UNIVERSITY OF SOUTHAMPTON

BROKEN DYNAMICAL SYMMETRIES IN QUANTUM MECHANICS AND PHASE TRANSITION PHENOMENA

by

Neil James Günther

A Thesis submitted in accordance with the requirements for the degree of Doctor of Philosophy.

Department of Physics

December 1979.

CONTENTS

TABLES AND FIGURES

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SCIENCE

PHYSICS

Doctor of Philosophy

BROKEN DYNAMICAL SYMMETRIES IN QUANTUM MECHANICS AND PHASE TRANSITION PHENOMENA.

by Neil James Günther

This thesis describes applications of dynamical symmetries to problems in quantum mechanics and many-body physics where the latter is formulated as a Euclidean scalar field theory in d-space dimensions. By invoking the concept of a dynamical symmetry group a unified understanding of apparently disparate results is achieved.

In the case of the quantum mechanical problem - a parametric oscillator with time-dependent parameters - the relevant dynamical symmetry group permits an explanation of the spectrum and an interpretation of the time-dependence as a symmetry breaking phenomenon. The breaking generator is found to transform as a $(\underline{6} \oplus \overline{\underline{6}})$ of SU(3).

In order to provide a foundation for our later discussion, the O(N) nonlinear sigma - model is reviewed as a field theory in the context of elementary particle physics and statistical mechanics. Spontaneous breaking of the internal O(N) symmetry implies the existence of (N-1) Goldstone modes. The effective Hamiltonian for these Goldstone modes furnishes a description of i) low energy pion interactions in elementary particle physics and ii) the statistical mechanics of the classical O(N) - invariant Heisenberg model of ferromagnetic systems. Loop diagrams in the effective Hamiltonian reveal that infra-red singularities are induced which inhibit a phase transition below two dimensions i.e. the critical temperature, T_c, moves to zero as the spatial dimension is lowered to 2.

Proceeding by analogy with the nonlinear sigma model we regard the Euclidean group as a dynamical symmetry applicable to critical phenomena in Ising-like systems with discrete internal symmetry. The Euclidean invariance is spontaneously broken below T_c by the existence of an interfacial boundary between thermodynamic phases in the Ising system. The corresponding Goldstone modes are identified with surface fluctuations in the boundary. An effective Hamiltonian is found in which the full Euclidean symmetry is nonlinearly realized on the Goldstone fields. In this way, several seemingly unrelated results for Ising systems can be understood in the same formalism. The ultra-violet renormalizability of these Hamiltonians above one-dimension, can be related to the absence of a phase transition in one-dimension. Also these effective Hamiltonians indicate the presence of a universal essential singularity at a first order phase transition in Ising systems.

PREFACE AND ACKNOWLEDGEMENTS

This thesis is the result of work performed under the supervision of Dr. D.J. Wallace in the Department of Physics, University of Southampton, between December 1976 and September 1979. It is a pleasure to thank him for his encouragement and advice during that period.

The work in Chapter Two is elaborated in several articles now published in the Journal of Mathematical Physics (see refs. [3-5]). The essential content of this chapter was presented at the VII International Conference on Group Theoretical Methods in Physics held at the University of Texas at Austin during September, 1978. I take this opportunity to record my thanks to Professor K.J. Barnes, Mr. E.D. Gordon (Senior Assistant Registrar) and the Advanced Studies Committee for the financial support which made possible my attendance at that conference.

Much of the content of Chapters Four and Five appears in a preprint entitled: "Goldstone Modes in Vacuum Decay and First Order Phase Transitions" written jointly with D.A. Nicole and D.J. Wallace and shortly to appear in Journal of Physics (A).

I am very grateful to Ian Affleck for making available an early copy of the manuscript of his Harvard Ph.D. Thesis.

My thanks are also due to those who are, and have been, members of the Theory Group for their assistance and friendship and, to several visitors, particularly Professors L.S. Schulman, M.T. Vaughn and R.K.P. Zia, for useful discussions.

The support of a Southampton University Research Award is gratefully acknowledged.

Special thanks to Hazel Paul for preparing the typescript.

CHAPTER ONE

PRELIMINARIES

§ I.1 Introduction:

The connection between a physical theory and experimental results may be established either through some dynamical principles contained in the theory or through some known invariance properties which the theory satisfies.These two routes for establishing the connection between theory and experiment need not be mutually exclusive. Symmetry concepts may prove useful when there is no dynamics available or they may provide restrictions on the form of the interaction dynamics. Also, known dynamics may give rise to larger symmetries than may be naively expected and thus explain a wider range of experimental data.

In this chapter it is our purpose to review in some detail the interplay between dynamics and invariance properties in quantum mechanics and many-body systems as described by quantum field theory. In particular we consider the concept of dynamical symmetries in quantum mechanics and quantum field theory. The value of group-theoretic concepts in working toward a coherent understanding of physical laws has been long appreciated and currently plays a quintessential role in attempts to formulate a unified view of particle interactions.

The material of this thesis is concerned with invoking the concept of dynamical symmetries in an effort to develop a unified explanation of some previously known results in quantum mechanics

and classical statistical mechanics.

The quantum mechanical problem in Chapter Two, when formulated in terms of the relevant dynamical symmetry group allows us to explain the known spectrum and further, to understand the existence of known invariances.

The other major problem which we investigate in this thesis is the existence of singular behaviour at a first-order phase transition in Ising-like systems in statistical mechanics. This phenomena can be formulated as a quantum field theory possessing a Euclidean invariance as a dynamical symmetry. The singular behaviour at the phase transition point can be consistently understood as a consequence of this dynamical symmetry. The details of a variety of results in phase transition phenomena for Ising systems are shown to be essentially understood when formulated in terms of dynamical symmetries.

We now turn to a preliminary discussion of the concepts which we shall employ in the rest of the thesis.

§ I.2 Dynamical Symmetries in Quantum Mechanics:

A symmetry operation is one which leaves a physical system invariant. It is well known that during the continuous time-development of a physical system, there are some dynamical quantities which remain constant. These quantities are often called constants of the motion. The description of point-particle dynamics in both classical mechanics and quantum mechanics is generally simplified when expressed in terms of these constants of the motion. The fact that such quantities exist at all is a manifestation of invariances of the system under some specified symmetry

operations. These constant properties are evidently universal among physical systems and are identified with conservation laws. In addition to those properties which are exactly conserved, there exist other properties which are only approximately conserved. Angular momentum and energy are examples of exact conservation laws. Isospin in nuclear systems is an example of a conservation law which holds only approximately.

The most convenient formulation in which to examine the connection between an invariance of a quantum mechanical system and the respective conservation law, is the Hamiltonian formalism. The existence of a conservation law implies that the Hamiltonian of the system is invariant under the corresponding (Lie) group of transformations. [For discrete transformations this statement is not true. The invariance of the Hamiltonian under some group of discrete transformations does not lead to a corresponding conservation law. Invariance under time reflection, $t \rightarrow -t$, is an example of this situation.]

A symmetry of the coordinate system alone is called a geometric symmetry. An example of this type of symmetry is provided by rotational invariance. When the symmetry depends explicitly on the coordinates and momenta expressing the forces of interaction between the point masses, then it is referred to as a dynamical symmetry. Conversely, the existence of a constant of motion or invariance principle may hold implications for the form which the dynamics can take. i.e. the allowable combination of coordinates and momenta in the Hamiltonian.

In quantum mechanics, such dynamical symmetry principles provide a complete understanding of conservation laws and play an

important role in classifying nuclear and atomic spectra and selection rules. The dynamical symmetries may be exact as is the case for the spectra of the nonrelativistic Hydrogen atom, or approximate as for example may be applied to the collective motion in nuclei. Essentially the Hamiltonian can be written in terms of the appropriate Casimir invariant whose eigenvalues furnish the corresponding energy eigenvalues of the original Hamiltonian.

Programmes of this type apply to systems where the force law is conservative. However, there are systems which are not closed, but the action of the external source results only in a time variation of the parameters in the Hamiltonian. The parametric harmonic oscillator is an example of such a system - an LC-circuit with variable capacitance. In Chapter Two we show that the unexpected presence of a dynamical symmetry permits an analysis of the quantum mechanical spectrum in a manner similar to that already described for closed systems.

§ I.3 Dynamical Symmetries in Quantum Field Theory:

Quantum field theory is a formulation of many-particle interactions described by classical fields carrying an infinite number of degrees of freedom. These fields are said to be local in the sense that they obey differential equations of propagation. The fields can be quantized in a manner similar to that used for dynamical variables in quantum mechanics - canonical quantization. Particle states, then correspond to quantum excitations of the fields defined as operators. This prescription is not unique for quantum field theory just as the canonical formalism is not a unique prescription for the development of quantum mechanics from classical mechanics.

In fact, the prescription we shall employ in this thesis is the functional integral formalism; a prescription based on the Feynman Path Integral formulation of quantum mechanics. The amplitude for a prescribed process is written as a functional average over the classical fields. Quantum mechanics is a field theory in one time and zero space dimensions. Partial integration of the functional average can be shown to correspond to the usual Wick expansion of the corresponding time-ordered product. In this way, the connection with conventional perturbation theory is established. In the functional method the fields are never defined as operators as in the canonical formalism.

Symmetries are also present in field theories. As in quantum mechanics, some of these symmetries are space-time dependent, e.g. reflections and Poincaré invariance, while others are referred to as internal symmetries, e.g., gauge invariance and chiral rotations In view of the remarks of § 1.2 we expect the content of a given invariance principle to impose constraints on the form which the field interactions can take and similarly this kind of invariance is conventionally referred to as a dynamical symmetry. Gauge covariance is such a principle where the invariance is realized nonlinearly on the classical fields treated as coordinates. Another example of a dynamical symmetry where the symmetry transformations are realized nonlinearly on the field coordinates is the so called "Non-linear σ-model" with O(N) - internal symmetry. The realization of the symmetry constrains the interaction of the fields to take on a particular form. The implications of this kind of dynamical symmetry for both particle physics and statistical mechanics will be discussed in detail in Chapter Three.

§ I.4 Dynamical Symmetries in Statistical Mechanics:

Equilibrium classical statistical mechanics is a formulation of equilibrium thermodynamics in terms of many -body interactions. The conventional starting point and fundamental thermodynamic quantity is the"Partition Function"which is a sum over thermodynamic configurations each weighted with a Boltzman probability factor, $\exp - E^n/k_BT$, where E^n is the energy of the respective n^{th} configuration.

An identification between the formalisms of statistical mechanics and relativistic quantum field theory can be established, particularly in the context of the functional formalism mentioned in § 1.3. They are both many-body theories. Essentially a statistical random variable may be associated with a scalar field, e.g., the magnetization density. The energy of the Boltzman exponent is replaced by a (reduced) Hamiltonian density defined as a Euclidean integral over d-spatial dimensions. The partition function then becomes a functional integral over all field configurations. Physical results are again computed from a Feynman graph expansion. In this sense statistical mechanics is equivalent to a Euclidean scalar field theory in d-space dimensions.

The O(N) nonlinear σ-model referred to in the last section is an N-component field theory which can also be expressed in functional formalism. Furthermore, it also corresponds to the statistical mechanics of the classical O(N) symmetric Heisenberg model of a ferromagnet. We explore the consequences of this model in Chapter Three and find that below the critical temperature, T_c , the original O(N)-symmetry is spontaneously broken and the ensuing (N-1) Goldstone modes control the low temperature behaviour of the

system. In particular for d = 2 the Goldstone modes induce infra-red singularities which destroy the possibility of long range order, thus inhibiting spontaneous magnetization for all temperatures.

In Chapters Four and Five we investigate an analogous approach for Ising-like systems, with a discrete internal symmetry, based on the notion that there is an underlying Euclidean dynamical symmetry for these systems. This Euclidean invariance is spontaneously broken by the interfacial boundary between the discrete coexisting phases described by the original Ising Hamiltonian. These surface fluctuations can be interpreted as the Goldstone modes of the spontaneously broken Euclidean symmetry. An effective Hamiltonian is introduced in which the Goldstone fields carry a nonlinear realization of the full Euclidean invariance. The form of these Hamiltonians determines, i) the interaction of the Goldstone modes which lower the critical temperature to zero as d is lowered to one (a result analogous to that for Heisenberg ferromagnets), ii) the nature of the universal essential singularity at a first order phase transition in models of critical nucleation which are Euclidean invariant.

CHAPTER TWO

DYNAMICAL SYMMETRIES FOR A TIME-DEPENDENT QUANTUM MECHANICAL PROBLEM

§ II.1 Geometric and Dynamical Symmetries:

By way of introduction to some of the group-theoretic concepts that will be employed in later chapters we present, in this chapter, some relevant results of recent and original investigations into the role of dynamical symmetries for an isotropic harmonic oscillator where the natural frequency, ω , of the oscillator is replaced by an arbitrary function of time, $\omega = \omega(t)$ [1-5].

In what follows, our attention will centre mostly on the 3-dimensional case although many of our results are immediately generalizable to the case of n-dimensions. We will open our discussion by recalling, briefly, the appropriate dynamical symmetries for the classic paradigms; the Kepler-Coulomb problem and the time-independent harmonic oscillator. The algebraic realization of the symmetry properties is conveniently expressed directly in terms of the canonical dynamical variables, $q(t)$ and $p(t)$, although we are aware that alternative realizations may also be appropriate e.g. ladder operator formalism. For the canonical realization, statements referring to symmetry properties in the context of classical mechanics are generally taken over into the quantum mechanical context by virtue of the replacement : Poisson bracket $\rightarrow$ Commutation relation. Our discussion will be quantum mechanical with the understanding that much of what is said applies to classical mechanics. We will not labour these technicalities. It will emerge that when viewed from the vantage point of dynamical symmetries, many seemingly disparate facets can be related in a very powerful way.

The hydrogen atom† (Kepler-Coulomb problem) and the isotropic harmonic oscillator are historically and conceptually the outstanding paradigms for systems which exhibit dynamical symmetry. The additional symmetry (beyond overt rotational symmetry) in these systems, arises as a consequence of the particle interactions belonging to the special cases of a conservative law of central force which permit closed orbits in the bound state. This is called Bertrand's theorem in classical mechanics.

More precisely the Hamiltonian specified in an n-dimensional coordinate system remains symmetric under a group of transformations larger than SO(n), the group of special rotations about the centre of force. For the bound states of the hydrogen atom, the larger symmetry group is SO(n+1) and for the oscillator it is SU(n). These larger symmetries are associated with the existence of further conserved quantities (constants of integration in the context of classical mechanics) which relate to the fixed, non precessive, orientation of the classical orbits.[6] In the Kepler-Coulomb problem [7] the centre of force is at one of the foci of the elliptical orbit and the orientation can be specified by an n-vector directed along the major axis with magnitude proportional to the eccentricity of the ellipse.†† In the classical oscillator the centre of force is at the centre of

† Here we are referring to the non-relativistic hydrogen atom as described by the Schödinger equation. The relativistic hydrogen atom, as described by the Dirac equation, also possesses dynamical symmetries, different from the SO(4) of the non relativistic case since the orbital angular momentum, $\underline{L}$, is no longer conserved. As an example, the sign degeneracy of the operator $K = \beta(\underline{\sigma}\cdot\underline{L} + \hbar)$ is associated with the constant psuedo-scalar

$$-\gamma_5(\gamma^0 K + i\,\underline{\alpha}\cdot\hat{\underline{r}} - K H)$$

The author has given a more complete discussion elsewhere.[1]

†† In the 3-dimensional Kepler-Coulomb problem the extra constant of the motion is most often referred to as the Runge-Lenz vector after C. Runge, Vektor analysis. Vol 1,70 (Hirzel, Leipzig, 1919) and W. Lenz, Z. Physik 24, 197 (1924), although this result was known previously to W.R. Hamilton in his rarely cited work. Proc. Roy. Irish Acad. 3, 441 (1847).

the ellipse and the orientation is specified by a second rank symmetric tensor aligned with the principal axes. [8]

These larger symmetries are not regarded as purely geometric in nature since they are not obtained by coordinate mappings alone. The extra dynamical invariants are explicit functions of the canonical momentum and the potential appearing in the original Hamiltonian and therefore must be represented in terms of the full phase-space. This explicit dependence on the form of the force law has led to the term "dynamical symmetry† " appearing in the literature.

Dynamical symmetries appear to be a signal for a further useful property. Their existence may be tantamount to the presence of closed form solutions for problems which,on superficial inspection,only admit "non-analytic" solution e.g. using perturbative or iterative methods.[10] This observation is a consequence of the work we shall describe in the following section and we will elaborate our remarks there.

Finally, it is worth noting that even if an exact dynamical symmetry does not exist, in the sense that all the appropriate generators commute with the Hamiltonian, an approximation to such a symmetry e.g. nearly all generators commute with the Hamiltonian, may permit useful classification of solutions [11] or algebraic evaluation of higher order corrections. [12]

We now turn to our results for the time-dependent harmonic oscillator.

† In the field-theoretic literature this terminology is applied generally to symmetries which are realized non-linearly on the field coordinates [9], as will be discussed in the following chapters.

§ II.2 Time-dependence as Symmetry Breaking:

The time dependent Hamiltonian

$$H(t) = \frac{1}{2}\sum_{i}^{3} (p_i p_i + \omega^2(t)\, q_i q_i) \; ; \; i,j = 1,2,\ldots,n \qquad (2.1)$$

in the one-dimensional case, (n=1), has been investigated by Lewis and Riesenfeld [13]. Our discussion will, for the most part, be restricted to $n = 3$ dimensions since some of the most interesting results occur in that case. For appropriate generalizations, the relevant literature [2,3] may be consulted.

Clearly, the Hamiltonian in equation (2.1) is not a dynamical invariant (constant of the motion) and historically the impulse to employ perturbative methods [10] has dominated the literature. However, the author [1, 2] has established the existence of a symmetric tensor invariant.

$$A_{ij}(q,p,\rho,\dot{\rho}) = \frac{1}{2}\left\{\rho^{-2} q_i q_j + (\rho p - \dot{\rho} q)_i\, (\rho p - \dot{\rho} q)_j\right\} \qquad (2.2)$$

where the auxiliary function $\rho(t)$ is required to be a solution of the equation

$$\rho^3(t)\left\{\ddot{\rho}(t) + \omega^2(t)\,\rho(t)\right\} = 1 \qquad (2.3)$$

and p_i and q_i are the usual canonical coordinates satisfying the commutation relations

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = i\,\delta_{ij}$$

The total time-derivative vanishes

$$i \frac{d}{dt} A_{ij}(q,p,\rho,\dot{\rho}) = \left[A_{ij}, H(t)\right] + \dot{\rho} \frac{\partial}{\partial \rho} A_{ij} + \ddot{\rho} \frac{\partial}{\partial \dot{\rho}} A_{ij} = 0 \quad (2.4)$$

and the algebraic properties of A_{ij} are recorded for completeness.

(a) $$A_{ij} = A_{ji}$$

(b) $$A_{ij} L_j = 0 = L_i A_{ij}$$

(c) $$\mathrm{Tr}\, A_{ij} = A$$

(d) $$A_{ik} A_{kj} = A A_{ij} - \frac{1}{4}\left\{L_i L_j - \delta_{ij} L^2 + 2i\left[L_i, L_j\right] - 2\delta_{ij}\right\}$$

(e) $$A_{ii} A_{jj} - A_{ij}^2 = \frac{1}{4}\left\{(\varepsilon_{kij} L_k) + 1\right\} \quad \text{(no sum on } i, j) \quad (2.5)$$

(f) $$q_i(A\delta_{ij} - A_{ij})q_j = \frac{1}{2} \rho^2 L^2$$

(g) $$p_i(A\delta_{ij} - A_{ij})q_j = q_j(A\delta_{ij} - A_{ij})p_j = \frac{1}{2} \rho\dot{\rho} L^2$$

(h) $$p_j(A\delta_{ij} - A_{ij})p_j = \frac{1}{2}(\rho^{-2} + \dot{\rho}^2)L^2$$

The antisymmetric $\underline{3}$ - tensor

$$L_k = i\, \varepsilon_{ijk}\, q_i\, p_j \quad (2.6)$$

together with the traceless symmetric $\underline{5}$ - tensor

$$B_{ij} = A_{ij} - \frac{\delta}{3} ij \sum_{k}^{3} A_{kk} \quad (2.7)$$

span the regular $\underline{8}$ - representation of an SU(3) algebra

$$\left[M_\mu, M_\nu\right] = 2i\, C_{\mu\nu k}\, M_k; \mu,\nu,k = 1,2,\ldots,8 \quad (2.8)$$

where

$$L_1 = M_7 \ , \ L_2 = - M_5 \ , \ L_3 = M_2$$

$$B_{12} = M_1 \ , B_{23} = M_6 \ , B_{13} = M_4$$

$$\frac{1}{\sqrt{3}} (B_{11} + B_{22} - 2 B_{33}) = M_8$$

$$B_{11} - B_{22} = M_3$$

An important point here is that this algebra closes on the $\mathrm{Tr}\, A_{ij}$ and not on the Hamiltonian in equation (2.1). This is a reflection of the statement in equation (2.4) and may be compared with the results for the $n = 3$ time independent ($\omega = \text{const}$) Hamiltonian (See Table 1).

The tensor invariant in equation (2.2) satisfies the eigenvalue equation

$$\sum_j^3 A_{ij}(q,p,\rho,\dot{\rho}) (X_j) = \xi \ (X_i) \tag{2.9}$$

with eigenvalues

$$\xi = \begin{pmatrix} \lambda^{\pm} \\ \lambda_o \end{pmatrix} \equiv \begin{pmatrix} A & \pm \sqrt{A^2 - L^2} \\ & 0 \end{pmatrix} \tag{2.10}$$

and associated (un-normalized) eigenvectors, (X_j)

$$\begin{pmatrix} \lambda^+ A_{31} + \frac{1}{4} L_3 L_1 \\ \lambda^+ A_{32} + \frac{1}{4} L_3 L_2 \\ \lambda^+ A_{33} - \frac{1}{4} (L^2 - L_3^2) \end{pmatrix} , \begin{pmatrix} \lambda^- A_{31} + \frac{1}{4} L_3 L_1 \\ \lambda^- A_{32} + \frac{1}{4} L_3 L_2 \\ \lambda^- A_{33} - \frac{1}{4} (L^2 - L_3^2) \end{pmatrix} , \ \underline{L} \tag{2.11}$$

Table 1. [3]

Time-independent	Time-dependent
H a constant of the motion	H(t) not a constant of the motion
Tr A_{ij} an SU(3) invariant	Tr $A_{ij}(\rho,\dot{\rho})$ an SU(3) invariant
$[H,L_i] = 0$	$[H(t),L_i] = 0$
$[H,A_{ij}] = 0$	$[H(t),A_{ij}(\rho,\dot{\rho})] \neq 0$
$\dot{A}_{ij} = 0$	$\dot{A}_{ij}(\rho,\dot{\rho}) = 0$
Tr $A_{ij} = H$	Tr $A_{ij}(\rho,\dot{\rho}) \neq H(t)$

A discussion of the relationship between the matrix A_{ij} and a description of the motion is simplified by a suitable orthogonal transformation of the coordinates q_i to $\tilde{q}_i$ such that the axes are aligned with the eigenvectors and the transformed matrix is

$$\tilde{A}_{ij} = \begin{pmatrix} \frac{1}{2}\left\{A + \sqrt{(A^2-L^2)}\right\} & 0 & 0 \\ 0 & \frac{1}{2}\left\{A - \sqrt{(A^2-L^2)}\right\} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In this coordinate system, equation (2. 5 f) becomes

$$\frac{\tilde{q}_1^2}{\rho^2\left\{(A + \sqrt{A^2-L^2})\right\}} + \frac{\tilde{q}_2^2}{\rho^2\left\{(A - \sqrt{A^2-L^2})\right\}} = 1 \qquad (2.12)$$

When $\omega(t)$ is constant, ω_o, and $\rho = \sqrt{\omega_o}$ equation (2.12) is the orbit equation of the time-independent oscillator, VIZ. an ellipse with semi axis of length

$$a^2 = \rho^2\left\{(A + \sqrt{A^2-L^2}\ )\right\}$$

$$b^2 = \rho^2\left\{(A - \sqrt{A^2-L^2}\ )\right\} \qquad (2.13)$$

For the time-dependent case equation (2.12) does not provide a complete description of the orbit since, although at any instant (2.12) describes an ellipse, at successive instants it describes a family of ellipses which are expanding or contracting in accordance with the variation in $\rho(t)$. At all times, because the eigenvalues and eigenvectors of A_{ij} are constant, the orientation of these ellipses

is unaltered. The path undertaken depends on the initial conditions and the time evolution of ρ. Thus SO(3) is maintained as an invariance symmetry because of the central force. The larger SU(3) group is not a true symmetry of the total Hamiltonian in the sense that the extra generators do not commute with $H(t)$. The term "noninvariance" symmetry has been adopted in this situation [2,3]. However, the above remarks show how the symmetry properties can be exploited to give concise description of the motion.

One could take the view that the introduction of an arbitrary time-dependent frequency, $\omega(t)$ into the 3-dimensional time-independent Hamiltonian has the effect of "breaking" the usual SU(3) dynamical symmetry to SO(3) and we have found a way of giving a group-theoretic description of this view point.

The form of the operators which are responsible for breaking the SU(3) invariance is perhaps not immediately obvious because of the "sourceless" form of the Hamiltonian i.e. the arbitrary function of time, $\omega(t)$ represents the result of some time-dependent dynamical effects e.g. contact with a thermal reservoir or electromagnetic interactions. This Hamiltonian is suggestive of so-called "open systems" [14]. However, as we shall show, the symmetry of the breaking generator is independent of the arbitrary function of time. This is to be expected since the symmetry algebra is constructed only from canonically conjugate variables.

Symmetry breaking effects are most conveniently expressed in the interaction picture with the total Hamiltonian written in the partitioned form $H = H_o + gH_1$ where g is an arbitrary constant and in general H, H_o, and H_1 do not commute with each other. If H_o is exactly invariant under a dynamical group G having subgroup G_o, and H is invariant under G_o then H_1 is also invariant under G_o. For our Hamiltonian, we have $G \sim SU(3)$ and $G_o \sim SO(3)$.

Note the branching rules given in the Appendix lead to $G \downarrow G_o : \underline{8} \supset \underline{5} \oplus \underline{3}$ confirming our identification of L_k and B_{ij} given in equation (2.6) and (2.7) respectively. We are now faced with the problem of finding the interaction expression for the Hamiltonian (2.1). Clearly we must have H_o time-independent with all explicit time-dependence relegated to the interaction term H_1 and, H_1 must be $SO(3)$ invariant. By writing the Hamiltonian, $H(t)$ in this form the symmetry breaking can be discussed along conventional lines. [11, 15] This is achieved by noting that since $\omega(t)$ in equation (2.1) is an arbitrary function, we may simply write $\omega^2(t) = \omega_o^2 + \Omega^2(t)$ to give an interaction expression for the $H(t)$ - Hamiltonian.

$$H = \frac{1}{2}(p_i^2 + \omega_o^2 q_i^2) + \frac{1}{2}\Omega^2(t) q_i^2 \tag{2.14}$$

The canonical variable in $H_1 = \frac{1}{2}\Omega^2(t) q_i^2$ evidently is not a realization of the $SU(3)$ generators, so that the total Hamiltonian cannot be expressed directly in terms of the quadratic Casimir invariant.

$$C_2 = L_k L_k + B_{ij} B_{ij} \tag{2.15}$$

as in the case of collective nuclear motion [11]. However, we can determine how H_1 transforms under the infinitesimal $SU(3)$ generators.

An intuitive feel for the possible $SU(3)$ content of H_1 is gained by recalling the familiar ladder operator realization.

$$\begin{aligned} a &= q + i\,p \\ a^+ &= q - i\,p \end{aligned} \tag{2.16}$$

identified with $\underline{3}$ and $\overline{\underline{3}}$ respectively in the fundamental representation. Then we write, $(\omega = 1)$

$$\begin{aligned} q^2 &\equiv \tfrac{1}{2}(q^2+p^2)+ \tfrac{1}{2}(q^2-p^2) \\ &\equiv \tfrac{1}{4}(q+i\ p)(q+i\ p) + \tfrac{1}{4}(q-i\ p)(q-i\ p) + \tfrac{1}{2}(q^2+p^2) \\ &= \tfrac{1}{2}(a\ a + a^+a^+) + H_o \\ &= \underline{6} \oplus \overline{\underline{6}} \oplus \underline{1} \end{aligned} \tag{2.17}$$

since q^2 is totally symmetric in a and a^+.

So we can anticipate $\underline{6}$ or $\overline{\underline{6}}$ or a linear combination of both to provide a suitable representation for H_1.

It is interesting to confirm this result using a method by which the irreducible $SU(3)$ representations, $M(\alpha,\beta)$ of an arbitrary operator $\Omega_b(M)$ may be found in terms of the eigenvalues (modulo signs) of the quadratic Casimir invariant [16].

$$C_2^{[\alpha,\beta]} = \tfrac{1}{3}(\alpha^2 + \alpha\beta + \beta^2 + 3\alpha + 3\beta) \tag{2.18}$$

Theorem : If G_a are the infinitesimal generators of $SU(3)$ in the regular representation then

$$\tfrac{1}{2}\sum_a \left[G_a,[G_a,\ \Omega_b(M)]\right] = C_2(M)\cdot\Omega_b(M) \tag{2.19}$$

where

$$C_2 \equiv \sum_a G_aG_a \ ;\ a = 1,2,\ldots,8 \tag{2.20}$$

The content of the theorem can be understood in the following way. The infinitesimal action of the group generators on an arbitrary operator is given by

$$U^{+}(w_a)\Omega_b\, U(w_a) = \Omega_b + i\, w_a [G_a, \Omega_b] + \ldots .$$

where we have written $U = \exp.\ i\, w_a\, G_a$. In general the infinitesimal group action on an operator defined in a particular representation is expressed in terms of commutators according to

$$\begin{aligned} G_a\ \Omega_b(M) &: \rightarrow [G_a,\ \Omega_b(M)] \\ G_a \cdot G_a\ \Omega_b(M) &: \rightarrow \left[G_a, [G_a,\ \Omega_b(M)]\right] \end{aligned} \tag{2.21}$$

By summing over repeated indices and using the definition in equation (2.20), the last expression in equation (2.21) becomes

$$\frac{1}{2} \sum_a \left[G_a, [G_a,\ \Omega_b]\right] \equiv \sum_a G_a G_a\ \Omega_b = C_2\ \Omega_b \tag{2.22}$$

where the factor of 2 provides a convenient normalization in agreement with equation (2.19).

It should be emphasised that since the time-dependence in H_1 stands as a factor only, we are free to go ahead and use the time-independent commutation relations in the same spirit as in equation (2.8).

The commutation relation $[L_i, q_j^2] = 0$ merely expresses the invariance of $q^2 (p^2$ is also an $SO(3)$ invariant). Consequently the value of the Casimir in equation (2.19) is determined by the generator B_{ij} alone. VIZ.

$$\frac{1}{2} \sum_{ijk} \left[B_{ij}, [B_{ij}, q_k^2]\right] \equiv C_2 \sum_k q_k^2 \tag{2.23}$$

Evaluating the inner commutator gives

$$\sum_k \left[B_{ij}, q_k^2\right] = -i\left\{(q_i p_j + p_i q_j) - \frac{1}{3}\delta_{ij}(qp + pq)\right\} \equiv -i\, T^{\circ}_{ij} \qquad (2.24)$$

Substituting this result in equation (2.23) gives

$$-\frac{i}{2}\sum_{ij}\left[B_{ij},\, T^{\circ}_{ij}\right] \qquad (2.25)$$

$$= -\frac{i}{4}\left\{\sum_{ij}\left[(q_i q_j + p_i p_j),(q_i p_j + p_i q_j)\right] - \frac{1}{3}\sum_j\left[(q^2 + p^2)_j,(qp + pq)_j\right]\right\}$$

The first term reduces to

$$i\sum_{ij}\left\{(q_j^2 + 3q_i^2) + (3q_j^2 + q_i^2) - (3p_j^2 + p_i^2) - (p_j^2 + 3p_i^2)\right\}$$

$$= 8i(q^2 - p^2) \qquad (2.26)$$

The second term in equation (2.25) reduces to

$$2\sum_j\left\{\left[q^2,\, qp\right] + \left[p^2,\, qp\right]\right\}$$

$$= 4i(q^2 - p^2) \qquad (2.27)$$

Combining equations (2.26), (2.27) and substituting into equation (2.25) we find

$$-\frac{i}{2}\sum_{ij}\left[B_{ij},\, T^{\circ}_{ij}\right] = 2(q^2 - p^2) - \frac{1}{3}(q^2 - p^2)$$

$$= \frac{10}{3}\cdot\frac{1}{2}(q^2 - p^2) \qquad (2.28)$$

This result looks anomalous, since we commenced the calculation with the operator q^2 in equation (2.23) but an additional p^2 term has developed on the right-hand side of equation (2.28). This apparent discrepancy is easily understood by recalling that the theorem requires the operator Ω_b to be in an irreducible representation already and this is not true of q^2 .

Recalling, the decomposition of q^2 in equation (2.17) we see that the SU(3) singlet, $\frac{1}{2}(q^2 + p^2)$ furnishes a zero eigenvalue in the double commutator of equation (2.23). The coefficient $\frac{10}{3}$ in equation (2.28) enables us to identify $\frac{1}{2}(q^2 - p^2)$ as belonging to $\underline{6}$ or $\overline{\underline{6}}$ corresponding to $C_2^{[2,0]} = C_2^{[0,2]} = \frac{10}{3}$.

Yet another confirmation of our result comes from the recognition of $\frac{1}{\Omega^2(t)} H_1$ as the trace of one of the bilinear generators [17].

$$(q_i q_j) \ , \ (p_i p_j) \ , \ (q_i p_j + p_i q_j)$$

which span the regular representation of $Sp(6,\mathbb{R})$. Using the branching rule method described in the Appendix we have the following Young tableaux expansion

$$Sp(6) \downarrow SU(3) :$$

$$\langle 2 \rangle \supset \{2,1\} \oplus \{2\} \oplus \{1^2\} \oplus \{1\}$$

$$= \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \oplus \boxed{3}\boxed{4} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \cline{1-1}\end{array} \oplus \boxed{3}$$

$$= \underline{8} \oplus \underline{6} \oplus \overline{\underline{6}} \oplus \underline{1} \ .$$

We have referred to this result as $(\underline{6} \oplus \overline{\underline{6}})$ - breaking [4, 5]

§ II.3 Projective Motion:

In this section we give some consideration to the physical nature of the dynamical invariant, A_{ij} discussed in the previous section.

In doing so we will establish a point of contact with other examples of systems involving dynamical symmetries. Let us return to the simple one-dimensional harmonic oscillator and its associated scalar constant of the motion first investigated by Lewis and Riesenfeld [13].

$$A = \frac{1}{2}\left\{\rho^{-2}q^2 + (\rho p - \dot{\rho} q)^2\right\} \tag{2.29}$$

Now we introduce the 2-dimensional equation of motion

$$\ddot{\underline{x}} + \omega^2(t)\,\underline{x} = 0 \tag{2.30}$$

referred to as the "auxil iary" equation of motion for the one-dimensional oscillator. On resolving the vector $\underline{x}$ into its polar components $(\rho \cos\theta, \rho \sin\theta)$ the radial equation of motion becomes

$$\ddot{\rho} + \omega^2(t)\rho = k^2 \rho^{-3} \tag{2.31}$$

where $k = \rho^2 \dot{\theta}$ is the magnitude of the angular momentum in the direction perpendicular to the plane of the auxilliary motion. Moreover, the invariant A reduces to

$$A = \frac{1}{2}\{k^2 \cos^2\theta + k^2 \sin^2\theta\} = \frac{1}{2}k^2 \tag{2.32}$$

This projection of the auxiliary motion into radial and angular components provides us with a simple interpretation of A as half the square of the auxiliary angular momentum [18]. Notice the welcome feature that the equation of constraint (and $\rho(t)$ in particular) is placed on a purely dynamical footing [19]. Furthermore, since the expression for A has no explicit dependence on the radial function $\rho(t)$ it is manifestly

time-independent. This approach can be generalized to higher dimensional motion [2]. For the n-dimensional oscillator Hamiltonian one must invoke a projection into the n+1 - dimensional (Euclidean) hyperspace.

The perspicacity rendered by the use of projective motions has arisen elsewhere in the context of dynamical symmetries in quantum mechanics. The quantum mechanical counterpart of an $SU(2) \otimes SU(2)$ invariant Lagrangian [20] was observed to correspond to the motion of an isotropic oscillator on the surface of a 3-dimensional sphere with curvature λ [21]. The radial equation of motion on the sphere is [20].

$$\ddot{\rho} + \frac{\lambda\, \rho\, (\dot{\rho})^2}{(1-\lambda\, \rho^2)} + \frac{\omega^2 \rho}{(1-\lambda\rho^2)} = (1-\lambda\rho^2)\, K^2 \rho^{-3} \qquad (2.33)$$

where

$$K^2 = k^2 + \rho^4\, \dot{\psi}^2$$

$$\omega = \text{const}$$

In Euclidean space $(\lambda=0)$ with $\omega = \omega(t)$ equation (2.33) corresponds to the 3-dimension analogue of equation (2.31).

§ II.4 Summary:

In this chapter we have discussed an example of a dynamical symmetry for the case of a particle Hamiltonian. We investigated how the dynamical symmetry may be exploited to yield a concise description of the motion. The introduction of an explicit time-dependence into the Hamiltonian was analyzed from the point of view of a symmetry breaking effect. Some lesser known group-theoretic techniques were found useful for this discussion. Some of the results in § II.2 of this chapter have been exploited and developed by other authors. [22]

The remaining chapters of this thesis will be concerned with exploiting dynamical symmetries in the context of quantum field theory.

APPENDIX

Branching Rules [23, 24]

The problem of determining those irreducible representations of multiplets in a given subgroup H of G may be resolved in the following way. Let λ be an irreducible representation of G and μ an irreducible representation of H. In general $G \downarrow H$ for the λ representation yields a set of matrices $\{\lambda(h):h\varepsilon H\}$ which are reducible so that

$$G\downarrow H \sim \lambda\downarrow \sum_{\mu} \oplus B^{\mu}_{\lambda}.\ \mu$$

where the μ are inequivalent irreducible representations and the B^{μ}_{λ} are the branching coefficients. We list the branching rules for the groups discussed.

(i) $U(n,\mathbb{C})\downarrow O(n,\mathbb{R})$:

Here the dimensionality $D_{\eta}[\lambda]$ is defined by

$$N_n[\lambda] = \prod_{(i,j)} \left[n-i+j+\delta_{ij\lambda}a_k-(1-\delta_{ij\lambda})(1+b_k)\right]$$

where $\delta_{ij\lambda} = \begin{cases} 1 \text{ if diagonal through } (i,j) \cap \text{ box at end of row.} \\ 0 \text{ otherwise} \end{cases}$

a_k is the arm length and b_k is the leg length measured from the (i,j) box.

Then

$$\{\lambda\} \supset \sum_{\{\delta\}} \oplus \left[\frac{\lambda}{\delta}\right]$$

is specified by $\{\delta\}= \{0\}, \{2\}, \{4\}, \{2^2\}, \{6\}, \{4,2\},....$

(ii) $\underline{Sp(2n, \mathbb{R}) \downarrow U(n,\mathbb{C})}$:

The dimensionality $D_n \langle\lambda\rangle$ is defined by

$$N_n \langle\lambda\rangle = \prod_{(i,j)} \left[n-i+j-\delta_{ij\tilde{\lambda}} b_k + (1-\delta_{ij\tilde{\lambda}})(1 + a_k)\right]$$

where

$$\delta_{ij\tilde{\lambda}} = \begin{cases} 1 \text{ if diag. through } (i,j) \cap \text{box at bottom of column.} \\ 0 \text{ otherwise} \end{cases}$$

Then

$$\langle\lambda\rangle \supset \sum_{\{\xi,\delta\}} \oplus \{\bar{\xi}; \frac{\lambda}{\delta\xi}\}$$

is specified by $\{\xi\}=\{0\}, \{1\}, \{1^2\}, \{2\},....$

CHAPTER THREE

O(N) NONLINEAR σ - MODEL

AND

GOLDSTONE SINGULARITIES

§ III.1 SU(2) ⊗ SU(2) Chiral Lagrangian for Pions:

The ideas we shall review in this chapter had their origin within the context of early attempts to understand the dynamics underlying strong interaction symmetries. One of the best approximated symmetries of the strong interactions is SU(3). In the absence of any symmetry breaking effects, the Hadrons can be assigned to mass multiplets belonging to linear representations of SU(3). Deviations from this symmetry ("explicit symmetry breaking") can then be understood in terms of corrections to these linear representations along the lines of conventional perturbation theory. (See § II.2).

The strong interactions appear also to display the approximate chiral SU(2) ⊗ SU(2) symmetry associated with parity and isospin.

Here, even in the absence of an explicit symmetry violating term, SU(2) ⊗ SU(2) is not a symmetry of the physical state spectrum in the sense that, the particle states no longer occur in degenerate mass multiplets which can be expressed as a linear representation of SU(2) ⊗ (SU(2).

It will prove useful to consider these remarks in more detail within the context of the so-called "Linear σ - model" [25]. In the case of a two-component field, (σ,π), the O(2) Lagrangian is given by:

$$L_o = \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi\partial^\mu\pi) + \frac{1}{2}\mu^2(\sigma^2 + \pi^2) - \frac{1}{4}g(\sigma^2 + \pi^2)^2 \qquad (3.1)$$

Conventionally, $g > 0$ ensures the positivity of the Hamiltonian. For $\mu^2 > 0$, there is a unique minimum of the potential (vacuum) at

$$\sigma\left[\mu^2 + g(\sigma^2 + \pi^2)\right] = \pi\left[\mu^2 + g(\sigma^2 + \pi^2)\right] = 0$$

i.e. $\sigma = \pi = 0$, so that the ground state transforms as a singlet under $O(2)$.

When $\mu^2 < 0$, the minimum resides on the circle $\sigma^2 + \pi^2 = -\mu^2/g$. Since the phase of the (σ,π) - multiplet is arbitrary under global $O(2)$, it is consistent to lift this vacuum degeneracy by the introduction of a small symmetry breaking term.

$$L_1 = c\sigma \quad , \quad (c = \text{const.}). \tag{3.2}$$

which chooses a particular direction in (σ,π) - space. This is " explicit symmetry breaking" or "PCAC phase" [26]. The minimum now occurs at

$$\sigma\left[\mu^2 + g(\sigma^2 + \pi^2)\right] = c, \; \pi\left[\mu^2 + g(\sigma^2 + \pi^2)\right] = 0$$

The solutions to these equations for $\mu^2 < 0$, are given by the vacuum expectation values (at lowest order in perturbation theory)

$$\langle\pi\rangle = 0 \quad \text{and} \quad \langle\sigma\rangle \overset{c\to 0}{=} \left(-\frac{\mu^2}{g}\right)^{\frac{1}{2}}$$

By defining the "shifted" field

$$\tilde{\sigma} = \sigma - \langle\sigma\rangle \tag{3.3}$$

the original Lagrangian in equation (3.1) can be rewritten as

$$\tilde{L} = \frac{1}{2}(\partial_\mu \tilde{\sigma}\partial^\mu \tilde{\sigma} + \partial_\mu \pi \partial^\mu \pi) + \mu^2 \tilde{\sigma}^2 - g\langle\sigma\rangle\sigma(\tilde{\sigma}^2 + \pi^2) - \frac{1}{4} g(\tilde{\sigma}^2 + \pi^2)^2 \tag{3.4}$$

The field $\tilde{\sigma}$ resembles that belonging to a particle with mass, $-2\mu^2$, while π is massless; it is the Goldstone boson of the spontaneously broken continuous $O(2)$ symmetry. In general there is one massless mode for each broken generator of the original Lagrangian symmetry. This is Goldstone's theorem [27].

To return to the internal $SU(2)_L \otimes SU(2)_R$ symmetry of the strong interactions, the multiplet $(\sigma,\underline{\pi})$ can be assigned to the $[\frac{1}{2},\frac{1}{2}]$ vector representation of $SO(4)$. The Lagrangian is correspondingly similar to equation (3.1) and the field components can be chosen to transform infinitesimaly (parameter $\underline{\omega}$) under the axial part of $SU(2) \otimes SU(2)$ as

$$\delta\sigma = -\underline{\omega}\cdot\underline{\pi} \quad \text{and} \quad \delta\underline{\pi} = \underline{\omega}\sigma \tag{3.5}$$

Hence the $SO(4)$ rotations mix the scalar and pseudoscalar components.

If we wish to have an effective Lagrangian for low momenta phenomena, involving only the massless pion fields, we must eliminate the σ-field from the dynamics. Now, the σ-field is massive because it includes changes in the length of the vector (σ,π) i.e. it takes the field configuration <u>out</u> of the sphere of minima of the Lagrangian. We can ensure that only the massless excitations contribute if we impose the $O(4)$ invariant constraint $\sigma^2 + \pi^2 = f^2$, i.e. $\sigma = \sqrt{f^2-\pi^2}$ (f can be identified with the pion decay constant). Substituting for the σ-field into the Lagrangian corresponding to eqn. (3.1) gives immediately an effective Lagrangian for the pion fields. It comes purely

from the derivative terms in equation (3.1) and has the form

$$L_{eff} = \frac{1}{2}\,\partial_\mu \pi_i\, g_{ij}\, \partial^\mu\, \pi_j \qquad (3.6)$$

where $$g^{ij} = \delta^{ij} + \frac{\pi^i\,\pi^j}{(f^2-\pi^2)}$$

In fact this g^{ij} is the metric tensor defined on the surface of a 4-dimensional sphere (S^3), the coset manifold of the factor space SO(4)/SO(3) [28].

Since equation (3.6) is coordinate independent, it holds true for any O(N) - invariant theory.

Substituting $\sigma = \sqrt{f^2-\underline{\pi}^2}$ in the axial transformations of equation (3.5) implies that the effective Lagrangian in equation (3.6) is now invariant under the non-linear transformation.

$$\delta\underline{\pi} = \underline{\omega}\,\sqrt{f^2-\underline{\pi}^2}$$

and correspondingly the form of the effective Lagrangian at low momenta is fixed by this invariance. In this way the massless pion fields are said to provide a nonlinear realization of the 3 broken axial generators [29-32].

Accordingly, the external field term in equation (3.2) produces an explicit mass term, $\frac{1}{2}\,c\,\underline{\pi}^2$ for the pion field.

This outlines the origin and structure of nonlinear σ-models† in the context of chiral symmetries in elementary particle physics. In the next section we shall review the relevance of Goldstone modes

† The nonlinear σ-model may also be considered as a formal limit of the Lagrangian in equation (3.1) when $g \to \infty$ [29,33].

in the context of phase transition phenomena in Statistical Mechanics. It will emerge that the long distance behaviour of classical Heisenberg ferromagnets below the transition temperature is described by an O(N) nonlinear σ-model defined by the same Lagrangian as equation (3.6).

The purpose of this review chapter then, is to establish the vocabulary and framework upon which we will base a parallel discussion of Goldstone singularities for systems possessing a discrete internal symmetry.

§ III.2 Low-temperature Expansion for Heisenberg Models:

The conventional Heisenberg model of ferromagnets, with O(N) - symmetric interaction, is described by the Hamiltonian [34,35]

$$\mathcal{H} = - \sum_{i \neq j} J_{ij} \left(\underline{S}_i \cdot \underline{S}_j \right) \qquad (3.7)$$

in which the $\underline{S}_i$ are unit N-component vectors associated with sites i on a periodic d-dimensional lattice and J_{ij} is a short ranged positive, translationally invariant interaction. The effect of an external field, $\underline{H}$, can also be incorporated into the Hamiltonian in equation (3.7) by including a term, $- \sum_i \underline{H} \cdot \underline{S}_i$.

This model has a phase transition above 2-dimensions and its long-distance behaviour may be investigated by means of a perturbative expansion around Mean Field Theory, with the result that the critical properties are given by a continuous field theory equivalent to the linear σ-model discussed in § III.1, equation (3.1).* In the context of Statistical Mechanics the corresponding Hamiltonian is called the Landau-Ginzburg - Wilson (LGW-) Hamiltonian [36] (Recall § I.4).

* The correlation length, ξ, provides a large length scale in the critical region since $\xi \xrightarrow[\mu_0 \to 0]{T \to T_c^+} \infty$. Since ξ dominates the lattice spacing it permits a continuum model description of critical behaviour. See also pp. 44 and 48.

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2}(\nabla \phi_i)^2 + \frac{1}{2} \mu_o^2 \phi_i^2 + \frac{1}{4} g_o (\phi_i^2)^2 \right\} \tag{3.8}$$

Here $\mathcal{H}$ is the dimensionless "reduced form" of the Hamiltonian being divided by a factor $1/k_B T$ and $\phi_{i=1,2,....N}(\underline{x})$ is a continuous function of the spatial coordinates $\underline{x}$. This Hamiltonian defines a Euclidean field theory.

Statistical averages may be formulated in terms of functional integrals which average over all possible field configurations. The partition function defined in this way is

$$Z = \int \mathcal{D}\, \phi_i \quad \exp \; -\mathcal{H}(\phi_i) \tag{3.9}$$

and the corresponding definition of the correlation function is

$$\langle \phi_{i_1}, \ldots, \phi_{i_N} \rangle = Z^{-1} \int \mathcal{D}\, \phi_i (\phi_{i_1}, \ldots, \phi_{i_N}) \exp - \mathcal{H} \tag{3.10}$$

With these definitions, we can elaborate on our previous remarks which alluded to singular behaviour at the critical temperature, T_c.

Two thermodynamic quantities of interest are the magnetization (spin field-density)

$$M_i = \langle \phi_i \rangle \tag{3.11}$$

and the magnetic susceptibility.

$$\chi_{ij} \equiv \frac{\partial M_i}{\partial H_j} = \int d^d x \left\langle \left[\phi_i(x) - \langle \phi_i(x) \rangle \right] \cdot \left[\phi_j(0) - \langle \phi_j(0) \rangle \right] \right\rangle \tag{3.12}$$

At the critical temperature, T_c, the magnetization vanishes

and the susceptibility diverges, for zero external magnetic field (Figure 1). The nature of the singular behaviour can be explored by a perturbation expansion in g_o based on the LGW-Hamiltonian in equation (3.8). To see this, let us consider just a single component field (Ising model), $\phi(x)$. The magnetization being zero, (for $T \gtrsim T_c$, $H = 0$) the susceptibility becomes

$$\chi = \int d^d x \, \langle \phi(\underline{x}) \, \phi(0) \rangle \tag{3.13}$$

The Fourier transform of this quantity,

$G^{(2)}(q^2) = \int d^d x \exp iq\cdot\underline{x} \, \langle \phi(\underline{x}) \, , \, \phi(0) \rangle$ is the 2-point function in field theory language. At lowest order, it is equivalent to the propagator defined by equation (3.8).

$$G^{(2)}(q^2) = \frac{1}{q^2+\mu_o^2} + O(g_o) \tag{3.14}$$

Clearly, the susceptibility, χ, in equation (3.13) corresponds to $G^{(2)}(q^2=0) = \frac{1}{\mu_o^2} + O(g_o)$, in the infra-red régime. Further, if μ_o^2 is assumed to be a function of temperature, $\mu(T)$, such that it can be Taylor expanded about T_c

$$\mu_o^2(T) = \mu_o^2(T_c) + A(T-T_c) + O\left[(T-T_c)^2\right]$$

then the expected divergence in χ will occur for $\mu_o^2(T_c)$ vanishing (massless theory) implying $\chi \alpha |T-T_c|^{-1}$. Consequently,we see that the infra-red singularities at lowest order are just those of mean field theory near the critical point.

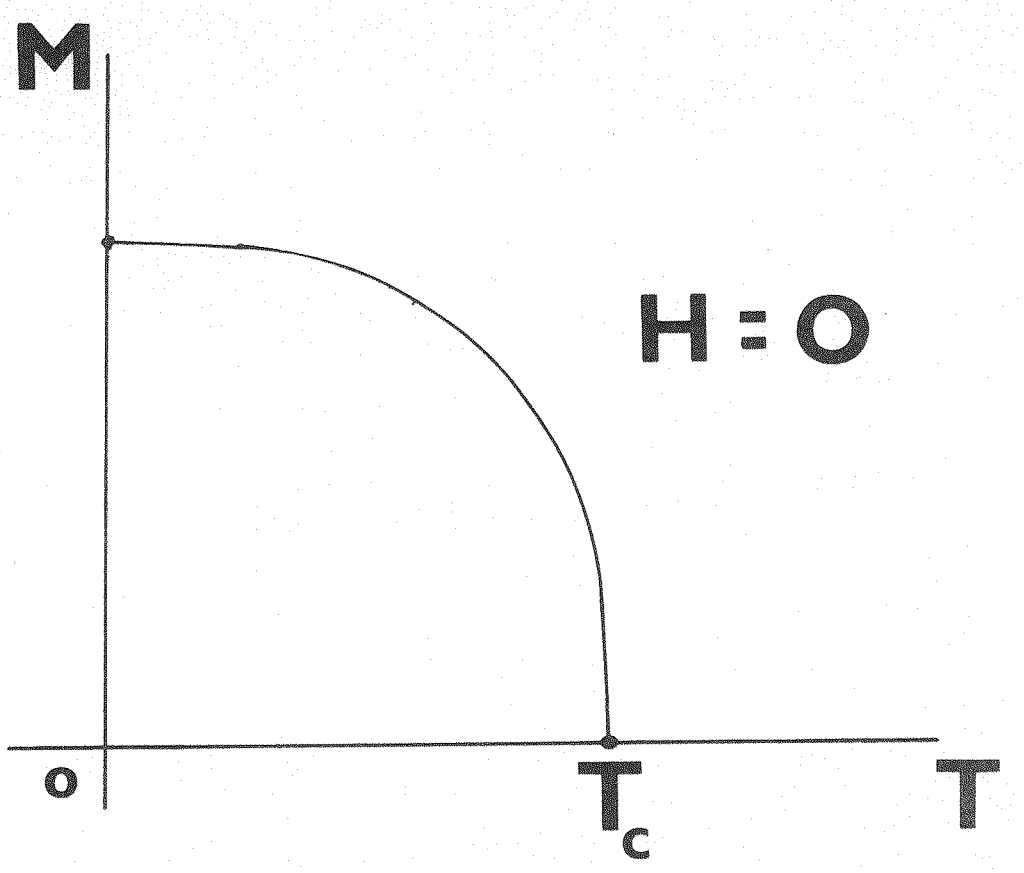


Figure 1.

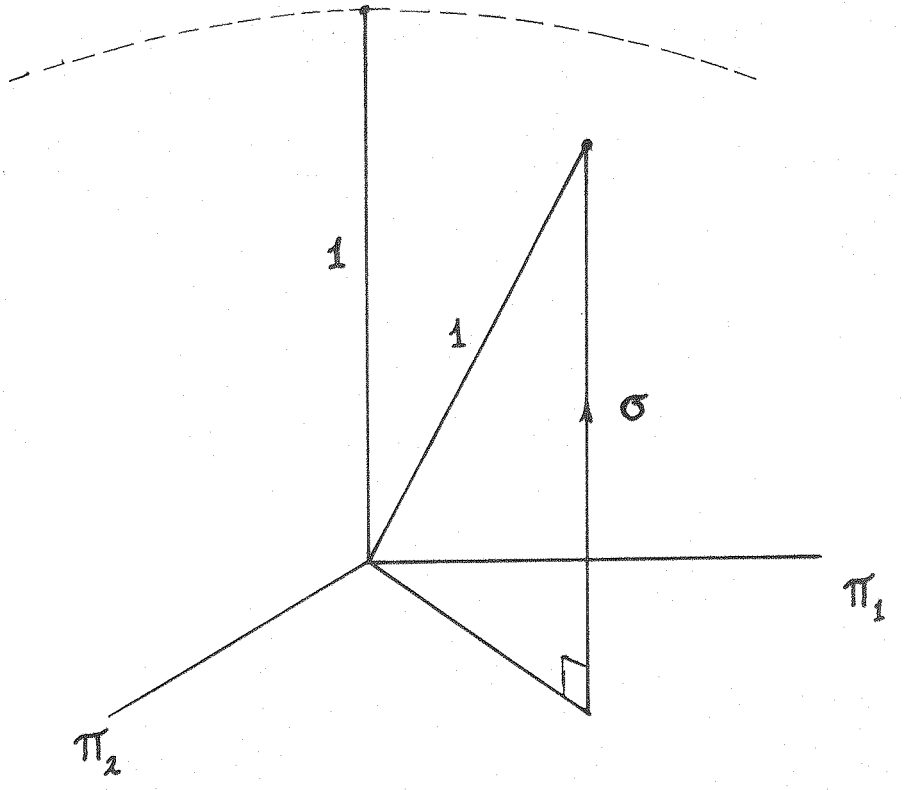


Figure 2.

By considering the presence of an external field in the "1" direction,

$$H \int d^d x \, \phi_1 \tag{3.15}$$

in the Hamiltonian equation (3.8), the continuous O(N) internal symmetry is broken and this brings into play the possiblity of generating infra-red singularities in the ordered phase for all the temperatures below T_c . In view of the remarks made in § III.1 about spontaneously broken continuous internal symmetries, it is possible to attribute these latter singularities to the (N-1) massless Goldstone modes arising out of this symmetry breaking. Since this situation corresponds to $\mu_o^2 < 0$, it is not naturally taken account of by the Hamiltonian in equation (3.8).

As may be anticipated, a suitable long-wavelength description as provided by the non-linear σ-model may be obtained directly from the Heisenberg model defined on a lattice, as in equation (3.7), in the presence of an external field.

In the ordered phase the $\underline{S}_i$ vectors are orientated along the direction, $\hat{\underline{H}}$, of the external field and at low temperatures the dominant fluctuations are small deviations away from this direction. From the geometry of the unit sphere (Figure 2) we can express $\underline{S}_i$ as

$$\underline{S}_i \cdot \hat{\underline{H}} = \sqrt{1-\underline{\pi}_i^2} = \sigma_i \tag{3.16}$$

$$\underline{S}_i - (\underline{S}_i \cdot \hat{\underline{H}})\hat{\underline{H}} = \underline{\pi}_i$$

which are respectively the component longitudinal and transverse to the $\hat{\underline{H}}$ - direction. In this parameterization the O(N) group of

rotations acts linearly on the N-vector $(\sigma,\underline{\pi})$.

In terms of the (N-1) - transverse modes, we can write the partition function for the Hamiltonian (3.7), including the H-field, as

$$Z = \int_{-\infty}^{\infty} \prod_{i}^{N-1} \frac{d^{N-1}\pi_i}{\sqrt{1-\underline{\pi}_i^2}} \exp \frac{1}{T} \sum_{ij} J_{ij} \left(\sqrt{(1-\underline{\pi}_i^2)(1-\underline{\pi}_j^2)} + \underline{\pi}_i \cdot \underline{\pi}_j \right) + H \sqrt{1-\underline{\pi}_i^2} \,, \qquad (3.17)$$

Then a standard loop expansion [35] of this functional integral generates an expansion in powers of T, where the order of T corresponds to the number of loops present at that order. The corresponding Feynman graphs involve propagators which are defined by the quadratic terms in the action in equation (3.17). In momentum space these terms will transform as

$$\begin{aligned} \pi_i^2 J_{ij} &\rightarrow \pi^2 J(0) \\ \pi_i J_{ij} \pi_j &\rightarrow \pi(q) J(q) \pi(-q) \end{aligned} \qquad (3.18)$$

where $J(q)$ is the Fourier transform

$$J(q) = \sum_{j}^{N} J_{ij} \quad \exp i\, q \cdot r_{ij} \qquad (3.19)$$

For sufficiently short-range interactions the vectors, r_{ij} (distance between sites i,j) are typically lattice vectors having magnitude of the order of the lattice spacing, Λ and a small $|q| < \Lambda$ expansion of the exponential in equation (3.18) to quadratic order is given by

$$J(q) = J\left[1 - (q/\Lambda)^2\right] + O(q^4) \qquad (3.20)$$

The propagators in equation (3.17)

$$G_{\alpha\beta}(q) = \frac{T\ \delta\alpha\beta}{J(0)(1+H)-J(q)}\ ;\ \alpha,\beta=1,2,\ldots,N-1$$

are then seen to be dominated by the most singular part in the long-wavelenth region, as expressed in equation (3.20). i.e.

$$G_{\alpha\beta}(q) \sim \frac{T\delta_{\alpha\beta}}{q^2+H} \tag{3.21}$$

The interaction terms in equation (3.17) are obtained from an expansion of the square roots and from the integration measure written as

$$\prod_i (1-\underline{\pi}_i^2)^{-\frac{1}{2}} = \exp -\frac{1}{2}\sum_i \ln(1-\underline{\pi}_i^2) \tag{3.22}$$

In momentum space, the π_i-field are functions of the momentum, $\pi_i(q)$, and after expansion using equation (3.20) the Euclidean action can be simplified to read

$$A_{eff} = \int d^d x \left\{ \left(\nabla\sqrt{1-\underline{\pi}^2} \right)^2 + (\nabla\underline{\pi})^2 + H\sqrt{1-\underline{\pi}^2} \right\} \tag{3.23}$$

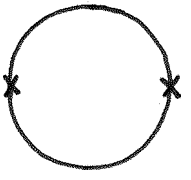
where the q^2 dependence of $J(q)$ has given rise to derivative couplings in equation (3.23). The Lagrangian appearing in this expression for the effective action is formally identical to equation (3.6) and justifies the use of the term "nonlinear σ-model" in this context.

An example of the nature of infra-red singular behaviour induced by the Goldstone modes is readily provided by the longitudinal susceptibility, in which the vector $\underline{H}$ of the external field is parallel to the magnetization, $\underline{M}$.

If we call this direction, 1, then from the definition of magnetic susceptibility given in equation (3.12) the longitudinal component is

$$\chi_L = \int d^d x \left\langle \left[\phi_1(x) - \langle\phi_1(x)\rangle\right] \cdot \left[\phi_1(0) - \langle\phi_1(0)\rangle\right] \right\rangle$$
$$= \int d^d x \left\langle \left[\sigma(x) - \langle\sigma(x)\rangle\right] \cdot \left[\sigma(0) - \langle\sigma(0)\rangle\right] \right\rangle \qquad (3.24)$$

In the nonlinear σ-model the integrand, eqn.(3.23) must generate connected correlation functions of the type, $\int d^d x \left\langle \left[\pi^2(x) - \langle\pi^2(x)\rangle\right] \cdot \left[\pi^2(0) - \langle\pi^2(0)\rangle\right] \right\rangle$, after expansion in π^2. Following our previous discussion of singular behaviour of propagators in mean field approximation we can see that disconnected graphs are cancelled by $\langle\pi^2\rangle$ terms in the susceptibility leaving connected graphs of the type



$$= \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+H)^2} \qquad (3.25)$$

This implies that a singular term, $H^{\frac{1}{2}(d-4)}$, in χ_L should exist up to the critical temperature, T_c [37-39].

In 2-dimensions this singular behaviour suggests that fluctuations in the longitudinal field components become as singular as those in the Goldstone fields and these infra-red singularities destroy the ordered phase, thus inhibiting spontaneous magnetization for all temperatures below T_c in Heisenberg models. [34,40,41].

Analogous singularities due to Goldstone modes in the case of a discrete internal symmetry will be the subject of Chapter Four.

§ III.3 Renormalization Theory:

The previous sections in this chapter have been concerned with the nonlinear σ-model as an effective Lagrangian for low energy, low momentum fluctuations in systems with a continuous internal symmetry below T_c . In this section we review the results of treating nonlinear σ-models as field theories in their own right i.e. we discuss also the control of ultra-violet (short distance) singularities. In the infra-red régime, the nonlinear σ-model has softer singular structure than the LGW - Hamiltonian due to the presence of gradient couplings at all orders in a series expansion. This improved infra-red (I.R.) behaviour naturally entails poorer ultra-violet behaviour than in the linear σ-model, and this must be regularized [34, 35].

In two dimensions the transverse, π - components are dimensionless and the model is renormalizable by power counting. Above two dimensions the nonlinear σ-model is naively not renormalizable, while below two dimensions it is super-renormalizable and we can use dimensional regularization to control the ultra-violet (U.V.) divergences of two dimensions. (Another O(N) - invariant regularization is provided by the Heisenberg model itself (i.e. the lattice Hamiltonian of eqn. (3.7)) but this approach is an inconvenient one for calculational purposes).

In addition to these standard U.V. divergences near two dimensions, there are also potential I.R. divergences which come from the q^{-2} of the propagators. A simple way of disengaging these two sources of divergences is to break the O(N) - symmetry via the introduction of an external magnetic field, H coupled to the σ-field. This has the effect of giving the π-field a mass and acts therefore as infra-red regulator.

With this introduction, we now turn to the problem of renormalization near two dimensions. The non-polynomial nature of the

interactions in the nonlinear σ-model is reflected in the fact that there are an infinite number of primitively divergent vertices. Therefore, one would anticipate an infinite number of kinds of coupling renormalization constants. However, it has been shown that for the paramet rization in eqns. (3.16), only two independent renormalization constants are required VIZ. a wavefunction renormalization, Z and a coupling renormalization, Z_1 [35]

The formalism which establishes this important result requires the use of a Ward identity and we outline now its derivation.

Firstly, it is convenient to replace the temperature T, which has dimension $[q]^{-(d-2)}$, by the dimensionless (bare) temperature

$$t_o = T \cdot \mu^{d-2} \tag{3.26}$$

where μ is an arbitrary mass scale for this massless theory. Then the generating functional is defined with an invariant measure as

$$Z(\underline{J}) = \int \frac{\mathcal{D}\,\pi(x)}{\sqrt{1-\pi^2(x)}} \exp.\left\{\frac{\mu^{d-2}}{t_o}\left[-A_{\text{eff.}} + \int d^d x\, \underline{J}\cdot\underline{\pi} + |H| \int d^d x\, \sigma\right]\right\} \tag{3.27}$$

where $\underline{J}(x)$ is a classical external source. Formally, the integration measure induces a further interaction term of the form

$$-\frac{1}{2}\,\delta^{(d)}(0) \int d^d x\, \ln(1-\underline{\pi}^2(x))$$

The use of dimensional regularization permits this problem to be ignored since $\delta^{(d)}(0)$ is then set to zero. With the assumption that the model can be regularized in this way, we consider the Ward identities and their implication for the structure of the renormalized action.

The general method, also appropriate to non Abelian gauge theories, can be found elsewhere [42]. For symmetry breaking of the type $O(N)/O(N-1)$, the fields transform as in eqn. (3.16)

$$\begin{aligned} \delta\ \underline{\pi}(x) &= \underline{\alpha}\ \sqrt{1-\underline{\pi}^2(x)} \\ \delta\ \sqrt{1-\pi^2(x)} &= -\ \underline{\alpha}\cdot\underline{\pi}(x) \end{aligned} \tag{3.28}$$

where $\underline{\alpha}$ is the parameter associated with the infinitesimal rotation which mixes the $\underline{\pi}$ and σ fields.

The invariance of the effective action, the measure and the regularization under such a transformation is contained in the Ward identity

$$\int d^d x \left[J_i(x) \frac{\delta Z}{\delta H(x)} - H(x) \frac{\delta Z}{\delta J_i(x)} \right] = 0 \tag{3.29}$$

The generator of the 1 P.I. vertices is obtained by the usual Legendre transform,

$$\Gamma(\underline{\pi}, |H|) = \int d^d x\ \underline{\pi}(x).\ \underline{J}(x) - W(\underline{J}, H)$$

where

$$W(\underline{J}, H) = t_o \ln Z$$

is the generating functional of the connected Green's functions of the $\underline{\pi}$-field.

$$\pi_i(x) = \frac{\delta W}{\delta J_i(x)}$$

In terms of the Γ's the Ward identity becomes

$$\int d^d x \left[\frac{\delta \Gamma}{\delta \pi_i(x)} \frac{\Gamma}{\delta H(x)} + H(x)\, \pi_i(x) \right] = 0 \qquad (3.30)$$

Invoking a loop expansion of the functional Γ, at lowest order $\Gamma^{(0)}$ is the regulated action, which necessarily satisfies the Ward identity.

At the one-loop level we have

$$\int d^d x \left[\frac{\delta \Gamma^{(0)}}{\delta \pi_i} \frac{\delta}{\delta H} + \frac{\delta \Gamma^{(0)}}{\delta H} \frac{\delta}{\delta \pi_i} \right] \Gamma^{(1)} = 0 \qquad (3.31)$$

where the factor in parentheses is simply the $O(N)$ generator defined on the coordinates, π_i, of the coset manifold $O(N)/O(N-1)$. Eqn. (3.31) holds for the bare regularized theory and since dimensional regularization is $O(N)$- invariant, it also holds for the U.V. divergent part of $\Gamma^{(1)}_{div}$ VIZ. the pole term with factor $(d-2)^{-1}$. Hence all 1-loop divergences may be cancelled by the addition to the effective action of a counter term, $-\Gamma_{div}$, whose structure is prescribed by eqn. (3.31).

Using inductive arguments, it has been shown [35] that to all orders in the coupling constant, the renormalized action (with this field parameterization) requires only two renormalization constants, Z and Z_1.

Thus the fully renormalized action reads

$$A = \frac{\mu^{d-2}}{2Z_1 t} \int d^d x \left\{ Z(\partial_\mu \underline{\pi})^2 + \left(\partial_\mu \sqrt{1-Z\underline{\pi}^2} \right)^2 - 2 \frac{HZ_1}{\sqrt{Z}} \sqrt{1-Z\underline{\pi}^2} \right\} \qquad (3.32)$$

where the introduction of the rescaled fields modifies the transformation

eqn. (3.28) under which the renormalized action is invariant. i.e.

$$\delta\, \underline{\pi}(x) = \underline{\alpha}\, \sqrt{1-Z\underline{\pi}^2} \tag{3.33}$$

§ III.4 Renormalization Group Properties:

In § III.3 we saw that the nonlinear σ-model is just renormalizable in two dimensions and requires only two renormalization constants. Above two dimensions the model is not renormalizable by power counting arguments.

To see if the nonlinear σ-model exhibits a phase transition above $d = 2$, it is necessary to invoke the Renormalization Group(R.G.). This is necessary because when we go above two dimensions the coupling constant develops powers of momentum in accordance with eqn. (3.26), i.e. it is no longer dimensionless. Consequently, we are in a strong coupling régime at high momenta and analysis therefore lies outside that of naive perturbation theory.

Moreover, the U.V. properties or short-distance behaviour is amenable to interpretation through the differential RG equations and the existence of U.V. -stable fixed points is intimately related to critical singularities. Defining the dimensionless renormalized temperature and external field, (T and H_o are "bare" quantities),

$$\left.\begin{aligned} T &= Z_1\, t/\mu^{d-2} \\ \frac{H_o}{T} &= \frac{H\,\mu^{d-2}}{t\,\sqrt{Z}} \end{aligned}\right\} \Rightarrow H_o = H\,\frac{Z_1}{\sqrt{Z}} \tag{3.34}$$

in terms of an arbitrary momentum scale μ, the renormalization group equations for the renormalized 1. P.I. - vertices

$$\Gamma_R^{(M)}(\underline{q},t,H,\mu) = Z^{M/2}\,\Gamma^{(M)}(\underline{q},\, T,\, H_o)$$

is obtained in the usual way [43]. For the Lagrangian given by (3.32) the resulting differential equation is

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(t)\frac{\partial}{\partial t} - \frac{M}{2}\gamma(t) + \gamma_H H\frac{\partial}{\partial H}\right)\Gamma_R^{(M)} = 0 \tag{3.35}$$

where the equations

$$\beta(t) = (d-2)t - t\mu\frac{\partial}{\partial\mu}\ln Z_1\Big|_{T,H_o}$$

$$\gamma(t) = \mu\frac{\partial}{\partial\mu}\ln Z\Big|_{T,H_o} \tag{3.36}$$

$$\gamma_H(t) = \mu\frac{\partial}{\partial\mu}\ln\frac{\sqrt{Z}}{Z_1}\Big|_{T,H_o} = \frac{1}{2}\gamma(t) + \frac{\beta(t)}{t} - (d-2)$$

can be deduced from the relations in (3.34). Using the fact that to first order in t the coupling renormalization constant is given by

$$Z_1 = 1 + \left(\frac{N-2}{d-2}\right)t \tag{3.37}$$

the Beta-function can be evaluated to quadratic order in t, to give

$$\beta(t) = (d-2)t - (N-2)t^2 + O(t^3) \tag{3.38}$$

The simple zeros of $\beta(t)$, for $N > 2$ are: the trivial I.R. stable fixed point, $t^* = 0 (\beta'(0) > 0)$, which controls the low momentum Goldstone mode singularities, and a U.V. stable fixed point

$(\beta'(0) < 0)$ [44-47] at

$$t_c = \frac{d-2}{N-2} + O\left((d-2)^2\right) \tag{3.39}$$

which is the effective coupling in the high momentum region.

The fixed point, t_c, can be interpreted as the renormalized critical temperature and support for such an interpretation comes from the behaviour of the correlation length, $\xi(t)$ as t approaches t_c.

The correlation length is defined to be invariant under a renormalization group transformation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(t) \frac{\partial}{\partial t}\right) \xi(\mu, t) = 0 \tag{3.40}$$

The solution of this differential equation is

$$\xi(\mu, t) = \mu^{-1} t^{1/(d-2)} \exp \int_0^t dt' \left[\beta^{-1}(t') - \left(t'(d-2)\right)^{-1} \right]$$

As t increases to t_c, $\xi(t)$ diverges as $(t_c - t)^{-\nu}$ where $\nu = -\frac{1}{\beta'(t_c)} = \frac{1}{(d-2)} + O(1)$. The diverging correlation length is one of the many signals that t_c should be interpreted as the transition temperature at which point there is restoration of the O(N) symmetry†, previously broken by the non vanishing expectation value of the σ-field in the low temperature phase.

† A.J. McKane and M. Stone, DAMTP preprint 13/(1979), give a R.G. analysis of symmetry restoration in the O(N)-nonlinear σ-model in d = 2.

CHAPTER FOUR

THE EUCLIDEAN GROUP AS A BROKEN DYNAMICAL SYMMETRY: THE INTERFACIAL PROFILE

§ IV.1 Surface Fluctuations and Critical Behaviour:

In this chapter we will be concerned with the statistical mechanics of the boundary separating two discrete coexisting thermodynamic phases [48].

Consider the phase diagram (Fig. 3) for a liquid-vapour phase transition (first order), the right-hand side corresponding to the vapour phase and the left-hand side corresponding to the liquid phase. In the two-phase region, liquid and vapour coexist and are in stable thermodynamic equilibrium with one another.

There are two fundamental aspects which have drawn much theoretical attention. One of these is concerned with local structure of the interface between the coexisting phases, i.e. it has a local width of the same order as the correlation length. The other is concerned with effects due to collective fluctuations away from the familiar planar interface [48-51].

A natural model for local structure is the Lattice gas model which is well-known to be equivalent to the Ising model [52]. This equivalence suggests that in field-theoretic terminology, for a one-component field, the ϕ^4 - Hamiltonian is the appropriate choice and this has been used with considerable success to describe critical behaviour of the interfacial width. So a natural question, and the one to which we address ourselves in this chapter is, whether there

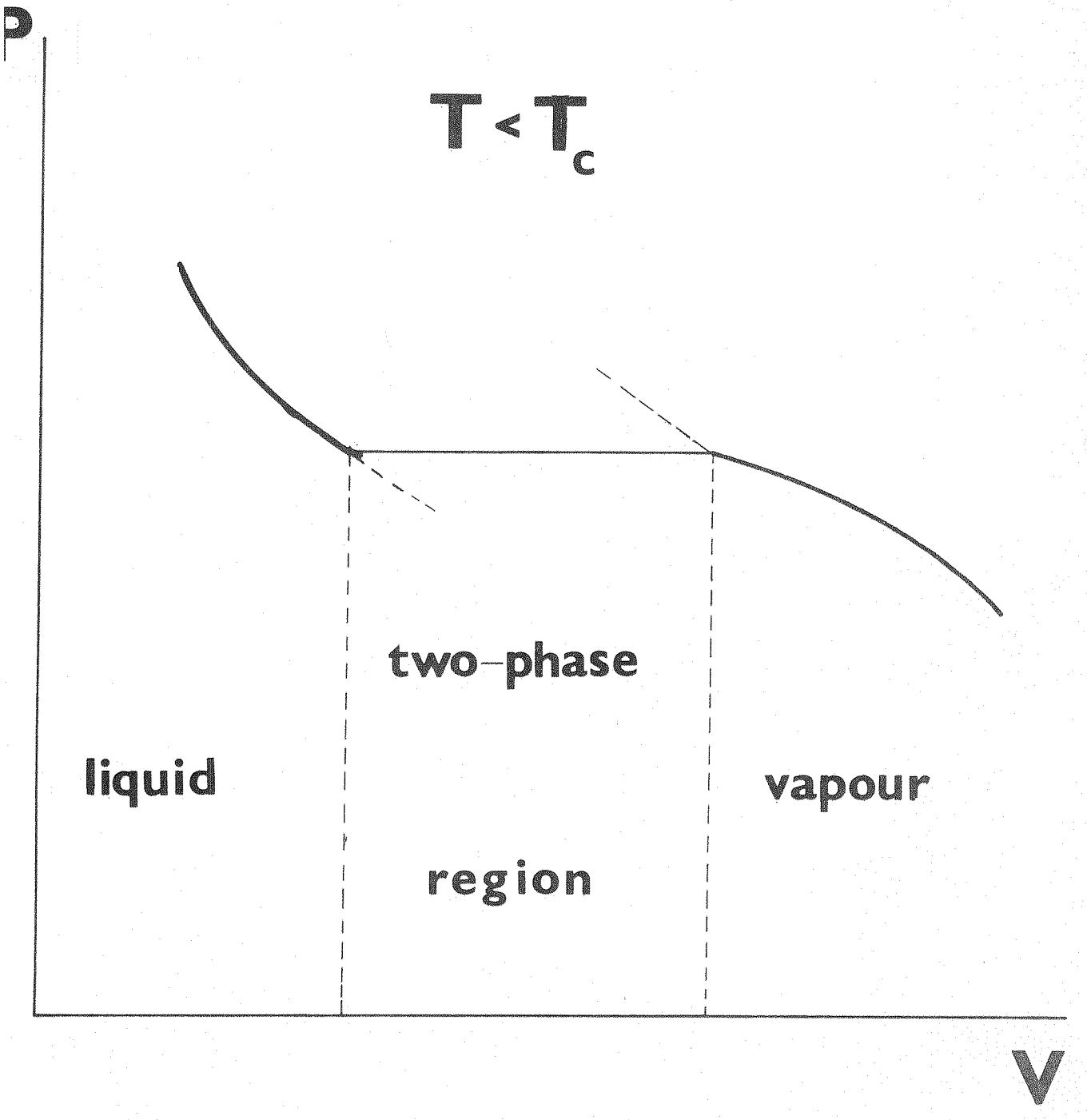


Figure 3.

exists an analogue of the nonlinear σ-model describing the critical behaviour below T_c in Ising systems which possess a discrete internal symmetry. Clearly the nonlinear σ-model itself cannot serve this purpose, since it is a continuous internal symmetry which is spontaneously broken thereby generating the massless Goldstone modes whose effective interactions control the critical behaviour below T_c, in Heisenberg models.

In Ising models it is not obvious that an analogue of the Goldstone modes exists at all; however in this chapter we will show that such analogues do exist and that their properties bear a strong resemblance to those of the nonlinear σ-model [48].

As already mentioned, it has been recognized that in systems with a discrete symmetry, the surface separating the discrete phases plays a crucial role below T_c [50]. The original system which is invariant under rotations and translations of the Euclidean group (in the continuum ϕ^4 theory) has this invariance broken by virtue of the surface of separation. In this sense we may regard the presence of coexisting phases in Ising systems, as being tantamount to a broken Euclidean invariance. Furthermore, since this is a continuous symmetry which is broken spontaneously by the interface, fluctuations in the position of the surface can be described by a field of zero mass - the Goldstone field of the spontaneously broken Euclidean invariance.

Following the discussion of the nonlinear σ-model in Chapter Three, we will construct the effective action for the Goldstone modes, identifying invariances and discussing the significance for phase transition phenomena in Ising - like models.

§ IV.2 Effective Action for Fluctuations :

The conventional field-theoretic formulation of critical phenomena in a one-component system is given by the Landau-Ginzburg-Wilson

Hamiltonian in d-space dimensions.

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} \mu_o^2 \phi^2 + \frac{1}{4} g_o \phi^4 \right\} \tag{4.1}$$

For convenience, we have redefined the mass in eqn. (3.8) $\mu_o > 0$ which implies that we are below T_c. The coexistence phenomena in which we are interested, occurs for $|H| = 0$, so that eqn. (4.1) has two degenerate minima.

The partition function is then defined in terms of the functional integral over the ϕ-fields as in eqn. (3.9),

$$Z = \int \mathcal{D}\phi \exp - \mathcal{H} \tag{4.2}$$

In order to set up a perturbation theory for the partition function in eqn. (4.2) we expand the field about the appropriate classical solution of the field equation [53].

$$\nabla^2\phi = \mu_o^2 \phi - g_o \phi^3 \tag{4.3}$$

which minimizes the action, $\frac{\delta \mathcal{H}}{\delta \phi} \Big|_{\phi=\phi_c} = 0$. The conventional choice to describe the ordered states is to take as the classical solutions, $\phi_\pm = \pm \mu_o/\sqrt{g_o}$, the classical minimum of the potential, thereby establishing a perturbation theory of up(down) ordered states. In our case we want to perturb about a classical solution which incorporates the interface and satisfies the boundary conditions at each of the phases, ϕ_+ and ϕ_-. Such a solution is well known and we write it as

$$\phi_c(x) = \frac{\mu_o}{\sqrt{g_o}} \tanh \frac{\mu_o}{\sqrt{2}} (z - z_o) \tag{4.4}$$

The sign of ϕ_c changes at $z = z_o$ which we identify as the position of the planar interface. This solution interpolates between the degenerate vacua of the ϕ^4 -model. [It also corresponds to finite barrier penetration in quantum mechanics, where the dominant contribution to the transmission amplitude is represented by a classical trajectory in imaginary Euclidean time [54].] The width of the interface is μ_o^{-1} and gives a classical description of the width, proportional to the correlation length ξ . In this way the primary physical characteristics are seen explicitly. Moreover, fluctuations about this classical solution are incorporated by writing

$$\phi = \phi_c + \hat{\phi} \tag{4.5}$$

in such a way that for small amplitude fluctuations we are led to a new perturbation expansion about the classical solution eqn. (4.4) resulting in the partition function being written as

$$Z = \exp -\mathcal{H}(\phi_c) \cdot \int \mathcal{D}\hat{\phi} \exp - \left\{ \frac{1}{2} \int d^d x \, \hat{\phi} \, M \, \hat{\phi} + O(\hat{\phi}^3) \right\} \tag{4.6}$$

where

$$M = \left. \frac{\delta^2 \mathcal{H}(\phi)}{\delta\phi \, \delta\phi} \right|_{\phi=\phi_c} \tag{4.7}$$

$$= - \nabla^2 - \mu_o^2 + 3g_o \, \phi_c^2 \tag{4.8}$$

The nature of the perturbation series is characterized by the spectrum of M. $\mathcal{H}(\phi_c)$ is extensive in the area of the hyperplane and represents the classical surface tension energy. This form of M permits a decomposition of its eigenfunctions into transverse and longitudinal

components. Firstly, we can check that $\nabla_{z_o} \phi_c(z, z_o)$ is a zero mode of M.

We rewrite eqn. (4.3) as, $\nabla^2\phi = -\frac{\delta}{\delta\phi}"H_{int}(\phi)"$, where $"H_{int}" = \frac{1}{2}\mu_o^2\phi^2 - \frac{1}{4}g_o\phi^4$. Then

$$\begin{aligned} \nabla^2 \partial_{z_o} \phi_c &= \partial_{z_o} \frac{\delta}{\delta\phi}"H_{int}"\Big|_{\phi=\phi_c} \\ &= \partial_{z_o} \phi_c \cdot \frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}"H_{int}"\Big|_{\phi=\phi_c} \end{aligned} \tag{4.9}$$

which according to eqn. (4.7) is equivalent to

$$M\partial_{z_o} \phi_c = 0$$

This is the well known zero mode of kink solution translation in one-dimension [55].

In our case it represents a uniform translation of the interface

$$\phi_c(z-a) \simeq \phi_c(z) - a \nabla_z \phi_c\Big|_{z=z_o} \tag{4.10}$$

More importantly for our purposes, the functions

$$\psi_q(s,z) = \exp i\, \underline{q}\cdot\underline{s}\, \psi_o(z) \tag{4.11}$$

are eigenfunctions of M, where $\psi_o(z) \equiv \nabla_{z_o} \phi_c$. From eqn. (4.8) and the fact that $\psi_o(z)$ is a zero mode of M, we see that

$$M\, \psi_{\underline{q}} = q^2\, \psi_{\underline{q}} \tag{4.12}$$

It follows that the eigenfunctions, $\psi_{\underline{q}}$, are the excitations of the massless Goldstone mode, $\psi_o(z)$. Moreover, the physical significance of these excited states can be seen by considering the field configuration, $\phi(x)$, built up from a super-position of these modes about the classical solution.

$$\begin{aligned} \phi(x) &= \phi_c + \sum_{\underline{q}} A(\underline{q}) \exp i\, \underline{q}\cdot\underline{s}.\psi_o \\ &\simeq \phi_c \left[z - \Phi(s) \right] \end{aligned} \tag{4.13}$$

where we have written

$$\Phi(s) = \sum_{\underline{q}} A(\underline{q}) \exp i\, \underline{q}\cdot\underline{s} \tag{4.14}$$

The configuration in eqn. (4.13) represents a field in which the interface at z_o is displaced by a length $\Phi(s)$ above the $\underline{s}$-plane perpendicular to the z-direction. The profile of the interface, given by the functional dependence of ϕ on z , remains unchanged, so that $\Phi(s)$ describes the collective motions of the interface away from planar [48].

Just as with the Goldstone modes discussed in Chapter Three, it is important to determine the effective action describing the interaction between these modes in order to see whether higher order terms become significant in controlling the potential infra-red singular behaviour well below T_c.

It is simplest again to resort to physical arguments to obtain the form of the effective action. Following our previous remarks, the energy with which we are concerned is the surface energy density parameterized by the field, $\Phi(s)$ describing the linear displacement away from planar. In the thin-wall or sharp interface approximation

(expressed in eqn.(4.6)), a transverse differential element $\underline{ds}$ is replaced by $\underline{ds}\left[1 + \left(\nabla\,\Phi(s)\right)^2\right]^{\frac{1}{2}}$. The Euclidean invariant volume in d-dimensions is then found by integrating over all transverse components [48]

$$A_{eff} = \frac{\gamma_o}{k_B T}\int d^{d-1}\underline{s}\left[1 + (\nabla\,\Phi)^2\right]^{\frac{1}{2}} \tag{4.15}$$

Here, γ_o is the bare surface tension/unit area and k_B is Boltzman's constant.

The reader will recall from § III.2 that the Goldstone modes of a spontaneously broken continuous symmetry provided a nonlinear realization of the broken generators, in the sense that the pion fields transformed nonlinearly under the action of the broken axial generators. Similarly, in the present case of a broken dynamical Euclidean symmetry it is a simple matter to show that the Goldstone field, $\Phi(s)$, describing fluctuations away from the planar surface, transforms nonlinearly under the action of rotations of the surface through an infinitesimal angle θ in the $(\underline{s}, z)$-plane.

Under the action of an infinitesimal translation, $\underline{a}$, in the z-direction,

$$\Phi'(s) = \Phi(s) + \underline{a} \tag{4.16}$$

Under the action of rotations by a small angle θ_i in the i,z-plane we have the field transformation (see Figure 4.)

$$\Phi'(s_i) = \Phi(s_i') + \theta_i\, s_i \tag{4.17}$$

and the coordinate transformation

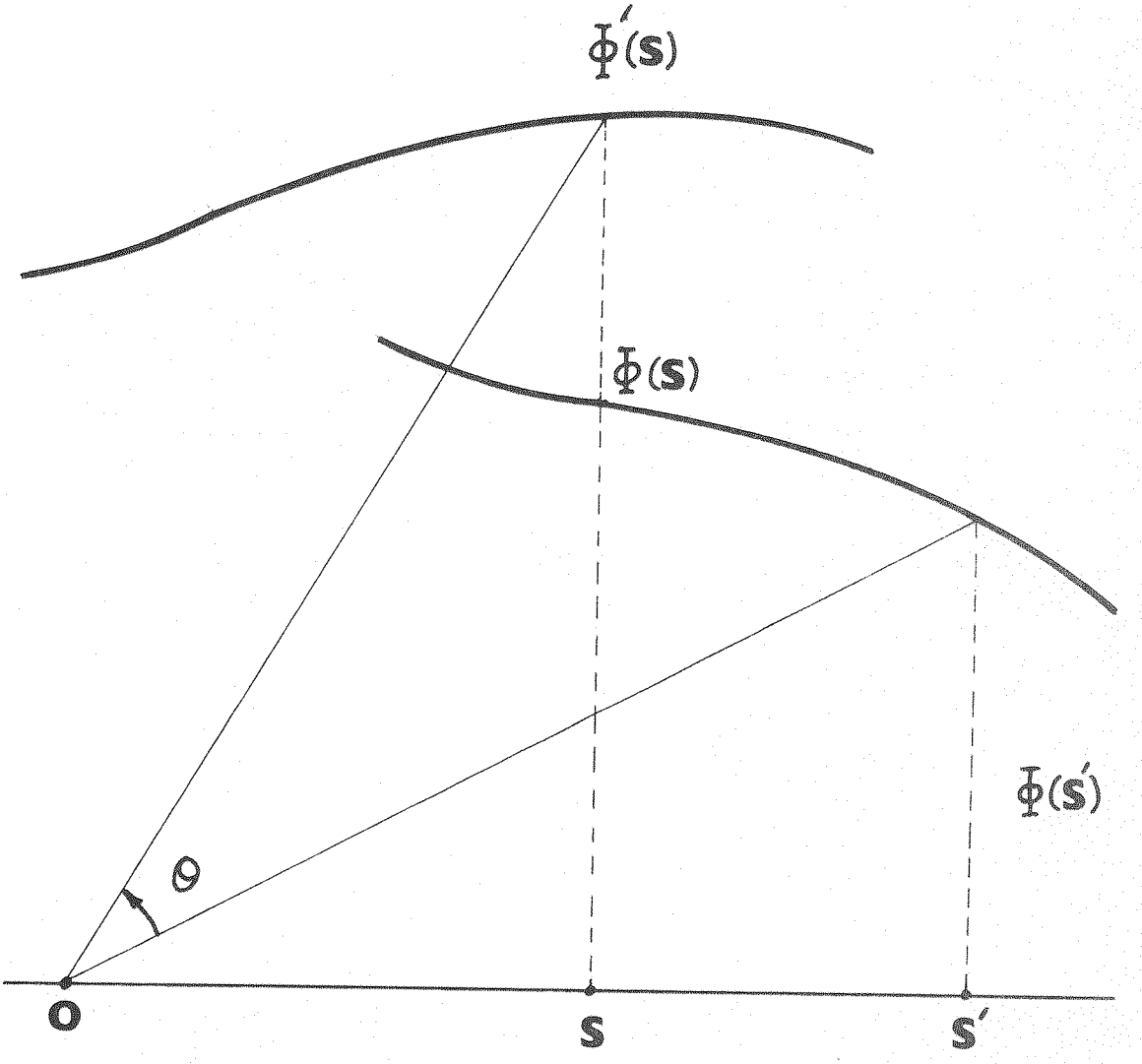


Figure 4.

$$s_i = s_i' - \theta_i \, \Phi(s_i) \tag{4.18}$$

A Taylor expansion to $O(\theta_i)$ of eqn. (4.17), using eqn. (4.18), establishes the following nonlinear transformation of Φ at the point s_i

$$\Phi'(s_i) = \Phi(s_i) + \theta_i(\Phi \frac{\partial}{\partial s_i} \Phi + s_i) \tag{4.19}$$

The Lagrangian in eqn. (4.15) is not invariant under the transformation in eqn. (4.19) since we have

$$\begin{aligned} \delta_\theta \sqrt{1+(\nabla\Phi)^2} &= \frac{\theta_i \nabla_j \Phi}{\sqrt{1+(\nabla\Phi)^2}} \cdot \nabla_j(\Phi \nabla_i \Phi + s_i) \\ &= \frac{\theta_i \nabla_j \Phi}{\sqrt{1+(\nabla\Phi)^2}} \cdot \left\{(\nabla_j\Phi)(\nabla_i\Phi) + \Phi \nabla_j(\nabla_i\Phi) + \nabla_j s_i\right\} \end{aligned} \tag{4.20}$$

which at order θ, can be written as

$$\nabla_i \Phi \sqrt{1+(\nabla\Phi)^2} + \frac{\nabla_i \Phi}{\sqrt{1+(\nabla\Phi)^2}} \cdot (\Phi \nabla_i(\nabla_i\Phi)) \tag{4.21}$$

The first term can be rewritten as

$$\begin{aligned} &\nabla_i\left(\Phi\sqrt{1+(\nabla\Phi)^2}\right) - \Phi \nabla_i \sqrt{1+(\nabla\Phi)^2} \\ &= \nabla_i\left(\Phi\sqrt{1+(\nabla\Phi)^2}\right) - \frac{\Phi \nabla_i \Phi (\nabla_j \nabla_j \Phi)}{\sqrt{1+(\nabla\Phi)^2}} \end{aligned} \tag{4.22}$$

Adding eqns. (4.21) and (4.22) leaves us with a total divergence which vanishes on integration over the transverse coordinates†. Hence $\delta_\theta A_{eff} = 0$.

† This result is reminiscent of the transformation properties of super-symmetric theories where it is the action which is regarded as the fundamental object, because supersymmetric transformations also involve an extra term which is a total divergence.

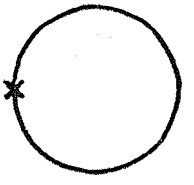
§ IV.3 Infra-red Singularities:

We are now in a position to examine how the interaction of these Goldstone modes controls the infra-red instabilities in the low energy configurations of the interface.

The collective contribution to the interfacial width, $\Delta\Phi$, can be expressed as $(\langle\Phi^2\rangle - \langle\Phi\rangle^2)^{\frac{1}{2}}$, where $\langle\Phi^2\rangle$ is the mean square deviation from planar and $\langle\Phi\rangle$ the mean deviation. Clearly, the average of all fluctuations away from planar will be zero, so it is the mean square contributions which are important.

Now $\langle\Phi^2\rangle$ is a particular type of correlation function whose explicit form can be obtained by considering a loop expansion of the effective action in eqn. (4.15). Since the field, $\Phi(s)$ characterizes a linear displacement, $\nabla\Phi(s)$ must be dimensionless and the effective action in this choice of paramet rization is thus purely a functional of the scalar $(\nabla\Phi)^2$.

At lowest order in the loop expansion the term, $\frac{1}{2T}(\nabla\Phi)^2$ corresponds to the momentum integral.



$$= \int \frac{d^{d-1}q}{(2\pi)^d} \cdot \frac{1}{q^2} \qquad (4.23)$$

Graphs of this type are infra-red divergent for $d \leq 3$ and such a term has been used by other authors [56] to demonstrate that the mean square deviation diverges in three dimensions, a result which bears a strong resemblance to the discussion in § III.2 in the context of the nonlinear σ-model.

Some important qualifications should be made concerning this last remark. Suppose we incorporate an additional term into the effective action of eqn. (4.15), which represents the extra potential energy density due to the influence of an external field e.g.

gravitation. What is the form of this extra term?

For a single mass-point, m, at a height, z, above some reference height, the mechanical potential energy is, mgz; but for an infinitesimally thin column of liquid of density, ρ and height Φ, the potential energy is $\rho g \int_0^{\Phi} z \, d z = \frac{1}{2} \rho g \, \Phi^2$. The complete contribution to the surface energy is then

$$\frac{1}{2} \rho g \int ds^{d-1} \, \Phi^2(s) \tag{4.24}$$

in our previous notation.

The integral in eqn. (4.23) is now modified to read

$$\frac{1}{\gamma_o T} \int_{2\pi/L}^{2\pi/\lambda_{min}} \frac{d^{d-1}q}{(2\pi)^d} \cdot \frac{1}{q^2 + B^2} \tag{4.25}$$

where $B^2 = \rho g/\gamma_o$ and L refers to the length in which the system is enclosed. Clearly, B^2 appears in the propagator of eqn. (4.25) because eqn. (4.24) is equivalent to a mass term in the Lagrangian.

The upper limit on the integral in eqn. (4.25) is expressed in terms of λ_{min} which has a magnitude of the order of the surface thickness, μ^{-1} and represents a short-distance cut-off.

In $d = 3$, the integral in eqn. (4.25) is equal to

$$\frac{1}{4\pi\gamma_o T} \ln \left[\frac{B^2 + (2\pi/\lambda)^2}{B^2 + (2\pi/L)^2} \right] \tag{4.26}$$

This expression illustrates two mechanisms which stabilize the planar interface in three dimensions and render finite the contribution to the interfacial width due to the wandering effect of the Goldstone modes.

Stabilization occurs

i) in a finite volume $(L < \infty)$

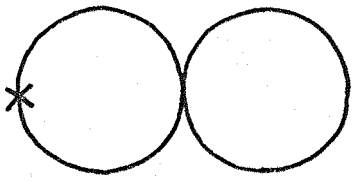
ii) in a gravitational field $(B^2 \neq 0)$ [57]

Naive dimensional analysis of the reduced effective action in eqn. (4.15) tells us the Lagrangian must be dimensionless. The coupling constant therefore has dimension $[q]^{-(d-1)}$ suggesting that A_{eff} is non-renormalizable above one-dimension. Here we have another striking parallel with the non linear σ-models which as we saw in Chapter Three, were naively non-renormalizable above two-dimensions. It is well known that continuum Ising systems(with short-range interactions) do not posses a phase transition in less than one-dimension for $T \neq 0$ [40,58,59].From our viewpoint, it is the non-existence of a renormalizable perturbation expansion above $d = 1$, which reflects the presence of an ordered phase. Below $d = 1$ there is perturbative disorder (weak coupling) which can be treated by a finite number of counter terms in the effective action.

§ IV.4 Higher Order Effects:

So far we have restricted our discussion to 1-loop effects in the low wave-vector region. It is therefore necessary to examine whether higher order terms in the effective action modify the nature of the infra-red behaviour and also consider short distance effects on the structure of the interface.

The reliability of the 1-loop approximation for the term, $\langle\Phi^2\rangle$, can be checked by incorporating the Φ-interactions in perturbation theory. The first non trivial graph contains the vertex $-\frac{1}{8T}\left[(\nabla\Phi)^2\right]^2$ from the loop-expansion of eqn.(4.15). The analytic expression for the graph has the form



$$= \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \cdot \frac{O(p^2)\ O(q^2)}{(p^2)^2\ q^2} \qquad (4.27)$$

Only the p integration has an infra-red divergence in $d \leq 3$, and its structure is exactly the same as that of the lowest order diagram given in eqn. (4.23). Thus the pure derivative interactions involving powers of $(\nabla\Phi)^2$ do not change the nature of the divergence of the width of the interface at low temperature.

Control over short distance effects has been investigated by Wallace and Zia [51], using a renormalization group approach, akin to that already described in Chapter Three, § III.4 for the nonlinear σ-model near d=2. They found that perturbative series in T were restricted to ε-expansions in $d = 1 + \varepsilon$ where the U.V. behaviour was governed by a U.V. stable fixed point. T_c was small and of order ε. The width of the interface was found to diverge when $T \to T_c$ directly as the correlation length, ξ, defined by analogy with eqn. (3.40). This conclusion supports the assumption of the interface being sharp for T well below T_c in Ising models.†

Finally, we highlight some of the other major differences between the two low temperature descriptions.

i) For the Ising system, only one renormalization constant is invoked since there is no anomalous dimension developed by a field having the dimensions of a length; as in the case of $\Phi(s)$ which describes a linear deviation away from planar.

† Correspondingly the critical behaviour cannot be accounted for by the nonlinear σ-model because the Heisenberg model with short range interaction does <u>not</u> possess a sharp interface. See e.g. [40].

ii) This parameterization of $\Phi(s)$ leads to Ward identities which are not closed, in contrast to the quadratic identities used in Chapter Three.

iii) All the broken generators of the Euclidean group are realized non linearly on <u>one</u> $\Phi(s)$ - field. Clearly, this represents a major difference between nonlinear realizations of internal and space-time symmetry groups.

CHAPTER FIVE

THE EUCLIDEAN GROUP

AS A

BROKEN DYNAMIC SYMMETRY:

CRITICAL NUCLEATION AND VACUUM DECAY

§ V.1 The Droplet Model :

We turn now to a problem which is related to the one discussed in the previous chapter; a problem that also exhibits a singularity at a first order phase transition:- critical nucleation at the condensation point well below T_c [48, 60]

The first attempt at an understanding of this phenomenon was expressed in the old fashioned "droplet model" [61] In this picture one considers a dilute "gas" of small clusters or "droplets" of the unfavoured phase (down spins in an Ising lattice picture) distributed amongst a background of favoured phase (up spins aligned with a small magnetic field below T_c). The system comes to equilibrium through association and disassociation of the droplets. For droplets of large enough size, ℓ , the free energy is assumed to take the form

$$H \ell^d + A \ell^{d-1} \qquad (5.1)$$

in d space dimensions. The first term in equation (5.1) is the magnetic energy required to flip ℓ^d-down spins. The second term

represents the energy of spin interaction at the droplet surface and is assumed to be proportional to the surface area of a spherical droplet. The free energy per droplet may be written in the cluster approximation as

$$\mathcal{J}(H) = \sum_{\ell=1}^{\infty} \exp - \frac{1}{k_B T} \left[H \ell^d + A \ell^{d-1} \right] \qquad (5.2)$$

At $H = 0$, in equation (5.2), $\mathcal{J}(H)$ exhibits an essential singularity controlled by the surface term [62].

Intuitively, the correction between the singularity in $\mathcal{J}(H)$ and the condensation mechanism can be understood in the following way. Crudely speaking, the number of droplets of size ℓ (proportional to the exponential factor in equation (5.2)) decreases rapidly as ℓ increases for positive H.

For H negative however, the summand in equation (5.2) has a minimum at some $\ell = \ell_c$ where the bulk energy, $- |H| \ell$ begins to decrease faster than the increase in the surface energy. Droplets larger than ℓ_c find it energetically favourable to grow, thereby nucleating condensation. Correspondingly for negative H in equation (5.2) the sum over ℓ does not converge for large ℓ i.e. $\mathcal{J}(H)$ is not analytic at $H = 0$. However, all derivatives, $\frac{\partial^n \mathcal{J}(H)}{\partial H^n}$ at $H = 0$ do exist and have the form, $\sum_{\ell} (- \ell^d)^n e^{-A \ell^{d-1}}$, which is finite. Hence the singularity present in equation (5.2) is an essential singularity.

Langer [60] gained further insight into the nature of the singularity in $\mathcal{J}(H)$ by considering an analytic continuation of the free energy describing the metastable phase within the context of the droplet model. $\mathcal{J}(H)$ has a branch cut along the negative

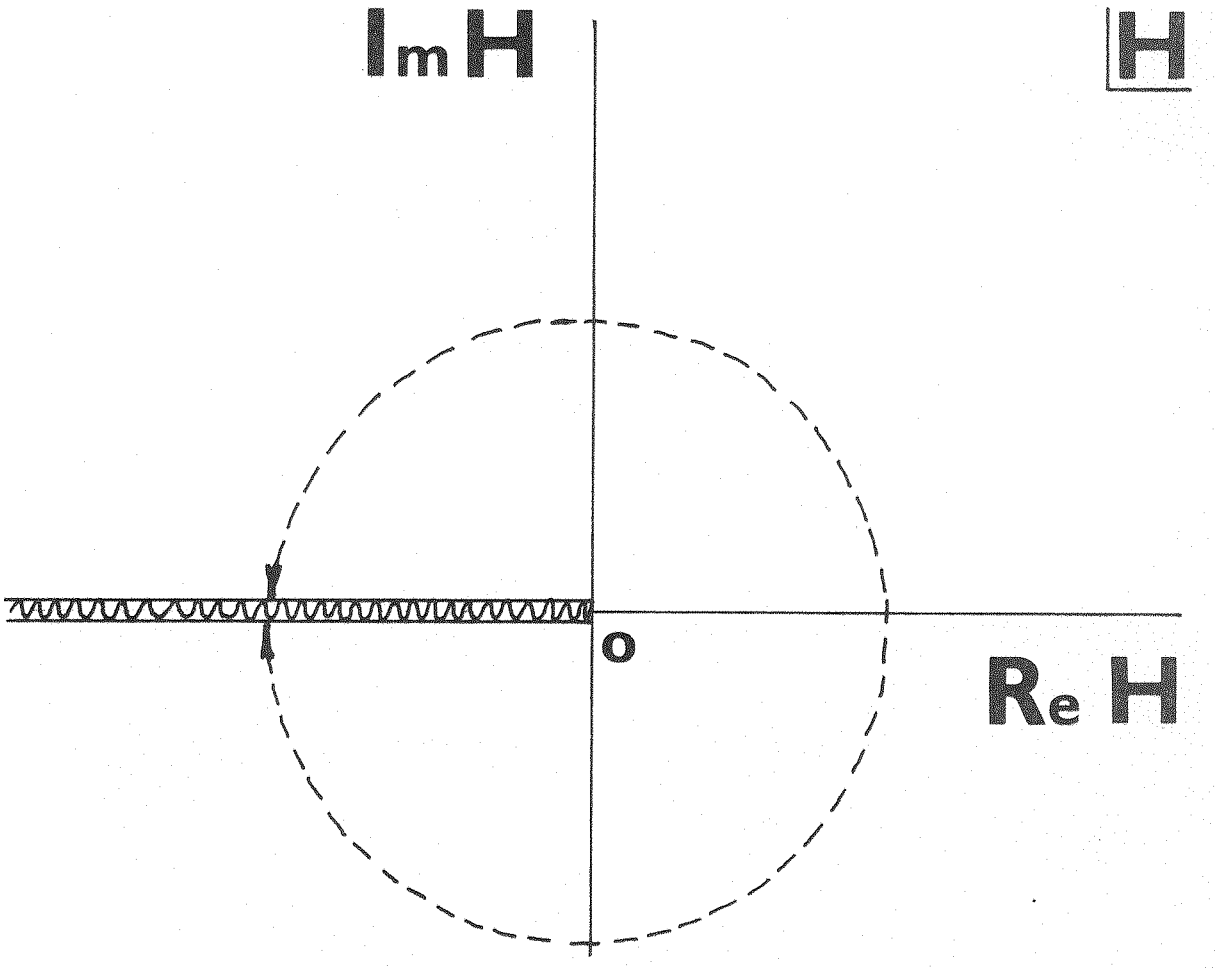


Figure 5.

H axis in the complex H-plane (Fig. 5). The total contribution to the free energy for negative H is related to the discontinuity across the cut (imaginary part) by

$$\mathcal{F}(-|H| \pm i\,\varepsilon) \underset{H\to 0}{=} \mathrm{Re}\,\mathcal{F}(H) \pm i\,\mathrm{Im}\,\mathcal{F}(H) \qquad (5.3)$$

The explicit form of the discontinuity can be obtained from equation (5.2).

Unfortunately, the results of the droplet model cannot be taken too seriously, even though certain intuitive insights may be gained about the microscopic behaviour at the condensation point. One major weakness is the ambiguity in defining the droplets themselves e.g. there is no constraint within the model which ensures the droplets should be spherical.

§ V.2 Functional Gaussian Approach:

An alternative phenomenology is based on the idea of a single droplet of radius r_o, containing the favoured phase and having a total free energy

$$E = A\, r_o^{d-1} - |H|\, r_o^d \qquad (5.4)$$

Here the negative sign means that the volume energy is attractive and the droplet must certainly be spherical in order to minimize the surface energy for a given volume. The droplet size may be determined by calculating the extremum of equation (5.4) VIZ. $dE/dr_o = 0$ which occurs at

$$r_o = \frac{(d-1)}{d} \cdot \frac{A}{|H|} \; ; \; E_o = \frac{A}{d} r_o^{d-1} \qquad (5.5)$$

i.e. the critical radius. At this value of the radius the energy in equation (5.5) behaves like H^{1-d} giving rise to an exponentially small term, exp -(const) H^{1-d} in equation (5.2). This result is reminiscent of tunnelling phenomena as remarked in § IV.2 [54].

Langer found a complete realization for this kind of phenomenological picture in a Gaussian functional model (LGW Hamiltonian) using an approximate classical solution (critical droplet) which locates a stationary point in the functional integrand near the metastable region.

In the programme of Langer, the functional integral is used to define the partition function as in equations (4.2) and (3.9). The free energy is related to the partition function by $\mathcal{F}(H) = \ln Z$. The relevant action, based on the model of a first-order phase transition for an Ising system in an external magnetic field, is that of equation (4.1) with addition of a term $- H \int d^d x \, \phi$. (Recall the discussion in Chapter III for a Heisenberg ferromagnet in an external field). The action corresponds to a "tilted ϕ^4 potential" or "modified Gaussian" having two nondegenerate minima.

The analytic properties of $\mathcal{F}(H)$ as H moves from positive to small negative values with the boundary condition that the expectation value , $\langle\phi\rangle$, remains positive, can be seen qualitatively by considering the behaviour of the inverted potential. For $H > 0$, one peak in the inverted potential is larger and one computes the functional integral in the neighbourhood of the dominant peak.

The other peak refers to the metastable phase. As H passes through zero, the roles of the peaks are interchanged, corresponding

to a phase transition. The integral determining the free energy is divergent in the metastable phase. $\mathcal{F}(H)$ has a branch point at $H = 0$, and an imaginary part for $H < 0$. The imaginary part is computed by considering extrema of the action near the metastable peak. The dominant contribution to the imaginary part comes from the critical droplet as can be seen by examining the form of the Hamiltonian near the stationary point i.e. $\mathbb{H}(\phi_c)$. Approximating the minima by $\phi_\pm \sim \pm \frac{\mu}{\sqrt{g}} + \frac{H}{2\mu^2}$, $\phi_c(r)$ must satisfy the asymptotic condition that it tends to the metastable minimum, ϕ_+, as $r \to \infty$. Then $\mathbb{H}(\phi_c) \simeq \mathbb{H}(\phi_+) +$ corrections, where $\mathbb{H}(\phi_+)$ is the classical free energy in the metastable phase. This term may be neglected in the (dilute gas) approximation for $\mathcal{F}(H)$. The corrections to the LGW-Hamiltonian are (Φ_c as in eqn (5.11))

$$\mathbb{H}(\phi_c) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dr r^{d-1} \left\{ \frac{1}{2}(\nabla\phi_c)^2 - \frac{1}{2}\mu^2(\phi_c^2 - \phi_+^2) + \frac{1}{4}g(\phi_c^4 - \phi_+^4) - H(\phi_c - \phi_+) \right\} \tag{5.6}$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \left\{ \frac{2\sqrt{2}}{3g} r_o^{d-1} - \frac{2\mu H}{d\sqrt{g}} r_o^d \right\} \tag{5.7}$$

which is of the same form as eqn. (5.4).

How the imaginary part arises in the functional integral will be examined again in the context of vacuum decay in § V.5. Essentially the critical droplet provides the dominant contribution to the free energy, as we have just seen. Corrections to the classical action take the form of a determinant referring to small oscillations about the droplet solution. In the metastable phase the small

oscillations determinant has a negative eigenvalue. The free energy is proportional to the square root of the small oscillations determinant which forms a pre-exponential factor. Hence the energy has an imaginary part and may be compared with the imaginary contribution as determined from the droplet model [48,60,63].

Both functions of H, have the same analytic structure at the condensation point with an essential singularity at $H = 0$. There is a smooth continuation of $\mathrm{Re}\,\mathfrak{F}(H)$ through $H = 0$ to describe the metastable phase for $H < 0$.

As we shall point out in § V.5 there is a great difference in the pre-exponential factors associated with the imaginary part as evaluated in the droplet model and the functional integral approach. In the functional integral formulation there is a logarithmic correction from the surface energy which is not present in the droplet model. However, it does not account for the difference in pre-factors, entirely.

Although the singularity at the condensation point is too weak to be observable experimentally, the results of the functional method are to be believed since certain features of the droplet model are recovered and the same pre-exponential factor for the universal logarithm is obtained when the small oscillations determinant is replaced by an effective Hamiltonian for the interaction of the Goldstone modes as we shall describe in the next section.

As a final remark in this section, we note that the old droplet model and the approach of Langer may be related in terms of a dispersion relation for $H > 0$ given by

$$\mathrm{Re}\,\mathcal{J}(H) = \frac{1}{\pi} \int_{H'<0} \frac{\mathrm{Im}\,\mathcal{J}(H')\, dH'}{H' - H} \qquad (5.8)$$

This integral can be interpreted as a sum over a contribution from "virtual droplets" having a continuum of radii - a "multi-length scale" contribution to $\mathcal{J}$, as in the droplet model. This contrasts with the calculation of $\mathrm{Im}\,\mathcal{J}(H)$ [48,63] which involves only a dominant single scale given by $r_o \sim H^{-1}$.

§ V.3 Effective Interaction of Fluctuations:

In this section we describe the role of the critical droplet in the context of a Landau-Ginzburg - Wilson Hamiltonian.

$$\mathcal{H} = \int d^d x \left[\frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}\mu^2 \phi^2 + \frac{1}{4} g \phi^4 - H\phi\right] \qquad (5.9)$$

Following the arguments of § V.2, we seek an extremum of the energy i.e. a classical solution which describes a bubble of the favoured phase, $\langle\phi\rangle < 0$ surrounded by the unfavoured phase, $\langle\phi\rangle > 0$; it is to be understood that this metastable state of the system is produced by an analytic continuation in H from $H > 0$ to $H < 0$.

Since we are seeking a radially symmetric solution, the classical field equation is

$$\frac{d^2\phi}{dr^2} + \frac{(d-1)}{r}\frac{d\phi}{dr} + \mu^2 \phi - g \phi^3 - |H| = 0 \qquad (5.10)$$

The classical solution alluded to in the previous section cannot be expressed in terms of elementary functions as was the case in § IV.2, equation (4.4). However for small $|H|$, the radius of the droplet increases like H^{-1}, according to equation (5.5), so that the local interface between phases is approximately that of the planar interface

$$\phi_c \simeq \frac{1}{2}(\phi_+ + \phi_-) + \frac{1}{2}(\phi_+ - \phi_-) \tanh \frac{\mu}{\sqrt{2}} (r - r_o) + O(H^2) \tag{5.11}$$

Notice that the original symmetry of the Hamiltonian density under the transformations $\phi \to -\phi$ together with $H \to -H$, is preserved by this choice of ϕ_c. This implies that corrections to ϕ_c, must be a power series in <u>even</u> functions of H.

Using equations (5.10) and (5.11) we can develop the spectrum of low lying angular excitations on the surface of the droplet. This spectrum has already been identified by Langer [60] For the d - fold degenerate eigen-function $\nabla \phi_c(r)$, corresponding to zero eigenvalue - the $\ell = 1$ translation mode; $E_{n,\ell} = E_{o,1}$. The excitations of this mode are conveniently obtained from an interaction picture, $M = M_o + M_1$, with

$$M_o = \frac{-d^2}{dr^2} + 3\mu^2 \tanh^2 \frac{\mu}{\sqrt{2}} (r - r_o)$$

and (5.12)

$$M_1 = \frac{-(d-1)}{r} \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + 3g\left[\phi_c - \frac{\mu^2}{g} \tanh^2 \frac{\mu}{\sqrt{2}} (r-r_o)\right]$$

The spectrum can now be generated by elementary perturbation theory; $M\,|n,\ell> = E_{n\ell}\,|n,\ell>$, such that

$$E_{n\,\ell} = E^{o}_{n,\ell} + E^{1}_{n,\ell} + E^{2}_{n,\ell} + \ldots .$$

where

$$E^{1}_{n,} = <n,\ell|M_1|n,\ell>$$

and

$$E^{2}_{n,\ell} = \sum_{n'\ell'} \frac{|<n,\ell\,|M_1|n\,'\ell'>|^2}{E_{n\,'\ell'} - E_{n\,\ell}}$$

Since we are considering only the rotational band in the neighbourhood of $E_{o,1} = 0$, for $E'_{o\ell}$ the corrections to the eigenvalues are all of $O(H^2)$ for $\ell << H^{-1}$. The eigenfunctions $|n,\ell>$ are factorizable into radial and angular contributions. The zero translation mode $|0\ ,\ 1>$ corresponds to the eigenfunction

$$\psi_o(r) = \frac{\mu^2}{\sqrt{2g}}\,\mathrm{sech}^2\,\frac{\mu}{\sqrt{2}}\,(r-r_o) \qquad (5.13)$$

which is peaked at $r = r_o$. So the dominant contribution to the eigenfunctions $|0,\ \ell>$ come from the spherical harmonics with eigenvalues determined directly from the angular momentum barrier $\ell(\ell+1)/r^2$. In general we may write $E_{o\ell} \sim \frac{\ell(\ell+1)}{r_o^{\,2}}$ and subtracting out the zero mode $E_{o,1} \sim \frac{2}{r_o^{\,2}}$ we have

$$E_{o\ell} = \frac{(\ell-1)(\ell+2)}{r_o^2}\left[1 + O(H^2)\right] \qquad (5.14)$$

which in d-dimensions becomes

$$E_{o\ell} = \frac{(\ell-1)(\ell+d-1)}{r_o^2}\left[1 + O(H^2)\right] \qquad (5.15)$$

Note that the energy of the non-degenerate lowest mode, $\ell = 0$, is <u>negative</u> and this is the single eigenvalue responsible for generating the imaginary part in the functional integral in the metastable phase as mentioned in § V.2. A negative eigenvalue in the Euclidean action means the integrand is now positive and therefore increases as we move away from ϕ_c. These displacements mean that the droplet is unstable to dilatations. In the context of analytic continuation, integration over these modes is performed by distorting the path of integration into the complex plane away from the saddle point ϕ_c.

However, the spectrum of equation (5.12) is discrete because the manifold is that of the surface of a d-dimensional sphere of radius $r_o \sim H^{-1}$. In the limit $H \to 0$, $r_o \to \infty$ and there is a quasi-continuous spectrum of Goldstone excitations corresponding to perturbative spherical harmonic modes of the droplet surface and once again it becomes important to determine the effective action describing the interaction between these long-wave length modes.

Recalling the derivation of the effective action for the interaction for the interaction of the Goldstone modes in the interfacial profile, we can describe deviations away from a purely spherical surface in the following way.

Consider the spherical surface of the undeformed droplet to be described in terms of the position unit vector, $\underline{r} = r_o \hat{\underline{r}}$ (Fig. 6). The "shifted" position vector, $\underline{r}' = \underline{r} - \underline{r}_o$ then represents any deviation away from spherical in the direction of $\hat{\underline{r}}$. The magnitude of this deviation is denoted by

$$\Phi(\hat{\underline{r}}) = r(\hat{\underline{r}}) - r_o \tag{5.16}$$

and is the Goldstone field for the spherical excitations of the critical droplet.

Physically, there are two contributions to the droplet energy, as explained in equation (5.4). VIZ. the bulk free energy, proportional to the volume of the droplet and the magnitude of the external field (in d-dimensions).

$$A_v = - \frac{|H|}{d} \int d|\underline{r}| \cdot (r_o + \Phi)^d \tag{5.17}$$

and the surface energy. In terms of the definition of the field in eqn. (5.16), the action for the surface corresponds to introducing the replacements,

$$\int d^{d-1}x \quad \rightarrow \int r^{d-1} \, d\Omega$$

$$|\nabla \Phi(\underline{x})|^2 \quad \rightarrow \frac{1}{2r^2} (L_{ij} \Phi)^2$$

into the action for the interfacial profile problem, equation (4.15) where

$$L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \; ; \; L^2 = \frac{1}{2} L_{ij} L_{ij} \tag{5.18}$$

To understand how this can come about, we recall our earlier argument that near $r = r_o$, the dominant contribution to

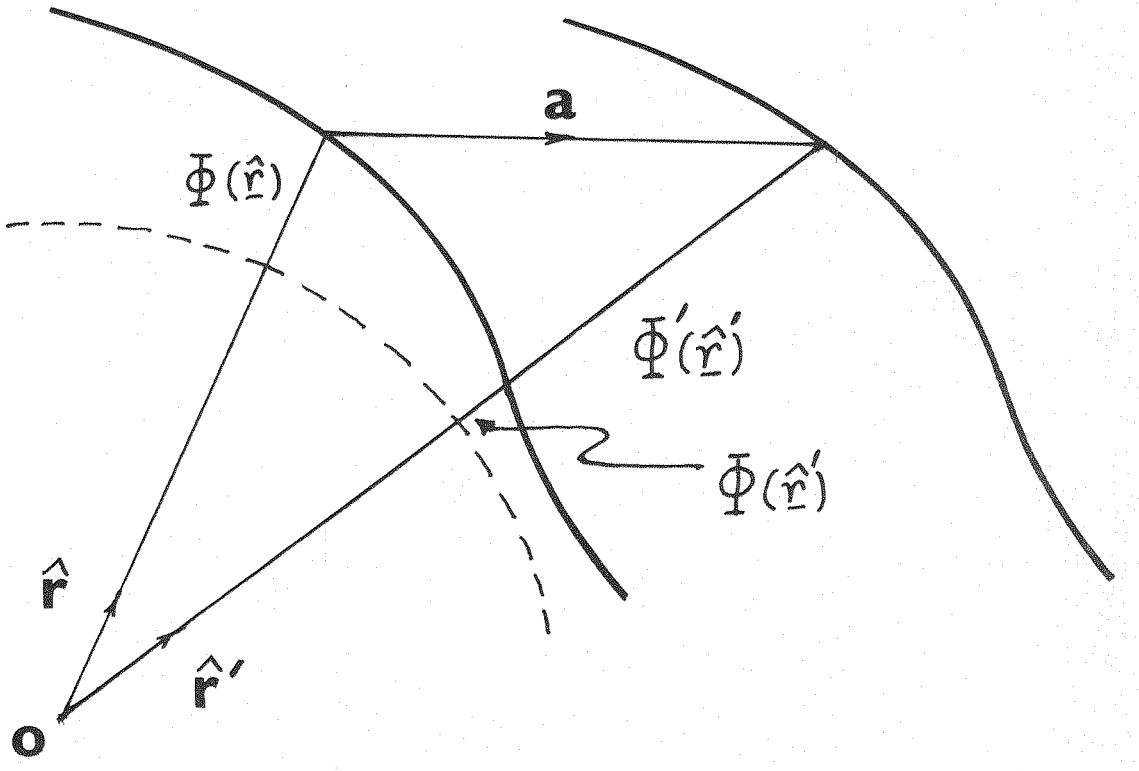


Figure 6.

the excitations come from the spherical harmonics in $\hat{\phi}$. Since $\Phi(\hat{\underline{r}})$ in equation (5.17) represents the low-wave vector modes, we expect that near $r = r_o$ we can write $r(\hat{\underline{r}}) = r_o g(\hat{r})$ and $\Phi(\underline{r}) \simeq r_o[g(\hat{\underline{r}}) - 1]$ so that the radial contributions to the Lagrangian are not significant. Further, we can expand the "kinetic" term in equation (4.15) (for d=3), as

$$(\nabla \Phi)^2 \equiv (\nabla \Phi)\cdot(\nabla \Phi) \tag{5.19}$$

$$= \left(\frac{\partial\Phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial\Phi}{\partial\phi}\right) \tag{5.20}$$

$$\simeq \frac{1}{r^2}\left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial\Phi}{\partial\phi}\right)^2 \tag{5.21}$$

Hence the elemental area

$$\sqrt{1 + (\nabla \Phi)^2} \tag{5.20}$$

can be written as,

$$\left\{1 + \frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial\Phi}{\partial\phi}\right)^2\right\}^{\frac{1}{2}} \tag{5.23}$$

We also have the expansion

$$\frac{1}{2}(L_{ij}\,\Phi)^2 = \frac{1}{2}L_{ij}(\Phi\, L_{ij}\,\Phi) - \frac{1}{2}\Phi L_{ij}(L_{ij}\,\Phi)$$

$$= \frac{1}{2}L_{ij}(\Phi\, L_{ij}\,\Phi) - \Phi L^2\,\Phi \tag{5.24}$$

Identifying respective terms in equations (5.21) and (5.24) we find

$$\frac{1}{2r^2}(L_{ij}\,\Phi)^2 \equiv \frac{1}{r^2}\left\{\left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\phi}\right)^2\right\} \tag{5.25}$$

so that the Lagrangian in equation (4.15) becomes

$$\left\{1 + \frac{1}{2r^2}(L_{ij}\,\Phi)^2\right\}^{\frac{1}{2}} \tag{5.26}$$

Integration over the surface corresponds to the replacement

$$\int d^2x \;\rightarrow\; \int d\varphi d\theta \sin\theta\, r^2 \tag{5.27}$$

The correct expression for the surface contribution to the action in d-dimensions is

$$A_s = \int d\,\Omega(\Phi+r_o)^{d-1}\left[1 + \frac{1}{2(\Phi+r_o)^2}(L_{ij}\,\Phi)^2\right]^{\frac{1}{2}} \tag{5.28}$$

A more explicit derivation will be presented in Chapter Six.

§ V.4 Behaviour of the Effective Action:

The Goldstone field, $\Phi(\hat{\underline{r}})$, transforms nonlinearly under the original full Euclidean group. The infinitesimal form of this transformation is conveniently obtained in the following way. We consider the infinitesimal translation of a point on the surface given by the position vector, $r(\hat{\underline{r}})$

From Fig. 6 we have

$$\underline{r}' = \underline{r} + \underline{a}$$

which on projection into the $\hat{\underline{r}}$-direction gives,

$$r' = r + \underline{a}\cdot\hat{\underline{r}} \quad ,$$

so that in terms of the field,

$$\Phi'(\hat{\underline{r}}') = \Phi(\hat{\underline{r}}) + \underline{a}\cdot\hat{\underline{r}} \quad . \tag{5.29}$$

Also we have

$$\hat{\underline{r}}' = \frac{\underline{r} + \underline{a}}{|\underline{r} + \underline{a}|} = \frac{\underline{r} + \underline{a}}{r}\left(1 - \frac{\underline{a}\cdot\hat{\underline{r}}}{r} + \ldots\right) \tag{5.30}$$

Using equation (5.30) we can write the transformed field, $\Phi'(\hat{\underline{r}}')$ as a Taylor expansion to $O(\underline{a})$

$$\begin{aligned}\Phi'(\hat{\underline{r}}') &= \Phi'(\hat{\underline{r}}) + \frac{1}{r}\left[\underline{r}\cdot\underline{r}(\underline{a}\cdot\nabla - (\underline{a}.\underline{r})(\underline{r}\cdot\nabla)\right]\Phi'(\hat{\underline{r}}) \\ &= \Phi'(\hat{\underline{r}}) + \frac{a_i\,\hat{r}_j}{r}\,L_{ij}\,\Phi'(\hat{\underline{r}})\end{aligned} \tag{5.31}$$

Substituting equation (5.31) into equation (5.29) gives to $O(\underline{a})$

$$\Phi'(\hat{\underline{r}}) = \Phi(\hat{\underline{r}}) + \underline{a}\cdot\hat{\underline{r}} - \frac{a_i \hat{r}_j}{r} L_{ij} \Phi(\hat{\underline{r}})$$

$$= \Phi(\hat{\underline{r}}) + \underline{a}\cdot\hat{\underline{r}} - \frac{a_i \hat{r}_j}{\Phi(\hat{\underline{r}})+r_o} L_{ij} \Phi(\hat{\underline{r}}) \qquad (5.32)$$

in agreement with ref. [48].

Both A_v and A_s are separately invariant under the nonlinear transformation (5.32). As was the case in § IV.2, the densities are invariant up to total divergences (involving L_{ij}) which vanish upon integration over all angles. The full effective action is the sum of A_v and A_s.

Choosing the value $|H| = \frac{(d-1)}{r_o}$, ensures that linear terms in Φ do not appear at lowest order. The free contribution to the full action is

$$A_o = \frac{1}{2} r_o^{d-3} \int d\Omega \left[\frac{1}{2}(L_{ij}\Phi)^2 - (d-1)\Phi^2\right]$$

$$= \frac{1}{2} r_o^{d-3} \int d\Omega \Phi \left[L^2 - (d-1)\right] \Phi \qquad (5.33)$$

The total angular momentum operator has eigenvalues

$$L^2 = \ell(\ell+d-2) \qquad (5.34)$$

in d-dimensions, and substitution into equation (5.33) reproduces correctly the energy spectrum given in equation (5.15), modulo a rescaling of A_o by a factor r_o^{d-1}.

Expanding A_{eff} to higher order to obtain the interaction

terms, we see again the appearance of derivative couplings involving $L_{ij}\,\Phi$. The U.V. divergences clearly arise from higher angular momentum states and are of the same character as those discussed in Chapter Four, i.e. they are nonrenormalizable above one-dimension.

On the other hand a discussion of the infra-red behaviour is now less trivial than it was for the planar interface. This comes about because an expansion of the action in equation (5.28) generates not only angular derivatives but also terms factored by the field Φ . Since this field must be scaled by r_o according to the decomposition equation (5.25),it carries with it a natural I.R. cut-off VIZ. the droplet radius. Thus non-derivative terms which potentially may create I.R. problems always appear with appropriate powers of r_o^{-1} so rendering them harmless. Furthermore, this suggests that the additional contribution to the singularity in $\mathrm{Im}\mathfrak{F}(H)$, which arises from the $\ln \mu r_o$ term in the small oscillations determinant, is unlikely to be affected by higher order contributions. This is analogous to the 1-loop contribution to $\langle\Phi^2\rangle$ in § IV.3.

§ V.5 Vacuum Decay:

The thermodynamic problem we have discussed in the last two sections, also has an important parallel in the context of a field theoretic formulation of particle physics.

It is now well established, since Dyson's original argument [64] concerning quantum electrodynamics, that field theories may not possess a unique "ground state" or vacuum. Hence the vacuum can decay due to the phenomenon of quantum mechanical tunnelling. In field theory it is the "instanton" [65] solutions

which describe vacuum tunnelling between classically stable vacua (e.g. in Yang-Mills theories) [66].

Let us review the development of the imaginary part in the case of a metastable vacuum. Consider the theory of a single scalar field with "tilted $g\phi^4$- potential" $V(\phi)$, as shown in Figs. 7 & 8. The dominant minimum action solutions are evaluated using the action S_o with inverted potential, - V. Tunnelling out of the false vacuum corresponds to what Coleman calls "the bounce" solution [54] and for lowest action is the spherically symmetric solution in equation (5.11). In the inverted potential, there is a classical turning point, where a classical particle having made an excursion away from $x = 0$, returns to $x = 0$. This turning point must correspond to a zero or node in the relevant eigenfunction, and as a consequence there should exist an eigenfunction without a node, possessing a negative eigenvalue. It has been demonstrated that there is only ever a single negative eigenvalue [67]. Since this negative energy solution is not an eigenvalue of the time-development operator (Hamiltonian) it must be defined through analytic continuation in V in the functional integral. This is the basis of the remarks in § V.2, regarding the imaginary part in the free energy of the metastable phase.

After saturating the functional integral with a dilute gas of bounces, the decay probability rate, Γ, per unit volume is given by

$$\frac{\Gamma}{V} = \text{Im } K \exp - So/\hbar \qquad (5.35)$$

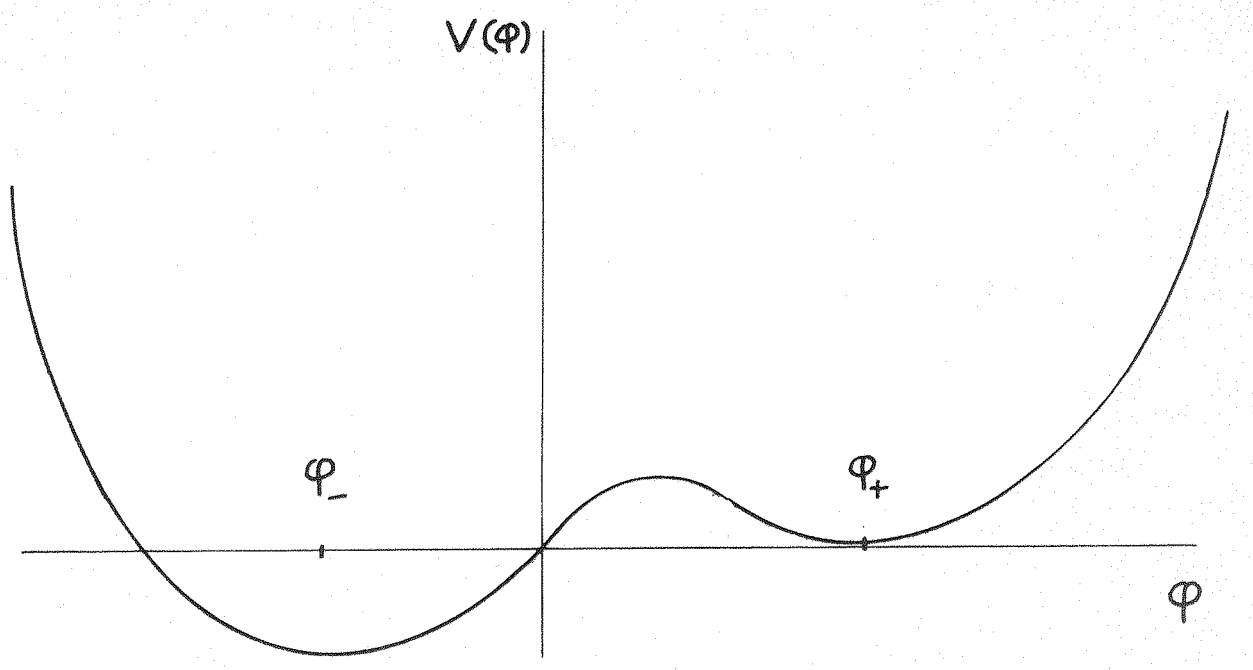


Figure 7.

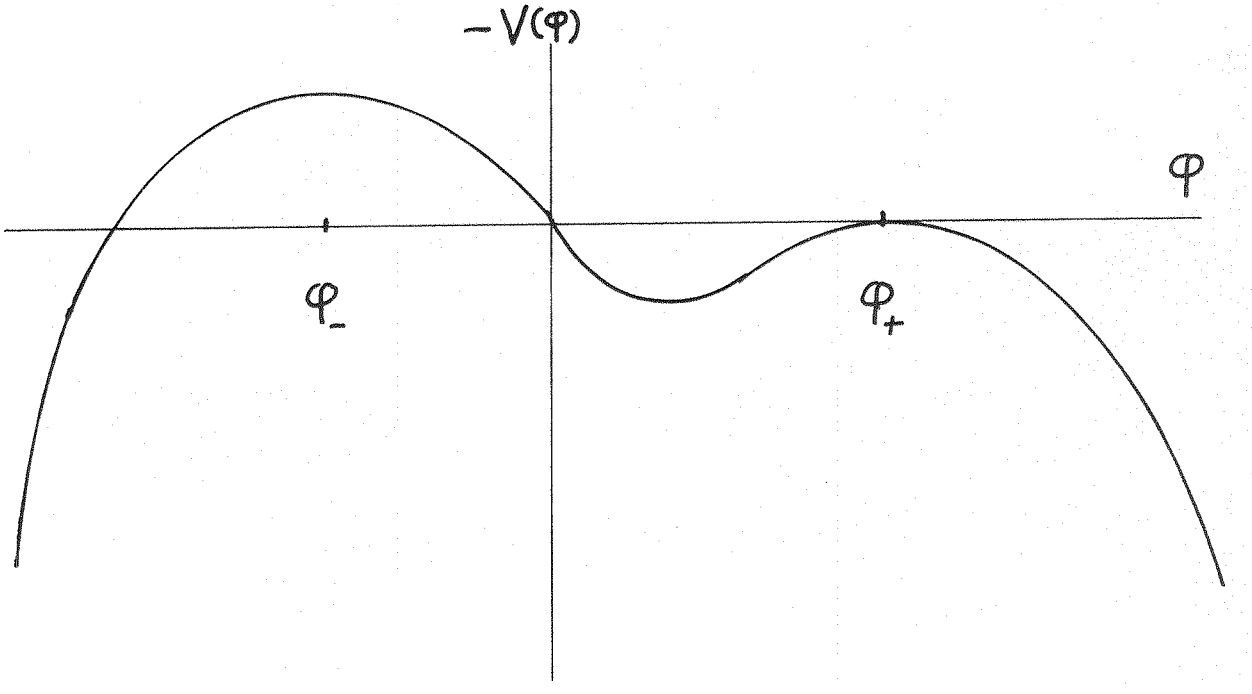


Figure 8.

Here

$$\text{Im } K = \left(\frac{So}{2\pi\hbar}\right)^2 \left[\frac{\text{Det}'\ \left(-\partial_\mu\partial^\mu + V''(\phi_c)\right)}{\text{Det}\ \left(-\partial_\mu\partial^\mu + V''(\phi_+)\right)}\right]^{-\frac{1}{2}} \cdot e^{-S^{(1)}(\phi_c)} \tag{5.36}$$

with Det' denoting 4 zero modes omitted and including one-loop renormalization counter-terms, $S^{(1)}(\phi_c)$. This quantity is finite since it is essentially the exponential of the renormalized one-loop effective action.

In $d = 3+1$, the one loop corrections have been explicitly evaluated in the "thin-wall approximation", [68] giving

$$\text{Im } K = \left(\frac{\mu^3 r_o}{3g\hbar}\right)^2 \exp-(\mu r_o)^3 \left\{\frac{-\pi}{9\sqrt{3}} + \zeta'(-2) + \frac{\text{const.}}{(\mu r_o)^2} + O\left(\frac{1}{g\hbar}\right)\right\} \cdot \left[1+O(g\hbar)\right]. \tag{5.37}$$

where $\zeta'(-2) = -0.3038$ is defined by the derivative of the Riemann-Zeta function, $\zeta(s) = \sum_{k=0}^{\infty} k^{-s}$.

In $d = 3$, a universal logarithmic correction, $\frac{2}{3} \ln \mu r_o$ can be carefully extracted from the exponential [68]. This is the same logarithm referred to in §.V.2. The coefficient is found to be in agreement with the calculation by Wallace [63] et al. [48] and disagrees with Langer's original result [60], which is larger by a factor of 2. The universal logarithm is a contribution from the low wave-vector modes in the bubble's surface i.e. the Goldstone modes of the broken Euclidean invariance described in §.V.3.

Coleman [54] has conjectured that apart from spontaneous decay of the false vacuum, there is also the possibility of induced vacuum

decay e.g. in a two-particle collision at high energy, there could exist a non-trivial cross-section for the production of a vacuum bubble. Affleck and De Luccia [69] have computed the additional decay amplitude for a one meson state. Semi-classically this situation is analogous to the first excited state in quantum mechanics. Their results indicate, that for sufficiently high meson density, the induced decay amplitude dominates the spontaneous decay amplitude [70].

In conclusion we remark that the results of Chapters Four and Five represent a unified formalism for the description of low-momentum effects of fluctuations in the surface of separation between thermodynamic phases. The success of this programme rests largely on interpreting these fluctuations as the Goldstone modes of a dynamical Euclidean invariance.

In Chapter Four we were concerned with the infra-red instabilities induced by these Goldstone modes in the absence of the external field ($H = 0$). In Chapter Five we examined the nature of the essential singularity in the free energy at the condensation point as $\mathcal{F}(H)$ was analytically continued to $H < 0$. The universal logarithmic contribution from the Goldstone modes, present at the phase transition point and the vacuum decay amplitude, carries a coefficient different to that evaluated by Langer.

CHAPTER SIX

NONLINEAR REALIZATIONS OF EUCLIDEAN SYMMETRIES AND GEOMETRIC EFFECTIVE ACTIONS

§ VI.1 Nonlinear Geometric Realizations:

In Chapters Four and Five, we have examined the merits of investigating the critical behaviour of the interface between two (or more) coexisting phases from the view point of a dynamically broken Euclidean symmetry and the associated Goldstone modes. Features analogous to those of nonlinear σ-models were highlighted, but we also noted that several major differences arose and in this Chapter we consider more carefully the group-theoretic content of nonlinear realizations of the Euclidean algebra and the structure of the effective actions.

For completeness we begin by recalling the essential properties of the Euclidean group [71].

The Lie algebra of the d-dimensional Euclidean group, $E(d)$, is spanned by two sets of generators; translations of the origin $P_i = \nabla_i$, and rotations of the position vector x_i, $L_{ij} = (x_i\nabla_j - x_j\nabla_i)$, where $i, j = 1,2,\ldots,d$. In this representation the commutation relations of the algebra are

$$[P_i,P_j] = 0 \; ; \; [P_i,L_{jk}] = \delta_{ik}P_j - \delta_{ij}P_k \qquad (6.1)$$

$$[L_{ij},L_{k\ell}] = \delta_{ik}L_{j\ell} - \delta_{i\ell}L_{jk} + \delta_{j\ell}L_{ik} - \delta_{jk}L_{i\ell}$$

A group element is specified by $\frac{1}{2}d(d+1)$ real parameters, a_i and ω_{ij} such that

$$g(a,\Lambda\omega) = \exp\,(a_i\,P_i + \tfrac{1}{2}\,\omega_{ij}\,L_{ij}) \qquad (6.2)$$

and any two elements satisfy the semi-direct product rule $g_{21} = g_2 \cdot g_1$ with $\Lambda = 1 + \omega$, $(\Lambda_2\Lambda_1\,,\ \Lambda_2\,a_1 + a_2) = (\Lambda_2\,,\ a_2)\cdot(\Lambda_1\,,\ a_1)$ The algebra, eqn. (6.1), has an Abelian invariant subalgebra and is consequently <u>non semi-simple</u>. Also, since the parameters, a_i, are unbounded, the group is <u>noncompact</u>.

Nonlinear realizations, as they pertain to continuous <u>internal</u> symmetries (e.g. the O(N) nonlinear σ-model discussed in Chapter Three) are based on the identification of the Goldstone field parameters with the coordinates of the appropriate coset (manifold) space. For a general compact, semi-simple Lie group, G, having a subgroup, H, (generally compact also), the Lie algebra can be decomposed into a set of generators $\{T_j\}$ associated with the subgroup H, and a set of generators $\{X_a\}$ each of which is associated with the left coset space, G/H.† Recall in the case of the nonlinear σ-model the coset space was O(N)/O(N-1).

Nonlinear realizations are realizations of the algebra in which the Goldstone fields, $\pi_a(x)$, transform into functions of themselves such that, these functions are <u>linear</u> under the action of the T_j, but are <u>nonlinear</u> under the action of the X_a. The action of a general G group element, g, on the element, $\exp i\,\pi_a\,X_a$, of the factor space, G/H, must also be a member of G by virtue of the closure property of groups. That is to say,

$$g\,\exp i\,\pi_a\,X_a = \exp i\,\alpha_a(\pi,g)X_a \cdot \exp i\,\beta_j(\pi,g)T_j \qquad (6.3)$$

$$= \exp i\,\pi_a'\,X_a'$$

† Considering the Goldstone field as an element of the homotopy group, $\Pi_3(SU_2) = Z_1$, leads to a dynamical realization relevant to Yang-Mills theories. V.N. Pervushin, Dubna preprint (E2-12514) (1979).

The canonical nonlinear realization [73] is then defined by requiring that the transformed field be equal to the function α_a , i.e.

$$\pi'_a(x) \equiv \alpha_a(\pi,g) \tag{6.4}$$

Now if g belongs to the subgroup H, then the generators, X_a , afford a representation, h_{ab} , of H : $g\, X_a\, g^{-1} = h_{ab}(g)\, X_b$, so that

$$g \cdot \exp i\, \pi_a\, X_a = \exp i\, \pi_a\, h_{ab}\, X_b \cdot g \tag{6.5}$$

But, any element of H can be expressed as

$$g = \exp i\, \beta_j(g)\, T_j \tag{6.6}$$

so that eqn. (6.5) is identical in form to eqn. (6.3), VIZ. $\beta_j(\pi,g) = \beta_j(g)$. Hence the Goldstone fields transform linearly under the action of the T_j as required, i.e.

$$\alpha_a(\pi,g) = h_{ab}(g)\pi_a \tag{6.7}$$

There is an important difference between this parameterization using coset representatives and nonlinear realizations as they have appeared in Chapters Four and Five. The d-dimensional Euclidean symmetry is broken by a (d-1) - dimensional physical surface. Accordingly, the Goldstone fields are parameterized as displacements away from the minimal physical surface (a length) so that the coset

space apparently plays no rôle in the parameterization of the Goldstone fields. This is in severe contrast to nonlinear realizations for continuous internal symmetries as indicated above.

Consequently, the noncompactness and non-semisimplicity of the Euclidean group seem to be of little relevance in determining the appropriate nonlinear realizations for the problems which we have been considering in the previous chapters.

§ VI.2 The Invariant Action:

The reader will recall from our remarks in Chapter Three, § III.1 that the O(N) - invariant nonlinear Lagrangian, eqn. (3.6) was defined through the metric tensor parameterized on an S^{N-1} manifold of the coset space. In view of our remarks in § VI.1, concerning the Goldstone modes of the dynamically broken Euclidean symmetry and their parameterization on the relevant physical surface, we are led to consider the effective action prescription, given previously in chapters Four and Five, as a consequence of defining the metric tensor on the physical manifold.

It is in this sense that we can reasonably refer to these effective actions as purely geometric in nature.

Formally, [74] the line element

$$ds^2 = dx_i \, g_{ij}(x) dx_j \; ; \; i,j=1,2\ldots(d-1) \qquad (6.8)$$

is invariant under coordinate transformations

$$\delta x_i = x_i' - x_i = \xi_i(x) \qquad (6.9)$$

where $\xi_i(x)$ are infinitesimal functions. Then the metric transforms as a rank-2 tensor,

$$\delta g_{ij} = \xi^k \partial_k(g_{ij}) + \partial_i(\xi^k g_{kj}) + \partial_j(\xi^k g_{ki}) \qquad (6.10)$$

A scalar density, $\Sigma(x)$, has the infinitesimal transformation

$$\delta\Sigma(x) = \partial_i(\xi^i \Sigma) \qquad (6.11)$$

so that an integral over the (d-1)-dimensional volume implies invariance for any quantity which transforms as a scalar density. In particular, the metric density $(i,j= 1,2,\ldots,n=d-1)$

$$\text{Det } g_{ij} = \frac{1}{n!} \varepsilon_{i_1 i_2 \ldots i_n} \varepsilon^{j_1,j_2,\ldots j_n} g_{i_1 j_1} \cdots g_{i_n j_n} \qquad (6.12)$$

fulfils this requirement and specifies the invariant volume of the surface in d-dimensional Euclidean space.†

$$A_{\text{eff}} \text{ (Vol)} = \int dx_1 . dx_2 \ldots dx_{(d-1)} \sqrt{\text{Det } g_{ij}(x)} \qquad (6.13)$$

§ VI.3 Geometric Densities:

Let us now use the formalism of § VI.2 to recover our previous results:

Planar Interface: The d-dimensional Euclidean space may be defined by

† A similar expression for the volume in Minkowski space has been used by P.A.M. Dirac, Proc. Roy. Soc. A268, 57 (1962) in an extensive model of the electron as a "bubble" in the electromagnetic field.

a Cartesian vector, $X_a \equiv (x_1, x_2, \ldots, x_d)$. Choosing the displacement, Φ, to lie in the X_d-direction and re-labelling the remaining orthogonal coordinates (transverse directions) as $s_1, s_2, \ldots, s_{d-1}$, we can rewrite the vector as $X_a \equiv (s_i, \Phi(s_i))$. The metric tensor then, can be determined according to eqn. (6.8) as

$$g_{ij} = \sum_{a=1}^{d} \frac{\partial X_a}{\partial s_i} \cdot \frac{\partial X_a}{\partial s_j} = \delta_{ij} + \nabla_i \Phi \cdot \nabla_j \Phi \qquad (6.14)$$

Eqn. (6.12) can be used to evaluate the determinant of the metric tensor, g_{ij}, as follows,

$$\text{Det } g_{ij} = \frac{1}{n!} \epsilon^{i_1, i_2, \ldots} \cdot \epsilon_{j_1, j_2 \ldots} (\delta_{i_1 j_1} + \nabla_{i_1} \Phi \nabla_{j_1} \Phi) \ldots (\delta_{i_n j_n} + \nabla_{i_n} \Phi \nabla_{j_n} \Phi) \qquad (6.15)$$

Expanding in ascending powers of $\nabla\Phi$ we find

$$= \frac{1}{n!} \epsilon^{i_1, i_2, \ldots} \cdot \epsilon_{j_1, j_2 \ldots} \left\{ \delta_{i_1 j_1} \ldots \delta_{i_n j_n} + \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \delta_{i_{n-1} j_{n-1}} \nabla_{i_n} \Phi \nabla_{j_n} \Phi + \right.$$

$$\left. \ldots\ldots\ldots + \nabla_{i_1} \Phi \nabla_{j_1} \Phi \ldots \nabla_{i_n} \Phi \nabla_{j_n} \Phi \right\} \qquad (6.16)$$

$$= \text{Det } \delta_{ij} + \delta^j{}_i \nabla_i \Phi \nabla_j \Phi + O[(\nabla\Phi)^4] \qquad (6.17)$$

$$= 1 + \sum_i \nabla_i \Phi \nabla_i \Phi \qquad (6.18)$$

where we have used contractions of the type, $\epsilon^{abc} \epsilon_{abf} = \delta^c{}_f$. In eqn. (6.17) all terms of $O[(\nabla\Phi)^4]$ and higher necessarily vanish since they involve contraction with the antisymmetric ϵ-tensor. Substitution of the result in eqn. (6.18) into the definition, eqn. (6.13), furnishes the effective action for the planar interface

employed in Chapter Four.

Vacuum Bubble: A consistent derivation of the effective action for the surface energy can be obtained from a metric density for the distorted surface expressed in terms of the original spherical surface. The line element, in 2-dimensions on the distorted surface in Fig. 9 is given by

$$ds^2 = dr^2 + r^2\, d\theta^2 \tag{6.19}$$

The field, $\Phi(\hat{\underline{r}})$, is defined as in eqn. (5.16) along the direction $\hat{\underline{r}}$. The angle, $d\psi$, is defined by

$$\tan d\psi = \frac{dr(\hat{\underline{r}})}{d\lambda} = \frac{1}{r}\frac{dr(\hat{\underline{r}})}{d\theta} \tag{6.20}$$

In terms of dr, the line element in eqn. (6.19) becomes

$$\begin{aligned} ds^2 &= r^2\, d\theta^2 \left[1 + \tan^2 \psi\right] \\ &= r^2\, d\theta^2 \left[1 + \frac{1}{r^2}\left(\frac{\partial \Phi(\hat{\underline{r}})}{\partial \theta}\right)^2\right] \end{aligned} \tag{6.21}$$

The corresponding metric tensor is most convenienty expressed in matrix form as

$$(g_{ij}) = \begin{bmatrix} 1 & \\ & r^2\left[1 + \frac{1}{r^2}\left(\frac{\partial \Phi}{\partial \theta}\right)^2\right] \end{bmatrix} \tag{6.22}$$

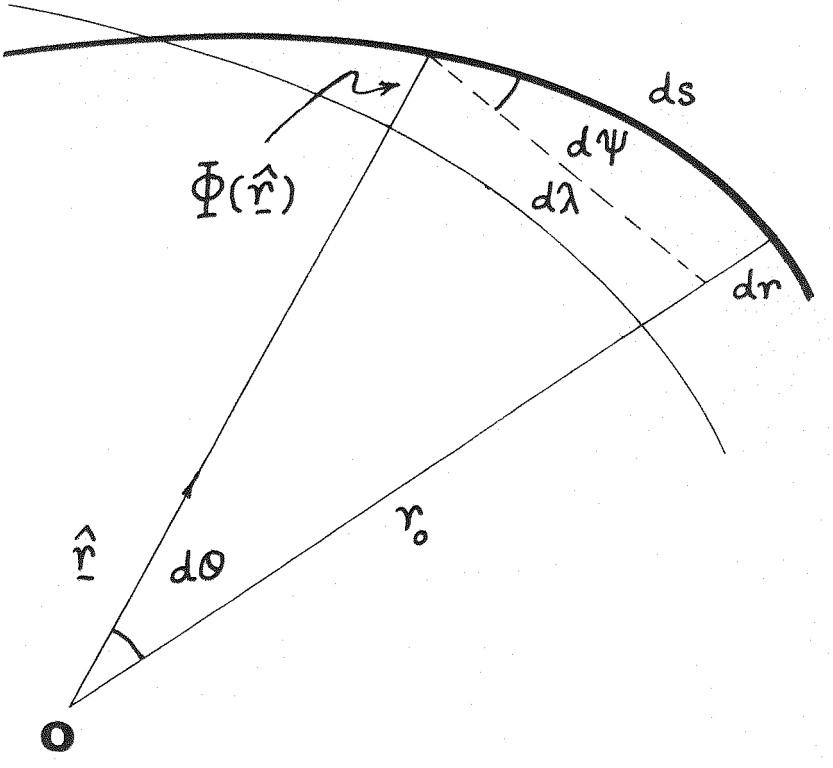


Figure 9.

The metric density (Jacobian) is then given by

$$(\text{Det } g_{ij})^{\frac{1}{2}} = r\left[1 + \frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right)^2\right]^{\frac{1}{2}} \tag{6.23}$$

and the effective action is

$$A_s = \int d\theta(r_o+\Phi(\hat{\underline{r}}))\left\{1 + \frac{1}{(r_o+\Phi)^2}\left(\frac{\partial\Phi}{\partial\theta}\right)^2\right\}^{\frac{1}{2}} \tag{6.24}$$

Now we have according to eqn. (5.18)

$$L_{12} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial\theta} \tag{6.25}$$

Clearly, eqn. (6.24) can be rewritten in terms of L_{12}

$$A_s = \int d\theta(r_o + \Phi(\hat{\underline{r}}))\left\{1 + \frac{1}{(r_o+\Phi)^2}(L_{12}\,\Phi)^2\right\}^{\frac{1}{2}} \tag{6.26}$$

which generalizes to d-dimensions as

$$A_s = \int d\Omega(r_o+\Phi)^{d-1}\left\{1 + \frac{1}{2(r_o+\Phi)^2}(L_{ij}\,\Phi)^2\right\}^{\frac{1}{2}} \tag{6.27}$$

The factor, 2 , appears in eqn. (6.27) as a consequence of the identity in eqn. (5.25).

Wallace [75] has found a derivation of eqn. (6.27) using a general method more closely allied to that given in deriving eqn. (6.18). We outline his method for completeness.

In the notation of Chapter Five, the distance to the surface is $r(\hat{\underline{r}})$

in the direction $\underline{\hat{r}}$. The vector displacement $\delta\underline{\hat{r}}$ between neighbouring points $\underline{r}(\underline{\hat{r}})$ and $\underline{r}'$ lies on the surface of the d-dimensional manifold where $\underline{r}' = (\underline{\hat{r}} + \delta\underline{\hat{r}})\ r\ (\underline{\hat{r}} + \delta\underline{\hat{r}})$. The line element on the surface is then

$$ds^2 \equiv (\delta\underline{r})^2 = r^2\ \delta\underline{\hat{r}} \cdot \delta\underline{\hat{r}} + (\delta r)^2 \qquad (6.28)$$

The infinitesimal generator of rotations in the (ij)-plane, $\exp \frac{1}{2} w_{ij} L_{ij}$, acting on the unit vector $\underline{\hat{r}}$ gives

$$\delta\hat{r}_k = \frac{1}{2} \omega_{ij} L_{ij} \hat{r}_k = - \omega_{ki} \hat{r}_i \qquad (6.29)$$

For the distance r , eqn. (6.29) implies

$$\delta r = \frac{1}{2} \omega_{ij} L_{ij} r \qquad (6.30)$$

Using eqn. (6.29) we can write eqn. (6.30, R.H.S.) as

$$\frac{1}{2} \omega_{ij} L_{ij} r = - \hat{r}_i \omega_{j\ell} \hat{r}_\ell L_{ij} r$$

$$= \delta\hat{r}_i \hat{r}_j L_{ij} r \qquad (6.31)$$

Substituting eqn. (6.31) into eqn. (6.30) gives for eqn. (6.28),

$$ds^2 = r^2 \delta_{ij} \delta\hat{r}_i \delta\hat{r}_j + \delta\hat{r}_i (\hat{r}_k L_{ik} r) \delta\hat{r}_j (\hat{r}_\ell L_{j\ell} r) \qquad (6.32)$$

and the metric tensor on the d-manifold is

$$G_{ij} = r^2 \delta_{ij} + \hat{r}_k \hat{r}_\ell L_{ik} r \cdot L_{j\ell} r \qquad (6.33)$$

which closely resembles the form given in eqn. (6.14). Det G_{ij} is defined by eqn. (6.12). Then

$$\text{Det } G_{ij} = r^2 \text{ Det } g_{ij} \tag{6.34}$$

where g_{ij} is the metric on (d-1) - manifold. In terms of eqn. (6.33), eqn. (6.34) can be expressed as

$$\text{Det } g_{ij} = r^{2(d-1)} \left\{1 + \frac{1}{r}(\hat{r}_k \cdot L_{ik}\, r)^2\right\} \tag{6.35}$$

Using the identity

$$(\hat{r}_k\, L_{ik}\, r) = \frac{1}{2}(L_{ij}\, r)^2 \tag{6.36}$$

eqn. (6.35) can be rewritten as

$$\text{Det } g_{ij} = r^{2(d-1)} \left\{1 + \frac{1}{2r^2}\,(L_{ij} r)^2\right\} \tag{6.37}$$

The effective Action is given by eqn. (6.13) where the integration is now over the solid angle $d\Omega$ in (d-1)-dimensions.

In this chapter we have presented a unified treatment of the effective actions, used in earlier chapters, and previously defined on the basis of more physical arguments. The unifying element is the recognition that the Lagrangian is just the Jacobian or metric density in d-dimensional Euclidean space. This interpretation is aided by the parameterization of the Goldstone field chosen to lie normal to the (d-1)-dimensional surface. The explicit dependence of the parameterization on the geometry leads us to refer to the integrals as geometric effective actions. Except for our presentation in Euclidean space , they are essentially the same action terms which

give rise to the "Cosmological constant" in General Relativity Theory [74]. In the formulation of this chapter, the difference between nonlinear realization for internal and geometric symmetries is manifest.

Finally, perhaps it is worth remarking that geometric actions of the type described in this chapter have appeared in other guises. The action for String Models of Dual Resonance Theory incorporate a nonlinear realization of Poincaré group on the string position written as a field variable in d = 2 [76-78]. Similarly, excitations of the spherical MIT Bag Model surface can usually be written as first approximations to actions of the type discussed for the vacuum bubble in § VI. 4 [79]. The complexity of boundary constraints renders this a difficult programme to pursue beyond lowest order [80], i.e. P-wave state.

REFERENCES

1. N.J. Günther, M.Sc. Thesis, Nat. Lib. Aust., Canberra (1977).

2. N.J. Günther and P.G.L. Leach, J. Math. Phys. 18, 572 (1977).

3. N.J. Günther, J. Math. Phys. 20, 1539 (1979).

4. N.J. Günther, J. Math. Phys. 20, in press (1979).

5. N.J. Günther, Group Theoretical Methods in Physics, Lecture Notes in Physics, ed. W. Beiglbock et al. 94, 75 (Springer, New York, 1979).

6. H.V. McIntosh, Group Theory and Its Applications ed. E.M. Loebl, 2, 75 (Academic, New York, 1968).

7. V.I. Karloukovski, "Classical Dynamical Systems with Symmetry of the Kepler Problem" Dubna preprint E2-11291 (1978).

8. D.M. Fradkin, Prog. Theoretical Phys. 37, 798 (1967).

9. See e.g. S. Weinberg in Lectures on Elementary Particles and Quantum Field Theory (M.I.T., Cambridge, 1970).

10. M. Kruskal, J. Math. Phys. 3, 806 (1962); C.L. Hammer and T.A. Weber, J. Math. Phys. 6, 1591 (1965); S. Solimeno, P. Di Porto and B. Crosignani, J. Math. Phys. 10, 1922 (1969); A.M. Perelmov and V.S. Papov, Sov. Phys. JETP 29, 738 (1969); B.K. Berger, Phys. Rev. D12, 368 (1975); L.S. Brown and L.J. Carson, U. of Washington preprint, June, (1979).

11. J.P. Elliot, Proc. Roy. Soc. (Lond.) A245, 128(1958).

12. Wu-Yang Tsai, Phys Rev. A9, 1081 (1974). This ref. contains a perturbative calculation of the Stark effect in hydrogen to third order based on the existence of the Runge-Lenz operator for the unperturbed hydrogen atom. The algebraic technique is an extension of that used by W. Pauli, Z. Physik 36, 336 (1926) to obtain the hydrogen spectrum and utilizes an operator calculus developed by J. Schwinger, Quantum Kinematics and Dynamics (Benjamin N.Y. 1970).

13. H.R. Lewis, Phys. Rev. Lett. 18, 510, 636 (1967); H.R. Lewis, J. Math. Phys. 9, 1976 (1968); H.R. Lewis and W.B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).

14. E.B. Davies, Quantum Theory of Open Systems. (Academic, N.Y. 1976).

15. J. Lànik, Nuc. Phys. B5, 523 (1968).

16. P. Carruthers, Introduction to Unitary Symmetry (Wiley, Sydney, 1966).

17. R. Gilmore, Lie Groups, Lie Algebras and Some of Their Applications (Wiley, N.Y., 1974).

18. C.J. Eliezer and A. Gray, S.I.A.M. 30, 463 (1976).

19. C.J. Eliezer and N.J. Günther, La Trobe University report, (1976).

20. M. Lakshmanan and K. Eswaran, J. Phys. (A) 8, 1658 (1975) and refs. contained therein.

21. P.W. Higgs, J. Phys. (A) 12, 309 (1979).

22. J.W. Macki, J. Math. Phys. 18, 1256 (1977); G.H. Katzin and J. Levine, J. Math. Phys. 18, 1267 (1977); K.B. Wolf, Communicaciones Técnicas 10, 203 (1979).

23. M.L. Whippman, J. Math. Phys. 6, 1534 (1965).

24. R.C. King, J. Math. Phys. 12, 1588 (1971) and J. Phys. (A) 8, 429 (1975).

25. M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960). B.W. Lee Chiral Dynamics (Gordon and Breach, N.Y. 1972).

26. W. Marciano and H. Pagels, Phys. Reps. 36C, 137 (1978).

27. J. Goldstone, Nuovo Cimento 19, 15 (1961); Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961), 124, 246 (1961); J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127, 246 (1962); Y. Nambu, Phys. Rev. Lett. 4, 380 (1960).

28. K. Meetz, J. Math. Phys. 10, 589 (1969); C.J. Isham, Nuovo Cimento, 59A, 356 (1969); 61A, 188, 729 (1969).

29. S. Gasiorowisz and D.A. Geffin, Rev. Mod. Phys. 41, 531 (1969); B. Zumino, in Lectures in Elementary Particles and Quantum Field Theory ed. S. Deser et al., Vol. 2. (M.I.T. Press, Cambridge, 1970).

30. S. Weinberg , Phys. Rev. 166, 1568 (1968); P. Chang and F. Gürsey, Phys. Rev. 164, 1752 (1967); S. Weinberg, Physica 96A, 327 (1979).

31. S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177, 2239 (1969); C.G. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. ibid p.2247.

32. K.J. Barnes, P.H. Dondi and S.C. Sarkar, Proc. Roy. Soc. 330A, 389 (1972).

33. D. Bessis and J.Zinn-Justin, Phys. Rev. D5, 1313 (1972); K.G. Wilson, Phys. Rev. B4, 3184 (1971).

34. E. Brézin and J.Zinn-Justin, Phys. Rev. 14B, 3110 (1976).

35. E. Brézin, J.Zinn-Justin and J.G. Le Guillou, Phys. Rev. 14D, 2615 (1976).

36. K.G. Wilson and J.B. Kogut, Phys. Reps. 12C, 75 (1974).

37. E. Brézin and D.J. Wallace, Phys. Rev. B7, 1967 (1973).

38. D.J. Wallace and R.K.P. Zia, Phys. Rev. B12, 5340 (1975).

39. H. Horner and L. Schäfer, Z. Phys. 29, 251 (1978).

40. R.B. Griffiths in Statistical Mechanics and Quantum Field Theory ed. C. De Witt and R. Stora (Gordon and Breach, N.Y. 1971) and refs. cited therein.

41. S. Coleman, Comm . Math. Phys. 31, 259 (1973); N.D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1960); P.C. Hohenberg, Phys, Rev. 158, 383 (1967).

42. J.C. Taylor, Gauge Theories of Weak Interactions (Cambridge U.P., Lond.) p.117ff (1976).

43. D.J. Amit, Field Theory, the Renormalization Group and Critical Phenomena.(McGraw-Hill, N.Y. (1978).

44. A.M. Polyakov, Phys. Lett. B59, 79(1975).

45. A.A. Migdal, Sov. Phys. JETP 42, 743 (1976).

46. D.R. Nelson and R.A. Pelcovits, Phys. Lett A57, 23 (1976); and Phys. Rev. B16, 2191 (1977).

47. S. Hikami and E. Brézin, J. Phys. (A) 11, 1141 (1978).

48. The main results are contained in : N.J. Günther, D.A. Nicole and D.J. Wallace, J. Phys. (A) to be published.

49. A general review containing many references relevant to the subject of this chapter is to be found in F.F. Abraham, Phys. Reports 53C, 95 (1979). See also, S. Fisk and B. Widom , J. Chem. Phys. 50, 3219 (1969); J.D. Weeks, J. Chem. Phys. 67 3106 (1977). M.S.Jhon, R.C. Desai and J.S. Dahler, J. Chem. Phys. 68, 5615 (1978).

50. D.B. Abraham and M. Issigoni, J. Phys. (A) 12, L125 (1979); D.B. Abraham and M.A. Robert, ibid p.L129; G. Gallavotti, Comm. Math. Phys. 27, 103 (1972); R.L. Dobrushin, Theory Prob. Its Appl. 17, 582 (1972) and ibid 18, 253 (1973).

51. D.J. Wallace and R.K.P. Zia, Phys. Rev. Lett. 43, 808 (1979).

52. R.B. Griffiths, Phase Transitions and Critical Phenomena, Vol. 1. eds. C. Domb and M.S. Green (Academic, Lond. 1972).

53. For a general review of solutions to differential equations of this type see J. Wong, S.I.A.M. Review 17, 339 (1975).

54. S. Coleman, "The Uses of Instantons". Lectures delivered at the International School of Subnuclear Physics, Ettoré Majorana, (1977).

55. R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D10, 4130, 4138 (1974); J.L. Gervais and B. Sakita, Phys. Rev. D11, 2943 (1975); R. Rajaraman, Phys. Reports 21C, 227 (1975); J.L. Gervais and A. Neveu, Phys. Reports 23C, (1976).

56. F.P. Buff, R.A. Lovett and F.H. Stillinger, Phys. Rev. Lett. 15, 621 (1965); J.D.Weeks, J. Chem. Phys. 67, 3106 (1977); H.T. Davies, J. Chem. Phys. 67, 3636 (1977); J. Bricmont, J.L. Lebowitz and C.E. Pfister, Comm. Math. Phys. 69, 267 (1979).

57. The experimental situation with regard to gravitational effects is discussed in; M.R. Moldovev, J.V. Sengers, R.W. Gammon and R.J. Hocken, Rev. Mod. Phys. 51, 79 (1979).

58. M. Aizenman, Phys. Rev. Lett 43, 407 (1979).

59. R. Peierls, Proc. Camb. Phil. Soc., 32, 477 (1936). D. Ruelle, Statistical Mechanics: Rigorous Results, (Benjamin, N.Y., 1969); F.J. Dyson, Comm. Math. Phys. 12, 91(1969); M.E. Fisher and B.U. Felderhof, Ann. Phys. (N.Y.) 58, 176 (1970).

60. J.S. Langer, Ann. Phys. 41, 108 (1967).

61. J. Frenkel, J. Chem. Phys. 7, 200, 538 (1939); J. Frenkel, Kinetic Theory of Liquids (Dover, N.Y. 1955). T.L. Hill, J. Chem. Phys. 23, 617 (1955); F.H. Stillinger, J. Chem. Phys. 38, 1486 (1963); R.B. Griffiths, in ref. [40].

62. A.F. Andreev, Sov. Phys. J.E.T.P. 18, 1415 (1964).

63. The imaginary contribution to the free energy in d-dimensions was first reported by D.J. Wallace in "Some Applications of Instantons in Statistical Mechanics", Symposium on Nonlinear Structure and Dynamics in Condensed Matter, Oxford, June 1978.

64. F.J. Dyson, Phys. Rev. 82, 428 (1951).

65. A.A. Belavin, A.M. Polyakov, A.S. Schwartz, Y.S. Tyupkin, Phys. Letters 59B, 85 (1975); G. 't Hooft, Phys. Rev. Lett. 37, 8(1976) and Phys. Rev. D14, 3432 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); C. Callan, R. Dashen and D. Gross, Phys. Lett. 63B, 334 (1976).

66. M.B. Voloshin, I. Yu Kobzarev and L.B. Okun, Sov. J. Nuc. Phys. 20, 644 (1975); S. Coleman, Phys. Rev. D15, 2929 (1977); C. Callan and S. Coleman. Phys.Rev. D16, 1762 (1977); M.Stone, Phys. Lett. 67B, 186 (1977); H. Katz, Phys. Rev. D17, 1056 (1978).

67. S. Coleman, V. Glaser and A. Martin, Comm. Math. Phys. 58, 211 (1978).

68. I.K.Affleck, Harvard Thesis (1979).

69. I.K. Affleck and F. De Luccia, Harvard preprint A020, (1979).

70. A. Linde, Phys. Lett. 70B, 306 (1977).

71. See e.g. J.D. Talman, Special Functions: A Group Theoretic Approach, (Benjamin, N.Y., 1968).

72. See e.g. refs. [28-32]. Also, J. Wess and B. Zumino, Phys. Lett. 74B, 51 (1978).

73. See ref. [31].

74. S. Weinberg, Gravitation and Cosmology (Wiley, N.Y., 1972); R. Arnowitt, P.Nath and B.Zumino, Phys. Lett. 56B, 81 (1975).

75. D.J. Wallace, private communication.

76. P. Goddard , J. Goldstone, C. Rebbi and C.B. Thorn, Nuc. Phys. B56, 109 (1973).

77. H.B. Nielson and P. Olesen, Nucl. Phys. B61, 45 (1973).

78. J. Scherk, Rev. Mod. Phys. 47, 123 (1975).

79. K. Johnson, Introduction to M.I.T. Bag Mechanics, in Proceedings of Scottish Summer School (1976).

80. C. Rebbi, Phys. Rev. D12,2407 (1975) and D14, 2362 (1976).