ON SPLIT GENERALIZED EQUILIBRIUM AND FIXED POINT PROBLEMS WITH MULTIPLE OUTPUT SETS IN REAL BANACH SPACES.

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Abstract. In this paper, we propose and study a modified inertial Halpern method for finding a common element of the set of solution of split generalized equilibrium problem which is also a fixed point of Bregman relatively nonexpansive mapping in p-uniformly convex Banach spaces which are also uniformly smooth. Our iterative method uses step-size which does not require prior knowledge of the operator norm and we prove a strong convergence result under some mild conditions. We display a numerical example to illustrate the performance of our result. The result presented in this article unifies and extends several existing results in the literature.

1. Introduction

Let C and Q be nonempty, closed and convex subsets of two real Banach spaces E_1 and E_2 with its dual spaces E_1^* and E_2^* respectively. Let $A : E_1 \to E_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) introduced and studied by Censor and Elving [15], consists of finding a point

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{1.1}$$

The SFP is known to have useful applications in many fields such as phase retrieval, medical image reconstruction, radiation therapy treatment planning, signal processing, among others, see [1, 15, 36]. In 2003, Byrne [13] in the framework of Euclidean spaces proposed a CQ algorithm of the form

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \ n \geq 1, \tag{1.2}$$

where $\gamma \in \left(0, \frac{2}{L}\right)$ with L being the spectral radius of the operator A^*A, A^* is the adjoint of A, P_C and P_Q are metric projections of C and Q, respectively. Byrne [13] proved that the sequence generated by (1.2) converges weakly to a solution of the SFP (1.1).

In 2008, Schopfer et al. [42] extended SFP (1.1) to the framework of p-uniformly convex real Banach spaces which are also uniformly smooth. Schopfer [42] proposed the following iterative scheme: for $x_1 \in E_1$, set

$$x_{n+1} = \Pi_C J_q^{E_1^*}\left[J_p^{E_1}(x_n) - \gamma_n A^* J_p^{E_2}(Ax_n - P_Q(Ax_n))\right], \ n \geq 1, \tag{1.3}$$

where Π_C denotes the Bregman projection of E_1 onto C and J_p^E is the duality mapping. Note that Algorithm (1.3) generalizes Algorithm (1.2). Several optimization problems such as Split Variational Inequality Problem (SVIP), Split Variational Inclusion Problem (SVIP), Split Minimization Problem (SMP), Split Equilibrium Problem (SEP), among others have been defined in terms of SFP (1.1), (see for example [1, 2, 3, 4, 22, 23, 29, 44] and the references therein).

Let C be a nonempty, closed and convex subset of a real Banach space E and $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The Equilibrium Problem (EP) is to find $z^* \in C$ such that

$$F(z^*, x) \geq 0, \ \forall \ x \in C. \tag{1.4}$$

We denote by $EP(F)$, the set of solutions of (1.4). Equilibrium problems is known to have a great influence on the development of several branches of pure and applied sciences. It has been shown that EP theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. Since the introduction of EP (1.4) by Blum and Oettli [8], many authors have used different iterative algorithms such as Halpern, viscosity, hybrid, cyclic, shrinking and many others to approximate solutions of EP (1.4) and related optimization problems in both Hilbert and Banach spaces, (see [1, 2, 3, 4, 10, 11, 8, 13, 22, 29] and other references contained therein).

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An important generalization of the EP (1.4) is the Generalized Equilibrium Problem (GEP) (see [24]) defined as: Find $z^* \in C$ such that

$$F(z^*, x) + \phi(z^*, x) - \phi(z^*, z^*) \geq 0, \ \forall \ x \in C; \tag{1.5}$$

where C is a nonempty, closed and convex subset of a real Banach space E, $F : C \times C \to \mathbb{R}$ is a bifunction and $\phi : C \times C \to \mathbb{R}$ is a mapping. We denote by $GEP(F, \phi)$ the set of solutions of (1.5).

Remark 1.1. The GEP (1.5) reduces to EP (1.4) when the mapping ϕ is taken to be the zero mapping i.e. $\phi \equiv 0$.

To solve EP (1.4) and GEP (1.5), we need the following assumptions:
Assumption 1.3:
(i) $F(x, x) = 0, \ \forall \ x \in C$;
(ii) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0, \ \forall \ x, y \in C$;
(iii) For each $x, y, z \in C$; $\limsup_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(iv) For each $x \in C, y \longmapsto F(x, y)$ is convex and lower semicontinuous.

Assumption 1.4:
(i) ϕ is skew-symmetric, i.e., $\phi(x, x) - \phi(x, y) - \phi(y, x) - \phi(y, y) \geq 0, \ \forall \ x, y \in C$;
(ii) ϕ is convex in the second argument;
(iii) ϕ is continuous.

Motivated by SFP (1.1), Kazmi and Rizvi [23] introduced the so called SEP in Hilbert spaces, which is to:

$$\text{find } u^* \in C \text{ such that } F(u^*, x) \geq 0, \ \forall \ x \in C; \tag{1.6}$$

and

$$v^* = Au^* \in Q \text{ solves } G(v^*, y) \geq 0, \ \forall \ y \in Q; \tag{1.7}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions with a bounded linear operator $A : H_1 \to H_2$.

Let $T : C \to C$ be a mapping, a point $x \in C$ is called a fixed point of T, if $Tx = x$. We denote by $Fix(T)$ the set of all fixed points of T. The fixed point problem has been studied due to its various applications to problems arising from differential and integral equations. Also, fixed point iterative schemes have been employed by researchers for obtaining a solution of some nonlinear optimization problems (see [1, 33, 38, 43]).

Recently, Abass et al. [1] introduced an iterative algorithm that does not require any knowledge of the operator norm for finding a common solution of split equilibrium problem and fixed point problem for an infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence result for approximating a common solution of split equilibrium problem and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings which also solves some variational inequality problem in real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively. The split generalized equilibrium problem is to find $u^* \in C$ such that

$$F(u^*, x) + \phi(u^*, x) - \phi(u^*, u^*) \geq 0, \ \forall \ x \in C, \tag{1.8}$$

and such that

$$y^* = Au^* \in Q \text{ solves } G(v^*, y) + \psi(v^*, y) - \psi(v^*, v^*) \geq 0, \ \forall \ y \in Q. \tag{1.9}$$

We denote the solution set of (1.8)-(1.9) by

$$SGEP(F, \phi, G, \psi) := \{u^* \in C : u^* \in GEP(F, \phi) \text{ and } Au^* \in GEP(G, \psi)\}.$$

In 2018, Phuengrattana and Lerkchayaphum [38] introduced the following shrinking projection method for solving the common solution of split generalized equilibrium problem and fixed point problem of multivalued nonexpansive mappings in real Hilbert spaces as follows:

$$\begin{cases} u_n = K_{r_n}^{F,\phi}(I - \gamma A^*(I - K_{r_n}^{G,\psi}))A)x_n, \\ z_n = \alpha_n^{(0)} x_n + \alpha_n^{(1)} y_n^{(1)} + \cdots \alpha_n^{(n)} y_n^{(n)}, \ y_n^{(i)} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : ||z_n - p|| \leq ||x_n - p||\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \in \mathbb{N}, \end{cases} \tag{1.10}$$

where $\{S_i\}$ is a countable family of nonexpansive multivalued mappings, $\{\alpha_n^{(i)}\}$, $\{r_n\} \in (0, \infty)$ and $\gamma \in (0, \frac{1}{L})$, where L is the spectral radius of A^*A, and A^* is the adjoint of A. They proved a strong convergence result for (1.10).

In the framework of a 2-uniformly convex and uniformly smooth real Banach space, Alansari et al. [7] introduced a hybrid algorithm together with an inertial method to approximate a common solution of GEP, variational inequality and fixed point problems. A strong convergence result was established. For more results on GEP, readers should consult ([20, 35] and the references therein).

Very recently, Reich and Tuyen [40] introduced two new self adaptive algorithms for solving split common null point problem with multiple output sets in Hilbert spaces.

One of the best ways to speed up the convergence rate of iterative algorithms is to combine the iterative scheme with the inertial term. This term which is represented by $\theta_n(x_n - x_{n-1})$, is a remarkable tool for improving the performance of algorithms and it is known to have some nice convergence characteristics. Thus, there are growing interests by authors working in this direction (see [3, 4, 7, 35]).

In this article, we consider the following problem:

$$x^* \in \bigcap_{j=1}^{m} Fix(S_j) \cap GEP(F, \phi) : \bigcap_{i=1}^{N} A_i x^* \in (GEP(F_i, \phi_i)). \tag{1.11}$$

Motivated by the results of [1], [40], [38] and other related results in literature, we introduce a self-adaptive modified inertial Halpern iterative method for finding a common element of the set of solution of split generalized equilibrium problem which is also a fixed point problem of Bregman relatively nonexpansive mapping in the framework of p-uniformly convex Banach spaces which are also uniformly smooth. We present some consequences of our result and also display some numerical examples to show the applicability of our result. The result discuss in this article extends and complements many related results in literature.
Our proposed method is endowed with the following features:

(1) We considered approximating the solution of problem (1.11) in a p-uniformly convex Banach spaces which is more general than the results of [7, 20, 35, 38].

(2) Our method uses self-adaptive stepsizes and the implementation of our method does not require the prior knowledge of the norm of the bounded linear operator A, (see [38]).

(3) The sequences generated by our proposed method converges strongly to the solution of the problem (1.11) in p-uniformly convex and uniformly smooth Banach spaces which is desirable to the weak convergence result obtained in [49].

(4) We were able to dispense with the condition $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$ which is often used when employing the inertial method during the course of obtaining our strong convergence result, (see [3, 4]).

(5) Another observation in the algorithm defined in our paper is that it does not require at each step of the iteration process, computation of subsets of C_n, Q_n and D_n (or C_{n+1}) as in the case of [7, 38] and the computation of the projection of the initial point onto their intersection, which leads to a high computational cost of iteration processes. All these also limit the usefulness of such algorithms in real world applications.

Remark 1.2. We will like to emphasize that approximating a common solution of GEPs, SFPs and fixed point problem have some possible applications to mathematical models whose constraints can be expressed as SFPs and GEPs. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, among others (see [21]).

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Let E be a real Banach space, given a function $f : E \to \mathbb{R}$,

(i) f is called Gâteaux differentiable at $x \in E$, if there exists an element of E, denoted by $f'(x)$ or $\nabla f(x)$ such that
$$\lim_{t\to 0}\frac{f(x+ty)-f(x)}{t}=\langle y, f'(x)\rangle, \quad y\in E,$$
where $f'(x)$ or $\nabla f(x)$ is called Gâteaux differential or gradient of f at x. We say f is Gâteaux differentiable on E if f is Gâteaux differentiable at every $x \in E$;

(ii) f is called weakly lower semicontinuous at $x \in E$, if $x_k \rightharpoonup x$ implies $f(x) \leq \liminf_{k\to\infty} f(x_k)$. We say that f is weakly lower semicontinuous on E, if f is weakly lower semicontinuous at every $x \in E$.

Let $K(E) := \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The modulus of convexity is the function $\delta_E : (0,2] \to [0,1]$ defined by
$$\delta_E(\epsilon)=\inf\left\{1-\frac{\|x+y\|}{2} : x,y\in K(E),\ \|x-y\|\geq\epsilon\right\}.$$
The space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. Let $p > 1$, then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0,2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by
$$\rho_E(\tau)=\sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1 : x,y\in K(E)\right\}.$$
The space E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$, then a Banach space E is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p}+\frac{1}{q}=1$, (see [17]).
Let $p > 1$ be a real number, the generalized duality mapping $J_p^E : E \to 2^{E^*}$ is defined by
$$J_p^E(x)=\{\overline{x}\in E^* : \langle x,\overline{x}\rangle=\|x\|^p, \|\overline{x}\|=\|x\|^{p-1}\},$$
where $\langle .,.\rangle$ denotes the duality pairing between elements of E and E^*. In particular, if $p = 2$, then J_2^E is called the normalized duality mapping. If E is p-uniformly convex and uniformly smooth, then E^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^*. Furthermore, if E is uniformly smooth then the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E, (see [18] for more details).

Let $f : E \to (-\infty,+\infty]$ be a proper, lower semicontinuous and convex function, then the Frenchel conjugate of f denoted by $f^* : E^* \to (-\infty,+\infty]$ is defined as
$$f^*(x^*)=\sup\{\langle x^*,x\rangle-f(x) : x\in E,\ x^*\in E^*\}.$$
Let the domain of f be denoted as $domf = \{x \in E : f(x) < +\infty\}$, hence for any $x \in int(domf)$ and $y \in E$, we define the right-hand derivative of f at x in the direction y by
$$f^0(x,y)=\lim_{t\to 0^+}\frac{f(x+ty)-f(x)}{t}.$$
Definition 2.1. [12] Let $f : E \to (-\infty,+\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : E\times E \to [0,+\infty)$ defined by
$$\Delta_f(x,y) := f(y)-f(x)-\langle\nabla f(x), y-x\rangle \tag{2.1}$$
is called the Bregman distance with respect of f, where $\langle\nabla f(x), y\rangle = f^0(x,y)$.

It is well-known that Bregman distance Δ_f does not satisfy the properties of a metric because Δ_f fail to satisfy the symmetric and triangular inequality property. Moreover, it is well known that the duality mapping J_E^p is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$, see [16]. From (2.1), one can show that the equality called three-point identity is satisfied:
$$\Delta_p(x,y)+\Delta_p(y,z)-\Delta_p(x,z)=\langle J_E^p(z)-J_E^p(y), x-y\rangle,\ \forall\ x,y,z\in E.$$

In addition, if $f(x) = \frac{1}{p}||x||^p$, where $\frac{1}{p} + \frac{1}{q} = 1$, then we obtain

$$
\begin{aligned}
\Delta_f(x,y) = \Delta_p(x,y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y - x, J_E^p(x)\rangle \\
&= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y, J_E^p(x)\rangle + \langle x, J_E^p(x)\rangle \\
&= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y, J_E^p(x)\rangle + \|x\|^p \\
&= \frac{1}{p}\|y\|^p + \frac{1}{q}\|x\|^p - \langle y, J_E^p(x)\rangle. \qquad (2.2)
\end{aligned}
$$

Let $T : C \to C$ be a nonlinear mapping,

(i) a point $p \in C$ is called an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim\limits_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{Fix}(T)$ the set of asymptotic fixed points of T;

(ii) T is said to be Bregman quasi-nonexpansive, if

$$Fix(T) \neq \emptyset \text{ and } \Delta_p(u, Tx) \leq \Delta_p(u, x),\ \forall\ x \in C,\ u \in Fix(T);$$

(iii) T is said to be Bregman relatively nonexpansive, if

$$\hat{Fix}(T) = Fix(T) \neq \emptyset \text{ and } \Delta_p(u, Tx) \leq \Delta_p(u, x),\ \forall\ x \in C,\ u \in Fix(T).$$

(iv) T is said to be Bregman firmly nonexpansive mapping (BFNE) if

$$\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty\rangle \leq \langle J_p^E(x) - J_p^E(y), Tx - Ty\rangle,\ \forall\ x, y \in C,$$

(v) T is said to be Bregman strongly nonexpansive mapping (BSNE) with $\hat{Fix}(T) \neq \emptyset$ if

$$\Delta_p(y, Tx) \leq \Delta_p(y, x),\ \forall\ y \in \hat{Fix}(T)$$

and for any bounded sequence $\{x_n\}_{n\geq 1} \subset C$,

$$\lim_{n\to\infty}(\Delta_p(y, x_n) - \Delta_p(y, Tx_n)) = 0$$

implies

$$\lim_{n\to\infty} \Delta_p(Tx_n, x_n) = 0.$$

Let C be a nonempty, closed and convex subset of E. The metric projection

$$P_C x := arg \min_{y\in E} ||x - y||,\ x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0,\ \forall\ z \in C. \qquad (2.3)$$

Also, the Bregman projection from E onto C denoted by Π_C also satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y\in C} \Delta_p(x, y),\ \forall\ x \in E. \qquad (2.4)$$

Let C is a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and $x \in E$. Then the following assertions hold: see [17],
$z = \Pi_C x$ if and only if

$$\langle J_E^p(x) - J_E^p(z),\ y - z\rangle \leq 0,\ \forall\ y \in C; \qquad (2.5)$$

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y),\ \forall\ y \in C. \qquad (2.6)$$

We now give some results that will help us in the proof of our main result.

Lemma 2.2. [16] *Let E be a Banach space and $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle J_E^q(x), y\rangle + C_q\|y\|^q.$$

Lemma 2.3. [42] *Let E be a p-uniformly convex Banach space, the metric and Bregman distance have the following relation for all $x, y \in E$*

$$\tau\|x-y\|^p \leq \Delta_p(x,y) \leq \langle x-y, J_E^p(x) - J_E^p(y)\rangle, \tag{2.7}$$

where $\tau > 0$ is a fixed number.

Lemma 2.4. [46] *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that $x, y \in E$ and $\alpha \in (0,1)$, we have*

$$||\alpha x + (1-\alpha)y||^q \leq \alpha||x||^q + (1-\alpha)||y||^q - [\alpha^q(1-\alpha) + \alpha(1-\alpha)^q]||x-y||^q.$$

Lemma 2.5. [43] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p, \ \forall\, x \in E, x^* \in E^*.$$

Then the following assertions hold:
(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(J_q^{E^*}(x^*), x) = V_p(x^*, x), \ \forall\, x \in E, \ x^* \in E^*$.
(iii) $V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle \leq V_p(x^* + y^*, x), \forall\, x \in E, \ x^*, y^* \in E^*$.
Also for all $x^* \in E$, we have

$$\Delta_p\bigg(x^*, J_q^{E^*}\Big(\sum_{i=1}^N t_i J_p^E(x_i)\Big)\bigg) \leq \sum_{i=1}^N t_i \Delta_p(x^*, x_i), \tag{2.8}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum\limits_{i=1}^N t_1 = 1$.

Lemma 2.6. [17] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then $\lim\limits_{n\to\infty} \Delta_p(x_n, y_n) = 0$ implies $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.*

Proposition 2.7. [42] *Let E be a p-uniformly convex Banach space. Then for $x, y \in E$, there exists a fixed contant τ_p such that*

$$\tau_p\|x-y\|^p \leq \Delta_p(x,y) \leq \langle x-y, J_E^p x - J_E^p y\rangle.$$

Lemma 2.8. [24] *Let C be a closed and convex subset of E. If $F : C \times C \to \mathbb{R}$ is a bifunction satisfying Assumption* **1.3** *and $\phi : C \times C \to \mathbb{R}$ satisfying Assumption* **1.4**, *then $dom(K_r^{F,\phi}) = E$.*

Lemma 2.9. [24] *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption* **1.3** *and $\phi : C \times C \to \mathbb{R}$ satisfying Assumption* **1.4**. *Let $K_r^{G,\phi} : E \to 2^C$ be the resolvent associated with F and ϕ defined as follows:*

$$\begin{aligned} K_r^{F,\phi}(x) = \{z \in C : F(z,y) &+ \langle J_E^p(z) - J_E^p(x), y - z\rangle \\ &+ \phi(z,y) - \phi(z,z) \geq 0, \ \forall\, y \in C\}, \ \forall\, x \in E. \end{aligned} \tag{2.9}$$

Then, the following holds:
(a) $K_r^{F,\phi}$ is single-valued;
(b) $K_r^{F,\phi}$ is Bregman firmly nonexpansive type mapping, i.e., for all $x, y \in E$,

$$\langle J_E^p(K_r^{F,\phi}x) - J_E^p(K_r^{F,\phi}y), K_r^{F,\phi}x - K_r^{F,\phi}y\rangle \leq \langle J_E^p(x) - J_E^p(y), K_r^{F,\phi}x - K_r^{F,\phi}y\rangle;$$

(c) $Fix(K_r^{F,\phi}) = Sol(GEP)$ is closed and convex;
(d) $\Delta_p(q, K_r^{F,\phi}x) + \Delta_p(K_r^{F,\phi}x, x) \leq \Delta_p(q, x), \ \forall\, q \in Fix(K_r^{F,\phi})$ and $x \in E$;
(e) $K_r^{F,\phi}$ is Bregman quasi-nonexpansive.

Lemma 2.10. [31] *Let E be a Banach space, $r > 0$ be a constant and $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then*

$$f\bigg(\sum_{k=0}^n \alpha_k x_k\bigg) \leq \sum_{k=0}^n \alpha_k f(x_k) - \alpha_i\alpha_j\rho_r(||x_i - x_j||),$$

for all $i, j \in \{0, 1, 2, \cdots, n\}, x_k \in rB, \ \alpha_k \in (0,1)$ and $k = 0, 1, 2 \cdots, n$ with $\sum_{k=0}^k \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.11. [39] *Let E be a uniformly convex and uniformly smooth Banach space. if $x_0 \in E$ and the sequence $\{\Delta_p(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.12. [47] *Let $\{a_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n n a_{n-1} + t_n s_n + \delta_n, \ \forall n \geq 0,$$

where $\sum\limits_{n=n_0}^{\infty} t_n = +\infty$, $\sum\limits_{n=n_0}^{\infty} \delta_n < +\infty$, for each $n \geq n_0$ (where n_0 is a positive integer) and $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup\limits_{n\to\infty} s_n \leq 0$. Then, the sequence $\{a_n\}$ converges strongly to zero.

Lemma 2.13. [28] *Let Γ_n be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_k}\}_{k\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_k} \leq \Gamma_{n_j+1}$ for all $j \geq 0$. Also, consider a sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) := \max\{k \leq n \mid \Gamma_{n_k} \leq \Gamma_{n_k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence satisfying $\lim_{n\to\infty} \tau(n) = \infty$. If it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$ then we have

$$\Gamma_\tau(n) \leq \Gamma_{\tau(n)+1}.$$

3. Main Result

First we give the following remark.

Remark 3.1. We adopt the notations $C = C_0,\ E = E_0,\ \phi = \phi_0,\ F = F_0$, and $A_0 = I$, where I is the identity mapping.

Now, we present our theorem and its proof.

Theorem 3.2. *Let $E, E_i,\ i = 1, 2, \cdots N$ be real Banach spaces and C, C_i be nonempty, closed and convex subsets of E and E_i respectively. For $i = 1, 2, \cdots N$, let $F : C \times C \to \mathbb{R},\ F_i : C_i \times C_i \to \mathbb{R}$ be bifunctions and $\phi : C \times C \to \mathbb{R},\ \phi_i : C_i \times C_i \to \mathbb{R}$ be nonlinear mappings. Suppose $A_i : E \to E_i,\ i = 1, 2, \cdots N$ be linear bounded operators and S_j be finite family of Bregman relatively nonexpansive mappings such that $\Gamma := \big\{x^* \in \bigcap\limits_{j=1}^{m} Fix(S_j) \cap GEP(F, \phi) \cap \bigcap\limits_{i=1}^{N} A_i^{-1}(GEP(F_i, \phi_i))\big\} \neq \emptyset$. Assume that $\{\theta_n\} \subset [0, \frac{1}{2}], \{\alpha_n\}, \{\beta_n\}, \{\mu_{i,n}\},\ \{\delta_{n,j}\}_{j=0}^{m}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\sum\limits_{i=0}^{N} \mu_{i,n} = 1,\ \alpha_n + \beta_n + \gamma_n = 1,\ \alpha_n \leq b < 1,\ (1-\alpha_n)a < \gamma_n,\ a \in (0, \frac{1}{2})$. Let $u, x_0, x_1 \in E$ and $\{x_n\}$ be the sequence generated as follows:*

$$\begin{cases} w_n = J^q_{E^*}\big[J^p_E(x_n) + \theta_n\big(J^p_E(x_{n-1}) - J^p_E(x_n)\big)\big], \\ y_n = J^q_{E^*}\Big[\sum\limits_{i=0}^{N} \mu_{i,n}\big(J^p_E(w_n) - \lambda_{i,n} A_i^* J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})A_i(w_n))\Big], \\ u_n = J^q_{E^*}\big[\delta_{n,0} J^p_E(y_n) + \sum\limits_{j=1}^{m} \delta_{n,j} J^p_E S_j(y_n)\big], \\ x_{n+1} = J^q_{E^*}\big(\alpha_n J^p_E(u) + \beta_n J^p_E(x_n) + \gamma_n J^p_E(y_n)\big). \end{cases} \tag{3.1}$$

Now, suppose for $\varepsilon > 0$, the step size $\lambda_{i,n}$ is chosen in such a way that

$$\lambda_{i,n} \in \Bigg(\varepsilon,\ \Big(\frac{q||A_i(w_n) - (K^{F_i,\phi_i}_{r_{i,n}})A_i(w_n)||^p}{C_q||A_i^* J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})A_i(w_n))||^q_*} - \varepsilon\Big)^{\frac{1}{q-1}}\Bigg);\ n \in \Omega, \tag{3.2}$$

where the index set $\Omega := \{n \in \mathbb{N} : (A_i w_n - (K^{F_i,\phi_i}_{r_{i,n}})A_i w_n) \neq 0\}$, otherwise $\lambda_{i,n} = \lambda$, λ is any nonnegative real number. The sequence $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}$ and $\{\theta_n\}$ satisfies the following conditions:

(1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(2) $0 < c < \theta_n < \gamma_n \leq \frac{1}{2}, \forall\ n \geq 1$,

(3) $0 < \liminf\limits_{n\to\infty} \gamma_n, \beta_n \leq \limsup\limits_{n\to\infty} \gamma_n, \beta_n < 1$,

(4) $\min_{i=0,1,\cdots N}\{\inf_n\{r_{i,n}\}\} = r > 0$,

(5) $\sum_{j=1}^{m} \delta_{n,j} = 1$ *and* $\liminf_{n\to\infty} \delta_{n,0}\delta_{n,j} > 0$. *Then,* $\{x_n\}$ *generated by* (3.1) *converges strongly to* $x^* \in \Gamma$*, where* $x^* = \Pi_\Gamma u$.

Proof. Let $x^* \in \Gamma$, then we have from (3.1) and (2.2) that

$$\begin{aligned}
\Delta_p(x^*, y_n) &= \Delta_p\Big(x^*, J^q_{E^*}\Big[\sum_{i=0}^{N}\mu_{i,n}(J^p_E(w_n) - \lambda_{i,n}A_i^*J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})))\Big]A_iw_n\Big)\\
&= \frac{||x^*||^p}{p} - \langle x^*, \sum_{i=0}^{N}\mu_{i,n}(J^p_E(w_n) - \lambda_{i,n}A_i^*J^p_{E_i}(I^{E_i} - (K^{F_i,\phi}_{r_{i,n}}))A_iw_n)\rangle\\
&+ \frac{||\sum_{i=0}^{N}\mu_{i,n}(J^p_E(w_n) - \lambda_{i,n}A_i^*J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})))A_iw_n)||^q_*}{q}\\
&\le \frac{||x^*||^p}{p} - \langle x^*, J^p_Ew_n\rangle + \sum_{i=0}^{N}\mu_{i,n}\lambda_{i,n}\langle A_ix^*, J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})))A_iw_n)\rangle\\
&+ \frac{\sum_{i=0}^{N}\mu_{i,n}||(J^p_E(w_n) - \lambda_{i,n}A_i^*J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})))A_iw_n)||^q_*}{q}\\
&\le \frac{||x^*||^p}{p} - \langle x^*, J^p_Ew_n\rangle + \frac{||w_n||^q_*}{q} + \sum_{i=0}^{N}\mu_{i,n}\lambda_{i,n}\langle A_ix^*, J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})))A_iw_n)\rangle\\
&- \sum_{i=0}^{N}\mu_{i,n}\lambda_{i,n}\langle A_iw_n, J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})))A_iw_n)\rangle\\
&+ \sum_{i=0}^{N}\mu_{i,n}\frac{C_q}{q}\lambda^q_{i,n}||A_i^*J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*\\
&= \Delta_p(x^*, w_n) - \sum_{i=0}^{N}\mu_{i,n}\lambda_{i,n}\langle A_ix^* - A_iw_n, J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})))A_iw_n\big)\\
&+ \sum_{i=0}^{N}\mu_{i,n}\frac{C_q}{q}\lambda^q_{i,n}||A_i^*J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*. \qquad (3.3)
\end{aligned}$$

Observe from (2.3) that for any $x^* \in \Delta$, where Δ is the solution set of GEP, we have

$$\begin{aligned}
\langle A_ix^* - A_iw_n, J^p_{E_i}\big(I^{E_i} - (K^{F_i,\phi}_{r_{i,n}})A_iw_n\rangle &= \langle A_ix^* - (K^{F_i,\phi}_{r_{i,n}})A_iw_n, J^p_{E_i}\big(I^{E_i} - (K^{F_i,\phi}_{r_{i,n}})A_iw_n\rangle\rangle\\
&+ \langle(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n - A_iw_n, J^p_{E_i}\big(I^{E_i} - (K^{F_i,\phi}_{r_{i,n}})A_iw_n\rangle\\
&= ||(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||^p\\
&+ \langle A_ix^* - (K^{F_i,\phi}_{r_{i,n}})A_iw_n, J^p_{E_i}\big(I^{E_i} - (K^{F_i,\phi}_{r_{i,n}})A_iw_n\rangle\\
&\le -||(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||^p. \qquad (3.4)
\end{aligned}$$

On substituting (3.4) into (3.3) and applying (3.2), we obtain

$$\begin{aligned}
\Delta_p(x^*, y_n) &\le \Delta_p(x^*, w_n) - \sum_{i=0}^{N}\mu_{i,n}\lambda_{i,n}\big(||(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||^p\\
&- \frac{C_q}{q}\lambda^{q-1}_{i,n}||A_i^*J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*\big) \qquad (3.5)\\
&\le \Delta_p(x^*, w_n). \qquad (3.6)
\end{aligned}$$

From (2.8), (3.1) and (3.6), we get

$$\begin{aligned}
\Delta_p(x^*,x_{n+1}) &= \Delta_p(x^*, J^q_{E^*}(\alpha_n J^p_E(u)+\beta_n J^p_E(x_n)+\gamma_n J^p_E(u_n))\\
&\le \alpha_n\Delta_p(x^*,u)+\beta_n\Delta_p(x^*,x_n)+\gamma_n\Delta_p(x^*,u_n)\\
&\le \alpha_n\Delta_p(x^*,u)+\beta_n\Delta_p(x^*,x_n)+\gamma_n\Big(\delta_{n,0}\Delta_p(x^*,y_n)+\sum_{j=1}^m\delta_{n,j}\Delta_p(x^*,S_jy_n)\Big)\\
&\le \alpha_n\Delta_p(x^*,u)+\beta_n\Delta_p(x^*,x_n)+\gamma_n\Big(\delta_{n,0}\Delta_p(x^*,y_n)+\sum_{j=1}^m\delta_{n,j}\Delta_p(x^*,y_n)\Big)\\
&\le \alpha_n\Delta_p(x^*,u)+\beta_n\Delta_p(x^*,x_n)+\gamma_n\Delta_p(x^*,y_n)\\
&\le \alpha_n\Delta_p(x^*,u)+\beta_n\Delta_p(x^*,x_n)+\gamma_n\Delta_p(x^*w_n)\\
&\le \alpha_n\Delta_p(x^*,u)+\beta_n\Delta_p(x^*,x_n)+\gamma_n((1-\theta_n)\Delta_p(x^*,x_n)+\theta_n\Delta_p(x^*,x_{n-1}))\\
&= \alpha_n\Delta_p(x^*,u)+(\beta_n+\gamma_n)\Delta_p(x^*,x_n)-\gamma_n\theta_n\Delta_p(x^*,x_n)+\gamma_n\theta_n\Delta_p(x^*,x_{n-1})\\
&= \alpha_n\Delta_p(x^*,u)+(1-\alpha_n)\Delta_p(x^*,x_n)-\gamma_n\theta_n\Delta_p(x^*,x_n)+\gamma_n\theta_n\Delta_p(x^*,x_{n-1})\\
&= \alpha_n\Delta_p(x^*,u)+(1-\alpha_n-\gamma_n\theta_n)\Delta_p(x^*,x_n)+\gamma_n\theta_n\Delta_p(x^*,x_{n-1})\\
&\le \max\{\Delta_p(x^*,u),\Delta_p(x^*,x_n),\Delta_p(x^*,x_{n-1})\},\ \forall\, n\ge 1
\end{aligned}$$

By induction,

$$\Delta_p(x^*,x_n)\le \max\{\Delta_p(x^*,u),\Delta_p(x^*,x_1),\Delta_p(x^*,x_0)\}.$$

Hence, $\{\Delta_p(x^*,x_n)\}$ is bounded. Consequently, $\{\Delta_p(x^*,y_n)\}$, $\{\Delta_p(x^*,u_n)\}$ and $\{\Delta_p(x^*,w_n)\}$ are bounded. Thus, we conclude from Lemma 2.11 that $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{w_n\}$ are bounded. Next, we show that for each $j\in\{1,2,\cdots,m\}$, $\lim\limits_{n\to\infty}||J^p_E(y_n)-J^p_ES_j(y_n)||=0$. Let $t_j:=\sup_{n\in\mathbb{N}}\{||J^p_E(y_n),||J^p_ES_j(y_n)||\}$ and let $\rho_{t_j}:[0,\infty)\to[0,\infty]$ be the gauge of uniform convexity of f_q. Then by Lemma 2.10, we obtain

$$\begin{aligned}
\Delta_p(x^*,u_n) &= \Delta_p\Big(x^*, J^q_{E^*}\big(\delta_{n,0}J^p_E(y_n)+\sum_{j=1}^m\delta_{n,j}J^p_ES_j(y_n)\big)\Big)\\
&= V_p\big(x^*,\delta_{n,0}J^p_E(y_n)+\sum_{j=1}^m\delta_{n,j}J^p_ES_j(y_n)\big)\\
&\le \frac{1}{p}||x^*||^p-\delta_{n,0}\langle x^*,J^p_E(y_n)\rangle-\sum_{j=1}^m\delta_{n,j}\langle x^*,J^p_ES_jy_n\rangle+\frac{\delta_{n,0}}{q}||J^p_E(y_n)||^q\\
&+\frac{1}{q}\sum_{j=1}^m\delta_{n,j}||J^p_ES_j(y_n)||^q-\delta_{n,0}\delta_{n,j}\rho^*_{t_j}||J^p_Ey_n-J^p_ES_jy_n||\\
&= \delta_{n,0}\frac{1}{p}\|x^*\|^p+\sum_{j=1}^m\delta_{n,j}\frac{1}{p}\|x^*\|^p+\frac{\delta_{n,0}}{q}\|y_n\|^p+\frac{1}{q}\sum_{j=1}^m\delta_{n,j}\|S_jy_n\|^p\\
&-\delta_{n,0}\langle x^*,J^p_Ey_n\rangle-\sum_{j=1}^m\delta_{n,j}\langle x^*,J^p_ES_jy_n\rangle-\delta_{n,0}\delta_{n,j}\rho^*_{t_j}\|J^p_Ey_n-J^p_ES_jy_n\|\\
&= \delta_{n,0}\Delta_p(x^*,y_n)+\sum_{j=1}^m\delta_{n,j}\Delta_p(x^*,S_jy_n)-\delta_{n,0}\delta_{n,j}\rho^*_{t_j}\|J^p_Ey_n-J^p_ES_jy_n\|\\
&\le \Delta_p(x^*,y_n)-\delta_{n,0}\delta_{n,j}\rho^*_{t_j}\|J^p_Ey_n-J^p_ES_jy_n\|. \qquad (3.7)
\end{aligned}$$

Now, we prove that the sequence $\{x_n\}$ converges strongly to an element in the solution set. Using (3.1), (3.7) and Lemma 2.5 (iii), we get

$$\begin{aligned}\Delta_p(x^*,x_{n+1}) &= \Delta_p(x^*, J^q_{E^*}(\alpha_n J^p_E(u)+\beta_n J^p_E(x_n)+\gamma_n J^p_E(u_n))\\ &= V_p(x^*,\alpha_n J^p_E(u)+\beta_n J^p_E(x_n)+\gamma_n J^p_E(u_n))\\ &\le V_p(x^*,\alpha_n J^p_E(u)+\beta_n J^p_E(x_n)+\gamma_n J^p_E(u_n))\\ &-\alpha_n(J^p_E(u)-J^p_E(x^*))-\langle-\alpha_n(J^p_E(u)-J^p_E(x^*)),J^q_{E^*}(\alpha_n J^p_E(x^*)\\ &+\beta_n J^p_E(x_n)+\gamma_n J^p_E(u_n))-x^*\rangle\\ &= V_p(x^*,\alpha_n J^p_E(u)+\beta_n J^p_E(x_n)+\gamma_n J^p_E(u_n)\\ &+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle\\ &= \Delta_p(x^*,J^q_{E^*}(\alpha_n J^p_E(x^*)+\beta_n J^p_E(x_n)+\gamma_n J^p_E(u_n))\\ &+\alpha_n\langle J^p_{E_1}(u)-J^p_E(x^*),x_{n+1}-x^*\rangle\\ &= \alpha_n\Delta_p(x^*,x^*)+\beta_n\Delta_p(x^*,x_n)+\gamma_n\Delta_p(x^*,u_n)\\ &+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle \qquad (3.8)\\ &\le \alpha_n\Delta_p(x^*,x^*)+\beta_n\Delta_p(x^*,x_n)+\gamma_n\Delta_p(x^*,w_n)\\ &+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle\\ &\le \alpha_n\Delta_p(x^*,x^*)+\beta_n\Delta_p(x^*,x_n)+\gamma_n[(1-\theta_n)\Delta_p(x^*,x_n)+\theta_n\Delta_p(x^*,x_{n-1})]\\ &+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle\\ &= (1-\alpha_n-\gamma_n\theta_n)\Delta_p(x^*,x_n)+\gamma_n\theta_n\Delta_p(x^*,x_{n-1})+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle. \qquad (3.9)\end{aligned}$$

Case 1: Assume that $\{\Delta_p(x^*,x_n)\}$ is monotone decreasing, that is $\Delta_p(x^*,x_{n+1})\le\Delta_p(x^*,x_n)$ since $\Delta_p(x^*,x_n)\le M$, for all $n\ge1$, where $M:=\max\{\Delta_p(x^*,u),\Delta_p(x^*,x_1),\Delta_p(x^*,x_0)\}$, which means $\{\Delta_p(x^*,x_n)\}$ is bounded. Then $\{\Delta_p(x^*,x_n)\}$ is convergent. Thus,

$$\lim_{n\to\infty}\big(\Delta_p(x^*,x_n)-\Delta_p(x^*,x_{n+1})\big)=\lim_{n\to\infty}\big(\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)\big)=0. \qquad (3.10)$$

From (3.5), (3.6), (3.7) and (3.8), we obtain that

$$\begin{aligned}\Delta_p(x^*,x_{n+1}) &\le \beta_n\Delta_p(x^*,x_n)+\gamma_n\Delta_p(x^*,u_n)+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle\\ &\le \beta_n\Delta_p(x^*,x_n)-\gamma_n\sum_{i=0}^N\mu_{i,n}\lambda_{i,n}\big(||(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||^p\\ &-\frac{C_q}{q}\lambda^{q-1}_{i,n}||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*\big)+\gamma_n\Delta_p(x^*,w_n)\\ &+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle-\gamma_n\delta_{n,0}\delta_{n,j}\rho^*_{t_j}\|J^p_Ey_n-J^p_ES_jy_n\|\\ &\le (1-\alpha_n)\Delta_p(x^*,x_n)+\gamma_n\theta_n(\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n))\\ &-\gamma_n\sum_{i=0}^N\mu_{i,n}\lambda_{i,n}\big(||(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||^p\\ &-\frac{C_q}{q}\lambda^{q-1}_{i,n}||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*\big)+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle\\ &-\gamma_n\delta_{n,0}\delta_{n,j}\rho^*_{t_j}\|J^p_Ey_n-J^p_ES_jy_n\|. \qquad (3.11)\end{aligned}$$

From (3.10) and (3.11), we get

$$\begin{aligned}&\gamma_n\sum_{i=0}^N\mu_{i,n}\lambda_{i,n}\big(||(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||^p\\ &-\frac{C_q}{q}\lambda^{q-1}_{i,n}||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*\big)-\gamma_n\delta_{n,0}\delta_{n,j}\rho^*_{t_j}\|J^p_Ey_n-J^p_ES_jy_n\|\\ &\le(1-\alpha_n)\Delta_p(x^*,x_n)-\Delta_p(x^*,x_{n+1})+\gamma_n\theta_n(\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n))\\ &+\alpha_n\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle. \qquad (3.12)\end{aligned}$$

By passing limit on (3.12), we obtain that

$$\lim_{n\to\infty}\big(||(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||^p-\frac{C_q}{q}\lambda^{q-1}_{i,n}||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*\big)=0. \tag{3.13}$$

Observe that by the choice of our step size, it holds that

$$\lambda^{q-1}_{i,n}<\frac{q||A_iw_n-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n||^p}{C_q||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*}-\varepsilon. \tag{3.14}$$

From (3.14), we get

$$\begin{aligned}\frac{\varepsilon C_q}{q}||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||^q_*&<\big(||A_iw_n-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n||^p\\&\quad-\frac{C_q}{q}\lambda^{q-1}_{i,n}||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n)||^q_*\big).\end{aligned} \tag{3.15}$$

By passing the limit as $n\to\infty$ on (3.13) and (3.15), we obtain that

$$\lim_{n\to\infty}||A_i^*J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||_*=0. \tag{3.16}$$

Similarly, from (3.13), (3.14) and condition (i) of (3.1), we have that

$$\lim_{n\to\infty}||A_iw_n-(K^{F_i,\phi_i}_{r_{i,n}}))A_iw_n||=0. \tag{3.17}$$

Also, from (3.12) and condition (5) of (3.2), we get

$$\lim_{n\to\infty}\rho^*_{t_j}\|J^p_Ey_n-J^p_ES_jy_n\|=0. \tag{3.18}$$

Then we prove that $\lim\limits_{n\to\infty}\|J^p_Ey_n-J^p_ES_jy_n\|=0$. In fact, if not, Suppose that $\epsilon_0>0$ and subsequence $\{n_k\}$ of $\{n\}$ such that $\|J^p_Ey_n-J^p_ES_jy_n\|\geq\epsilon_0$. Since $\rho^*_{t_j}$ is nondecreasing, we have $\rho^*_{t_j}(\epsilon_0)\leq\rho^*_{t_j}(\|J^p_Ey_n-J^p_ES_jy_n\|)$ for all $k\in\mathbb{N}$. Letting $k\to\infty$, we have $\rho^*_{t_j}(\epsilon_0)\leq 0$. This contradicts to the uniform convexity of f_q on bounded sets, hence

$$\lim_{n\to\infty}\|J^p_Ey_n-J^p_ES_jy_n\|=0. \tag{3.19}$$

Since $J^q_{E^*}$ is uniformly continuous on bounded subset of E^*, we obtain

$$\lim_{n\to\infty}\|y_n-S_jy_n\|=0. \tag{3.20}$$

Also, we have

$$\|J^p_E(u_n)-J^p_E(y_n)\|\leq\sum_{j=1}^m\delta_{n,j}\|J^p_E(S_jy_n)-J^p_E(y_n)\|\to 0. \tag{3.21}$$

Since $J^q_{E^*}$ is uniformly continuous on bounded subset of E^*, we obtain

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.22}$$

Let $v_n := J^q_{E^*}\big(\frac{\beta_n}{1-\alpha_n}J^p_E x_n + \frac{\gamma_n}{1-\alpha_n}J^p_E(u_n)\big)$, we obtain from (2.2) and Lemma 2.4 that

$$\begin{aligned}\Delta_p(x^*, v_n) &= \frac{1}{p}||x^*||^p - \Big\langle \frac{\beta_n}{1-\alpha_n}J^p_E x_n + \frac{\gamma_n}{1-\alpha_n}J^p_E(u_n), x^*\Big\rangle\\ &+ \frac{1}{q}\Big|\Big|\frac{\beta_n}{1-\alpha_n}J^p_E x_n + \frac{\gamma_n}{1-\alpha_n}J^p_E(u_n)\Big|\Big|\\ &\le \frac{1}{p}||x^*||^p - \Big\langle \frac{\beta_n}{1-\alpha_n}J^p_E x_n + \frac{\gamma_n}{1-\alpha_n}J^p_E(u_n), x^*\Big\rangle\\ &+ \frac{1}{q}\Big[\Big|\Big|\frac{\beta_n}{1-\alpha_n}J^p_E x_n + \frac{\gamma_n}{1-\alpha_n}J^p_E(u_n)\Big|\Big|\\ &- \Big((\frac{\beta_n}{1-\alpha_n})^q(1-\frac{\beta_n}{1-\alpha_n}) + \frac{\gamma_n}{1-\alpha_n}(1-\frac{\gamma_n}{1-\alpha_n})\Big)\Big]||J^p_E(x_n) - J^p_E(u_n)||^q\\ &= \frac{\beta_n}{1-\alpha_n}\Big(\frac{1}{p}||x^*||^p - \langle J^p_E(x_n), x^*\rangle + \frac{1}{q}||x_n||^p\Big) + \frac{\gamma_n}{1-\alpha_n}\Big(\frac{1}{p}||x^*||^p - \langle J^p_E(u_n), x^*\rangle\\ &+ \frac{1}{q}||(u_n)||^p\Big) - \frac{2\gamma_n\beta_n^q}{(1-\alpha_n)^{q+1}}\Big|\Big|J^p_E(x_n) - J^p_E(u_n)\Big|\Big|^q\\ &= \frac{\beta_n}{1-\alpha_n}\Delta_p(x^*, x_n) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(x^*, u_n)\\ &- \frac{2\gamma_n\beta_n^q}{(1-\alpha_n)^{q+1}}\Big|\Big|J^p_E(x_n) - J^p_E(u_n)\Big|\Big|^q\\ &\le \frac{\beta_n}{1-\alpha_n}\Delta_p(x^*, x_n) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(x^*, x_n) + \frac{\gamma_n\theta_n}{1-\alpha_n}\big(\Delta_p(x^*, x_{n-1}) - \Delta_p(x^*, x_n)\big)\\ &- \frac{2\gamma_n\beta_n^q}{(1-\alpha_n)^{q+1}}\Big|\Big|J^p_E(x_n) - J^p_E(u_n)\Big|\Big|^q\\ &= \Delta_p(x^*, x_n) + \frac{\gamma_n\theta_n}{1-\alpha_n}\big(\Delta_p(x^*, x_{n-1}) - \Delta_p(x^*, x_n)\big)\\ &- \frac{2\gamma_n\beta_n^q}{(1-\alpha_n)^{q+1}}\Big|\Big|J^p_E(x_n) - J^p_E(u_n)\Big|\Big|^q. \end{aligned} \tag{3.23}$$

From (3.1) and (3.23), we obtain

$$\begin{aligned}\Delta_p(x^*, x_{n+1}) &\le \alpha_n\Delta_p(x^*, u) + (1-\alpha_n)\Delta_p(x^*, v_n)\\ &\le \alpha_n\Delta_p(x^*, u) + (1-\alpha_n)\Delta_p(x^*, x_n) + \gamma_n\theta_n\big(\Delta_p(x^*, x_{n-1}) - \Delta_p(x^*, x_n)\big)\\ &- \frac{2\gamma_n\beta_n^q}{(1-\alpha_n)^{q+1}}\Big|\Big|J^p_E(x_n) - J^p_E(u_n)\Big|\Big|^q,\end{aligned}$$

which implies that

$$\begin{aligned}0 &\le \frac{2\gamma_n\beta_n^q}{(1-\alpha_n)^{q+1}}\Big|\Big|J^p_E(x_n) - J^p_E(u_n)\Big|\Big|^q\\ &\le \alpha_n\Delta_p(x^*, u) + (1-\alpha_n)\Delta_p(x^*, x_n) - \Delta_p(x^*, x_{n+1})\\ &+ \gamma_n\theta_n\big(\Delta_p(x^*, x_{n-1}) - \Delta_p(x^*, x_n)\big) \to\ 0,\ \text{as } n \to\ \infty.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\Big|\Big|J^p_E(x_n) - J^p_E(u_n)\Big|\Big|^q = 0.$$

Since $J^q_{E^*}$ is uniformly norm-to-norm continuous on bounded subsets, we obtain

$$\lim_{n\to\infty}\Big|\Big|x_n - u_n\Big|\Big| = 0, \tag{3.24}$$

which implies from Proposition 2.7 that

$$\lim_{n\to\infty}\Delta_p(x_n, u_n) = 0. \tag{3.25}$$

Also, from (3.1) and (3.25), we obtain that

$$\Delta_p(x_n,x_{n+1})\leq \alpha_n\Delta_p(x_n,u)+\beta_n\Delta_p(x_n,x_n)+\gamma_n\Delta_p(x_n,u_n))\to 0 \text{ as } n\to\infty.$$

Hence, from Lemma 2.6, we obtain that

$$\lim_{n\to\infty}||x_n-x_{n+1}||=0. \tag{3.26}$$

Let $w_n=J^q_{E^*}\big(J^p_E x_n+\theta_n(J^p_E(x_{n-1})-J^p_E(x_n)\big)$. It then follows that

$$J^p_E w_n-J^p_{E_1}x_n=\theta_n\big(J^p_E(x_{n-1})-J^p_E(x_n)\big).$$

Then by the uniform continuity of J^p_E on bounded subsets of E, we get that

$$\begin{aligned}||J^p_E w_n-J^p_E x_n||_* =&||\theta_n\big(J^p_{E_1}(x_{n-1})-J^p_{E_1}(x_n)\big)||_*\\ \leq& \theta_n||J^p_{E_1}(x_{n-1})-J^p_E(x_n)||_*\to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.27}$$

By the uniform continuity of $J^q_{E^*}$ on bounded subsets of E^* and (3.27), we obtain that

$$\lim_{n\to\infty}||w_n-x_n||=0. \tag{3.28}$$

Moreso, from (3.1) and (3.16), we obtain that

$$||J^p_E(y_n)-J^p_E(w_n)||\leq\sum_{i=0}^N\mu_{i,n}\lambda_{i,n}||A_i^*||\;||J^p_{E_i}(I^{E_i}-(K^{F_i,\phi_i}_{r_{i,n}})A_iw_n)||\to\;0 \text{ as } n\to\infty.$$

By the uniform continuity of $J^q_{E^*}$ on bounded subsets of E^*, we obtain that

$$\lim_{n\to\infty}||y_n-w_n||=0. \tag{3.29}$$

Thus, from (3.28) and (3.29), we obtain that

$$\lim_{n\to\infty}||y_n-x_n||=0. \tag{3.30}$$

Also, using (3.22) and (3.30), we get

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.31}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ which converges weakly to $z\in E$. From (3.28), (3.30) and (3.31), there exist subsequences $\{w_{n_k}\}$ of $\{w_n\}$ which converges weakly to $z\in E$, $\{y_{n_k}\}$ of $\{y_n\}$ which also converges weakly to $z\in E$ and lastly, $\{u_{n_k}\}$ of $\{u_n\}$ which converges weakly to $z\in E$. Using the fact that A_i for $i=0,1,2,\cdots N$ is a bounded linear operator, we get $A_iw_{n_k}\rightharpoonup A_iz\in E_i$ as $k\to\infty$. By applying (3.17), we obtain that $A_iz\in Fix(K^{F_i,\phi_i}_{r_{i,n}})$ for all $i=0,1,\cdots N$. Now by the Bregman relatively nonexpansivity of each $S_j, j=1,2,\cdots m$, we obtain that $z\in\bigcap_{j=1}^m Fix(S_j)$. Hence, we conclude that $z\in\Gamma$.

Next, since $x_{n_k}\rightharpoonup z\in\Gamma$, then for any $z=\Pi_\Gamma u$ we get from (2.5) that

$$\begin{aligned}\limsup_{n\to\infty}\langle J^p_E(u)-J^p_E(x^*),x_n-x^*\rangle &=\lim_{j\to\infty}\langle J^p_E(u)-J^p_E(x^*),x_{n_j}-x^*\rangle\\ &=\langle J^p_E(u)-J^p_E(x^*),z-x^*\rangle\\ &\leq 0.\end{aligned} \tag{3.32}$$

Furthermore, since

$$\begin{aligned}\langle J^p_E(u)-J^p_E(x^*),x_{n+1}-x^*\rangle &=\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n+1}-x_n\rangle\\ &+\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_n-x^*\rangle.\end{aligned}$$

Hence, from (3.26) and (3.32), we obtain that

$$\limsup_{n\to\infty}\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n+1}-x^*\rangle\leq 0. \tag{3.33}$$

By applying Lemma 2.12, (3.9) and (3.33), we obtain that $\{x_n\}$ converges strongly to x^*.

Case 2: Assume that $\{\Delta_p(x^*,x_n)\}$ is non-monotone. Set $\Upsilon_n=\Delta_p(x^*,x_n)$ as stated in Lemma 2.13 and let $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\geq n_0$, (for some n_0 large enough) defined by $\tau(n):=\max\{k\in\mathbb{N}:k\leq n,\Upsilon_k\leq\Upsilon_{k+1}\}$. Then τ is non-decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$. Thus

$$0\leq\Upsilon_{\tau(n)}\leq\Upsilon_{\tau(n)+1},\;\forall\; n\geq n_0,$$

this implies that

$$\Delta_p(x^*, x_{\tau(n)}) \leq \Delta_p(x^*, x_{\tau(n)+1}), n \geq n_0.$$

Now following the process as estimated in Case 1 above, we obtain that

$$\begin{cases} \lim\limits_{\tau(n)\to\infty} ||y_{\tau(n)} - u_{\tau(n)}|| = 0 \\ \lim\limits_{\tau(n)\to\infty} ||u_{\tau(n)} - x_{\tau(n)}|| = 0 \\ \lim\limits_{\tau(n)\to\infty} ||w_{\tau(n)} - x_{\tau(n)}|| = 0 \\ \lim\limits_{\tau(n)\to\infty} ||(I - S_i(K^{F_i,\phi_i}_{r_{i,\tau(n)}})A_i w_{\tau(n)}|| = 0 \\ \lim\limits_{\tau(n)\to\infty} \langle J^p_E(u) - J^p_E(x^*), x_{\tau(n)+1} - x^*\rangle \leq 0. \end{cases} \tag{3.34}$$

From (3.1) and $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$, we have

$$\begin{aligned} \Delta_p(z, x_{\tau(n)+1}) \leq (1 - \alpha_n - \theta_{\tau(n)}\gamma_{\tau(n)})\Delta_p(z, x_{\tau(n)}) + \mu_{\tau(n)}\theta_{\tau(n)}\Delta_p(v, x_{\tau(n)-1}) \\ + \alpha_{\tau(n)}\langle J^p_{E_1}(u) - J^p_E(z), x_{\tau(n)+1} - z\rangle. \end{aligned}$$

Hence, we obtain

$$\Delta_p(z, x_{\tau(n)}) \leq \Delta_p(z, x_{\tau(n)+1}) \leq \langle J^p_E(u) - J^p_E(z), x_{\tau(n)+1} - z\rangle.$$

Hence, from (3.34), we get

$$\lim_{\tau(n)\to\infty} \Delta_p(z, x_{\tau(n)}) = 0,$$

and

$$\Delta_p(z, x_{\tau(n)+1}) = 0.$$

Thus,

$$\lim_{\tau(n)\to\infty} \Upsilon_{\tau(n)} = \lim_{\tau(n)\to\infty} \Upsilon_{\tau(n)+1} = 0, \tag{3.35}$$

for all $n \geq n_0$, we have that $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$, if $n \neq \tau(n)$ (that is, $\tau(n) < n$), since $\Upsilon_{k+1} \leq \Upsilon_k$ for some $\tau(n) \leq k \leq n$. Hence, we obtain for all $n \geq n_0$,

$$0 \leq \Upsilon_n \leq \max\left\{\Upsilon_{\tau(n)}, \Upsilon_{\tau(n)+1}\right\} = \Upsilon_{\tau(n)+1}.$$

This implies that $\lim\limits_{n\to\infty} \Upsilon_n = 0$ which implies that $\lim\limits_{n\to\infty} \Delta_p(z, x_n) = 0$ $n \to \infty$. Hence $\{x_n\} \to z = \Pi_\Gamma u$ as $n \to \infty$. □

Corollary 3.3. *Let $E, E_i,\ i = 1, 2, \cdots N$ be real Banach spaces and C, C_i be nonempty, closed and convex subsets of E and E_i respectively. For $i = 1, 2, \cdots N$, let $F : C \times C \to \mathbb{R},\ F_i : C_i \times C_i \to \mathbb{R}$ be bifunctions and $\phi : C \to \mathbb{R},\ \phi_i : C \to \mathbb{R}$ be nonlinear mappings. Suppose $A_i : E \to E_i,\ i = 1, 2, \cdots N$ be linear bounded operators such that $\Gamma := \left\{x^* \in GEP(F, \phi) \cap \bigcap_{i=1}^{N} A_i^{-1}(GEP(F_i, \phi_i))\right\} \neq \emptyset$. Assume that $\{\theta_n\} \subset [0, \frac{1}{2}], \{\alpha_n\}, \{\beta_n\}, \{\mu_{i,n}\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\sum_{i=0}^{N} \mu_{i,n} = 1,\ \alpha_n + \beta_n + \gamma_n = 1,\ \alpha_n \leq b < 1,\ (1 - \alpha_n)a < \gamma_n,\ a \in (0, \frac{1}{2})$. Let $x_0, x_1 \in E$ and $\{x_n\}$ be sequence generated as follows:*

$$\begin{cases} w_n = J^q_{E^*}\left[J^p_E(x_n) + \theta_n\left(J^p_E(x_{n-1}) - J^p_E(x_n)\right)\right], \\ y_n = J^q_{E^*}\left[\sum\limits_{i=0}^{N} \mu_{i,n}\left(J^p_E(w_n) - \lambda_{i,n}A_i^* J^p_{E_i}(I^{E_i} - (K^{F_i,\phi_i}_{r_{i,n}})A_i(w_n)\right)\right], \\ x_{n+1} = J^q_{E^*}\left(\alpha_n J^p_E(u) + \beta_n J^p_E(x_n) + \gamma_n J^p_E(y_n)\right). \end{cases} \tag{3.36}$$

Now, suppose for $\varepsilon > 0$, the step size $\lambda_{i,n}$ is chosen in such a way that

$$\lambda_{i,n} \in \left(\varepsilon,\ \left(\frac{q||A_i(w_n) - (K^{F_i,\phi_i}_{r_{i,n}})A_i(w_n)||^p}{C_q||A_i^* J^p_{E_i}(I^{E_i} - ((K^{F_i,\phi_i}_{r_{i,n}})A_i(w_n))||^q_*} - \varepsilon\right)^{\frac{1}{q-1}}\right);\ n \in \Omega, \tag{3.37}$$

where the index set $\Omega := \{n \in \mathbb{N} : A_i w_n - ((K^{F_i,\phi_i}_{r_{i,n}})A_i w_n) \neq 0\}$, otherwise $\lambda_{i,n} = \lambda$, λ is any nonnegative real numbers. Then the sequence $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}$ and $\{\theta_n\}$ satisfies the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $0 < c < \theta_n < \gamma_n \leq \frac{1}{2}, \forall\ n \geq 1$,

(3) $0 < \liminf_{n\to\infty} \gamma_n, \beta_n \leq \limsup_{n\to\infty} \gamma_n, \beta_n < 1$,

(4) $\min_{i=0,1,\cdots N}\{\inf_n\{r_{i,n}\}\} = r > 0$. *Then,* $\{x_n\}$ *generated by* (3.36) *converges strongly to* $x^* \in \Gamma$, *where* $x^* = \Pi_\Gamma u$.

Corollary 3.4. *Let* $E, E_i,\ i = 1, 2, \cdots N$ *be real Banach spaces and* C, C_i *be nonempty, closed and convex subsets of* E *and* E_i *respectively. For* $i = 1, 2, \cdots N$, *let* $F : C \times C \to \mathbb{R},\ F_i : C_i \times C_i \to \mathbb{R}$ *be bifunctions. Suppose* $A_i : E \to E_i,\ i = 1, 2, \cdots N$ *be linear bounded operators such that* $\Gamma := \{x^* \in GEP(F) \cap \bigcap_{i=1}^{N} A_i^{-1} \cap GEP(F_i))\} \neq \emptyset$. *Assume that* $\{\theta_n\} \subset [0, \frac{1}{2}], \{\alpha_n\}, \{\beta_n\}, \{\mu_{i,n}\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *such that* $\sum_{i=0}^{N} \mu_{i,n} = 1,\ \alpha_n + \beta_n + \gamma_n = 1,\ \alpha_n \leq b < 1,\ (1 - \alpha_n)a < \gamma_n,\ a \in (0, \frac{1}{2})$. *Let* $x_0, x_1 \in E$ *and* $\{x_n\}$ *be sequence generated as follows:*

$$\begin{cases} w_n = J_{E^*}^q\left[J_E^p(x_n) + \theta_n\big(J_E^p(x_{n-1}) - J_E^p(x_n)\big)\right], \\ y_n = J_{E^*}^q\left[\sum_{i=0}^{N} \mu_{i,n}\big(J_E^p(w_n) - \lambda_{i,n}A_i^* J_{E_i}^p(I^{E_i} - (K_{r_{i,n}}^{F_i})A_i(w_n)\big)\right], \\ x_{n+1} = J_{E^*}^q\big(\alpha_n J_E^p(u) + \beta_n J_E^p(x_n) + \gamma_n J_E^p(y_n)\big). \end{cases} \tag{3.38}$$

Now, suppose for $\varepsilon > 0$, *the step size* $\lambda_{i,n}$ *is chosen in such a way that*

$$\lambda_{i,n} \in \left(\varepsilon,\ \Big(\frac{q||A_i(w_n) - (K_{r_{i,n}}^{F_i,\phi_i})A_i(w_n)||^p}{C_q||A_i^* J_{E_i}^p(I^{E_i} - (S_i(K_{r_{i,n}}^{F_i})A_i(w_n))||_{(*)}^q} - \varepsilon\Big)^{\frac{1}{q-1}}\right);\ n \in \Omega, \tag{3.39}$$

where the index set $\Omega := \{n \in \mathbb{N} : A_i w_n - ((K_{r_{i,n}}^{F_i})A_i w_n) \neq 0\}$, *otherwise* $\lambda_{i,n} = \lambda$, λ *is any nonnegative real numbers. Then the sequence* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}$ *and* $\{\theta_n\}$ *satisfies the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $0 < c < \theta_n < \gamma_n \leq \frac{1}{2}, \forall\ n \geq 1$,

(3) $0 < \liminf_{n\to\infty} \gamma_n, \beta_n \leq \limsup_{n\to\infty} \gamma_n, \beta_n < 1$,

(4) $\min_{i=0,1,\cdots N}\{\inf_n\{r_{i,n}\}\} = r > 0$. *Then,* $\{x_n\}$ *generated by* (3.38) *converges strongly to* $x^* \in \Gamma$, *where* $x^* = \Pi_\Gamma u$.

4. Numerical example

In this section, we present a numerical example to illustrate the performance of our method.

Example 4.1. Let $E, E_i = \mathbb{R}^2$ for $i = 0, 1, 2$ with $E = E_0$. We define the mappings $F = F_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $F_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ respectively by $F(x, y) = -3x^2 + xy + 2y^2$, $F_1(x, y) = -4x^2 + xy + 3y^2$ and $F_2(x, y) = -5y^2 + 2y + 5xy - 5xy^2$ for each $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Also, for $i = 0, 1, 2$, let $\phi_0 = \phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\phi_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\phi(x, y) = x^2 - xy$, $\phi_1(x, y) = 2x(x - y)$ and $\phi_2(x, y) = 5y^2 - 2x$ respectively for each $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. By some simple calculations, we obtain the following for some $r > 0$:

$$v = K_r^{F,\phi}u = \frac{1}{4r+1}u,\ y = K_r^{F_1,\phi_1}x = \frac{1}{1+5r}x \text{ and } w = K_r^{F_2,\phi_2}z = \frac{z-2r}{1+5r}.$$

Now, for $j = 1, 2, \cdots, N$, let $S_j : \mathbb{N}^2 \to \mathbb{R}^2$ be defined by

$$S_j(x) = P_{Q_j}(x) = \begin{cases} b_i + r\frac{x-b_j}{\|x-b_j\|}, & \text{if} \|x - b_j\| < r, \\ x, & \text{otherwise} \end{cases}$$

where Q_j are close balls in $\mathbb{R}^2$ centered at $b_j \in \mathbb{R}^2$ with radius $r > 0$, that is

$$Q_j = \{x \in \mathbb{R}^2 : \|x - b_j\| < r\}.$$

It is easy to see that P_{Q_j} is nonexpansive, hence Bregman relatively nonexpansive. Let $A_i : \mathbb{R}^2 \to \mathbb{R}$ be given by $A_i(x) = \frac{x}{i+1}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. For this experiment, let $\alpha_n = \frac{1}{150n+1}$, $\beta_n = \frac{1}{2n+13}$, $\gamma_n = 1 - \alpha_n - \beta_n$, $r_{i,n} = r = 0.5$ $\lambda_{i,n} = \frac{3.5n(i+1)}{n+1}$. and $\beta_{i,n} = \frac{6}{7^i} + \frac{1}{1029}$. Let $E_n = \|x_{n+1} - x_n\|^2 = 10^{-4}$, be the stopping criterion. We consider the following cases for initial values of x_0 and x_1 :

Case 1 $x_0 = (0.5, 0.35)$ and $x_1 = (0.78, 1.25)$;
Case 2 $x_0 = (1.5, 2.35)$ and $x_1 = (3.78, 1.25)$;
Case 3 $x_0 = (0, 3)$ and $x_1 = (4, 2)$;
Case 4 $x_0 = (-4, -4)$ and $x_1 = (-10, -20)$;

The results of this experiment are reported in Figure 1.

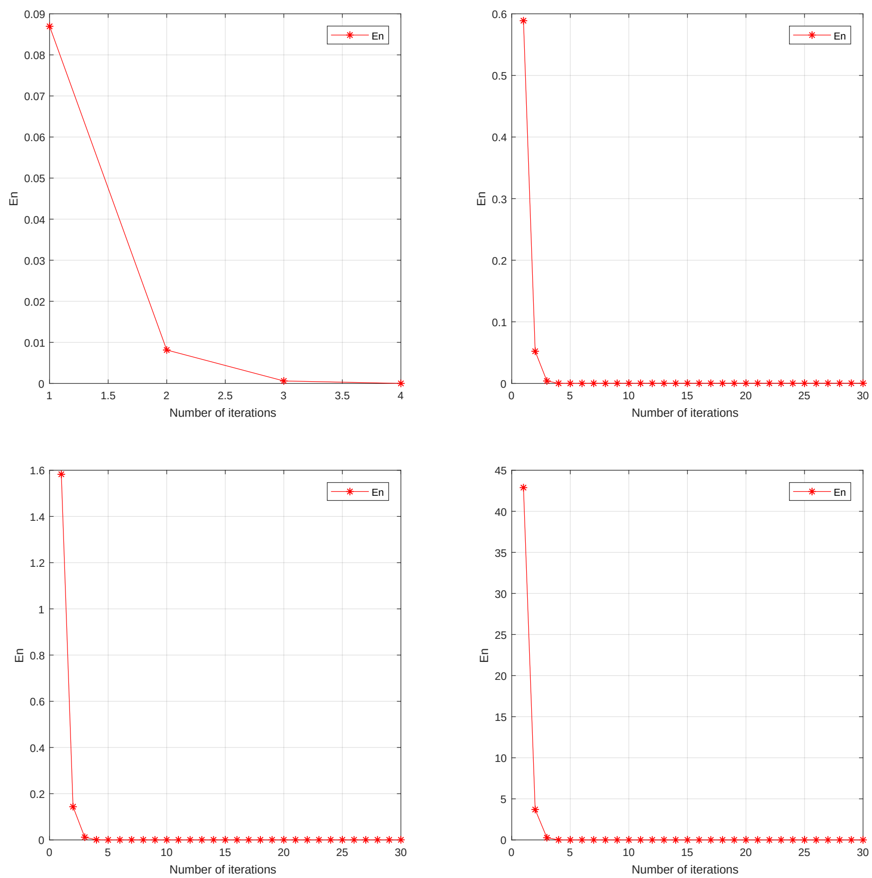

FIGURE 1. Example 4.1. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

We also display an example in ℓ_3 space which is a uniformly convex and 2-uniformly smooth Banach space but not a Hilbert space.

Example 4.2. Let $E, E_i = \ell_3$ for $i = 0, 1, 2$ with $E = E_0$. We define the mappings $F = F_0 : \ell_3 \times \ell_3 \to \mathbb{R}$, $F_1 : \ell_3 \times \ell_3 \to \mathbb{R}$ and $F_2 : \ell_3 \times \ell_3 \to \mathbb{R}$ respectively by $F(x, y) = -3x^2 + xy + 2y^2$, $F_1(x, y) = -4x^2 + xy + 3y^2$ and $F_2(x, y) = -5y^2 + 2y + 5xy - 5xy^2$ for each $x = (x_1, x_2, x_3, \cdots) \in \ell_3$ and $y = (y_1, y_2, y_3, \cdots) \in \ell_3$. Also, for $i = 0, 1, 2$, let $\phi_0 = \phi : \ell_3 \times \ell_3 \to \mathbb{R}$, $\phi_1 : \ell_3 \times \ell_3 \to \mathbb{R}$ and $\phi_2 : \ell_3 \times \ell_3 \to \mathbb{R}$ be defined by $\phi(x, y) = x^2 - xy$, $\phi_1(x, y) = 2x(x - y)$ and $\phi_2(x, y) = 5y^2 - 2x$ respectively for each $x = (x_1, x_2, x_3, \cdots) \in \ell_3$

and $y = (y_1, y_2, y_3, \cdots) \in \ell_3$. By some simple calculations, we obtain the following for some $r > 0$:

$$v = K_r^{F,\phi} u = \frac{1}{4r+1} u, \ y = K_r^{F_1,\phi_1} x = \frac{1}{1+5r} x \text{ and } w = K_r^{F_2,\phi_2} z = \frac{z-2r}{1+5r}.$$

Now, for $j = 1, 2, \cdots, N$, let $S_j : \ell_3 \to \ell_3$ be defined by

$$S_j(x) = P_{Q_j}(x) = \begin{cases} b_j + r\frac{x-b_j}{\|x-b_j\|}, & \text{if} \|x-b_j\| < r, \\ x, & \text{otherwise} \end{cases}$$

where Q_j are close balls in ℓ_3 centered at $b_j \in \ell_3$ with radius $r > 0$, that is

$$Q_j = \{x \in \ell_3 : \|x - b_j\| < r\}.$$

It is easy to see that P_{Q_j} is nonexpansive, hence Bregman relatively nonexpansive. Let $A_i : \ell_3 \to \ell_3$ be given by $A_i(x) = \frac{x}{i+1}$ where $x = (x_1, x_2, x_3, \cdots) \in \ell_3$. For this experiment, let $\alpha_n = \frac{1}{150n+1}$, $\beta_n = \frac{1}{2n+13}$, $\gamma_n = 1-\alpha_n-\beta_n$, $r_{i,n} = r = 0.5$ $\lambda_{i,n} = \frac{3.5n(i+1)}{n+1}$. and $\beta_{i,n} = \frac{6}{7^i} + \frac{1}{1029}$. Let $E_n = \|x_{n+1} - x_n\|^2 = 10^{-4}$, be the stopping criterion

5. Application

Let C be a nonempty, closed and convex subset of a Banach space E with dual space E^*. The indicator function is denoted i_C and defined by

$$i_C = \begin{cases} 0, & \text{if } x \in C \\ \infty, & \text{if } x \notin C. \end{cases}$$

It is well known that the subdifferential ∂i_C is maximal monotone. Moreover,

$$\partial i_C(x) = N_C(x) = \{y \in E^* : \langle x - z, y\rangle \geq 0, \ \forall z \in C\},$$

where $N_C(x)$ is called the normal cone at x.

For $r > 0$, denote by K_r the resolvent of ∂i_C and $y = K_r(x)$ for $x \in E$, that is

$$\frac{1}{r}(x - y) \in \partial i_C(y) = N_C(y).$$

Then, we obtain

$$\langle Jx - y, y - z\rangle \geq 0, \ \forall z \in C.$$

It follows that $y = P_C(x)$ where P_C is the metric projection onto C. Under this settings, we obtain a corollary of our main theorem for approximating a solution of split feasibility problem with multiple output set. That is the solution of the problem:

$$\text{find } x \in C \text{ and } A_i x \in Q_i, \ \forall \ i = 1, 2, \cdots N.$$

Corollary 5.1. *Let $E, E_i, \ i = 1, 2, \cdots N$ be real Banach spaces and $C = C_0, C_i$ be nonempty, closed and convex subsets of E and E_i respectively. Suppose $\Upsilon := \{x^* \in C \cap \bigcap_{i=1}^{N} A_i^{-1} C_i\} \neq \emptyset$. Assume that $\{\theta_n\} \subset [0, \frac{1}{2}], \{\alpha_n\}, \{\beta_n\}, \{\mu_{i,n}\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\sum_{i=0}^{N} \mu_{i,n} = 1, \ \alpha_n + \beta_n + \gamma_n = 1, \ \alpha_n \leq b < 1, \ (1 - \alpha_n)a < \gamma_n, \ a \in (0, \frac{1}{2})$. Let $u, x_0, x_1 \in E$ and $\{x_n\}$ be the sequence generated as follows:*

$$\begin{cases} w_n = J_{E^*}^q \left[J_E^p(x_n) + \theta_n\big(J_E^p(x_{n-1}) - J_E^p(x_n)\big)\right], \\ y_n = J_{E^*}^q \left[\sum_{i=0}^{N} \mu_{i,n}\big(J_E^p(w_n) - \lambda_{i,n} A_i^* J_{E_i}^p (I^{E_i} - P_{C_i}) A_i(w_n)\big)\right], \\ x_{n+1} = J_{E^*}^q\big(\alpha_n J_E^p(u) + \beta_n J_E^p(x_n) + \gamma_n J_E^p(y_n)\big). \end{cases} \tag{5.1}$$

Now, suppose for $\varepsilon > 0$, the step size $\lambda_{i,n}$ is chosen in such a way that

$$\lambda_{i,n} \in \left(\varepsilon, \ \Big(\frac{q||A_i(w_n) - P_{C_i} A_i(w_n)||^p}{C_q||A_i^* J_{E_i}^p (I^{E_i} - P_{C_i}) A_i(w_n))||_{(*)}^{(q)}} - \varepsilon\Big)^{\frac{1}{q-1}}\right); \ n \in \Omega, \tag{5.2}$$

where the index set $\Omega := \{n \in \mathbb{N} : A_i w_n - P_{C_i} A_i w_n \neq 0\}$, otherwise $\lambda_{i,n} = \lambda$, λ is any nonnegative real number. The sequence $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}$ and $\{\theta_n\}$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $0 < c < \theta_n < \gamma_n \le \frac{1}{2}, \forall\, n \ge 1$,

(3) $0 < \liminf_{n\to\infty} \gamma_n, \beta_n \le \limsup_{n\to\infty} \gamma_n, \beta_n < 1$,

(4) $\min_{i=0,1,\cdots N} \{\inf_n\{r_{i,n}\}\} = r > 0$. *Then, $\{x_n\}$ generated by (5.1) converges strongly to $x^* \in \Upsilon$, where $x^* = P_\Upsilon u$.*

5.1. **Application to image restortion problems.** In this section, we apply our method to image deblurring and denoising. General image reovery problem can be formulated by the inversion of the following observation model:

$$b = Ax + v, \tag{5.3}$$

where $x \in \mathbb{R}^n$, x, v and b are unknown original image, unknown additive random noise and known degrded observation, respectively, and A is a linear operator that depends on the concerned image recovery problem. This model (5.3), is approximately equivalent to several different formulations available for optimization problems. In the literature, there is a growing interest in using the l_1 norm in solving these types of problems. The l_1 regularization problem is given by

$$\min_x \frac{1}{2}\|Ax - b\|_2^2 \text{ such that } \|x\|_1 \le t, \tag{5.4}$$

where t is a positive constant, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$ and A is a $k \times n$ matrix.
Next, we use our algorithm to approximate the solution of the following convex minimization problem:

$$\text{find } x \in \arg\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 \text{ such that } \|x\|_1 \le t, \tag{5.5}$$

where b is the degraded image, and A is an operator representing the mask. We start the process by setting $t = m$.

Let $C = \{x \in \mathbb{R}^k : \|x\|_1 \le t\}$ and $Q_i = \{y_i\}$. Then the minimization problem can be seen as the problem (SFP). We make use of the subgradient projection since the projection onto the closed convex set C does not have a close form. Let $c(x) = \|x\|_1 - t$ and denote the level set C_n by

$$C_n = \{x \in C : c(x_n) + \langle d_n, x - x_n\rangle \le 0\},$$

where $d_n \in \partial c(x_n)$. Then the projection onto C_n can be calculated by the following formula;

$$P_{C_n}(u) = \begin{cases} u, & \text{if} x \in C_n \\ u - \frac{c(x_n)+\langle d_n, x-x_n\rangle}{\|d_n\|^2} d_n. & \end{cases}$$

The subdifferential at x_n is given by

$$\partial c(x_n) = \begin{cases} 1, & x_n > 0, \\ [-1, 1], & x_n = 0, \\ -1, & x_n < 0. \end{cases}$$

Example 5.2. For this experiment we use [6, Algorithm 4] and Algorithm 5.1 to solve (5.5). We consider the blur function in MATLAB "special ('motion', 30, 60)" and add random noise. The test images are cameraman, kids and pout (see 2) and the stopping criterion of the algorithm is $\frac{\|x_{n+1}-x_n\|}{\|x_{n+1}\|} < 10^{-4}$. Each subplot contains four images, the original image, the blurred image, the restored image by [**?**, Algorithm] and the restored image by Algorithm 5.1. As we can see from Figure 3, our proposed algorithm is competitive and promising.

The results of this experiment for the cameraman for various values of n are reported in Figure 3. The signal to noise ration (SNR) is defined as

$$EN = SNR = 10 \log \frac{\|x\|^2}{\|x - x_n\|^2}$$

where x and x_n are the original and estimated image at iteration. All codes were written with MATLAB 2022a on a personal dell latitude laptop with 8gig/256gig, RAM/ROM respectively and 2.4ghz processor speed.

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FIGURE 2. Example 5.2. First, Cameraman, Second: Kids, Third: Pout.

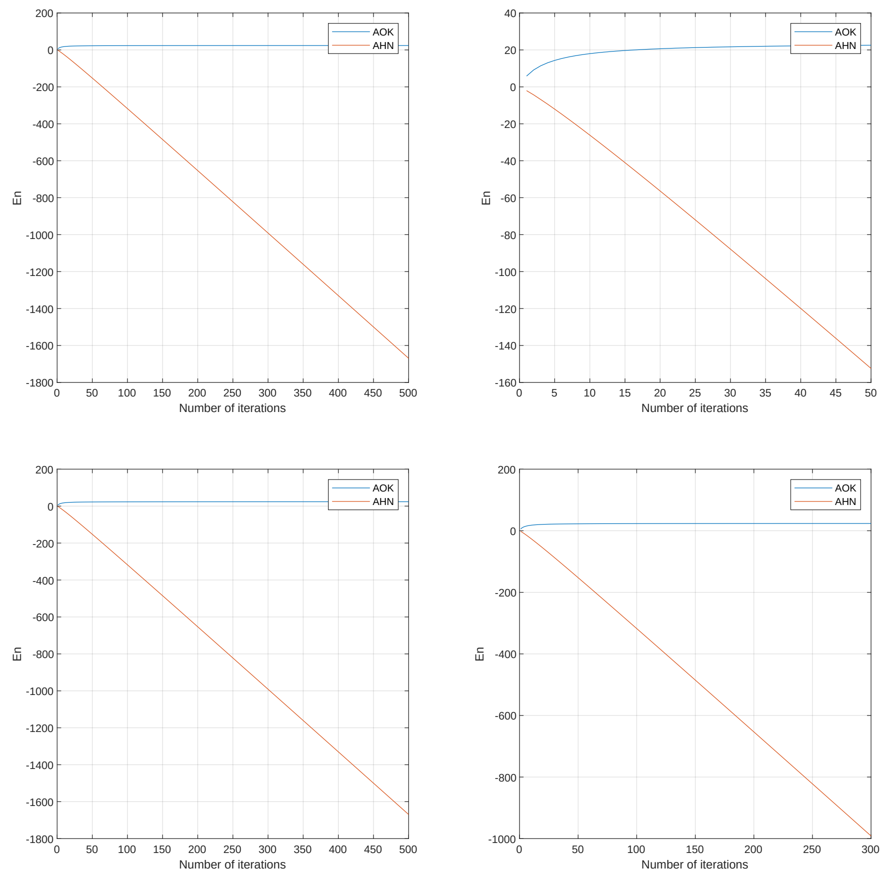

FIGURE 3. Example 5.2. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

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