E}\left(D^2_{m,[ms]}|\mathcal{F}_{\tau_{[ms]-1}}\right) \to_p \omega^2_D(s),$$

so that

$$\sum_{j=1}^{J_m} \sigma^4(\tau_{j-1})[\tau_j - \tau_{j-1}] \frac{\mathrm{E}\left(D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right)}{\mathrm{E}\left(D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right)} \to_p \int_0^1 \sigma^4(t) \frac{\omega_D^2(t)}{\mu_D(t)} dt,$$

giving the limit theory

$$\sqrt{m} \left\{ \sum_{j=1}^{J_m} [p^*(\tau_j) - p^*(\tau_{j-1})]^2 - \int_0^1 \sigma^2(t) dt \right\} \Rightarrow \mathcal{MN}\left(0, 2\int_0^1 \sigma^4(t) \frac{\omega_D^2(t)}{\mu_D(t)} dt\right),$$

in (1.14) as stated. The result under Assumption D1 follows when $\omega_D^2(t) = \omega_D^2$, $\mu_D(t) = \mu_D$ *a.s.* for all t.

To prove (1.15), we have the following decomposition of the estimated quarticity using (1.73)

$$\begin{aligned} m \sum_{j=1}^{J_m} [p^*(\tau_j) - p^*(\tau_{j-1})]^4 &= m \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) [B(\tau_j) - B(\tau_{j-1})]^4 \left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\} \\ &= m \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) [\tau_j - \tau_{j-1}]^2 \chi_{1j}^4 \left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\} \\ &= \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j}^2}{m} \chi_{1j}^4 \left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\} \\ &= \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j}^2}{m} \mathrm{E}\left(\chi_{1j}^4\right) \\ &\quad + \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j}^2}{m} \left\{\chi_{1j}^4 - \mathrm{E}\left(\chi_{1j}^4\right)\right\} + o_p(1). \end{aligned} \quad (1.79)$$

The second term of (1.79) is a martingale and of order $o_p(1)$, and since $D_{m,j}^2 - \mathrm{E}\left\{D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right\}$ and $D_{m,j} - \mathrm{E}\left\{D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right\}$ are both martingale differences we have

$$\begin{aligned} \mathrm{E}\left(\chi_{1j}^4\right) \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j}^2}{m} &= 3 \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{\mathrm{E}\left\{D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right\}}{m} + 2 \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j}^2 - \mathrm{E}\left\{D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right\}}{m} \\ &= 3 \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{\mathrm{E}\left\{D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right\}}{m} \frac{\mathrm{E}\left\{D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right\}}{\mathrm{E}\left\{D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right\}} + o_p(1) \\ &= 3 \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j}}{m} \frac{\mathrm{E}\left\{D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right\}}{\mathrm{E}\left\{D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right\}} \\ &\quad - 3 \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j} - \mathrm{E}\left\{D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right\}}{m} \frac{\mathrm{E}\left\{D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right\}}{\mathrm{E}\left\{D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right\}} + o_p(1) \end{aligned}$$

$$= \quad 3\sum_{j=1}^{J_m} \sigma^4(\tau_{j-1})[\tau_j - \tau_{j-1}] \frac{\mathrm{E}\left\{D_{m,j}^2 | \mathcal{F}_{\tau_{j-1}}\right\}}{\mathrm{E}\left\{D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right\}} + o_p(1)$$

$$\to_p \; 3\int_0^1 \sigma^4(t) \frac{\omega_D^2(t)}{\mu_D(t)} dt.$$

Hence, $\frac{2m}{3}\sum_{j=1}^{J_m}[p^*(\tau_j) - p^*(\tau_{j-1})]^4$ is a consistent estimate of the weighted quarticity functional $2\int_0^1 \sigma^4(t)\frac{\omega_D^2(t)}{\mu_D(t)}dt$, and it follows by joint convergence that

$$\sqrt{\frac{3}{2}} \frac{\sum_{j=1}^{J_m}[p^*(\tau_j) - p^*(\tau_{j-1})]^2 - \int_0^1 \sigma^2(t)dt}{\left\{\sum_{j=1}^{J_m}[p^*(\tau_j) - p^*(\tau_{j-1})]^4\right\}^{1/2}} \Rightarrow \mathcal{N}(0,1),$$

as required.

Proof of (1.19) Using (1.73), write

$$\frac{1}{m}\sum_{j=1}^{J_m} \frac{\{p^*(\tau_j) - p^*(\tau_{j-1})\}^2}{\tau_j - \tau_{j-1}}$$

$$= \frac{1}{m}\sum_{j=1}^{J_m} \frac{\sigma(\tau_{j-1})^2[B(\tau_j) - B(\tau_{j-1})]^2\left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\}}{\tau_j - \tau_{j-1}}$$

$$= \frac{1}{m}\sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})\chi_{1j}^2\left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\}$$

$$= \mathrm{E}\left(\chi_{1j}^2\right)\sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})\frac{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{m}\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}\left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\}$$

$$= \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})\frac{D_{m,j}}{m}\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}\left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\}$$

$$+ \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})\frac{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right) - D_{m,j}}{m}\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}\left\{1 + o_p\left(\frac{1}{\sqrt{m}}\right)\right\}$$

$$= \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1})(\tau_j - \tau_{j-1})\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)} + o_p(1)$$

$$\to_p \quad \int_0^1 \sigma^2(t)\mu(dt) = \int_0^1 \sigma^2(t)\frac{1}{\mu_D(t)}dt.$$

Proof of (1.20)
Proceeding as in (1.75) we find

$$\frac{1}{m}\sum_{j=1}^{J_m} \frac{\{p^*(\tau_j) - p^*(\tau_{j-1})\}^2}{\tau_j - \tau_{j-1}} - \int_0^1 \sigma^2(t)\mu(dt)$$

$$
\begin{aligned}
&= \sum_{j=1}^{J_m} \left\{ \frac{\{p^*(\tau_j) - p^*(\tau_{j-1})\}^2}{(\tau_j - \tau_{j-1})\, m} - \int_{\tau_{j-1}}^{\tau_j} \sigma^2(t) \frac{dt}{\mu_D(t)} \right\} - \int_{\tau_{J_m}}^{1} \sigma^2(t) \frac{dt}{\mu_D(t)} \\
&= \sum_{j=1}^{J_m} \sigma^2(\tau_{j-1}) \left\{ \frac{\chi^2_{1j}}{m} - \frac{\tau_j - \tau_{j-1}}{\mu_D(\tau_{j-1})} \right\} + o_p\left(\frac{1}{\sqrt{m}}\right). \qquad (1.80)
\end{aligned}
$$

In (1.80), note that the component in braces is a martingale difference because χ^2_{1j} is independent of $\mathcal{F}_{\tau_{j-1}}$ and

$$
\mathrm{E}\left\{ \left(\frac{\chi^2_{1j}}{m} - \frac{\tau_j - \tau_{j-1}}{\mu_D(\tau_{j-1})} \right) | \mathcal{F}_{\tau_{j-1}} \right\} = 0.
$$

The conditional variance is

$$
\begin{aligned}
\mathrm{E}\left\{ \left(\frac{\chi^2_{1j}}{m} - \frac{\tau_j - \tau_{j-1}}{\mu_D(\tau_{j-1})} \right)^2 | \mathcal{F}_{\tau_{j-1}} \right\} &= \mathrm{E}\left(\frac{\chi^2_{1j}}{m} \right)^2 - 2\mathrm{E}\left(\frac{\chi^2_{1j}}{m} \right) \frac{\mathrm{E}\left(D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right)}{m\mu_D(\tau_{j-1})} + \frac{\mathrm{E}\left(D^2_{m,j} | \mathcal{F}_{\tau_{j-1}}\right)}{m^2 \mu_D(\tau_{j-1})^2} \\
&= \frac{3}{m^2} - \frac{2}{m^2} + \frac{\omega^2_D(\tau_{j-1})}{m^2 \mu_D(\tau_{j-1})^2} \\
&= \frac{1}{m^2} \frac{\omega^2_D(\tau_{j-1}) + \mu_D(\tau_{j-1})^2}{\mu_D(\tau_{j-1})^2}.
\end{aligned}
$$

It follows that

$$
\sqrt{m} \left\{ \frac{1}{m} \sum_{j=1}^{J_m} \frac{\{p^*(\tau_j) - p^*(\tau_{j-1})\}^2}{\tau_j - \tau_{j-1}} - \int_0^1 \sigma^2(t) \mu(dt) \right\}
$$

is a local martingale with conditional variance

$$
\begin{aligned}
& \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{1}{m} \frac{\omega^2_D(\tau_{j-1}) + \mu_D(\tau_{j-1})^2}{\mu_D(\tau_{j-1})^2} \\
=& \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{\mathrm{E}\left(D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right)}{m} \frac{\omega^2_D(\tau_{j-1}) + \mu_D(\tau_{j-1})^2}{\mu_D(\tau_{j-1})^3} \\
=& \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{D_{m,j}}{m} \frac{\omega^2_D(\tau_{j-1}) + \mu_D(\tau_{j-1})^2}{\mu_D(\tau_{j-1})^3} \\
& + \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) \frac{\mathrm{E}\left(D_{m,j} | \mathcal{F}_{\tau_{j-1}}\right) - D_{m,j}}{m} \frac{\omega^2_D(\tau_{j-1}) + \mu_D(\tau_{j-1})^2}{\mu_D(\tau_{j-1})^3} \\
=& \sum_{j=1}^{J_m} \sigma^4(\tau_{j-1}) (\tau_j - \tau_{j-1}) \frac{\omega^2_D(\tau_{j-1}) + \mu_D(\tau_{j-1})^2}{\mu_D(\tau_{j-1})^3} + o_p(1) \\
\to_p & \int_0^1 \sigma^4(t) \frac{\omega^2_D(t) + \mu_D(t)^2}{\mu_D(t)^2} \mu(dt).
\end{aligned}
$$

By the martingale central limit theorem we deduce that

$$\sqrt{m}\left\{\frac{1}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j)-p^*(\tau_{j-1})\}^2}{\tau_j-\tau_{j-1}}-\int_0^1\sigma^2(t)\mu(dt)\right\}$$
$$\Rightarrow\quad \mathcal{MN}\left(0,\int_0^1\sigma^4(t)\frac{\omega_D^2(t)+\mu_D(t)^2}{\mu_D(t)^2}\mu(dt)\right),$$

as required.

Proof of (1.24): The empirical standardized quarticity is

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{J_m}\frac{\{p^*(\tau_j)-p^*(\tau_{j-1})\}^4}{(\tau_j-\tau_{j-1})^2}\\
=\;&\frac{1}{m}\sum_{j=1}^{J_m}\frac{\sigma(\tau_{j-1})^4[B(\tau_j)-B(\tau_{j-1})]^4\left\{1+o_p\left(\frac{1}{\sqrt{m}}\right)\right\}}{(\tau_j-\tau_{j-1})^2}
\end{aligned}$$

$$\begin{aligned}
=\;&\frac{1}{m}\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})\chi_{1j}^4\left\{1+o_p\left(\frac{1}{\sqrt{m}}\right)\right\}\\
=\;&\mathrm{E}\left(\chi_{1j}^4\right)\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})\frac{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}{m}\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}\left\{1+o_p\left(\frac{1}{\sqrt{m}}\right)\right\}\\
=\;&3\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})\frac{D_{m,j}}{m}\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}\left\{1+o_p\left(\frac{1}{\sqrt{m}}\right)\right\}\\
&+3\sum_{j=1}^{J_m}\sigma^2(\tau_{j-1})\frac{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)-D_{m,j}}{m}\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}\left\{1+o_p\left(\frac{1}{\sqrt{m}}\right)\right\}\\
=\;&3\sum_{j=1}^{J_m}\sigma^4(\tau_{j-1})(\tau_j-\tau_{j-1})\frac{1}{\mathrm{E}\left(D_{m,j}|\mathcal{F}_{\tau_{j-1}}\right)}+o_p(1)\\
\to_p\;&3\int_0^1\sigma^4(t)\mu(dt)=3\int_0^1\sigma^4(t)\frac{1}{\mu_D(t)}dt,
\end{aligned}$$

as stated.