A SELF-ADAPTIVE INERTIAL SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR SOLVING BILEVEL EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce an inertial subgradient extragradient method with a self-adaptive technique for solving bilevel equilibrium problem in real Hilbert spaces. The algorithm is designed such that its stepsize is chosen without the need for prior estimates of the Lipschitz-like constants of the upper level bifunction nor a line searching procedure. This provides computational advantages to the algorithm compared with other similar methods in the literature. We prove a strong convergence result for the sequences generated by our algorithm under suitable conditions. We also provide some numerical experiments to illustrate the performance and efficiency of the proposed method.

1. Introduction

The Bilevel Optimization Problem (shortly, BOP) is defined as a mathematical program in which a problem contains another problem as a constraint. Mathematically, the BOP is formulated as follows:

$$\begin{gathered}\text{Find } x^\dagger \in \mathcal{M} \subset H \text{ such that } x^\dagger \text{ solves Problem } \mathbf{P1} \text{ installed in a space } H,\\ \text{where } \mathcal{M} \subset H \text{ is the solution set of the problem:}\\ \text{Find } x^* \in \mathcal{N} \subset H \text{ such that } x^* \text{ solves Problem } \mathbf{P2} \text{ installed in a space } H.\end{gathered}$$

The Problem **P1** is called the upper level problem, while Problem **P2** is called lower level problem. In the last few years, BOP has attracted the interest of many researchers due to its wide applications in several fields of applied sciences such as engineering, economics, management, network design, optimal chemical equilibrium, etc.; see, e.g. [22, 33, 42]. A wide collection of algorithms for solving BOP can be seen in, for instance, for bilevel variational inequalities [1, 42, 45, 57], for bilevel equilibrium problem [18, 23, 25, 26, 48], and for bilevel minimization problem [21, 51, 52].

In this paper, we study the approximation of solution of Bilevel Equilibrium Problem (BEP) in real Hilbert spaces. Let C be a nonempty, closed and convex subset of a real Hilbert space H, and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for all $x \in H$. The Equilibrium Problem (EP) is defined as finding $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the solution set of the EP (1.1) by $Sol(f, C)$. The EP (1.1) generalizes many other optimization problems such as variational inequalities, convex minimization, fixed point, Nash equilibrium problems among others (see, [10, 47]). The basic methods for solving the EP (1.1) includes regularization methods, proximal method, extragradient method, projection methods, gap-function methods, Bregman distance method, etc., see [2–6, 8, 9, 11, 12, 19, 27–32, 34–41, 49, 53, 54, 58] for details.

Definition 1.1. A bifunction $f : C \times C \to \mathbb{R}$ is said to be:

(i) strongly monotone on C if there exists a constant $\tau > 0$ such that

$$f(u, v) + f(v, u) \leq -\tau\|u - v\|^2, \ \forall u, v \in C;$$

(ii) monotone on C if $f(u, v) + f(v, u) \leq 0, \ \forall u, v \in C$;

(iii) strongly pseudomonotone on C if there exists a constant $\gamma > 0$ such that

$$f(u, v) \geq 0 \Rightarrow f(v, u) \leq -\gamma\|u - v\|^2, \ \forall x, y \in C;$$

(iv) pseudomonotone on C $f(u, v) \geq 0 \Longrightarrow f(v, u) \leq 0, \ \forall u, v \in H$;

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(v) satisfying a Lipschitz-like condition if there exist two positive constants L_1, L_2 such that

$$f(u,v) + f(v,w) \geq f(u,w) - L_1\|u-v\|^2 - L_2\|v-w\|^2, \quad \forall u,v,w \in H. \tag{1.2}$$

From the definitions above, it is easy to see that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) and (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). However, the converse implications do not hold in general, see, e.g. [34, 35].

The BEP is defined as finding a point $x^* \in Sol(f,C)$ such that

$$g(x^*, y) \geq 0 \quad \forall y \in Sol(f,C), \tag{1.3}$$

where $g : H \times H \to \mathbb{R}$ is another bifunction satisfying $g(x,x) = 0$ for all $x \in H$. The motivation for studying the BEP can be derived from (for instance) the supply-chain management problem (see [17]) and power control problems of CDMA networks (see, [33]). The BEP was first studied by Chadli et al. [14] in 2000 for generalized monotone bifunctions. Also, Moudafi [46] introduced a proximal method and proved a weak convergence result for solving monotone BEP in real Hilbert spaces. Later, Quy [56] combined the Halpern's method and proximal method to obtain a strong convergence result for solving the BEP in real Hilbert spaces. It should be observed that the convergence of the proximal method requires the bifunctions to satisfy monotone and para-pseudomonotone conditions. When this conditions are relaxed, the proximal method fails to converge to a solution of the BEP (1.3).

Recently, Yujing et al. [59] introduced an extragradient method for solving the BEP when f is strongly monotone and g satisfies pseudomonotone and Lipschitz-like condition as follows:

Algorithm 1.2 (Extragradient Method (EGM))**.**

Initialization: *Choose* $x_0 \in H$, $0 < \mu < \frac{2\beta}{L^2}$, $\{\alpha_n\} \subset [0,1], \{\eta_n\}, \{\lambda_n\}$ *satisfying*

$$\begin{cases} \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \\ 0 \leq \eta_n \leq 1 - \alpha \quad \lim_{n\to\infty} \eta_n = \eta < 1, \quad \forall n \geq 0, \\ 0 \leq \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \min\left(\frac{1}{2L_1}, \frac{1}{2L_2}\right). \end{cases}$$

Set $n = 0$ and go to Step 1.
Step 1: *Compute*

$$\begin{cases} y_n = \underset{y \in C}{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}, \\ z_n = \underset{y \in C}{argmin}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}. \end{cases}$$

Step 2: *Compute $w_n \in \partial_2 f(z_n, z_n)$ and*

$$x_{n+1} = \eta_n x_n + (1-\eta_n) z_n - \alpha_n \mu w_n. \tag{1.4}$$

Set $n = n+1$ and go back to Step 1.

The authors proved that the sequence $\{x_n\}$ generated by Algorithm 1.2 converges strongly to a solution of the BEP (1.3). It should be observed that in Algorithm 1.2, one needs to solve two strongly convex optimization problems over the entire feasible set C. This can be very complicated if the set C is not simple. Moreover, the stepsize λ_n of Algorithm 1.2 depends on the prior estimate of the Lipschitz-like constants L_1 and L_2 which are very difficult to determine. In order to improve the efficiency of Algorithm 1.2, the authors in [59] also introduced the following extragradient method with line search:

Algorithm 1.3 (Extragradient Method with Linesearch (EML))**.**

Initialization: *Choose* $x_0 \in C$, $0 < \mu < \frac{2\beta}{L^2}$, $\rho \in (0,2)$, $\gamma \in (0,1)$, $\{\alpha_n\}, , \{\xi_n\}, \{\lambda_n\} \subset (0,1)$ *satisfying*

$$\begin{cases} \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \alpha_n^2 < \infty, \\ \lambda_n \in [\underline{\lambda}, \bar{\lambda}] \subset (0,\infty), \quad \xi_n \in [\underline{\xi}, \bar{\xi}] \subset (0,2). \end{cases}$$

Set $n = 0$ and go to Step 1.
Step 1: *Compute*

$$y_n = \underset{y \in C}{argmin} \left\{ \lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}.$$

If $y_n = x_n$, then set $u_n = \xi_n$ and go to Step 4. Otherwise, go to Step 2.
Step 2: *(Armijo line search): Find m being the smallest positive integer number satisfying*

$$\begin{cases} z_{n,m} = (1 - \gamma^m)x_n + \gamma^m y_n, \\ g(z_{n,m}, x_n) - g(z_{n,m}, y_n) \geq \frac{\rho}{2\lambda_n}\|x_n - y_n\|^2. \end{cases}$$

Set $z_n = z_{n,m}$ and $\gamma_n = \gamma^m$.
Step 3: *Choose $t_n \in \partial_2 g(z_n, x_n)$ and compute $P_C(x_n - \xi_n \sigma_n t_n)$ where $\sigma_n = \frac{g(z_n, x_n)}{\|t_n\|^2}$.*
Step 4: *Compute $w_n \in \partial_2 f(u_n, u_n)$ and*

$$x_{n+1} = P_C(u_n - \alpha_n \mu w_n).$$

Set $n = n + 1$ and go back to Step 1.

The authors proved that the sequence $\{x_n\}$ generated by Algorithm 1.3 converges strongly to a solution of the BEP (1.3). Though, Algorithm 1.3 improves Algorithm 1.2, it however incurred the following setbacks:

- The algorithm requires computing two projection onto the feasible set C per each iteration. This can be computationally expensive if the set C is not so simple.
- The line search procedure uses an inner-iteration which posses additional computation and execution time.

In view of the above, in this paper, we introduce a new inertial self-adaptive subgradient extragradient method for solving the BEP in real Hilbert spaces. The inertial extrapolation term is regarded as a means of accelerating the convergence rate of optimization algorithms and has been implemented in many recent methods; see for instance [13, 15, 16, 24, 34, 35]. Furthermore, our algorithm solves only one strongly convex optimization problem over the feasible set while the second optimization problem is solved over a constructible half-space which can easily be calculated using available techniques in convex optimization. More so, the stepsize of the algorithm is determined by a step-adaptive process and does not require the prior estimate of the Lipschitz-like constants. We prove a strong convergence theorem for approximating the solution of the BEP in real Hilbert spaces. We also provide some numerical experiments to illustrate the performance of the proposed method and compare it with other related methods in the literature.

The rest of the paper is organized as follows: In Section 2, we recall some essential definitions and fundamental results. In Section 3, we present our algorithm and its convergence analysis. In Section 4, we discuss some numerical experiments to show the applicability and efficiency of the proposed method.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert spaces H. We denote the strong convergence of the sequence $\{x_n\}$ to p by $x_n \to p$ and the weak convergence of $\{x_n\}$ to p by $x_n \rightharpoonup p$. For each $x \in H$, the metric projection $P_C : H \to C$ is defined by $P_C(x) = argmin\{\|x - w\| : w \in C\}$. The metric projection satisfies the following identities (see, e.g. [50]):

(i) $\langle x - y, P_C x - P_C y \rangle \geq ||P_C x - P_C y||^2$, for every $x, y \in H$;
(ii) for $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \ \forall y \in C; \tag{2.1}$$

(iii) for $x \in H$ and $y \in C$,

$$||y - P_C(x)||^2 + ||x - P_C(x)||^2 \leq ||x - y||^2. \tag{2.2}$$

The following identities hold in any Hilbert space:

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle, \quad \forall \ x, y \in H, \tag{2.3}$$

and

$$||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \quad \forall\, x, y \in H. \tag{2.4}$$

A subset D of H is called proximal if for each $x \in H$, there exists $y \in D$ such that

$$||x-y|| = d(x, D).$$

In what follows, we define the Hausdorff metric on H as follows

$$\mathcal{H}(A,B) := \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

for all A and B and to be subsets of H. The normal cone N_C to C at a point $x \in C$ is defined by $N_C(x) = \{w \in H : \langle w, x-y\rangle \geq 0, \forall y \in C\}$.

Lemma 2.1. [20] *Let C be a convex subset of a real Hilbert space H and $\varphi : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then x^* is a solution to the convex problem: minimize$\{\varphi(x) : x \in C\}$ if and only if $0 \in \partial\varphi(x^*) + N_C(x^*)$, where $\partial\varphi(x^*)$ denotes the subdifferential of φ and $N_C(x^*)$ is the normal cone of C at x^*.*

Lemma 2.2. *[59] Let $f : H \times H \to \mathbb{R}$ be a β-strongly monotone bifunction and satisfies k-Lipschitz continuous condition, i.e., for each $x, y \in H$, $u \in \partial_2 f(x,\cdot)(x)$ and $v \in \partial_2 f(y,\cdot)(y)$. Suppose $0 < \alpha \leq 1$, $0 \leq \delta \leq 1-\alpha$ and $0 < \mu < \frac{2\beta}{k^2}$, then*

$$\|(1-\delta)x - \alpha\mu w - [(1-\delta)y - \alpha\mu v]\| \leq (1-\delta-\alpha\tau)\|x-y\|,$$

where $\tau = 1 - \sqrt{1-\mu(2\beta-\mu k^2)} \in (0,1]$.

Lemma 2.3. *[43] Let $\{\alpha_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\delta_n)\alpha_n + \beta_n + \gamma_n, \qquad n \geq 1,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{\beta_n\}$ is a real sequence. Assume that $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then, the following results hold:

(i) *If $\beta_n \leq \delta_n M$ for some $M \geq 0$, then $\{\alpha_n\}$ is a bounded sequence.*
(ii) *If $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{\beta_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} \alpha_n = 0$.*

Lemma 2.4. *[44] Let $\{a_n\}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{a_{n_i}\}$ of $\{a_n\}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large number $k \in \mathbb{N}$): $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.*

3. Main results

In this section, we propose an inertial subgradient extragradient method with a self-adaptive stepsize for solving the BEP (1.3). The self-adaptive technique we adopt is suitably updated at each iteration, it is independent of the Lipschitz-like constants of the bifunction g and without any line search procedure. We begin by stating some basic assumptions on the bifunctions f and g.

Assumption 1. We assume that the bifunction $f : H \times H$ satisfies the following:

(A1) f is β-strongly monotone.
(A2) $f(u,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $u \in H$.
(A3) $f(\cdot, v)$ is weakly upper semicontinuous on H for every fixed $v \in H$.
(A4) f is k-Lipschitz-continuous, i.e., for each $u, v \in H$, there exists a constant $k > 0$ such that

$$\mathcal{H}(\partial_2 f(u,u), \partial_2 f(v,v)) \leq k\|u-v\|.$$

Assumption 2. Also, we assume that the bifunction $g : H \times H$ satisfies the following:

(B1) g is pseudomonotone on C with respect to $Sol(f,C)$, i.e.,

$$g(x,p) \leq 0, \quad \forall x \in C, p \in Sol(f,C).$$

(B2) $g(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
(B3) $g(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.

(B4) g satisfies Lipschitz-like condition (1.2) and the constants L_1, L_2 do not necessary need to be known.

Remark 3.1. Note that when the bifunction g satisfies Assumption (B1), (B2) and (B3), then the solution set $Sol(g,C)$ is closed and convex; see [7, Proposition 3.1, 3.2]. Also, if f satisfies Assumption 1 and g satisfies Assumption 2, and in addition, $Sol(g,C)$ is nonempty, then the BEP (1.3) has a unique solution; see [56].

Next, we present our algorithm as follows:

Algorithm 3.2. *An Inertial Subgradient Extragradient Method with Self-adaptive technique (ISEMS)*

Initialization: *Choose* $x_0, x_1 \in H$, $\alpha \geq 3$, $\lambda_0 > 0$, $\sigma, \theta \in (0,1)$, $0 < \mu < \frac{2\beta}{k^2}$, $\{\alpha_n\}, \{\delta_n\}, \{\varepsilon_n\} \subset (0,1)$ *satisfying*

$$\begin{cases} \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \\ 0 < \delta_n \leq 1 - \alpha_n, \lim_{n\to\infty} \delta_n = \delta < 1, \quad \forall n \geq 0, \\ \varepsilon_n = \circ(\alpha_n) \text{ which means } \lim\limits_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0. \end{cases} \tag{3.1}$$

Set $n = 0$ and go to Step 1.
Step 1: *Given the $(n-1)$-th and n-th iterates, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$ where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\max\{\|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\|\}}\right\}, & \text{if} \quad x_n \neq x_{n-1}, \\ \theta, \quad \text{otherwise}. \end{cases} \tag{3.2}$$

Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2: *Compute*

$$\begin{cases} y_n = \underset{y\in C}{argmin} \left\{\lambda_n g(w_n, y) + \frac{1}{2}\|w_n - y\|^2\right\}, \\ z_n = \underset{y\in T_n}{argmin} \left\{\lambda_n g(y_n, y) + \frac{1}{2}\|w_n - y_n\|^2\right\}, \end{cases}$$

where $T_n = \{x \in H : \langle x_n - \lambda_n\xi_n - y_n, x - y_n\rangle \leq 0\}$ and $\xi_n \in \partial_2 g(w_n, y_n)$ is chosen such that $w_n - \lambda_n\xi_n - y_n \in N_C(y_n)$.
Step 3: *Pick $\rho_n \in \partial_2 f(z_n, \cdot)(z_n)$ and compute the $(n+1)$-th iterate via*

$$x_{n+1} = \delta_n w_n + (1 - \delta_n)z_n - \alpha_n\mu\rho_n,$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\sigma(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(g(w_n,z_n) + g(w_n,y_n) - g(y_n,z_n))}\right\}, \\ \qquad \text{if} \quad g(w_n, z_n) + g(w_n, y_n) - g(y_n, z_n) > 0, \\ \lambda_n, \quad \text{otherwise}. \end{cases} \tag{3.3}$$

Set $n = n + 1$ and go back to Step 1.

Remark 3.3. From (3.2), we note that

$$\theta_n\|x_n - x_{n-1}\|^2 \leq \bar{\theta}_n\|x_n - x_{n-1}\|^2 \leq \varepsilon_n, \ n \geq 1$$

also,

$$\theta_n\|x_n - x_{n-1}\| \leq \bar{\theta}_n\|x_n - x_{n-1}\| \leq \varepsilon_n, \ n \geq 1.$$

Therefore, it follows from $\lim\limits_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$, that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|^2 \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$$

and

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0.$$

Next, we show that the stepsize $\{\lambda_n\}$ defined by (3.3) is well-defined.

Lemma 3.4. *The sequence* $\{\lambda_n\}$ *generated by* (3.3) *is monotonically non-increasing and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \frac{\sigma}{2\max\{L_1, L_2\}}.$$

Proof. Clearly, $\{\lambda_n\}$ is monotonically non-increasing. Also, since g satisfies condition (B4), we have

$$\begin{aligned}\frac{\sigma(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(g(w_n, z_n) - g(w_n, y_n) - g(y_n, z_n))} &\geq \frac{\sigma(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(L_1\|w_n - y_n\|^2 + L_2\|y_n - z_n\|^2)} \\ &\geq \frac{\sigma}{2\max\{L_1, L_2\}}.\end{aligned}$$

Hence $\{\lambda_n\}$ is bounded below by $\frac{\sigma}{2\max\{L_1,L_2\}}$. This implies that there exists

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \frac{\sigma}{2\max\{L_1, L_2\}}.$$

□

Lemma 3.5. *Suppose* $Sol(g, C)$ *is nonempty. Let* $\{x_n\}, \{y_n\}, \{z_n\}$ *be a sequences generated by Algorithm* 3.2*. Then, the following estimate holds for all* $n \geq 0$ *and* $p \in Sol(g, C)$*.:*

$$\|z_n - p\|^2 \leq \|w_n - y\|^2 - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)\|w_n - y_n\|^2 - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)\|y_n - z_n\|^2.$$

Proof. First we show that $C \subseteq T_n$. Since $y_n \in C$, it follows from Lemma 2.1 and Step 2 of Algorithm 3.2 that

$$0 \in \partial_2\left(\lambda_n g(w_n, \cdot) + \frac{1}{2}\|w_n - y\|^2\right)(y_n) + N_C(y_n), \quad \forall y \in C.$$

Then, there exists $\xi_n' \in \partial_2 g(w_n, \cdot)(y_n)$ and $\rho \in N_C(y_n)$ such that

$$\lambda_n\xi_n' + y_n - w_n + \rho = 0.$$

Since $\rho \in N_C(y_n)$, then $\langle \rho, y - y_n\rangle \leq 0$, for all $y \in C$. Therefore

$$\langle w_n - \lambda_n\xi_n' - y_n, y - y_n\rangle \leq 0, \quad \forall y \in C.$$

This implies that $C \subseteq T_n$. Also since $z_n \in T_n$, then from Lemma 2.1, we get

$$0 \in \partial_2\left(\lambda_n g(y_n, y) + \frac{1}{2}\|w_n - y\|^2\right)(z_n) + N_{T_n}(z_n), \quad \forall y \in H.$$

Thus, there exists $\bar{\xi}_n \in \partial_2 g(y_n, \cdot)(z_n)$ and $\bar{\rho} \in N_{T_n}(z_n)$ such that

$$\lambda_n\bar{\xi}_n + z_n - w_n + \bar{\rho} = 0.$$

Note that $\langle \bar{\rho}, y - z_n\rangle \leq 0$, for all $y \in T_n$. Hence

$$\lambda_n\langle\bar{\xi}_n, y - z_n\rangle \geq \langle w_n - z_n, y - z_n\rangle \quad \forall y \in T_n.$$

Since $\bar{\xi}_n \in \partial_2 g(y_n, z_n)$, then

$$g(y_n, y) - g(y_n, z_n) \geq \langle\bar{\xi}_n, y - z_n\rangle \forall y \in H.$$

Substituting $y = p$ into the last two inequalities, we obtain

$$\lambda_n\langle\bar{\xi}_n, p - z_n\rangle \geq \langle w_n - z_n, p - z_n\rangle. \tag{3.4}$$

Also, since $\bar{\xi}_n \in \partial_2 g(y_n, \cdot)(z_n)$, then

$$g(y_n, y) - g(y_n, z_n) \geq \langle\bar{\xi}_n, y - z_n\rangle \quad \forall y \in H. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\lambda_n(g(y_n, p) - g(y_n, z_n)) \geq \langle w_n - z_n, p - z_n\rangle.$$

Since g satisfies condition (B1), we get $g(y_n, p) \leq 0$, hence

$$-\lambda_n g(y_n, z_n) \geq \langle w_n - z_n, p - z_n\rangle. \tag{3.6}$$

Furthermore, since $z_n \in T_n$, then

$$\langle w_n - \lambda_n\xi_n - y_n, z_n - y_n\rangle \leq 0.$$

Hence

$$\langle w_n - y_n, z_n - y_n\rangle \leq \lambda_n\langle\xi_n, z_n - y_n\rangle.$$

Since $\xi_n \in \partial_2 g(w_n, \cdot)(y_n)$, then

$$g(w_n, y) - g(w_n, y_n) \geq \langle \xi_n, y - y_n \rangle, \quad \forall y \in H.$$

Therefore

$$\begin{aligned} \lambda_n(g(w_n, z_n) - g(w_n, y_n)) &\geq \lambda_n \langle \xi_n, z_n - y_n \rangle \\ &\geq \langle w_n - y_n, z_n - y_n \rangle. \end{aligned} \tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$\begin{aligned} 2\lambda_n(g(w_n, z_n) - g(w_n, y_n) - g(y_n, z_n)) &\geq 2\langle w_n - y_n, z_n - y_n \rangle + 2\langle w_n - z_n, p - z_n \rangle \\ &= \|w_n - y_n\|^2 + \|z_n - y_n\|^2 - \|w_n - z_n\|^2 \\ &\quad + \|w_n - z_n\|^2 + \|z_n - p\|^2 - \|w_n - p\|^2. \end{aligned}$$

Thus

$$\begin{aligned} \|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 - \|z_n - y_n\|^2 \\ &\quad + 2\lambda_n(g(w_n, z_n) - g(w_n, y_n) - g(y_n, z_n)). \end{aligned}$$

Using the definition of λ_{n+1} in (3.3), we have

$$\begin{aligned} \|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 - \|z_n - y_n\|^2 \\ &\quad + 2\lambda_n(g(w_n, z_n) - g(w_n, y_n) - g(y_n, z_n)) \\ &\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 - \|z_n - y_n\|^2 \\ &\quad + \frac{\lambda_n}{\lambda_{n+1}}\sigma(\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &= \|w_n - p\|^2 - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)\|w_n - y_n\|^2 \\ &\quad - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)\|y_n - z_n\|^2. \end{aligned}$$

This completes the proof. □

Lemma 3.6. *Suppose $Sol(g, C)$ is nonempty and let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then $\{x_n\}$ is bounded and consequently, the sequences $\{y_n\}, \{z_n\}$ are bounded.*

Proof. Let $p \in Sol(g, C)$. First, since $\lambda_n = \lambda$ exists (from Lemma 3.4), then there exists $n_0 \in \mathbb{N}$ such that $\lim_{n\to\infty}\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right) = 1 - \sigma$ for every $n \geq n_0$. Hence, from Lemma 3.5, we have

$$\|z_n - p\|^2 \leq \|w_n - p\|^2.$$

Also, from Algorithm 3.2, we have

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned}$$

Using (A1), (A4) and Lemma 2.2, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\delta_n w_n + (1 - \delta_n)z_n - \alpha_n\mu\rho_n - p\| \\ &= \|\delta_n w_n + (1 - \delta_n)z_n - \alpha_n\mu\rho_n - \delta_n - (1 - \delta_n)p + \alpha_n\mu v - \alpha_n\mu v\| \\ &\leq \|(1 - \delta_n)z_n - \alpha_n\mu\rho_n - [(1 - \delta_n)p - \alpha_n\mu p]\| + \delta_n\|w_n - p\| + \alpha_n\mu\|v\| \\ &\leq (1 - \delta_n - \alpha_n\bar{\tau})\|z_n - p\| + \delta_n\|w_n - p\| + \alpha_n\mu\|v\| \\ &\leq (1 - \delta_n - \alpha_n\bar{\tau})\|w_n - p\| + \delta_n\|w_n - p\| + \alpha_n\mu\|v\| \\ &\leq (1 - \alpha_n\bar{\tau})[\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] + \alpha_n\mu\|v\| \\ &\leq (1 - \alpha_n\bar{\tau})\|x_n - p\| + \theta_n\|x_n - x_{n-1}\| + \alpha_n\mu\|v\| \\ &= (1 - \alpha_n\bar{\tau})\|x_n - p\| + \alpha_n\bar{\tau}\left(\frac{\theta_n}{\alpha_n} \times \frac{\|x_n - x_{n-1}\|}{\bar{\tau}} + \frac{\mu\|v\|}{\bar{\tau}}\right), \end{aligned} \tag{3.8}$$

where $\bar{\tau} = 1 - \sqrt{1 - \mu(2\beta - \mu k^2)} \in (0, 1]$. Now, putting $M = \max\left\{\sup_{n\geq 0}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|, \frac{\mu\|v\|}{\bar{\tau}}\right\}$. Then from (3.8), we have

$$\|x_{n+1} - p\| \leq (1 - \alpha_n\bar{\tau})\|x_n - p\| + \alpha_n\bar{\tau}M.$$

Hence, by induction and Lemma 2.3 (i), we obtain that $\{\|x_n - p\|\}$ is bounded. This implies that $\{x_n\}$ is bounded. Consequently, we have $\{y_n\}$ and $\{z_n\}$ are bounded. □

Lemma 3.7. *Suppose $Sol(g, C)$ is nonempty and let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then $\{x_n\}$ satisfies the following estimate:*

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n,$$

where $s_n = \|x_n - p\|^2$, $a_n = \alpha_n \bar{\tau}$, $b_n = 2\langle v, p - x_{n+1}\rangle + \frac{\theta_n}{\alpha_n} \cdot \frac{\|x_n - x_{n-1}\|}{\bar{\tau}} M^*$ for some $M^* > 0$ and $\forall n \geq 0$, $p \in Sol(g, C)$.

Proof. Let $p \in Sol(g, C)$, using (2.4) and the fact that $\theta_n + \theta_n^2 \leq 2\theta_n$, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n(\|x_n - p\|^2 + \|x_n - x_{n-1}\|^2 - \|x_{n-1} - p\|^2) + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n(\|x_n - p\| + \|x_{n-1} - p\|)\|x_n - x_{n-1}\| + 2\theta_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M^*, \qquad (3.9)
\end{aligned}$$

where $M^* = \sup_{n\geq 0}\Big(\|x_n - p\| + \|x_{n-1} - p\| + 2\|x_n - x_{n-1}\|\Big)$. Hence, from (2.3) and Lemma 2.2, we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\delta_n w_n + (1 - \delta_n)z_n - \alpha_n\bar{\tau}\rho_n - p\|^2 \\
&= \|(1 - \delta_n)z_n - \alpha_n\bar{\tau}\rho_n - [(1 - \delta_n)p - \alpha_n\bar{\tau}v] + \delta_n(w_n - p) - \alpha_n\bar{\tau}v\|^2 \\
&\leq \|(1 - \delta_n)z_n - \alpha_n\bar{\tau}\rho_n - [(1 - \delta_n)p - \alpha_n\bar{\tau}v] + \delta_n(w_n - p)\|^2 + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&\leq \|(1 - \delta_n)z_n - \alpha_n\bar{\tau}\rho_n - [(1 - \delta_n)p - \alpha_n\bar{\tau}v]\|^2 + \|\delta_n(w_n - p)\|^2 \\
&\quad + 2\delta_n\|(1 - \delta_n)z_n - \alpha_n\bar{\tau}\rho_n - [(1 - \delta_n)p - \alpha_n\bar{\tau}v]\|\|w_n - p\| + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&\leq (1 - \delta_n - \alpha_n\bar{\tau})^2\|z_n - p\|^2 + \delta_n^2\|w_n - p\|^2 \\
&\quad + 2(1 - \delta_n - \alpha_n\bar{\tau})\delta_n\|z_n - p\|\|w_n - p\| + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&\leq (1 - \delta_n - \alpha_n\bar{\tau})^2\|z_n - p\|^2 + \delta_n^2\|w_n - p\|^2 \\
&\quad + (1 - \delta_n - \alpha_n\bar{\tau})\delta_n(\|z_n - p\|^2 + \|w_n - p\|^2) + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&= (1 - \delta_n - \alpha_n\bar{\tau})(1 - \alpha_n\bar{\tau})\|z_n - p\|^2 + \delta_n(1 - \alpha_n\bar{\tau})\|w_n - p\|^2 + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&\leq (1 - \alpha_n\bar{\tau})^2\|w_n - p\|^2 + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&\leq (1 - \alpha_n\bar{\tau})[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M^*] + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&= (1 - \alpha_n\bar{\tau})\|x_n - p\|^2 + (1 - \alpha_n\bar{\tau})\theta_n\|x_n - x_{n-1}\|M^* + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle \\
&\leq (1 - \alpha_n\bar{\tau})\|x_n - p\|^2 + \alpha_n\bar{\tau}\left(2\langle v, p - x_{n+1}\rangle + \frac{\theta_n}{\alpha_n} \times \frac{\|x_n - x_{n-1}\|}{\bar{\tau}}M^*\right).
\end{aligned}$$

This completes the proof.

□

We now present our main convergence theorem.

Theorem 3.8. *Suppose $Sol(g, C)$ is nonempty. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to a solution of the BEP (1.3).*

Proof. Let $p \in Sol(g, C)$, then $f(p, y) \geq 0$ for all $y \in Sol(g, C)$. This implies that p is a minimum of the convex function $f(p, y)$ over $Sol(g, C)$. Hence, by Lemma 2.2, we obtain

$$0 \in \partial_2 f(p, \cdot)(p) + N_{Sol(g,C)}(p).$$

Hence, there exists $v \in \partial_2 f(p, \cdot)(p)$ such that

$$\langle v, z - p\rangle \geq 0 \quad \forall z \in Sol(g, C). \qquad (3.10)$$

Now, let $s_n = \|x_n - p\|^2$. We consider the following possible cases.

Case I: Assume that there exists $n_0 \in \mathbb{N}$ such that $\{s_n\}$ is monotonically decreasing for all $n \geq n_0$. Since $\{s_n\}$ is bounded, then $\lim_{n\to\infty} s_n$ exists which implies that $s_n - s_{n+1} \to 0$ as $n \to \infty$. Now from Lemma 3.5, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\delta_n w_n + (1-\delta_n)z_n - \alpha_n \bar{\tau}\rho_n - p\|^2\\
&\leq (1-\delta_n - \alpha_n\bar{\tau})(1-\alpha_n\bar{\tau})\|z_n - p\|^2 + \delta_n(1-\alpha_n\bar{\tau})\|w_n - p\|^2 + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle\\
&\leq (1-\delta_n - \alpha_n\bar{\tau})(1-\alpha_n\bar{\tau})\left[\|w_n - p\|^2 - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)(\|w_n - y_n\|^2 + \|y_n - z_n\|^2)\right]\\
&\quad +\delta_n(1-\alpha_n\bar{\tau})\|w_n - p\|^2 + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle\\
&\leq (1-\alpha_n\bar{\tau})\|w_n - p\|^2\\
&\quad -(1-\delta_n - \alpha_n\bar{\tau})(1-\alpha_n\bar{\tau})\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)(\|w_n - y_n\|^2 + \|y_n - z_n\|^2)\\
&\quad +2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle\\
&\leq (1-\alpha_n\bar{\tau})\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M^*\\
&\quad -(1-\delta_n - \alpha_n\tau - \Gamma_n)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)(\|w_n - y_n\|^2 + \|y_n - z_n\|^2)\\
&\quad +2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle,
\end{aligned}$$

where $\Gamma_n = \alpha_n\bar{\tau}(1-\delta_n - \alpha_n\bar{\tau})$. This implies that

$$\begin{aligned}
&(1-\delta_n - \alpha_n\tau - \Gamma_n)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right)(\|w_n - y_n\|^2 + \|y_n - z_n\|^2)\\
&\qquad\qquad \leq s_n - s_{n+1} - \alpha_n\bar{\tau}s_n + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|M^* + 2\alpha_n\bar{\tau}\langle v, p - x_{n+1}\rangle.
\end{aligned}$$

Note that $\Gamma_n \to 0$ (since $\alpha_n \to 0$) as $n \to \infty$ and $\lim_{n\to\infty}\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\sigma\right) = 1 - \sigma$. Also, since $\lim_{n\to\infty}\delta_n = \delta < 1$ and $\sigma \in (0,1)$, then passing limit as $n \to \infty$ to the last inequality above, we obtain

$$\lim_{n\to\infty}(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) = 0.$$

This implies that

$$\lim_{n\to\infty}\|w_n - y_n\| = \lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{3.11}$$

Hence

$$\lim_{n\to\infty}\|w_n - z_n\| = 0. \tag{3.12}$$

Moreover

$$\begin{aligned}
\|w_n - x_n\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x_n\|\\
&= \theta_n\|x_n - x_{n-1}\|.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Since $\|w_n - x_n\| \to 0$, then $w_{n_k} \rightharpoonup \hat{x}$, hence from (3.11), we have $y_{n_k} \rightharpoonup \hat{x}$ and $z_{n_k} \rightharpoonup \hat{x}$. We now show that $\hat{x} \in Sol(g, C)$. From the definition of $\{y_n\}$ and Lemma 2.2, we have

$$0 \in \partial_2\left(\lambda_n g(w_n, y) + \frac{1}{2}\|w_n - y\|^2\right)(y_n) + N_C(y_n).$$

Then, there exists $\bar{\rho} \in N_C(y_n)$ and $\xi_n \in \partial_2 g(w_n, \cdot)(y_n)$ such that

$$\lambda_n\xi_n + y_n - w_n + \bar{\rho} = 0. \tag{3.13}$$

Moreover, $\bar{\rho} \in N_C(y_n)$ implies that $\langle \bar{\rho}, y - y_n\rangle \leq 0$, for all $y \in C$. Then, it follows from (3.13) that

$$\lambda_n\langle \rho, y - y_n\rangle \geq \langle w_n - y_n, y - y_n\rangle, \quad \forall y \in C.$$

Also, since $\xi_n \in \partial_2 g(w_n, \cdot)(y_n)$, then

$$g(w_n, y) - g(w_n, y_n) \geq \langle \bar{\rho}, y - y_n\rangle, \quad \forall y \in H.$$

Therefore

$$\lambda_n\Big(g(w_n, y) - g(w_n, y_n)\Big) \geq \langle w_n - y_n, y - y_n\rangle, \quad \forall y \in C.$$

This means that

$$\lambda_{n_k}\Big(g(w_{n_k},y)-g(w_{n_k},y_{n_k})\Big)\geq\langle w_{n_k}-y_{n_k},y-y_{n_k}\rangle,\quad \forall y\in C.$$

Passing limit to the above inequality, using condition (B2) and (B3), and since $\|w_{n_k}-y_{n_k}\|\to 0$, we have

$$g(\hat{x},y)\geq 0,\quad \forall y\in C.$$

Hence $\hat{x}\in Sol(g,C)$. Next, we show that $\limsup_{n\to\infty}\langle v,p-x_{n_k+1}\rangle\leq 0$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle v,p-x_{n+1}\rangle=\lim_{k\to\infty}\langle v,p-x_{n_k}\rangle.$$

Then, from (3.10), we have

$$\limsup_{n\to\infty}\langle v,p-x_{n+1}\rangle=\lim_{k\to\infty}\langle v,p-x_{n_k}\rangle=\langle v,p-\hat{x}\rangle\leq 0. \tag{3.14}$$

From Lemma 3.7, we have $b_n=2\langle v,p-x_{n+1}\rangle+\frac{\theta_n}{\alpha_n}\cdot\frac{\|x_n-x_{n-1}\|}{\bar{\tau}}M^*$. Since $\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\to 0$ as $n\to\infty$, then we have from (3.14) that $\limsup_{n\to\infty}b_n\leq 0$. Using Lemma 2.3 (ii) and Lemma 3.7, we have that $\lim_{n\to\infty}\|x_n-p\|=0$. This implies that $\{x_n\}$ converges strongly to p as $n\to\infty$.

Case II: Assume that $\{s_n\}$ is not monotonically decreasing. That is, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $s_{n_k}\leq s_{n_k+1}$ for all $k\in\mathbb{N}$. By Lemma 2.4, there exists a non-decreasing sequence $\{\tau(n)\}$ of $\mathbb{N}$ such that $\tau(n)\to\infty$, $s_{\tau(n)}\leq s_{\tau(n)+1}$ and $s_n\leq s_{\tau(n)}$ for sufficiently large $n\in\mathbb{N}$. More so, since $\{s_{\tau(n)}\}$ is bounded, following similar argument as in Case I, we obtain

$$\lim_{n\to\infty}\|w_{\tau(n)}-y_{\tau(n)}\|=\lim_{n\to\infty}\|y_{\tau(n)}-z_{\tau(n)}\|=\lim_{n\to\infty}\|w_{\tau(n)}-x_{\tau(n)}\|=0.$$

Also

$$\limsup_{n\to\infty}\langle v,p-x_{\tau(n)+1}\rangle\leq 0.$$

Hence, from Lemma 3.7, we get

$$\begin{aligned}0&\leq s_{\tau(n)+1}-s_{\tau(n)}\\&\leq(1-a_{\tau(n)})s_{\tau(n)}+a_{\tau(n)}b_{\tau(n)}-s_{\tau(n)},\end{aligned}$$

where $a_{\tau(n)}=\alpha_{\tau(n)}\bar{\tau}$, $b_{\tau(n)}=2\langle v,p-x_{\tau(n)+1}\rangle+\frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\cdot\frac{\|x_{\tau(n)}-x_{\tau(n)-1}\|}{\bar{\tau}}M^*$ for some $M^*>0$ and $\forall n\geq n_0$. Thus, we have

$$s_{\tau(n)}\leq b_{\tau(n)}.$$

This implies that

$$\limsup_{n\to\infty}\|x_{\tau(n)}-p\|=0.$$

As a consequence, we obtain that for all $n\geq n_0$,

$$0\leq\|x_n-p\|^2\leq\|x_{\tau(n)+1}-p\|^2.$$

Therefore $\lim_{n\to\infty}\|x_n-p\|=0$. This implies that $\{x_n\}$ converges strongly to p. This completes the proof. □

4. Numerical experiments

In this section, we present to numerical experiments to illustrate the performance of the proposed algorithm. We compare the convergence behaviour of Algorithm 3.2 with Algorithm 1.2 (namely, EGM) and Algorithm 1.3 (namely, EGML).

Example 4.1. This example is an equilibrium problem that comes from Nash-Cournot oligopolistic electricity market equilibrium model which has been considered as real-world problem by many authors (see, for example, [55, 57]).
Suppose there are n^c (here we take $n^c=3$) generating companies, each company i ($i=1,2,3$) (Com.) has several I_i generating units (Gen.) (here, we take $I_1=\{1\},I_2=\{2,3\}$ and $I_3=\{4,5,6\}$). We take n^g (here, $n^g=6$) to be the number of all generating units and x the vector whose entry x_j ($j=1,\ldots,6$) stands for the

TABLE 1. The lower and upper bounds of the power generation of the generating units, companies and other parameters.

Com.	Gen.	$x^g_{\min}$	$x^g_{\max}$	$x^c_{\min}$	$x^c_{\max}$	α^0_j	β^0_j	γ^0_j	α^1_j	β^1_j	γ^1_j
1	1	0	80	0	80	0.0400	2.00	0	2.00	1.00	25.0000
2	2	0	80	0	130	0.0350	1.75	0	1.75	1.00	28.5714
2	3	0	50	0	130	0.1250	1.00	0	1.00	1.00	8.0000
3	4	0	55	0	125	0.0116	3.25	0	3.25	1.00	86.2069
3	5	0	30	0	125	0.0500	3.00	0	3.00	1.00	20.0000
3	6	0	40	0	125	0.0500	3.00	0	3.00	1.00	20.0000

power generation of unit j, as shown in Table 1. Using the same ideas in [55, 57], we assume that the price p is a decreasing affine function of σ, where $\sigma = \sum_{j=1}^{n^g} x_j$. Hence,

$$p(x) = 378.4 - 2\sum_{j=1}^{n^g} x_j = p(\sigma).$$

The cost of generating unit j is given as

$$c_j(x_j) := \max\{c^0_j(x_j),\ c^1_j(x_j)\},\ j = 1, 2, \dots, n^g,$$

where $c^0_j(x_j) := \dfrac{\alpha^0_j}{2}x_j^2 + \beta^0_j x_j + \gamma^0_j$ and $c^1_j(x_j) := \alpha^1_j x_j + \dfrac{\beta^1_j}{\beta^1_j + 1}\gamma_j^{\frac{-1}{\beta^1_j}}(x_j)^{\frac{\beta^1_j+1}{\beta^1_j}}$,
$\alpha^k_j, \beta^k_j, \gamma^k_j$ $(k = 0, 1;\ j = 1, \dots, n^g)$ are parameters given in Table 4.1.

The profit gained by company i that owns I_i generating units is

$$f_i(x) = p(\sigma)\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j), \tag{4.1}$$

subject to the constraints $(x^g_{\min})_j \le x_j \le (x^g_{\max})_j$ $(j = 1, \dots, n^g)$, where $(x^g_{\min})_j$ and $(x^g_{\max})_j$ are the lower and upper bounds for the power generation of unit j.
Furthermore, the strategy set of the model is given by

$$C := \{x = (x_1, \cdots, x_{n^g})^T : (x^g_{\min})_j \le x_j \le (x^g_{\max})_j,\ j = 1, \dots, n^g\}.$$

Now, suppose $q^i := (q^i_1, \cdots, q^i_{n^g})^T$ with

$$q^i_j = \begin{cases} 1, & \text{if } j \in I_i \\ 0, & \text{if } j \notin I_i, \end{cases}$$

and $\bar{q}^i := (\bar{q}^i_1, \cdots, \bar{q}^i_{n^g})^T$ with $\bar{q}^i_j := 1 - q^i_j$, $(j = 1, 2, \dots, n^g)$; then, define

$$A := 2\sum_{i=1}^{n^c} \bar{q}^i (q^i)^T, \qquad B := 2\sum_{i=1}^{n^c} q^i (q^i)^T,$$

$$a := -387.4\sum_{i=1}^{n^c} q^i, \text{ and } c(x) := \sum_{j=1}^{n^g} c_j(x_j).$$

Thus, we have that the oligopolistic equilibrium model under consideration can be written as (see [55, Page 155] for details)

$$\text{Find } x^* \in C : g(x^*, y) = [(A + B)x^* + By + a]^T (y - x^*) + c(y) - c(x^*) \ge 0,\ \forall y \in C.$$

Note that g satisfies (A1)-(A4) with $L_1 = L_2 = \frac{1}{2}\|A - B\|$. In addition, we assume that the equilibrium point x^* satisfies an environmental condition $0 \le A(x^*) \le b$, where a_{ij} of matrix $A \in \mathbb{R}^{n_g \times n_c}$ is the environmental pollution caused by company i using j generating unit with constraint set

$$S = \{u \in \mathbb{R}^{n_c} : u^c_{\min} \le u_i \le u^c_{\max}\},$$

where $u^c_{\min}$ and $u^c_{\max}$ are listed in Table 2. Consequently, the total environmental pollution caused by company i is $\sum_{i=1}^{n_c} a_{ij} x_i$. However, since it is possible for a solution of the EP say $x^\dagger$ not to satisfy the environment

TABLE 2. The lower and upper bounds of the pollution constraint.

	1	2	3
$u^c_{\min}$	0	0	0
$u^c_{\max}$	25	15	20

TABLE 3. Computational result for Example 4.1.

		Algorithm 3.2	Algorithm 1.2	Algorithm 1.3
Case I	Iter.	30	67	49
	Time (s)	0.4841	1.0354	0.5851
Case II	Iter.	30	66	58
	Time (s)	0.5831	1.8095	0.6896
Case III	Iter.	28	140	58
	Time (s)	0.6399	3.1590	1.6427
Case IV	Iter.	28	68	45
	Time (s)	0.4654	1.9537	1.1402

constraint $0 \leq A(x^*) \leq b$, our interest is then to find an equilibrium point x^* satisfying the constraint which is nearest to $x^\dagger$. In this case, we can defined an operator $B : \mathbb{R}^6 \to \mathbb{R}^6$ such that $B(x) = x - x^\dagger$ and seek

$$f(x^*, y) = \langle B(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C := \{x \in \mathbb{R}^{n_g} : (x^g_{\min})_j \leq x_j \leq (x^g_{\max})_j, \ j = 1, \dots, n^g\}.$$

It is easy to see that f is 1-strongly monotone and 1-Lipschitz continuous (see, e.g. [55]), More so, $f(x, x) = 0$ for all $x \in C$ and Assumptions (A2)-(A4) are satisfied. We compare the performance of Algorithm 3.2 with Algorithm 1.2 and 1.3 choosing the following parameters: For Algorithm 3.2, we take $\alpha_n = \frac{1}{n+1}, \delta_n = \frac{n}{2n+3}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\alpha = 3$, $\sigma = 0.26, \mu = 1$; For Algorithm 1.2, we take $\frac{1}{n+1}, \eta_n = \frac{n}{2n+3}$, $\lambda_n = \frac{1}{6c_1}$; and for Algorithm 1.3, we take $\mu = 1, \rho = 1.14, \gamma = 0.25, \alpha_n = \frac{1}{n+1}$, $\lambda = \frac{1}{66}, \xi_n = \frac{1}{80}$. We use different initial points which are generated randomly in $\mathbb{R}^6$ for each algorithm and $D_n = \|x_{n+1} - x_n\| < 10^{-4}$ as stopping criterion in the numerical computation. We perform the experiment for four different cases and compare the number of iteration and the time taken by each algorithm in each case. Table 3 shows that our Algorithm ISEMS performs better than Algorithm 1.2 and 1.3. Figure 1 show the graph of D_n against number of iterations for each algorithm.

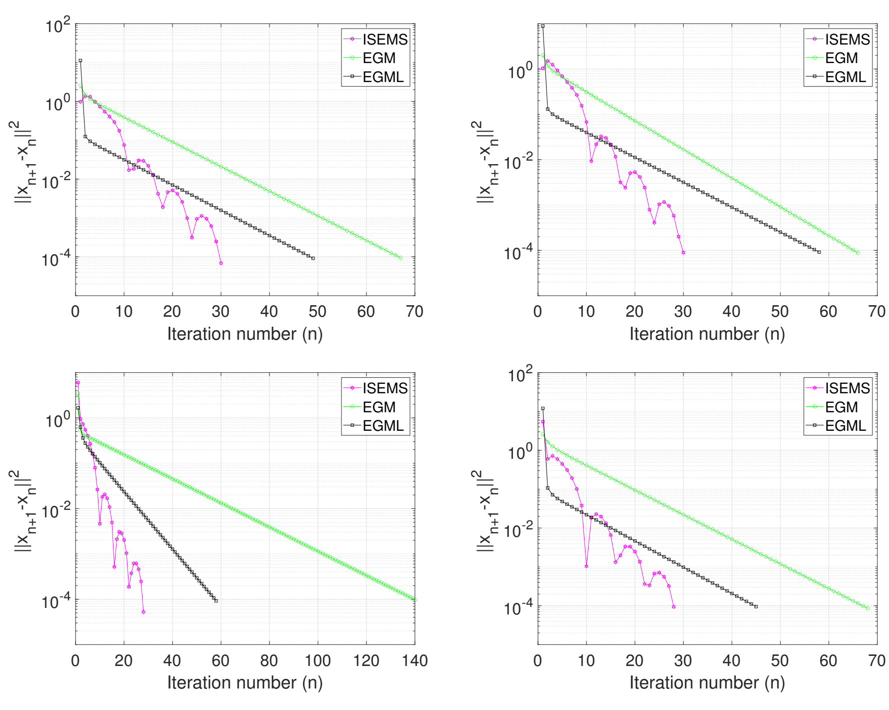

FIGURE 1. Example 4.1, Case I – Case IV.

TABLE 4. Computational result for Example 4.2.

		Algorithm 3.2	Algorithm 1.2	Algorithm 1.3
$m = 5$	Iter.	8	108	46
	Time (s)	0.0357	3.1297	1.2685
$m = 10$	Iter.	8	158	46
	Time (s)	0.0677	4.4182	1.2022
$m = 30$	Iter.	9	204	46
	Time (s)	0.2691	4.4182	1.3518
$m = 50$	Iter.	10	231	46
	Time (s)	0.3357	12.0182	1.3547

Example 4.2. Let $H = \mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n : -5 \le x_i \le 5, \quad \forall i = 1, 2, \dots, n\}$. We define the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$g(x, y) = \langle Px + Qy, y - x\rangle \forall x, y \in \mathbb{R}^n,$$

where P and Q are randomly symmetric positive semidefinite matrices such that $P - Q$ is positive definite. It is easy to see that g is pseudomonotone and Lipschitz-type continuous with $L_1 = L_2 = \frac{1}{2}\|P - Q\|$. More so, g satisfies condition (B1)-(B4). Furthermore, let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x, y) = \langle Sx + Ty, y - x\rangle \quad \forall x, y \in \mathbb{R}^n,$$

where S and T are positive definite matrices defined by

$$S = N^T N + nI_n \quad \text{and} \quad T = S + M^T M + nI_n,$$

M, N are $n \times n$ matrices and I_n is the identity matrix. It is clear that f satisfies condition (A1) - (A4); see, e.g. [59]. We choose the following parameters and compare the performance of our Algorithm 3.2 with Algorithms 1.2 and 1.3 respectively. For Algorithm 3.2, we take $\alpha_n = \frac{1}{\sqrt{n+1}}, \delta_n = \frac{5n}{15n+2}, \epsilon_n = \frac{1}{(3n+1)}, \alpha = 3, \sigma = 0.26, \mu = 1$; For Algorithm 1.2, we take $\alpha_n = \frac{1}{\sqrt{n+1}}, \eta_n = \frac{5n}{15n+2}, \lambda_n = \frac{1}{2c_1}$; and for Algorithm 1.3, we take $\mu = 1, \rho = 0.099, \gamma = 0.35, \alpha_n = \frac{1}{\sqrt{n+1}}, \lambda = \frac{1}{8}, \xi_n = \frac{1}{4}$. The initial points are generated randomly using $x_0 = \texttt{rand(n,1)}, x_1 = \texttt{rand(n,1)}$, where $n = 5, 10, 30, 50$. We use $D_n = \|x_{n+1} - x_n\| < 10^{-6}$ as stopping criterion in the computations. The numerical results are shown in Table 4 and Figure 2.

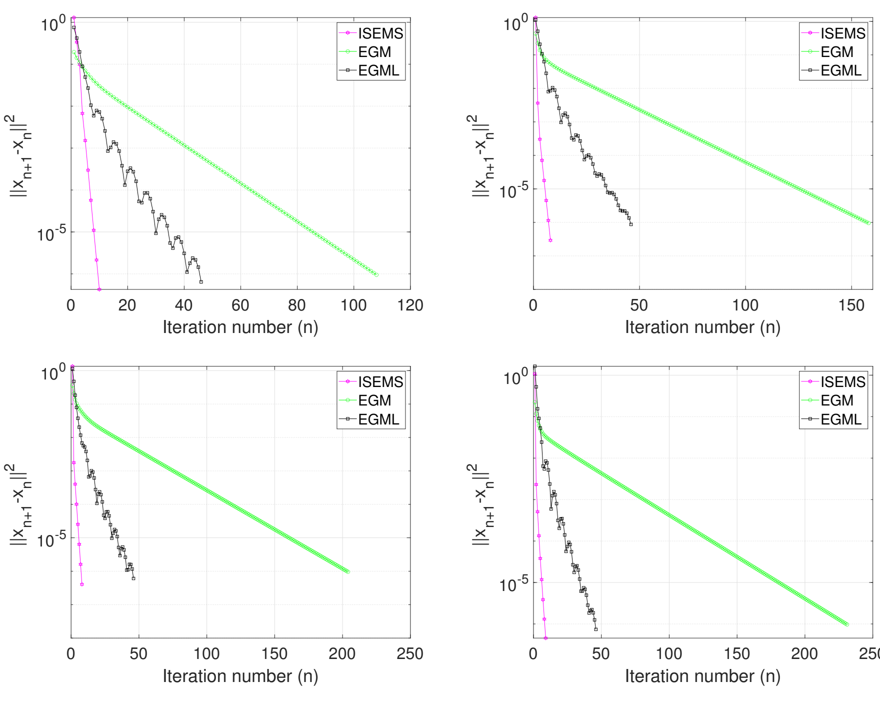

FIGURE 2. Example 4.2, $m = 5, 10, 30, 50$.

TABLE 5. Computational result for Example 4.3.

		Algorithm 3.2	Algorithm 1.2	Algorithm 1.3
Case I	Iter.	8	18	14
	Time (s)	1.6932	10.2086	4.0792
Case II	Iter.	3	20	15
	Time (s)	6.0210	19.8968	9.8621
Case III	Iter.	4	20	14
	Time (s)	1.0219	12.0933	5.1435
Case IV	Iter.	4	22	14
	Time (s)	2.0316	20.3860	5.0876

Example 4.3. Here, we consider an infinite-dimensional Hilbert space. Let $H = L^2([0,1])$ with norm $\|x\| = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ for all $x, y \in L^2([0,1])$. Assume that $C = \{x \in L^2([0,1]) : \int_0^1 \frac{t}{2}x(t)dt = 1\}$. Then the projection onto C is given by

$$P_C x(t) = x(t) - \frac{\int_0^1 \frac{t}{2}x(t)dt - 1}{\int_0^1 \frac{t^2}{2}dt}, \quad x \in L^2([0,1]),\ t \in [0,1].$$

Consider the bifunction $g : L^2([0,1]) \times L^2([0,1]) \to \mathbb{R}$ defined by

$$g(x,y) = \langle T(x), y - x\rangle, \quad \forall x, y \in L^2([0,1])$$

and $T(x) = \int_0^1 \frac{x(t)}{2}dt$ for all $x \in L^2([0,1])$. It is easy to see that g is monotone (hence, pseudomonotone) and satisfies Lipschitz-type continuous with $L_1 = L_2 = \frac{1}{\pi}$. Also, define a mapping $B : L^2([0,1]) \to L^2([0,1]) \to \mathbb{R}$ by $f(x,y) \in \langle Bx, y - x\rangle$, where $B : L^2([0,1]) \to L^2([0,1])$ is given by $Bx(t) = x(t) - x_0$. Then B is 1-strongly monotone and 1-Lipschitz continuous. We chose the following parameters and compare the performance of our Algorithm 3.2 with Algorithm 1.2 and 1.3. For Algorithm 3.2, we take $\alpha_n = \frac{1}{\sqrt{n+1}}, \delta_n = \frac{3n}{5n+7}$, $\epsilon_n = \frac{1}{(n+1)}$, $\alpha = 3$, $\sigma = 0.38, \mu = 1$; For Algorithm 1.2, we take $\alpha_n = \frac{1}{\sqrt{n+1}}, \eta_n = \frac{3n}{5n+7}$, $\lambda_n = \frac{1}{9c_1}$; and for Algorithm 1.3, we take $\mu = 1, \rho = 0.99, \gamma = 0.45, \alpha_n = \frac{1}{\sqrt{n+1}}$, $\lambda = \frac{1}{99}, \xi_n = \frac{1}{8}$. We test the algorithms using the following initial points and $D_n = \|x_{n+1} - x_n\| < 10^{-4}$ as stopping criterion:

Case I: $x_0 = t^2 - 1$, $x_1 = \frac{1}{3}\exp(3t)$;
Case II: $x_0 = t\exp(-2t)$, $x_1 = t^3 + 2t - 1$;
Case III: $x_0 = \frac{1}{2}t^2$, $x_1 = \cos(2t)$;
Case IV: $x_0 = \frac{1}{7}(t^3 - 1)$, $x_1 = \exp(2t)$.

The numerical results are shown in Table 5 and Figure 3.

5. Conclusion

In this paper, we introduced a subgradient extragradient method with self-adaptive technique for solving bilevel equilibrium problem in real Hilbert spaces. The algorithm is designed such that its convergence does not require the prior estimate of the Lipschitz-like constant of the upper level bifunction. More so, the first strongly convex optimization problem is solved over the feasible set while a second strongly convex optimization problem is solved over a constructible half-space which can easily be calculated explicitly. Furthermore, we proved a strong convergence result under some mild conditions and provided some numerical experiments to show the accuracy and efficiency of the proposed method. This improves some existing results on solving bilevel pseudomonotone equilibrium problems in the literature.

Availability of data and materials: Not applicable.

Competing interest: The authors declare that there is not competing interest on the paper.

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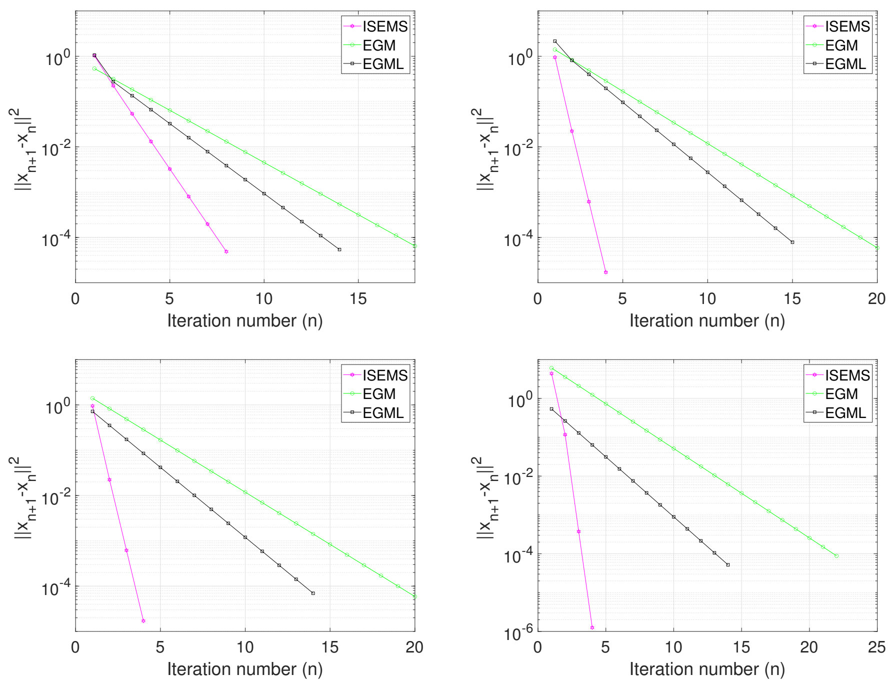

FIGURE 3. Example 4.3, Case I – Case IV.

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