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The Homotopy Theory of Polyhedral Products

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Abstract

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Polyhedral products are topological spaces which unify and generalise many fundamental constructions. Their study is of significant interest in homotopy theory, where they underpin the combinatorics inherent in a broad variety of key objects, such as the join product, the Cartesian product, the smash product, and the Whitehead filtration. More broadly, polyhedral products are of interest in fields including algebraic geometry, geometric group theory, number theory, and representation theory, where they enable the study of familiar objects through underlying combinatorial structures.

In this thesis, we study homotopy-theoretic properties of polyhedral products. Our results lie within two major areas. The first of these is the study of duality. Duality phenomena are exhibited by fundamental objects across mathematics, such as manifolds in topology, generalised homology spheres in combinatorics, and Gorenstein rings in commutative algebra. Through a study of polyhedral products known as moment-angle complexes, we equivocate homotopy-theoretic, combinatorial, and algebraic dualities. We show that this equivocation has both algebraic and homotopy-theoretic applications, to the study of moment-angle complexes and to broader families of polyhedral products.

The second area in which our results lie is the study of homotopy classes of maps. Generalising known results, we show that a new family of relations is satisfied by certain homotopy classes of maps of polyhedral products. This brings new insight into the rich algebraic structures behind sets of homotopy classes of maps, and as an application we show that this work enables us to analyse the algebraic structure of the homotopy groups of odd spheres from an entirely new angle.

Contents

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I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

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7. Parts of this work have been published as: J. Grbić and M. Staniforth. Duality in toric topology, 2021.

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To my wife Beth,
The love and light of my life.

Introduction

Polyhedral products are topological spaces constructed from the combinatorial data of a simplicial complex together with a tuple of pairs of CW-complexes. By taking products of spaces prescribed by the faces, and attaching them according to the face-category, we obtain a space whose topological properties are intertwined with the combinatorics of the underlying simplicial complex. From a homotopy-theoretic point of view, their study is motivated by the fact that many objects central to the study of homotopy theory are polyhedral products, including the join and smash products, the cartesian product, and the Whitehead-filtration of a cartesian product. Polyhedral products unify and generalise these constructions, enabling their study through underlying combinatorial structure.

The study of polyhedral products takes place at the intersection of the fields of homotopy theory, combinatorics, and commutative algebra. Through the application of an interplay of techniques from these disciplines, new connections between distinct fields can be created, and problems made more tractable. To date, extensive progress has been made in the study of topics such as Massey products and non-formal spaces in homotopy theory [13], right-angled Artin and Coxeter groups in geometric group theory [2, 14], and toric varieties in algebraic geometry [12].

In this thesis, we study the homotopy-theoretic properties of polyhedral products. Our work falls into two key areas, of the study of duality, and of the study of homotopy classes of maps.

Chapter one consists of pre-requisite background for the work in chapters two and three. We clarify definitions and introduce notation, and state fundamental theoretical results which we make use of later on. The intended purpose of the background chapter is to serve as a dictionary of sorts, to be referred to as necessary.

In chapter two, we study duality in the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$. Our first main result is an equivocation of Poincaré duality in $\mathcal{Z}_{\mathcal{K}}$, Alexander duality in the simplicial complex $\mathcal{K}$, and Gorenstein duality in the face-ring $\mathbb{Z}[\mathcal{K}]$, revealing an interplay of homotopy-theoretic, combinatorial, and algebraic dualities. Our methods motivate the study of a new algebraic construction, which we refer to as the polyhedral ring product $(\underline{R}, \underline{S})^{\mathcal{K}}$, and which we use to study algebraic phenomena through the underlying combinatorics. As an immediate application, we construct Gorenstein rings which are polyhedral ring products of rings which are themselves non-Gorenstein. This leads to our second main result, which is the equivocation of Gorenstein duality in $(\underline{\mathbb{Z}[\mathcal{K}_i]}, \underline{\mathbb{Z}[\mathcal{L}_i]})^{\mathcal{K}}$, Alexander duality in $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$, and Poincaré duality in $(\boldsymbol{\mathcal{Z}}_{\mathcal{K}_{\mathrm{i}}}, \boldsymbol{\mathcal{Z}}_{\mathcal{L}_{\mathrm{i}}})^{\mathcal{K}}$, suggesting a deeper categorical connection between dualities in polyhedral products, polyhedral ring products, and polyhedral join products.

In chapter three, we study homotopy classes of maps between polyhedral products called higher Whitehead maps, a generalisation of Whitehead products. First, we characterise the existence and non-triviality of the higher Whitehead map in terms of the inclusion of certain subcomplexes in the associated simplicial complex. Secondly, we break new ground by proving that new families of relations are present among higher Whitehead maps, including those of differing arity. This brings new insight into the rich algebraic structure of groups of homotopy classes of maps, where these are the first results of this kind, and sets the stage for the development of an entirely new area of research in the homotopy theory of polyhedral products.

Chapter 1

Background

In this chapter, we state the definitions and results which are pre-requisite to the later material in this thesis, and fix our notational conventions and standing assumptions.

The purpose of this chapter is to serve as a reference for later material, and as such we keep exposition to a minimum.

1.1 Combinatorics

For a set S, we denote by 2^S the set of all subsets of S.

Definition 1.1. Let S be a finite set. A simplicial complex $\mathcal{K}$ on the vertex set S is a non-empty collection of subsets $\mathcal{K} \subseteq 2^S$, such that if $\sigma \in \mathcal{K}$ and $\tau \subseteq \sigma$ then $\tau \in \mathcal{K}$.

For simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$ on the vertex sets S and S' respectively, a map of simplicial complexes is a pair of maps $f\colon \mathcal{K} \longrightarrow \mathcal{K}'$ and $v\colon S \longrightarrow S'$ such that for every $\sigma \in \mathcal{K}$, $f(\sigma) = \{v(i) \mid i \in \sigma\}$, and $f(\sigma) \in \mathcal{K}'$.

The complexes $\mathcal{K}$ and $\mathcal{K}'$ on the vertex sets S and S' are isomorphic if $|S| = |S'|$, and there exist maps of simplicial complexes $f\colon \mathcal{K} \longrightarrow \mathcal{K}'$ and $g\colon \mathcal{K}' \longrightarrow \mathcal{K}$ such that $f \circ g = Id_{\mathcal{K}'}$ and $g \circ f = Id_{\mathcal{K}}$.

$\mathcal{K}'$ is a subcomplex of $\mathcal{K}$ if $S' \subseteq S$, and $\mathcal{K}' \subseteq \mathcal{K}$.

For a subset $J \subseteq [m]$, the full subcomplex on J is the simplicial complex with vertex set J, defined by $\mathcal{K}_J = \{\sigma \in \mathcal{K} \mid \sigma \subseteq J\}$.

We emphasise for any simplicial complex $\mathcal{K}$ that $\emptyset \in \mathcal{K}$, and we do not assume that a simplicial complex contains the elements of its vertex set. Unless it is necessary to emphasise it, we omit the map of underlying vertex sets from the notation, and refer to a map of simplicial complexes by the map $f\colon \mathcal{K} \longrightarrow \mathcal{K}'$. We often assume that the vertex set S is equipped with a well order, when for $|S| = m$, we denote the vertex set by $[m] = \{1, \ldots, m\}$.

We denote by

$$\begin{aligned}
&\bullet = \{\emptyset, \{1\}\} && \text{on the vertex set } \{1\},\\
&\circ = \{\emptyset\} && \text{on the vertex set } \{1\},\\
&\bullet_{[m]} = \{\emptyset, \{1\}, \ldots, \{m\}\} && \text{on the vertex set } [m],\\
&\circ_{[m]} = \{\emptyset\} && \text{on the vertex set } [m],\\
&\Delta^{m-1} = 2^{[m]} && \text{on the vertex set } [m],\\
&\partial\Delta^{m-1} = 2^{[m]}\backslash\{[m]\} && \text{on the vertex set } [m],
\end{aligned}$$

and for any simplicial complex $\mathcal{K}$ on vertex set $[m]$, for $0 \leq n \leq m$, we denote by $\operatorname{sk}^n \mathcal{K} = \{\sigma \in \mathcal{K} \mid |\sigma| \leq n+1\}$ the n-skeleton of $\mathcal{K}$.

Definition 1.2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes on the vertex sets $[m_1]$ and $[m_2]$ respectively. The join of $\mathcal{K}_1$ and $\mathcal{K}_2$ is the simplicial complex on $[m_1] \sqcup [m_2]$ defined by

$$\mathcal{K}_1 * \mathcal{K}_2 = \{\sigma_1 \sqcup \sigma_2 \mid \sigma_1 \in \mathcal{K}_1, \sigma_2 \in \mathcal{K}_2\}.$$

Definition 1.3. Let $\mathcal{K}$ be a simplicial complex on the vertex set $[m]$. The Alexander dual to $\mathcal{K}$ is the simplicial complex on $[m]$ defined by

$$\widehat{\mathcal{K}} = \{I \subseteq [m] \mid [m]\backslash I \notin \mathcal{K}\}.$$

Definition 1.4. Let $\mathcal{K}$ be a simplicial complex on the vertex set $[m]$. For a simplex $\sigma \in \mathcal{K}$, the star of σ is defined by

$$\operatorname{st}(\sigma) = \{\tau \in \mathcal{K} \mid \sigma \cup \tau \in \mathcal{K}\}.$$

In the case that σ is a lone vertex, we abbreviate by denoting its star by $\operatorname{st}(i)$.

Definition 1.5. Let $\mathcal{K}$ be a simplicial complex on $[m]$. Denote by $\operatorname{core}([m]) = \{i \in [m] \mid \mathcal{K} \neq \operatorname{st}(i)\}$. The core of $\mathcal{K}$ is the full subcomplex defined by

$$\operatorname{core}(\mathcal{K}) = \mathcal{K}_{\operatorname{core}([m])}.$$

Observe that by definition, if $\mathcal{K} \neq \operatorname{core}(\mathcal{K})$, then there exists $\{i\} \in \mathcal{K}$ such that $\mathcal{K} = \operatorname{st}(i) = \{i\} * \mathcal{K}_{[m]\backslash i}$, implying that $\mathcal{K}$ is contractible.

1.2 Homotopy theory

We assume that all spaces have the homotopy type of based CW-complexes, and that all maps preserve the basepoint. We denote the unit interval by $[0,1] = I$.

Definition 1.6. For a space X, the cone on X, denoted CX is the homotopy pushout of the diagram

$$X \xleftarrow[Id]{} X \longrightarrow *,$$

the suspension of X, denoted ΣX is the homotopy pushout of the diagram

$$* \longleftarrow X \longrightarrow *,$$

and the based loop space on X, denoted ΩX is the homotopy pullback of the diagram

$$* \hookrightarrow X \hookleftarrow *,$$

These spaces are modelled by

$$CX = X \times I/(X \times \{1\}), \quad \Sigma X = CX/(X \times \{0\}), \quad \text{and} \quad \Omega X = \{f\colon S^1 \longrightarrow X\}$$

with basepoints $*_{CX} = (*_X, 0)$, $*_{\Sigma X} = (*_X, \frac{1}{2})$ and $*_{\Omega X} = \omega_*$, where $\omega_*(t) = *_X$ for all $t \in S^1$.

For $n \geq 2$, we denote by $\Sigma^n X = \Sigma(\Sigma^{n-1} X)$, where $\Sigma^1 X = \Sigma X$.

Definition 1.7. Let X and Y be spaces. The join $X * Y$ is the homotopy pushout of the diagram

$$X \xleftarrow[p_1]{} X \times Y \xrightarrow{p_2} Y$$

the wedge product $X \vee Y$ is the homotopy pushout of the diagram

$$X \hookleftarrow * \hookrightarrow Y$$

the smash product $X \wedge Y$ is the homotopy pushout of the diagram

$$* \longleftarrow X \vee Y \hookrightarrow X \times Y$$

the left half-smash product $X \ltimes Y$ is the homotopy pushout of the diagram

$$* \longleftarrow X \times *_Y \hookrightarrow X \times Y$$

and the right half-smash product $X \rtimes Y$ is the homotopy pushout of the diagram

$$* \longleftarrow *_X \times Y \hookrightarrow X \times Y \ .$$

For $m \geq 2$, given a family of spaces $\{X_i\}_{i\in[m]}$, these spaces are modelled by

$$\begin{aligned}
\mathop{\ast}_{i=1}^{m} X_i &= \bigcup_{i=1}^{m} CX_1 \times \cdots \times CX_{i-1} \times X_i \times CX_{i+1} \times \cdots \times CX_m \\
\bigvee_{i=1}^{m} X_i &= \bigcup_{i=1}^{m} \ast_{X_1} \times \cdots \times \ast_{X_{i-1}} \times X_i \times \ast_{X_{i+1}} \times \cdots \times \ast_{X_m} \\
\bigwedge_{i=1}^{m} X_i &= \prod_{i=1}^{m} X_i \Big/ FW_{i=1}^{m}(X_1, \ldots, X_m)
\end{aligned}$$

respectively, where

$$FW_{i=1}^{m}(X_1, \ldots, X_m) = \{(x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i \mid x_i = \ast_{X_i} \text{ for at least one } i \in [m]\}$$

is the fat-wedge. The basepoint in each of these spaces is the m-fold product of the basepoints of the spaces X_i, or its equivalence class.

Similarly, the left and right half-smash products are modelled by

$$\begin{aligned}
X \ltimes Y &= (X \sqcup \{\ast\}) \wedge Y = X \times Y/(X \times \ast_Y) \qquad \text{and} \\
X \rtimes Y &= X \wedge (Y \sqcup \{\ast\}) = X \times Y/(\ast_X \times Y)
\end{aligned}$$

respectively, where the basepoint of both $X \sqcup \{\ast\}$ and $Y \sqcup \{\ast\}$ is the disjoint point, and the basepoints of $X \ltimes Y$ and $X \rtimes Y$ are the equivalence class of $\ast_X \times \ast_Y$ in each case.

Observe that if X_i is a co-H space for any $i \in [m]$, then $\bigwedge_i X_i$ is a co-H space. Also observe that $X \ltimes Y$ is a co-H space if Y is a co-H space, and $X \rtimes Y$ is a co-H space if X is co-H.

Proposition 1.8 ([21, Theorem 4]). *Let, $m \geq 2$. For a family of spaces $\{X_i\}_{i\in[m]}$, there is a homotopy equivalence*

$$\mathop{\ast}_{i=1}^{m} X_i \simeq \Sigma^{m-1}\left(\bigwedge_{i=1}^{m} X_i\right). \tag{1.1}$$

Proposition 1.9 ([23, Theorem 7.9.5, Theorem 7.9.6]).

1. *A space X is an H-space if and only if there exists a map $\phi\colon \Omega\Sigma X \longrightarrow X$ such that the composite*

$$X \hookrightarrow \Omega\Sigma X \xrightarrow{\phi} X$$

is homotopic to the identity, and

2. *A space X is a co-H-space if and only if there exists a map $\psi\colon X \longrightarrow \Sigma\Omega X$ such that the composite*

$$X \xrightarrow{\psi} \Sigma\Omega X \xrightarrow{ev} X$$

is homotopic to the identity.

Proposition 1.10. *Let X be a co-H-space, and denote by $q_1\colon X \rtimes Y \longrightarrow X$ and $q_2\colon X \rtimes Y \longrightarrow X \wedge Y$ the obvious identification maps. Then the composite*

$$h\colon X \rtimes Y \longrightarrow (X \rtimes Y) \vee (X \rtimes Y) \xrightarrow{q_1 \vee q_2} X \vee (X \wedge Y)$$

where the first map is the co-multiplication, is a homotopy equivalence.

Proof. The pair $(\Sigma Y, \{*_{\Sigma Y}\})$ has the homotopy extension property. Therefore, the map

$$\begin{aligned} \Sigma Y \times \{0\} \vee \{*_{\Sigma Y}\} \times I &\longrightarrow \Sigma(Y \sqcup \{*\}) \\ ((y,t),0) &\mapsto (y,t) \\ (*_{\Sigma Y}, s) &\mapsto \begin{cases} (*_Y, \frac{1}{2} + s) & \text{for } s \in [0, \frac{1}{2}] \\ (*, 1 - (s - \frac{1}{2}) & \text{otherwise.} \end{cases} \end{aligned}$$

admits an extension up to homotopy to a map $F : \Sigma Y \times I \longrightarrow \Sigma(Y \sqcup \{*\})$, such that $F(-,0)\colon \Sigma Y \longrightarrow \Sigma(Y \sqcup \{*\})$ is the inclusion, and $F(*_{\Sigma Y}, 1) = *$.

Denote by $i_1\colon X \hookrightarrow X \rtimes Y$ the inclusion, and by $i_2\colon X \wedge Y \longrightarrow X \rtimes Y$ the composite

$$\begin{aligned} X \wedge Y \xrightarrow{\psi \wedge id} \Sigma\Omega X \wedge Y \cong \Omega X \wedge \Sigma Y &\xrightarrow{Id \wedge F(-,1)} \Omega X \wedge \Sigma(Y \sqcup \{*\}) \\ &\cong \Sigma\Omega X \wedge (Y \sqcup \{*\}) \xrightarrow{ev \wedge Id} X \wedge (Y \sqcup \{*\} = X \rtimes Y. \end{aligned}$$

Then the composites

$$\begin{aligned} &h \circ (\nabla \circ (i_1 \vee i_2))\colon X \vee (X \wedge Y) \longrightarrow X \rtimes Y \longrightarrow X \vee (X \wedge Y) \qquad \text{and} \\ &(\nabla \circ (i_1 \vee i_2)) \circ h\colon X \rtimes Y \longrightarrow X \vee (X \wedge Y) \longrightarrow X \rtimes Y \end{aligned}$$

are each homotopic to the identity by the universal property of the homotopy pushout. □

Proposition 1.11 ([23, Theorem 7.2.7]). *Suppose that we are given an inclusion of CW-complexes $A \longrightarrow X$ and a CW-complex W. Then the following sequence of sets of homotopy classes of maps*

$$\begin{aligned} \cdots \longrightarrow [\Sigma^n W, A] \xrightarrow{i} [\Sigma^n W, X] \xrightarrow{j} &[(C\Sigma^{n-1}W, \Sigma^{n-1}W), (X, A)] \\ &\xrightarrow{\partial} [\Sigma^{n-1}W, A] \longrightarrow \cdots \longrightarrow [W, X] \end{aligned} \tag{1.2}$$

is a long exact sequence, where i denotes the map induced by the inclusion, where j is the map induced by writing $[f] \in [\Sigma^n W, X]$ as the homotopy class of a map of pairs $[\widehat{f}] \in [(C\Sigma^{n-1}W, \Sigma^{n-1}W), (X, *)]$ and composing $\widehat{f}$ with the inclusion of pairs, and where ∂ is the map induced by sending $[g] \in [(C\Sigma^{n-1}W, \Sigma^{n-1}W), (X, A)]$ to the homotopy class of the restriction.

1.3 Category theory, colimits and homology colimtis

Recall that a category is small if its set of objects is small, and its set of morphisms is small.

Definition 1.12. Let $\mathcal{C}$ be a category and $\mathcal{J}$ be a small category. A diagram of shape $\mathcal{J}$ in $\mathcal{C}$ is a functor

$$\mathcal{D}\colon \mathcal{J} \longrightarrow \mathcal{C}.$$

Definition 1.13. For a diagram $\mathcal{D}\colon \mathcal{J} \longrightarrow \mathcal{C}$, a co-cone under $\mathcal{D}$ is an object $N \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\Phi = \{\phi_j\colon \mathcal{D}(j) \longrightarrow N\}_{j\in \mathrm{Ob}\,\mathcal{J}}$, such that for all pairs of objects $j_1, j_2 \in \mathrm{Ob}(\mathcal{J})$ and all morphisms $f\colon j_1 \longrightarrow j_2$, $\phi_{j_2} \circ \mathcal{D}(f) = \phi_{j_1}$.

A colimit of $\mathcal{D}$ is a co-cone (N', Φ') such that for any other co-cone (N, Φ), there exists a unique morphism $u\colon N \longrightarrow N'$ such that $u \circ \phi'_j = \phi_j$.

We now define the homotopy colimit. This requires significant preliminaries in setting up the required machinery. The reason for this is that intuitively, requiring only homotopy commutativity among maps in a diagram allows for situations where higher homotopies, or homotopies between homotopies, are ill-behaved. We require a notation of "Commutativity of all higher homotopies", which is precisely the notion of homotopy coherence which we now define. The definition of homotopy colimit then follows naturally.

We first define the homotopy coherent co-cone. For that, we define the simplicial resolution of a small category.

Definition 1.14. The simplex category Δ is the category whose objects are the non-empty sets $[n] = \{0, 1, \ldots, n\}$, and whose morphisms are the order-preserving functions between these sets.

Definition 1.15. A simplicial set is a contravariant functor $\Delta \longrightarrow \mathbf{Set}$.

Definition 1.16. A simplicially enriched category, or S-category, is a category where the set of morphisms between any two objects is a simplicial set.

We now define the simplicial resolution of a small category.

Recall that, given a categories $\mathcal{C}$ and $\mathcal{D}$ and functors $G\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $F\colon \mathcal{D} \longrightarrow \mathcal{C}$, if for all $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ there exists a natural bijection $\mathrm{Hom}_{\mathcal{C}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, GY)$, then F is called a left-adjoint to G, and G is called a right-adjoint to F.

We denote by **Cat** the category of small categories, and by **Grph** the category of directed graphs with distinguished loops at the vertices. We denote by

$U\colon \mathbf{Cat} \longrightarrow \mathbf{Grph}$ the obvious forgetful functor, and by $V\colon \mathbf{Grph} \longrightarrow \mathbf{Cat}$ the free functor which takes a directed graph to the category whose objects are vertices of the graph, and whose morphisms are paths between objects. Observe that V is the left-adjoint to U.

For a small category A, we denote by $F_i(A) = (VU)^{i+1}(A) \in \mathrm{Ob}(\mathbf{Cat})$. Denote by η the unit of the aforementioned adjunction, given by $\eta_G\colon G \longrightarrow UV(G)$ for $G \in \mathbf{Grph}$, and denote by ε the co-unit, given by $\varepsilon_A\colon VU(A) \longrightarrow A$. We denote by $\delta = V\eta U\colon VU \longrightarrow (VU)^2$. Then, for all $n \geq 0$, for all i such that $0 \leq i \leq n$, we denote by

$$d_i = F_{n-i}\varepsilon F_i(A)\colon F_{n+1}(A) \longrightarrow F_n(A), \qquad \text{and}$$
$$s_i = F_{n-i-1}\delta F_i\colon F_n(A) \longrightarrow F_{n+1}(A)$$

Definition 1.17. Let A be a small category. The simplicial resolution of A, denoted $S(A)$ is the simplicially enriched category obtained by inductively applying the aforementioned procedure to A.

We now define the notion of a homotopy coherent diagram in **Top**, noting that the definition is the same if replacing **Top** by any simplicially enriched category.

Definition 1.18. Let A be a small category. A homotopy coherent diagram of shape A in **Top** is a simplicially enriched functor $\mathcal{D}\colon S(A) \longrightarrow \mathbf{Top}$.

Definition 1.19. Let $\mathcal{D}\colon S(A) \longrightarrow \mathbf{Top}$ be a homotopy coherent diagram. The homotopy colimit of $\mathcal{D}$ is homotopy coherent co-cone N under $\mathcal{D}$ such that for any other homotopy coherent co-cone N', there exists a unique morphism up to homotopy $N \longrightarrow N'$ such that the result diagram is homotopy coherent.

1.4 Commutative and homological algebra

Here, we recall the necessary prerequisite machinery from commutative algebra for the later work in this thesis. We also clarify the precise definitions which we work with of the objects in question, in order to avoid later ambiguity.

All rings are assumed to be commutative and be equipped with a multiplicative identity. For a ring R, an R-module M is an abelian group equipped with an action which distributes over addition in M, and such that $1_R m = m$, and $(rs)m = r(sm)$ for all $r, s \in R, m \in M$. A module M is finitely generated if there exists a finite subset $S \subseteq M$ such that every element of M can be expressed as a finite linear combination of elements of S with coefficients in R.

Definition 1.20. The tensor product of R-modules M_1 and M_2, denoted $M_1 \otimes_R M_2$ is the quotient of the free R-module generated by the set $M_1 \times M_2$ by the relations

$$\begin{aligned}(m_1 + n_1, m_2) &= (m_1, m_2) + (n_1, m_2), \\ (m_1, m_2 + n_2) &= (m_1, m_2) + (m_1, n_2), \\ (rm_1, m_2) &= r(m_1, m_2) = (m_1, rm_2).\end{aligned} \tag{1.3}$$

for all $m_1, n_1 \in M_1$, $m_2, n_2 \in M_2$, and $r \in R$.

We denote elements of the tensor product by $m_1 \otimes m_2$.

An R-algebra A is an R-module equipped with an associative multiplication which distributes over addition in A, and such that if $r_1, r_2 \in R$, $a_1, a_2 \in A$, then $(r_1a_1)(r_2a_2) = (r_1r_2)(a_1a_2)$. A commutative algebra is an algebra whose multiplication is commutative. Under our standing assumption that all rings are commutative with unit, a ring is the same as an associative, commutative $\mathbb{Z}$-algebras with unit. An algebra A is finitely generated over R if there exists a finite subset $S \subseteq A$ such that every element of A can be expressed as a polynomial in elements of S with coefficients in R.

Definition 1.21. The tensor product of R-algebras A_1 and A_2, denoted $A_1 \otimes_R A_2$, is their tensor product as R-modules equipped with componentwise multiplication.

Given rings R and S, their direct product RS is the cartesian product of the underlying sets equipped with componentwise addition and multiplication. The tensor product $R \otimes S$ is their tensor product as $\mathbb{Z}$-algebras. For ideals $I, J \subseteq R$, we denote by $IJ = \{\sum_{k=1}^{n} i_k j_k \mid n \in N, i_k \in I, j_k \in J \text{ for } k = 1 \ldots, n\}$ their product, and by $I + J = \{i + j \mid i \in I, j \in J\}$ their sum.

The ring R is Noetherian if all ideals $I \subseteq R$ are finitely generated as R-modules. An ideal I is maximal if there does not exist an ideal J such that $I \subseteq J \subseteq R$ and $I \neq J \neq R$. R is local if there exists a unique maximal ideal.

Suppose that R is a local Noetherian ring, and let $M \neq 0$ be a finitely-generated R-module. The element $r \in R$ is M-regular if for all $m \in M$, $rm = 0$ implies that $m = 0$. An M-sequence is a sequence of elements $r_1, ..., r_n \in R$ such that $M/(r_1, ..., r_n)M \neq 0$, and r_i is an $M/(r_1, ..., r_{i-1})M$-regular element for all i.

Definition 1.22. For an ideal I of R, where $M \neq IM$, the grade of I on M is the common length of all maximal M-sequences in I. The depth of M is the grade of $\mathfrak{m}$ on M.

The depth of a ring R is the depth of R as an R-module.

For a prime ideal $P \subseteq R$, the localisation of R at P is $R_P = R \times D/\sim$, where $D = R \backslash P$ and $(r_1, d_1) \sim (r_2, d_2)$ if and only if $x(er - ds) = 0$ for some $x \in D$. Addition and multiplication in R_P are defined by $(r_1, d_1) + (r_2, d_2) = (r_1 d_2 + r_2 d_2, d_1 d_2)$ and $(r_1, d_1) \times (r_2, d_2) = (r_1 r_2, d_1 d_2)$ respectively. For an R-module M, the localisation M_P is the R_P-module defined similarly.

Definition 1.23. The Krull dimension of an R-module M, denoted $\dim M$, is the supremum of the lengths of strictly descending chains $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq ... \supsetneq \mathfrak{p}_t$, where for all i, $\mathfrak{p}_i$ is a prime ideal of R, and the localisation $M_{\mathfrak{p}_i}$ is non-zero. The Krull dimension of a ring R is its Krull dimension as an R-module.

Definition 1.24. A Cohen-Macaulay R-module M is a module such that $\operatorname{depth} M = \dim M$.

A short exact sequence of R-modules is a sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where $\ker f = 0$, $\operatorname{im} g = C$, and $\ker g = \operatorname{im} f$. The sequence is split if $B = A \oplus C$, and f and g are the inclusion and projection, respectively.

An R-module M is projective if every short exact sequence of R-modules of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$$

is isomorphic to a split exact sequence. A projective resolution of an R-module M is an exact sequence

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of projective R-modules.

For R-modules M and N, $\operatorname{Ext}^i_R(M, N)$ is the abelian group defined in the following way. Choose a projective resolution of M

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and replace the object M with 0, and the associated two maps with the zero map. Then, apply the functor $\operatorname{Hom}_R(-, B)$ to obtain

$$0 \longrightarrow \operatorname{Hom}(P_0, B) \longrightarrow \operatorname{Hom}(P_1, B) \longrightarrow \operatorname{Hom}(P_2, B) \longrightarrow \dots$$

and $\operatorname{Ext}^i_R(M, N)$ is defined to be the cohomology of this complex at position i.

Chapter 2

Duality in Toric Topology

2.1 Introduction

Duality phenomena are studied in areas of mathematics ranging from mathematical physics to algebraic geometry. They appear in many guises, in pairings of dual objects, in duality properties satisfied by certain families of objects, and in the realisability of dual versions of definitions and theorems. In the field of homotopy theory, a duality fundamental to the study of manifolds and their homotopy-theoretic analogues is Poincaré duality.

An integral Poincaré duality space X is one whose action of its integral cohomology algebra on its integral homology satisfies Poincaré duality. That is, there exists $n \in \mathbb{N}$ and $[\mu] \in H_n(X)$ such that the cap product

$$[\mu] \frown (-) \colon H^l(X) \longrightarrow H_{n-l}(X) \tag{2.1}$$

is an isomorphism for all l. Poincaré duality spaces include both manifolds and homology manifolds, but are a significantly broader family of spaces. For example, any space which is homotopy equivalent to a manifold is a Poincaré duality space. Moreover, the converse is not true in general, and there exist many Poincaré duality spaces which are not homotopy equivalent to manifolds. Phrased in different language, after Poincaré duality, there are in general further obstructions to a space being homotopy equivalent to a manifold. See e.g. [20].

In this chapter, we investigate an interplay of dualities in combinatorics, homotopy theory, and algebra, through an equivocation of Poincaré duality in the moment-angle complex $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ with Alexander duality in $\mathcal{K}$, a duality condition on the homology and cohomology groups of full subcomplexes of $\mathcal{K}$, and Gorenstein duality in $\mathbb{Z}[\mathcal{K}]$, a property for a d-dimensional Noetherian ring R that is measured by the functor $\operatorname{Ext}^{d-t}(-, R)$.

In Theorem 2.10, we show that Poincaré duality in the moment-angle complex $\mathcal{Z}_\mathcal{K}$ is equivalent to Alexander duality in the underlying simplicial complex $\mathcal{K}$. Fan and Wang's [11] work enables us to recognise Alexander duality complexes by directly inspecting a geometric realisation of $\mathcal{K}$, enabling the construction of many examples. Combining our result with work of Stanley [24], and the aforementioned results of Fan and Wang, in Theorem 2.14 we show that Poincaré duality in $\mathcal{Z}_\mathcal{K}$ is equivalent to Gorenstein duality in $\mathbb{Z}[\mathcal{K}]$, realising a connection between algebraic, combinatorial and topological dualities. This complements the result proven by Buchstaber and Panov [8, Theorem 4.6.8] in the case of coefficients over a field.

Combining our results with Cai's [9] classification of moment-angle manifolds, we obtain as a corollary that any Poincaré duality moment-angle complex is homotopy equivalent to a manifold. This result distinguishes moment-angle complexes from general spaces from a surgery-theoretic point of view; since any finite Poincaré duality space is the base space of a Spivak normal fibration, and for a manifold this fibration comes from a vector bundle, this implies that the Spivak normal fibration associated to Poincaré duality moment-angle complex is fibre homotopy equivalent to a vector bundle. We apply this in constructing a descending series of manifolds embedded in odd spheres, corresponding to a descending series of Gorenstein quotients of the polynomial ring, controlled by the geometry of the associated simplicial complex.

It is natural to ask to which polyhedral products our aforementioned results extend. We address this question next, extending our characterisation of Poincaré duality in $\mathcal{Z}_\mathcal{K}$ via the polyhedral join product of simplicial complexes [26]. As a cartesian product of simplicial complexes is not a simplicial complex, a polyhedral product $(\boldsymbol{\mathcal{K}_i}, \boldsymbol{\mathcal{L}_i})^\mathcal{K}$ of pairs $(\mathcal{K}_i, \mathcal{L}_i)$ of geometric realisations of simplicial complexes is not itself the geometric realisation of a simplicial complex. The polyhedral join product is constructed as a union of join products of simplicial complexes. In Theorem 2.18, we characterise a family of polyhedral products which satisfy Poincaré duality, in terms of Alexander duality of the polyhedral join.

These results motivate the study of a new combinatorial construction in commutative algebra, of which the Stanley-Reisner rings of polyhedral join products are a special case. For a simplicial complex $\mathcal{K}$ on vertex set $[m]$, and an m-tuple of pairs (R_i, I_i) of rings R_i and ideals $I_i \subseteq R_i$, we define the polyhedral ring product $(\underline{R}, \underline{I})$ to be a quotient of the tensor product $\bigotimes_\mathbb{Z} R_i$ by an ideal constructed from the ideals I_i according to the minimal missing faces of $\mathcal{K}$. In Section 2.6 we introduce the polyhedral ring product, and explore its basic properties. We show that the Stanley-Reisner ring of the polyhedral join product arises as a special case, and exploit this connection to construct Gorenstein rings as polyhedral ring products of rings which are themselves non-Gorenstein.

This section is based on the joint work G-S [15].

2.2 The moment-angle complex $\mathcal{Z}_{\mathcal{K}}$

We begin by defining the polyhedral product, of which the moment-angle complex is a special case. Recall that for a positive integer m, a simplicial complex $\mathcal{K}$ on the vertex set $[m] = \{1, ..., m\}$ is a subset of $2^{[m]}$ which is closed under taking subsets, and where $\emptyset \in \mathcal{K}$. We do not assume that $\mathcal{K}$ contains the elements of its vertex set $[m]$.

Definition 2.1. Let $\mathcal{K}$ be a simplicial complex on vertex set $[m]$, and denote by $(\mathbf{X}, \mathbf{A}) = \{(X_i, A_i)\}_{i=1}^m$ an m-tuple of CW-pairs. The polyhedral product is defined as

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}} (\mathbf{X}, \mathbf{A})^{\sigma} \subseteq \prod_{i=1}^{m} X_i, \text{ where } (\mathbf{X}, \mathbf{A})^{\sigma} = \prod_{i=1}^{m} Y_i, \; Y_i = \begin{cases} X_i & \text{for } i \in \sigma \\ A_i & \text{for } i \notin \sigma. \end{cases} \tag{2.2}$$

If $(X_i, A_i) = (X, A)$ for all i, we denote the polyhedral product by $(X, A)^{\mathcal{K}}$. When $(X_i, A_i) = (D^2, S^1)$ for all i, the polyhedral product is denoted by $\mathcal{Z}_{\mathcal{K}}$, and referred to as the moment-angle complex on $\mathcal{K}$.

The polyhedral smash product, denoted $(\mathbf{X}, \mathbf{A})^{\wedge\mathcal{K}}$, is obtained by replacing the cartesian product in the definition of the polyhedral product with the smash product.

Definition 2.2. Let $\mathcal{K}$ be a simplicial complex on vertex set $[m]$, and denote by $(\mathbf{X}, \mathbf{A}) = \{(X_i, A_i)\}_{i=1}^m$ an m-tuple of CW-pairs. The polyhedral smash product is defined as

$$(\mathbf{X}, \mathbf{A})^{\wedge\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}} (\mathbf{X}, \mathbf{A})^{\wedge\sigma} \subseteq \bigwedge_{i=1}^{m} X_i, \text{ where } (\mathbf{X}, \mathbf{A})^{\sigma} = \bigwedge_{i=1}^{m} Y_i, \; Y_i = \begin{cases} X_i & \text{for } i \in \sigma \\ A_i & \text{for } i \notin \sigma. \end{cases} \tag{2.3}$$

Before giving some examples, we state three key homotopy-theoretic results which we use throughout this thesis.

The face-category $\mathrm{CAT}(\mathcal{K})$ is the category whose objects are the simplices of $\mathcal{K}$, and whose morphisms are the inclusions of simplices. The definition of the polyhedral product, recast in category theoretic terms, is

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}) \tag{2.4}$$

where $\mathcal{D}_{\mathcal{K}}(\mathbf{X}, \mathbf{A})$ is the diagram

$$\mathcal{D}_{\mathcal{K}}(\mathbf{X}, \mathbf{A})\colon \mathrm{CAT}(\mathcal{K}) \longrightarrow \mathrm{TOP}_{CW}, \qquad \sigma \mapsto (\mathbf{X}, \mathbf{A})^{\sigma} \tag{2.5}$$

where inclusions of simplices $\tau \hookrightarrow \sigma$ are mapped to inclusions $(\mathbf{X}, \mathbf{A})^{\tau} \hookrightarrow (\mathbf{X}, \mathbf{A})^{\sigma}$.

Proposition 2.3 ([8, Proposition 8.1.1]). *If for all $i \in [m]$, the inclusion $A_i \hookrightarrow X_i$ is a cofibration, then there is a homotopy equivalence*

$$\operatorname{hocolim} \mathcal{D}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}) \simeq \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}) \tag{2.6}$$

Proposition 2.4 ([8, Lemma 8.2.3]). *Let Q denote the homotopy colimit of the diagram*

$$\begin{array}{ccc} A \times B & \xrightarrow{f \times Id_B} & C \times B \\ {\scriptstyle Id_A \times g}\downarrow & & \\ A \times D & & \end{array}$$

where the maps $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow D$ are trivial. Then

$$Q \simeq (A * B) \vee (C \rtimes B) \vee (A \ltimes D). \tag{2.7}$$

In the following proposition, for $J \subseteq [m]$, we denote by $(\mathbf{X}_J, \mathbf{A}_J)$ the tuple $\{(X_j, A_j)\}_{j \in J}$.

Proposition 2.5 ([8, Theorem 8.3.2]). *There is a natural homotopy equivalence*

$$\Sigma(\mathbf{X}, \mathbf{A})^{\mathcal{K}} \simeq \Sigma \left(\bigvee_{J \subseteq [m]} (\mathbf{X}_J, \mathbf{A}_J)^{\wedge \mathcal{K}} \right) \tag{2.8}$$

The following examples demonstrate how enhancing the usual homotopy-theoretic techniques with our understanding of the underlying combinatorial structure of $\mathcal{Z}_{\mathcal{K}}$ enables us to make explicit computations.

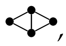
Example 2.6.

1. *If $\mathcal{K} =$ , then*

$$\begin{aligned} (D^2, S^1)^{\{1,2\}} &= D^2 \times D^2 \times S^1, \\ (D^2, S^1)^{\{1,3\}} &= D^2 \times S^1 \times D^2, \\ (D^2, S^1)^{\{2,3\}} &= S^1 \times D^2 \times D^2 \end{aligned}$$

so that

$$\mathcal{Z}_{\mathcal{K}} = D^2 \times D^2 \times S^1 \cup D^2 \times S^1 \times D^2 \cup S^1 \times D^2 \times D^2 = \partial D^6 = S^5.$$

2. *If* $\mathcal{K} =$ , *then*

$$\begin{aligned}
\mathcal{Z}_{\mathcal{K}} &= (D^2 \times D^2 \times S^1 \times S^1) \cup (D^2 \times S^1 \times D^2 \times S^1) \cup (S^1 \times D^2 \times D^2 \times S^1) \\
&\qquad \cup (S^1 \times D^2 \times S^1 \times D^2) \cup (S^1 \times S^1 \times D^2 \times D^2) \\
&= ((D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2)) \times S^1 \\
&\qquad \cup S^1 \times ((D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2)) \\
&= S^5 \times S^1 \cup S^1 \times S^5.
\end{aligned}$$

This is the pushout of the diagram

$$\begin{array}{ccc}
S^1 \times D^2 \times D^2 \times S^1 & \hookrightarrow & S^5 \times S^1 \\
\big\downarrow & & \\
S^1 \times S^5 & &
\end{array}$$

and is homotopy equivalent to the homotopy pushout of the same diagram, by Proposition 2.3. Therefore, $\mathcal{Z}_{\mathcal{K}}$ *is homotopy equivalent to the homotopy pushout of the diagram*

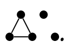
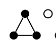
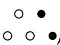
$$\begin{array}{ccc}
S^1 \times S^1 & \xrightarrow{* \times Id} & S^5 \times S^1 \\
\big\downarrow{\scriptstyle Id \times *} & & \\
S^1 \times S^5 & &
\end{array}$$

so that by Proposition 2.4,

$$\mathcal{Z}_{\mathcal{K}} \simeq (S^1 * S^1) \vee (S^5 \rtimes S^1) \vee (S^1 \ltimes S^5)$$

and since S^5 *is a co-H-space, it follows from Proposition 1.10 that*

$$\mathcal{Z}_{\mathcal{K}} \simeq S^3 \vee S^5 \vee S^5 \vee S^6 \vee S^6.$$

3. *Let* $\mathcal{K} =$ . *Then,* $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ *where* $\mathcal{K}_1 =$  *and* $\mathcal{K}_2 =$ , *so that*

$$\mathcal{Z}_{\mathcal{K}} = \mathcal{Z}_{\mathcal{K}_1} \cup \mathcal{Z}_{\mathcal{K}_2} = (S^5 \times T^2) \bigcup_{T^3 \times T^2} (T^3 \times S^3)$$

Then, since $T^3 \hookrightarrow S^5$ *and* $T^2 \hookrightarrow S^3$ *are trivial, it follows from Proposition 1.8 and Proposition 2.4 that*

$$\begin{aligned}
\mathcal{Z}_{\mathcal{K}} &\simeq (T^3 * T^2) \vee (S^5 \rtimes T^2) \vee (T^3 \ltimes S^3) \\
&\simeq (T^3 * T^2) \vee S^3 \vee 3S^4 \vee 4S^5 \vee 5S^6 \vee S^7 \\
&\simeq (T^3 \wedge (S^2 \vee S^2 \vee S^3)) \vee S^3 \vee 3S^4 \vee 4S^5 \vee 5S^6 \vee S^7 \\
&\simeq \Sigma^2 T^3 \vee \Sigma^2 T^3 \vee \Sigma^3 T^3 \vee S^3 \vee 3S^4 \vee 4S^5 \vee 5S^6 \vee S^7 \\
&\simeq 7S^3 \vee 12S^4 \vee 9S^5 \vee 6S^6 \vee S^7
\end{aligned}$$

2.3 Poincaré duality of $\mathcal{Z}_{\mathcal{K}}$

In this section we turn our focus to the aim of this chapter, an investigation of duality in $\mathcal{Z}_{\mathcal{K}}$. We begin with a description of the integral cellular cochain algebra $C^*(\mathcal{Z}_{\mathcal{K}};\mathbb{Z})$ due to Panov and Buchstaber [8, Section 4.4].

Let $\mathbb{D}^m$ denote the m-dimensional unit ball in $\mathbb{C}^m$. The disk $\mathbb{D}^1$ admits a decomposition into 3 cells: the basepoint $*$, the boundary circle S, and the 2-cell D.

Taking products, we obtain a cellular decomposition of $\mathbb{D}^m$. A cell e of $\mathbb{D}^m$ is a product of cells of $\mathbb{D}$ of the form $\prod_{i=1}^m Y_i$, where for each i, Y_i is either $*$, S or D. Phrasing this in terms of subsets of $[m]$, cells of $\mathbb{D}^m$ are in one-to-one correspondence with pairs (J, I) of subsets $J, I \subseteq [m]$, with $J \cap I = \emptyset$. The subset J corresponds to the 1-cells in the product, I to the 2-cells, and $[m]\backslash(J \cup I)$ to the 0-cells. We denote the cell corresponding to the pair (J, I) by $\kappa(J, I)$. The dimension of such a cell is $\dim \kappa(J, I) = |J| + 2|I|$.

This CW-structure on $\mathbb{D}^m$ induces a CW-structure on $\mathcal{Z}_{\mathcal{K}} \subseteq \mathbb{D}^m$; for $J \subseteq [m]$, we denote by $\mathcal{K}_J = \{\sigma \in \mathcal{K} \mid \sigma \subseteq J\}$ the full subcomplex of $\mathcal{K}$ on J. Then, $\kappa(J\backslash\sigma, \sigma)$ is a cell of $\mathcal{Z}_{\mathcal{K}}$ if and only if $\sigma \in \mathcal{K}_J$

We denote by $C_*(\mathcal{K})$ and $C_*(\mathcal{Z}_{\mathcal{K}})$ the simplicial and cellular chain complexes of $\mathcal{K}$ and $\mathcal{Z}_{\mathcal{K}}$, respectively. Here and throughout, we assume that cellular chain and cochain complexes of $\mathcal{K}$ are augmented; $C_{-1}(\mathcal{K}) \cong C^{-1}(\mathcal{K}) \cong \langle \emptyset_* \rangle \cong \langle \emptyset^* \rangle = \mathbb{Z}$, where $\partial(\{i\}_*) = \emptyset_*$ and $d(\emptyset^*) = \sum_{\{i\}\in\mathcal{K}} \{i\}^*$ for all $\{i\} \in \mathcal{K}$. There is an isomorphism of graded modules

$$h_* : \bigoplus_{J\subseteq[m]} C_*(\mathcal{K}_J) \xrightarrow{\cong} C_*(\mathcal{Z}_{\mathcal{K}}), \quad \sigma \mapsto \kappa(J\backslash\sigma, \sigma) \tag{2.9}$$

where the grading on the left hand side is given by $\deg \sigma = 2|\sigma| + |J|$ for $\sigma \in \mathcal{K}_J$.

By theorems of Hochster, Baskakov, Panov and Buchstaber [8, Theorem 4.5.7], the map h in (2.9) induces the isomorphism of cochain algebras

$$h^* : C^*(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{J\subseteq[m]} C^*(\mathcal{K}_J) \tag{2.10}$$

where the ring structure on the right hand side is induced by the Baskakov product

$$\begin{aligned} C^{p-1}(\mathcal{K}_I) \otimes C^{q-1}(\mathcal{K}_J) &\to C^{p+q-1}(\mathcal{K}_{I\cup J}), \\ \sigma^* \otimes \tau^* &\mapsto \begin{cases} (-1)^{\delta}(\sigma \cup \tau)^* & \text{if } I \cap J = \emptyset, \quad \sigma \cup \tau \in \mathcal{K}_{I\cup J} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.11}$$

where $(\sigma \cup \tau)^*$ denotes the cochain dual to $(\sigma \cup \tau)_* \in C_*(\mathcal{K}_{I \cup J})$, and where $\delta = q|I| + \sum_{i \in I} |\{j \in J \mid j < i\}|$.

As a corollary, we obtain the following.

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J \subseteq [m]} H(C^*(\mathcal{K}_J)) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}^*(\mathcal{K}_J). \tag{2.12}$$

The decomposition (2.12) describes both the cohomology groups and the cup product structure in the cohomology of moment-angle complexes in purely combinatorial terms. We demonstrate the utility of this result in Example 2.8, where upon recognizing the space in question as a moment-angle complex, we read off the algebraic structure of its cohomology ring. We first state the following result, which enables us to recognise the moment-angle complex therein.

Theorem 2.7 ([7, Theorem 6.3]). *Let $\mathcal{K}$ be be boundary of an m-gon for $m \geq 4$. Then*

$$\mathcal{Z}_\mathcal{K} \cong \#_{k=3}^{m-1} (S^k \times S^{m+2-k})^{\#(k-2)\binom{m-2}{k-1}} \tag{2.13}$$

Example 2.8. 1. *Let $\mathcal{K}$ be the boundary of a 5-gon. By Theorem 2.7, $\mathcal{Z}_\mathcal{K} \cong \#_{i=1}^5 (S^3 \times S^4)$. Then by (2.12), $H^*(\mathcal{Z}_\mathcal{K})$ is the graded algebra with generators $\{u_i\}_{i=1}^5$ of degree two and $\{v_i\}_{i=1}^5$ of degree three, where all pairwise products are trivial other than $u_i v_i$ for $i = 1, \ldots, 5$, and where $u_i v_i = u_j v_j$ for all $i \neq j$.*

This example demonstrates that not only can the study of the moment-angle complex $\mathcal{Z}_\mathcal{K}$ be applied to moment-angle complexes which we construct, but the philosophy can be, in a loose sense, reversed; recognising well known spaces as moment-angle complexes enables us to effectively compute their invariants.

Poincaré duality in a topological space X is not solely a property of the algebra $H^*(X)$, but of its action on the module $H_*(X)$. The cap product, the action in question, is the operation induced on the homological level by the dual of the cochain-level cup product.

For fixed $\phi \in C^*(X)$, the dual of the cup product $(-) \smile \phi \colon C^*(X) \longrightarrow C^*(X)$ is the map

$$(-) \frown \phi \colon C_*(X) \longrightarrow C_*(X), \qquad \alpha \mapsto \alpha \frown \phi \tag{2.14}$$

where $\alpha \frown \phi \in C_*(X)$ is defined by $\langle \alpha, \phi \smile \psi \rangle = \langle \alpha \frown \phi, \psi \rangle$, where $\langle -, - \rangle$ denotes the evaluation pairing.

For $d \in \mathbb{N}$, a CW-complex X is an *n-Poincaré duality space* if there exists a class $[\mu] \in H_n(X)$ such that the cap product

$$[\mu] \frown (-) \colon H^l(X) \to H_{n-l}(X)$$

is an isomorphism for all l. Here, n is referred to as the Poincaré duality-dimension of X, and $[\mu]$ is referred to as the fundamental class.

We reframe statements about Poincaré duality of $\mathcal{Z}_\mathcal{K}$ in terms of duality of the simplicial chains and cochains of full subcomplexes of $\mathcal{K}$, by using the decomposition of the structure of the cellular chains and cochains of $\mathcal{Z}_\mathcal{K}$ (2.10). We first describe the cap product in $\mathcal{Z}_\mathcal{K}$ on the cellular level, in combinatorial terms, and then exploit this description to obtain a characterisation of Poincaré duality of $\mathcal{Z}_\mathcal{K}$ in terms of $\mathcal{K}$, and also in terms of the Stanley-Reisner ring $\mathbb{Z}[\mathcal{K}]$.

Proposition 2.9. *Let $\kappa(J\backslash\sigma,\sigma) \in C_*(\mathcal{Z}_\mathcal{K})$ and $\kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^* \in C^*(\mathcal{Z}_\mathcal{K})$, corresponding to simplices $\sigma \in \mathcal{K}_J$ and $\widehat{\sigma} \in \mathcal{K}_{\widehat{J}}$, respectively. Then the cap product is given by*

$$\kappa(J\backslash\sigma,\sigma) \frown \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^* = \begin{cases} 0 & \widehat{J} \nsubseteq J \\ 0 & \widehat{\sigma} \nsubseteq \sigma \\ \kappa\left((J\backslash\sigma)\backslash(\widehat{J}\backslash\widehat{\sigma}),\sigma\backslash\widehat{\sigma}\right) & \textit{otherwise.} \end{cases}$$

Proof. For any CW-complex, the cellular chain-level cup $\smile$ and cap $\frown$ products satisfy

$$\langle \alpha, \phi \smile \psi \rangle = \langle \alpha \frown \phi, \psi \rangle$$

for $\alpha \in C_{k+l}(\mathcal{Z}_\mathcal{K}), \phi \in C^l(\mathcal{Z}_\mathcal{K}), \psi \in C^k(\mathcal{Z}_\mathcal{K})$, where $\langle -,- \rangle$ denotes the evaluation pairing.

Let $\kappa(J\backslash\sigma,\sigma) \in C_*(\mathcal{Z}_\mathcal{K})$ and $\kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^* \in C^*(\mathcal{Z}_\mathcal{K})$. We write the cap product $\kappa(J\backslash\sigma,\sigma) \frown \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^*$ in terms of generators $C_*(\mathcal{Z}_\mathcal{K})$. For $L \subseteq [m]$ and $\tau \in \mathcal{K}_L$, the coefficient of a generator $\kappa(L\backslash\tau,\tau) \in C_*(\mathcal{Z}_\mathcal{K})$ in $\kappa(J\backslash\sigma,\sigma) \frown \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^*$ is given by $\langle(\kappa(J\backslash\sigma,\sigma) \frown \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^*), \kappa(L\backslash\tau,\tau)^*\rangle = \langle(\kappa(J\backslash\sigma,\sigma), \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^* \smile \kappa(L\backslash\tau,\tau)^*\rangle$.
Now,

$$\langle(\kappa(J\backslash\sigma,\sigma), \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^* \smile \kappa(L\backslash\tau,\tau)^*\rangle \neq 0$$

is equivalent to

$$(J\backslash\sigma,\sigma) = ((\widehat{J}\backslash\widehat{\sigma}) \cup (L\backslash\tau), \widehat{\sigma} \cup \tau).$$

Thus $\kappa(J\backslash\sigma,\sigma) \frown \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^*$ is non-trivial if and only if $\widehat{J}\backslash\widehat{\sigma} \subseteq J\backslash\sigma$ and $\widehat{\sigma} \subseteq \sigma$, whence $\kappa(J\backslash\sigma,\sigma) \frown \kappa(\widehat{J}\backslash\widehat{\sigma},\widehat{\sigma})^* = \kappa((J\backslash\sigma)\backslash(\widehat{J}\backslash\widehat{\sigma}),\sigma\backslash\widehat{\sigma})$.

□

We show that n-Poincaré duality spaces $\mathcal{Z}_\mathcal{K}$ are characterised by a duality in $\mathcal{K}$ referred to as combinatorial n-Alexander duality, which is an equivalent condition to being an n-dimensional generalised homology sphere (GHS^n); a space X is a GHS^n if it is a homology n-manifold with the homology of S^n. Fan and Wang [11, Theorem 3.4] showed that for $\mathcal{K}$ a simplicial complex of dimension n on vertex set $[m]$, $\mathcal{K}$ is a GHS^n if and only if

$$\widetilde{H}^l(\mathcal{K}_J) \cong \widetilde{H}_{n-l-1}(\mathcal{K}_{[m]\backslash J})$$

for all $J \subseteq [m]$, $0 \leq l \leq n$. In this case we say that $\mathcal{K}$ has n-dimensional combinatorial Alexander duality.

We are now ready to give our combinatorial classification of Poincaré duality moment-angle complexes.

Theorem 2.10. *Let $\mathcal{K}$ be a simplicial complex on $[m]$ with non-trivial cohomology. Then $\mathcal{Z}_{\mathcal{K}}$ is an $(n+m)$-Poincaré duality space if and only if $\mathcal{K}$ satisfies $(n-1)$-dimensional combinatorial Alexander duality.*

Proof. The sufficient implication is settled by a result of Cai [9, Corollary 2.10]; if $\mathcal{K}$ satisfies $(n-1)$-dimensional combinatorial Alexander duality and has non-trivial cohomology, then $\mathcal{Z}_{\mathcal{K}}$ is an $(n+m)$-dimensional manifold.

We show the necessary implication. Let $\mathcal{K}$ be a simplicial complex on $[m]$, with non-trivial cohomology, and suppose that $\mathcal{Z}_{\mathcal{K}}$ is an $(n+m)$-Poincaré duality space. We show first that $\mathcal{K}$ has the homology of S^{n-1}. We subsequently utilise this fact in showing that a certain chain is a representative of the fundamental class $[\mu] \in H_{n+m}(\mathcal{Z}_{\mathcal{K}})$.

As the simplicial complex $\mathcal{K}$ has non-trivial cohomology, for some l, there exists $0 \neq [\tau] \in \tilde{H}^l(\mathcal{K})$. Let $\tau = \sum_j \alpha_j \tau_j^*$, where $\tau_j^* \in C^l(\mathcal{K})$ are basis cochains, corresponding to simplices τ_j. We show that l must equal $n-1$.

The image of $[\tau]$ under isomorphism (2.12) is the class

$$h^*([\tau]) = \left[\sum_j A_j \kappa([m]\backslash\tau_j, \tau_j)^*\right] \in H^{l+m+1}(\mathcal{Z}_{\mathcal{K}})$$

where $A_j = \operatorname{sgn}(\tau_j, [m])\alpha_j$.

Let $0 \neq [\mu] \in H_{n+m}(\mathcal{Z}_{\mathcal{K}})$ denote the fundamental class, represented by $\mu = \sum_i a_i \kappa(J_i\backslash\sigma_i, \sigma_i)$. We evaluate the product $0 \neq [\mu] \frown h^*([\tau]) = \left[\sum_{i,j} a_i A_j \left(\kappa(J_i\backslash\sigma_i, \sigma_i) \frown \kappa([m]\backslash\tau_j, \tau_j)^*\right)\right] \in H_{n-(l+1)}(\mathcal{Z}_{\mathcal{K}})$. By Proposition 2.9,

$$\kappa(J_i\backslash\sigma_i, \sigma_i) \frown \kappa([m]\backslash\tau_j, \tau_j)^* \neq 0$$

implies that

$$\tau_j \subseteq \sigma_i, \text{ and } [m]\backslash\tau_j \subseteq J_i\backslash\sigma_i$$

and therefore

$$J_i = [m] \text{ and } \tau_j = \sigma_i.$$

Thus, the non-triviality of $\kappa(J_i\backslash\sigma_i,\sigma_i)\frown\kappa([m]\backslash\tau_j,\tau_j)^*$ implies that

$$\kappa(J_i\backslash\sigma_i,\sigma_i)\frown\kappa([m]\backslash\tau_j,\tau_j)^* = \kappa(\emptyset,\emptyset).$$

Therefore

$$\begin{aligned}[\mu]\frown h^*([\tau]) &= \left[\sum_{i,j}\operatorname{sgn}(\tau_j,[m])a_i\alpha_{\tau_j}\left(\kappa(J_i\backslash\sigma_i,\sigma_i)\frown\kappa([m]\backslash\tau_j,\tau_j)^*\right)\right]\\ &= [A\kappa(\emptyset,\emptyset)]\in H_0(\mathcal{Z}_\mathcal{K})\end{aligned}$$

where $A\neq 0$. It follows that $h^*([\tau])\in H^{n+m}(\mathcal{Z}_\mathcal{K})$, so that $[\tau]\in\tilde{H}^{n-1}(\mathcal{K})$ by the definition of the isomorphism h^*.

We have obtained that the $(n-1)$-st cohomology group of $\mathcal{K}$ is the only non-trivial cohomology group. It remains to show that $\tilde{H}^{n-1}(\mathcal{K})\cong\mathbb{Z}$. By Poincaré duality, we have that $H^{n+m}(\mathcal{Z}_\mathcal{K})\cong H_0(\mathcal{Z}_\mathcal{K})\cong\mathbb{Z}$. By (2.12), $\tilde{H}^{n-1}(\mathcal{K})$ includes into $H^{n+m}(\mathcal{Z}_\mathcal{K})\cong\mathbb{Z}$ as a component of a direct sum. It follows that $h^*\colon H^{n+m}(\mathcal{Z}_\mathcal{K})\to\tilde{H}^{n-1}(\mathcal{K})\cong\mathbb{Z}$ is an isomorphism. Therefore $\mathcal{K}$ has the homology of S^{n-1}, as claimed. It follows that the fundamental class $[\mu]\in H_{n+m}(\mathcal{Z}_\mathcal{K})$ can be represented by a cellular chain of the form $\mu=\sum_i a_i\kappa([m]\backslash\sigma_i,\sigma_i)$.

We now show that $\mathcal{K}$ has combinatorial Alexander duality, that is, for any $J\subseteq[m]$, and $0\leq l\leq n+m$,

$$\tilde{H}^l(\mathcal{K}_J)\cong\tilde{H}_{n-l-2}(\mathcal{K}_{[m]\backslash J}).$$

By Poincaré duality, we have the sequence of isomorphisms

$$\bigoplus_{J\subseteq[m]}\tilde{H}^{\hat{l}-|J|-1}(\mathcal{K}_J)\cong H^{\hat{l}}(\mathcal{Z}_\mathcal{K})\cong H_{n+m-\hat{l}}(\mathcal{Z}_\mathcal{K})\cong\bigoplus_{L\subseteq[m]}\tilde{H}_{(n+m-\hat{l})-|L|-1}(\mathcal{K}_L)$$

given by the composition $(h_*)^{-1}\circ([\mu]\frown(-))\circ h^*$, where $[\mu]\in H_{n+m}(\mathcal{Z}_\mathcal{K})$ denotes the fundamental class of $\mathcal{Z}_\mathcal{K}$. Substituting $\widehat{l}=l+|J|+1$,
$\bigoplus_{J\subseteq[m]}\tilde{H}^l(\mathcal{K}_J)\cong H^{l+|J|+1}(\mathcal{Z}_\mathcal{K})\cong H_{n+m-(l+|J|+1)}(\mathcal{Z}_\mathcal{K})\cong\bigoplus_{L\subseteq[m]}\tilde{H}_{n+m-l-|J|-|L|-2}(\mathcal{K}_L)$.
Denoting the composite isomorphism by Φ, we show that Φ respects the direct sum decompositions. In particular, for all $J\subseteq[m]$,

$$\Phi(\tilde{H}^l(\mathcal{K}_J))\subseteq\tilde{H}_{n+m-l-|J|-(m-|J|)-2}(\mathcal{K}_{[m]\backslash J})=\tilde{H}_{n-l-2}(\mathcal{K}_{[m]\backslash J}).$$

Suppose that for $J\subseteq[m]$, $\mathcal{K}_J$ has non-trivial cohomology. Otherwise, the statement follows vacuously. Let $0\neq[\tau]\in\tilde{H}^l(\mathcal{K}_J)$ with representative cochain $\tau=\sum_j\alpha_j\tau_j^*$. Then

$$h^*([\tau])=\left[\sum_j\operatorname{sgn}(\tau_j,J)\alpha_j\kappa(J\backslash\tau_j,\tau_j)^*\right]\in H^{l+|J|+1}(\mathcal{Z}_\mathcal{K}).$$

Let $[\mu]$ denote the fundamental class of $\mathcal{Z}_\mathcal{K}$ with representative chain $\mu = \sum_i a_i \kappa([m]\backslash\sigma_i, \sigma_i)$. Evaluating the cap product gives

$$\begin{aligned} [\mu] \frown h_c([\tau]) &= \left[\sum_{i,j} \mathrm{sgn}(\tau_j, J) a_i \alpha_j \kappa([m]\backslash\sigma_i, \sigma_i) \frown \kappa(J\backslash\tau_j, \tau_j)^*\right] \\ &= \left[\sum_{\hat{i},\hat{j}} A_{\hat{i},\hat{j}} \kappa([m]\backslash\sigma_{\hat{i}}, \sigma_{\hat{i}}) \frown \kappa(J\backslash\tau_{\hat{j}}, \tau_{\hat{j}})^*\right] \\ &= \left[\sum_{\hat{i},\hat{j}} A_{\hat{i},\hat{j}} \kappa(([m]\backslash\sigma_{\hat{i}})\backslash(J\backslash\tau_{\hat{j}}), \sigma_{\hat{i}}\backslash\tau_j)\right] \\ &= \left[\sum_{\hat{i},\hat{j}} A_{\hat{i},\hat{j}} \kappa(([m]\backslash J)\backslash(\sigma_{\hat{i}}\backslash\tau_{\hat{j}}), \sigma_{\hat{i}}\backslash\tau_j)\right] \in H_{n+m-(l+|J|+1)}(\mathcal{Z}_\mathcal{K}) \end{aligned}$$

where $\hat{i}, \hat{j}$ are the pairs for which the cap product $\kappa([m]\backslash\sigma_{\hat{i}}, \sigma_{\hat{i}}) \frown \kappa(J\backslash\tau_{\hat{j}}, \tau_{\hat{j}})^*$ is non-trivial, and $A_{\hat{i},\hat{j}} = \mathrm{sgn}(\tau_j, J) a_i \alpha_j \neq 0$. The last equality follows since both J and σ_i contain τ_j, and $J \cap \sigma_i = \tau_j$ since the cap product is non-trivial.

The image of $[\mu] \frown h^*([\tau])$ under the inverse of the homology isomorphism of (2.12) is

$$(h_*)^{-1}([\mu] \frown h_c([\tau]) = \left[\sum_{\hat{i},\hat{j}} A_{\hat{i},\hat{j}} \sigma_{\hat{i}}\backslash\tau_{\hat{j}}\right] \in \tilde{H}_{n-l-2}(\mathcal{K}_{[m]\backslash J}).$$

We therefore have that under the composition of isomorphisms

$$(h_h)^{-1} \circ ([\mu] \frown (-)) \circ h_c \colon \bigoplus_{J\subseteq[m]} \tilde{H}^{l-|J|-1}(\mathcal{K}_J) \to \bigoplus_{L\subseteq[m]} \tilde{H}_{(n+m-l)-|L|-1}(\mathcal{K}_L)$$

the image of each of the groups $\tilde{H}^l(\mathcal{K}_J)$ is contained in $\tilde{H}_{n-l-2}(\mathcal{K}_{[m]\backslash J})$. These groups are therefore isomorphic, and $\mathcal{K}$ therefore satisfies $(n-1)$-dimensional combinatorial Alexander duality. □

Corollary 2.11. *Let $\mathcal{K}$ be a simplicial complex on $[m]$ with non-trivial cohomology. Then the following are equivalent:*

1. *$\mathcal{Z}_\mathcal{K}$ is an $(n+m)$-Poincaré duality space over $\mathbb{Z}$*
2. *$\mathcal{K}$ has $(n-1)$-dimensional combinatorial Alexander duality*
3. *$\mathcal{Z}_\mathcal{K}$ is an $(n+m)$-dimension manifold.*

We defer examples until the end of the next section.

2.4 Gorenstein duality of $\mathbb{Z}[\mathcal{K}]$

In this section, we relate Gorenstein duality of Stanley-Reisner rings of simplicial complexes to Poincaré duality of moment-angel complexes $\mathcal{Z}_{\mathcal{K}}$.

Definition 2.12. Let $\mathcal{K}$ be a simplicial complex on vertex set $[m]$, and R a commutative ring. The Stanley-Reisner ring is

$$R[\mathcal{K}] = R[v_1, ..., v_m]/\mathcal{I}_{\mathcal{K}}$$

where

$$\mathcal{I}_{\mathcal{K}} = (v_{i_1}...v_{i_j} \mid \{i_1, ..., i_j\} \notin \mathcal{K})$$

is the Stanley-Reisner ideal, that is, the ideal generated by monomials corresponding to missing faces of $\mathcal{K}$.

A Noetherian ring satisfies Gorenstein duality if its localisation at every maximal ideal exhibits a certain form of self duality, which we describe next.

Definition 2.13. A local Noetherian ring R of Krull-dimension d has *Gorenstein duality* if for any Cohen-Macaulay R-module M of Krull-dimension t,

i) $\mathrm{Ext}_R^{d-t}(M, R)$ is Cohen-Macaulay of dimension t,

ii) $\mathrm{Ext}_R^{i}(M, R) = 0$ for all $i \neq d - t$,

iii) There exists an isomorphism of R-modules $M \to \mathrm{Ext}_R^{d-t}(\mathrm{Ext}_R^{d-t}(M, R), R)$.

An arbitrary Noetherian ring has Gorenstein duality if its localisation at every maximal ideal has Gorenstein duality.

To paraphrase this definition, a d-dimensional local Noetherian ring is Gorenstein if for t-dimensional Cohen-Macaulay modules, the functor $\mathrm{Ext}_R^{d-t}(-, R)$ exhibits duality.

By a result of Stanley [24, Theorem 5.1], the Stanley-Reisner ring $\mathbb{Z}[\mathcal{K}]$ having Gorenstein duality is equivalent to $\mathrm{core}(\mathcal{K})$ being an integral generalised homology d-sphere, where $\mathrm{core}(\mathcal{K}) = \mathcal{K}_{\{v \in [m] \mid st_{\mathcal{K}}(v) \neq \mathcal{K}\}}$ is the core of $\mathcal{K}$, and d is the dimension of $\mathrm{core}(\mathcal{K})$.

Notice that if $\mathcal{K}$ has non-trivial cohomology, then $\mathcal{K} = \mathrm{core}(\mathcal{K})$. Theorem 2.10 together with Stanley's [24, Theorem 5.1] relates Poincaré duality of moment-angle complexes, Gorenstein duality of Stanley-Reisner rings, and combinatorial Alexander duality of simplicial complexes. We obtain an interplay between algebraic, combinatorial and topological dualities.

Theorem 2.14. *Let $\mathcal{K}$ be a simplicial complex on $[m]$ with non-trivial cohomology. The following are equivalent:*

1. *$\mathcal{Z}_{\mathcal{K}}$ is an $(n+m)$-Poincaré duality space*

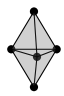
2. *$\mathcal{K}$ has $(n-1)$ dimensional combinatorial Alexander duality*

3. *$\mathbb{Z}[\mathcal{K}]$ has Gorenstein duality.*

Example 2.15. *i) Let $\mathcal{K} = \partial\Delta^1$. $\mathcal{K}$ has two vertices, and has 0-dimensional combinatorial Alexander duality, since $\mathcal{K} = S^0$. Then $\mathcal{Z}_{\mathcal{K}} = S^3$, and we verify that $\mathcal{Z}_{\mathcal{K}}$ is a Poincaré duality space of dimension 3, and $\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, v_2]/(v_1v_2)$ has Gorenstein duality.*

ii) We construct a sequence of quotients of the polynomial ring $\mathbb{Z}[v_1, \ldots, v_5] = \mathbb{Z}[\Delta^4]$, where each stage of the quotient corresponds to deleting simplices from Δ^4.

Let $\mathcal{K} = \partial\Delta^4$. Then $\mathcal{Z}_{\mathcal{K}} = S^9$, and $\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, v_2, v_3, v_4, v_5]/(v_1v_2v_3v_4v_5)$ is Gorenstein. Consider the subcomplex $\mathcal{K}' \subseteq \mathcal{K}$ given by a 5-vertex triangulation of S^2:

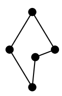
$\mathcal{K}' =$ 

Then $\mathcal{Z}_{\mathcal{K}'} = S^1 \times S^5 \times D^2 \cup D^2 \times S^5 \times S^1 = S^3 \times S^5 \subseteq S^9$ is a Poincaré duality space, and

$$\mathbb{Z}[\mathcal{K}'] = \mathbb{Z}[\mathcal{K}]/(\{v_1v_2v_3v_4v_5/v_i \mid i \in [5]\} \sqcup \{v_1v_2v_3, v_4v_5\})$$

is Gorenstein. Moreover, for any simplicial complex $\mathcal{L}$ such that $\mathcal{K}' \subsetneq \mathcal{L} \subsetneq \mathcal{K}$, the corresponding Stanley-Reinser rings are non-Gorenstein, and the corresponding moment-angle complexes are not Poincaré duality spaces.

Consider the subcomplex $\mathcal{K}'' \subseteq \mathcal{K}'$, given by a 5-vertex triangulation of S^1:

$\mathcal{K}'' =$ 

The corresponding Stanley-Reisner ring, a quotient of $\mathbb{Z}[\mathcal{K}']$ is Gorenstein, the associated moment-angle complex $\mathcal{Z}_{\mathcal{K}''} = (S^3 \times S^4)^{\#5}$, by Theorem 2.7, is a Poincaré duality space, and we obtain a sequence of inclusions of Poincaré duality spaces

$$(S^3 \times S^4)^{\#5} \longrightarrow S^3 \times S^5 \longrightarrow S^9$$

where for any complexes $\mathcal{L}$ and $\mathcal{L}'$ such that $\mathcal{K}' \subsetneq \mathcal{L} \subsetneq \mathcal{K}$ and $\mathcal{K}'' \subsetneq \mathcal{L}' \subsetneq \mathcal{K}'$, the moment-angle complexes $\mathcal{Z}_{\mathcal{L}}$ and $\mathcal{Z}_{\mathcal{L}}$ are not Poincaré duality spaces.

2.5 The polyhedral join product

In this section, we extend our characterisation of Poincaré duality in $\mathcal{Z}_\mathcal{K}$ using the polyhedral join product of simplicial complexes. We apply this result in specifying examples of Poincaré duality polyhedral products, including a polyhedral product of spaces which themselves do not have Poincaré duality.

Definition 2.16. Let $\mathcal{K}$ be a simplicial complex on $[m]$, and for $1 \leq i \leq m$, let $(\mathcal{K}_i, \mathcal{L}_i)$ be a simplicial pair on $[l_i]$, where the sets $[l_i]$ are pairwise disjoint. The *polyhedral join product* is the simplicial complex on vertex set $[l_1] \sqcup ... \sqcup [l_m]$, defined as

$$(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}} (\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\sigma} \text{ where } (\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\sigma} = \mathop{*}_{i=1}^{m} \mathcal{Y}_i,\ \mathcal{Y}_i = \begin{cases} \mathcal{K}_i & i \in \sigma \\ \mathcal{L}_i & \text{otherwise.} \end{cases}$$

The polyhedral join product recovers the join of simplicial complexes as a special case, and given an m-tuple of pairs $(\mathcal{K}_i, \mathcal{L}_i)$, interpolates between $\mathop{*}_{i=1}^{m} \mathcal{L}_i$ and $\mathop{*}_{i=1}^{m} \mathcal{K}_i$. In particular, we note the following special cases.

i) if $\mathcal{K} = \Delta^{m-1}$, then $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} = \mathop{*}_{i=1}^{m} \mathcal{K}_i$,

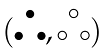
ii) if $\mathcal{K} = \circ_{[m]}$, then $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} = \mathop{*}_{i=1}^{m} \mathcal{L}_i$

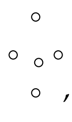
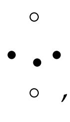
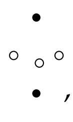
iii) if $(\mathcal{K}_i, \mathcal{L}_i) = (\circ, \circ)$ for all i, then $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} = \circ_{[m]}$,

iv) if $(\mathcal{K}_i, \mathcal{L}_i) = (\bullet, \circ)$ for all i, then $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} = \mathcal{K}$,

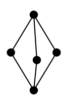
v) if $(\mathcal{K}_i, \mathcal{L}_i) = (\bullet, \bullet)$ for all i, then $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} = \Delta^{m-1}$

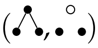
In the following examples, we demonstrate how familiar triangulations of topological spaces arise as polyhedral join products.

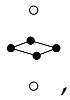
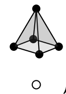
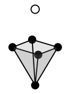
Example 2.17. *i) Let $\mathcal{K} = \Delta^1$, let $(\mathcal{K}_1, \mathcal{L}_1) = ($$)$ and let $(\mathcal{K}_2, \mathcal{L}_2) = ($$)$. Then*

$(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\emptyset} =$  , $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\{1\}} =$  , $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\{2\}} =$  ,

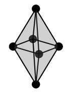
and in this case, since $\mathcal{K} = \Delta^1$ is the full simplex,

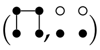
$(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\{1,2\}} = (\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} =$ 

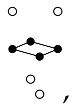
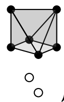
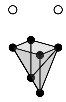
ii) Let $\mathcal{K} = \partial\Delta^1 = \{\emptyset, \{1\}, \{2\}\}$, and let $(\mathcal{K}_1, \mathcal{L}_1) = (\mathcal{K}_2, \mathcal{L}_2) = ($$)$, where $\circ$ denotes a ghost vertex. Then

$(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\emptyset} =$  , $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\{1\}} =$  , *and* $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\{2\}} =$ 

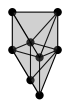
so that

$(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}} =$  $\cong S^2$

iii) Let $\mathcal{K} = \partial\Delta^1$, and $(\mathcal{K}_1, \mathcal{L}_1) = (\mathcal{K}_2, \mathcal{L}_2) = ($$)$. Then

$(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\emptyset} =$  , $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\{1\}} =$  , *and* $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\{2\}} =$ 

so that

$(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}} =$  $\cong S^2$.

Let $l = \sum_{i=1}^{m} l_i$ where $l_i \geq 1\ \forall i$, and let $(\mathbf{X}, \mathbf{A})$ be an l-tuple of CW complexes, partitioned into m distinct l_i tuples with $\mathbf{X_i} = \{X_{ij}\}_{j=1}^{l_i}$ and $\mathbf{A_i} = \{A_{ij}\}_{j=1}^{l_i}$. It was proven by Vidaurre [26, Theorem 2.9] that the polyhedral join product and the polyhedral product interact in the following way

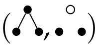
$$(\mathbf{X}, \mathbf{A})^{(\mathcal{K}_{\mathrm{i}}, \mathcal{L}_{\mathrm{i}})^{*\mathcal{K}}} = \left((\mathbf{X_i}, \mathbf{A_i})^{\mathcal{K}_{\mathrm{i}}}, (\mathbf{X_i}, \mathbf{A_i})^{\mathcal{L}_{\mathrm{i}}}\right)^{\mathcal{K}}. \tag{2.15}$$

We make use of this fact in extending our classification of Poincaré duality moment-angle complexes to polyhedral products of tuples of pairs of spaces which are themselves moment-angle complexes.

Theorem 2.18. *Let $\mathcal{K}$ be a simplicial complex on $[m]$, and for $1 \leq i \leq m$, $(\mathcal{K}_i, \mathcal{L}_i)$ a simplicial pair on $[l_i]$. Suppose that the polyhedral join $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$ has non-trivial cohomology. Then the following are equivalent.*

1. *The polyhedral product $(\mathcal{Z}_{\mathcal{K}_{\mathrm{i}}}, \mathcal{Z}_{\mathcal{L}_{\mathrm{i}}})^{\mathcal{K}}$ is a Poincaré duality space.*
2. *The polyhedral join product $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$ has combinatorial Alexander duality.*
3. *The Stanley-Reisner ring $\mathbb{Z}[(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}})]$ has Gorenstein duality.*

Proof. The equivalence of i) and ii) follows from (2.15) together with Theorem 2.14. The equivalence of ii) and iii) follows from Theorem 2.14. □

Example 2.19. *i)* *Let* $\mathcal{K} = \partial\Delta^1$, *and let* $(\mathcal{K}_1, \mathcal{L}_1) = (\mathcal{K}_2, \mathcal{L}_2) = ($$)$, *where* $\circ$ *denotes a ghost vertex. Then* $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}}$ *is a* 6*-vertex triangulation of* S^2, *and in particular is a generalised homology sphere, so that*

$$(\boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{K}}_\mathbf{i}}, \boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{L}}_\mathbf{i}})^{\mathcal{K}} = (S^3 \times D^2, S^3 \times S^1)^{\partial\Delta^1}$$

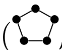
is a Poincaré duality space. Indeed, applying (2.15), *and realising this* 6*-vertex triangulation of* S^2 *as* $\partial\Delta^1 * \partial\Delta^1 * \partial\Delta^1$, *we have*

$$(S^3 \times D^2, S^3 \times S^1)^{\partial\Delta^1} = \mathcal{Z}_{\partial\Delta^1} \times \mathcal{Z}_{\partial\Delta^1} \times \mathcal{Z}_{\partial\Delta^1} = S^3 \times S^3 \times S^3.$$

ii) *Generalising the previous example, let* $\mathcal{K} = \partial\Delta^1$, *and let* $(\mathcal{K}_1, \mathcal{L}_1) = (\mathcal{K}_2, \mathcal{L}_2) = (\partial\Delta^{n-1} * \{v\}, \partial\Delta^n * \{\circ\})$, *where* $\circ$ *denotes a ghost vertex. Then* $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}}$ *is a* $(2n+2)$*-vertex triangulation of the* $(n+1)$*-sphere, and thus*

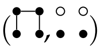
$$(\boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{K}}_\mathbf{i}}, \boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{L}}_\mathbf{i}})^{\mathcal{K}} = (S^{2n-1} \times D^2, S^{2n-1} \times S^1)^{\partial\Delta^1}$$

is a Poincaré duality space.

iii) *Let* $\mathcal{K} = \partial\Delta^1 * \{j\}$, $(\mathcal{K}_1, \mathcal{L}_1) = (\mathcal{K}_2, \mathcal{L}_2) = (\{v\}, \{\emptyset\})$, *and* $(\mathcal{K}_j, \mathcal{L}_j) = ($$, \emptyset)$. *Then* $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}}$ *is a* 7*-vertex triangulation of* S^2. *Here,* $(\mathcal{Z}_{\mathcal{K}_1}, \mathcal{Z}_{\mathcal{L}_1}) = (S^3, T^2)$ *and* $(\mathcal{Z}_{\mathcal{K}_2}, \mathcal{Z}_{\mathcal{L}_2}) = ((S^3 \times S^4)^{\#5}, T^5)$, *so that*

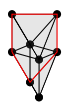
$$\Big((S^3, T^2), ((S^3 \times S^4)^{\#5}, T^5)\Big)^{\Delta^1}$$

is a Poincaré duality space.

iv) *Let* $\mathcal{K} = \partial\Delta^1$, *and* $(\mathcal{K}_1, \mathcal{L}_1) = (\mathcal{K}_2, \mathcal{L}_2) = ($$)$. *Then* $\mathcal{Z}_{\mathcal{K}_1} = \mathcal{Z}_{\mathcal{K}_2} = S^3 \vee (S^3 \rtimes S^1) \vee (S^1 \ltimes S^3) = 3S^3 \vee 2S^4$, *and* $\mathcal{Z}_{\mathcal{L}_1} = \mathcal{Z}_{\mathcal{L}_2} = S^3 \times T^2$.

Recall from Example 2.17 iii) that the resulting polyhedral join product is an 8*-vertex triangulation of* S^2, *so that* $(\boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{K}}_\mathbf{i}}, \boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{L}}_\mathbf{i}})^{\mathcal{K}}$ *is an* 11*-dimensional Poincaré duality space, and thus is homotopy equivalent to an* 11*-dimensional manifold, whilst* $\mathcal{Z}_{\mathcal{K}_1}$ *and* $\mathcal{Z}_{\mathcal{K}_2}$ *are not Poincaré duality spaces.*

Moreover, the 5*-gon outlined in red below*

$$(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} =$$ 

is a full subcomplex, and therefore the moment-angle complex associated to this full subcomplex $(S^3 \times S^4)^{\#5}$ *is a homotopy retract of* $(\boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{K}}_\mathbf{i}}, \boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{L}}_\mathbf{i}})^{\mathcal{K}}$. *Similarly, considering for example the full subcomplex given by the equatorial* 4*-vertex triangulation of* S^1 *we observe that* $S^3 \times S^3$ *is a retract by the same argument.*

The last of the above examples suggests that a full classification of polyhedral join products which are generalised homology spheres, and which thus give rise to Poincaré duality spaces $\mathcal{Z}_{\mathcal{K}}$, is a complicated question in general. We address the special case of composition complexes, where the classification of polyhedral join products which are generalised homology spheres in this special case enables us to extend our results on duality. Recall that composition complexes are the special case of the polyhedral join product where for all i, $\mathcal{K}_i = \Delta^{n_i}$, $n_i \geq 1$. Ayzenberg [5, Theorem 6.6] proved that the composition complex $\mathcal{K}(\mathcal{K}_1, ..., \mathcal{K}_m) = (\Delta^{n_i}, \mathcal{K}_i)^{*\mathcal{K}}$ is a generalised homology sphere if and only if $\mathcal{K}$ is a generalised homology sphere, and for any non-ghost vertex i of $\mathcal{K}$, $\mathcal{K}_i = \partial\Delta^{l_i - 1}$, and for any ghost vertex i of $\mathcal{K}$, $\mathcal{K}_i$ is a generalised homology sphere.

Utilising this result together with (2.15), and the fact that the polyhedral product is a homotopy functor [8, Proposition 8.1.1], we obtain the following corollary.

Corollary 2.20. *Let $\mathcal{K}$ be a complex on $[m]$ with no ghost vertices, and let $\mathcal{K}_1, ..., \mathcal{K}_m$ be complexes on $[l_1], ..., [l_m]$, respectively. Then, $(\mathbf{C}\boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{K}}i}, \boldsymbol{\mathcal{Z}}_{\boldsymbol{\mathcal{K}}i})^{\mathcal{K}}$ is a Poincaré duality space if and only if $\mathcal{K}$ is a generalised homology sphere, and for all $i \in [m]$, $\mathcal{K}_i = \partial\Delta^{[l_i]}$.* □

We end this section by applying this corollary in the following two examples, which address implicitly the question of to which polyhedral products Theorem 2.14 generalises.

One possible question is whether, upon observing that S^1 is a manifold and $D^2 = CS^1$, the statement remains true upon replacing S^1 by any other manifold. The first of the two examples confirms that this is not the case. Indeed, upon replacing S^1 by any torus T^{l_i} for $l_i > 1$, the statement fails.

Another natural question is whether, upon replacing S^1 by any higher odd sphere, the statement remains true. The second example confirms that this generalisation holds. In order to address a possible generalisation to even spheres, which cannot be realised as moment-angle complexes, work using an alternative method would be required. For example, any even sphere is a real moment-angle complex $\mathcal{R}_{\mathcal{K}}$, where the cap product structure is more complicated.

Example 2.21.

1. *Let $\mathcal{K}$ be a simplicial complex on $[m]$. For $1 \leq i \leq m$, let $l_i \geq 2$. Then $(CT^{l_i}, T^{l_i})^{\mathcal{K}}$ is a Poincaré duality space if and only if $\mathcal{K}$ consists solely of ghost vertices.*
2. *Let $\mathcal{K}$ be a simplicial complex on $[m]$, and for $1 \leq i \leq m$, let $l_i \geq 1$. Then $(D^{2l_i}, S^{2l_i - 1})^{\mathcal{K}}$ is a Poincaré duality space if and only if $\mathcal{K}$ is a generalised homology sphere.*

2.6 The polyhedral ring product

In this section, we introduce a new combinatorial algebraic construction, which we refer to as the polyhedral ring product. We explore its properties, and show that it can be used to study algebraic phenomena combinatorially. The Stanley Reisner ring of $(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}}$ arises as a special case of the polyhedral ring product, and we exploit this fact in constructing Gorenstein rings as polyhedral ring products of non-Gorenstein rings.

Recall our standing assumption that all rings are commutative with identity. We define a ring-ideal pair to be a pair (R, J) where R is a ring and $J \subseteq R$ is an ideal of R.

Definition 2.22. Let $\mathcal{K}$ be a simplicial complex on $[m]$. For each $i \in [m]$, let (R_i, J_i) denote a ring-ideal pair. The polyhedral ring product is defined by

$$(\underline{R}, \underline{J})^{\mathcal{K}} = \bigotimes_{i=1}^{m} R_i \Big/ \left(\sum_{L \in MF(\mathcal{K})} \bigotimes_{i \in L} J_i \right) \tag{2.16}$$

where the tensor product is the tensor product of $\mathbb{Z}$-algebras in the numerator, and of $\mathbb{Z}$-modules in the denominator.

The polyhedral ring product is a bi-functor. In analogy with the polyhedral product and polyhedral join product, it is covariant with respect to maps of m-tuples of ring-ideal pairs, and in contrast with the polyhedral join and polyhedral product, the polyhedral ring product is contravariant with respect to inclusions of simplicial complexes.

More precisely, consider the category whose objects are m-tuples of ring-ideal pairs $(\underline{R}, \underline{J}) = \{(R_i, J_i)\}_{i \in [m]}$, and whose morphisms are m-tuples of homomorphisms

$$\underline{f} = \{f_i\}_{i \in [m]} \colon \{R_i\}_{i \in [m]} \longrightarrow \{R'_i\}_{i \in [m]}$$

where for all $i \in [m]$, the homomorphism $f_i \colon R_i \longrightarrow R'_i$ is such that $f_i(J_i) \subseteq J'_i$. Then, for a fixed simplicial complex $\mathcal{K}$, define the polyhedral ring product of a tuple $(\underline{R}, \underline{J})$ to be $(\underline{R}, \underline{J})^{\mathcal{K}}$, and the polyhedral ring product of a morphism $\underline{f}$ by

$$\begin{gathered} \underline{f}^{\mathcal{K}} \colon (\underline{R}, \underline{J})^{\mathcal{K}} \longrightarrow (\underline{R}', \underline{J}')^{\mathcal{K}}, \\ (r_1 \otimes \cdots \otimes r_m) + \sum_{L \in MF(\mathcal{K})} \bigotimes_{i \in L} J_i \longmapsto (f_1(r_1) \otimes \cdots \otimes f_m(r_m)) + \sum_{L \in MF(\mathcal{K})} \bigotimes_{i \in L} J'_i \end{gathered} \tag{2.17}$$

which is well-defined since $f_i(J_i) \subseteq J'_i$ for all i. Since $\underline{Id}^{\mathcal{K}} = Id \colon (\underline{R}, \underline{J})^{\mathcal{K}} \longrightarrow (\underline{R}, \underline{J})^{\mathcal{K}}$ and $(\underline{g \circ f})^{\mathcal{K}} = \underline{g}^{\mathcal{K}} \circ \underline{f}^{\mathcal{K}}$, this operation is covariant functor.

On the other hand, for a fixed m-tuple $(\underline{R}, \underline{J})$, we define the polyhedral ring product of a simplicial complex $\mathcal{K}$ by $(\underline{R}, \underline{J})$. Given an inclusion of simplicial complexes $\mathcal{K} \hookrightarrow \mathcal{K}'$, define the polyhedral ring product of $\mathcal{K} \hookrightarrow \mathcal{K}'$ to be

$$\begin{gathered}(\underline{R}, \underline{J})^{\mathcal{K}'} \longrightarrow (\underline{R}, \underline{J})^{\mathcal{K}}, \\ (r_1 \otimes \cdots \otimes r_m) + \sum_{L' \in MF(\mathcal{K}')} \bigotimes_{i \in L} J_i \longmapsto (r_1 \otimes \cdots \otimes r_m) + \sum_{L \in MF(\mathcal{K})} \bigotimes_{i \in L} J_i\end{gathered}$$

which is well-defined since $MF(\mathcal{K}') \subseteq MF(\mathcal{K})$. This is a contravariant functor from the category of simplicial complexes and inclusions of simplicial complexes to the category of rings.

Example 2.23. *i) Let $\{(R_i, J_i)\}_{i=1}^m$ be an m-tuple of ring-ideal pairs. Then,*

$$\begin{aligned}(\underline{R}, \underline{J})^{\Delta^{m-1}} &= \bigotimes_{i=1}^m R_i \\ (\underline{R}, \underline{J})^{\circ_{[m]}} &= \bigotimes_{i=1}^m (R_i / J_i) \\ (\underline{R}, \underline{J})^{\partial\Delta^{m-1}} &= \bigotimes_{i=1}^m R_i \Big/ \bigotimes_{i=1}^m J_i\end{aligned}$$

ii) Let $(R_i, J_i) = (\mathbb{Z}[x], (x))$ for all $i \in [m]$. Then for any simplicial complex $\mathcal{K}$,

$$(\underline{\mathbb{Z}[x]}, \underline{(x)})^{\mathcal{K}} = \mathbb{Z}[\mathcal{K}].$$

iii) Let $(\mathcal{K}_1, \mathcal{L}_1) = (\mathcal{K}_2, \mathcal{L}_2) = (\partial\Delta^1, \circ_{[2]})$, and let $\mathcal{K} = \partial\Delta^1$. Then, denoting by $\mathcal{I}_{\mathcal{K}_i}$ the Stanley-Reisner ideal of $\mathcal{L}_i$,

$$\begin{aligned}(\underline{\mathbb{Z}[\mathcal{K}_i]}, \underline{\mathcal{I}_{\mathcal{K}_i}})^{\mathcal{K}} &= (\underline{\mathbb{Z}[\mathcal{K}_i]}, \underline{(x_{i_1}, x_{i_2})})^{\mathcal{K}} \\ &= \mathbb{Z}[\mathcal{K}_1] \otimes \mathbb{Z}[\mathcal{K}_2] / ((x_{1_1}, x_{1_2}) \otimes (x_{2_1}, x_{2_2})) \\ &\cong \mathbb{Z}[x_1, x_2, x_3, x_4] \Big/ \left(\{ \prod_{i \neq j} x_i x_j \} \right) \\ &= \mathbb{Z} \left[\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix} \right] \\ &= \mathbb{Z}[(\mathcal{K}_{\mathbf{i}}, \mathcal{L}_{\mathbf{i}})^{\mathcal{K}}]\end{aligned}$$

We now show that the Stanley-Reisner ring of a polyhedral join product arises as a special case of the polyhedral ring product. For this, we describe the missing faces of the polyhedral join in general.

Proposition 2.24. *Let $\mathcal{K}$ be a simplicial complex on $[m]$, and let $(\mathcal{K}_1, \mathcal{L}_1), ..., (\mathcal{K}_m, \mathcal{L}_m)$ be simplicial pairs on vertex sets $[l_1], ..., [l_m]$ respectively. Then,*

$$MF\left((\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}\right) = \{J \in MF(\mathcal{K}_i) \mid i \in \mathcal{K}\} \sqcup \left\{\bigsqcup_{i\in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{L}_i), J_i \in \mathcal{K}_i\right\}.$$

Proof. We first show that

$$\begin{aligned}\{J \in MF(\mathcal{K}_i) \mid i \in \mathcal{K}\} \sqcup \{\bigsqcup_{i\in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{L}_i), J_i \in \mathcal{K}_i\}\\ \subseteq MF\left((\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})_*^{\mathcal{K}}\right).\end{aligned}$$

For any $\{i\} \in \mathcal{K}$, $J \in MF(\mathcal{K}_i)$ implies that $J \in MF\left((\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}\right)$. Now consider $\bigsqcup_{i\in L} J_i$, where $L \in MF(\mathcal{K})$, $J_i \in MF(\mathcal{L}_i)$ and $J_i \in \mathcal{K}_i$ for all $i \in L$. This is a missing face of $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$ by definition of the polyhedral join. Moreover, it is minimal, since for any $i \in L$ and $s \in J_i$,

$$\bigsqcup_{i\neq k\in L} J_k \sqcup J_i \backslash s = \bigsqcup_{k\in\tau} J_k \sqcup \sigma_i \in (\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\tau}$$

where $\tau \in \mathcal{K}$, since L is a minimal missing face, and $\sigma_i \in \mathcal{L}_i$, since J_i is a minimal missing face.

Now we show that

$$\begin{aligned}MF\left((\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}\right) \subseteq \{J \in MF(\mathcal{K}_i) \mid i \in \mathcal{K}\}\\ \sqcup \{\bigsqcup_{i\in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{L}_i), J_i \in \mathcal{K}_i\}\end{aligned}$$

Let $F \in MF((\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}})$. We show that either $F \in \{J \in MF(\mathcal{K}_i) \mid i \in \mathcal{K}\}$ or $F \in \{\bigsqcup_{i\in L} F_i \mid L \in MF(\mathcal{K}), F_i \in MF(\mathcal{L}_i), F_i \in \mathcal{K}_i\}$.

If for any i, the restriction $F_i := F|_{[l_i]} \in MF(\mathcal{K}_i)$, then $F = F_i \in MF(\mathcal{K})$. Otherwise F would not be minimal, since $F_i \subseteq F$ is a missing face.

On the other hand, suppose that $F_i \in \mathcal{K}_i$ for all i. Denote by $\sigma = \{i \in [m] \mid F|_i \neq \emptyset\}$. Firstly, $\sigma \notin \mathcal{K}$, as otherwise $F \in \mathcal{K}$, since $F = \bigsqcup_{i\in\sigma} F_i$ where $F_i \in \mathcal{K}_i$ for all i. For all $i \in \sigma$, F_i is a non-face of $\mathcal{L}_i$. Otherwise, by removing vertices from F_i we obtain a smaller non-face of $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$. For such i, $F_i \in MF(\mathcal{L}_i)$. It follows that $\sigma \in MF(\mathcal{K})$, as otherwise we restrict to a minimal missing face $\tau \in MF(\mathcal{K})$ with $\tau \subseteq \sigma$, and $\bigsqcup_{i\in\tau} F_i$ is a missing face of $(\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$. Finally, for all i, F_i is a minimal missing face of $\mathcal{L}_i$, since otherwise, for $\widehat{F}_i \subsetneq F_i$ with $\widehat{F}_i \in MF(\mathcal{K}_i)$, $(F\backslash F_i) \sqcup \widehat{F}_i \in (\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$. □

It follows that

$$\begin{aligned}
\mathbb{Z}[(\boldsymbol{\mathcal{K}}_{\mathbf{i}},\boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}] &\cong \mathbb{Z}[x_{1_1},\ldots,x_{m_{l_m}}] \Big/ \left(\sum_{i=1}^{m}\mathcal{I}_{\mathcal{K}_i} + \sum_{L\in MF(\mathcal{K})}\prod_{i\in L}\mathcal{I}_{\mathcal{L}_i}\right)\\
&\cong \left(\bigotimes_{i=1}^{m}\mathbb{Z}[x_{i_1},\ldots,x_{i_{l_i}}]\right) \Big/ \left(\sum_{i=1}^{m}\mathcal{I}_{\mathcal{K}_i} + \sum_{L\in MF(\mathcal{K})}\prod_{i\in L}\mathcal{I}_{\mathcal{L}_i}\right)\\
&\cong \left(\bigotimes_{i=1}^{m}\mathbb{Z}[\mathcal{K}_i]\right) \Big/ \sum_{L\in MF(\mathcal{K})}\prod_{i\in L}\mathcal{I}_{\mathcal{L}_i}\\
&\cong (\underline{\mathbb{Z}[\mathcal{K}_i]},\underline{\mathbb{Z}[\mathcal{L}_i]})^{\mathcal{K}}. \qquad (2.18)
\end{aligned}$$

In the case of a polyhedral ring product of the form of (2.18), algebraic properties of the ring $(\underline{\mathbb{Z}[\mathcal{K}_i]},\underline{\mathbb{Z}[\mathcal{L}_i]})^{\mathcal{K}}$ can be understood in terms of equivalent combinatorial properties of the polyhedral join product $(\boldsymbol{\mathcal{K}}_{\mathbf{i}},\boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$.

Example 2.25. *i) Let $\mathcal{K}=\Delta^1$, and for $i=1,2$, let $(R_i,S_i)=(\mathbb{Z}[x_1,x_2]/x_1x_2,\mathbb{Z})$. Then*

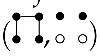
$$(\underline{R},\underline{S})^{\mathcal{K}} = (\mathbb{Z}[x_1,x_2]/(x_1x_2))\otimes(\mathbb{Z}[x_1',x_2']/x_1'x_2') \cong \mathbb{Z}[x_1,x_2,x_3,x_4]/(x_1x_2,x_3x_4).$$

For $i=1,2$, $R_i=\mathbb{Z}[\mathcal{K}_i]$ where $\mathcal{K}_i=\partial\Delta^1$, and therefore $(\underline{R},\underline{S})^{\mathcal{K}}\cong\mathbb{Z}[(\boldsymbol{\mathcal{K}}_{\mathbf{i}},\boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}]$. Using that $(\boldsymbol{\mathcal{K}}_{\mathbf{i}},\boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}=\mathcal{K}_1*\mathcal{K}_2$ is a 4-vertex triangulation of S^1 and therefore has combinatorial Alexander-duality, we obtain that $(\underline{R},\underline{S})^{\mathcal{K}}$ has Gorenstein duality.

ii) Let $\mathcal{K}=$ , with vertex set $[3]$ labelled from left to right. Let $(R_1,S_1)=(R_3,S_3)=(\mathbb{Z}[x],\mathbb{Z}[x]/(x))$, and let $R_2=(\mathbb{Z}[x_1,x_2,x_3]/(x_1x_2x_3),\mathbb{Z})$. Then $MF(\mathcal{K})=\{\{1,3\}\}$, and therefore

$$\begin{aligned}
(\underline{R},\underline{S})^{\mathcal{K}} &= (\mathbb{Z}[x]\otimes(\mathbb{Z}[x_1,x_2,x_3]/x_1x_2x_3)\otimes\mathbb{Z}[x'])/(xx')\\
&\cong \mathbb{Z}[x_1,x_2,x_3,x_4,x_5]/(x_1x_2x_3)+(xx').
\end{aligned}$$

We claim that $(\underline{R},\underline{S})^{\mathcal{K}}$ has Gorenstein duality. Since for $i=1,3$, $(R_i,S_i)\cong(\mathbb{Z}[\mathcal{K}_i],\mathbb{Z}[\mathcal{L}_i])$ where $(\mathcal{K}_i,\mathcal{L}_i)=(\bullet,\circ)$, and $(R_2,S_2)\cong(\mathbb{Z}[\partial\Delta^2],\mathbb{Z}[\circ_{[3]}])$, then $(\underline{R},\underline{S})^{\mathcal{K}}$ is Gorenstein if and only if $(\boldsymbol{\mathcal{K}}_{\mathbf{i}},\boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$ has Alexander duality. Since the latter is a 5-vertex triangulation of S^2, proving the claim.

iii) Recall Example 2.19. In the fourth of these examples, $\mathcal{K}=\partial\Delta^1$, and $(\mathcal{K}_1,\mathcal{L}_1)=(\mathcal{K}_2,\mathcal{L}_2)=($$)$. Since $\mathcal{K}_1=\mathcal{K}_2$ is not a generalised homology sphere, $\mathbb{Z}[\mathcal{K}_1]=\mathbb{Z}[\mathcal{K}_2]$ is non-orenstein. However, since $(\boldsymbol{\mathcal{K}}_{\mathbf{i}},\boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}$ is an 8-vertex triangulation of S^2, the polyhedral ring product $(\underline{\mathbb{Z}[\mathcal{K}_i]},\underline{\mathbb{Z}[\mathcal{L}_i]})^{\mathcal{K}}$ is Gorenstein. We therefore obtain that Gorenstein rings can be constructed as polyhedral ring products of non-Gorenstein rings.

Chapter 3

The higher Whitehead map

3.1 Introduction

Understanding the relationship between two or more given objects is an objective which is central to many mathematical disciplines. In the realm of homotopy theory, the most fundamental form of relationship between two spaces X and Y takes the form of a homotopy class of maps $X \longrightarrow Y$, the set of which is denoted by $[X, Y]$. A broad variety of techniques have now been established with a view to gaining an understanding of these sets, and the development of theory continues to be driven by this goal.

A natural question is whether the homotopy-theoretic properties of the spaces X and Y give rise to algebraic structures with which the set $[X, Y]$ can be equipped. In the case that either X is a simply-connected co-associative co-H-space, or Y is a connected associative H-space, there is an induced multiplication on $[X, Y]$. It was proven by Arkowitz [4] in the first case, and James [17] in the second, that this multiplication endows the set $[X, Y]$ with the structure of a group. The presence of this algebraic structure enables us to study relations among elements of $[X, Y]$, a knowledge of which brings insight into the global structure of $[X, Y]$.

In this chapter, we analyse relations among homotopy classes of a family of maps called higher Whitehead maps. Higher Whitehead maps generalise several fundamental homotopy-theoretic constructions, including Whitehead products [27], generalised Whitehead products [10], and inner n-ary Whitehead products [16], to the setting of polyhedral products. These operations and the relations among them have been studied since the middle of the twentieth century, and an understanding of their properties has applications to a broad variety of homotopy-theoretic phenomena. In order to give a sense of where our work sits within the research in this area, we provide a brief overview of the key historical developments.

The first result regards relations among Whitehead products, and was independently conjectured by Blakers and Massey [6], and Samelson [22], who supposed that the Whitehead product satisfies the graded Jacobi identity. This conjecture was first proven by Nakaoka and Toda [19], and later independently by Massey and Uehara [25]. This later proof is particularly noteworthy for the historical interest of its methodology, which included the introduction of the state-of-the-art algebraic machinery which we now know as the Massey product. Arkowitz [3] subsequently proved that the graded Jacobi identity is also satisfied by the generalised Whitehead product. These results were of major significance, since they showed that the graded module of homotopy classes of maps $\bigoplus_{n\geq 0}[\Sigma^n, Y]$ has the additional algebraic structure of a graded Lie algebra when equipped with the Whitehead product.

These results culminated in the work of Hardie [16], who proved that the inner n-ary Whitehead product satisfies an analogous relation to the graded Jacobi identity, which it recovers as a special case in the binary setting.

The common thread running through the aforementioned results is that the relations in question are among homotopy classes of maps whose codomains all lie in a certain stage of the Whitehead filtration, and whose domains are all certain polyhedral products. By recognising the Whitehead filtration as a polyhedral product, we obtain a natural framework which unifies these results, and within which we can investigate generalisations by probing the underlying combinatorics.

Our main result is the construction of new families of relations among non-trivial homotopy classes of higher Whitehead maps. We prove that there exist extensive further families of relations among higher Whitehead maps than those already known, including among higher Whitehead maps of differing arity. We recover as special cases the aforementioned relations among Whitehead products, generalised Whitehead products and inner Whitehead products, placing them in their natural context as relations among homotopy classes of maps of polyhedral products. As an initial application, we demonstrate how these results can be utilised to analyse the algebraic structure of homotopy groups of odd spheres in a completely new way.

Higher Whitehead products, defined by Porter [21], are a generalisation of Whitehead and generalised Whitehead products in another direction. Panov and Abramyan [1] studied higher Whitehead products in moment-angle complexes, and asked whether there are trivial higher Whitehead products with non-trivial elements. Here we address this question, and construct an an infinite family of trivial Whitehead products with non-trivial indeterminacy.

The work in this chapter is joint work of Grbić, Simmons and myself, and the results here are to appear in a joint paper, and in the thesis of Simmons.

3.2 The higher Whitehead map

In this section we define the higher Whitehead map, and explore its properties.

We begin by recalling the definition of the polyhedral product, and a crucial associated homotopy-theoretic property.

Definition 2.1. Let $\mathcal{K}$ be a simplicial complex on vertex set $[m]$, and denote by $(\mathbf{X}, \mathbf{A}) = \{(X_i, A_i)\}_{i=1}^{m}$ an m-tuple of CW-pairs. The polyhedral product is defined as

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \bigcup_{\sigma\in\mathcal{K}} (\mathbf{X}, \mathbf{A})^{\sigma} \subseteq \prod_{i=1}^{m} X_i, \text{ where } (\mathbf{X}, \mathbf{A})^{\sigma} = \prod_{i=1}^{m} Y_i,\ Y_i = \begin{cases} X_i & \text{for } i \in \sigma \\ A_i & \text{for } i \notin \sigma. \end{cases} \tag{3.1}$$

Proposition 2.3 ([8, Proposition 8.1.1]). If for all $i \in [m]$, the inclusion $A_i \hookrightarrow X_i$ is a cofibration, then there is a homotopy equivalence

$$\operatorname{hocolim} \mathcal{D}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}) \simeq \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}) \tag{3.2}$$

Throughout this thesis thus far, we have implicitly referred to spaces up to homotopy, without specifying concrete models. We proceed on this basis in this chapter until Section 3.7, when we switch to using concrete models of spaces for the purposes of the proof. We digress further on that change when it arises.

We now define the higher Whitehead map.

Definition 3.1. Let $m \geq 2$. For $i = 1, \ldots, m$, suppose that we are given maps $f_i \colon \Sigma X_i \longrightarrow Y_i$. Let $\mathbf{Y}$, $\mathbf{CX}$, $\mathbf{\Sigma X}$ and $\mathbf{X}$ denote the m-tuples $\{Y_i\}_{i\in[m]}$, $\{CX_i\}_{i\in[m]}$, $\{\Sigma X_i\}_{i\in[m]}$ and $\{X_i\}_{i\in[m]}$ respectively, and denote by

$$\rho\colon \mathop{\ast}_{i=1}^{m} X_i \longrightarrow (\mathbf{\Sigma X}, \ast)^{\partial\Delta^{m-1}} \tag{3.3}$$

the map of polyhedral products induced by the maps of pairs $p_i \colon (CX_i, X_i) \longrightarrow (\Sigma X_i, \ast)$. The higher Whitehead map is the composite

$$h_w(f_1, \ldots, f_m)\colon \mathop{\ast}_{i=1}^{m} X_i \xrightarrow{\rho} (\mathbf{\Sigma X}, \ast)^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Y}, \ast)^{\partial\Delta^{m-1}} \tag{3.4}$$

where the second map is induced by the maps $f_i \colon \Sigma X_i \longrightarrow Y_i$.

By Proposition 2.3, the homotopy class of the second map in (3.4), and thus the homotopy class of the composite, depends only on the homotopy classes of the maps $f_1, \ldots, f_m$.

Before discussing the properties of the higher Whitehead map, we give some examples. In the following and henceforth, unless stated otherwise, spheres S^n are equipped with the CW-structure consisting of exactly two cells.

Example 3.2. *i) Let $m = 2$, and suppose that $f_1 = f_2 = Id\colon S^2 \longrightarrow S^2$. Then*

$$h_w(f_1, f_2)\colon (D^2 \times S^1) \cup (S^1 \times D^2) = S^1 * S^1 = S^3 \longrightarrow S^2 \vee S^2$$

is the attaching map for the 4-cell of $S^2 \times S^2$ when equipped with the product CW-structure; that is, consisting of one 0-cell, two 2-cells, and one 4-cell.

ii) Generalising the previous example, let $m \geq 2$ and for all $i \in [m]$ suppose that $f_i = Id\colon S^{n_i} \longrightarrow S^{n_i}$, where each sphere has the usual CW-structure consisting of two cells. Then the higher Whitehead map is

$$h_w(f_1, \dots, f_m)\colon \mathop{\ast}_{i=1}^{m} S^{n_i - 1} = S^{n-1} \longrightarrow FW(S^{n_1}, \dots, S^{n_m})$$

where $n = n_1 + \cdots + n_m$, is the attaching map for the n-cell of $\prod_i S^{n_i}$, when equipped with the product CW-structure.

iii) Let $m = 2$, $f_1 = Id\colon S^2 \longrightarrow S^2$ and $f_2 = h_w(f_{2_1}, f_{2_2})\colon S^3 \longrightarrow S^2 \vee S^2$, where $f_{2_1} = f_{2_2} = Id\colon S^2 \longrightarrow S^2$. Then

$$h_w(f_1, f_2)\colon S^1 * S^2 = S^4 \xrightarrow{\rho} ((S^2, *), (S^3, *))^{\partial\Delta^1} \xrightarrow{Id \vee f_2} S^2 \vee S^2 \vee S^2$$

iv) Let $m = 2$, and suppose that we are given maps $f_1\colon \Sigma X_1 \longrightarrow Y$ and $f_2\colon \Sigma X_2 \longrightarrow Y$. Then the higher Whitehead map composed with the fold map

$$X^1 * X^2 \xrightarrow{h_w(f_1, f_2)} Y \vee Y \xrightarrow{\nabla} Y$$

is the generalised Whitehead product.

v) Let $m \geq 2$, and let $f_i\colon \Sigma X_i \longrightarrow \Sigma X_i$ denote the identity for each $i \in [m]$. Then the homotopy cofibre of the higher Whitehead map

$$X_1 * \cdots * X_m \xrightarrow{h_w} FW(\Sigma X_1, \dots, \Sigma X_m) \longrightarrow \prod_{i=1}^{m} \Sigma X_i$$

is the Cartesian product.

We now explore the properties of the higher Whitehead map. In the following and throughout this thesis, we say that a map is trivial if it is null-homotopic, and non-trivial if it is not null-homotopic. We begin with the following theorem; the non-triviality of the higher Whitehead map is controlled by the non-triviality of the maps f_i.

Theorem 3.3. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ for $i = 1, \dots, m$. The higher Whitehead map*

$$h_w(f_1, \dots, f_m)\colon (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Y}, *)^{\partial\Delta^{m-1}}$$

is trivial if and only if there exists i such that $f_i\colon \Sigma X_i \longrightarrow Y_i$ is trivial. Moreover, the composition

$$\iota \circ h_w(f_1, \dots, f_m)\colon (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Y}, *)^{\partial\Delta^{m-1}} \hookrightarrow (\mathbf{Y}, *)^{\Delta^{m-1}}$$

is always trivial, where ι is the map on polyhedral products induced by the inclusion $\partial\Delta^{m-1} \hookrightarrow \Delta^{m-1}$ of simplicial complexes.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
\prod_{i=1}^m \Omega Y_i & \longrightarrow & (\mathbf{C\Omega Y}, \mathbf{\Omega Y})^{\partial\Delta^{m-1}} & \longrightarrow & (\mathbf{Y}, *)^{\partial\Delta^{m-1}} & \longrightarrow & \prod_{i=1}^m Y_i \\
 & \nwarrow & \uparrow{\scriptstyle *\widehat{f}} & & {\scriptstyle h_w}\uparrow & & \uparrow \\
 & & (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} & \xrightarrow{=} & (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} & \longrightarrow & *
\end{array} \tag{3.5}$$

where both rows are homotopy fibration sequences, the first vertical map is the join of the adjoint maps $\widehat{f_i}\colon X \longrightarrow \Omega Y$, and the second vertical map is the higher Whitehead map. The existence of dashed arrow, making the left-hand triangle commute, is equivalent to the triviality of the adjacent square, and thus of the higher Whitehead map.

Observe that $\prod_{i=1}^m \Omega Y_i \longrightarrow (\mathbf{C\Omega Y}, \mathbf{\Omega Y})^{\partial\Delta^{m-1}}$ is always trivial, since it factors through $CX_1 \times \cdots \times X_i \times \cdots \times CX_m$ for all $i \in [m]$. Therefore, the existence of the dashed arrow is equivalent to the triviality of the map $\widehat{\mathbf{f}}\colon (\mathbf{CX}, \mathbf{X})^{\partial\Delta} \longrightarrow (\mathbf{C\Omega Y}, \mathbf{\Omega Y})^{\partial\Delta^{m-1}}$, which is trivial if and only if at least one of the maps $f_i\colon \Sigma X_i \longrightarrow Y_i$ is trivial.

Moreover, by the fact that the top row is a fibration sequence, the composite with the inclusion into the product $(\mathbf{Y}, *)^{\Delta^{m-1}} = \prod Y_i$ is trivial, by the fact that the middle square commutes. □

The higher Whitehead map exhibits multilinearity in the sense of the following proposition.

Proposition 3.4. *Let $m \geq 2$, let $f_i\colon \Sigma X_i \longrightarrow Y_i$ for all $i = 1, \dots, m$ and for some $j \in [m]$ let $f'_j\colon \Sigma X_j \longrightarrow Y_j$ where $X_j = \Sigma \widetilde{X}_j$ for a CW-complex $\widetilde{X}_j$. Then*

$$h_w(f_1, \dots, f_j \vee f'_j, \dots, f_m) = h_w(f_1, \dots, f_j, \dots, f_m) + h_w(f_1, \dots, f'_j, \dots, f_m)$$

in $\left[(\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}}, (\mathbf{Y}, *)^{\partial\Delta^{m-1}}\right]$.

Proof. By [21], the homotopy equivalence $(\mathbf{CX},\mathbf{X})^{\partial\Delta^{m-1}} \simeq \bigwedge_{i=1}^{m-1} S^1 \wedge \bigwedge_{i=1}^{m} X_i$, is a homotopy functor with respect to the m-tuple $\mathbf{X}$. Then, by the fact that there is a homeomorphism

$$\left(\bigwedge_{i=1}^{m-1} S^1\right) \wedge X_1 \wedge \cdots \wedge \left(X_j \vee X_j\right) \wedge \cdots \wedge X_m \cong \left(\bigwedge_{i=1}^{m-1} S^1 \wedge \bigwedge_{i=1}^{m} X_i\right) \vee \left(\bigwedge_{i=1}^{m-1} S^1 \wedge \bigwedge_{i=1}^{m} X_i\right)$$

together with the fact that the co-multiplication in the smash product induced by the co-multiplication in any of the factors is the same up to homotopy, the result follows. □

We later make use of the following technical lemma and proposition.

Lemma 3.5. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ for $i = 1,\ldots,m$, and let $\sigma\colon (i_1,\ldots,i_m) \mapsto (1,\ldots,m)$ be a permutation. Let Z be a space and suppose there are maps $\iota\colon FW(Y_1,\ldots,Y_m) \longrightarrow Z$ and $\iota'\colon FW(Y_{i_1},\ldots,Y_{i_m}) \longrightarrow Z$ such that $\iota'(Y_{i_j}) = \iota(Y_{\sigma(i_j)})$ for $j = 1,\ldots,m$. Denoting*

$$h_w^Z(f_1,\ldots,f_m) = \iota \circ h_w(f_1,\ldots,f_m)\colon \mathop{\ast}_{i=1}^{m} X_i \longrightarrow Z$$

and

$$h_w^Z(f_{i_1},\ldots,f_{i_m}) = \iota' \circ h_w(f_{i_1},\ldots,f_{i_m})\colon \mathop{\ast}_{i=1}^{m} X_{j_i} \longrightarrow Z$$

and $\xi\colon X_{i_1} * \cdots * X_{i_m} \longrightarrow X_1 * \cdots * X_m$ the restriction of the coordinate permutation

$$CX_{i_1} \times \cdots \times CX_{i_m} \longrightarrow CX_1 \times \cdots \times CX_m,$$

then

$$h_w^Z(f_{i_1},\ldots,f_{i_m}) = h_w^Z(f_1,\ldots,f_m) \circ \xi. \tag{3.6}$$

Proof. The result follows directly from the definition of the higher Whitehead map. □

Proposition 3.6. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$, $g_i\colon Y_i \longrightarrow Z_i$, and $h_i\colon W_i \longrightarrow X_i$ for all $i \in [m]$. Denote by $g\colon (\mathbf{Y},*)^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Z},*)^{\partial\Delta^{m-1}}$ the induced map of polyhedral products. Then,*

$$h_w(g_1 \circ f_1,\ldots,g_m \circ f_m) = g \circ h_w(f_1,\ldots,f_m) \tag{3.7}$$

and

$$h_w(f_1 \circ \Sigma h_1,\ldots,f_m \circ \Sigma h_m) = h_w(f_1,\ldots,f_m) \circ \mathop{\ast}_{i=1}^{m} h_i. \tag{3.8}$$

Proof. The first equality (3.7) follows from the functoriality of the fat-wedge, and the second (3.8) from the functoriality of the join. □

3.3 The substituted higher Whitehead map

In this section, we study the special case of higher Whitehead map of maps $f_i\colon \Sigma X_i \longrightarrow (\mathbf{Y}_i, *)^{\mathcal{K}_i}$ where $\mathcal{K}_1, \dots, \mathcal{K}_m$ are simplicial complexes on $[l_1], \dots, [l_m]$ respectively.

A substituted higher Whitehead map is intuitively a higher Whitehead map where there is extra underlying combinatorial structure in at least some of the codomains of the maps f_i. This extra structure enables us to recognise the codomain of the higher Whitehead map as a polyhedral join product. This in turn implies the existence of inclusions into strictly bigger polyhedral products, where composing with these inclusions does not trivialise the higher Whitehead map.

We begin by recalling the definition of the polyhedral join product.

Definition 2.16. Let $\mathcal{K}$ be a simplicial complex on $[m]$, and for $1 \leq i \leq m$, let $(\mathcal{K}_i, \mathcal{L}_i)$ be a simplicial pair on $[l_i]$, where the sets $[l_i]$ are pairwise disjoint. The *polyhedral join product* is the simplicial complex on vertex set $[l_1] \sqcup ... \sqcup [l_m]$, defined as

$$(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}} (\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\sigma} \text{ where } (\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\sigma} = \mathop{\ast}_{i=1}^{m} \mathcal{Y}_i,\ \mathcal{Y}_i = \begin{cases} \mathcal{K}_i & i \in \sigma \\ \mathcal{L}_i & \text{otherwise.} \end{cases}$$

Two special cases of the polyhedral join product which will arise frequently henceforth are the substitution complex and composition complex, studied by Abramyan and Panov, [1] and Ayzenberg [5] respectively. We abbreviate notation by denoting the substitution complex by $(\mathcal{K}_\mathbf{i}, \varnothing)^{*\mathcal{K}}$ by $\mathcal{K}\langle \mathcal{K}_1, \dots, \mathcal{K}_m \rangle$, and the composition complex by $\mathcal{K}(\mathcal{K}_1, \dots, \mathcal{K}_m)$. In particular, when $\mathcal{K}$ is the boundary of the full simplex, we abbreviate the associated substitution and composition complexes by $\partial\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_m \rangle$ and $\partial\Delta(\mathcal{K}_1, \dots, \mathcal{K}_m)$ respectively.

Recall (2.15)

$$(\mathbf{X}, \mathbf{A})^{(\boldsymbol{\mathcal{K}}_\mathbf{i}, \boldsymbol{\mathcal{L}}_\mathbf{i})^{*\mathcal{K}}} = \left((\mathbf{X}_\mathbf{i}, \mathbf{A}_\mathbf{i})^{\boldsymbol{\mathcal{K}}_\mathbf{i}}, (\mathbf{X}_\mathbf{i}, \mathbf{A}_\mathbf{i})^{\boldsymbol{\mathcal{L}}_\mathbf{i}}\right)^{\mathcal{K}}.$$

By Theorem 3.3, a higher Whitehead map $h_w(f_1, \dots, f_m)\colon \mathop{\ast}_{i=1}^{m} X_i \longrightarrow (\mathbf{Y}, *)^{\partial\Delta^{m-1}}$ is trivial if and only if $f_i\colon \Sigma X_i \longrightarrow Y_i$ is trivial for some $i \in [m]$, whilst the composition with the inclusion

$$h_w(f_1, \dots, f_m)\colon \mathop{\ast}_{i=1}^{m} X_i \longrightarrow (\mathbf{Y}, *)^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Y}, *)^{\Delta^{m-1}}$$

is always trivial. In general, for a space Z such that $(\mathbf{Y}, *)^{\partial\Delta^{m-1}} \subsetneq Z \subsetneq (\mathbf{Y}, *)^{\Delta^{m-1}}$, the question of whether the composition

$$h_w(f_1, \dots, f_m)\colon \mathop{\ast}_{i=1}^{m} X_i \longrightarrow (\mathbf{Y}, *)^{\partial\Delta^{m-1}} \longrightarrow Z$$

is trivial is dependent on internal properties of the space Z.

In the case that the codomain of the maps f_i are polyhedral products $(\mathbf{Y}_i,*)^{\mathcal{K}_i}$ with $|\mathcal{K}_i| \geq 2$ for all $i \in [m]$, and there exists $i \in [m]$ such that $|\mathcal{K}_i| > 2$, it follows from (2.15) that the codomain of the higher Whitehead map is the polyhedral product $(\mathbf{Y},*)^{\partial\Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle}$, and therefore there exist complexes $\mathcal{K}$ such that $\partial\Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle \subsetneq \mathcal{K} \subsetneq \Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle$. In this case, there exists a sequence of strict inclusions

$$\left((\mathbf{Y_i},*)^{\mathcal{K}_i},*\right)^{\partial\Delta^{m-1}} = (\mathbf{Y},*)^{\partial\Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle} \subsetneq (\mathbf{Y},*)^{\mathcal{K}} \subsetneq (\mathbf{Y},*)^{\partial\Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle} = \left((\mathbf{Y_i},*)^{\mathcal{K}_i},*\right)^{\Delta^{m-1}}$$

and the triviality of the composite

$$h_w^{\mathcal{K}}(f_1,\ldots,f_m) = \iota\circ h_w(f_1,\ldots,f_m)\colon (\mathbf{CX},\mathbf{X})^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Y},*)^{\partial\Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle} \longrightarrow (\mathbf{Y},*)^{\mathcal{K}}$$

is controlled by the combinatorics of the complex $\mathcal{K}$.

Definition 3.7. A substituted higher Whitehead map is a composite

$$h_w^{\mathcal{K}}(f_1,\ldots,f_m)\colon (\mathbf{CX},\mathbf{X})^{\partial\Delta^{m-1}} \xrightarrow{h_w(f_1,\ldots,f_m)} (\mathbf{Y},*)^{\partial\Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle} \longrightarrow (\mathbf{Y},*)^{\mathcal{K}} \tag{3.9}$$

where $m \geq 2$, and for each $i \in [m]$ the map $f_i\colon \Sigma X_i \longrightarrow (\mathbf{Y}_i,*)^{\mathcal{K}_i}$ is such that $|\mathcal{K}_i| \geq 2$, and there exists $i \in [m]$ with $|\mathcal{K}_i| > 2$, and where $\partial\Delta\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle \subseteq \mathcal{K}$.

The next theorem is a characterisation of when the substituted higher Whitehead map is trivial, in terms of the combinatorics of $\mathcal{K}$.

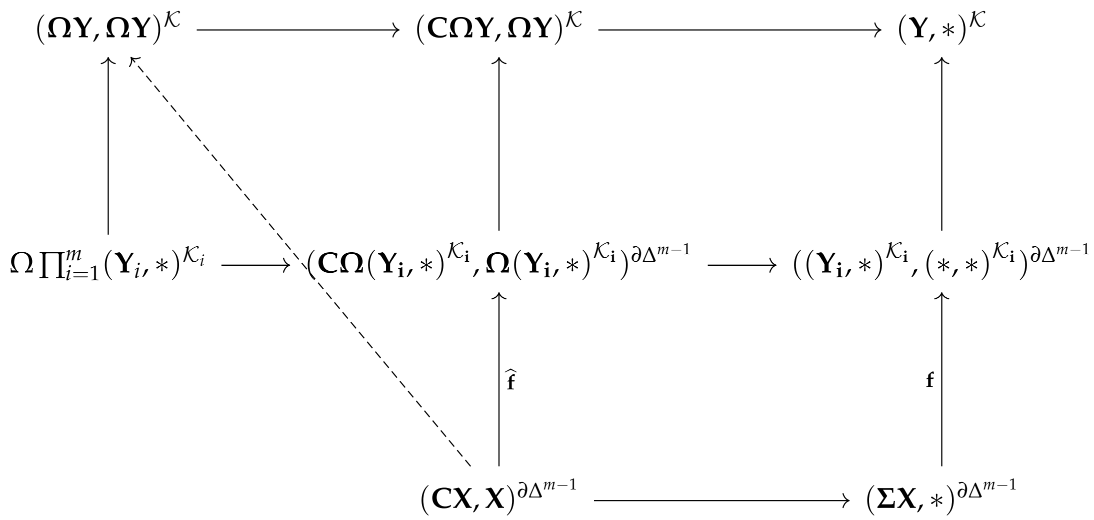
We introduce the following terminology; given a map $f\colon \Sigma X \longrightarrow (\mathbf{Y},*)^{\mathcal{K}}$ which is non-trivial, and a simplicial complex $\mathcal{K}'$ on $[m]$ such that $\mathcal{K} \subseteq \mathcal{K}'$, if the composite $f'\colon \Sigma X \longrightarrow (\mathbf{Y},*)^{\mathcal{K}} \longrightarrow (\mathbf{Y},*)^{\mathcal{K}'}$ is trivial, we say that $\mathcal{K}'$ is a trivialising complex for f. We note that if $\mathcal{K}'$ is a trivialising complex for f, then any simplicial complex $\mathcal{K}''$ such that $\mathcal{K}' \subseteq \mathcal{K}''$ is also a trivialising complex for f. This can be seen by composing with the inclusion induced by the inclusion of simplicial complexes $\mathcal{K}' \longrightarrow \mathcal{K}''$.

Theorem 3.8. *Let $\mathcal{K}$ be a complex on $[l] = [l_1] \sqcup \cdots \sqcup [l_m]$ such that $\partial\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_m\rangle \subseteq \mathcal{K}$. For all $i \in [m]$, suppose that $f_i \colon \Sigma X_i \longrightarrow (\mathbf{Y}_i, *)^{\mathcal{K}_i}$. Then $h_w^{\mathcal{K}}(f_1, \ldots, f_m)$ is trivial if and only if at least one of the following three conditions is satisfied*

1) *The map $f_i \colon \Sigma X_i \longrightarrow (\mathbf{Y}_i, *)^{\mathcal{K}_i}$ is trivial for some $i \in [m]$.*
2) *$\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_m\rangle \subseteq \mathcal{K}$.*
3) *For some $i \in [m]$, $\partial\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_i', \ldots, \mathcal{K}_m\rangle \subseteq \mathcal{K}$, where $\mathcal{K}_i'$ is a trivialising complex for f_i.*

Proof. The proof is analogous to that of Theorem 3.3.

Consider the following diagram



where the top row and middle row are homotopy fibrations, and the composite of the three maps along the bottom and up the right hand side is the substituted higher Whitehead map $h_w^{\mathcal{K}}(f_1, \ldots, f_m)$. Observe that the right-hand rectangle commutes because the bottom square commutes, together with the fact that both vertical arrows are inclusions.

The triviality of the higher Whitehead map is equivalent to the existence of the dashed arrow, which is equivalent to the triviality of the composite

$$\widehat{\mathbf{f}} \colon (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{C\Omega}(\mathbf{Y_i}, *)^{\mathcal{K}_\mathrm{i}}, \mathbf{\Omega}(\mathbf{Y_i}, *)^{\mathcal{K}_\mathrm{i}})^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{C\Omega Y}, \mathbf{\Omega Y})^{\mathcal{K}}$$

and the result follows.

□

3.4 The folded higher Whitehead map

Given maps $f_i \colon \Sigma X_i \longrightarrow Y$, each with the same codomain Y, the higher Whitehead map does not have the codomain Y, but rather the codomain $(\mathbf{Y}, *)^{\partial\Delta^{m-1}}$. In the case of the 2-ary higher Whitehead map, we obtain a map with codomain Y by composing with the fold map

$$\nabla \circ h_w(f_1, f_2) \colon X_1 * X_2 \longrightarrow Y \vee Y \xrightarrow{\nabla} Y$$

when we recover the generalised Whitehead product as defined by Arkowitz (see [3, Definition 2.2]). Whether or not such a Whitehead product is trivial is in general an open problem, see for example [18]. In the special case that Y is an H-space, then $[f_1, f_2]$ is the homotopy class of the composite

$$X_1 * X_2 \xrightarrow{\rho} \Sigma X_1 \vee \Sigma X_2 \hookrightarrow \Sigma X_1 \times \Sigma X_2 \xrightarrow{f_1 \times f_2} Y \times Y \xrightarrow{\mu} Y$$

where μ is the H-multiplication map $\mu \colon Y \times Y \longrightarrow Y$. Therefore $[f_1, f_2]$ is trivial since $\Sigma X_1 \times \Sigma X_2$ is the homotopy cofibre of $X_1 * X_2 \xrightarrow{\rho} \Sigma X_1 \vee \Sigma X_2$, see [21, Theorem 2.3].

In the case of the higher and substituted higher Whitehead maps where $m > 2$, the existence of a map $(Y, *)^{\partial\Delta^{m-1}} \longrightarrow (Y, *)^{\Delta^{m'-1}}$ extending the fold $(Y, *)^{\bullet[m]} \longrightarrow (Y, *)^{\bullet[m']}$, where $1 \leq m' < m$, is equivalent to the condition that Y is an H-space; if Y is not an H-space, no such map exists.

In this section, we analyse the composition of higher and substituted higher Whitehead maps with a fold map. In the case of a (non-substituted) higher Whitehead map, the composite is trivial. In the case of a fold composed with a substituted higher Whitehead map, we characterise the triviality of the resulting composite

$$h_w^{\mathcal{K}}(f_1, \dots, f_m) \colon (\mathbf{\Sigma X}, *)^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}} \xrightarrow{\nabla} (\mathbf{Y}, *)^{\mathcal{K}_\nabla}$$

in terms of the combinatorics of the simplicial complex $\mathcal{K}$.

We begin by defining a fold of simplicial complexes.

Definition 3.9. Let $\mathcal{K}$ be a simplicial complex on $[m]$. For $I \subseteq [m]$ and $j \in [m], j \notin I$, $\mathcal{K}_{\nabla(I,j)}$, is the simplicial complex on $[m] \backslash I$ defined by

$$\mathcal{K}_{\nabla(I,j)} = \{\sigma \subseteq [m] \backslash I \mid \sigma \in \mathcal{K} \text{ or } (\sigma \backslash j) \sqcup i \in \mathcal{K} \text{ for } i \in I\}.$$

We equip the vertex set of $\mathcal{K}_{\nabla(I,j)}$ with the ordering obtained by deletion of the elements of I from $[m]$. We denote by $\nabla(I,j) \colon \mathcal{K} \longrightarrow \mathcal{K}_{\nabla(I,j)}$ the map of simplicial complexes induced by the identification of vertices $i \mapsto j$ for all $i \in I$. We make the abbreviation of $\mathcal{K}_{\nabla(\{i\},j)}$ to $\mathcal{K}_{\nabla(i,j)}$ when $I = \{i\}$ consists of just one element.

The following properties follow immediately from the definition.

Proposition 3.10.

1. *If* $i, j \in [m]$ *with* $i \neq j$*, then* $\mathcal{K}_{\nabla(i,j)} = \mathcal{K}_{\nabla(j,i)}$.
2. *If* $I = \{i_1, \dots, i_n\} \subseteq [m]$*, and* $j \in [m]$ *with* $j \notin I$*, then*
$$\mathcal{K}_{\nabla(I,j)} = \mathcal{K}_{\nabla(i_1,j)_{\nabla(i_2,j)_{\dots\nabla(i_n,j)}}}.$$
3. *If* $I_1, I_2 \subseteq [m]$ *are such that* $I_1 \cap I_2 = \emptyset$*, and* $j \notin I_1 \sqcup I_2$*, then*
$$\mathcal{K}_{\nabla(I_1,j)_{\nabla(I_2,j)}} = \mathcal{K}_{\nabla(I_2,j)_{\nabla(I_1,j)}}.$$
4. *Let* $\mathcal{K}_1, \mathcal{K}_2$ *be simplicial complexes on* $[m_1], [m_2]$ *respectively. Let* $\mathcal{L}_1 \cong \mathcal{L}_2 \cong \mathcal{L}$ *be isomorphic full subcomplexes of* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *respectively, with vertex sets* $\{l_{1_1}, \dots, l_{n_1}\}$ *and* $\{l_{1_2}, \dots, l_{n_2}\}$ *respectively, where* $l_{n_1} = l_{n_2}$*. Let* $\mathcal{K} = \mathcal{K}_1 \sqcup \mathcal{K}_2$*. Then*
$$\mathcal{K}_1 \bigcup_{\mathcal{L}} \mathcal{K}_2 = \mathcal{K}_{\nabla(l_{1_1},l_{1_2})_{\nabla(l_{2_1},l_{2_2})_{\dots\nabla(l_{n_1},l_{n_2})}}}.$$

We now define a fold of full subcomplexes. Suppose that for $J_1 = \{j_{1_1}, \dots, j_{1_{n_1}}\}, J_2 = \{j_{2_1}, \dots, j_{2_{n_2}}\} \subseteq [m]$ with $J_1 \cap J_2 = \emptyset$, a surjective map $\psi\colon J_1 \longrightarrow J_2$, $j_{1_l} \mapsto j_{2_l}$ is given. Then the fold of $\mathcal{K}_{J_1}$ onto $\mathcal{K}_{J_2}$ by ψ is defined by

$$\mathcal{K}_{\nabla(J_1,J_2)_\psi} = \mathcal{K}_{\nabla(j_{1_1},\psi(j_{1_1}))_{\dots\nabla(j_{1_n},\psi(j_{1_n}))}} \tag{3.10}$$

We denote by $\nabla(J_1, J_2)_\psi\colon \mathcal{K} \longrightarrow \mathcal{K}_{\nabla(J_1,J_2)_\psi}$ the induced map of simplicial complexes. Where the map ψ does not need to be emphasised, we omit it from the notation.

We now define the folded higher Whitehead map.

Definition 3.11. Consider a substituted higher Whitehead map $h_w^{\mathcal{K}}(f_1, \dots, f_m)$ with codomain $(\mathbf{Y}, *)^{\mathcal{K}}$, where the vertex set of $\mathcal{K}$ is $[l]$. Let $I, J \subseteq [l]$ with $I \cap J = \emptyset$, and let $\psi\colon I \longrightarrow J$ be such that for all $i \in I$, $Y_i = Y_{\psi(i)}$, and Y_i is an H-space. Let $\mathcal{K}_{\nabla(I,J)}$ denote the image of $\mathcal{K}$ under the fold by ψ and denote the induced map of polyhedral products by $\nabla_{(I,J)}\colon (\mathbf{Y}, *)^{\mathcal{K}} \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_{\nabla(I,J)}}$. The folded higher Whitehead map is the composite

$$\nabla_{(I,J)} \circ h_w^{\mathcal{K}}(f_1, \dots, f_m)\colon (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}} \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_{\nabla(I,J)}}.$$

For the statement of the main theorem in this section, we begin with two preparatory Lemmata.

Lemma 3.12. *Let $m \geq 2$. For $i \in [m]$, let $f_i \colon \Sigma X_i \longrightarrow (Y,*)^{\mathcal{K}_i}$ where the vertex set of $\mathcal{K}_i$ is denoted $[l_i]$. Suppose for simplicial complexes $\mathcal{K}, \mathcal{K}'$ on $[l] = [\sum l_i]$ such that $\partial\Delta^{m-1}\langle \mathcal{K}_1, \ldots, \mathcal{K}_m \rangle \subseteq \mathcal{K}, \mathcal{K}'$, that $\mathcal{K}_{\nabla(I,J)} = \mathcal{K}'_{\nabla(I,J)}$. Then*

$$\nabla_{(I,J)} \circ h_w^{\mathcal{K}}(f_1, \ldots, f_m) = \nabla_{(I,J)} \circ h_w^{\mathcal{K}'}(f_1, \ldots, f_m).$$

Proof. The claim is equivalent to the claim that the following diagram commutes, where the top row is $\nabla_{(I,J)} \circ h_w^{\mathcal{K}}(f_1, \ldots, f_m)$, and the bottom row is $\nabla_{(I,J)} \circ h_w^{\mathcal{K}'}(f_1, \ldots, f_m)$.

$$\begin{array}{ccccccc}
X_1 * \cdots * X_m & \xrightarrow{h_w(f_1,\ldots,f_m)} & (\mathbf{Y},*)^{\partial\Delta^{m-1}\langle \mathcal{K}_1,\ldots,\mathcal{K}_m\rangle} & \hookrightarrow & (\mathbf{Y},*)^{\mathcal{K}} & \xrightarrow{\nabla_{(I,J)}} & (\mathbf{Y},*)^{\mathcal{K}_{\nabla(I,J)}} \\
\| & & \| & & & & \| \\
X_1 * \cdots * X_m & \xrightarrow{h_w(f_1,\ldots,f_m)} & (\mathbf{Y},*)^{\partial\Delta^{m-1}\langle \mathcal{K}_1,\ldots,\mathcal{K}_m\rangle} & \hookrightarrow & (\mathbf{Y},*)^{\mathcal{K}'} & \xrightarrow{\nabla_{(I,J)}} & (\mathbf{Y},*)^{\mathcal{K}'_{\nabla(I,J)}}
\end{array}$$

Commutativity of the left-hand square is immediate, and commutativity of the right-hand rectangle follows from the asssumption that $\mathcal{K}_{\nabla(I,J)} = \mathcal{K}'_{\nabla(I,J)}$. □

We define the following complex. Let $I \subseteq [l]$ and let $j \in [l]$ where $j \notin I$. We define

$$\mathcal{L} = \mathcal{K}_{\nabla(I,j)}\langle \bullet, \ldots, \bullet, \Delta^{|I|-1}, \bullet, \ldots, \bullet \rangle$$

where the substitution occurs at the vertex $\{j\}$.

To paraphrase, the next lemma is the statement that $\mathcal{L}$ is the largest complex among all pre-images of the folded complex $\mathcal{K}_{\nabla(I,j)}$, when the fold is considered as an operation from simplicial complexes on vertex set $[l]$ to simplicial complexes on vertex set $[l - |I| + 1|$.

Lemma 3.13. *Suppose that a complex $\mathcal{K}'$ is such that $\mathcal{K}'_{\nabla(I,j)} \subseteq \mathcal{K}_{\nabla(I,j)}$ Then $\mathcal{K}' \subseteq \mathcal{L}$.*

Proof. We show that for all $\sigma \in \mathcal{K}'$, then $\sigma \in \mathcal{L}$. Let $\sigma = \sigma_I \sqcup \sigma_{[m]\backslash I} \in \mathcal{K}'$, where $\sigma_I \subseteq I$ and $\sigma_{[m]\backslash I} \subseteq [m]\backslash I$. If $\sigma_I = \emptyset$, the claim follows immediately. Suppose that $\sigma_I \neq \emptyset$. Since $\mathcal{L}'_{\nabla(I,j)} \subseteq \mathcal{K}_{\nabla(I,j)}$, then $\nabla(I,j)(\sigma) = \{j\} \sqcup \sigma_{[m]\backslash I} \in \mathcal{K}_{\nabla(I,j)}$. Since by construction $I \sqcup \{j\} \sqcup \sigma_{[m]\backslash I} \in \mathcal{L}$, then since $\sigma_I \subseteq I$, it follows that $\sigma \in \mathcal{L}$. □

The following Theorem characterises the triviality of the folded higher Whitehead map in terms of the underlying combinatorics.

For $m \geq 2$, let $\mathcal{K}$ be a simplicial complex on $[l] = [l_1] \sqcup \cdots \sqcup [l_m]$ such that $\partial\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_m \rangle \subseteq \mathcal{K}$.

Theorem 3.14. *Suppose that for all $i \in I$, $Y_i = Y_j$ is an H-space. Then, the following are equivalent.*

1. *The folded higher Whitehead map*

$$\nabla_{(I,j)} \circ h_w^{\mathcal{K}}(f_1, \ldots, f_m) \colon X_1 * \cdots * X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_{\nabla(I,j)}}$$

   is trivial,

2. *Either*
   - *(i) There exists $\mathcal{K}'$ such that $\mathcal{K}_{\nabla(I,j)} = \mathcal{K}'_{\nabla(I,j)}$, where $\mathcal{K}'$ is such that for some $i \in [m]$, $\mathcal{K}' \supseteq \partial\Delta^{m-1}\langle \mathcal{K}_1, \ldots, \mathcal{K}'_i, \ldots, \mathcal{K}_m \rangle$ where $\mathcal{K}'_i$ is a trivialising complex for $f_i \colon \Sigma X_i \longrightarrow (\mathbf{Y}_i, *)^{\mathcal{K}_i}$, or*
   - *(ii) There exists $\mathcal{K}'$ such that $\mathcal{K}_{\nabla(I,j)} = \mathcal{K}'_{\nabla(I,j)}$, where $\mathcal{K}' \supseteq \Delta^{m-1}\langle \mathcal{K}_1, \ldots, \mathcal{K}_m \rangle$*

3. *The map $h_w^{\mathcal{L}}(f_1, \ldots, f_m)$ is trivial.*

Proof. We first show that *(1)* $\iff$ *(2)*.

We prove the forward implication first. The map $\nabla_{(I,j)} \circ h_w^{\mathcal{K}}$ is trivial if and only if there exists an extension $\overline{\phi}$ for which the following diagram commutes up to homotopy

$$\begin{array}{ccccccc}
(\mathbf{\Sigma X}, *)^{\partial\Delta^{m-1}} & \longrightarrow & (\mathbf{Y}, *)^{\partial\Delta^{m-1}\langle \mathcal{K}_1, \ldots, \mathcal{K}_m \rangle} & \stackrel{\iota}{\hookrightarrow} & (\mathbf{Y}, *)^{\mathcal{K}} & \longrightarrow & (\mathbf{Y}, *)^{\mathcal{K}_{\nabla(I,j)}} \\
\big\downarrow & & & & & \overset{\overline{\phi}}{\nearrow} & \\
\prod_{i=1}^{m} \Sigma X_i & & & & & &
\end{array} \tag{3.11}$$

Composing with the inclusion $\iota \colon (\mathbf{Y}, *)^{\mathcal{K}_{\nabla(I,j)}} \longrightarrow \prod_{i \in [l] \setminus I} Y_i$, it follows from the universal property of the product that $\iota \circ \overline{\phi} \simeq \nabla_{(I,j)} \circ (f_1 \times \cdots \times f_m)$. This implies that either $\mathcal{K}_{\nabla(I,J)} \supseteq \Delta^{m-1}\langle \mathcal{K}_1, \ldots, \mathcal{K}_m \rangle$, or that there exists $i \in [m]$ such that

$$\nabla_{(I,j)} \circ (f_1 \times \cdots \times f_m) \simeq \nabla_{(I,j)} \circ (f_1 \times \cdots \times f_{i-1} \times * \times f_{i+1} \times \cdots \times f_m)$$

which implies that $\mathcal{K}_{\nabla(I,J)} = \mathcal{K}'_{\nabla(I,J)}$, for some $\mathcal{K}' \supseteq \partial\Delta^{m-1}\langle \mathcal{K}_1, \ldots, \mathcal{K}'_i, \ldots, \mathcal{K}_m \rangle$, where $\mathcal{K}'_i$ is a trivialising complex for f_i.

We now prove the backwards implication. Suppose that there exists $\mathcal{K}'$ such that $\mathcal{K}_{\nabla(I,j)} = \mathcal{K}'_{\nabla(I,j)}$, where $\mathcal{K}'$ is such that for some $i \in [m]$, $\mathcal{K}' \supseteq \partial\Delta^{m-1}\langle \mathcal{K}_1, \ldots, \mathcal{K}'_i, \ldots, \mathcal{K}_m \rangle$ where $\mathcal{K}'_i$ is a trivialising complex for $f_i \colon \Sigma X_i \longrightarrow (\mathbf{Y}_i, *)^{\mathcal{K}_i}$. Then, by Lemma 3.12 $\nabla_{(I,j)} \circ h_w^{\mathcal{K}} = \nabla_{(I,j)} \circ h_w^{\mathcal{K}'}$, and since $h_w^{\mathcal{K}'}$ is trivial by Theorem 3.8, then $\nabla_{(I,j)} \circ h_w^{\mathcal{K}}$ is trivial.

On the other hand, if $\mathcal{K}_{\nabla(I,j)} \supseteq \Delta^{m-1}\langle \partial\Delta^{l_1-1}, \dots, \partial\Delta^{l_m-1}\rangle$, then the map $\overline{\phi} = \nabla_{(I,j)} \circ (f_1 \times \cdots \times f_m)$ makes the diagram (3.11) commute up to homotopy, and therefore $\nabla_{(I,j)} \circ h_w^{\mathcal{K}}$.

We now show that *(2)* $\iff$ *(3)*. We prove the forward implication first. Suppose that either *(2)(i)* or *(2)(ii)* holds. Then by Lemma 3.13, $\mathcal{K}' \subseteq \mathcal{L}$. Then $h_w^{\mathcal{L}}$ is equal to the composite

$$h_w^{\mathcal{L}} \colon X_1 * \cdots * X_m \xrightarrow{h_w^{\mathcal{K}'}} (\mathbf{Y}, *)^{\mathcal{K}'} \hookrightarrow (\mathbf{Y}, *)^{\mathcal{L}},$$

and since $h_w^{\mathcal{K}'}$ is trivial, then $h_w^{\mathcal{L}}$ is trivial.

We now prove the backwards implication. Suppose that $h_w^{\mathcal{L}}$ is trivial. Then, since $\mathcal{L}_{\nabla(I,j)} = \mathcal{K}_{\nabla(I,j)}$, it follows from Lemma 3.12 that $\nabla_{(I,j)} \circ h_w^{\mathcal{K}}$ is trivial. Since *(1)* $\iff$ *(2)*, the claim follows. □

Example 3.15. *i) Let $\mathcal{K} = \partial\Delta^2 \sqcup \bullet$ on the vertex set $[4]$. For each $i \in [4]$, suppose that $X_i = S^1$, $Y_i = \mathbb{C}P^\infty$ and $f_i \colon \Sigma S^1 = S^2 \longrightarrow \mathbb{C}P^\infty$ is the inclusion of the bottom cell. Denote by $\nabla_{(4,1)}$ the fold of vertex 4 into vertex 1.*

Then by Theorem 3.14, the folded higher Whitehead map

$$\nabla_{(4,1)} \circ h_w(h_w(f_1, f_2, f_3, f_4) \colon S^6 \longrightarrow (\mathbb{C}P^\infty, *)^{\partial\Delta^2}$$

is non-trivial.

Since there is a homotopy fibration sequence

$$\prod_{i=1}^{m} S^1 \longrightarrow \mathcal{Z}_{\mathcal{K}} \longrightarrow DJ_{\mathcal{K}} \longrightarrow \prod_{i=1}^{m} \mathbb{C}P^\infty$$

where $DJ_{\mathcal{K}} = (\mathbb{C}P^\infty, *)^{\mathcal{K}}$ is the Davis-Januszkiewics space, we obtain that for $n \geq 3$, $\pi_n(DJ_{\mathcal{K}}) \cong \pi_n(\mathcal{Z}_{\mathcal{K}})$. In this case, we obtain that there is a non-trivial map $S^6 \longrightarrow \mathcal{Z}_{\mathcal{K}} \simeq S^5$.

ii) Let $\mathcal{K} = \bullet_{[5]}$. For each $i \in [5]$, suppose that $X_i = S^1, Y_i = \mathbb{C}P^\infty$, and $f_i \colon S^2 \longrightarrow \mathbb{C}P^\infty$ is again the inclusion of the bottom cell.

Denote by $\nabla_{\{3,4,5\},\{1,2\}}$ a choice of fold which folds each of the vertices in $\{3,4,5\}$ into either vertex $\{1\}$ or $\{2\}$. The resulting folded higher Whitehead map

$$\nabla_{\{3,4,5\},\{1,2\}} \circ h_w(h_w(h_w(h_w(f_1, f_2), f_3), f_4), f_5) \colon S^6 \longrightarrow \mathbb{C}P^\infty \vee \mathbb{C}P^\infty$$

is non-trivial. By the same argument as in the previous example, for each choice of fold $\nabla_{\{3,4,5\},\{1,2\}}$ we obtain a non-trivial map $S^6 \longrightarrow S^3$.

3.5 Relations

In this section we examine relations among the homotopy classes of higher Whitehead maps. These relations are realised in polyhedral products associated to a family of simplicial complexes which we refer to as identity complexes. We begin by defining these complexes. We then studying their combinatorial properties, before proceeding with our study of the relations themselves.

Given a vertex set $[m]$, a k-partition Π is a collection $\{P_1, \ldots, P_k\}$ of pairwise disjoint subsets of $[m]$ such that $\bigcup_{i=1}^{k} P_i = [m]$.

Definition 3.16. Let $\Pi = \{P_1, \ldots, P_k\}$ denote a k-partition of of $[m]$, where for $i \in [k]$, $P_i = \{p_{i_1}, \ldots, p_{i_{n_i}}\}$. The identity complex $\mathcal{K}_\Pi$ is defined by

$$\mathcal{K}_\Pi = \mathrm{sk}^{k-3}\,\Delta^{k-1}(\partial\Delta^{n_1-1}, \ldots, \partial\Delta^{n_k-1}).$$

We make use of two further equivalent definitions of identity complexes, in order to streamline our subsequent discussion of their properties. In order to prove the equivalence of these three definitions, we recall Proposition 2.24, which describes the minimal missing faces of polyhedral join products.

Proposition 2.24. *Let $\mathcal{K}$ be a simplicial complex on $[m]$, and let $(\mathcal{K}_1, \mathcal{L}_1), \ldots, (\mathcal{K}_m, \mathcal{L}_m)$ be simplicial pairs on vertex sets $[l_1], \ldots, [l_m]$ respectively. Then,*

$$MF\left((\boldsymbol{\mathcal{K}}_{\mathbf{i}}, \boldsymbol{\mathcal{L}}_{\mathbf{i}})^{*\mathcal{K}}\right) = \{J \in MF(\mathcal{K}_i) \mid i \in \mathcal{K}\} \sqcup \left\{\bigsqcup_{i \in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{L}_i), J_i \in \mathcal{K}_i\right\}.$$

It follows from Proposition 2.24 that the minimal missing faces of $\mathcal{K}(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ are given by

$$MF(\mathcal{K}(\mathcal{K}_1, \ldots, \mathcal{K}_m)) = \left\{\bigsqcup_{i \in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{K}_i)\right\}$$

so that

$$MF(\mathrm{sk}^{k-3}\,\Delta^{k-1}(\partial\Delta^{n_1-1}, \ldots, \partial\Delta^{n_k-1})) = \{\{[m] \backslash \{i_1, \ldots, i_{n_i}\}\} \mid i = 1, \ldots, k\}\,. \qquad (3.12)$$

Since a simplicial complex is determined by its set of minimal missing faces, an equivalent definition of $\mathcal{K}_\Pi$, the second of our three equivalent definitions, is to specify

$$MF(\mathcal{K}_\Pi) = \{[m] \backslash P_i \mid i \in [k]\}. \qquad (3.13)$$

The following is the third equivalent definition of $\mathcal{K}_\Pi$.

For a simplicial complex $\mathcal{K}$ on $[m]$, given a subset $J \subseteq [m]$, if $\partial\Delta^{|J|-1} \subseteq \mathcal{K}_J$, we denote by $\partial\Delta[j_1, \ldots, j_m]$ the subcomplex $\partial\Delta^{|J|-1} \subseteq \mathcal{K}_J$. Observe that this is not necessarily a full subcomplex.

Proposition 3.17. *Let $[m] = [n_1] \sqcup \cdots \sqcup [n_k]$ and denote the vertex set $[n_i]$ by $\{i_1, \ldots, i_{n_i}\}$. Then,*

$$\mathrm{sk}^{k-3}\,\Delta^{k-1}(\partial\Delta^{n_1}, \ldots, \partial\Delta^{n_k}) = \bigcup_{i=1}^{k} \partial\Delta\langle \partial\Delta[j_1, \ldots, j_{r_i}], i_1, \ldots, i_{n_i}\rangle$$

where $\{j_1, \ldots, j_{r_i}\} = [m]\backslash\{i_1, \ldots, i_{n_i}\}$.

Proof. Recall that two finite simplicial complexes are equal if and only if their Alexander duals are equal, and that the maximal simplices of the Alexander dual $\widehat{\mathcal{L}}$ of any simplicial complex $\mathcal{L}$ are the complements to the minimal missing faces of $\mathcal{L}$. Therefore by (i), the Alexander dual of $\mathrm{sk}^{k-3}\,\Delta^{k-1}(\partial\Delta^{n_1}, \ldots, \partial\Delta^{n_k})$ is given by $\bigsqcup_{i=1}^{k} \Delta[i_1, \ldots, i_{n_i}]$.

On the other hand, let $\mathcal{K}^i = \partial\Delta\langle\partial\Delta[j_1, \ldots, j_{r_i}], i_1, \ldots, i_{n_i}\rangle$. Then by Proposition 2.24, for $i = 1, \ldots, k$,

$$\begin{aligned}\widehat{\mathcal{K}^i}_{max} &= \{[m]\backslash\{j_1, \ldots, j_{r_i}\} \sqcup \{[m]\backslash\{j, i_1, \ldots, i_{n_i}\} \mid j \in J_i\} \\ &= \{i_1, \ldots, i_{n_i}\} \sqcup \{J_i\backslash\{j\} \mid j \in J_i\}.\end{aligned}$$

Then

$$\widehat{\bigcup_{i=1}^{k} \mathcal{K}^i} = \bigcap_{i=1}^{k} \widehat{\mathcal{K}^i} = \bigcap_{i=1}^{k} \left(\widehat{\mathcal{K}^i}_{max}\right) = \bigsqcup_{i=1}^{k} \Delta[i_1, \ldots, i_{n_i}]$$

and the result follows. □

We are now ready to define the spaces which are the codomains of the higher Whitehead maps which we will prove satisfy certain families of relations.

Let $\Pi = \{P_1, \ldots, P_k\}$ be a k-partition of $[m]$, and for each $i \in [k]$ denote by $P_i = \{p_{i_1}, \ldots, p_{i_{n_i}}\}$ and by $Q_i = [m]\backslash P_i = \{q_{i_1}, \ldots, q_{i_{r_i}}\}$. Suppose that we are given maps $f_l \colon \Sigma X_l \longrightarrow Y_l$ for each $l \in [m]$. It follows from Proposition 3.17 that for each $i \in [k]$, $\partial\Delta\langle\partial\Delta[q_{i_1}, \ldots, q_{i_{r_i}}], p_{i_1}, \ldots, p_{i_{n_i}}\rangle \subseteq \mathcal{K}_\Pi$, so that by Definition 3.1, for all $i \in [k]$, the substituted higher Whitehead maps

$$\xi_i \circ h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_{q_{i_1}}, \ldots, f_{q_{i_{r_i}}}\right), f_{p_{i_1}}, \ldots, f_{p_{i_{n_i}}}\right) \circ \sigma_i \colon \Sigma^{m-2} X_1 \wedge \cdots \wedge X^m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi} \tag{3.14}$$

are well-defined, where for each $i \in [k]$, ξ_i is the restriction of the permutation

$$Y_{q_{i_1}} \times \cdots \times Y_{q_{i_{r_i}}} \times Y_{p_{i_1}} \times \cdots \times Y_{p_{i_{n_i}}} \longrightarrow Y_1 \times \cdots \times Y_m$$

and where σ_i is the permutation

$$\Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow \Sigma^{n_i}\left(\Sigma^{r_i-2}X_{q_{i_1}} \wedge \cdots \wedge X_{q_{i_{r_i}}}\right) \wedge X_{p_{i_1}} \wedge \cdots \wedge X_{p_{i_{n_i}}}.$$

To keep the notation from becoming too cumbersome, for each $i \in [k]$, we denote the map (3.14) by ${}_ih_w^{\mathcal{K}_\Pi}$.

With a view to analysing the relations among these higher Whitehead maps, we begin by proving that they're non-trivial.

Proposition 3.18. *Suppose that for each $l \in [m]$, $f_l\colon \Sigma X_l \longrightarrow Y_l$ is non-trivial. Then for each $i = 1, \ldots, k$, the map*

$${}_ih_w^{\mathcal{K}_\Pi}\colon \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi}$$

is non-trivial.

Proof. Fix a choice of $i \in [k]$. By Theorem 3.8, since ξ_i and σ_i are homeomorphisms, then ${}_ih_w^{\mathcal{K}_\Pi}$ is trivial if and only if either $\Delta\langle\partial\Delta[q_{i_1}, \ldots, q_{i_{r_i}}], p_{i_1}, \ldots, p_{i_{n_i}}\rangle \subseteq \mathcal{K}_\Pi$ or $\partial\Delta\langle\Delta[q_{i_1}, \ldots, q_{i_{r_i}}], p_{i_1}, \ldots, p_{i_{n_i}}\rangle \subseteq \mathcal{K}_\Pi$. Since by (3.12) $Q_i = \{q_{i_1}, \ldots, q_{i_{r_i}}\}$ is a missing face of $\mathcal{K}_\Pi$, then $\partial\Delta\langle\Delta[j_1, \ldots, j_{r_i}], i_1, \ldots, i_{n_i}\rangle$ is not a subcomplex. Furthermore, since for all $i' \in [k]$, $Q_{i'}$ is a minimal missing face of $\mathcal{K}_\Pi$, but the only minimal missing face of $\Delta\langle\partial\Delta[q_{i_1}, \ldots, q_{i_{r_i}}], p_{i_1}, \ldots, p_{i_{n_i}}\rangle \subseteq \mathcal{K}_\Pi$ is Q_i, $\Delta\langle\partial\Delta[q_{i_1}, \ldots, q_{i_{r_i}}], p_{i_1}, \ldots, p_{i_{n_i}}\rangle$ is also not a subcomplex of $\mathcal{K}_\Pi$. □

We now proceed to the statement of our main theorem, which is that the homotopy classes of these maps satisfy a relation encoded by the combinatorics of $\mathcal{K}_\Pi$.

Theorem 3.19. *Let $\Pi = \{P_1, \ldots, P_k\}$ be a k-partition of $[m]$, and for each $i \in [k]$ denote by $P_i = \{p_{i_1}, \ldots, p_{i_{n_i}}\}$ and by $Q_i = [m]\backslash P_i = \{q_{i_1}, \ldots, q_{i_{r_i}}\}$. Suppose that we are given maps $f_l\colon \Sigma X_l \longrightarrow Y_l$ for each $l \in [m]$ such that $X_l = \Sigma\widetilde{X_l}$. Then the map*

$$\sum_{i=1}^{k} {}_ih_w^{\mathcal{K}_\Pi}\colon \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi} \tag{3.15}$$

is trivial, where each summand is non-trivial.

We postpone the proof of Theorem 3.19 to Section 3.7.

We proceed with examples after the following proposition, which proves that the relation of Theorem 3.19 is minimal, in an appropriate sense.

Proposition 3.20. *Under the assumptions of Theorem 3.19, for a subset $J \subsetneq [k]$, the map*

$$\sum_{i\in J} {}_ih_w^{\mathcal{K}_\Pi} \colon \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi}$$

is non-trivial.

Proof. For contradiction, suppose that there exists $J \subsetneq [k]$ such that $\sum_{i\in J} {}_ih_w^{\mathcal{K}_\Pi}$ is trivial.

Recall that the minimal missing faces of $\mathcal{K}_\Pi$ are $MF(\mathcal{K}_\Pi) = \{[m]\backslash P_i \mid i \in [k]\}$. Fix a choice of $s \in [k]\backslash J$. Define the complex $\mathcal{L} = \mathcal{K}_\Pi \sqcup \{P_i \mid i \in [k]\backslash J, i \neq s\}$, whose minimal missing faces are $\{[m]\backslash P_i \mid i \in J \sqcup \{s\}\}$.

Composing with the inclusion induced by the inclusion of simplicial complexes $\mathcal{K}_\Pi \longrightarrow \mathcal{L}$, it follows that for each $i \in [k]\backslash J$ with $i \neq s$, the composite

$$\iota \circ {}_ih_w^{\mathcal{K}_\Pi} \colon \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi} \longrightarrow (\mathbf{Y}, *)^{\mathcal{L}}$$

is trivial, whilst the same composite for $i \in J \sqcup \{s\}$ is non-trivial. Therefore

$$\begin{aligned}\sum_{i\in[k]} \iota \circ {}_ih_w^{\mathcal{K}_\Pi} &= \sum_{i\in J\sqcup\{s\}} \iota \circ {}_ih_w^{\mathcal{K}_\Pi} \colon \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi} \longrightarrow (\mathbf{Y}, *)^{\mathcal{L}} \\ &= \left(\sum_{i\in J} \iota \circ {}_ih_w^{\mathcal{K}_\Pi}\right) + \iota \circ {}_sh_w^{\mathcal{K}_\Pi} \colon \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{L}}.\end{aligned}$$

Since $\sum_{i\in[k]} \iota \circ {}_ih_w^{\mathcal{K}_\Pi}$ is trivial, and $\left(\sum_{i\in J} \iota \circ {}_ih_w^{\mathcal{K}_\Pi}\right)$ is trivial by assumption, it follows that the map $\iota \circ_s h_w^{\mathcal{K}_\Pi}$ is trivial, which is a contradiction.

□

In the following examples, we show how the theorem gives rise to relations among higher Whitehead maps of varying arity.

Example 3.21. *Let $m \geq 3$. For all $i \in [m]$, let $f_i \colon \Sigma X_i \longrightarrow Y_i$ such that $X_i = \Sigma\widetilde{X}_i$.*

1) *Let $\Pi_1 = \{\{1\}, \{2\}, \{3\}\}$. Then $J_1 = \{2, 3\}$, $J_2 = \{1, 3\}$ and $J_3 = \{1, 2\}$, and $\mathcal{K}_{\Pi_1} = \bullet_{[3]}$. Then*

$$\begin{aligned}&h_w^{\mathcal{K}_{\Pi_1}}(h_w(f_2, f_3), f_1) \circ \sigma_1 + h_w^{\mathcal{K}_{\Pi_1}}(h_w(f_1, f_3), f_2) \circ \sigma_2 \\ &\qquad + h_w^{\mathcal{K}_{\Pi_1}}(h_w(f_1, f_2), f_3) \circ \sigma_3 \colon \Sigma X_1 \wedge X^2 \wedge X^3 \longrightarrow Y_1 \vee Y_2 \vee Y_3\end{aligned}$$

is trivial.

In the case that $Y_1 = Y_2 = Y_3$, then by composing with the fold map, we recover the graded Jacobi-identity for generalised Whitehead products.

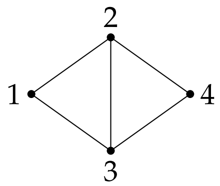
2) *Generalising the previous example, let* $\Pi_2 = \{\{1\}, \dots, \{m\}\}$. *Then* $k = m$, *and for each* $i \in [m]$, $Q_i = \{1, \dots, \hat{i}, \dots, m\}$ *for* $i \in [m]$ *and* $\mathcal{K}_{\Pi_2} = \mathrm{sk}^{m-3}\,\Delta^{m-1}$. *Then*

$$
\begin{aligned}
&{}_1h_w^{\mathcal{K}_{\Pi_2}}\left(h_w\left(f_2, \dots, f_m\right), f_1\right) + \cdots \\
&\qquad + {}_m\,h_w^{\mathcal{K}_{\Pi_2}}\left(h_w\left(f_1, \dots, f_{m-1}\right), f_m\right) : \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_{\Pi_2}}
\end{aligned}
$$

is trivial. This recovers Hardie's relation [16], which he proved in the special case that each X_i *is a sphere.*

3) *Let* $\Pi_3 = \{\{1\}, \{2,3\}, \{4\}\}$. *Then* $Q_1 = \{2,3,4\}$, $Q_2 = \{1,4\}$ *and* $Q_3 = \{1,2,3\}$. *Then* $\mathcal{K}_{\Pi_3}$ *is depicted in Figure 3.1, and*

FIGURE 3.1: A drawing of $\mathcal{K}_{\Pi_3}$



$$
\begin{aligned}
&{}_1h_w^{\mathcal{K}_{\Pi_3}}(h_w(f_2, f_3, f_4), f_1) + {}_2h_w^{\mathcal{K}_{\Pi_3}}(h_w(f_1, f_4), f_2, f_3) \\
&\qquad + {}_3h_w^{\mathcal{K}_{\Pi_3}}(h_w(f_1, f_2, f_3), f_4) \colon \Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4 \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_{\Pi_3}}
\end{aligned}
$$

is trivial.

4) *Let* $\Pi = \{\{1\}, \{2,3\}, \{4\}\}$ *and let* $\mathcal{K}_1 = \partial\Delta^2$ *and* $\mathcal{K}_4 = \partial\Delta^1$. *Let* $g_{1_1}, g_{1_2}, g_{1_3}, g_{4_1}, g_{4_2}$ *be non-trivial maps, and let*

$$
\begin{aligned}
g_1 &= h_w(g_{1_1}, g_{1_2}, g_{1_3}) \colon \Sigma^2 X_{1_1} \wedge X_{1_2} \wedge X_{1_3} \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_1} \qquad \text{and} \\
g_4 &= h_w(g_{4_1}, g_{4_2}) \colon \Sigma X_{4_1} \wedge X_{4_2} \longrightarrow (\mathbf{Y}, *)^{\partial\Delta^1}
\end{aligned}
$$

and let $g_i \colon \Sigma X_i \longrightarrow Y_i$ *be non-trivial maps for* $i = 2, 3$.

Consider the fold $\nabla_{4_1, 1_1}$. *Then*

$$
\begin{aligned}
&\nabla_{(4_1,1_1)} \circ h_w(h_w(g_2, g_3, h_w(g_{4_1}g_{4_2}), h_w(g_{1_1}, g_{1_2}, g_{1_3}) \circ \sigma_1 \\
&+\nabla_{(4_1,1_1)} \circ h_w(h_w(h_w(g_{1_1}, g_{1_2}, g_{1_3}), h_w(g_{4_1}, g_{4_2})), g_2, g_3) \circ \sigma_2 \\
&+\nabla_{(4_1,1_1)} \circ h_w(h_w(h_w(g_{1_1}, g_{1_2}, g_{1_3}), g_2, g_3), h_w(g_{4_1}, g_{4_2})) \colon \\
&\qquad \Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4 \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{L}}
\end{aligned}
$$

is trivial, where $\mathcal{L} = \partial\Delta\langle\partial\Delta^2, 2, 3, 4_2\rangle$. *Furthermore, each term is non-trivial, since* $1_1 \in J$ *is contained in the missing face* $\{1_1, 1_2, 1_3\}$ *of* $\mathcal{K}_1$.

3.6 Applications to the higher Whitehead product

In this section, we apply our results to the study of the higher Whitehead product of Porter [21].

We begin with the definition.

Definition 3.22. Suppose that we are given maps $f_i\colon \Sigma X_i \longrightarrow Y$ for each $i \in [m]$, $m \geq 2$. Denote by

$$\rho\colon \mathop{\ast}_{i=1}^{m} X_i \longrightarrow FW(\Sigma X_1, \ldots, \Sigma X_m)$$

the map (3.3). The higher Whitehead product is the set

$$[f_1, \ldots, f_m] = \{[\phi \circ \rho] \; |\phi|_{\Sigma X_1 \vee \cdots \vee \Sigma X_m} \simeq f_1 \vee \cdots \vee f_m\}$$

Observe that in the case that $m = 2$, then the higher Whitehead product $[f_1, f_2]$ contains exactly one element, and composing $[f_1, f_2] \in [X_1 * X_2, Y \vee Y]$ with the fold map recovers the generalised Whitehead product.

If $m \geq 3$, the higher Whitehead product map be empty, may contain a single class, or may contain many classes. In the latter case, the higher Whitehead product is said to have non-trivial indeterminacy.

If a higher Whitehead product contains the trivial class, then it is said to be trivial. Notice that with this definition, a higher Whitehead product containing non-trivial classes may nevertheless be trivial.

Abramyan and Panov's [1] work established the study of higher Whitehead products in moment-angle complexes. By constructing and analysing particular classes within certain higher Whitehead products, they characterise[1, Theorem 5.2] both the non-emptiness and non-triviality of these higher Whitehead products in $\pi_*(\mathcal{Z}_\mathcal{K})$ in terms of the underlying combinatorics. The particular classes which Abramyan and Panov analyse are higher Whitehead maps of inclusions $\mu_i\colon S^2 \hookrightarrow \mathbb{C}P^\infty$.

In the Example 3.24, we answer a question posed in [1, Example 7.3]. Using the higher Whitehead map, we construct an infinite family of trivial higher Whitehead products with non-trivial indeterminacy. Applying it to the special case of a higher Whitehead of maps $\mu_i\colon S^2 \hookrightarrow \mathbb{C}P^\infty$, we show that there exist examples of moment-angle complexes whose wedge summands are non-trivial elements of trivial higher Whitehead products.

We first have the following preliminary proposition.

Proposition 3.23. *Let $\mathcal{K}$ be a complex on $[l_1] \sqcup \cdots \sqcup [l_m]$ such that $\partial\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle \subseteq \mathcal{K}$. Then*

$$[h_w^{\mathcal{K}}(f_1, \dots, f_m)] \in [\iota_1 \circ f_1, \dots, \iota_m \circ f_m].$$

where $\iota_i \colon (\mathbf{Y_i}, *)^{\mathcal{K}_i} \hookrightarrow (\mathbf{Y}, *)^{\mathcal{K}}$ denotes the map of polyhedral products induced by the simplicial inclusion $\mathcal{K}_i \hookrightarrow \mathcal{K}$.

Proof. The following diagram commutes

$$\begin{array}{ccccccc}
\bigvee_{i=1}^m \Sigma X_i & = & (\mathbf{\Sigma X}, *)^{\bullet^{m-1}} & \longrightarrow & (\mathbf{Y}, *)^{\bullet^{m-1}} & \longrightarrow & \mathbf{Y}, *)^{\mathcal{K}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\| \\
FW(\Sigma X_1, \dots, \Sigma X_m) & = & (\mathbf{\Sigma X}, *)^{\partial\Delta^{m-1}} & \longrightarrow & (\mathbf{Y}, *)^{\partial\Delta^{m-1}} & \longrightarrow & (\mathbf{Y}, *)^{\mathcal{K}}
\end{array}$$

by naturality of the polyhedral product with respect to simplicial inclusions and continuous maps of pairs. The map $h_w^{\mathcal{K}}(f_1, \dots, f_m)$ is the composite of $\rho \colon \mathop{*}_{i=1}^{m} X_i \longrightarrow FW(\Sigma X_i)$ with the bottom row of the diagram, while the composite along the top row is the map $\bigvee_{i=1}^m (\iota_i \circ f_i)$. Thus $h_w^{\mathcal{K}}(f_1, \dots, f_m)$ is the composite of ρ with an extension of the map $\bigvee_{i=1}^m (\iota_i \circ f_i)$ to $FW(\Sigma X_1, \dots, \Sigma X_m)$. □

Example 3.24. *Let $n_1 \geqslant 2$ and $k \geqslant 3$, $\mathcal{K}_1 = \partial\Delta^{n_1-1}$ on vertex set $[n_1]$ and for all $2 \leqslant i \leqslant k$, let $\mathcal{K}_i = \bullet$.*

Define a complex $\mathcal{K}$ on vertex set $[m] = [n_1] \sqcup [k-1]$ by

$$\begin{aligned}
\mathcal{K} &= \partial\Delta^{k-1}\langle \mathcal{K}_1, \dots, \mathcal{K}_q\rangle \cup \Delta^{n_i-1} \\
&= \partial\Delta^{k-1}\langle \partial\Delta^{n_1-1}, \bullet, \dots, \bullet\rangle \cup \Delta^{n_i-1}
\end{aligned}$$

For each $i \in [n_1]$, let $f_{1_i} \colon \Sigma X_{1_i} \longrightarrow Y_{1_i}$ be non-trivial, and define $f_1 = h_w^{\partial\Delta^{n_1-1}}(f_{1_1}, \dots, f_{1_p}) \colon (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{n_1-1}} \longrightarrow (\mathbf{Y}_1, *)^{\partial\Delta^{n_1-1}}$. For $2 \leqslant i \leqslant k$, let $f_i \colon \Sigma X_i \longrightarrow (\mathbf{Y_i}, *)^{\mathcal{K}_i}$ be non-trivial. By Proposition 3.23, the map $h_w^{\mathcal{K}}(f_1, \dots, f_k)$ is an element of the higher Whitehead product $[\iota_1 \circ f_1, \dots, \iota_k \circ f_k]$. Since the map $\iota_1 \circ f_1$ is trivial, then $0 \in [\iota_1 \circ f_1, \dots, \iota_k \circ f_k]$ by Theorem 3.3. On the other hand, by Theorem 3.8, the map $h_w^{\mathcal{K}}(f_1, \dots, f_k)$ is non-trivial since $\mathcal{K}$ contains neither $\Delta^{k-1}\langle \partial\Delta^{n_1-1}, \bullet, \dots, \bullet\rangle$ nor $\partial\Delta^{k-1}\langle \Delta^{n_1-1}, \bullet, \dots, \bullet\rangle$. Therefore, the higher Whitehead product $[\iota \circ f_1, \dots, \iota \circ f_k]$ has non-trivial indeterminacy.

The case where $n_1 = k = 3$ was considered in [1] in the special case of the map $\mathbb{Z}_{\mathcal{K}} \longrightarrow DJ_{\mathcal{K}}$. In particular it was observed that there is a wedge summand S^{10} of $\mathbb{Z}_{\mathcal{K}}$ such that the composite $S^{10} \hookrightarrow \mathbb{Z}_{\mathcal{K}} \longrightarrow DJ_{\mathcal{K}}$ is not a representative of an element of a non-trivial higher Whitehead product. The previous example shows that this composite is the higher Whitehead map $h_w^{\mathcal{K}}(f_1, \dots, f_k)$, which is a non-trivial element of the trivial higher Whitehead product $[\iota_1 \circ f_1' \dots, \iota_k f_k]$.

3.7 Proof of main theorem

In this section, we prove Theorem 3.19. We begin by recalling the statement.

Theorem 3.19. Let $m \geq 3$. Let $\Pi = \{P_1, \dots, P_k\}$ be a k-partition of $[m]$. Suppose that we are given maps $f_l \colon \Sigma X_l \longrightarrow Y_l$ for each $l \in [m]$ such that $X_l = \Sigma \widetilde{X}_l$. Then the map

$$\sum_{i=1}^{k} {}_i h_w^{\mathcal{K}_\Pi} \colon \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi} \tag{3.16}$$

is trivial, and for any subset $\emptyset \neq J \subsetneq [k]$, the sum $\sum_{i \in J} {}_i h_w^{\mathcal{K}_\Pi}$ is non-trivial.

Before outlining our approach to the proof, we set up notation and clarify terminology.

Throughout the proof, we assume that we have been given an m-tuple of CW complexes X_l such that $X_l = \Sigma \widetilde{X_l}$, and a k-partition of m denoted Π. For each $i \in [k]$ we denote by $P_i = \{p_{i_1}, \dots, p_{i_{n_i}}\}$ and by $Q_i = [m] \backslash P_i = \{q_{i_1}, \dots, q_{i_{r_i}}\}$. Depending on context, we interchange between indexing spaces and maps by their position $l \in [m]$, or by the partition that they are in and their position in that partition. Doing the latter is necessary at times to avoid significant ambiguity in our notation, but has the downside that it results in notation like $X_{p_{i_{n_i}}}$, or $f_{q_{i_t}}$.

Given an m-tuple of pairs $(\mathbf{X}, \mathbf{A})$ and a simplicial complex $\mathcal{K}$, for a simplicial subcomplex $\mathcal{L} \subseteq \mathcal{K}$ on a subset of the vertex set $J \subseteq [m]$, where it does not cause ambiguity, we abuse notation by denoting by $(\mathbf{X}, \mathbf{A})^{\mathcal{L}}$ the polyhedral product on the tuple obtained by deleting the pairs in the tuple $(\mathbf{X}, \mathbf{A})$ which correspond to elements of $[m] \backslash J$.

Until this point in this thesis, when referring to a space, we have implicitly been referring to that space up to homotopy, and have freely interchanged between models where needed. This proof necessitates working with specific topological models at times, whilst referring to spaces up to homotopy at others. We will clarify specifically when we are working with a specific model of a space, and otherwise it can be assumed that we are working up to homotopy.

We introduce the following notation.

$$\begin{aligned}
\widehat{\Sigma W} &= \bigcup_{i \in [m]} CX_1 \times \cdots \times X_i \times \cdots \times CX_m \\
&= (\mathbf{CX}, \mathbf{X})^{\partial \Delta^{m-1}} \\
\widehat{C\Sigma W} &= CX_1 \times \cdots \times CX_m \\
&= (\mathbf{CX}, \mathbf{X})^{\Delta^{m-1}}.
\end{aligned}$$

For each $i \in [k]$, we denote by

$$
\begin{aligned}
\widetilde{W}_i &= \Sigma^{r_i-2} X_{q_{i_1}} \wedge \cdots \wedge X_{q_{i r_i}} \\
\widehat{\Sigma W}_i &= \bigcup_{t \in Q_i} \left(CX_{q_{i_1}} \times \cdots \times X_t \times \cdots \times CX_{q_{i r_i}} \right) \\
&= (\mathbf{CX}, \mathbf{X})^{\partial(\Delta^{m-1}_{Q_i})} \\
\widehat{C\Sigma W}_i &= CX_{q_{i_1}} \times \cdots \times CX_{q_{i r_i}} \\
&= (\mathbf{CX}, \mathbf{X})^{(\Delta^{m-1}_{Q_i})} \\
\widehat{W}_i &= \bigcup_{s \in P_i} \left(C\widetilde{W}_i \times CX_{p_{i_1}} \times \cdots \times X_s \times \cdots \times CX_{p_{i n_i}} \right) \cup \left(\widetilde{W}_i \times CX_{p_{i_1}} \times \cdots \times CX_{p_{i n_i}} \right) \\
&= W_i * X_{p_{i_1}} * \cdots * X_{p_{i n_i}}.
\end{aligned}
$$

We denote by $Y = FW(Y_1, \ldots, Y_m) = (\mathbf{Y}, *)^{\partial \Delta^{m-1}}$ and $Z = (\mathbf{Y}, *)^{\mathcal{K}_\Pi} \subseteq Y$.

Recall that ξ_i is the restriction of the permutation

$$
Y_{q_{i_1}} \times \cdots \times Y_{q_{i r_i}} \times Y_{p_{i_1}} \times \cdots \times Y_{p_{i n_i}} \longrightarrow Y_1 \times \cdots \times Y_m
$$

and σ_i is the permutation

$$
\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m \longrightarrow \Sigma^{n_i} \left(\Sigma^{r_i-2} X_{q_{i_1}} \wedge \cdots \wedge X_{q_{i r_i}} \right) \wedge X_{p_{i_1}} \wedge \cdots \wedge X_{p_{i n_i}}.
$$

For each $i \in [k]$, we denote by ${}_i h_w^{\mathcal{K}_\Pi}$ the map

$$
\xi_i \circ h_w^{\mathcal{K}_\Pi} \left(h_w \left(f_{q_{i_1}}, \ldots, f_{q_{i r_i}} \right), f_{p_{i_1}}, \ldots, f_{p_{i n_i}} \right) \circ \sigma_i \colon \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi}
$$

With this set up, our statement becomes abbreviated to the statement that the map

$$
\sum_{i=1}^{k} {}_i h_w^{\mathcal{K}_\Pi} \colon W \longrightarrow Z
$$

is trivial, and for any subset $\emptyset \neq J \subsetneq [k]$, the sum $\sum_{i \in J} {}_i h_w^{\mathcal{K}_\Pi}$ is non-trivial.

3.7.1 Overview

Before proceeding with the proof itself, we provide a preliminary overview of the argument, outlining the key steps at each stage.

Our method is to show that $\sum_{i=1}^{k}[{}_ih_w^{\mathcal{K}_{\Pi}}] = \partial(j([h_w(f_1,\dots,f_m)]))$, where ∂ and j are maps in a particular long exact sequence, when it follows by exactness that this homotopy class is zero.

The segment of the long exact sequence (see (1.11)) in question is

$$\cdots \longrightarrow [\Sigma W, Y] \xrightarrow{\ j\ } [(CW,W),(Y,Z)] \xrightarrow{\ \partial\ } [W,Z] \longrightarrow \cdots \tag{3.17}$$

where j is the map induced by the inclusion, and ∂ is induced by the restriction.

We begin by defining the relative higher Whitehead map, which we make use of later in the proof. This comprises step one.

The second step is to analyse $j([h_w(f_1,\dots,f_m)])$. For this we model ΣW by a space $\widehat{\Sigma W}$. We then define a family of CW-pairs $\{(F_i,Z_i)\}_{i\in[k]}$, each homotopy equivalent as a pair to (CW,W), and where $F_i \subseteq \widehat{\Sigma W}$, satisfying certain conditions with respect to the higher Whitehead map $h_w(f_1,\dots,f_m)\colon \widehat{\Sigma X} \longrightarrow Y$. We show that in these conditions,

$$\begin{aligned} j([h_w(f_1,\dots,f_m)]) &= \sum_i [h_w(f_1,\dots,f_m)|_{F_i} \circ h_i] \\ &= \sum_i [h_w(f_1,\dots,f_m)|_{F_i}] \in [(CW,W),(Y,Z)] \end{aligned}$$

where for each $i \in [k]$, $h_i\colon (CW,W) \longrightarrow (F_i,Z_i)$ is a homotopy equivalence whose notation we suppress.

For each $i \in [k]$ we show that the map $h_w(f_1,\dots,f_m)|_{F_i}\colon (CW,W)) \longrightarrow (Y,Z)$ is homotopic to a particular relative higher Whitehead map pre-composed with a coordinate permutation. We apply the boundary map $\partial : [(CW,W),(Y,Z)] \longrightarrow [W,Z]$ to each relative Whitehead map in the sum, and the result follows by exactness.

We now give a detailed breakdown of our approach to each step of proof.

Step one: i) We define the relative higher Whitehead map; for a map of pairs $f\colon (C\Sigma X, \Sigma X) \longrightarrow (Z, B)$, and maps $f_l\colon \Sigma X_l \longrightarrow Y_l$ for $l = 1, \dots, m$ the relative higher Whitehead map is a map of pairs

$$
\begin{aligned}
{}^{Rel}h_w(f, f_1, \dots, f_m)\colon (CX * X_1 * \cdots * X_m, X * X_1 * \cdots * X_m) \\
\longrightarrow (FW(Z, Y_1, \dots, Y_m), FW(B, Y_1, \dots, Y_m))
\end{aligned}
$$

which restricts to the higher Whitehead map

$$h_w(f|_{\Sigma X}, f_1, \dots, f_m)\colon X * X_1 * \cdots * X_m \longrightarrow FW(B, Y_1, \dots, Y_m)$$

ii) We show that

$$\partial([{}^{Rel}h_w(f, f_1, \dots, f_m)]) = [h_w(f|_{\Sigma X}, f_1, \dots, f_m)]. \tag{3.18}$$

Step two: We show that

$$
\begin{aligned}
\widehat{j}([h_w(f_1, \dots, f_m)]) &= \sum_{i=1}^{k} [h_w(f_1, \dots, f_m)|_{F_i}] \\
&= \sum_{i=1}^{k} ([{}^{Rel}h_w(\psi_i, f_1, \dots, f_m) \circ \sigma_i]) \in \Big[(\widehat{CW}, \widehat{W}), (Y, Z)\Big]
\end{aligned} \tag{3.19}
$$

where ψ_i is a map which we outline below.

In order to summarise our approach to proving statement (3.19), we first clarify notation, and make preliminary observations.

For all $l \in [m]$, we denote by $\widehat{f_l}\colon (CX_l, X_l) \longrightarrow (Y_l, *)$ the map of pairs obtained from $f_l\colon \Sigma X_l \longrightarrow Y_l$, by defining $\widehat{f_l}(x, t) \mapsto f_l(x)$ for $t > 0$, and $\widehat{f_l}(x, 0) \mapsto *_{Y_l}$. Recall that

$$
\begin{aligned}
\widehat{C\Sigma W} &= (\mathbf{CX}, \mathbf{X})^{\Delta^{m-1}} \\
\widehat{\Sigma W} &= (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} \\
\widehat{C\Sigma W}_i &= (\mathbf{CX}, \mathbf{X})^{\Delta^{r_i-1}} \\
\widehat{\Sigma W}_i &= (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{r_i-1}}
\end{aligned}
$$

and observe that there is a homotopy equivalence of pairs $(\widehat{C\Sigma W}, \widehat{\Sigma W}) \simeq (C\Sigma W, \Sigma W)$. We define the map of pairs

$$\psi = \prod_{l \in [m]} \widehat{f_l}\colon \left(\widehat{C\Sigma W}, \widehat{\Sigma W}\right) \longrightarrow \left((\mathbf{Y}, *)^{\Delta^{m-1}}, (\mathbf{Y}, *)^{\partial\Delta^{m-1}}\right)$$

and for all $i \in [k]$, we define the map of pairs

$$\psi_i = \prod_{t \in Q_i} \widehat{f}_t \colon \left(\widehat{C\Sigma W}_i, \widehat{\Sigma W}_i\right) \longrightarrow \left((\mathbf{Y}, *)^{\Delta^{r_i-1}}, (\mathbf{Y}, *)^{\partial\Delta^{r_i-1}}\right).$$

We observe that

$$\psi|_{(\mathbf{CX},\mathbf{X})^{\partial\Delta^{m-1}}} = h_w(f_1, \ldots, f_m)$$

and that for all $i \in [k]$, $\psi_i|_{\widehat{\Sigma W}_i} = h_w(f_{q_1}, \ldots, f_{q_{r_i}})$.

We now proceed with the outline of the proof of (3.19).

i) We specify a family of pairs $\{(F_i, Z_i)\}_{i \in [k]}$, where $Z_i \subseteq F_i \subseteq \widehat{\Sigma W}$, which satisfy the following properties:

  I) For each $i \in [k]$, there is a homotopy equivalence of pairs $(F_i, Z_i) \simeq (CW, W)$,

  II) For all $i, i' \in [k]$ such that $i \neq i'$, $(F_i \backslash Z_i) \cap (F_{i'} \backslash Z_{i'}) = \emptyset$

  III) For each $i \in [k]$, the following maps are homotopic maps of pairs

$$h_w(f_1, \ldots, f_m)|_{F_i} \simeq {}^{Rel}h_w(\psi_i, f_{p_{i_1}}, \ldots, f_{p_{i_{n_i}}}) \circ \sigma_i \colon (F_i, Z_i) \longrightarrow \left((\mathbf{Y}, *)^{\Delta^{m-1}}, (\mathbf{Y}, *)^{\partial\Delta^{m-1}}\right) \qquad (3.20)$$

  where we as before omit the homotopy equivalence $(F_i, Z_i) \simeq (CW, W)$. Note that by definition, ${}^{Rel}h_w(\psi_i, f_{p_{i_1}}, \ldots, f_{p_{i_{n_i}}})|_{Z_i} = h_w(h_w(f_{q_{i_1}}, \ldots, f_{q_{i_{r_i}}}), f_{p_{i_1}}, \ldots, f_{p_{i_{n_i}}}) \circ \sigma_i$.

ii) We prove Lemma 3.36:

Lemma 3.36. Let $n \geq 2$. For each $i \in [n]$, let $F_i \subseteq \widehat{\Sigma W}$ be contractible and such that there exists $Z_i \subseteq F_i$ such that $(F_i, Z_i) \simeq (CW, W)$, where $W = \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m$. Suppose that for all $i \in [n]$, the subspaces $F_i \backslash Z_i$ are pairwise disjoint.

Let (Y, Z) be a CW-pair, and suppose that $f \colon \widehat{\Sigma W} \longrightarrow Y$ is such that $f((\widehat{\Sigma W} \backslash \bigcup_i F_i) \cup (\bigcup_i Z_i)) \subseteq Z$ for all i. Denote by $f_i = f|F_i \colon (F_i, Z_i) \longrightarrow (Y, Z)$, so that $[f_i] \in [(F_i, Z_i), (Y, Z)] \cong [(CW, W), (Y, Z)]$. Let $j \colon [\widehat{\Sigma W}, X] \longrightarrow [CW, W]$ denote the map from the long exact sequence. Then $j([f]) = \sum_i [f_i]$.

iii) We apply Corollary 3.37 to the homotopy class $j([h_w(f_1, \ldots, f_m]) \in [(CW, W), (X, A)]$, when

$$\begin{aligned} j([h_w(f_1, \ldots, f_m)] &= \sum_{i=1}^{k} [h_w(f_1, \ldots, f_m)|_{F_i}] \\ &= \sum_{i=1}^{k} [\xi_i \circ {}^{Rel}h_w(\psi_i, f_{p_{i_1}}, \ldots, f_{p_{i_{n_i}}}) \circ \sigma_i] \qquad (3.21) \end{aligned}$$

The theorem then follows by exactness, together with (3.18).

3.7.2 Step one: The relative higher Whitehead map

Before defining the relative higher Whitehead map, we fix notation and make preliminary observations

Suppose that we are given CW-complexes $X, X_1, \dots, X_m$ for $m \geq 1$. Then we denote by

$$\begin{aligned}
(\mathbf{CX}, \mathbf{X}) &= ((CX, X), (CX_1, X_1), \dots, (CX_m, X_m)) \\
(\mathbf{\Sigma X}, *) &= ((\Sigma X, *), (\Sigma X_1, *), \dots, (\Sigma X_m, *)) \\
(\mathbf{CX_C}, \mathbf{X_C}) &= ((CCX, CX), (CX_1, X_1), \dots, (CX_m, X_m)) \\
(\mathbf{\Sigma X_C}, *_\mathbf{C}) &= ((C\Sigma X, \Sigma X), (\Sigma X_1, *), \dots, (\Sigma X_m, *))
\end{aligned} \tag{3.22}$$

and we denote by

$$^{Rel}\rho\colon \left((\mathbf{CX_C}, \mathbf{X_C})^{\partial\Delta^m}, (\mathbf{CX}, \mathbf{X})^{\partial\Delta^m}\right) \longrightarrow \left((\mathbf{\Sigma X_C}, *_\mathbf{C})^{\partial\Delta^m}, (\mathbf{\Sigma X}, *)^{\partial\Delta^m}\right) \tag{3.23}$$

the map of pairs induced by the maps of pairs $(CCX, CX) \longrightarrow (C\Sigma X, \Sigma X)$ and $(CX_i, X_i) \longrightarrow (\Sigma X_i, *)$. Observe that

$$\begin{aligned}
\left((\mathbf{CX_C}, \mathbf{X_C})^{\partial\Delta^m}, (\mathbf{CX}, \mathbf{X})^{\partial\Delta^m}\right) &= (CX * X_1 * \cdots * X_m, X * X_1 * \cdots * X_m), \\
\left((\mathbf{\Sigma X_C}, *_\mathbf{C})^{\partial\Delta^m}, (\mathbf{\Sigma X}, *)^{\partial\Delta^m}\right) &= (FW(C\Sigma X, \Sigma X_1, \dots, \Sigma X_m), FW(\Sigma X, \Sigma X_1, \dots, \Sigma X_m))
\end{aligned}$$

and that

$$^{Rel}\rho|_{(\mathbf{CX},\mathbf{X})^{\partial\Delta^m}} = \rho\colon (\mathbf{CX}, \mathbf{X})^{\partial\Delta^m} \longrightarrow (\mathbf{\Sigma X}, *)^{\partial\Delta^m}.$$

We now define the relative higher Whitehead map.

Definition 3.25. Suppose that we are given a map of pairs $f\colon (C\Sigma X, \Sigma X) \longrightarrow (Z, B)$, and maps $f_l\colon \Sigma X_l \longrightarrow Y_l$ for $l = 1, \dots, m$. The relative higher Whitehead map is the map of pairs defined by

$$\begin{aligned}
{}^{Rel}h_w(f, f_1, \dots, f_m)\colon &\left((\mathbf{CX_C}, \mathbf{X_C})^{\partial\Delta^m}, (\mathbf{CX}, \mathbf{X})^{\partial\Delta^m}\right) \\
&\xrightarrow{^{Rel}\rho} \left((\mathbf{\Sigma X_C}, *_\mathbf{C})^{\partial\Delta^m}, (\mathbf{\Sigma X}, *)^{\partial\Delta^m}\right) \\
&\longrightarrow (FW(Z, Y_1, \dots, Y_m), FW(B, Y_1, \dots, Y_m)).
\end{aligned} \tag{3.24}$$

where the final map is one induced by the maps $f, f_1, \dots, f_m$.

Observe firstly that by Proposition 2.3, the relative higher Whitehead map is invariant up to choice of representative of the homotopy class of the maps $f, f_1, \dots, f_m$. Secondly, observe that the restriction of (3.24) to $(\mathbf{CX}, \mathbf{X})^{\partial\Delta^m}$ is the higher Whitehead map $h_w(f|_{\widehat{\Sigma X}}, f_1, \dots, f_m)$.

Proposition 3.26. *The relative higher Whitehead map* ${}^{Rel}h_w(f, f_1, \dots, f_m)$ *of* (3.24) *satisfies*

$$\partial([{}^{Rel}h_w(f, f_1, \dots, f_m)]) = [h_w(f|_{\widehat{\Sigma X}}, f_1, \dots, f_m)].$$

Proof. Since by definition of the boundary map ∂,

$$\partial([h_w(f, f_1, \dots, f_m)]) = [h_w(f, f_1, \dots, f_m)|_{(\mathbf{CX},\mathbf{X})^{\partial\Delta^m}}]$$

the result follows immediately by the definition of $h_w(f, f_1, \dots, f_m)$. □

3.7.3 Step two: Subspaces (F_i, Z_i)

Recall that $\widehat{\Sigma W} = (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}}$.

Our aim now is to define the subspaces $Z_i \subseteq F_i \subseteq \widehat{\Sigma W}$. Their construction requires significant preliminary leg work; we first construct, for each $l \in [m]$, a family of subspaces of CX_l. We subsequently use this construction to define, for each $i \in [k]$, spaces F_i and $Z_i \subseteq F_i$, which are subspaces of $\widehat{\Sigma W}$.

Recall that by assumption, for all $l \in [m]$ $X_l = \Sigma\widetilde{X}_l$. In the following, we work with the following concrete models of the suspension and cone on the suspension.

$$\begin{aligned} \Sigma\widetilde{X}_l &= I \times \widetilde{X}_l/(\{0,1\} \times \widetilde{X}_l), \qquad \text{and} \\ C\Sigma\widetilde{X}_l &= I' \times \Sigma\widetilde{X}_l/(\{1'\} \times \Sigma\widetilde{X}_l). \end{aligned}$$

We first define, for each $l \in [m]$, the following subspaces of $C\Sigma\widetilde{X}_l$,

$$\begin{aligned} D_l^+ &= \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_l \mid t \geqslant \frac{1}{2}\right\}, \qquad D_l^- = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_l \mid t \leqslant \frac{1}{2}\right\}, \\ D_l^1 &= \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_l \mid s \geqslant \frac{3}{4} - t\right\}, \quad D_l^2 = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_l \mid s \leqslant \frac{3}{4} - t\right\}. \end{aligned}$$

We observe that $C\Sigma\widetilde{X}_l = D_l^- \cup D_l^+ = D_l^1 \cup D_l^2$.

We now define the following collection of subspaces of D_l^- and D_l^1

$$
\begin{aligned}
(B_2)_l^- &= \left\{(s,(t,x)) \in D_l^- \mid t \leqslant \frac{1}{2}, s=0\right\}, & (B_2)_l^1 &= \left\{(s,(t,x)) \in D_l^1 \mid s = \frac{3}{4} - t\right\} \\
(B_1)_l^- &= \left\{(s,(t,x)) \in D_l^- \mid t = \frac{1}{2}\right\}, & (B_1)_l^1 &= \left\{(s,(t,x)) \in D_l^1 \mid t \geqslant \frac{3}{4}, s=0\right\}, \\
E_l^- &= \left\{(s,(t,x)) \in D_l^- \mid t \leqslant \frac{1}{8}\right\} & E_l^1 &= \left\{(s,(t,x)) \in D_l^1 \mid t \geqslant \frac{7}{8}\right\}, \\
R_l^- &= \left\{(s,(t,x)) \in D_l^- \mid t = \frac{1}{8}, s=0\right\} & R_l^1 &= \left\{(s,(t,x)) \in D_l^1 \mid t = \frac{7}{8}, s=0\right\}, \\
S_l^- &= \left\{(s,(t,x)) \in D_l^- \mid t = \frac{1}{8}\right\} & S_l^1 &= \left\{(s,(t,x)) \in D_l^1 \mid t = \frac{7}{8}\right\}, \\
T_i^- &= \left\{(s,(t,x)) \in D_l^- \mid t \leqslant \frac{1}{8}, s=0\right\} & T_l^1 &= \left\{(s,(t,x)) \in D_l^1 \mid t \geqslant \frac{7}{8}, s=0\right\}, \\
C_l^- &= \left\{(s,(t,x)) \in D_l^- \mid t \geqslant \frac{1}{8}\right\} & C_l^1 &= \left\{(s,(t,x)) \in D_l^1 \mid t \leqslant \frac{7}{8}\right\}
\end{aligned}
$$

and denote by $B_l^- = (B_1)_l^- \cup (B_2)_l^-$ and $B_l^1 = (B_1)_l^1 \cup (B_2)_l^1$.

We make some preliminary observations before proceeding. We observe that $(B_1)_l^- \cong (B_2)_l^- \cong C\widetilde{X}_l$, and $(B_1)_l^- \cap (B_2)_l^- \cong \widetilde{X}_l$, from which it follows that $B_l^- \cong \Sigma\widetilde{X}_l$. Similarly, $(B_1)_l^1 \cong (B_2)_l^1 \cong C\widetilde{X}_l$, and $(B_1)_l^1 \cap (B_2)_l^1 \cong \widetilde{X}_l$, so that $B_l^1 \cong \Sigma\widetilde{X}_l$. We also observe that $E_l^- \cong E_l^+ \cong CC\widetilde{X}_l$, that $R_l^- \cong R_l^1 \cong \widetilde{X}_l$, that $S_l^- \cong S_l^1 \cong C\widetilde{X}_l$, and that $T_l^- \cong T_l^1 \cong C\widetilde{X}_l$. Finally, $C_l^- \cong C_l^+ \cong CC\widetilde{X}_l$.

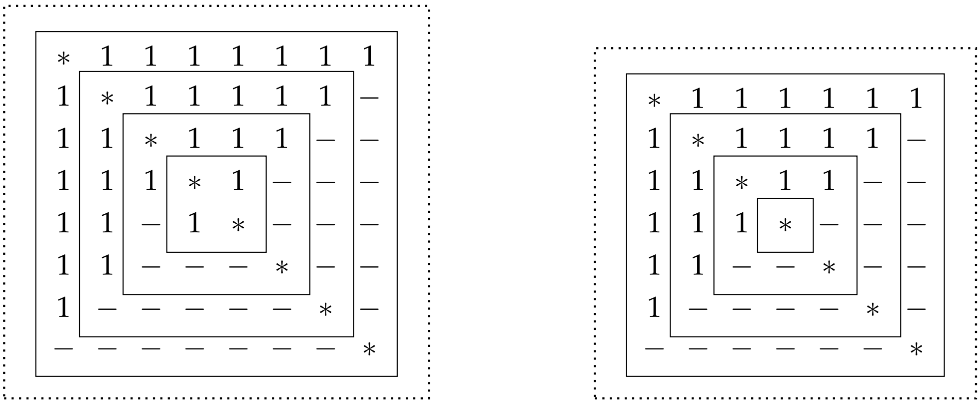
We utilise the construction of the above subspaces in defining the forthcoming spaces, which will be used to define the pair (F_i, Z_i).

Recall that

$$
\begin{aligned}
\widehat{C\Sigma W} &= (\mathbf{CX}, \mathbf{X})^{\Delta^{m-1}} \\
\widehat{\Sigma W} &= (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}} \\
\widehat{C\Sigma W_i} &= (\mathbf{CX}, \mathbf{X})^{\Delta^{r_i-1}} \\
\widehat{\Sigma W_i} &= (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{r_i-1}}.
\end{aligned}
$$

Observe that $\widehat{\Sigma W_i} \simeq \Sigma^{r_i-1} X_{q_{i_1}} \wedge \cdots \wedge X_{q_{i r_i}} = \Sigma\widetilde{W}_i$

For $i = 1, \ldots, k$ we define subspaces $G_i \subseteq \widehat{C\Sigma W_i}$ and $G_i^* \subseteq G_i$ as follows. Consider the infinite arrays shown in Figure 3.2. If k is even choose the centrally located $k \times k$ array from the array in Figure 3.2a, and if k is odd choose the centrally located $k \times k$ array from Figure 3.2b. Let $\eta_k(i,j)$ denote the (i,j)th entry in the selected array, where $(1,1)$ is the top left entry. For now, we do not digress on the reason for constructing these arrays in this way, and simply utilise them in the forthcoming definitions. The relevance of the choice of construction becomes clear at Lemma 3.32. Briefly, this



(A) k even (B) k odd

FIGURE 3.2: Construction of arrays

choice of construction ensures that certain subspaces which we define later are pairwise disjoint.

We now define subspaces $G_i^* \subseteq G_i \subseteq \widehat{C\Sigma W}_i$. For each $i \in [k]$, denote by $\mathbf{D_i}$ and $\mathbf{B_i}$ the m-tuples $\left(D_{p_{j_s}}^{\eta_k(i,j)}\right)_{j\in[k],s\in[n_j]}$ and $\left(B_{p_{j_s}}^{\eta_k(i,j)}\right)_{j\in[k],s\in[n_j]}$ respectively, and by $\mathbf{D_{\hat{i}}}$ and $\mathbf{B_{\hat{i}}}$ the r_i-tuples $\left(D_{p_{j_s}}^{\eta_k(i,j)}\right)_{j\in[k]\backslash i,s\in[n_j]}$ and $\left(B_{p_{j_s}}^{\eta_k(i,j)}\right)_{j\in[k]\backslash i,s\in[n_j]}$ respectively. We define

$$
\begin{aligned}
G_i &= \prod_{j\in[k]\backslash\{i\}} \prod_{s\in[n_j]} D_{p_{j_s}}^{\eta_k(i,j)} \\
&= (\mathbf{D_{\hat{i}}}, \mathbf{B_{\hat{i}}})^{\Delta^{r_i-1}} \\
G_i^* &= \bigcup_{t\in[r_i]} \left(\left(\prod_{j\in[k]\backslash\{i\}} \left(\prod_{\substack{s\in[n_j]\\ s\neq t}} D_{p_{j_s}}^{\eta_k(i,j)} \right) \right) \times B_{p_{j_t}}^{\eta(i,j)} \right) \\
&= (\mathbf{D_{\hat{i}}}, \mathbf{B_{\hat{i}}})^{\partial\Delta^{r_i-1}}
\end{aligned}
$$

Observe that since $D_l^- = E_l^- \cup C_l^-$ and $D_l^1 = E_l^1 \cup C_l^1$, then

$$
G_i = \prod_{j\in[k]\backslash\{i\}} \prod_{s\in[n_j]} D_{p_{j_s}}^{\eta_k(i,j)} = \prod_{j\in[k]\backslash i} \prod_{s\in[n_j]} \left(E_{p_{j_s}}^{\eta_k(i,j)} \cup C_{p_{j_s}}^{\eta_k(i,j)} \right) = \bigcup_{\substack{\{t_{1_1},\ldots,t_{n_k}\}\\ \in\{0,1\}^{m-n_i}}} \left(\prod_{j\in[k]\backslash i} \prod_{s\in[n_j]} \left(Y_{t_{j_s}}\right)_{p_{j_s}}^{\eta_k(i,j)} \right)
$$

where $(Y_0)_{j_s}^{\eta_k(i,j)} = E_{p_{j_s}}^{\eta_k(i,j)}$ and $(Y_1)_s^{\eta_k(i,j)} = C_{p_{j_s}}^{\eta_k(i,j)}$.

We define the following subspaces of G_i^*.

$$\tau_i = G_i^* \cap \prod_{j\in[k]\backslash i}\prod_{s=1}^{n_j} E_{p_{j_s}}^{\eta_k(i,j)},$$

$$\kappa_i = G_i^* \cap \left(\bigcup_{\substack{\{t_{1_1},\ldots,t_{n_k}\}\in\{0,1\}^{m-n_i}\\ \{t_{1_1},\ldots,t_{n_k}\}\neq\{0,\ldots,0\}}} \left(\prod_{i\in[k]\backslash j}\prod_{s\in n_j} \left(Y_{t_{j_s}}\right)_{j_s}^{\eta_k(i,j)} \right) \right),$$

$$\sigma_i = \tau_i \cap \kappa_i.$$

Recall that ξ_i denotes the permutation map

$$X_{q_{i_1}} \times \cdots \times X_{q_{i_{r_i}}} \times X_{p_{i_1}} \times \cdots \times X_{p_{i_{n_i}}} \longrightarrow X_1 \times \cdots \times X_m.$$

We define

$$F_i = \xi_i\left(\left(G_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i_1}} \times \cdots \times X_{p_{i_s}} \times \cdots \times CX_{p_{i_{n_i}}}\right)\right) \cup \left(\tau_i \times \left(CX_{p_{i_1}} \times \cdots \times CX_{p_{i_{n_i}}}\right)\right)\right) \tag{3.25}$$

$$Z_i = \xi_i\left(\left(\kappa_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i_1}} \times \cdots \times X_{p_{i_s}} \times \cdots \times CX_{p_{i_{n_i}}}\right)\right) \cup \left(\sigma_i \times \left(CX_{p_{i_1}} \times \cdots \times CX_{p_{i_{n_i}}}\right)\right)\right).$$

We now prove that there is a homotopy-equivalence of pairs $(CW, W) \longrightarrow (F_i, Z_i)$. For this, we have the following preparatory lemmata.

We denote by $\gamma_i\colon \widehat{C\Sigma W}_i \longrightarrow G_i$ the deformation retract defined as the product of the deformation retracts $C\Sigma\widetilde{X}_{q_{i_t}} \longrightarrow D_{q_{i_t}}^{\eta_k(i,j)}$.

Lemma 3.27. *For all $i = 1, \ldots, k$, the restriction*

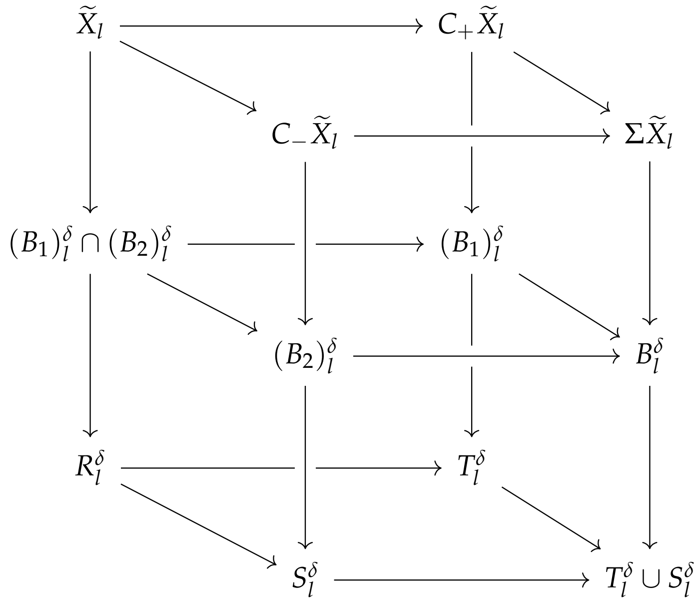
$$\gamma_i^* = \gamma_i|_{\widehat{\Sigma W}_i}\colon \widehat{\Sigma W}_i \longrightarrow G_i^*$$

is a homotopy equivalence.

Proof. Recall that $\widehat{\Sigma W}_i = (\mathbf{C\Sigma\widetilde{X}}, \mathbf{\Sigma\widetilde{X}})^{\partial\Delta^{r_i-1}}$. Then the result follows from the fact that for all $j \in [k]\backslash i$, and all $t \in [r_j]$, the deformation retract $C\Sigma\widetilde{X}_{q_{j_t}} \longrightarrow D_{q_{j_t}}^{\eta_k(i,j)}$ can be chosen to be a homotopy equivalence of pairs $(C\Sigma\widetilde{X}_{q_{j_t}}, \Sigma\widetilde{X}_{q_{j_t}}) \simeq (D_{q_{j_t}}^{\eta_k(i,j)}, B_{q_{j_t}}^{\eta_k(i,j)})$.

□

Lemma 3.28. *Let $\delta \in \{1, -\}$. For all $l \in [m]$, there exists a homotopy equivalence $f\colon C\Sigma\widetilde{X}_l \longrightarrow D_l^\delta \longrightarrow E_l^\delta$ whose restrictions $f|_{C_-\widetilde{X}_l}, f|_{C_+\widetilde{X}_l}$ and $f|_{\widetilde{X}_l}$ are homotopy equivalences, so that the following diagram*



where the horizontal squares are pushouts and all vertical arrows are the restriction of the map f to the respective subspaces, is homotopy commutative.

Proof. Recall that $\widetilde{X}_i \cong (B_1)_l^\delta \cap (B_2)_l^\delta \cong R_l^\delta \cong \widetilde{X_l}$. Define $g_i^\delta\colon \widetilde{X}_i \longrightarrow (B_1)_i^\delta \cap (B_2)_i^\delta \longrightarrow R_i^\delta$ to be the composite of these homeomorphisms. Define g_{-i}^δ and g_{+i}^δ similarly. It follows from the universal property of the pushout that the pushouts are homeomorphic, and therefore the diagram is in particular homotopy commutative.

□

Lemma 3.29. *Recall that we denote by $\widetilde{W}_i = \Sigma^{r_i-2}X_{q_{i_1}} \wedge \cdots \wedge X_{q_{i_{r_i}}}$, so that $\Sigma\widetilde{W}_i \simeq \widehat{\Sigma W}_i$, and recall that by Lemma 3.27, $\gamma_i^*\colon \widehat{\Sigma W}_i \longrightarrow G_i^*$ is a homotopy equivalence. We denote by $\overline{\gamma_i^*}\colon \Sigma\widetilde{W}_i \longrightarrow \widehat{\Sigma W}_i \longrightarrow G_i^*$ the composite homotopy equivalence. Then the restrictions*

$$\overline{\gamma_i^*}|_{\widetilde{W}_i}\colon \widetilde{W}_{Q_i} \longrightarrow \sigma_i,$$

$$\overline{\gamma_i^*}|_{C_+\widetilde{W}_i}\colon C_+\widetilde{W}_i \longrightarrow \tau_i, \qquad \text{and } C_+\widetilde{W}_i \longrightarrow \kappa_i$$

are homotopy equivalences.

Proof. We begin by proving that $\overline{h_i^*}|_W \colon W \longrightarrow \sigma_i$ is a homotopy equivalence.

Since $E_i^- \cap C_i^- = S_i^-$ and $E_i^1 \cap C_i^1 = S_i^1$,

$$\begin{aligned}
\sigma_i &= \tau_i \cap \kappa_i \\
&= G_i^* \cap \left(\prod_{j\in[k]\setminus\{i\}} \prod_{s=1}^{n_j} E_{p_{j_s}}^{\eta_k(i,j)} \cap \bigcup_{\substack{\{t_{1_1},\dots,t_{n_k}\}\in\{0,1\}^{m-n_i} \\ \{t_{1_1},\dots,t_{n_k}\}\neq\{0,\dots,0\}}} \prod_{j\in[k]\setminus\{i\}} \prod_{s=1}^{n_j} \left(Y_{t_{j_s}}\right)_{j_s}^{\eta_k(i,j)} \right) \\
&= G_i^* \cap \left(\bigcup_{\substack{\{t_{1_1},\dots,t_{n_k}\}\in\{0,1\}^{m-n_i} \\ \{t_{1_1},\dots,t_{n_k}\}\neq\{0,\dots,0\}}} \prod_{j\in[k]\setminus\{i\}} \left(\prod_{s=1}^{n_j} \left(Y_{t_{j_s}}\right)_{j_s}^{\eta_k(i,j)} \cap \prod_{j\in[k]\setminus\{i\}} \prod_{s=1}^{n_j} E_{p_{j_s}}^{\eta_k(i,j)} \right) \right) \\
&= G_i^* \cap \left(\bigcup_{\substack{\{t_{1_1},\dots,t_{n_k}\}\in\{0,1\}^{m-n_i} \\ \{t_{1_1},\dots,t_{n_k}\}\neq\{0,\dots,0\}}} \prod_{j\in[k]\setminus\{i\}} \prod_{s=1}^{n_j} \left(Z_{t_{j_s}}\right)_{j_s}^{\eta_k(i,j)} \right)
\end{aligned}$$

where $(Z_0)_{j_s}^{\eta_k(i,j)} = E_{p_{j_s}}^{\eta_k(i,j)}$ and $(Z_1)_{j_s}^{\eta_k(i,j)} = S_{p_{j_l}}^{\eta_k(i,j)}$. Since $S_i^- \subset E_i^-$ and $S_i^1 \subset E_i^1$, then

$$\begin{aligned}
\sigma_i = G_i^* \cap \Big(& S_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times E_{p_{1_{n_1}}}^{\eta_k(i,1)} \times \cdots \times E_{p_{k_1}}^{\eta_k(i,k)} \times \cdots \times E_{p_{k_{n_k}}}^{\eta_k(i,k)} \\
& \cup \cdots \cup E_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times E_{p_{1_{n_1}}}^{\eta_k(i,1)} \times \cdots \times E_{p_{k_1}}^{\eta_k(i,k)} \times \cdots \times S_{p_{k_{n_k}}}^{\eta_k(i,k)} \Big).
\end{aligned}$$

Since $S_i^- \cap B_i^- = R_i^-$, $S_i^1 \cap B_i^1 = R_i^1$, $B_i^- \cap E_i^- = T_i^-$, and $B_i^1 \cap E_i^1 = T_i^1$, then

$$\begin{aligned}
\sigma_i &= \left(B_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times D_{p_{k_{n_k}}}^{\eta_k(i,k)} \right) \cap \left(S_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times E_{p_{k_{n_k}}}^{\eta_k(i,k)} \cup \cdots \cup E_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times S_{p_{k_{n_k}}}^{\eta_k(i,k)} \right) \\
&\quad \cup \cdots \cup \left(D_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times B_{p_{k_{n_k}}}^{\eta_k(i,k)} \right) \cap \left(S_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times E_{p_{k_{n_k}}}^{\eta_k(i,k)} \cup \cdots \cup E_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times S_{p_{k_{n_k}}}^{\eta_k(i,k)} \right) \\
&= R_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times E_{p_{k_{n_k}}}^{\eta_k(i,k)} \\
&\qquad \cup T_{p_{1_1}}^{\eta_k(i,1)} \times \left(S_{p_{1_2}}^{\eta_k(i,1)} \times \cdots \times E_{p_{k_{n_k}}}^{\eta_k(i,k)} \cup \cdots \cup E_{p_{1_2}}^{\eta_k(i,1)} \times \cdots \times S_{p_{k_{n_k}}}^{\eta_k(i,k)} \right) \\
&\quad \cup \cdots \cup E_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times R_{p_{k_{n_k}}}^{\eta_k(i,k)} \\
&\qquad \cup \left(S_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times E_{p_{k_{n_k-1}}}^{\eta_k(i,k)} \cup \cdots \cup E_{p_{1_1}}^{\eta_k(i,1)} \times \cdots \times S_{p_{k_{n_k-1}}}^{\eta_k(i,k)} \right) \times T_{p_{k_{n_k}}}^{\eta_k(i,k} \Big).
\end{aligned}$$

Then, by Lemma 3.28,

$$\begin{aligned}
\sigma_i = \widetilde{X}_{p_{1_1}} \times CC\widetilde{X}_{p_{1_2}} \times \cdots \times CC\widetilde{X}_{p_{k_{n_k}}} \\
\cup C_-\widetilde{X}_{p_{1_1}} \times \left(C_+\widetilde{X}_{p_{1_2}} \times \cdots \times CC\widetilde{X}_{p_{k_{n_k}}} \cup \cdots \cup CC\widetilde{X}_{p_{1_2}} \times \cdots \times C_+\widetilde{X}_{p_{k_{n_k}}} \right) \\
\cup \cdots \cup CC\widetilde{X}_{p_{1_1}} \times \cdots \times CC\widetilde{X}_{p_{k_{n_k-1}}} \times \widetilde{X}_{p_{k_{n_k}}} \\
\cup \left(C_+\widetilde{X}_{p_{1_1}} \times \cdots \times CC\widetilde{X}_{p_{k_{n_k-1}}} \cup \cdots \cup CC\widetilde{X}_{p_{1_1}} \times \cdots \times C_+\widetilde{X}_{p_{k_{n_k-1}}} \right) \times C_-\widetilde{X}_{p_{k_{n_k}}} \\
= \widetilde{X}_{q_{i1}} * C_+\widetilde{X}_{q_{i2}} * \cdots * C_+\widetilde{X}_{q_{ir_i}} \cup \cdots \cup C_+\widetilde{X}_{q_{i1}} * C_+\widetilde{X}_{q_{i2}} * \cdots * \widetilde{X}_{q_{ir_i}} \qquad (3.26)
\end{aligned}$$

To conclude that $\sigma_i = W$, we use the following observation. Suppose that B_1 and B_2 are contractible spaces containing A_1 and A_2, respectively. Since

$$\begin{aligned}
(A_1 * B_2) \cap (B_1 * A_2) &= (CA_1 \times B_2 \cup A_1 \times CB_2) \cap (CB_1 \times A_2 \cup B_1 \times CA_2) \\
&= CA_1 \times A_2 \cup A_1 \times CA_2 \\
&= A_1 * A_2
\end{aligned}$$

then $(A_1 * B_2) \cup (B_1 * A_2) = \Sigma A_1 * A_2$. Now suppose for $i = 1, \ldots, k$ that B_i is a contractible space containing A_i and let

$$D = (A_1 * B_2 * \cdots * B_k) \cup (B_1 * A_2 * \cdots * B_k) \cup \cdots \cup (B_1 * B_2 * \cdots * A_k).$$

Applying the above observation inductively we obtain that

$$\begin{aligned}
D &= (A_1 * B_2 * \cdots * B_k) \cup (B_1 * A_2 * \cdots * B_k) \cup \cdots \cup (B_1 * B_2 * \cdots * A_k) \\
&= A_1 * (B_2 * \cdots * B_k) \cup B_1 * ((A_2 * \cdots * B_k) \cup \cdots \cup (B_2 * \cdots * A_k)) \\
&= \Sigma(A_1 * \Sigma^{k-2}(A_2 * \cdots * A_k)) \\
&= \Sigma^{k-1} A_1 * \cdots * A_k.
\end{aligned}$$

Applying this to (3.26), we obtain

$$\begin{aligned}
\sigma_i &= \widetilde{X}_{q_{i1}} * C_+\widetilde{X}_{q_{i2}} * \cdots * C_+\widetilde{X}_{q_{ir_i}} \cup \cdots \cup C_+\widetilde{X}_{q_{i1}} * C_+\widetilde{X}_{q_{i2}} * \cdots * \widetilde{X}_{q_{ir_i}} \\
&= \Sigma^{r_i-1}\widetilde{X}_{q_{i1}} * \cdots * \widetilde{X}_{q_{ir_i}} \\
&= \Sigma^{2r_i-2}\widetilde{X}_{q_{i1}} \wedge \cdots \wedge \widetilde{X}_{q_{ir_i}} \\
&= \Sigma^{r_i-2} X_{q_{i1}} \wedge \cdots \wedge X_{q_{ir_i}} = W.
\end{aligned}$$

Now we prove that the map $\overline{\gamma_i^*}|_{C_+W} \colon C_+W \longrightarrow \tau_i$ is a homotopy equivalence. The proof of the map $\overline{\gamma_i^*}|_{C_-W} \colon C_-W \longrightarrow \kappa_i$ being a homotopy equivalence is mutatis mutandis the same.

Since $B_i^- \cap E_i^- = T_i^-$ and $B_i^1 \cap E_i^1 = T_i^1$ then

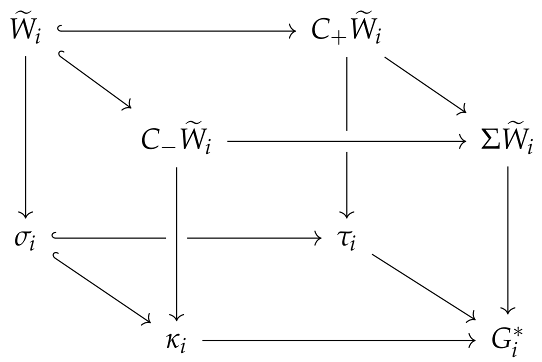
$$\begin{aligned}
\tau_i &= G_i^* \cap \left(E_{1_1}^{\eta_k(i,1)} \times \cdots \times E_{1_{n_1}}^{\eta_k(i,1)} \times \cdots \times E_{k_1}^{\eta_k(i,k)} \times \cdots \times E_{k_{n_k}}^{\eta_k(i,k)} \right) \\
&= G_i^* \cap \Big(T_{1_1}^{\eta_k(i,1)} \times \cdots \times E_{1_{n_1}}^{\eta_k(i,1)} \times \cdots \times E_{k_1}^{\eta_k(i,k)} \times \cdots \times E_{k_{n_k}}^{\eta_k(i,k)} \\
&\qquad \cup \cdots \cup E_{1_1}^{\eta_k(i,1)} \times \cdots \times E_{1_{n_1}}^{\eta_k(i,1)} \times \cdots \times E_{k_1}^{\eta_k(i,k)} \times \cdots \times T_{k_{n_k}}^{\eta_k(i,k)} \Big)
\end{aligned}$$

so by Lemma 3.28 again,

$$\begin{aligned}
\tau_i &= G_i^* \cap \Big(C_- \widetilde{X}_{1_1} \times \cdots \times C\Sigma \widetilde{X}_{1_{n_1}} \times \cdots \times C\Sigma \widetilde{X}_{k_1} \times \cdots \times C\Sigma \widetilde{X}_{k_{n_k}} \\
&\qquad \cup \cdots \cup C\Sigma \widetilde{X}_{1_1} \times \cdots \times C\Sigma \widetilde{X}_{1_{n_1}} \times \cdots \times C\Sigma \widetilde{X}_{k_1} \times \cdots \times C_- \widetilde{X}_{k_{n_k}} \Big) \\
&= C_- \widetilde{X}_{q_{i_1}} * \cdots * C_- \widetilde{X}_{q_{i_{r_i}}}
\end{aligned}$$

from which it follows that $\tau_i \simeq CW$, as claimed. □

Corollary 3.30. *Recall that $\widetilde{W}_i = \Sigma^{r_i - 2} X_{q_{i_1}} \wedge \cdots \wedge X_{q_{i_{r_i}}}$. Then there is a homotopy equivalence $C_- \widetilde{W}_i \longrightarrow \kappa_i$ such that the following diagram is homotopy commutative*



where the top and bottom squares are pushouts and the vertical maps are homotopy equivalences.

Lemma 3.31. *There is a homotopy equivalence of pairs*

$$(CW, W) \longrightarrow (F_i, Z_i)$$

for each $i = 1, \ldots, k$.

Proof. Recall that $\widetilde{W}_i = \Sigma^{r_i - 2} X_{q_{i_1}} \wedge \cdots \wedge X_{q_{i_{r_i}}}$. Then by Lemma 3.30, there are coherent homotopy equivalences of pairs $(C\Sigma\widetilde{W}_i, C_+\widetilde{W}_i) \longrightarrow (G_i, \tau_i)$ and

$(C_-\widetilde{W}_i, \widetilde{W}_i) \longrightarrow (\kappa_i, \sigma_i)$. Therefore

$$\begin{aligned}
F_i &= \xi_i\left(\left(\left(G_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i1}} \times \cdots \times X_{p_{is}} \times \cdots \times CX_{p_{in_i}}\right)\right)\right) \cup \left(\tau_i \times \left(CX_{p_{i1}} \times \cdots \times CX_{p_{in_i}}\right)\right)\right) \\
&= \xi_i\left(C\Sigma\widetilde{W}_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i1}} \times \cdots \times X_{p_{is}} \times \cdots \times CX_{p_{in_i}}\right) \cup C_+W_i \times \left(CX_{p_{i1}} \times \cdots \times CX_{p_{in_i}}\right)\right) \\
&= \xi_i\left(C_+\widetilde{W}_i * X_{p_{i1}} * \cdots * X_{p_{in_i}}\right) \\
&= \xi_i\left(C(X_{q_{i1}} * \cdots * X_{q_{ir_i}}) * X_{p_{i1}} * \cdots * X_{p_{in_i}}\right) \\
&= CW
\end{aligned}$$

and

$$\begin{aligned}
Z_i &= \xi_i\left(\left(\left(\kappa_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i1}} \times \cdots \times X_{p_{is}} \times \cdots \times CX_{p_{in_i}}\right)\right)\right) \cup \left(\sigma_i \times \left(CX_{p_{i1}} \times \cdots \times CX_{p_{in_i}}\right)\right)\right) \\
&= \xi_i\left(C_-\widetilde{W}_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i1}} \times \cdots \times X_{p_{is}} \times \cdots \times CX_{p_{in_i}}\right) \cup \widetilde{W}_i \times \left(CX_{p_{i1}} \times \cdots \times CX_{i_{n_i}}\right)\right) \\
&= \xi_i\left(\widetilde{W}_i * X_{p_{i1}} * \cdots * X_{p_{in_i}}\right) \\
&= W
\end{aligned}$$

which establishes the claimed homotopy equivalence of pairs. □

To recap, we have now achieved our first aim, which was to show that there is a homotopy equivalence of pairs $(CW, W) \simeq (F_i, Z_i)$.

We now turn our focus to our second aim, which is to prove that for each $i \in [k]$, the spaces $F_i \backslash Z_i$ are pairwise disjoint.

We first have the following technical lemmata.

Lemma 3.32. *Suppose that $1 \leqslant i < j \leqslant k$. Then there exists $i \neq r \neq j$ such that G_i contains a factor D_r^1 and G_j contains a factor D_r^-.*

Proof. Observe from Figure 3.2 that given any two rows i and j of a k-array, there exists $i \neq r \neq j$ such that $\eta_k(i, r) = 1$ and $\eta_k(j, r) = -$. Then by construction G_i has a factor D_r^1 and G_j has a factor D_r^-. □

Lemma 3.33. *Let $i, j \in [k]$ such that $i < j$. Then,*

$$G_i \cap G_j = \widehat{C\Sigma W}_i \cap \widehat{C\Sigma W}_j \cap \left(D_{1_1}^{\eta_k(i,1)} \times \cdots \times D_{1_{n_1}}^{\eta_k(i,1)} \times \cdots \times D_{k_1}^{\eta_k(i,k)} \times \cdots \times D_{k_{n_k}}^{\eta_k(i,k)} \right)$$

Proof. This follows immediately from the definition of G_i and G_j. □

We now proceed with the main statement which we prove in this step.

Lemma 3.34. *Let $W = \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m$. If $i, j \in [k]$ such that $i \neq j$, then*

$$(F_i \backslash Z_i) \cap (F_j \backslash Z_j) = \emptyset.$$

Proof. Fix i and j with $i \neq j$, and without loss of generality, suppose that $i < j$. We show that $F_i \cap F_j \subset Z_i \cap Z_j$, from which the claim follows. Define

$$H = \xi_i \left(G_i \times \bigcup_{s=1}^{n_i} CX_{i_1} \times \cdots \times X_{p_{1_s}} \times \cdots \times CX_{p_{i_{n_i}}} \right) \cap \xi_j \left(G_j \times \bigcup_{s=1}^{n_j} CX_{p_{j_1}} \times \cdots \times X_{p_{j_s}} \times \cdots \times CX_{p_{j_{n_j}}} \right).$$

Comparing coordinates, since $D_k^- \cap X_k \subset B_k^-$ and $D_k^1 \cap X_k \subset B_k^1$, then

$$H \subset \xi_i \left(G_i^* \times \bigcup_{s=1}^{n_i} CX_{p_{i_1}} \times \cdots \times X_{p_{i_s}} \times \cdots \times CX_{p_{i_{n_i}}} \right) \cap \xi_j \left(G_j^* \times \bigcup_{s=1}^{n_j} CX_{p_{j_1}} \times \cdots \times X_{p_{j_s}} \times \cdots \times CX_{p_{j_{n_j}}} \right).$$

By Lemma 3.32, there is $i \neq r \neq j$ such that G_i has a factor D_r^1 and G_j has a factor D_r^-. Then, denoting by π_r the projection onto the r-th coordinate, since $\pi_r(\tau_i) \cap D_r^- = \pi_r(\tau_j) \cap D_r^1 = \emptyset$, it follows that for $\alpha \in \{i, j\}$,

$$H \cap \xi_\alpha \left(\tau_\alpha \times \bigcup_{s=1}^{n_\alpha} CX_{q_{\alpha_1}} \times \cdots \times X_{q_{\alpha_s}} \times \cdots \times CX_{q_{\alpha_{n_\alpha}}} \right) = \emptyset.$$

Therefore, observing that $G_i^* \backslash \tau_i \subseteq \kappa_i$,

$$H \subset \xi_i \left((G_i^* \backslash \tau_i) \times \bigcup_{s=1}^{n_i} CX_{p_{i_1}} \times \cdots \times X_{p_{i_s}} \times \cdots \times CX_{p_{i_{n_i}}} \right) \cap \xi_j \left((G_j^* \backslash \tau_j) \times \bigcup_{s=1}^{n_j} CX_{p_{j_1}} \times \cdots \times X_{p_{j_s}} \times \cdots \times CX_{p_{j_{n_j}}} \right) \subseteq Z_i \cap Z_j.$$

Finally since $\pi_r(G_i) = D^1_r$ and $\pi_r(\tau_j) \cap D^1_r = \emptyset$ then

$$\xi_i \left(G_i \times \bigcup_{s=1}^{n_i} CX_{p_{i1}} \times \cdots \times X_{p_{il}} \times \cdots \times CX_{p_{in_i}} \right) \cap \xi_j \left(\tau_j \times CX_{p_{j1}} \times \cdots \times CX_{p_{jn_j}} \right) = \emptyset$$

and the same holds after interchanging i and j, and since $\pi_r(\tau_i) = E^1_r$ and $\pi_r(\tau_j) = E^-_r$, then

$$\xi_i \left(\tau_i \times CX_{p_{i1}} \times \cdots \times CX_{p_{in_i}} \right) \cap \xi_j \left(\tau_j \times CX_{p_{i1}} \times \cdots \times CX_{p_{in_i}} \right) = \emptyset.$$

Therefore $F_i \cap F_j \subset Z_i \cap Z_j$.

□

Lemma 3.35. *Suppose that for a CW-pair (X, A), a homotopy equivalence pair $(X', A') \simeq (X, A)$ and a deformation retract $h\colon X \longrightarrow X'$ are given. Suppose that a map $f\colon (X, A) \longrightarrow (Y, B)$ satisfies $f((X - A) \cup A') \subset B$. Then $f|_{X'} \circ h = f\colon (X, A) \longrightarrow (Y, B)$.*

Proof. Since (X', A') is a deformation retract of (X, A) there is a map $H_t\colon X \longrightarrow X$ such that $H_0((X, A)) = (X, A)$, $H_1(X', A') \in (X', A')$ and $H_1((X, A)) = (X', A')$. Then we can define a homotopy $f_t\colon (X, A) \longrightarrow (Y, B)$ by $f_t = f \circ H_t$. Then $f_0 = f$, $f_1 = f|_{X'}$ and $f_t(A) = f \circ H_t(A) \subset f((X - A) \cup A') \subset B$. □

Lemma 3.36. *Let $n \geq 2$. For each $i \in [n]$, let $F_i \subseteq \Sigma W$ be contractible and such that there exists $Z_i \subseteq F_i$ such that $(F_i, Z_i) \simeq (CW, W)$, where $W = \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m$. Suppose that for all $i \in [n]$, the subspaces $F_i \backslash Z_i$ are pairwise disjoint.*

Let (Y, Z) be a CW-pair, and suppose that $f\colon \Sigma W \longrightarrow Y$ is such that $f((\Sigma W \backslash \bigcup_i F_i) \cup (\bigcup_i Z_i)) \subseteq Z$ for all i. Denote by $f_i = f|F_i\colon (F_i, Z_i) \longrightarrow (Y, Z)$, so that $[f_i \in [(F_i, Z_i), (Y, Z)] \cong [(CW, W), (Y, Z)]$. Let $j\colon [\Sigma W, X] \longrightarrow [CW, W]$ denote the map from the long exact sequence. Then $j([f]) = \sum_i [f_i]$.

Proof. Let $W = \Sigma Y = C_+Y \cup C_-Y$. Suppose that $* \simeq F_1, F_2 \subseteq \Sigma W$, that $W \simeq Z_1 \subseteq F_1$ and $W \simeq Z_2 \subseteq f_2$, and $(F_1 - Z_1) \cap (F_2 - Z_2) = \emptyset$. Let (X, A) be a CW-pair, denote by X_A the homotopy fibre of $A \hookrightarrow X$, and let $f\colon \Sigma W \longrightarrow X$ be such that $f((\Sigma W - \cup F_i) \cup (\cup Z_i)) \subseteq A$ for all i.

We prove the result for $n = 2$. That is, we show that $j[f] = f_1 + f_2$, where $f_1 = f|_{F_1}$ and $f_2 = f|_{F_2}$. Without loss of generality, we assume that $F_1 = C_+Y$ and $F_2 = C_-Y$.

Denote by

$$G_+ = CC_+Y = \{(s,(t,w)) \in CW \mid t \geqslant \frac{1}{2}\} \subseteq CW$$

and

$$G_- = CC_-Y = \{(s,(t,w)) \in CW \mid t \leqslant \frac{1}{2}\} \subseteq CW$$

whence $G_+ \cap G_- = CY = \{(s,(t,w) \mid t = \frac{1}{2}\} \subseteq CW$, and $G_+ \cup G_- = CW$.

Let $g\colon (CW,W) \longrightarrow (X,A)$ be a map such that $g(G_+ \cap G_-) \subseteq A$. For $[f] \in [W, X_A]$, let $f_+ = f|_{G_+}$ and $f_- = f|_{G_-}$. Since $G_+ \cap G_- = CY \simeq *$, there exists a homotopy $g_t\colon (CY,Y) \longrightarrow (X,A)$ such that $g_0 = g$, $g_t(G_+ \cap G_-) \subseteq A$ and $g_1(G_+ \cap G_-) = * \in A$, and therefore $[f] = [f_+] + [f_-]$.

Denote by $\Sigma W = C_+W \cup C_-W$. Since $F_1 \simeq F_2 \simeq *$ and $Z_1 \simeq Z_2 \simeq W$ where $Z_1 \subseteq F_1$ and $Z_2 \subseteq F_2$, there are homotopy equivalences of pairs $h_1\colon (F_1,Z_1) \xrightarrow{\simeq} (C_+W,W)$ and $h_2\colon (F_2,Z_2) \xrightarrow{\simeq} (C_-W,W)$.

Denote by $\theta\colon (CW,W) \longrightarrow (\Sigma W,*)$ the map

$$(s,(t,w)) \mapsto \begin{cases} (*,(*,*)) & s = 0 \\ (s,(t,w)) & \text{otherwise.} \end{cases}$$

so that $\theta|_{G_+}\colon G_+ \longrightarrow C_+W \longrightarrow F_1$, $\theta|_{G_-}\colon G_- \longrightarrow C_-W \longrightarrow F_2$, $\theta|_{G_+\cap G_-}\colon G_+ \cap G_- \longrightarrow W \longrightarrow Z_1 \cap Z_2$ and $\theta(CW - W) = \Sigma W - *$. Then $g = f\theta\colon (CW,W) \longrightarrow (X,*)$ satisfies $g(G_+ \cap G_-) \subseteq A$, $g|_{G_+} = f_1$ and $g|_{G_-} = f_2$. Then $j[f] = [f_1] + [f_2]$.

The result then follows by induction on n. □

In the following, we model the space ΣW by $\widehat{\Sigma W} = (\mathbf{CX}, \mathbf{X})^{\partial\Delta^{m-1}}$.

Corollary 3.37. *Let $n \geq 2$. For each $i \in [n]$, let $F_i \subseteq \widehat{\Sigma W}$ be contractible and such that there exists $Z_i \subseteq F_i$ such that $(F_i, Z_i) \simeq (CW,W)$, where $W = \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m$. Suppose that for all $i \in [n]$, the subspaces $F_i \backslash Z_i$ are pairwise disjoint.*

Let (Y,Z) be a CW-pair, and suppose that $f\colon \widehat{\Sigma W} \longrightarrow Y$ is such that $f((\widehat{\Sigma W} \backslash \bigcup_i F_i) \cup (\bigcup_i Z_i)) \subseteq Z$ for all i. Denote by $f_i = f|F_i\colon (F_i,Z_i) \longrightarrow (Y,Z)$, so that $[f_i \in [(F_i,Z_i),(Y,Z)] \cong [(CW,W),(Y,Z)]$. Let $j\colon [\widehat{\Sigma W}, X] \longrightarrow [CW,W]$ denote the map from the long exact sequence. Then $j([f]) = \sum_i [f_i]$.

We are now ready to prove the main theorem. We recall the statement once more.

Theorem 3.19. Let $m \geq 3$. Let $\Pi = \{P_1, \ldots, P_k\}$ be a k-partition of $[m]$, and for each $i \in [k]$ denote by $P_i = \{p_{i_1}, \ldots, p_{i_{n_i}}\}$ and by $Q_i = [m] \backslash P_i = \{q_{i_1}, \ldots, q_{i_{r_i}}\}$. Suppose that we are given maps $f_l\colon \Sigma X_l \longrightarrow Y_l$ for each $l \in [m]$ such that $X_l = \Sigma\widetilde{X}_l$. Then the map

$$\sum_{i=1}^{k} {}_i h_w^{\mathcal{K}_\Pi}\colon \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m \longrightarrow (\mathbf{Y}, *)^{\mathcal{K}_\Pi} \tag{3.27}$$

is trivial, and for any subset $\varnothing \neq J \subsetneq [k]$, the sum $\sum_{i\in J} {}_i h_w^{\mathcal{K}_\Pi}$ is non-trivial.

Proof. Recall that

$$D_l^+ = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_l \mid t \geqslant \frac{1}{2}\right\}, \qquad \text{and}$$
$$D_l^2 = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_l \mid s \leqslant \frac{3}{4} - t\right\}.$$

Observe that $X_l = \Sigma\widetilde{X}_l \simeq D_l^+ \cup D_l^2$. Since $X_l \subseteq D_l^+ \cup D_l^2$, under the isomorphism $\theta\colon [\Sigma X_j, Y_j] \cong [(CX_j, X_j), (Y, *)]$ the class $\theta([f_l])$ is represented by a map of pairs $\widehat{f}_l\colon (C\Sigma\widetilde{X}_l, \Sigma\widetilde{X}_l) \longrightarrow (Y_l, *)$ such that $\widehat{f}_l(D_l^+ \cup D_l^2) = *$.

Recall that $\widehat{C\Sigma W} = (\mathbf{C\Sigma\widetilde{X}}, \mathbf{\Sigma\widetilde{X}})^{\Delta^{m-1}}$, and that $\widehat{\Sigma W} = (\mathbf{C\Sigma\widetilde{X}}, \mathbf{\Sigma\widetilde{X}})^{\partial\Delta^{m-1}}$. Define the map of pairs $\psi\colon (\widehat{C\Sigma X}, \widehat{\Sigma X}) \longrightarrow ((\mathbf{Y},*)^{\Delta^{m-1}}, (\mathbf{Y},*)^{\partial\Delta^{m-1}})$ as the maps of pairs of polyhedral products induced by the maps of pairs $\widehat{f_1}, \ldots, \widehat{f_m}$.

Recall that $\widehat{C\Sigma W}_i = (\mathbf{C\Sigma\widetilde{X}}, \mathbf{\Sigma\widetilde{X}})^{\Delta^{m-1}_{Q_i}}$, and that $\widehat{\Sigma W}_i = (\mathbf{C\Sigma\widetilde{X}}, \mathbf{\Sigma\widetilde{X}})^{\partial(\Delta^{m-1}_{Q_i})}$. For all $i \in [k]$ define maps of pairs $\psi_i\colon (\widehat{C\Sigma W}_i, \widehat{\Sigma W}_i) \longrightarrow ((\mathbf{Y},*)^{\Delta^{m-1}_{Q_i}}, (\mathbf{Y},*)^{\partial\Delta^{m-1}_{Q_i}})$ by the restriction of ψ to these subspaces.

Observe that

$$(D_l^+ \cup D_l^2) \cup (D_l^1 \cap D_l^-) = C\Sigma\widetilde{X}_l, \qquad \text{and}$$
$$(D_l^+ \cup D_l^2) \cap (D_l^1 \cap D_l^-) = ((B_1)_l^- \cap D_l^2) \cup ((B_2)_l^1 \cap D_l^-).$$

Since $\widehat{f}_l(D_l^+ \cup D_l^2) = *$, then for $x_l \in C\Sigma\widetilde{X}_l$, $\widehat{f}_l(x_l) \neq *_{C\Sigma\widetilde{X}_l}$ if and only if $x_l \in (D_l^1 \cap D_l^-)\backslash(((B_1)_l^- \cap D_l^2) \cup ((B_2)_l^1 \cap D_l^-))$.

In particular, $\psi_i\left(x_{q_{i_1}}, \ldots, x_{q_{i_{r_i}}}\right) \in (\mathbf{Y},*)^{\Delta^{m-1}_{Q_i}} \Big\backslash (\mathbf{Y},*)^{\partial(\Delta^{m-1}_{Q_i})}$ only if $\left(x_{q_{i_1}}, \ldots, x_{q_{i_{r_i}}}\right) \in \prod_{t=1}^{r_i} D^1_{q_{i_t}} \cap D^-_{q_{i_t}} \subseteq G_i$

Equivalently, $\psi_i((\widehat{C\Sigma W}_i \backslash G_i) \cup G_i^*) \subseteq (\mathbf{Y},*)^{\partial(\Delta^{m-1}_{Q_i})}$. Therefore by Lemma 3.35, since $(\widehat{C\Sigma W}_i, \widehat{\Sigma W}_i) \simeq (G_i, G_i^*)$ and $G_i \subseteq \widehat{C\Sigma W}_i$, then $\psi_i|_{G_i}$ is homotopic to ψ_i.

Now consider

$$\begin{aligned}
\psi\xi_i|_{\xi_i^{-1}F_i}\colon G_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i1}} \times \cdots \times X_{p_{is}} \times \cdots \times CX_{p_{in_i}}\right) \cup \tau_i \times \left(CX_{p_{i1}} \times \cdots \times CX_{p_{in_i}}\right)\\
\longrightarrow \psi_i(G_i) \times \left(\bigcup_{s=1}^{n_i} \Sigma X_{p_{i1}} \times \cdots \times * \times \cdots \times \Sigma X_{p_{in_i}}\right) \cup * \times \left(\Sigma X_{p_{i1}} \times \cdots \times \Sigma X_{p_{in_i}}\right)\\
\longrightarrow \left(\prod_{t=1}^{r_i} Y_{q_{it}} \times (\mathbf{Y},*)^{\partial(\Delta_{P_i}^{m-1})}\right) \cup \left(* \times \prod_{s=1}^{n_i} Y_{p_{is}}\right)\\
= FW\left(\prod_{t=1}^{r_i} Y_{q_{it}}, Y_{p_{i1}}, \ldots, Y_{p_{in_i}}\right)\\
= (\mathbf{Y},*)^{\partial\Delta\langle\Delta_{Q_i}^{m-1}, p_{i1},\ldots,p_{in_i}\rangle} \hookrightarrow (\mathbf{Y},*)^{\mathcal{L}}
\end{aligned}$$

and the restriction

$$\begin{aligned}
\psi\xi_i|_{\xi_i^{-1}Z_i}\colon \kappa_i \times \left(\bigcup_{s=1}^{n_i} CX_{p_{i1}} \times \cdots \times X_{p_{is}} \times \cdots \times CX_{p_{in_i}}\right) \cup \left(\sigma_i \times \left(CX_{p_{i1}} \times \cdots \times CX_{p_{in_i}}\right)\right)\\
\longrightarrow \left((\mathbf{Y},*)^{\partial(\Delta_{Q_i}^{m-1}} \times (\mathbf{Y},*)^{\partial(\Delta_{P_i}^{m-1})}\right) \cup \left(* \times \prod_{s=1}^{n_i} Y_{p_{is}}\right)\\
= FW\left((\mathbf{Y},*)^{\partial\Delta_{Q_i}^{m-1}}, Y_{p_{i1}}, \ldots, Y_{p_{in_i}}\right)\\
= (\mathbf{Y},*)^{\partial\Delta\langle\partial\Delta_{Q_i}^{m-1}, p_{i1},\ldots,p_{in_i}\rangle} \hookrightarrow (\mathbf{Y},*)^{\mathcal{K}}
\end{aligned}$$

It follows that $\psi\xi_i|_{\xi_i^{-1}F_i}\colon (\xi_i^{-1}F_i, \xi_i^{-1}Z_i) \longrightarrow ((\mathbf{Y},*)^{\partial\Delta^{m-1}}, (\mathbf{Y},*)^{\mathcal{K}})$ is the relative higher Whitehead map $h_w(\psi_i|_{G_i}, f_{p_{i1}}, \ldots, f_{p_{in_i}})$. Therefore
$\psi|_{F_i}\colon (F_i, Z_i) \longrightarrow ((\mathbf{Y},*)^{\partial\Delta^{m-1}}, (\mathbf{Y},*)^{\mathcal{K}})$ is the composite $h_w(\psi_i|_{G_i}, f_{p_{i1}}, \ldots, f_{p_{in_i}}) \circ \xi_i^{-1}$.
Finally since $\psi_i|_{G_i}$ is homotopic to ψ_i, then $\psi|_{F_i}$ is homotopic to $h_w(\psi_i, f_{p_{i1}}, \ldots, f_{p_{in_i}}) \circ \sigma_i$.

Let $W = \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m$, $Y = (\mathbf{Y},*)^{\mathcal{L}}$ and $Z = (\mathbf{Y},*)^{\mathcal{K}}$. Appealing to the long exact sequence

$$\cdots \longrightarrow [\Sigma W, Y] \xrightarrow{\ j\ } [(CW, W), (Y, Z)] \xrightarrow{\ \partial\ } [W, Z] \longrightarrow \cdots.$$

By Proposition 3.26 and Lemma 3.37,

$$\begin{aligned}
\partial(j([h_w(f_1,\dots,f_m)])) &= \partial\left(\sum_{i=1}^{k}[h_w(f_1,\dots,f_m)|_{F_i}]\right)\\
&= \sum_{i=1}^{k}\partial([\xi_i\circ h_w(\psi_i, f_{p_{i1}},\dots,f_{p_{in_i}})\circ\sigma_i]\\
&= \sum_{i=1}^{k}[\xi_i h_w(\psi_i|_{\widehat{\Sigma W}_{i*}}, f_{p_{i1}},\dots,f_{p_{in_i}})\circ\sigma_i]\\
&= \sum_{i=1}^{k}[\xi_i\circ h_w(h_w(f_{q_{i1}},\dots,f_{q_{ir_i}}), f_{p_{i1}},\dots,f_{p_{in_i}})\circ\sigma_i]\\
&= \sum_{i=1}^{k}[{}_i h_w^{\mathcal{K}}]\\
&= 0
\end{aligned}$$

where the final equality follows from exactness. □

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